UNIVERSAL PORTFOLIOS GENERATED BY PROBABILITY DISTRIBUTIONS

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By

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ABSTRACT

UNIVERSAL PORTFOLIOS GENERATED BY PROBABILITY DISTRIBUTIONS

Lim Choon Seng

The idea of using a probability distribution to generate a universal portfolio is due to Cover (1991) and Cover and Ordentlich (1996). We generalize this idea to generate a wider class of universal portfolios different from the Dirichlet universal portfolios due to Cover and Ordentlich (1996). The well-known Dirichlet joint distribution can be derived from a transformation of a m random variables, each of which has the gamma($\alpha_i, 1$) distribution for $i = 1, 2, \ldots, m$. The non-Dirichlet Cover-Ordentlich universal portfolio can be generated using any set of m independent distributions different from the gamma distribution. For this purpose, we need to use the Monte Carlo method to simulate the m random variables to generate the universal portfolio. The portfolio is run on some real stock data sets selected from the Kuala Lumpur Stock Exchange to evaluate its empirical performance.

The restriction of the variates lying in the simplex of the vectors in the Cover-Ordentlich universal portfolio can be removed to generate a general class of finite order and moving-order universal portfolios. The low-order universal portfolios contribute to the saving of computer memory and computational time in their implementation. The comparative performance of the universal

portfolios of order 1, 2, 3 generated by some common probability distributions is studied. These portfolios can outperform the Dirichlet Cover-Ordentlich universal portfolio for some data sets, thereby demonstrating the practical significance of the memory and time saving implementation of the such portfolios.

An algorithm to generate the moving-order universal portfolio is proposed for efficient implementation of the portfolio. The moving-order universal portfolios are generated for a few probability distributions and run on some selected three-stock data sets. The ratio of the capitals achieved by the best-constant-rebalanced portfolio to the universal portfolio as a function of the number of trading days is computed for the moving-order universal portfolio to evaluate its performance

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(LIM CHOON SENG)

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APPROVAL SHEET

This dissertation/thesis entitled "**UNIVERSAL PORTFOLIOS GENERATED BY PROBABILITY DISTRIBUTIONS**" was prepared by LIM CHOON SENG and submitted as partial fulfillment of the requirements for the degree of Master of Sciences at Universiti Tunku Abdul Rahman.

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SUBMISSION OF DISSERTATION

It is hereby certified that Lim Choon Seng(ID No: 10UEM02157) has completed this thesis entitled "Universal Portfolios Generated By Probability Distributions" under the supervision of Dr. Tan Choon Peng (Supervisor) from the Department of Mathematical and Actuarial Science, Faculty of Engineering and Sciences, and Dr. Chua Kuan Chin (Co-Supervisor) from the Department of Mathematical and Actuarial Science, Faculty of Engineering and Sciences.

I understand that University will upload softcopy of my thesis in pdf format into UTAR Institutional Repository, which may be made accessible to UTAR community and public.

Yours truly,

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DECLARATION

I hereby declare that the dissertation is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UTAR or other institutions.

(LIM CHOON SENG)

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LIST OF ABBREVIATIONS

BCRP	Best constant rebalanced portfolio
YTL Corp.	Yeo Tiong Lay Corporation
RHB Cap.	Rashid Hussain Berhad Capital
BAT	British American Tobacco
CIMB	Commence Industrial Malaysian Berhad
IOI	Industrial Oxygen Industries
PBB	Public Bank Berhad

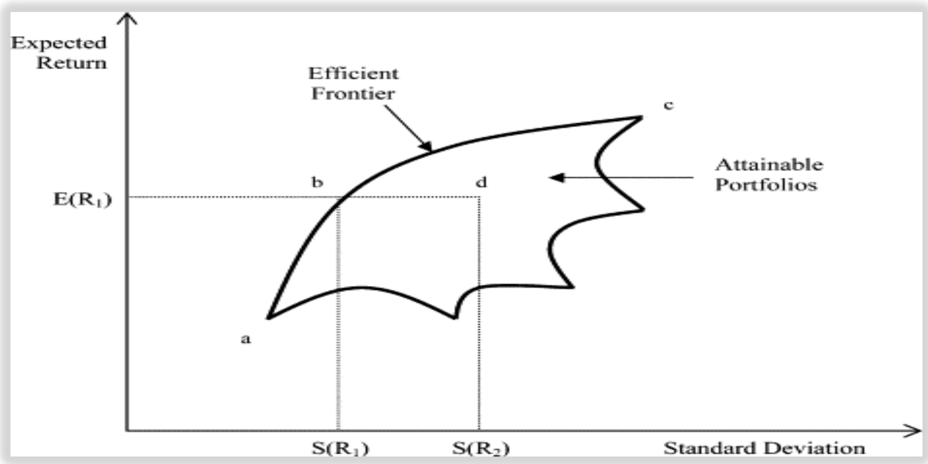
CHAPTER 1

INTRODUCTION

A portfolio is an investment strategy for more than one asset. In this thesis, without loss of generality, we assume that the assets are stocks. Generally, the investment goal is to increase capital return, reduce the risk of investment or both. We invest a portion of the total capital in each asset in the portfolio to achieve the goal. The key concern in portfolio theory is how to allocate the different proportions of the portfolio in everyday investment.

Harry Markowitz introduced the mathematical portfolio theory in 1952. In Markowitz (1952), the expected returns and variances of return are calculated for different brackets of portfolio. It searches for the portfolio that shows the maximum expected return, given that the variance of return is fixed at certain level which is acceptable by the investor; or vise versa, minimize the variance of return by holding the portfolio expected return constant. The Markowitz portfolio theory shows that the selected portfolio can achieve the goal of return by ensuring that it does not exceed the tolerance risk level of the investor. This idea is later known as the expected return-variance rule (EV-rule). The concept is widely used in financial sector for investment and it can be described by a diagram known as the Efficient Frontier curve (Figure 1.1).

Figure 1.1: Efficient frontier curve



However, there are some criteria for the EV-rule to be valid. The first market behavior assumption is that all investors are risk averse but somehow in practice it may not be applicable to all investors. The investors are solely making their decision based on the risk and reward of their investment based on the mean and variance of the returns. Therefore, another important assumption to be taken into account is the probability distribution of the asset prices in the portfolio over the holding period of the mean and variance calculation. In Markowitz (1959), it assumes the normal distribution for investment returns over a single period case.

The study is extended to the multi-period case where the mean and variance of the returns could be varying in time space. Mossin (1969), Fama (1970) and Hakansson (1974) make different assumptions in analyzing the problem. Independence of returns in between periods is one of the common assumptions. In Merton (1990), a more realistic assumption such as lognormal distribution for returns is considered when analyzing the portfolio problem in continuous time.

From the previous research in portfolio theory, we realise that the selection of proportion of assets in the portfolio is not unique which depends on the special characteristics of the market. The relation and movement of assets in the portfolio will directly affect the expected return and variance for analysis. Besides, the assumption in distribution and the independence properties may not always hold for a multi-period investment portfolio.

In 1991, Cover (1991) introduces the uniform universal portfolio in which, unlike the optimal portfolio theory, no assumption is made on the distribution of the asset prices and returns. Instead of the mean and variance of the returns, it focuses on determining the proportions of investment for each asset in the portfolio. It is based on the rebalancing the portfolios, developed in Kelly (1956), Mossin (1969), Thorp (1971) and etc. We allocate different proportions of investment in the assets contained in the portfolio to generate a higher wealth return, which is better than investment in a single asset

The benchmark of the Cover universal portfolio is to generate a portfolio with the total wealth achievable approaching that of the *best constant rebalance portfolio* (BCRP). Cover (1991) showed empirically that based on the New York Stock Exchange data over a 22 year period, universal portfolios perform well on two-stock portfolios. However, in this model of the universal portfolio, the transaction cost is not included. Two-stock universal portfolios are also studied by Tan (2004b).

Another study on universal portfolio is by Helmbold et al. (1998) in employing the multiplicative update rule in the portfolio derived from a frame rule in Kivinen and Warmuth (1997). The algorithm is easier to be implemented and requires less computer memory for implementation compared to Cover's universal portfolio. A study by Cover and Ordentlich (1996) includes the utilization of side information to increase the universal wealth achievable. They focus on the universal portfolio generated by the Dirichlet probability distribution. It was proven that the ratio of the wealth of the BCRP to the universal wealth with side information is bounded under by a polynomial in the number of trading days of the assets.

Ishijima (2001) employs a numerical method in computing the Dirichlet universal portfolio and shows empirically that the results obtained are comparable to the theoretical results. These simulation algorithms give us an idea to generate a wider class of universal portfolios different from the Dirichlet universal portfolio. In Chapter 3, we shall show how to generate universal portfolios using Monte Carlo simulation.

In Cover and Ordentlich (1996), the focus is on the universal portfolio generated by Dirichlet distribution due to theoretical difficulties in considering other distributions. For the universal portfolio generated by a distribution different from the Dirichlet distribution, namely the non-Dirichlet universal portfolio, we propose to compute the portfolio by employing the computation algorithm suggested in Ishijima (2001). We are basically using Monte Carlo simulation to generate the random variables from different probability distributions for the portfolio computation. The details will be presented in Chapter 3. The performance results on some data sets will be studied and compared with the wealth achieved by the Dirichlet universal portfolio.

In Chapter 4, we introduce the finite-order universal portfolio generated by some probability distributions due to Tan (2013). This type of universal portfolio depends only on the positive moments of the generating probability distributions.

The finite-order universal portfolio of order ν depends on the most recent ν days of the stock-price data, assuming that our assets are stocks. In order to reduce the extensive memory and long computational time, we choose the low order universal portfolio, where the order ν is a small integer, say $\nu = 1,2,3$. The universal wealth achieved can be compared with the BCRP wealth and also among the wealths generated by different probability distributions

The finite-order universal portfolio can be generalized to a moving-order universal portfolio, where the order $\nu = n$ and in which n days of past stock-price data is required for computing the portfolio on day $n+1$. This moving-order portfolio is the same as the μ-weighted universal portfolio defined by Cover and Ordentlich (1996). It is more general than the Cover-Ordentlich universal portfolio because the support of the generating probability distributions is not restricted to the simplex of the portfolio vectors.

The moving-order universal portfolio on day $n + 1$ is computed as the normalized sum of all products of the raw moments of the generating distributions and the price-relatives for n days. As the number of trading days increases, the computation will be burdensome as the order of the moments increases. To overcome this problem, we use a recursive algorithm to update the raw moments each day. In summary, we use Tan's (2013) algorithm to update the moment function and product of the price relatives recursively. This algorithm in calculating the moving-order portfolio has the advantage that the normalized sum of the product of the moments and the price-relatives can be calculated recursively, thereby saving the memory-implementation requirements and computation time.

In Chapter 5, we study the ratio of wealth achieved by the BCRP to that of the universal wealth as a function of the number of trading days. This ratio seems to behave like a polynomial in the number of trading days for come stock-price data sets. This asymptotic behavior is useful in evaluating the theoretical performance of a distribution-generated universal portfolio.

CHAPTER 2

REVIEW OF THE COVER-ORDENTLICH UNIVERSAL PORTFOLIO, THE NUMERICAL METHOD AND ALGORITHM FOR ITS IMPLEMENTATION

2.1 Some Basic Definitions

In universal portfolio investment, no assumption is made on the stochastic model of the stock prices. Let $\boldsymbol{x}_n = (x_{n1}, x_{n2}, \dots, x_{nm})$ be the stock-price-relative vector on the n^{th} trading day in an m-stock market, where the x_{ni} is the price relative of the i^{th} stock which is the ratio of the closing price of the i^{th} stock to the opening price on day n, for $i = 1,2, \dots, m$. Clearly $\boldsymbol{x}_n \geq \boldsymbol{0}$. Let $\boldsymbol{x}^n$ denote the sequence of price-relative vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n$. Let $\widehat{\boldsymbol{b}}_n = (\hat{b}_{n1}, \hat{b}_{n2}, \dots, \hat{b}_{n3})$ be the portfolio strategy used on day n, where $\hat{b}_{ni}$ is the proportion of the current wealth invested on stock i on day n, $0 \leq \hat{b}_{ni} \leq 1$ for $i = 1,2, \dots, m$ and $\sum_{i=1}^{m} \hat{b}_{ni} = 1$. The sequence of portfolios is known as a constant rebalanced portfolio if $\hat{b}_{ni} = b_i$ for all $n = 1,2, \dots$, independent of n. We let $\boldsymbol{b} = (b_i)$ denote a constant rebalanced portfolio. The wealth achieved after n trading days by a constant rebalanced portfolio $\boldsymbol{b}$ is given by (2.1) assuming an initial wealth of 1 unit, after knowing the price-relative sequence $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n$:

$$S_n(\boldsymbol{b}) = \prod_{\boldsymbol{i=1}}^{\boldsymbol{n}} \boldsymbol{b}^t \boldsymbol{x}_i. \tag{2.1}$$

(Remark: t refer to the transpose of a vector)

Given these n days performance, the best-constant-rebalanced-portfolio (BCRP) wealth $S_n^*(\boldsymbol{x}^n)$ is defined as:

$$S_n^*(\boldsymbol{x}^n) = \max_{\boldsymbol{b}} S_n(\boldsymbol{b}) \tag{2.2}$$

and the portfolio $\boldsymbol{b}^*$ achieving the maximum in (2.2) is known as the BCRP, namely, $S_n^*(\boldsymbol{x}^{\mathrm{n}}) = S_n(\boldsymbol{b}^*)$.

2.2 The Cover-Ordentlich Universal Portfolio

Cover and Ordentlich (1996) shows that there exists a universal portfolio having the same asymptotic exponential growth rate as the BCRP for any stock vector sequence $\boldsymbol{x}^n$ with or without side information as $n \to \infty$.

Let $\mu(\boldsymbol{b})$ be a probability measure defined on the simplex of portfolio vectors B, where $B = \{\boldsymbol{b} \in R^m : b_i \geq 0, \sum_{i=1}^m b_i = 1\}$, with $\int_{\boldsymbol{B}} d\mu(\boldsymbol{b}) = 1$. The portfolio vector is assumed to be:

$$\widehat{\boldsymbol{b}}_1 = \left(\frac{1}{m}, \frac{1}{m}, \dots, \frac{1}{m}\right). \tag{2.3}$$

When $\mu(\boldsymbol{b})$ is the Dirichlet$\left(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right)$ distribution, $d\boldsymbol{\mu}$ is given by:

$$d\mu = \frac{\Gamma\left(\frac{m}{2}\right)}{\left[\Gamma\left(\frac{1}{2}\right)\right]^m} \prod_{j=1}^{m} b_j^{-\left(\frac{1}{2}\right)} \, d\boldsymbol{b}. \tag{2.4}$$

Then $\{\widehat{\boldsymbol{b}}_{n+1}\}$ generated by $\mu(\boldsymbol{b})$ or the μ-weighted universal portfolio is defined as:

$$\widehat{\boldsymbol{b}}_{n+1} = \frac{\int_{\boldsymbol{B}} \boldsymbol{b} S_n(\boldsymbol{b})\, d\mu(\boldsymbol{b})}{\int_{\boldsymbol{B}} S_n(\boldsymbol{b})\, d\mu(\boldsymbol{b})} \tag{2.5}$$

where $S_0(\boldsymbol{b}) = 1$ and $S_n(\boldsymbol{b}) = \prod_{i=1}^{n} \boldsymbol{b}^t \boldsymbol{x}_i$ given the market sequence $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n$. Then, the total wealth achieved after n trading days by the universal portfolio can be expressed as the average of $S_n(\boldsymbol{b})$ with respect to the measure $\mu(\boldsymbol{b})$:

$$\begin{aligned}
\hat{S}_n(\boldsymbol{x}^n) &= \prod_{i=1}^{n} \widehat{\boldsymbol{b}}_i^t \boldsymbol{x}_i \\
&= \prod_{i=1}^{n} \frac{\left(\int \boldsymbol{b}^t S_{i-1}(\boldsymbol{b})\, d\mu(\boldsymbol{b})\right)\boldsymbol{x}_i}{\int S_{i-1}(\boldsymbol{b}) d\mu(\boldsymbol{b})} \\
&= \prod_{i=1}^{n} \frac{\int S_i(\boldsymbol{b}) d\mu(\boldsymbol{b})}{\int S_{i-1}(\boldsymbol{b}) d\mu(\boldsymbol{b})} \\
&= \int_{\boldsymbol{B}} S_n(\boldsymbol{b})\, d\mu(\boldsymbol{b}).
\end{aligned} \tag{2.6}$$

It was shown in Cover and Ordentlich (1996) that:

$$\lim_{n\to\infty} \sup_{x^n} \left[\frac{1}{n} \ln S_n^* - \frac{1}{n} \ln \hat{S}_n\right] = 0.$$

Cover and Ordentlich (1996) studied the performance of the universal portfolio (2.5) with the availability of the side information. Side information is modeled as a finite-valued variable y that is available at the starting time of each investment period where $y = 1,2, \dots, k$ is known as the state. The constant rebalanced portfolio is extended to the state dependent rebalanced portfolio since the side information is available for each period and the wealth function is:

$$S_n(\boldsymbol{b}(\cdot), \boldsymbol{x}^n | y^n) = \prod_{i=1}^{n} \boldsymbol{b}^t(y_i)\boldsymbol{x}_i \tag{2.7}$$

where $\boldsymbol{b}(1), \boldsymbol{b}(2), \dots, \boldsymbol{b}(k) \in \boldsymbol{\mathcal{B}}$ and $y_i \in \mathcal{Y} = \{1,2,\dots,k\}$. The BCRP wealth is given by:

$$S_n^*(\boldsymbol{x}^n | \boldsymbol{y}^n) = \max_{\boldsymbol{b}(\cdot)\in\mathcal{B}^k} S_n(\boldsymbol{b}(\cdot), \boldsymbol{x}^n | y^n) \tag{2.8}$$

where the exponential growth rate of wealth at time n is:

$$W^*(\boldsymbol{x}^n | y^n) = \frac{1}{n} \log S_n^*(\boldsymbol{x}^n | y^n). \tag{2.9}$$

Again, let $\mu(\boldsymbol{b})$ be a probability measure defined on the simplex B of the portfolio vectors. The μ-weighted universal portfolio with side information is defined as:

$$\widehat{\boldsymbol{b}}_n(y) = \frac{\int_{\mathcal{B}} \boldsymbol{b} S_{n-1}(\boldsymbol{b}|y)\, d\mu(\boldsymbol{b})}{\int_{\mathcal{B}} S_{n-1}(\boldsymbol{b}|y)\, d\mu(\boldsymbol{b})}, y \in \mathcal{Y}, n = 2,3, \dots \,. \tag{2.10}$$

The wealth function with side information is given by:

$$\hat{S}_n(\boldsymbol{x}^n | y^n) = \prod_{i=1}^{n} \widehat{\boldsymbol{b}}_i^t(y_i)\boldsymbol{x}_i. \tag{2.11}$$

Due to (2.6), the wealth function can be expressed as the following along each subsequence $\{i: y_i = y\}$:

$$\hat{S}_n(\boldsymbol{x}^n | y^n) = \prod_{y=1}^{k} \int_B S_n(\boldsymbol{b}|y) d\mu(\boldsymbol{b}) \tag{2.12}$$

where the corresponding exponential growth rate of wealth is:

$$\widehat{W}(\boldsymbol{x}^n|y^n) = \frac{1}{n}\log \hat{S}_n(\boldsymbol{x}^n|y^n). \tag{2.13}$$

From (2.9) and (2.13), for the special measure $\mu(\boldsymbol{b})$ chosen to be the uniform (Dirichlet $(1,1,\dots,1)$) and Dirichlet $\left(\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2}\right)$ distributions, Cover and Ordentlich shows that:

$$\begin{aligned}
&\lim_{n\to\infty}\sup_{x^n,y^n}\frac{1}{n}\log\frac{S_n^*(\boldsymbol{x}^n|y^n)}{\hat{S}_n(\boldsymbol{x}^n|y^n)} \\
&= \lim_{n\to\infty}\sup_{x^n,y^n}\left(W^*(\boldsymbol{x}^n|y^n) - \widehat{W}(\boldsymbol{x}^n|y^n)\right) \\
&= 0
\end{aligned} \tag{2.14}$$

The proof of (2.14) is based on the following bounds for the ratio of wealths:

$$\frac{S_n^*(\boldsymbol{x}^n|y^n)}{\hat{S}_n(\boldsymbol{x}^n|y^n)} \le (n+1)^{k(m-1)} \tag{2.15}$$

for the Dirichlet$(1,1,\dots,1)$ distribution and

$$\frac{S_n^*(\boldsymbol{x}^n|y^n)}{\hat{S}_n(\boldsymbol{x}^n|y^n)} \le 2^k(n+1)^{\frac{k(m-1)}{2}} \tag{2.16}$$

for the Dirichlet$\left(\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2}\right)$ distribution.

Without loss of generality, we study the Cover-Ordentlich universal portfolio (2.5) without side information in this thesis. The focus of our thesis is to study the performance of the Cover-Ordentlich universal portfolio (2.5) generated by the general Dirichlet$(\alpha_1,\alpha_2,\dots,\alpha_m)$ distribution and a non-Dirichlet distribution defined on the simplex B. For this purpose, we use Ishijima (2001) numerical method.

2.3 The Ishijima Numerical Method

Ishijima (2001) proposed to compute the Dirichlet universal portfolio by using a numerical method. The general Dirichlet$(\alpha_1, \alpha_2, \dots, \alpha_m)$ distribution is defined by:

$$d\mu(\boldsymbol{b}) = \frac{\Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_m)}{\Gamma(\alpha_1)\Gamma(\alpha_2) \dots \Gamma(\alpha_m)} b_1^{\alpha_1 - 1} b_2^{\alpha_2 - 1} \dots b_{m-1}^{\alpha_{m-1}-1} b_m^{\alpha_m - 1} d\boldsymbol{b} \tag{2.17}$$

where $0 \leq b_i \leq 1$ for $i = 1,2, \dots, m$ and $\sum_{i=1}^{m} b_i = 1$. Note that the parameters $\alpha_i > 0$ for $i = 1,2, \dots, m$.

A Monte Carlo simulation technique is used to generate the Dirichlet random variables. In Wilks (1962), an efficient way to generate the Dirichlet variables is to start from the gamma variables. Let u_i have the gamma$(\alpha_i, 1)$ distribution with probability density function (p.d.f):

$$f_i(u_i) = \frac{1}{\Gamma(\alpha_i)} u_i^{\alpha_i - 1} e^{-u_i}, u_i > 0 \tag{2.18}$$

for $i = 1,2, \dots, m$ where $u_1, u_2, \dots, u_m$ are mutually independent.

The join p.d.f of $u_1, u_2, \dots, u_m$ is given by:

$$f(u_1, u, \dots, u_m) = \frac{1}{\prod_{i=1}^{m} \Gamma(\alpha_i)} u_1^{\alpha_1 - 1} u_2^{\alpha_2 - 1} \dots u_m^{\alpha_m - 1} e^{-(u_1 + u_2 + \cdots + u_m)}. \tag{2.19}$$

By the transformation of:

$$b_i = \frac{u_i}{\sum_{j=1}^{m} u_j}$$

$$
\begin{aligned}
&or \quad u_i = \frac{zb_i}{b_m}, \text{for } i = 1,2,\dots,m-1 \\
&and\ u_m = z,
\end{aligned} \tag{2.20}
$$

it is well-known that $(b_1, b_2, \dots, b_{m-1})$ has the Dirichlet $(\alpha_1, \alpha_2, \dots, \alpha_m)$ distribution (2.17) (see for example, Wilks (1962)). In other words, by simulating $u_1, u_2, \dots, u_m$, we can generate the Dirichlet $(\alpha_1, \alpha_2, \dots, \alpha_m)$ distribution.

We shall focus on generating the non-Dirichlet distribution by simulating m independent random variables $u_1, u_2, \dots, u_m$ which are not gamma $(\alpha_i, 1)$ variates, through the transformation $b_i = \frac{u_i}{\sum_{j=1}^{m} u_j}$ for $i = 1,2,\dots,m-1$.

In order to generate the universal portfolio by this numerical method, it is necessary to write (2.5) in the form:

$$
\widehat{\boldsymbol{b}}_{n+1} = \frac{E_{\mu(\boldsymbol{b})}[\boldsymbol{b}S_n(\boldsymbol{b})]}{E_{\mu(\boldsymbol{b})}[S_n(\boldsymbol{b})]} \tag{2.21}
$$

and using sampling with Monte Carlo simulation,

$$
\hat{b}_{n+1,i} = \frac{\frac{1}{T}\sum_{t=1}^{T} b_i^{(t)} S_n\left(\boldsymbol{b}^{(t)}\right)}{\frac{1}{T}\sum_{t=1}^{T} S_n\left(\boldsymbol{b}^{(t)}\right)}, i = 1,2,\dots,m \tag{2.22}
$$

where T is the total number of simulations. The details will be discussed in Chapter 3.

2.4 An Algorithm for Computing the Moving-Order Dirichlet Universal Portfolio

Tan $(2004a)$ proposed an algorithm for computing the (moving-order) universal portfolio (2.5) generated by the Dirichlet$(\alpha_1, \alpha_2, \dots, \alpha_m)$ distribution (2.17). Writing $\boldsymbol{b} = (b_i), \boldsymbol{x}_i = (x_{ij})$ and expanding the product in (2.1) as a sum, we obtain :

$$S_n(\boldsymbol{b}, \boldsymbol{x}^n)$$

$$= \sum_{i_1=0}^{n} \dots \sum_{i_{m-1}=0}^{n} b_1^{i_1} \dots b_{m-1}^{i_{m-1}} b_m^{(n-i_1-\dots-i_{m-1})} X_n(i_1, \dots, i_{m-1}) \tag{2.23}$$

where
$$X_n(i_1, \dots, i_{m-1}) = \sum_{\boldsymbol{k}^n \in T_n(\boldsymbol{i})} \prod_{j=1}^{n} \boldsymbol{x}_{j_{k_j}} \tag{2.24}$$

for $i_j = 0,1, \dots, n \; and \; j = 1,2, \dots, m-1$. $T_n(\boldsymbol{i})$ is the set of all sequences $\boldsymbol{k}^n = (k_1, k_2, \dots, k_n) \in \{1,2, \dots, m\}^n$ with i_1 1's, i_2 2's,…,$i_{m-1}(m-1)$'s and $(n - i_1 - i_2 - \dots - i_{m-1})$ m's and $\boldsymbol{i} = (i_1, i_2, \dots, i_{m-1})$. We define the moment function $C_n(\cdot)$ by:

$$C_n(i_1, \dots, i_{m-1}) = \int_{\mathcal{B}} b_1^{i_1} \dots b_{m-1}^{i_{m-1}} b_m^{(n-i_1-\dots-i_{m-1})} \, d\mu(\boldsymbol{b}) \tag{2.25}$$

where $d\mu(\boldsymbol{b})$ is given by (2.17). Therefore, from (2.23), (2.24) and (2.25), the universal wealth (2.7) can be expressed as:

$$\hat{S}_n(\boldsymbol{x}^n)$$

$$= \int_{\boldsymbol{\mathcal{B}}} S_n(\boldsymbol{b}, \boldsymbol{x}^n) \, d\mu(\boldsymbol{b})$$

$$= \int_{\mathcal{B}} \sum_{i_1=0}^{n} \dots \sum_{i_{m-1}=0}^{n} b_1^{i_1} \dots b_{m-1}^{i_{m-1}} b_m^{(n-i_1-\dots-i_{m-1})} X_n(i_1, \dots, i_{m-1}) \, d\mu(\boldsymbol{b})$$

$$= \sum_{i_1=0}^{n} \dots \sum_{i_{m-1}=0}^{n} X_n(i_1, \dots, i_{m-1}) C_n(i_1, \dots, i_{m-1}) \tag{2.26}$$

To compute (2.26) recursively, define:

$$Q_n(i_1, \dots, i_{m-1}) = X_n(i_1, \dots, i_{m-1}) C_n(i_1, \dots, i_{m-1}). \tag{2.27}$$

From (2.26) and (2.27), we have:

$$\hat{S}_n(\boldsymbol{x}^n) = \sum_{i_1=0}^{n} \dots \sum_{i_{m-1}=0}^{n} Q_n(i_1, \dots, i_{m-1}). \tag{2.28}$$

The two quantities $X_n(\cdot)$ and $C_n(\cdot)$ in (2.27) are computed recursively. From (2.17) and (2.25), we have

$$\begin{aligned}
&C_n(i_1, \dots, i_{m-1}) \\
&= \frac{\Gamma(\alpha_1 + \cdots + \alpha_m)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_m)} \times \int_{\mathcal{B}} b_1^{i_1+\alpha_1-1} \dots b_{m-1}^{i_{m-1}+\alpha_{m-1}-1} b_m^{(n-i_1-\cdots-i_{m-1}+\alpha_m-1)} \\
&= \frac{\Gamma(\alpha_1 + \cdots + \alpha_m)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_m)} \\
&\quad \times \frac{\Gamma(i_1+\alpha_1) \dots \Gamma(i_{m-1}+\alpha_{m-1}) \Gamma(n - i_1 - \cdots - i_{m-1} + \alpha_m)}{\Gamma(n + \alpha_1 + \cdots + \alpha_{m-1} + \alpha_m)}
\end{aligned} \tag{2.29}$$

Obviously from (2.29), we have

$$\begin{aligned}
&\frac{C_n\left(i_1, \dots, i_j + 1, \dots, i_{m-1}\right)}{C_{n-1}\left(i_1, \dots, i_j, \dots, i_{m-1}\right)} \\
&= \frac{\Gamma\left(i_j + 1 + \alpha_j\right)}{\Gamma\left(i_j + \alpha_j\right)} \times \frac{\Gamma(n - 1 + \alpha_1 + \cdots + \alpha_{m-1} + \alpha_m)}{\Gamma(n + \alpha_1 + \cdots + \alpha_{m-1} + \alpha_m)} \\
&= \frac{\left(i_j + \alpha_j\right)}{(n - 1 + \alpha_1 + \cdots + \alpha_m)}
\end{aligned} \tag{2.30}$$

for $j = 1,2, \dots, m-1$. Similarly,

$$\frac{C_n(i_1,\dots,i_{m-1})}{C_{n-1}(i_1,\dots,i_{m-1})}$$

$$=\frac{\Gamma(n-i_1-\cdots-i_{m-1}+\alpha_m)}{\Gamma(n+\alpha_1+\cdots+\alpha_{m-1}+\alpha_m)}\times\frac{\Gamma(n-1+\alpha_1+\cdots+\alpha_{m-1}+\alpha_m)}{\Gamma(n-1-i_1-\cdots-i_{m-1}+\alpha_m)}$$

$$=\frac{n-1-i_1-\cdots-i_{m-1}+\alpha_m}{(n-1+\alpha_1+\cdots+\alpha_m)} \tag{2.31}$$

On the other hand, the recursion formula for X_n is given by:

$$X_n(i_1,\dots i_{m-1})$$

$$=\sum_{\substack{j=1\\ i_j\neq 0}}^{m-1} x_{n,j}\,X_{n-1}\big(i_1,\dots,i_j-1,\dots,i_{m-1}\big)+x_{n,m}X_{n-1}(i_1,\dots,i_{m-1}) \tag{2.32}$$

with the end-point conditions of:

(i) $X_n(0,\dots,0)=x_{n,m}X_{n-1}(0,\dots,0)$ for all $i_j=0, j=1,2,..,m-1$,

(ii) $X_n(0,\dots,n,\dots,0)=x_{n,j}X_{n-1}(0,\dots,n-1,\dots,0)$ for $i_j=n, i_k=0$ for all $k\neq j$.

By multiplying (2.32) with C_n and apply the recursion formula for C_n in (2.30) and (2.31), the quantity $Q_n(\cdot)$ in (2.27) can be computed recursively as:

$$Q_n(i_1,\dots,i_{m-1})$$

$$=\sum_{\substack{j=1\\ i_j\neq 0}}^{m-1} x_{n,j}\,X_{n-1}\big(i_1,\dots,i_j-1,\dots,i_{m-1}\big)C_n\big(i_1,\dots i_j,\dots,i_{m-1}\big)$$

$$+x_{n,m}X_{n-1}(i_1,\dots,i_{m-1})C_n(i_1,\dots,i_{m-1})$$

$$= \sum_{\substack{j=1 \\ i_j \neq 0}}^{m-1} x_{n,j} \frac{(i_j - 1 + \alpha_j)}{(n - 1 + \alpha_1 + \cdots + \alpha_m)} Q_{n-1}(i_1, \dots, i_j - 1, \dots, i_{m-1})$$

$$+ x_{n,m} \frac{n - 1 - i_1 - \cdots - i_{m-1} + \alpha_m}{(n - 1 + \alpha_1 + \cdots + \alpha_m)} Q_{n-1}(i_1, \dots, i_{m-1}) \tag{2.33}$$

for $i_1 + i_2 + \cdots + i_{m-1} \neq n$ and remove the last term in (2.33) for $i_1, i_2, \dots, i_{m-1} = n$. The end-point conditions of Q_n are given by:

(i) $Q_n(0, \dots, 0) = \frac{x_{n,m}(n-1+\alpha_m)}{n-1+\alpha_1+\cdots+\alpha_m} Q_{n-1}(0, \dots, 0)$

(ii) $Q_n(0, \dots n, \dots, 0) = \frac{x_{n,j}(n-1+\alpha_j)}{n-1+\alpha_1+\cdots+\alpha_m} Q_{n-1}(0, \dots, n-1, \dots, 0)$ for

$i_j = n, i_k = 0$ for all $k \neq j$.

To compute $\widehat{\boldsymbol{b}}_{n+1}$ in (2.5) recursively, we use the recursive formula for calculating $\hat{S}_n(\boldsymbol{x}^n)$ in (2.26) and apply a similar type of recursion formula for calculating $\int_{\mathcal{B}} \boldsymbol{b}\, S_n(\boldsymbol{b}) d\mu(\boldsymbol{b})$.

Tan (2013) has generalized the algorithm for computing the moving-order universal portfolio generated by the independent probability distributions. This algorithm will be used in Chapter 5 of the thesis.

CHAPTER 3

PERFORMANCE OF THE DIRICHLET AND NON-DIRICHLET UNIVERSAL PORTFOLIOS GENERATED FROM SIMULATION

In Section 2.3, we have mentioned the Ishijima numerical method for computing the Dirichlet universal portfolio by using a transformation of m random variables, where each variable has the gamma$(\alpha_i, 1)$ distribution for $i = 1,2,\dots,m$. We can generate a μ-weighted universal portfolio which is non-Dirichlet by choosing a set of m non-gamma$(\alpha, 1)$ random variables using the Ishijima numerical method. The performance of the non-Dirichlet Cover-Ordentlich universal portfolios will be studied in this chapter. Furthermore, we will compare the performance among the best-constant-rebalanced portfolio (BCRP), the Dirichlet universal portfolio and the non-Dirichlet universal portfolio by running the Dirichlet and non-Dirichlet universal portfolios on selected stock-price data sets.

In Section 2.3, we observe that the Cover-Ordentlich universal portfolio generated by the probability measure $\mu(\boldsymbol{b})$ defined on the portfolio simplex B can be computed a ratio of expectations, namely,

$$\hat{\boldsymbol{b}}_{n+1} = \frac{E_{\mu(\boldsymbol{b})}[\boldsymbol{b}S_n(\boldsymbol{b})]}{E_{\mu(\boldsymbol{b})}[S_n(\boldsymbol{b})]}, \tag{3.1}$$

where $E_{\mu(\boldsymbol{b})}[\boldsymbol{b}S_n(\boldsymbol{b})]$ and $E_{\mu(\boldsymbol{b})}[S_n(\boldsymbol{b})]$ are the expectations of $\boldsymbol{b}S_n(\boldsymbol{b})$ and $S_n(\boldsymbol{b})$ respectively with respect to the probability measure $\mu(\boldsymbol{b})$. We can compute expected values of random variables using the law of large numbers. Hence the Monte Carlo Simulation technique may be used.

3.1 Generation of the Dirichlet Universal Portfolio

According to a theorem in Wilks (1962), if Z_i, for $i = 1,2, \dots, m$ are mutually independent random variables drawn from the gamma $(\alpha_i, 1)$ distribution where the probability density function (p.d.f) of Z_i is:

$$f(z_i) = \frac{1}{\Gamma(\alpha_i)} z_i^{\alpha_i - 1} e^{-z_i} \, , z_i > 0, \tag{3.2}$$

and $\Gamma(\cdot)$ is the gamma function, then the Dirichlet$(\alpha_1, \alpha_2, \dots, \alpha_m)$ distribution is the joint p.d.f. of $b_1, b_2, \dots, b_{m-1}$ through the following transformation:

$$b_i = \frac{z_i}{\sum_{i=1}^{m} z_i} \tag{3.3}$$

for $i = 1,2, \dots, m-1$. It is proven in Wilks (1962) and Arnason (1972) that it is an efficient way to generate the Dirichlet$(\alpha_1, \alpha_2, \dots, \alpha_m)$ distribution.

Given m stocks in the portfolio and a vector $\boldsymbol{Z} = [z_1, z_2, \dots, z_m]$ is generated independently from the gamma$(\alpha_i, 1)$ distributions for $i = 1,2, \dots, m$ where $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \dots, \alpha_m]$ is the parameter vector the Dirichlet$(\boldsymbol{\alpha})$ distribution. The generated $\mathbf{Z}$ is then transformed into the Dirichlet variable $\boldsymbol{b}^{(t)} = [b_1, b_2, \dots, b_m]$ where t denotes the t^{th} time simulation. The sampling process will be repeated T times and the algorithm is summarized below for one cycle:

1. Generate the independent random variables, z_i from the gamma$(\alpha_i, 1)$ for $i = 1,2, \dots, m$.
2. Transform the generated gamma variable into the Dirichlet variable by:

$$b_i^{(t)} = \frac{z_i}{\sum_{j=1}^{m} z_j}, \text{for } i = 1,2, \dots, m-1$$

$$\text{and } b_m^{(t)} = 1 - \sum_{j=1}^{m-1} b_j^{(t)}$$

where t denotes the t^{th} sampling.

3. Resample the Dirichlet variable, z_i again for the next cycle.

After the cycle is repeated a total of T times, the Dirichlet universal portfolio is computed by:

$$\hat{b}_{n+1,i} = \frac{\sum_{t=1}^{T} b_i^{(t)} S_n\left(\boldsymbol{b}^{(t)}\right)}{\sum_{t=1}^{T} S_n(\boldsymbol{b}^{(t)})} \text{ and}$$

$$S_n(\boldsymbol{b}^{t}) = \prod_{j=1}^{n} \left(\boldsymbol{b}^{(t)}\right)^t \boldsymbol{x}_j .$$

(A remark on the notation (t) *and* t *is in order here. We refer* (t) *as the quantity obtained from the* t^{th} *simulation, whereas* t *refers to the transpose of a vector.)*

The number of simulations, T will be set at 10000 in order to ensure that the approximation from the Monte Carlo simulation is close to the actual value.

Remark. Note that in the sampling process, if the random variable Z_i is generated from gamma$(1,1)$ for all $i = 1,2, \dots, m$, then the measure $\mu(\cdot)$ becomes the uniform distribution (Dirichlet$(1,1, \dots,1)$).

3.2 Generation of the non-Dirichlet Universal Portfolio

The non-Dirichlet universal portfolio is the focus of our study in this Chapter. We shall modify the algorithm in constructing the *Dirichlet universal portfolio* by generating the vector $\boldsymbol{Z}$ from non-gamma probability distributions.

The algorithm for a simulation is:

1. Generate the m independent random variables z_i from some given distribution which is other than the gamma$(\alpha_i, 1)$ distribution for $i = 1,2,\dots,m$.
2. Transform the generated variable z_i into the pseudo portfolio variable by:

$$b_i^{(t)} = \frac{z_i}{\sum_{j=1}^{m} z_j}\ \text{, for } i = 1,2,\dots,m$$

$$\text{and } b_m^{(t)} = 1 - \sum_{j=1}^{m-1} b_j^t$$

   where t denotes the t^{th} sampling.

3. Resample the variable z_i again for the next cycle.

After the cycle is repeated a total of T times, the non-Dirichlet universal portfolio is computed by:

$$\hat{b}_{n+1,i} = \frac{\sum_{t=1}^{T} b_i^{(t)} S_n\left(\boldsymbol{b}^{(t)}\right)}{\sum_{t=1}^{T} S_n\left(\boldsymbol{b}^{(t)}\right)} \text{ and}$$

$$S_n(\boldsymbol{b}^t) = \prod_{j=1}^{n} \left(\boldsymbol{b}^{(t)}\right)^t \boldsymbol{x}_j.$$

Again the number of simulations T will be set at 10,000.

Note that the random variables z_i can be drawn from same common probability distributions with different parameters for $i = 1,2,\dots,m$. The random variables are generated through the inverse function in Matlab and the inverse transform method. We shall focus our study in two-parameter distribution. One of the parameter will be fixed and adjust the other to achieve higher performance in terms of total wealth achieved. The joint p.d.f of $b_1, b_2, \dots, b_{m-1}$ for a non-Dirichlet universal portfolio may not have a closed form.

3.3 Performance of the Dirichlet and non-Dirichlet Universal Portfolio

The performance results for both Dirichlet universal portfolio and non-Dirichlet universal portfolio are presented in this section. For our study, 3-stock portfolios are selected form the Kuala Lumpur Stock Exchange(KLSE). These portfolios designated as A, B and C consists of the companies listed in Table 3.1. The stock-price data sets cover the period of 1 January 2003 until 30 November 2004, consisting of 500 trading days.

Table 3.1: Malaysia companies in the 3-stock portfolios in A, B and C

Data Set	Company 1	Company 2	Company 3
Set A	Maybank	Genting	Amway
Set B	Public Bank	Sunrise	YTL Corp
Set C	Hong Leong Bank	RHB Capital	YTL Corp

The abbreviations YTL and RHB in Table 3.1 represent Yeo Tiong Lay and Rashid Hussain Berhad respectively.

Table 3.2 lists out some common probability distributions with the corresponding probability density functions (p.d.f) and the methods of computation in generating the random variables. We shall generate Z_i for $i = 1,2,3$ using the distributions selected from Table 3.2.

Table 3.2: Some common probability distributions with probability density functions and their computation method

Distribution	P.d.f, f(y)	Computation Method
gamma (α, β)	$\frac{1}{\beta^\alpha\,\Gamma(\alpha)} y^{\alpha-1} e^{-\frac{y}{\beta}}$, $\Gamma(.)$ is the gamma function	Built-in inverse transform function in Matlab
lognormal (μ, σ)	$\frac{1}{y\sigma\sqrt{2\pi}} e^{-\frac{(\ln y-\mu)^2}{2\sigma^2}}$	Built-in inverse transform function in Matlab
Pareto (α, β)	$\frac{\alpha\beta^\alpha}{y^{\alpha+1}}$	Inverse transform sampling by letting $Y = \frac{\beta}{U^{1/\alpha}}$,where U is the random variable generated from the uniform distribution over (0,1).
Weibull (κ, λ)	$\frac{\kappa}{\lambda}\left(\frac{y}{\lambda}\right)^{\kappa-1} e^{-\left(\frac{y}{\lambda}\right)^\kappa}$	Inverse transform sampling by letting $Y = \lambda(-lnU)^{-\frac{1}{\kappa}}$, where U is the random variable generated from the uniform distribution over (0,1).
inverse Gaussian (μ, λ)	$\sqrt{\frac{\lambda}{2\pi y^3}}\, e^{-\frac{\lambda(y-\mu)^2}{2y\mu^2}}$	Inverse transform sampling by letting $Y = \mu + \frac{\mu^2 U^2}{2\lambda}$ $-\frac{\mu}{\lambda}\sqrt{4\mu\lambda U + \mu^2 U^4}$, where U is the random variable generated from the uniform distribution over (0,1).

We assume throughout this thesis that the initial wealth for the investment is 1 unit. For each portfolio, we compute the total wealth achieved after 500 trading days by the BCRP for the purpose of comparing with the wealth achieved by the Dirichlet universal portfolio and the non-Dirichlet universal portfolio generated by different probability distributions. The BCRP wealths are summarized in Table 3.3.

Table 3.3: Total wealth achieved, $S_{500}^{*}(b^{*})$ by the BCRP after 500 trading days

Data	$S_{500}^{*}(b^{*})$
Set A	1.853389825
Set B	4.297023754
Set C	4.297023754

Table 3.4 shows the wealths achieved by the Dirichlet$(1,1,1)$ and Dirichlet$\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ universal portfolios.

Table 3.4: The total wealths S_{500} achieved by the Dirichlet$(1,1,1)$ and Dirichlet$\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ universal portfolios generated from the gamma$(\alpha_i,1)$ variable for data sets A, B and C

α_1	α_2	α_3	Set A	Set B	Set C
0.5	0.5	0.5	1.5660	2.1582	1.8364
1	1	1	1.5673	2.1579	1.8335

In terms of total wealth achieved, the Dirichlet universal portfolios $(1,1,1)$ and $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ perform far below that of the BCRP for the three data sets in Table 3.4. The wealths achieved are less than 50% of the BCRP wealth for sets B and C. For the general Dirichlet$(\alpha_1, \alpha_2, \alpha_3)$ universal portfolios, the welaths achieved are shown in Table 3.5. The performance of the portfolios B and C are better than that of the Dirichlet$(1,1,1)$ and $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ portfolios in Table 3.4.

Table 3.5: The total wealth S_{500} achieved by the Dirichlet universal portfolios generated from the gamma$(\alpha_i, 1)$ variables

α_1	α_2	α_3	Set A	Set B	Set C
5	7	9	1.5464	2.4104	2.0441
0.3	0.5	0.7	1.5393	2.5235	2.1344
0.2	0.4	0.8	1.5545	2.8120	2.4466
0.2	0.3	0.8	1.5836	2.9345	2.6154
0.1	1.1	2.1	1.5060	3.0402	2.5795
1	1	50	1.6505	4.1371	4.0966
0.01	0.01	50	1.6539	4.2953	4.2949
2	3	4	1.5434	2.4518	2.0794
2	6	10	1.5222	2.7815	2.3570
1	2	30	1.6357	3.9386	3.8188
2	4	8	1.5534	2.8107	2.4410
0.1	0.1	30	1.6532	4.2695	4.2618
0.1	0.1	40	1.6535	4.2760	4.2708
10	20	30	1.5342	2.6137	2.2155
1	1	40	1.6495	4.1001	4.0508
0.1	0.2	0.3	1.5360	2.6060	2.2074

1.5	15	55	1.5628	3.4630	3.1166
0.1	0.1	0.9	1.6365	3.5828	3.4344

Two types of non-Dirichlet universal portfolios depending on the distribution of the generating Z_1, Z_2 and Z_3 are studied. First, Z_1, Z_2 and Z_3 come from the same family of distribution with different parameters. Second, each of Z_1, Z_2 and Z_3 may come from different families of distribution.

Tables 3.6-3.9 show the wealths achieved by the first type of non-Dirichlet universal portfolios for the lognormal, Pareto, Weibull and inverse Gausian distributions. The distributions referred to in the captions are the distributions of $Z_i, i = 1,2,3$.

Table 3.6: The total wealth S_{500} achieved by the non-Dirichlet universal portfolios generated from the lognormal$(\mu_i, 0.05)$ variables

μ_1	μ_2	μ_3	**Set A**	**Set B**	**Set C**
2	3	4	1.5610	3.0863	2.7290
0.1	0.5	5	1.6509	4.2181	4.1941
10	30	60	1.6539	4.2970	4.2970
1	5	100	1.6539	4.2970	4.2970
5	25	125	1.6539	4.2970	4.2970
0.01	10	20	1.6539	4.2968	4.2967
5	1	5	1.7672	2.4567	2.4474
50	100	150	1.6539	4.2970	4.2970

Table 3.7: The total wealth S_{500} achieved by the non-Dirichlet universal portfolios generated from the Pareto$(\alpha_i,1)$ variables

α_1	α_2	α_3	Set A	Set B	Set C
1	2	3	1.6043	1.9484	1.6865
1	5	10	1.6377	1.8918	1.6711
0.1	0.1	0.9	1.5073	1.5348	1.2188
0.1	0.6	50	1.7429	1.4654	1.3639
25	50	100	1.5666	2.1486	1.8255
25	2	10	1.5006	2.0874	1.6873
5	0.5	3	1.3816	1.9189	1.4181
50	5	1	1.5798	2.6533	2.3351
8	88	30	1.5774	2.1349	1.8261
0.5	10	0.01	1.6591	4.1879	4.1845

Table 3.8: The total wealth S_{500} achieved by the non-Dirichlet universal portfolios generated from the Weibull$(\lambda_i, 10)$ variables

κ1	κ2	κ3	Set A	Set B	Set C
1.2	1.4	3.2	1.5697	2.2246	1.9031
1	0.8	2.5	1.5702	2.2383	1.9152
0.05	0.5	5	1.6096	2.2701	1.9927
3	3.8	4.7	1.5626	2.1800	1.8502
1.2	1.5	1.8	1.5650	2.1838	1.8546
1.9	1.5	1	1.5669	2.1245	1.8047
4.6	3	0.5	1.5653	2.0882	1.7706
3	5	4	1.5585	2.1655	1.8303
0.5	1.8	1	1.5498	2.1460	1.7992

4	9	5	1.5570	2.1573	1.8225

Table 3.9: The total wealth S_{500} achieved by the non-Dirichlet universal portfolios generated from the inverse Gaussian $(\mathbf{100}, \boldsymbol{\lambda_i})$ variables

λ1	λ2	λ3	Set A	Set B	Set C
2	3	4	1.5507	2.2852	1.9458
0.1	0.5	5	1.5738	3.1390	2.8055
10	30	60	1.5401	2.4364	2.0565
1	5	100	1.5822	3.1562	2.8393
5	25	125	1.5432	2.6926	2.3087
0.001	10	20	1.4607	2.8449	2.3051
5	1	5	1.6581	2.2966	2.0944
50	100	150	1.5489	2.2836	1.9313
6	6	10	1.5730	2.2871	1.9599
2.5	10	15	1.5228	2.4461	2.0394

The non-Dirichlet universal portfolios (in Tables 3.6-3.9) generated from different distributions perform better for some sets of parameter compared to the Dirichlet universal portfolio (in Tables 3.4,3.5). For certain sets of parameter, the portfolios achieve wealths close to that of the BCRP.

For Set A, the portfolio performs well in the lognormal distributions and Pareto distributions with wealths 1.7672 (in Table 3.6) and parameter $(5,1,5)$, and 1.7429 (in Table 3.7) and parameter $(0.1,0.6,50)$ respectively. The performances are below the BCRP wealth but it is closer to the BCRP wealths compared to that of Dirichlet universal portfolio in Tables 3.4 and 3.5.

In sets B and C, it is worth noting that the Dirichlet universal portfolio with parameter $(0.01, 0.01, 50)$ in Tables 3.5 and the non-Dirichlet universal portfolios generated from the lognormal distribution in Table 3.6 are performing as well as the BCRP, achieving a total wealth of 4.2970 which is the approximate BCRP wealth. The empirical results here show that there are non-Dirichlet universal portfolios that outperform the Dirichlet universal portfolios, thereby establishing the importance of these parametric families of universal portfolios.

Tables 3.10-3.13 show the wealths achieved by the non-Dirichlet universal portfolios of the second type where the generating variables Z_1, Z_2 and Z_3 come from different families of the probability distributions.

Table 3.10: The wealth S_{500} achieved by the non-Dirichlet universal portfolios generated from the gamma$(\alpha_i, 10)$ variable for $i = 1$ and the lognormal$(\mu_i, 5)$ variables for $i = 2, 3$

α_1	μ_2	μ_3	Set A	Set B	Set C
0.5	5	10	1.5447	3.3974	2.9930
0.2	8	6	1.3673	2.3640	1.7606
5	20	40	1.6524	4.2840	4.2791
0.1	5	50	1.6539	4.2970	4.2970
0.01	10	100	1.6539	4.2970	4.2970
2	10	20	1.6150	3.9887	3.8169
0.1	1	5	1.5305	3.1943	2.7660
2	25	75	1.6539	4.2970	4.2970
5	6	8	1.5001	2.8205	2.3554

2	10	15	1.5400	3.4143	3.0228

Table 3.11: The wealth S_{500} achieved by the non-Dirichlet universal portfolios generated from the lognormal$(\alpha_i, 50)$ variables for $i = 1, 2$ and the inverse Gaussian$(50, \lambda_3)$ variable

α_1	α_2	λ_3	Set A	Set B	Set C
2	3	4	1.5487	1.9624	1.6313
0.1	0.5	5	1.5497	2.0005	1.6737
10	30	60	1.4528	1.7607	1.3516
1	5	100	1.5381	1.9878	1.6359
5	25	125	1.4679	1.8207	1.4097
0.01	10	20	1.5084	1.9454	1.5833
5	1	5	1.5682	1.9500	1.6317
50	100	150	1.3232	1.5443	1.0701
6	6	10	1.5467	1.9183	1.5855
2.5	10	15	1.5194	1.9171	1.5612

Table 3.12: The wealth S_{500} achieved by the non-Dirichlet universal portfolios generated from the Pareto$(\alpha_1, 100)$ variable and the gamma$(\alpha_i, 100)$ variables for $i = 2, 3$

α_1	α_2	α_3	Set A	Set B	Set C
1	0.5	0.5	1.7447	1.6373	1.5393
1	1	1	1.6843	1.8107	1.6451
0.1	7	9	1.7660	1.6394	1.5572
0.1	0.5	0.7	1.8313	1.4359	1.4168
25	0.4	0.8	1.7036	2.0061	1.8664
25	0.3	0.8	1.7252	2.0154	1.9101

5	1.1	2.1	1.6176	2.4146	2.1535
50	1	50	1.6508	4.1332	4.0936
8	0.01	50	1.6558	4.1574	4.1380
0.5	3	4	1.6437	2.0546	1.8329

Table 3.13: The wealth S_{500} achieved by the non-Dirichlet universal portfolios generated from the gamma $(\alpha_1, 10)$ variable, Pareto$(\alpha_2, 10)$ variable and the lognormal$(\mu_3, 5)$ variable

α_1	α_2	μ_3	**Set A**	**Set B**	**Set C**
0.5	2	4	1.5088	2.8421	2.3826
1	5	5	1.5839	3.0246	2.7019
5	0.1	60	1.6519	4.2849	4.2759
0.3	0.6	100	1.6539	4.2970	4.2970
0.2	50	125	1.6539	4.2970	4.2970
0.2	2	20	1.6538	4.2944	4.2934
0.1	0.5	5	1.4477	2.8109	2.2629
1	5	150	1.6539	4.2970	4.2970
0.01	88	10	1.6192	4.0154	3.8632
2	10	15	1.6536	4.2339	4.2216

For data sets B and C, universal portfolios generated by a mixture of gamma and lognormal distributions in Table 3.10 and a mixture of gamma, Pareto and lognormal distributions in Table 3.13 achieve the best wealth, that is, the BCRP wealth of 4.2970 units for certain parametric values. This again establishes the importance of non-Dirichlet universal portfolios generated by a mixture of different parametric families of distributions. Another well-performing universal portfolio is that generated by the Pareto$(0.1,100)$, gamma$(0.5,100)$ and gamma$(0.7,100)$ distributions in Table 3.12, achieving a wealth of 1.8313 units which is close to the BCRP wealth of 1.8534 units. Universal portfolios generated by certain mixtures of probability distributions may perform below expectations, as exemplified by Table 3.11. Some of the empirical results here are presented in Tan and Lim (2012b).

An inventory of available universal portfolios is important to the potential investor. Since the parameters of the portfolio are chosen at the beginning of the investment period, the problem of choosing a good parametric vector becomes critical. In Tan and Lim (2013), a method for mixing different types of universal portfolios is proposed, whereby the investor uses the best current parametric vector. This method seems to work well for certain data sets and the mixture portfolio can even outperform the BCRP.

3.4 Discussion on Achieving Better Performance

We summarize the best numerical results generated from different distributions for sets A, B and C in Table 3.14. For data set A, the closest performance to BCRP is 1.8313, which is generated from the Pareto$(0.1,100)$, gamma $(0.5,100)$ and gamma $(0.7,100)$ distributions for stock $i = 1,2,3$ respectively. Other distributions with different sets of parameter do not give good performance, with total wealth achieved below 1.8 after 500 trading days. In order to increase the portfolio performance, we try other distributions with different parameter sets to search for better distributions generating higher wealths.

Table 3.14: A summary of the generating probability distributions achieving the best wealths for the universal portfolios discussed in Chapter 3

Data Set	**Probability Distributions**	**Parameter for stock i,** i=1	i=2	i=3	**Wealth, S_{500}**
Set A	Pareto$(\alpha_i, 100)$ variable for $i = 1$ and gamma$(\alpha_i, 100)$ variables for $i = 2,3$.	0.1	0.5	0.7	1.8313
Set B	Lognormal$(\mu_i, 0.05)$ variables for $i = 1,2,3$.	10	30	60	4.2970
		1	5	100	
		5	25	125	
		50	100	150	
	Gamma$(\alpha_i, 10)$ variable for $i = 1$ and lognormal$(\mu_i, 5)$ variables for $i = 2,3$.	0.1	5	50	
		0.01	10	100	
		2	25	75	
	Gamma$(\alpha_1, 10)$ variable, Pareto$(\alpha_2, 10)$ variable and the lognormal$(\mu_3, 5)$ variable	0.3	0.6	100	
		0.2	50	125	
		1	5	150	

Set C	Lognormal$(\mu_i, 0.05)$ variables for $i = 1,2,3.$	10	30	60	
		1	5	100	
		5	25	125	
		50	100	150	
	Gamma$(\alpha_i, 10)$ variable for $i = 1$ and lognormal$(\mu_i, 5)$ variables for $i = 2,3.$	0.1	5	50	4.2970
		0.01	10	100	
		2	25	75	
	Gamma$(\alpha_1, 10)$ variable, Pareto$(\alpha_2, 10)$ variable and the lognormal$(\mu_3, 5)$ variable	0.3	0.6	100	
		0.2	50	125	
		1	5	150	

Another alternative is to select different families of distributions generating the universal portfolios. The wealth of 1.8313 for data set A is below the BCRP wealth of 1.8534 where the wealth of 4.2970 for data sets B and C correspond to the BCRP wealth. From Table 3.14, it is the BCRP wealth of 4.2970 can be achieved by using different families of distributions with appropriate parameters selected. The key to generate good-performance universal portfolios for data sets B and C is to focus on the third generating distribution which is lognormal by putting heavier weights on the parameter μ_3.

CHAPTER 4

PERFORMANCE OF THE LOW ORDER UNIVERSAL PORTFOLIOS

It is clear from Section 2.2 that the μ-weighted Cover-Ordentlich universal portfolio is restricted by the simplex B of portfolio vectors which is the support of the probability measure μ. By removing the restriction on the support of the μ, a more general type of universal portfolio generated by one or more probability distributions can be defined. Tan (2013) adopted this approach and introduced the theory of finite and moving order universal portfolios generated by probability distributions. We shall study the empirical performance of low order universal portfolios in this chapter. A significant amount of computational time and computer memory is saved when implementing the low order universal portfolio in comparison with implementing the Cover-Ordentlich universal portfolio. For this reason, it is a more practical portfolio for use in investment than the Cover-Ordentlich universal portfolio.

4.1 The Finite Order Universal Portfolio

Suppose $Y_1, Y_2, \dots, Y_m$ are mutually independent random variables with probability density functions $f_{Y_1}(y_1), f_{Y_2}(y_2), \dots, f_{Y_m}(y_m)$ respectively. Then the joint probability density function of $Y_1, Y_2, \dots, Y_m$ is :

$$f(y_1, y_2, \dots, y_m) = f_{Y_1}(y_1) f_{Y_2}(y_2) \dots f_{Y_m}(y_m) \tag{4.1}$$

for $\boldsymbol{y} = (y_1, y_2, \dots, y_m) \in D$, where D is defined by:

$$D = \{(y_1, y_2, \dots, y_m): f_{Y_i}(y_i) > 0, \text{for all } i = 1,2, \dots m\}. \qquad (4.2)$$

Let $\boldsymbol{x} = (x_{ni})$ denote the price-relative vector on day n (see Section 2.1). Then the inner product $\boldsymbol{y}^t \boldsymbol{x}_j = \sum_{j=1}^{m} y_i \, x_{ji}$ for $j = 1,2, \dots$ is well defined for an m-stock market. Let ν be a fixed positive integer. Then the order ν universal portfolio generated by the m independent random variables $Y_1, Y_2, \dots, Y_m$ is the sequence $\{\widehat{\boldsymbol{b}}_{n+1}\}$ given by:

$$\hat{b}_{n+1,k} = \frac{\int_D \, y_k (\boldsymbol{y}^t \boldsymbol{x}_n) \dots (\boldsymbol{y}^t \boldsymbol{x}_{n-(\nu-1)}) f(\boldsymbol{y}) \, d\boldsymbol{y}}{\int_D \, (y_1 + y_2 + \cdots + y_m)(\boldsymbol{y}^t \boldsymbol{x}_n) \dots (\boldsymbol{y}^t \boldsymbol{x}_{n-(\nu-1)}) f(\boldsymbol{y}) \, d\boldsymbol{y}} \qquad (4.3)$$

for $k = 1,2, \dots, m; n = 1,2, \dots$. We assume that the moments $E[Y_i^j]$ are positive for $i = 1,2, \dots, m; j = 1,2, \dots, \nu + 1$.

4.2 Low Order Universal Portfolio

The order ν of the universal portfolio (4.3) is the number of days of the past stock-price information to be taken into account in calculating the next-day portfolio. Computer memory and computational time can be saved significantly for implementing the portfolio if the order ν is small, say $\nu = 1,2,3$.

In this chapter, orders 1,2,3 universal portfolios generated by some common probability distributions will be studied. The specific formula of (4.3) for each $\nu = 1,2,3$ will be obtained in the following sub-sections. It is clear that the formula (4.3) involves the moments of the generating variables $Y_1, Y_2, \dots, Y_m$.

4.2.1 Order 1 Universal Portfolio

From (4.3), the portfolio proportion $\hat{b}_{n+1,k}$ for stock k on day $n+1$ for the order 1universal portfolio is given by:

$$\hat{b}_{n+1,k} = \frac{\int_D y_k(\boldsymbol{y}^t\boldsymbol{x}_n)f(\boldsymbol{y})d\boldsymbol{y}}{\int_D (y_1 + y_2 + \cdots + y_m)(\boldsymbol{y}^t\boldsymbol{x}_n)f(\boldsymbol{y})d\boldsymbol{y}}$$

$$= \frac{\int_D \sum_{i=1}^m y_k y_i x_{n,i} f(y_1, y_2, \dots, y_m)dy_1dy_2 \dots dy_m}{\int_D (y_1 + y_2 + \cdots + y_m)\sum_{i=1}^m (y_i x_{n,i})\, f(y_1, y_2, \dots, y_m)dy_1dy_2 \dots dy_m}$$

$$= \frac{\sum_{i=1}^m \left[x_{n,i} \int_D y_k y_i\, f(y_1, y_2, \dots, y_m)dy_1dy_2 \dots dy_m\right]}{\sum_{i=1}^m \left[x_{n,i} \int_D (y_1 + y_2 + \cdots + y_m)y_i\, f(y_1, y_2, \dots, y_m)dy_1dy_2 \dots dy_m\right]}$$

$$= \frac{x_{n,1}E[Y_kY_1] + x_{n,2}E[Y_kY_2] + \cdots + x_{n,m}E[Y_kY_m]}{\sum_{k=1}^m \{x_{n,1}E[Y_kY_1] + x_{n,2}E[Y_kY_2] + \cdots + x_{n,m}E[Y_kY_m]\}}$$

$$= \zeta_{n,1}\left(\sum_{i=1}^m x_{n,i}E[Y_kY_i]\right), \text{for } k = 1,2, \dots, m \qquad (4.4)$$

where the normalizing constant,

$$\zeta_{n,1} = \left[\sum_{k=1}^m \left(\sum_{i=1}^m x_{n,i}E[Y_kY_i]\right)\right]^{-1}. \qquad (4.5)$$

We note that by the independence of $Y_1, Y_2, \dots, Y_m$,

$$E[Y_kY_i] = \begin{cases} E[Y_k]E[Y_i]\,, \text{if } i \neq k \\ E[Y_i^2] \qquad , \text{if } i = k \end{cases}$$

4.2.2 Order 2 Universal Portfolio

From (4.3), the order 2 portfolio proportion $\hat{b}_{n+1,k}$ for stock k on day $n+1$ is given by:

$$\hat{b}_{n+1,k}$$

$$= \frac{\int_D y_k(\boldsymbol{y}^t\boldsymbol{x}_n)(\boldsymbol{y}^t\boldsymbol{x}_{n-1})f(\boldsymbol{y})d\boldsymbol{y}}{\int_D (y_1+y_2+\cdots+y_m)(\boldsymbol{y}^t\boldsymbol{x}_n)(\boldsymbol{y}^t\boldsymbol{x}_{n-1})f(\boldsymbol{y})d\boldsymbol{y}}$$

$$= \frac{\int_D y_k\left(\sum_{i_1=1}^m y_{i_1}x_{n,i_1}\right)\left(\sum_{i_2=1}^m y_{i_2}x_{n-1,i_2}\right)f(y_1,y_2,\ldots,y_m)dy_1dy_2\ldots dy_m}{\begin{bmatrix}\int_D (y_1+y_2+\cdots+y_m)\left(\sum_{i_1=1}^m y_{i_1}x_{n,i_1}\right)\left(\sum_{i_2=1}^m y_{i_2}x_{n-1,i_2}\right)\\ \times f(y_1,y_2,\ldots,y_m)dy_1dy_2\ldots dy_m\end{bmatrix}}$$

$$= \frac{\begin{bmatrix}\sum_{i_1=1}^m\sum_{i_2=1}^m x_{n,i_1}x_{n-1,i_2}\int_D y_k y_{i_1}y_{i_2}\\ \times f(y_1,y_2,\ldots,y_m)dy_1dy_2\ldots dy_m\end{bmatrix}}{\begin{bmatrix}\sum_{i_1=1}^m\sum_{i_2=1}^m x_{n,i_1}x_{n-1,i_2}\int_D (y_1+y_2+\cdots+y_m)y_{i_1}y_{i_2}\\ \times f(y_1,y_2,\ldots,y_m)dy_1dy_2\ldots dy_m\end{bmatrix}}$$

$$= \frac{\begin{bmatrix}x_{n,1}x_{n-1,1}E[Y_kY_1Y_1]+x_{n,1}x_{n-1,2}E[Y_kY_1Y_2]+\cdots\\ +x_{n,m}x_{n-1,m}E[Y_kY_mY_m]\end{bmatrix}}{\sum_{k=1}^m\begin{Bmatrix}x_{n,1}x_{n-1,1}E[Y_kY_1Y_1]+x_{n,2}x_{n-1,2}E[Y_kY_1Y_2]+\cdots\\ +x_{n,m}x_{n-1,m}E[Y_kY_mY_m]\end{Bmatrix}}$$

$$= \zeta_{n,2}\left(\sum_{i_1=1}^m\sum_{i_2=1}^m x_{n,i_1}x_{n-1,i_2}E[Y_kY_{i_1}Y_{i_2}]\right), \text{for } k=1,2,\ldots,m \qquad (4.6)$$

where the normalizing constant,

$$\zeta_{n,2} = \left[\sum_{k=1}^m\left(\sum_{i_1=1}^m\sum_{i_2=1}^m x_{n,i_1}x_{n-1,i_2}E\left[Y_kY_{i_1}Y_{i_2}\right]\right)\right]^{-1}. \qquad (4.7)$$

We not that by the independence of $Y_1, Y_2, \dots, Y_m$,

$$E\left[Y_k Y_{i_1} Y_{i_2}\right] = \begin{cases} E[Y_k]E\left[Y_{i_1}\right]E\left[Y_{i_2}\right] & , k \neq i_1 \neq i_2 \ \big((k, i_1, i_2) \text{ distinct}\big) \\ E[Y_k^2]E\left[Y_{i_2}\right] & , k = i_1, k \neq i_2 \\ E[Y_k]E\left[Y_{i_1}^2\right] & , i_1 = i_2, k \neq i_1 \\ E[Y_k^3] & , k = i_1 = i_2. \end{cases}$$

4.2.3 Order 3 Universal Portfolio

From (4.3), the order 3 portfolio proportion $\hat{b}_{n+1,k}$ for stock k on day $n+1$ is given by:

$$\hat{b}_{n+1,k} = \frac{\int_D \ y_k(\boldsymbol{y}^t \boldsymbol{x}_n)(\boldsymbol{y}^t \boldsymbol{x}_{n-1})(\boldsymbol{y}^t \boldsymbol{x}_{n-2}) f(\boldsymbol{y}) d\boldsymbol{y}}{\int_D \ (y_1 + y_2 + \cdots + y_m)(\boldsymbol{y}^t \boldsymbol{x}_n)(\boldsymbol{y}^t \boldsymbol{x}_{n-1})(\boldsymbol{y}^t \boldsymbol{x}_{n-2}) f(\boldsymbol{y}) d\boldsymbol{y}}$$

$$= \frac{\begin{bmatrix} \int_D \ y_k\left(\sum_{i_1=1}^m y_{i_1} x_{n,i_1}\right)\left(\sum_{i_2=1}^m y_{i_2} x_{n-1,i_2}\right)\left(\sum_{i_3=1}^m y_{i_3} x_{n-2,i_3}\right) \\ \times f(y_1, y_2, \dots, y_m) dy_1 dy_2 \dots dy_m \end{bmatrix}}{\begin{bmatrix} \int_D \ (y_1 + y_2 + \cdots + y_m)\left(\sum_{i_1=1}^m y_{i_1} x_{n,i_1}\right)\left(\sum_{i_2=1}^m y_{i_2} x_{n-1,i_2}\right)\left(\sum_{i_3=1}^m y_{i_3} x_{n-2,i_3}\right) \\ \times f(y_1, y_2, \dots, y_m) dy_1 dy_2 \dots dy_m \end{bmatrix}}$$

$$= \frac{\begin{bmatrix} \sum_{i_1=1}^m \sum_{i_2=1}^m \sum_{i_3=1}^m x_{n,i_1} x_{n-1,i_2} x_{n-2,i_3} \int_D \ y_k y_{i_1} y_{i_2} y_{i_3} \\ \times f(y_1, y_2, \dots, y_m) dy_1 dy_2 \dots dy_m \end{bmatrix}}{\begin{bmatrix} \sum_{i_1=1}^m \sum_{i_2=1}^m \sum_{i_3=1}^m x_{n,i_1} x_{n-1,i_2} x_{n-2,i_3} \int_D \ (y_1 + y_2 + \cdots + y_m) y_{i_1} y_{i_2} y_{i_3} \\ \times f(y_1, y_2, \dots, y_m) dy_1 dy_2 \dots dy_m \end{bmatrix}}$$

$$= \zeta_{n,3} \left(\sum_{i_1=1}^m \sum_{i_2=1}^m \sum_{i_3=1}^m x_{n,i_1} x_{n-1,i_2} x_{n-2,i_3} E\left[Y_k Y_{i_1} Y_{i_2} Y_{i_3}\right] \right), \tag{4.8}$$

for $k = 1,2, \dots, m$.

where the normalizing constant,

$$\zeta_{n,3} = \left[\sum_{k=1}^m \left(\sum_{i_1=1}^m \sum_{i_2=1}^m \sum_{i_3=1}^m x_{n,i_1} x_{n-1,i_2} x_{n-2,i_3} E\left[Y_k Y_{i_1} Y_{i_2} Y_{i_3}\right] \right) \right]^{-1}. \tag{4.9}$$

We note that by the independence of $Y_1, Y_2, \ldots, Y_m$,

$$E\left[Y_k Y_{i_1} Y_{i_2} Y_{i_3}\right] =$$

$$\begin{cases} E[Y_k]E\left[Y_{i_1}\right]E\left[Y_{i_2}\right]E\left[Y_{i_3}\right] & ,k \neq i_1 \neq i_2 \neq i_3 \ \left((k, i_1, i_2, i_3) \text{ distinct}\right) \\ E[Y_k^2]E\left[Y_{i_2}\right]E\left[Y_{i_3}\right] & ,k = i_1, k \neq i_2, k \neq i_3, i_2 \neq i_3 \\ E[Y_k]E\left[Y_{i_1}^2\right]E\left[Y_{i_3}\right] & ,i_1 = i_2, k \neq i_1, k \neq i_3, i_1 \neq i_3 \\ E[Y_k^2]E\left[Y_{i_1}^2\right] & ,k = i_3, i_1 = i_2, k \neq i_1 \\ E[Y_k^3]E[Y_3] & ,k = i_1 = i_2, k \neq i_3 \\ E[Y_k^4] & ,k = i_1 = i_2 = i_3 \end{cases}.$$

4.3 Wealth Function of the Order ν Universal Portfolio

The wealth function $\hat{S}_n(\boldsymbol{x}^n)$ can be calculated recursively as follows:

$$\begin{aligned} \widehat{\boldsymbol{S}}_{\boldsymbol{n}+\boldsymbol{1}} &= \prod_{\boldsymbol{j}=\boldsymbol{1}}^{\boldsymbol{n}+\boldsymbol{1}} \widehat{\boldsymbol{b}}_j^t \boldsymbol{x}_j \\ &= \left(\widehat{\boldsymbol{b}}_{n+1}^t \boldsymbol{x}_{n+1}\right) \prod_{\boldsymbol{j}=\boldsymbol{1}}^{\boldsymbol{n}} \widehat{\boldsymbol{b}}_j^t \boldsymbol{x}_j \\ &= \left(\widehat{\boldsymbol{b}}_{n+1}^t \boldsymbol{x}_{n+1}\right) \hat{S}_n \end{aligned} \tag{4.10}$$

where

$$\widehat{\boldsymbol{b}}_{n+1}^t \boldsymbol{x}_{n+1} = \sum_{k=1}^{m} \hat{b}_{n+1,k} x_{n+1,k}.$$

From (4.3), the wealth increase on day $n+1$, namely $\widehat{\boldsymbol{b}}_{n+1}^t \boldsymbol{x}_{n+1}$ can be evaluated as follows for the order ν universal portfolio:

$\widehat{\boldsymbol{b}}_{n+1}^t \boldsymbol{x}_{n+1}$

$$= \frac{\sum_{k=1}^{m} x_{n+1,k} \int_D (y_k)(\boldsymbol{y}^t\boldsymbol{x}_n)(\boldsymbol{y}^t\boldsymbol{x}_{n-1}) \dots (\boldsymbol{y}^t\boldsymbol{x}_{n-(v-1)}) f(\boldsymbol{y})\, d\boldsymbol{y}}{\int_D (y_1 + y_2 + \cdots + y_m)(\boldsymbol{y}^t\boldsymbol{x}_n)(\boldsymbol{y}^t\boldsymbol{x}_{n-1}) \dots (\boldsymbol{y}^t\boldsymbol{x}_{n-(v-1)}) f(\boldsymbol{y})\, d\boldsymbol{y}}$$

$$= \frac{\int_D (\boldsymbol{y}^t\boldsymbol{x}_{n+1})(\boldsymbol{y}^t\boldsymbol{x}_n)(\boldsymbol{y}^t\boldsymbol{x}_{n-1}) \dots (\boldsymbol{y}^t\boldsymbol{x}_{n-(v-1)}) f(\boldsymbol{y})\, d\boldsymbol{y}}{\int_D (y_1 + y_2 + \cdots + y_m)(\boldsymbol{y}^t\boldsymbol{x}_n)(\boldsymbol{y}^t\boldsymbol{x}_{n-1}) \dots (\boldsymbol{y}^t\boldsymbol{x}_{n-(v-1)}) f(\boldsymbol{y})\, d\boldsymbol{y}} \tag{4.11}$$

From (4.4), for the special case of the order 1 universal portfolio for 3 stocks, the wealth increase $\widehat{\boldsymbol{b}}_{n+1}^t \boldsymbol{x}_{n+1}$ on day $n + 1$ is given by:

$$\sum_{k=1}^{3} \hat{b}_{n+1,k} x_{n+1,k}$$

$$= \frac{\sum_{k=1}^{3} x_{n+1,k} \{x_{n,1}E[Y_kY_1] + x_{n,2}E[Y_kY_2] + \cdots + x_{n,m}E[Y_kY_m]\}}{\sum_{k=1}^{3}\{x_{n,1}E[Y_kY_1] + x_{n,2}E[Y_kY_2] + \cdots + x_{n,m}E[Y_kY_m]\}}$$

where $m = 3$,

$$\begin{aligned} = \zeta_{n,1} \Big\{ & x_{n+1,1} \left[x_{n,1}E[Y_1^2] + x_{n,2}E[Y_1Y_2] + x_{n,3}E[Y_1Y_3]\right] \\ & + x_{n+1,2} \left[x_{n,1}E[Y_2Y_1] + x_{n,2}E[Y_2^2] + x_{n,3}E[Y_2Y_3]\right] \\ & + x_{n+1,3} \left[x_{n,1}E[Y_3Y_1] + x_{n,2}E[Y_3Y_2] + x_{n,3}E[Y_3^2]\right] \Big\} \end{aligned} \tag{4.12}$$

where from (4.5),

$$\begin{aligned} \zeta_{n,1} = \{ & x_{n,1}E[Y_1^2] + x_{n,2}E[Y_1Y_2] + x_{n,3}E[Y_1Y_3] + x_{n,1}E[Y_2Y_1] \\ & + x_{n,2}E[Y_2^2] + x_{n,3}E[Y_2Y_3] + x_{n,1}E[Y_3Y_1] \\ & + x_{n,2}E[Y_3Y_2] + x_{n,3}E[Y_3^2]\}^{-1}. \end{aligned} \tag{4.13}$$

The wealth increase $\widehat{\boldsymbol{b}}_{n+1}^t \boldsymbol{x}_{n+1}$ on day $n + 1$ for order 2 and 3 universal portfolios are given in Appendix A.

4.4 Performance of Order 1, 2, 3 Universal Portfolios on Three-Stock Data Sets A, B and C

In Section 3.3 we have introduced some selected 3-stock portfolios with designated stock-price data sets A, B and C. The 500-days trading period of A, B and C is over the period 2003 until 2004. In this chapter, we introduce some more recent 5-stock data sets selected from the Kuala Lumpur Stock Exchange (KLSE). There are 3 selected stock-price data sets designated as 1, 2 and 3 where each data set are consists of a 5-stock portfolio. The five Malaysian companies in each data set are listed in Table 4.1. The trading period is from 1st May 2005 until 31th December 2011, consisting of a total of 1728 trading days. Note that the abbreviations BAT, CIMB, IOI, YTL, PBB in

Table 4.1: Malaysian companies in the 5-stock portfolio in Sets 1, 2, 3

Data Set	Companies				
	1	2	3	4	5
Set 1	BAT	CIMB	Genting	IOI	AirAsia
Set 2	CIMB	Digi	IOI	BAT	Nestle
Set 3	Berjaya Corp.	HupSeng	YTL	Nestle	PBB

Table 4.1 represent the companies British American Tobacco, Commerce Industrial Malaysian Bank, Industrial Oxygen Industries, Yeo Tiong Lay and Public Bank Berhad respectively. The performance of the order 1,2 and 3 universal portfolios on data sets 1,2 and 3 will be discussed in Section 4.4.

First, we discuss the performance of the order 1,2 and 3 universal portfolios on data sets A, B, C. We select some common probability distributions listed in Table 3.2 as the generating distributions Y_1, Y_2 and Y_3. For this purpose, we need the first four moments of the generating distributions which are listed in Table 4.2.

Table $4.3 - 4.7$ show the total wealth S_{500} achieved by the low order universal portfolios generated by the same probability distribution with different parameters for the data sets. The computation time for one set of parameters is just less than second and it is very useful in computing the wealths of large portfolios or for large number of trading days.

Recall from Table 3.3 that the BCRP wealths achieved by Sets A, B and C are 1.8534, 4.2970 and 4.2970 units respectively and the highest wealths from the Monte Carlo simulation are 1.8313, 4.2970 and 4.2970 units respectively. The common probability distributions generating the low order universal portfolios in Tables $4.3 - 4.7$ are the lognormal distribution, gamma distribution, Pareto distribution, Weibull distribution and the inverse Gaussian distribution. The highest wealths achieved are close to the BCRP wealth, namely, (i) 1.8524 for order 1 universal portfolio generated by the inverse Gaussian$(100,0.01), (100,10), (100,20)$ distributions for Set A (Table 4.7), (ii) 4.1959 for the order 1 universal portfolio generated by the lognormal $(5,5), (25,5), (125,5)$ distributions for the two Sets B and C (Table 4.3). Another high wealth of 1.7668 is achieved for set A in Table 4.3 by the order 1 universal portfolio generated by the lognormal (5,5), (1,5), (5,5) distributions.

Table 4.2: First four moments of some common probability distributions where the corresponding probability density function are given in Table 3.2

Distribution	$E(X)$	$E(X^2)$	$E(X^3)$	$E(X^4)$
gamma (α, β)	$\alpha\beta$	$\alpha(\alpha+1)\beta^2$	$\alpha(\alpha+1)(\alpha+2)\beta^3$	$\alpha(\alpha+1)(\alpha+2)$ $\times(\alpha+3)\beta^4$
lognormal (μ, σ)	$e^{\mu+(\frac{\sigma^2}{2})}$	$e^{2\mu+(2\sigma^2)}$	$e^{3\mu+(\frac{3^2\sigma^2}{2})}$	$e^{4\mu+(8\sigma^2)}$
Pareto (α, β)	$\frac{\alpha\beta}{\alpha-1}$	$\frac{\alpha\beta^2}{\alpha-2}$	$\frac{\alpha\beta^3}{\alpha-3}$	$\frac{\alpha\beta^4}{\alpha-4}$
Weibull (κ, λ)	$\lambda\Gamma(1+\frac{1}{\kappa})$	$\lambda^2\Gamma(1+\frac{2}{\kappa})$	$\lambda^3\Gamma(1+\frac{3}{\kappa})$	$\lambda^4\Gamma(1+\frac{4}{\kappa})$
inverse Gaussian (μ, λ)	μ	$\frac{\mu^2(\lambda+\mu)}{\lambda}$	$\frac{\mu^3(\lambda^2+3\lambda\mu+3\mu^2)}{\lambda^2}$	$\frac{5\mu^2E(X^3)}{\lambda}+\mu^2E(X^2)$

Table 4.3: **Total wealth achieved by the low order universal portfolios generated from the lognormal distribution with different parameter sets for 3-stock portfolios in Sets A, B and C**

	$\sigma = 5$		Set A			Set B			Set C		
μ_1	μ_2	μ_3	**Order 1**	**Order 2**	**Order 3**	**Order 1**	**Order 2**	**Order 3**	**Order 1**	**Order 2**	**Order 3**
2	3	4	1.5977	1.6225	1.6473	3.7283	3.9490	3.9869	3.5005	3.8476	3.9471
0.1	0.5	5	1.6417	1.6417	1.6539	4.1952	4.1120	4.0451	4.1950	4.1120	4.0451
10	30	60	1.6417	1.6417	1.6539	4.1959	4.1120	4.0451	4.1959	4.1120	4.0451
5	25	125	1.6417	1.6417	1.6539	4.1959	4.1120	4.0451	4.1959	4.1120	4.0451
0.01	10	20	1.6417	1.6417	1.6539	4.1959	4.1120	4.0451	4.1959	4.1120	4.0451
5	1	5	1.7668	1.7332	1.7285	2.4436	2.3977	2.3653	2.4239	2.4122	2.3713
6	6	10	1.6417	1.6417	1.6539	4.1933	4.1120	4.0451	4.1925	4.1119	4.0451
2.5	10	15	1.6417	1.6417	1.6539	4.1958	4.1120	4.0451	4.1957	4.1120	4.0451

Table 4.4: Total wealth achieved by the low order universal portfolios generated from the gamma distribution with different parameter sets for 3-stock portfolios in Sets A, B and C

$\beta = 10$			Set A			Set B			Set C		
α_1	α_2	α_3	**Order 1**	**Order 2**	**Order 3**	**Order 1**	**Order 2**	**Order 3**	**Order 1**	**Order 2**	**Order 3**
0.5	0.5	0.5	1.5652	1.5481	1.5687	2.1477	2.1369	2.1219	1.8018	1.7986	1.7670
1	1	1	1.5647	1.5484	1.5701	2.1461	2.1348	2.1204	1.8016	1.7980	1.7659
5	7	9	1.5455	1.5354	1.5609	2.3844	2.3699	2.3541	2.0050	1.9949	1.9587
0.3	0.5	0.7	1.5391	1.5292	1.5531	2.4919	2.4778	2.4591	2.0928	2.0816	2.0455
0.2	0.3	0.8	1.5788	1.5704	1.5900	2.8917	2.8631	2.8333	2.5594	2.5361	2.4932
0.1	1.1	2.1	1.5019	1.5019	1.5311	2.9980	2.9777	2.9544	2.5219	2.4956	2.4520
1	1	50	1.6390	1.6380	1.6510	4.0433	3.9657	3.9035	4.0021	3.9261	3.8618
0.01	0.01	50	1.6417	1.6417	1.6539	4.1943	4.1105	4.0436	4.1939	4.1100	4.0432

Table 4.5: **Total wealth achieved by the low order universal portfolios generated from the Pareto distribution with different parameter sets for 3-stock portfolios in Sets A, B and C**

$\beta = 100$			Set A			Set B			Set C		
α_1	α_2	α_3	**Order 1**	**Order 2**	**Order 3**	**Order 1**	**Order 2**	**Order 3**	**Order 1**	**Order 2**	**Order 3**
5	5	5	1.5640	1.5488	1.5720	2.1436	2.1318	2.1183	1.8015	1.7973	1.7644
10	10	90	1.5604	1.5447	1.5683	2.0953	2.0825	2.0675	1.7514	1.7460	1.7113
25	50	100	1.5656	1.5503	1.5736	2.1319	2.1198	2.1061	1.7928	1.7887	1.7557
100	50	5	1.5714	1.5590	1.5838	2.2682	2.2722	2.2874	1.9311	1.9442	1.9431
5000	500	5	1.5728	1.5604	1.5851	2.2758	2.2799	2.2956	1.9405	1.9539	1.9535
10000	1000	10	1.5678	1.5535	1.5768	2.1999	2.1890	2.1770	1.8605	1.8575	1.8265
100	66	5	1.5720	1.5596	1.5843	2.2693	2.2734	2.2887	1.9331	1.9463	1.9454

Table 4.6: **Total wealth achieved by the low order universal portfolios generated from the Weibull distribution with different parameter sets for 3-stock portfolios in Sets A, B and C**

$\lambda = 50$			Set A			Set B			Set C		
κ_1	κ_2	κ_3	**Order 1**	**Order 2**	**Order 3**	**Order 1**	**Order 2**	**Order 3**	**Order 1**	**Order 2**	**Order 3**
1.2	1.4	3.2	1.5684	1.5527	1.5767	2.0525	1.9794	1.9139	1.7195	1.6627	1.5889
1	0.8	2.5	1.5062	1.4520	1.4460	1.9365	1.8438	1.7799	1.5383	1.4117	1.2870
3	3.8	4.7	1.5643	1.5499	1.5738	2.1447	2.1248	2.1037	1.8029	1.7925	1.7541
1.2	1.5	1.8	1.5742	1.5624	1.5883	2.0835	2.0322	1.9806	1.7561	1.7254	1.6691
19	15	0.1	1.6417	1.6417	1.6539	4.1959	4.1120	4.0451	4.1959	4.1120	4.0451
4.6	3	0.5	1.6256	1.6347	1.6513	3.4440	3.8394	3.9168	3.2631	3.7677	3.8793
3	5	4	1.5635	1.5507	1.5756	2.1422	2.1270	2.1098	1.7995	1.7957	1.7631

Table 4.7: Total wealth achieved by the low order universal portfolios generated from the inverse Gaussian distribution with different parameter sets for 3-stock portfolios in Sets A, B and C

$\mu = 100$			Set A			Set B			Set C		
λ_1	λ_2	λ_3	**Order 1**	**Order 2**	**Order 3**	**Order 1**	**Order 2**	**Order 3**	**Order 1**	**Order 2**	**Order 3**
2	3	4	1.6003	1.6109	1.6520	1.9275	1.7322	1.5834	1.6307	1.5093	1.4086
0.1	0.5	5	1.7357	1.7675	1.7717	1.4585	1.3735	1.3422	1.3213	1.3529	1.3395
10	30	60	1.6388	1.6895	1.7341	1.8166	1.5542	1.4254	1.5735	1.4404	1.3707
1	5	100	1.7256	1.7637	1.7702	1.4821	1.3811	1.3462	1.3355	1.3562	1.3414
5	25	125	1.6919	1.7461	1.7632	1.6375	1.4268	1.3658	1.4566	1.3824	1.3516
0.01	10	20	1.8524	1.7916	1.7762	1.3791	1.3581	1.3386	1.3551	1.3666	1.3422
5	1	5	1.3511	1.2236	1.2440	1.8266	1.6509	1.6124	1.2827	1.0316	0.9452
50	100	150	1.5850	1.5918	1.6360	2.0375	1.8963	1.7505	1.7244	1.6406	1.5371

A low wealth of 1.2236 is achieved for set A in Table 4.7 by the order 2 universal portfolio generated by the inverse Gaussian (100,5), (100,1), (100,5) distributions.

High wealths of 4.19 and above for sets B and C are achieved by order 1 universal portfolios generated by the lognormal, gamma and Weibull distributions in Table 4.3, 4.4 and 4.6 respectively. In these tables, most of the order 1 portfolios perform better than the order 2 and order 3 portfolios. An example of an order 3 portfolio performing better than order 1 and 2 portfolio is given in Table 4.5 for the portfolio generated by the Pareto (5000,100), (500,100) and (5,100) distributions, for sets A and B. In the same table with the same generating distributions, the order 2 portfolio is better than the orders 1 and 3 portfolios for set C.

In Table 4.3, the variation of the lognormal parameters does not lead to a big variation in the wealth achieved. In other words, the wealth achieved by the lognormal universal portfolio is more stable with respect to a variation in parameters. Although most of the wealths achieved by the lognormal universal portfolio in Table 4.3 are close to the BCRP, there is a case of poor performance by the lognormal order 3 portfolios with parameters (5,5), (1,5), (5,5) achieving wealths of 2.3653 and 2.3713 for sets B and C respectively.

The Pareto universal portfolio in Table 4.5 performs poorly, achieving wealths below 1.6 for set A, below 2.3 for set B and below 2 for set C. One way of improving performance is to use mixtures of different distribution families to generate the universal portfolio. Table $4.8 - 4.10$ show the wealths achieved by low order universal portfolios generated from mixture of different distribution families.

An improved performance of the Pareto universal portfolio is observed in Table 4.9, where the wealths achieved for sets B and C are above 3.3 when the generating Pareto (100,1) and (66,1) distributions are mixed with the Weibull (1,10) distribution. In Table 4.8, the mixture gamma-lognormal universal portfolio achieves wealths close to the BCRP wealths for the three data sets A, B and C. The inverse-Gaussian-lognormal misture universal portfolios in Table 4.10 provide significant improvements in the wealth achieved compared to the non-mixture inverse Gaussian universal portfolio in Table 4.7.

A local search in the parameter space may result in a better parametric vector generating a higher achieved wealth. Figure 4.1 shows the total wealth achieved from the inverse Gaussian $(\mu,\lambda_1),(\mu,\lambda_2),(\mu,\lambda_3)$ order 1 universal portfolio with $\mu = 100, \lambda_1 = 0.1$ and different combinations of λ_2 and λ_3. The parameter vector $(\lambda_1,\lambda_2,\lambda_3) = (0.1,70,29.9)$ of the inverse Gaussian universal portfolio gives a better wealth return of 1.8495 after a search in the region $\{(\lambda_2,\lambda_3): 0 < \lambda_2 < 100, 0 < \lambda_3 < 100\}$.

Table 4.8: Total wealth achieved by the low order universal portfolios generated from the gamma (0.2,10), lognormal (10,2) and lognormal (20,2) distributions for 3-stock portfolios in Sets A, B and C

$\beta = 10, \sigma = 2$			Set A			Set B			Set C		
α_1	μ_2	μ_3	Order 1	Order 2	Order 3	Order 1	Order 2	Order 3	Order 1	Order 2	Order 3
0.2	10	20	1.6417	1.6417	1.6539	4.1959	4.1120	4.0451	4.1959	4.1120	4.0451

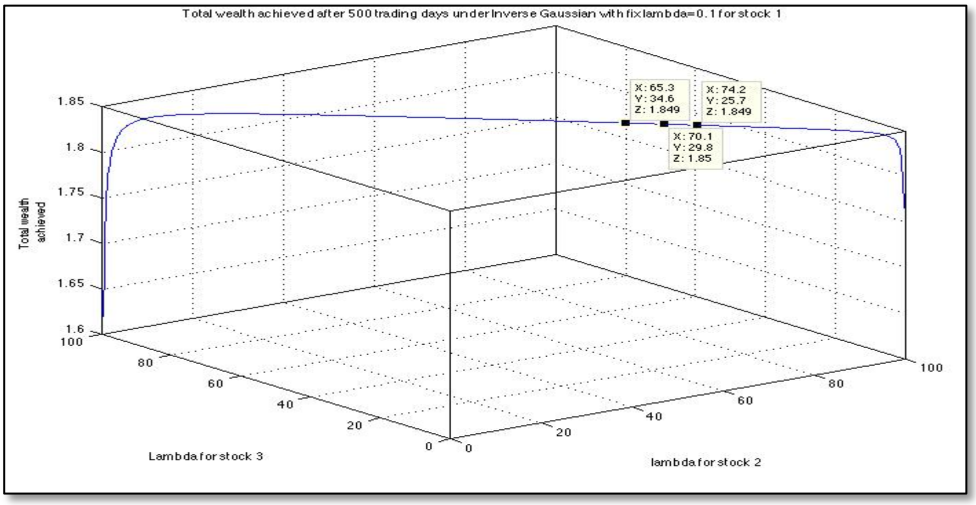
Table 4.9: Total wealth achieved by the low order universal portfolios generated from the Pareto (100,1), Pareto (66,1) and Weibull (1,10) distributions for 3-stock portfolios in Sets A, B and C

$\beta = 1, \lambda = 10$			Set A			Set B			Set C		
α_1	α_2	κ_3	Order 1	Order 2	Order 3	Order 1	Order 2	Order 3	Order 1	Order 2	Order 3
100	66	1	1.6282	1.6236	1.6395	3.5684	3.4935	3.4646	3.4140	3.3408	3.3083

Table 4.10: Total wealth achieved by the low order universal portfolios generated from inverse Gaussian (50,100), inverse Gaussian (50,66) and lognormal (1,2) distributions for 3-stock portfolios in Sets A, B and C

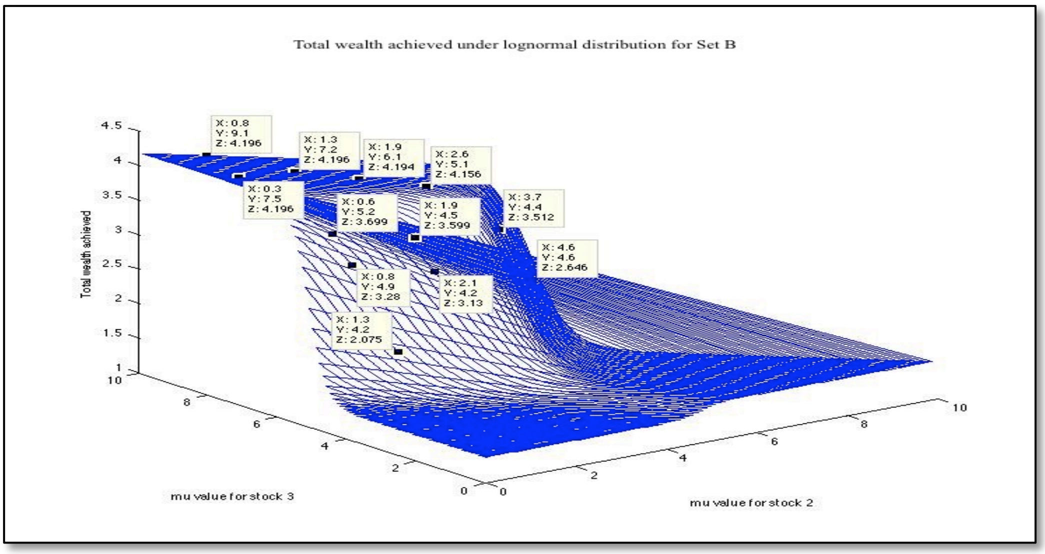
$\mu = 50, \sigma = 2$			Set A			Set B			Set C		
λ_1	λ_2	μ_3	Order 1	Order 2	Order 3	Order 1	Order 2	Order 3	Order 1	Order 2	Order 3
100	66	1	1.6282	1.6236	1.6395	3.5684	3.4935	3.4646	3.4140	3.3408	3.3083

Figure 4.1: **3D-graph of total wealth achieved after 500 trading days in Set A under the inverse Gaussian distribution with the parameters $\mu = 100, \lambda_1 = 0.1$ and different combination sets of λ_2 and λ_3**



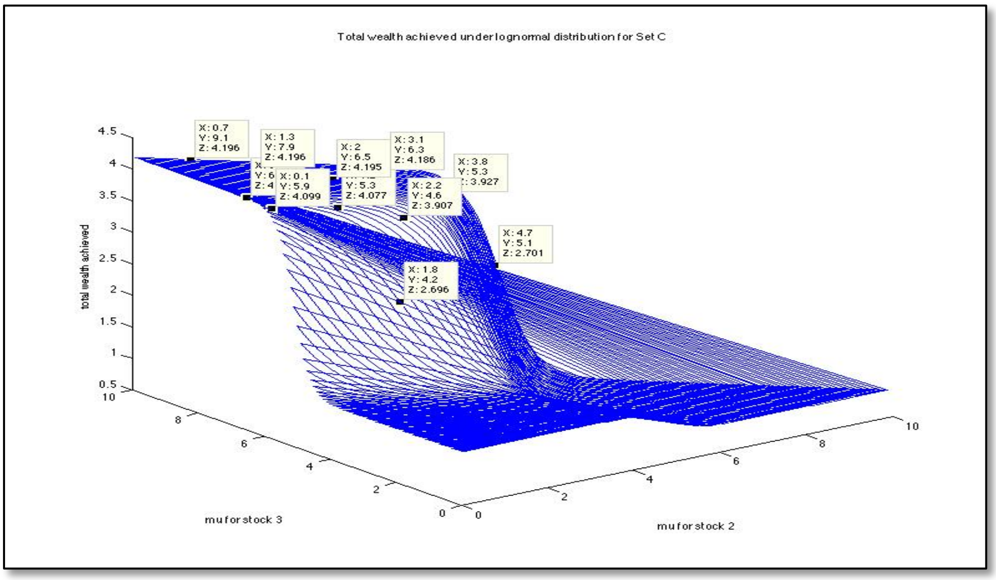
For set B, we search for a higher wealth for the lognormal $(\mu_1, \sigma), (\mu_2, \sigma), (\mu_3, \sigma)$ order 1 universal portfolio where $\sigma = 5, \mu_1 = 0.1$. Figure 4.2 shows the total wealth achieved by the lognormal universal portfolio for $0 < \mu_2 < 10, 0 < \mu_3 < 10$. A higher wealth of 4.196 is obtained for the

Figure 4.2: **3D-graph of total wealth achieved after 500 trading days in Set B under the lognormal distribution with the parameters $\sigma = 5, \mu_1 = 0.1$ and different combination sets of μ_2 and μ_3**



lognormal universal portfolio after a search in the region $\{(\mu_2, \mu_3): 0 < \mu_2 < 10, 0 < \mu_3 < 10\}$. The situation is more or less the same for set C, displayed in Figure 4.3 for the lognormal universal portfolio. A search in the same parametric region $\{(\mu_2, \mu_3): 0 < \mu_2 < 10, 0 < \mu_3 < 10\}$ produces a higher wealth of 4.196 for set C.

Figure 4.3: 3D-graph of total wealth achieved after 500 trading days in Set C under the lognormal distribution with the parameters $\boldsymbol{\sigma = 5, \mu_1 = 0.1}$ and different combination sets of $\boldsymbol{\mu_2}$ and $\boldsymbol{\mu_3}$



4.5 Performance of Orders 1, 2, 3 Universal Portfolios on Five-Stock Data Sets 1,2 and 3

We refer to the five-stock data sets 1,2 and 3 introduced in Section 4.3. The low order universal portfolios are run on the sets 1,2 and 3 and the performance is analyzed in this section.

The BCRP wealths are 2.6281, 3.6798 and 14.7707 for sets 1,2 and 3 respectively. Tables $4.11 - 4.13$ display the wealths achieved over a 1728-day trading period by the distribution-generated low order universal portfolios on data sets 1,2, and 3 respectively. One of the parameters of the distribution is fixed and the second parameter is varied to obtain a higher wealth return. For set 1, the highest wealth achieved is 2.5552 by the order 1 lognormal and Weibull universal portfolios in Table 4.11 which is close to the BCRP wealth of 2.6281. For set 2, the highest wealth achieved is 3.6817 by the orders 2 and 3 lognormal, Weibull and inverse Gaussian universal portfolios in Table 4.12 which exceeds the BCRP wealth of 3.6798 slightly. This is a very good result. Finally, for set 3, the highest wealth achieved is 13.8103 by the order 3 inverse Gaussian universal portfolios in Table 4.13 which is close to the BCRP wealth of 14.7707.

The order ν universal portfolio for a fixed ν is no longer a Cover-Ordentlich universal portfolio. Hence it is possible for a order ν universal portfolio to achieve a wealth higher that the BCRP wealth. This is an advantage over the Cover-Ordentlich universal portfolio where it is well-known than it cannot achieve a wealth higher than the BCRP wealth.

There is not enough evidence to say that order 1 universal portfolios can perform well in most data sets. However, it is clear that it is convenient to use in terms of the time and memory saved. Some of the empirical results in Sections 4.3 are presented in Tan and Lim (2012a). The finite-order universal

Table 4.11: Total wealth achieved by the low order universal portfolios generated from different distributions with good parameter sets for 5-stock portfolio in Data Set 1

Probability Distribution	Parameters: Fixed (One parameter)	Second Parameters 1	2	3	4	5	Order 1	2	3
lognormal(μ, σ)	$\sigma = 5$	5	1	5	100	5	2.5552	2.5261	2.5166
gamma(α, β)	$\beta = 10$	50	30	1	100	1	2.4612	2.4361	2.4318
Pareto(α, β)	$\beta = 1$	5	5	10	50	100	1.5918	1.5792	1.5799
Weibull(κ, λ)	$\lambda = 10$	1	0.8	2.5	0.1	5	2.5552	2.5262	2.5166
inverse Gaussian(μ, λ)	$\mu = 100$	5	1	5	1	5	2.0574	2.3167	2.3850

Table 4.12: Total wealth achieved by the low order universal portfolios generated from different distributions with good parameter sets for 5-stock portfolio in Data Set 2

Probability Distribution	Parameters						Order		
	Fixed (One parameter)	Second Parameters							
		1	2	3	4	5	1	2	3
lognormal(μ, σ)	$\sigma = 5$	0.1	0.5	5	10	50	3.6492	3.6817	3.6817
gamma(α, β)	$\beta = 10$	0.1	0.1	0.1	0.1	50	3.6443	3.6764	3.6764
Pareto(α, β)	$\beta = 1$	100	50	5	80	9	2.8859	2.8767	2.8757
Weibull(κ, λ)	$\lambda = 10$	0.5	5	50	3	0.1	3.6492	3.6817	3.6817
inverse Gaussian(μ, λ)	$\mu = 100$	50	1	5	2	0.01	3.6412	3.6817	3.6817

Table 4.13: Total wealth achieved by the low order universal portfolios generated from different distributions with good parameter sets for 5-stock portfolio in Data Set 3

Probability Distribution	Parameter						Order		
	Fixed (One parameter)	Second Parameters							
		1	**2**	**3**	**4**	**5**	**1**	**2**	**3**
lognormal(μ, σ)	$\sigma = 5$	100	5	1	20	0.1	11.5000	11.6806	11.6806
gamma(α, β)	$\beta = 10$	50	30	1	100	1	12.9514	13.1066	13.0247
Pareto(α, β)	$\beta = 1$	5	5	10	50	100	9.1311	9.2387	9.1959
Weibull(κ, λ)	$\lambda = 10$	0.4	9	0.4	1	2	12.0723	12.9440	13.3551
inverse Gaussian(μ, λ)	$\mu = 100$	2	3	4	5	6	10.4903	12.6943	13.8103

portfolio generated by dependent variables from the Dirichlet distribution has been studied by Tan, Chu and Lim(2012).

4.6 Further Discussion on the Poor Performance of the Pareto Universal Portfolio

From Tables 4.11, 4.12 and 4.13, it is observed that the low performance portfolios are generated from the Pareto distribution for orders 1,2 and 3. The comparison of the difference in wealths achieved by the Pareto and other distributions generating the universal portfolios is shown in Table (4.14)

For set 1, the total wealth achieved by the universal portfolio generated by the Pareto distribution for all orders is at least 29.25% lower than that of the universal portfolios generated from the inverse Gaussian distribution. For set 2, the wealth difference is 26.17% for the inverse Gaussian generating distribution of order 1, the wealth difference is 27.80% for the gamma distribution of orders 2 and 3. For set 3, the wealth difference is 14.88% for the inverse Gaussian generating distribution of order 1, the wealth differences are 26.43% and 27.02% for the lognormal distributions of orders 2 and 3 respectively. The inverse Gaussian, gamma and lognormal universal portfolios can achieve higher wealths than that of the Pareto universal portfolios. Using different probability distributions to generate the universal portfolios can improve the performance of the universal portfolio.

Table 4.14 : Comparison of the difference in wealths achieved by the Pareto and other distributions generating the universal portfolios

Data Set	Generating Probability Distribution	Order 1	Difference in Wealths (Percentage)	Generating Probability Distribution	Order 2	Difference in Wealths (Percentage)	Generating Probability Distribution	Order 3	Difference in Wealths (Percentage)
1	Pareto	1.5918	0.4656 (29.25%)	Pareto	1.5792	0.7375 (46.70%)	Pareto	1.5799	0.8051 (50.96%)
	Inverse Gaussian	2.0574		Inverse Gaussian	2.3167		Inverse Gaussian	2.3850	
2	Pareto	2.8859	0.7553 (26.17%)	Pareto	2.8767	0.7997 (27.80%)	Pareto	2.8757	0.8007 (27.80%)
	Inverse Gaussian	3.6412		gamma	3.6764		gamma	3.6764	
3	Pareto	9.1311	1.3592 (14.88%)	Pareto	9.2387	2.4419 (26.43%)	Pareto	9.1959	2.4847 (27.02%)
	Inverse Gaussian	10.4903		lognormal	11.6806		lognormal	11.6806	

CHAPTER 5

EMPIRICAL STUDY OF THE RATIO OF WEALTHS FOR SOME MOVING-ORDER UNIVERSAL PORTFOLIOS

When the order ν of a universal portfolio equals to the number of trading days, n it is known as a moving-order universal portfolio. The Cover-Ordentlich (1996) universal portfolio is a moving-order universal portfolio generated by the Dirichlet distribution of dependent random variables. We have seen in the last chapter that the finite order universal portfolio requires very much less memory and computational time for its implementation.

In this chapter, we study the ratio of wealths of the BCRP to that of the universal portfolio as a function of the number of trading days. For the Cover-Ordentlich Dirichlet universal portfolio, it is well-known that this ratio of wealths is bounded above by a polynomial in the number of trading days with degree depending on m, the number of stocks in the portfolio (see Cover and Ordentlich (1996) and Tan (2002)). In this chapter, we shall consider a moving-order universal portfolio generated by m independent random variables. Empirically, we shall run the moving-order universal portfolios generated by three independent gamma variables on three-stock data sets selected from the Kuala Lumpur Stock Exchange (KLSE).

5.1 An Algorithm for Computing the Moving-Order Universal Portfolio

In this section, we describe an algorithm for computing the moving-order universal portfolio generated by m independent random variables. This algorithm is due to Tan (2013). Consider a m-stock market. Let $Y_1, Y_2, \dots, Y_m$ be m independent random variables where Y_j has the probability density function $f_Y(y_j|\theta_j)$ for $j = 1,2,\dots,m$. Then the moving order universal portfolio generated by $Y_1, Y_2, \dots, Y_m$ is the sequence $\{\widehat{\boldsymbol{b}}_{n+1}\}$ of the portfolios where the portfolio component for stock k on day $n+1$ is given by:

$$\hat{b}_{n+1,k}$$

$$= \frac{\int_D y_k (\mathbf{y}^t\mathbf{x}_n)(\mathbf{y}^t\mathbf{x}_{n-1}) \dots (\mathbf{y}^t\mathbf{x}_1) \prod_{j=1}^{m} f_{Y_j}\left(y_j|\theta_j\right) dy_1 dy_2 \dots dy_m}{\begin{array}{c}\int_D (y_1 + y_2 + \cdots + y_m)(\mathbf{y}^t\mathbf{x}_n)(\mathbf{y}^t\mathbf{x}_{n-1}) \dots (\mathbf{y}^t\mathbf{x}_1) \\ \times \prod_{j=1}^{m} f_{Y_j}\left(y_j|\theta_j\right) dy_1 dy_2 \dots dy_m\end{array}}$$

$$= \frac{\int_D y_k (\prod_{i=1}^{n} \mathbf{y}^t\mathbf{x}_i)(\prod_{j=1}^{m} f_{Y_j}(y_j|\theta_j)) d\mathbf{y}}{\int_D (y_1 + y_2 + \cdots + y_m)(\prod_{i=1}^{n} \mathbf{y}^t\mathbf{x}_i)(\prod_{j=1}^{m} f_{Y_j}(y_j|\theta_j)) d\mathbf{y}} \qquad (5.1)$$

for $k = 1,2,\dots,m$, where $\boldsymbol{y} = (y_1, y_2, \dots, y_m)$, $\boldsymbol{x}_n = (x_{n,1}, x_{n,2}, \dots, x_{n,m})$, $d\boldsymbol{y} = dy_1 dy_2 \dots dy_m$; integration in (5.1) is over $D = \{\boldsymbol{y}: \prod_{i=1}^{m} f_{Y_i}(y_i|\theta_i) > 0\}$.

We note that (5.1) can be simplified by interchanging integration and summation:

$$\hat{b}_{n+1,k}$$

$$= \frac{\int_D y_k(y_1x_{1,1}+y_2x_{1,2}+\cdots+y_mx_{1,m})(y_1x_{2,1}+y_2x_{2,2}+\cdots+y_mx_{2,m})\ldots \times(y_1x_{n,1}+y_2x_{n,2}+\cdots+y_mx_{n,m})(\prod_{j=1}^{m} f_{Y_j}(y_j|\theta_j))d\mathbf{y}}{\int_D (y_1+y_2+\cdots+y_m)(y_1x_{1,1}+y_2x_{1,2}+\cdots+y_mx_{1,m}) \times(y_1x_{2,1}+y_2x_{2,2}+\cdots+y_mx_{2,m})\ldots(y_1x_{n,1}+y_2x_{n,2}+\cdots+y_mx_{n,m}) \times(\prod_{j=1}^{m} f_{Y_j}(y_j|\theta_j))d\mathbf{y}}$$

$$= \zeta_n^{-1}\left\{\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_n=1}^{m}(x_{1,i_1}x_{2,i_2}\ldots x_{n,i_n})\prod_{j=1}^{m}E\left[Y_j^{a_j(k;\boldsymbol{i})}\right]\right\} \tag{5.2}$$

for $k = 1,2,\ldots,m$, where the normalizing constant,

$$\zeta_n = \left\{\sum_{k=1}^{m}\left\{\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_n=1}^{m}(x_{1,i_1}x_{2,i_2}\ldots x_{n,i_n})\prod_{j=1}^{m}E\left[Y_j^{a_j(k;\boldsymbol{i})}\right]\right\}\right\} \tag{5.3}$$

for $\boldsymbol{i} = (i_1, i_2, \ldots, i_n)$ which $1 \le i_l \le m$ for $l = 1,2,\ldots,n$ and the $a_j(k;\ \boldsymbol{i})$ is the number of y_j's in the product of $(y_k y_{i_1} y_{i_2} \ldots y_{i_n})$ for $j = 1,2,\ldots,m$. As n grows larger, the products $x_{1i_1}x_{2i_2}\ldots x_{ni_n}$ and $\prod_{j=1}^{m}E\left[Y_j^{a_j(k;\,\boldsymbol{i})}\right]$ grow longer together with the number of terms to be summed. Therefore we need recursive formulae for calculating the products $x_{1i_1}x_{2i_2}\ldots x_{ni_n}$ and $\prod_{j=1}^{m}E\left[Y_j^{a_j(k;\,\boldsymbol{i})}\right]$.

We note that the numerator of $\hat{b}_{n+1,k}$ in (5.1) can be written as

$$\int_D y_k(\prod_{i=1}^{n}\mathbf{y}^t\mathbf{x}_i)(\prod_{j=1}^{m} f_{Y_j}(y_j|\theta_j))d\mathbf{y}$$

$$= \sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_n=1}^{m}(x_{1,i_1}x_{2,i_2}\ldots x_{n,i_n})\prod_{j=1}^{m}E\left[Y_j^{a_j(k;\boldsymbol{i})}\right]$$

$$= \sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_n=1}^{m}(x_{1,i_1}x_{2,i_2}\ldots x_{n,i_n})E\left[Y_k^{a_k(\boldsymbol{i})+1}\right]\prod_{j=1,j\neq k}^{m}E\left[Y_j^{a_j(\boldsymbol{i})}\right] \tag{5.4}$$

because $a_k(k;\boldsymbol{i}) = a_k(\boldsymbol{i}) + 1$, $a_j(k;\boldsymbol{i}) = a_j(\boldsymbol{i})$ for $j \neq k$ and $Y_1, Y_2, \dots, Y_m$ are mutually independent. Now (5.4) can be evaluated as:

$$\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_n=1}^{m}\left(x_{1,i_1}x_{2,i_2}\dots x_{n,i_n}\right)E\left[Y_k^{a_k(\boldsymbol{i})+1}\right]\prod_{j=1,j\neq k}^{m}E\left[Y_j^{a_j(\boldsymbol{i})}\right]$$

$$= \sum_{a_1(\boldsymbol{i})+a_2(\boldsymbol{i})+\cdots+a_m(\boldsymbol{i})=n} \begin{array}{c} X_n\left(a_1(\boldsymbol{i}), a_2(\boldsymbol{i}), \dots, a_m(\boldsymbol{i})\right) \\ \times E\left[Y_k^{a_k(\boldsymbol{i})+1}\right]\prod_{j=1,j\neq k}^{m}E\left[Y_j^{a_j(\boldsymbol{i})}\right] \end{array} \tag{5.5}$$

where $X_n\left(a_1(\boldsymbol{i}), a_2(\boldsymbol{i}), \dots, a_m(\boldsymbol{i})\right)$ is defined as:

$$\sum_{a_1(\boldsymbol{i})+a_2(\boldsymbol{i})+\cdots+a_m(\boldsymbol{i})=n} x_{1,i_1}x_{2,i_2}\dots x_{n,i_n} = X_n\left(a_1(\boldsymbol{i}), a_2(\boldsymbol{i}), \dots, a_m(\boldsymbol{i})\right) \tag{5.6}$$

where $a_j(\boldsymbol{i})$ is the number of y_j 's in the product $y_{i_1}y_{i_2}\dots y_{i_n}$ and $\boldsymbol{i} = (i_1, i_2, \dots, i_n)$. Define

$$R_n\left(a_1(\boldsymbol{i}), a_2(\boldsymbol{i}), \dots, a_m(\boldsymbol{i})\right) = E\left[Y_1^{a_1(i)}\right]E\left[Y_2^{a_2(i)}\right]\dots E\left[Y_m^{a_m(i)}\right], \tag{5.7}$$

where $a_1(\boldsymbol{i}) + a_2(\boldsymbol{i}) + \cdots + a_m(\boldsymbol{i}) = n$. Then

$$E\left[Y_k^{a_k(\boldsymbol{i})+1}\right]\prod_{j=1,j\neq k}^{m}E\left[Y_j^{a_j(\boldsymbol{i})}\right] = \frac{E\left[Y_k^{a_k(\boldsymbol{i})+1}\right]}{E\left[Y_k^{a_k(\boldsymbol{i})}\right]}\prod_{j=1}^{m}E\left[Y_j^{a_j(\boldsymbol{i})}\right]$$

$$= \frac{E\left[Y_k^{a_k(\boldsymbol{i})+1}\right]}{E\left[Y_k^{a_k(\boldsymbol{i})}\right]}R_n(a_1, a_2, \dots a_k, \dots, a_m)$$

$$= R_{n+1}(a_1, a_2, \dots, a_k + 1, \dots, a_m) \tag{5.8}$$

where $a_k(\boldsymbol{i}) \geq 1$. The recursive formula for calculating the moment function $R_n(a_1, a_2, \dots, a_m)$ in (5.7) is given by (5.8). So, the recursive algorithm in updating the moment from day n to $n+1$ for portfolio k is to multiply the ratio of $\frac{E\left[Y_k^{a_k(i)+1}\right]}{E\left[Y_k^{a_k(i)}\right]}$, which depends on the probability distribution of the random variable Y_k.

Therefore, (5.5), the numerator of $\hat{b}_{n+1,k}$ can be written

$$\int_{\mathrm{D}} \mathrm{y_k}\Big(\prod_{\mathrm{i=1}}^{\mathrm{n}} \mathbf{y}^{\mathrm{t}}\mathbf{x}_{\mathrm{i}}\Big)\Big(\prod_{\mathrm{j=1}}^{\mathrm{m}} \mathrm{f}_{\mathrm{Y_j}}(\mathrm{y_j}|\theta_{\mathrm{j}})\Big)\mathrm{d}\mathbf{y}$$

$$= \sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_n=1}^{m}\left(x_{1,i_1}x_{2,i_2}\dots x_{n,i_n}\right)E\left[Y_k^{a_k(\boldsymbol{i})+1}\right]\prod_{j=1,j\neq k}^{m} E\left[Y_j^{a_j(\boldsymbol{i})}\right]$$

$$= \sum_{a_1(\boldsymbol{i})+a_2(\boldsymbol{i})+\cdots+a_m(\boldsymbol{i})=n} \begin{matrix} X_n\big(a_1(\boldsymbol{i}),a_2(\boldsymbol{i}),\dots,a_m(\boldsymbol{i})\big) \\ \times R_{n+1}(a_1(\boldsymbol{i}),\dots,a_k(\boldsymbol{i})+1,\dots,a_m(\boldsymbol{i})) \end{matrix}. \tag{5.9}$$

The denominator of $\hat{b}_{n+1,k}$ in (5.1) and from (5.3) is given by:

$$\int_{\mathrm{D}} (\mathrm{y_1}+\mathrm{y_2}+\cdots+\mathrm{y_m})\Big(\prod_{\mathrm{i=1}}^{\mathrm{n}} \mathbf{y}^{\mathrm{t}}\mathbf{x}_{\mathrm{n}}\Big)\Big(\prod_{\mathrm{j=1}}^{\mathrm{m}} \mathrm{f}_{\mathrm{Y_j}}(\mathrm{y_j}|\theta_{\mathrm{j}})\Big)\mathrm{d}\mathbf{y}$$

$$= \sum_{k=1}^{m}\left\{\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\cdots\sum_{i_n=1}^{m}\left(x_{1,i_1}x_{2,i_2}\dots x_{n,i_n}\right)\prod_{j=1}^{m} E\left[Y_j^{a_j(k;\boldsymbol{i})}\right]\right\}$$

$$= \sum_{k=1}^{m}\left\{\sum_{a_1(\boldsymbol{i})+a_2(\boldsymbol{i})+\cdots+a_m(\boldsymbol{i})=n} \begin{matrix} X_n\big(a_1(\boldsymbol{i}),a_2(\boldsymbol{i}),\dots,a_m(\boldsymbol{i})\big) \\ \times R_{n+1}(a_1(\boldsymbol{i}),\dots,a_k(\boldsymbol{i})+1,\dots,a_m(\boldsymbol{i})) \end{matrix}\right\} \tag{5.10}$$

Hence, in summary, from (5.9) and (5.10),

$\hat{\mathrm{b}}_{\mathrm{n+1,k}}$

$$= \frac{\int_{\mathrm{D}} \mathrm{y_k}(\prod_{\mathrm{i=1}}^{\mathrm{n}} \mathbf{y}^{\mathrm{t}}\mathbf{x}_{\mathrm{i}})(\prod_{\mathrm{j=1}}^{\mathrm{m}} \mathrm{f}_{\mathrm{Y_j}}(\mathrm{y_j}|\theta_{\mathrm{j}}))\mathrm{d}\mathbf{y}}{\int_{\mathrm{D}} (\mathrm{y_1}+\mathrm{y_2}+\cdots+\mathrm{y_m})(\prod_{\mathrm{i=1}}^{\mathrm{n}} \mathbf{y}^{\mathrm{t}}\mathbf{x}_{\mathrm{n}})(\prod_{\mathrm{j=1}}^{\mathrm{m}} \mathrm{f}_{\mathrm{Y_j}}(\mathrm{y_j}|\theta_{\mathrm{j}}))\mathrm{d}\mathbf{y}}$$

$$= \frac{\sum_{a_1+a_2+\cdots+a_m=n} X_n(a_1,a_2,\dots,a_m)R_{n+1}(a_1,a_2,\dots a_k+1,\dots,a_m)}{\sum_{k=1}^{m}\left\{\sum_{a_1+a_2+\cdots+a_m=n} \begin{matrix} X_n(a_1,a_2,\dots,a_m) \\ \times R_{n+1}(a_1,a_2,\dots a_k+1,\dots,a_m) \end{matrix}\right\}} \tag{5.11}$$

for $k = 1,2,\dots,m$, where for simplicity, we suppress the vector $\boldsymbol{i}$ in $a_j(\boldsymbol{i})$ for $j = 1,2,\dots,m$, $X_n(a_1,a_2,\dots,a_m)$ and $R_n(a_1,a_2,\dots,a_m)$ are defined by (5.6) and (5.7) respectively. The recursive formula for calculating $R_n(a_1, a_2, \dots, a_m)$ is given by (5.8). Similarly $X_n(a_1, a_2, \dots, a_m)$ can be calculated recursively as:

$$X_n(a_1, a_2, \dots, a_m) = \sum_{j=1}^{m} x_{nj} X_{n-1}(a_1, .., a_j - 1, \dots, a_m). \tag{5.12}$$

5.2 Recursive Calculation of the Moment Function $R_n(a_1, a_2, \dots, a_m)$ for the Gamma, Lognormal and the Pareto Distributions

In (5.8), the ratio of $\frac{E\left[Y_k^{a_k+1}\right]}{E\left[Y_k^{a_k}\right]}$ for updating the moment function $R_n(n_1, n_2, \dots, n_m)$ depends on the parameters of the probability distribution of the random variable Y_k. In general, we have the recurrence relationship given that $a_1 + a_2 + \cdots + a_m = n$ and l is any positive integer,

$$R_{n+l}(a_1, a_2, \dots, a_k + l, \dots, a_m)$$

$$= \left(\frac{E\left[Y_k^{a_k+l}\right]}{E\left[Y_k^{a_k}\right]} \right) R_n(a_1, a_2, \dots, a_k, \dots, a_m) \tag{5.13}$$

is the more general recursive relationship.

For some probability distributions, the ratio $\left(\frac{E\left[Y_k^{a_k+l}\right]}{E\left[Y_k^{a_k}\right]} \right)$ may also depend on a_k and l besides the distribution parameters. For example, consider the following Laplace probability density function of Y_k:

$$f(y_k) = \frac{1}{2\beta_k} e^{-\frac{|y_k|}{\beta_k}}, \text{for } -\infty < y_k < \infty, \beta_k > 0.$$

Then the moment is given by:

$$E\left[Y_k^{a_k}\right] = \begin{cases} 0, & for\ a_k = odd \\ a_k!\,\beta_k^{a_k}, & for\ a_k = even \end{cases}$$

For a_k an even integer, $\left(\frac{E\left[Y_k^{a_k+2}\right]}{E\left[Y_k^{a_k}\right]}\right) = (a_k + 2)(a_k + 1)\beta_k^2$.

Next, we compute the updating ratio $\left(\frac{E\left[Y_k^{a_k+1}\right]}{E\left[Y_k^{a_k}\right]}\right)$ for 3 common probability distributions, namely, the gamma, lognormal and Pareto distributions.

(a) Gamma $(\boldsymbol{\alpha_k}, \boldsymbol{\beta_k})$ distribution

The probability density function of the gamma (α_k, β_k) distribution is:

$$f(y_k) = \frac{1}{\Gamma(\alpha_k)\beta_k^{\alpha_k}} e^{-\frac{y_k}{\beta_k}} y^{\alpha_k - 1}$$

where $y_k > 0, \alpha_k > 0, \beta_k > 0$, for $k = 1,2, \ldots, m$. The a_k^{th} moment is:

$$E\left[Y_k^{a_k}\right] = \int_0^\infty \frac{1}{\Gamma(\alpha_k)\beta_k^{\alpha_k}} e^{-\frac{y_k}{\beta_k}} y^{\alpha_k + a_k - 1}\ dy_k$$

$$= \frac{\Gamma(\alpha_k + a_k)\beta_k^{\alpha_k + a_k}}{\Gamma(\alpha_k)\beta_k^{\alpha_k}}$$

$$= \frac{\Gamma(\alpha_k + a_k)}{\Gamma(\alpha_k)}\beta_k^{a_k}.$$

Then,

$$\frac{E\left[Y_k^{a_k+1}\right]}{E\left[Y_k^{a_k}\right]} = \frac{\Gamma(\alpha_k + a_k + 1)}{\Gamma(\alpha_k)}\beta_k^{a_k+1} \cdot \frac{\Gamma(\alpha_k)}{\Gamma(\alpha_k + a_k)\beta_k^{a_k}}$$

$$= (\alpha_k + a_k)\beta_k.$$

(b) Lognormal (μ_k, σ_k) distribution

The probability density function of the lognormal (μ_k, σ_k) distribution is:

$$f(y_k) = \frac{1}{\sigma_k y_k \sqrt{2\pi}} e^{-\frac{(\ln y_k - \mu_k)^2}{2\sigma_k^2}}$$

where $y_k > 0, \mu_k > 0, \sigma_k > 0$, for $k = 1,2,\dots,m$. The a_k^{th} moment is:

$$E\left[Y_k^{a_k}\right] = \int_0^\infty \frac{1}{\sigma_k \sqrt{2\pi}} e^{-\frac{(\ln y_k - \mu_k)^2}{2\sigma_k^2}} y_k^{a_k - 1}\, dy_k$$

$$= e^{a_k \mu_k + \left(\frac{a_k^2 \sigma_k^2}{2}\right)}$$

Then,

$$\frac{E\left[Y_k^{a_k+1}\right]}{E\left[Y_k^{a_k}\right]} = \frac{e^{(a_k+1)\mu_k + \left(\frac{(a_k+1)^2 \sigma_k^2}{2}\right)}}{e^{a_k \mu_k + \left(\frac{a_k^2 \sigma_k^2}{2}\right)}}$$

$$= e^{(a_k-1)\mu_k - a_k\mu_k} \cdot e^{\frac{(a_k^2 + 2a_k + 1)\sigma_k^2 - a_k^2\sigma_k^2}{2}}$$

$$= e^{\mu_k + \frac{(2a_k+1)\sigma_k^2}{2}}.$$

(c) Pareto (α_k, β_k) distribution

The probability density function of the Pareto (α_k, β_k) distribution is:

$$f(y_k) = \frac{\alpha_k {\beta_k}^{\alpha_k}}{y_k^{\alpha_k + 1}}$$

where $y_k > \beta_k, \alpha_k > 0, \beta_k > 0$, for $k = 1,2,\dots,m$. The a_k^{th} moment for the Pareto distribution only exists for $\alpha_k > a_k$ and is denoted by:

$$E\left[Y_k^{a_k}\right] = \int_0^{\infty} \frac{\alpha_k {\beta_k}^{\alpha_k}}{y_k^{\alpha_k - a_k + 1}}\, dy_k$$

$$= \frac{\alpha_k \beta_k^{a_k}}{\alpha_k - a_k}.$$

Then,

$$\frac{E\left[Y_k^{a_k+1}\right]}{E\left[Y_k^{a_k}\right]} = \frac{\alpha_k \beta_k^{a_k+1}}{\alpha_k - (a_k + 1)} \times \frac{\alpha_k - a_k}{\alpha_k \beta_k^{a_k}}$$

$$= \frac{\beta_k(\alpha_k - a_k)}{\alpha_k - a_k - 1}.$$

The summary of the recurrence relationships in the moment functions for the three probability distributions (a), (b) and (c) are listed in Table 5.1.

Table 5.1: Summary of the recursive formulae for $R_{n+1}(a_1, a_2, \dots, a_k + 1, \dots, a_m)$ for the gamma, lognormal and Pareto distributions

Probability Distribution	$R_{n+1}(a_1, a_2, \dots, a_k + 1, \dots, a_m)$
gamma (α_k, β_k)	$(\alpha_k + a_k)\beta_k R_n(a_1, a_2, \dots, a_k, \dots, a_m)$
lognormal (μ_k, σ_k)	$e^{\mu_k + \frac{(2a_k+1)\sigma_k^2}{2}} R_n(a_1, a_2, \dots, a_k, \dots, a_m)$
Pareto (α_k, β_k)	$\frac{\beta_k(\alpha_k - a_k)}{\alpha_k - a_k - 1} R_n(a_1, a_2, \dots, a_k, \dots, a_m), a_k < \alpha_k$

5.3 Recursive Calculation of the Wealth Function

The wealth $\hat{S}_n(\boldsymbol{x}^n)$ at the end of the n^{th} trading day can be calculated recursively as：

$$\hat{S}_{n+1} = \prod_{i=1}^{n+1} \hat{\boldsymbol{b}}_i^t \boldsymbol{x}_i = (\hat{\boldsymbol{b}}_{n+1}^t \boldsymbol{x}_{n+1}) \hat{S}_n \tag{5.14}$$

where $\hat{S}_n = \prod_{i=1}^{n} \hat{\boldsymbol{b}}_i^t \boldsymbol{x}_i$. We now derive a formula based on $X_n(a_1, \dots, a_m)$ and $R_n(a_1, \dots, a_m)$ for update $\hat{\boldsymbol{b}}_{n+1}^t \boldsymbol{x}_{n+1}$.

First, note that the recurrence for $X_n(a_1, \dots, a_m)$ in (5.12) can also be written in another form. Given $a_1 + a_2 + \cdots + \cdots a_k + \cdots + a_m = n$ and the knowledge of $\boldsymbol{x}_{n+1}$, then

$$\begin{aligned} &X_{n+1}(a_1, a_2, \dots, a_k + 1, \dots, a_m) \\ &\quad = \sum_{j=1, j \neq k}^{m} x_{n+1,j} X_n(a_1, a_2, \dots, a_j - 1, \dots, a_m) \\ &\qquad\qquad + x_{n+1,k} X_n(a_1, a_2, \dots, a_k, \dots, a_m). \end{aligned} \tag{5.15}$$

From (5.1) and (5.11), we obtain

$$\begin{aligned} &\hat{b}_{n+1,k} x_{n+1,k} \\ &= \frac{\int_D y_k x_{n+1,k} \left(\prod_{i=1}^{n} \boldsymbol{y}^t \boldsymbol{x}_i\right) f(\boldsymbol{y}) d\boldsymbol{y}}{\int_{\boldsymbol{D}} (y_1 + y_2 + \cdots + y_m) \int_D \left(\prod_{i=1}^{n} \boldsymbol{y}^t \boldsymbol{x}_i\right) f(\boldsymbol{y}) d\boldsymbol{y}} \\ &= \frac{\sum_{a_1 + a_2 + \cdots + a_m = n} x_{n+1,k} X_n(a_1, a_2, \dots, a_m) R_{n+1}(a_1, a_2, \dots a_k + 1, \dots, a_m)}{\sum_{k=1}^{m} \left\{\sum_{a_1 + a_2 + \cdots + a_m = n} X_n(a_1, a_2, \dots, a_m) R_{n+1}(a_1, a_2, \dots a_k + 1, \dots, a_m)\right\}} \end{aligned}$$

and hence,

$$\begin{aligned} &\hat{\boldsymbol{b}}_{n+1} \boldsymbol{x}_{n+1} \\ &\quad = \frac{\sum_{a_1 + a_2 + \cdots + a_m = n+1} X_{n+1}(a_1, a_2, \dots, a_m) R_{n+1}(a_1, a_2, \dots, a_m)}{\sum_{k=1}^{m} \left\{\sum_{a_1 + a_2 + \cdots + a_m = n} \begin{matrix} X_n(a_1, a_2, \dots, a_m) \\ \times R_{n+1}(a_1, a_2, \dots a_k + 1, \dots, a_m) \end{matrix}\right\}} \end{aligned} \tag{5.16}$$

The wealth function $\hat{S}_{n+1}(\boldsymbol{x}^{n+1})$ is calculated recursively using (5.14) and (5.16).

5.4 Empirical Study of the Ratio of Wealths for the Moving-Order Universal Portfolios Generated by Three Independent Gamma Variables

Let $S_n^*(\boldsymbol{x}^n)$ be the BCRP wealth given the market information $\boldsymbol{x}^n$. Cover and Ordentlich (1996) showed that if $\hat{S}_n(\boldsymbol{x}^n)$ is the universal wealth achieved by the Dirichlet $\left(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}\right)$ universal portfolio, then the ratio of wealths $\frac{S_n^*(\boldsymbol{x}^n)}{\hat{S}_n(\boldsymbol{x}^n)}$ satisfies the following upper bound, independent of $\boldsymbol{x}^n$, where m is the number of stocks: :

$$\frac{S_n^*(\boldsymbol{x}^n)}{\hat{S}_n(\boldsymbol{x}^n)} \leq 2(n+1)^{\frac{m-1}{2}} \tag{5.17}$$

In this section, we study the ratio of wealths $\frac{S_n^*(\boldsymbol{x}^n)}{\hat{S}_n(\boldsymbol{x}^n)}$ where $\hat{S}_n(\boldsymbol{x}^n)$ is the wealth of a moving-order universal portfolio generated by three independent gamma random variables with parameters $(\alpha_1, \beta), (\alpha_2, \beta), (\alpha_3, \beta)$. We fix β at a certain value and hence for simplicity, we refer to the parameter vector as $(\alpha_1, \alpha_2, \alpha_3)$.

Our data sets for this empirical study are chosen from the subsets of the data sets 1,2 and 3 introduced in Table 4.1 (see section 4.3). We reduce the 5-stock portfolios in Table 4.1 to 3-stock portfolios in Table 5.2. Data sets D and F are subsets of sets 1 and 3 respectively. Set E consists of subsets of sets 2 and 3. The stocks cover an 800-day trading period starting from 1st May 2005.

Table 5.2: Malaysian companies in the 3-stock portfolios in sets D, E, F which are subsets of 1, 2 and 3 in Table 4.1

Set	**Company**		
	1	**2**	**3**
D	BAT	CIMB	IOI
E	DIGI	NESTLE	PBB
F	BERJAYA CORP	HUPSENG	YTL

Using the algorithm described in Section 5.1, we run the moving-order universal portfolios generated by three independent gamma random variables on the data sets D, E and F. For simplicity, we refer to the generating distributions as gamma$(\alpha_1, \alpha_2, \alpha_3)$ with β fixed at a certain value.

Figures 5.2, 5.4 and 5.6 display the ratio of the BCRP wealth to the universal portfolio wealth as a function of the number of trading days for the universal portfolios generated by the gamma$(\alpha_1, \alpha_2, \alpha_3)$ distribution for four different sets of $(\alpha_1, \alpha_2, \alpha_3)$ with β fixed at 2, 2, 2 respectively for sets D, E and F respectively. The 3-dimensional views of the four functions shown in Figures 5.2, 5.4 and 5.6 are shown in Figures 5.1, 5.3 and 5.5 respectively.

The maximum ratio of $\frac{S_n^*(x^n)}{\hat{S}_n(x^n)}$ over the 800 trading days are 3.3734 for Set D at $\boldsymbol{\alpha} = (0.1,100,3)$ in Figure 5.2, 11.3292 for Set E at $\boldsymbol{\alpha} = (2,4,50)$ in Figure 5.4 and 11.2542 for Set F at $\boldsymbol{\alpha} = (3,100,3)$ in Figure 5.6.

Figure 5.1: The 3-dimensional view of four functions of the ratio wealths against the number of trading days given in Figure 5.2 for set D

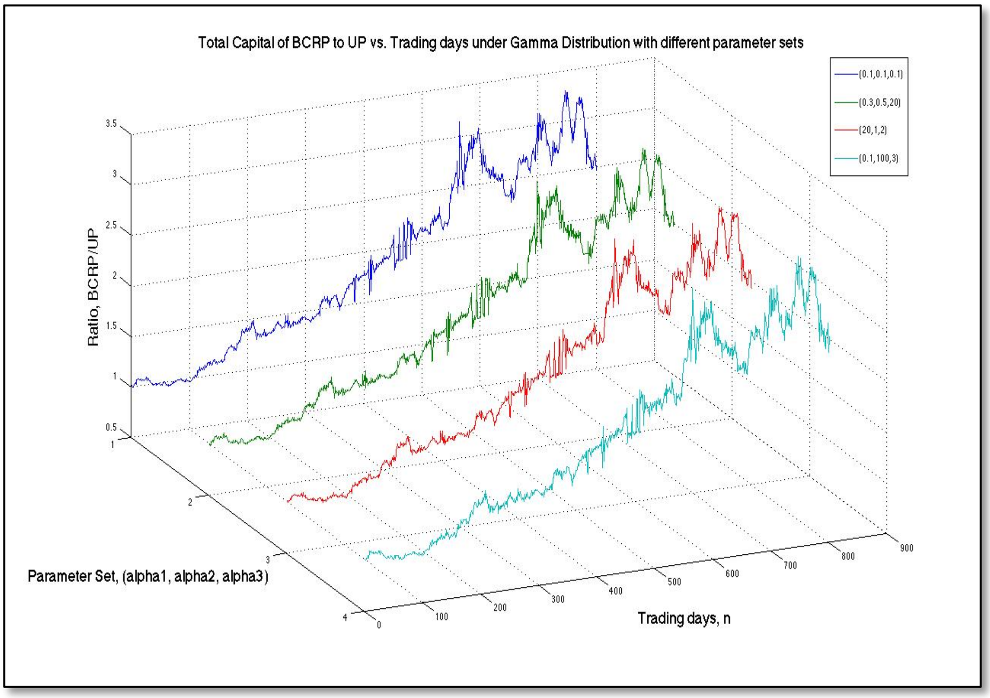


Figure 5.2: Ratio of BCRP wealth to universal portfolio wealth for the universal portfolios generated by the gamma$(\alpha_1, \alpha_2, \alpha_3)$ distribution for four sets of $(\alpha_1, \alpha_2, \alpha_3)$ with β fixed at 2 for set D

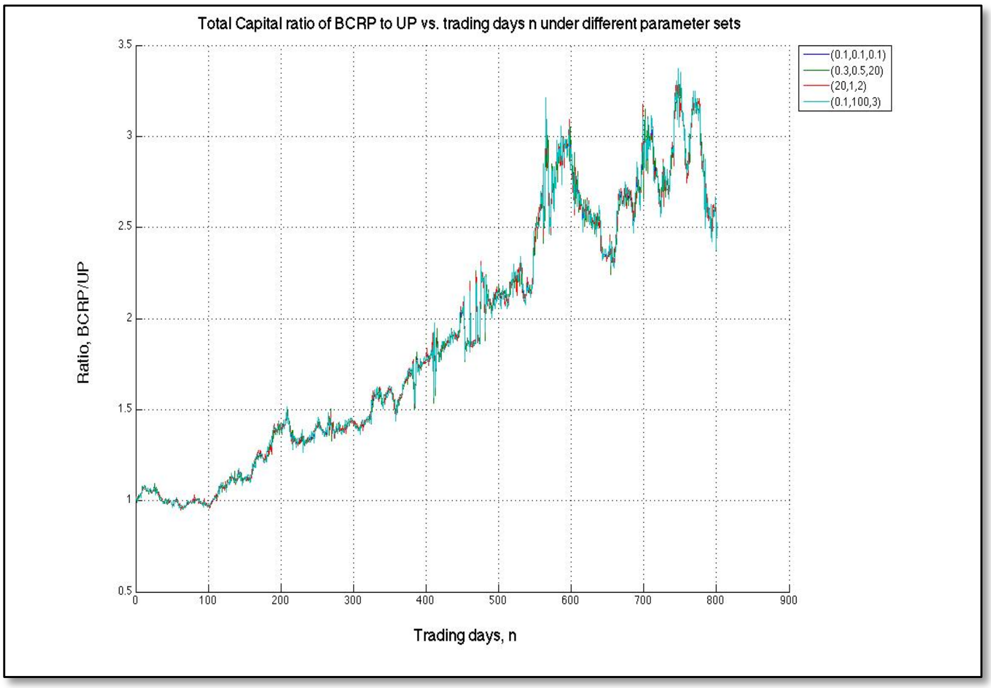


Figure 5.3: **The 3-dimensional view of four functions of the ratio wealths against the number of trading days given in Figure 5.4 for set E**

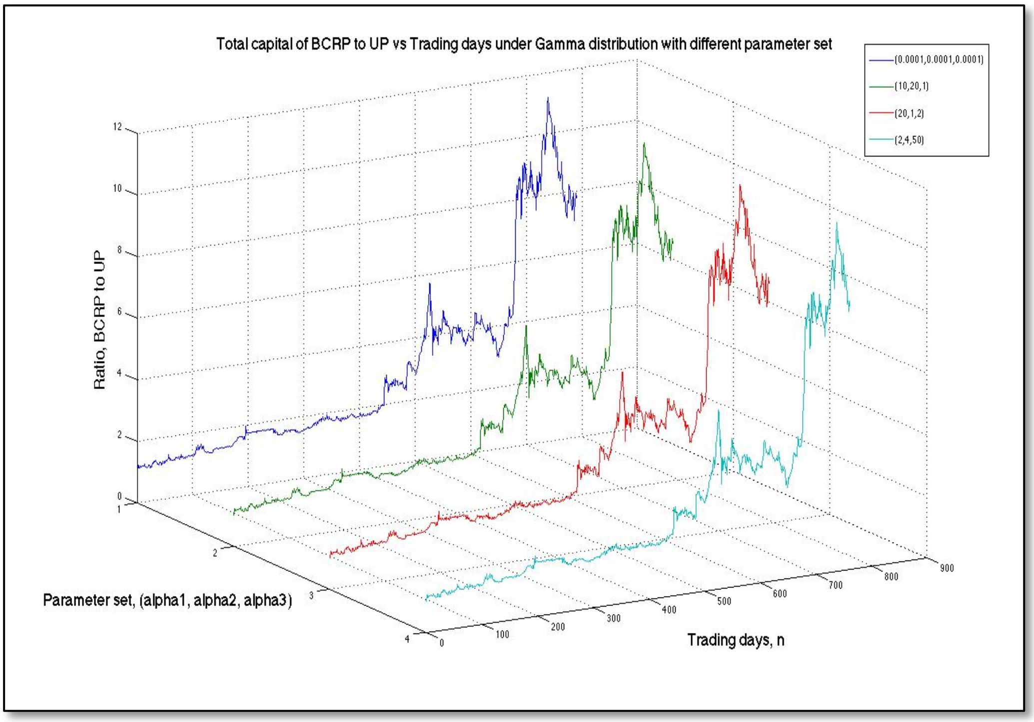


Figure 5.4: **Ratio of BCRP wealth to universal portfolio wealth for the universal portfolios generated by the gamma$(\alpha_1, \alpha_2, \alpha_3)$ distribution for four sets of $(\alpha_1, \alpha_2, \alpha_3)$ with β fixed at 2 for set E**

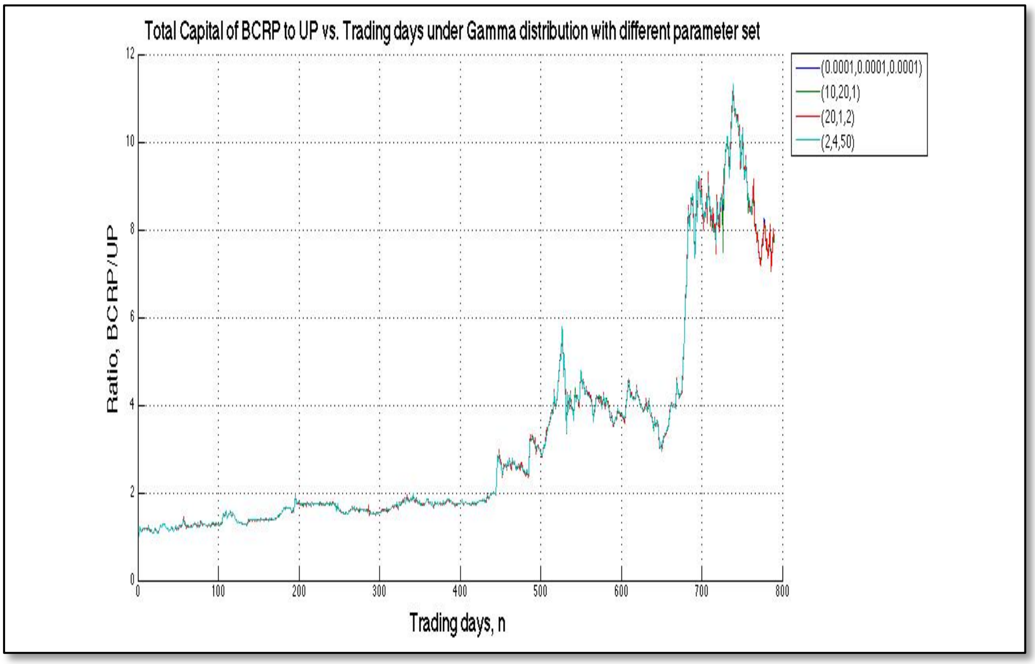


Figure 5.5: The 3-dimensional view of four functions of the ratio wealths against the number of trading days given in Figure 5.6 for set F

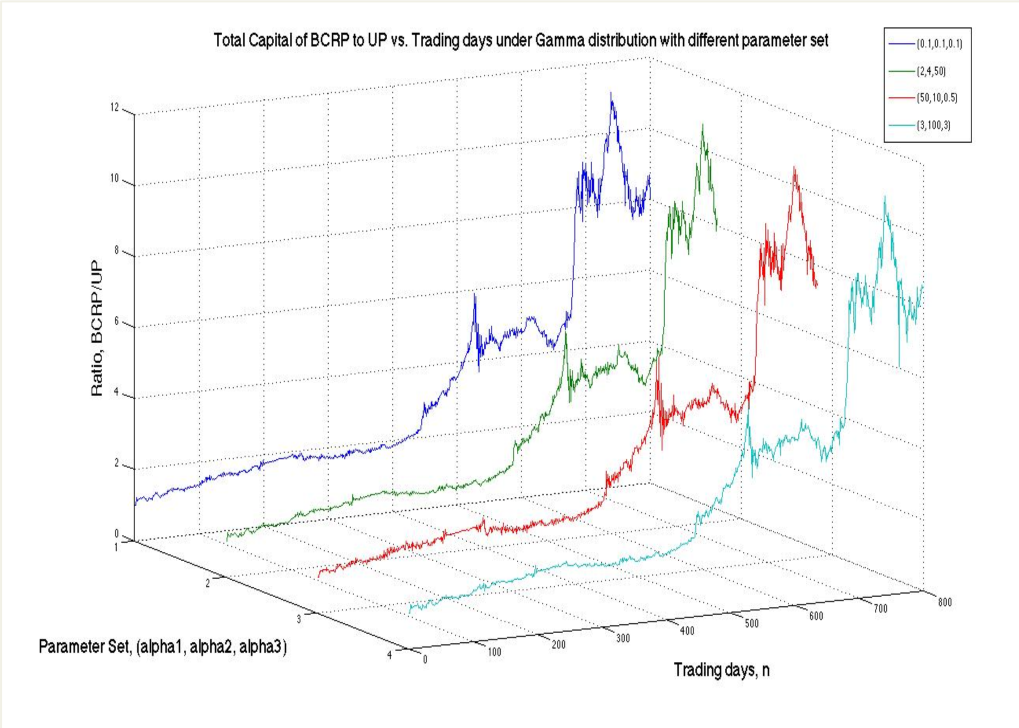
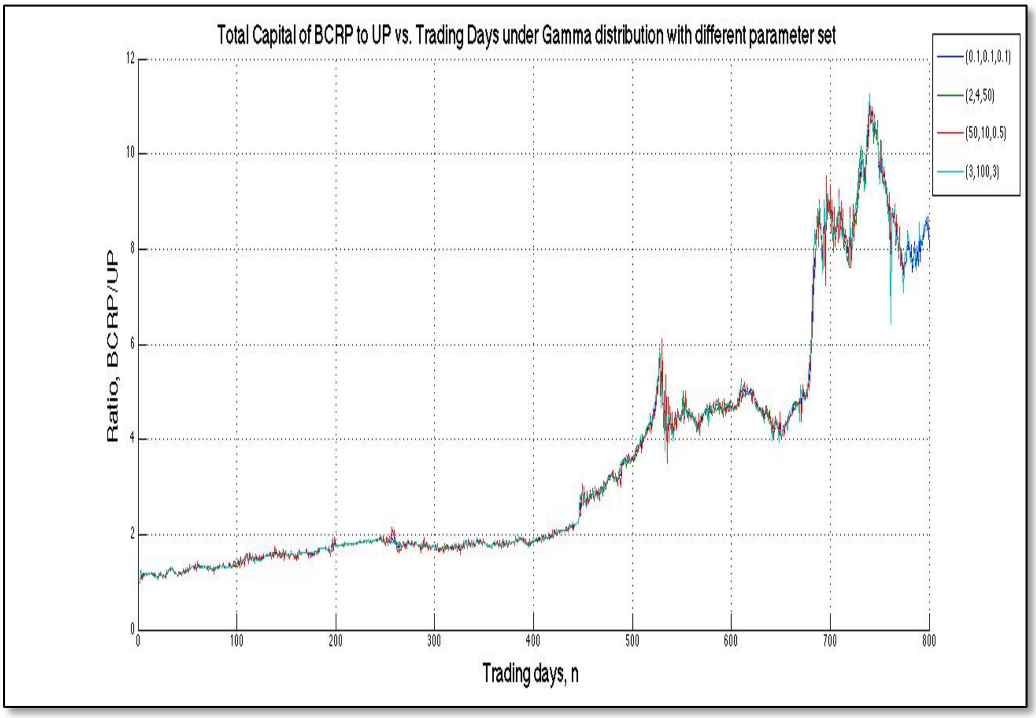
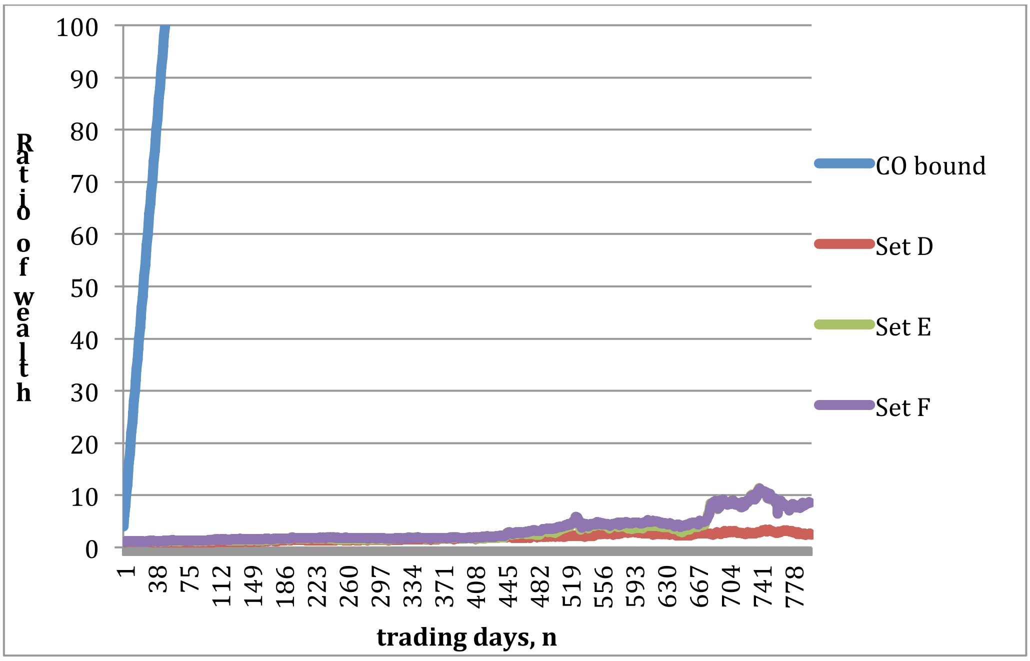


Figure 5.6: Ratio of BCRP wealth to universal portfolio wealth for the universal portfolios generated by the gamma$(\alpha_1, \alpha_2, \alpha_3)$ distribution for four sets of $(\alpha_1, \alpha_2, \alpha_3)$ with β fixed at 2 for set F



The trend of the functions displayed in Figures 5.2, 5.4 and 5.6 is upward. The comparison of the ratio of wealths with the Cover-Ordentlich bound (5.17) for the Dirichlet universal portfolio is shown in Figures 5.7, where $m = 3$ stocks. It is clear that the gamma-distribution-generated universal portfolios all have ratio of wealths far below the Cover-Ordentlich bound $2(n+1)$. A better bound for the ratio of wealths under empirical study maybe $c \ln n$ where c is some constant independent of n because the ratio of wealths grows slowly, close to logarithmic growth.

Figure 5.7: Comparison of the ratio of the BCRP wealth to the universal wealth for sets D, E and F with the Cover-Ordentlich bound $\boldsymbol{2(n+1)}$



5.5 Concluding Remarks

Future work in this direction may be to developing theoretic bounds for the ratio of the BCRP wealth to the universal portfolio wealth for any finite or moving order universal portfolio. Polynomial bounds in the number of trading days are well-known for the Dirichlet moving-order universal portfolios (Cover and Ordentlich (1996), Tan (2002)).

We have seen in Chapter 4 that low order universal portfolios are practical to use, in terms of substantial savings in computer memory and computational time in their implementation. On the whole, the order 1 portfolio can perform as well as the orders 2 and 3 portfolios for most data sets. The choice of parameters to use at the beginning of investment is a difficult problem. An attempt to solve this problem by mixing universal portfolios is given in Tan and Lim (2013).