PERFORMANCE OF THE FINITE-ORDER DIRICHLET AND SPECIAL ADDITIVE UPDATE UNIVERSAL PORTFOLIOS

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By

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ABSTRACT

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Chu Sin Yen

Universal portfolios of order 1, 2, and 3 generated by probability distributions require very much lesser computer memory and time to implement than the Cover-Ordentlich (1996) portfolio because we only use a fixed number of past data to implement the portfolios. In our study, we focus on the performance of the Dirichlet universal portfolios of order 1, 2 and 3 by running the portfolios on some selected data sets. The performance of the finite-order Dirichlet universal portfolios is shown to be better than that of the moving-order Cover-Ordentlich universal portfolios when it is run on the selected stock-price data sets.

An additive-update universal portfolio generated by the Mahalanobis squared divergence was proposed by Tan and Lim in 2011. We extend their research on the Mahalanobis universal portfolios by studying the universal portfolios generated by a matrix and its inverse which are called a complementary pair with regard to the wealth achieved by the portfolios. The wealth achieved by universal portfolios generated by special symmetric positive definite matrices are studied in our research. In particular the performance of universal portfolios generated by three-band and nine-band symmetric Toeplitz matrices is discussed. Pseudo Mahalanobis Toeplitz universal portfolios are introduced with a promising increase of investment returns.

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(CHU SIN YEN)

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APPROVAL SHEET

This dissertation entitled "**PERFORMANCE OF THE FINITE-ORDER DIRICHLET AND SPECIAL ADDITIVE UPDATE UNIVERSAL PORTFOLIOS"** was prepared by CHU SIN YEN and submitted as partial fulfillment of the requirements for the degree of Master of Science at Universiti Tunku Abdul Rahman.

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SUBMISSION OF DISSERTATION

It is hereby certified that Chu Sin Yen (ID No: (11UEM06751) has completed this dissertation entitled “PERFORMANCE OF THE FINITE-ORDER DIRICHLET AND SPECIAL ADDITIVE UPDATE UNIVERSAL PORTFOLIOS” under the supervision of Dr Tan Choon Peng from the Department of Mathematical and Actuarial Sciences, Faculty of Engineering and Sciences, and Dr Pan Wei Yeing from the Department of Mathematical and Actuarial Sciences, Faculty of Engineering and Science.

I understand that the University will upload softcopy of my thesis in pdf format into UTAR Institutional Repository, which may be made accessible to UTAR community and public.

Yours truly,

____

(Chu Sin Yen)

DECLARATION

I Chu Sin Yen hereby declare that the dissertation is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UTAR or other institutions.

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LIST OF ABBREVIATIONS

BCRP	**Best constant rebalanced portfolio**
CSD	**Chi-square divergence**

CHAPTER ONE

INTRODUCTION

Portfolio is a pool of assets such as stocks, bonds or cash equivalent held by a financial institution or private investor. The focus of this thesis is on the portfolio of stocks. The performance of the stock market is influenced by corporate performance, the economic indicators such as recession and unemployment rates, geo-political changes like changes in trade policy or change in tax policy and investor sentiment. All of these factors are mostly qualitative and unpredictable. Therefore, portfolio selection is always a difficult problem for the investor who aims for a higher investment return and beating the market.

The universal portfolio is a strategy of trading on stocks that does not assume any probability model for the stock prices. Cover (1991) introduces the uniform universal portfolio (or Dirichlet (1,1,…,1) portfolio) has shown that the investment return of this portfolio can be close to that of the best constant rebalanced portfolio (BCRP). Cover and Ordentlich (1996) introduces the Dirichlet-weighted universal portfolio with special emphasis on the beta-weighted (1,1) and (1/2,1/2) portfolios and also studied the presence of side information. Although the performance of these two types of universal portfolio can get close to the performance of the BCRP, the computer memory capacity growing exponentially with the number of stocks invested in the market. This research mainly focuses on finite-order Dirichlet and some special additive-update universal portfolios which require much lesser memory requirements in their implementation. The performance of the universal portfolios is studied by running these universal portfolios on some selected stocks from the Kuala Lumpur Stock Exchange (KLSE).

This thesis consists of four chapters. An introduction is given in the first chapter which states the objectives and basic definitions of the research followed by a literature review on the area. In Chapter Two, finite–order universal portfolios generated by probability distributions are studied. We compare the performance of Ordentlich-Cover universal portfolios which require extensive implementation time and computational memory with the finite-order universal portfolios generated by the Dirichlet distribution. The finite-order universal portfolios requiring much lesser implementation time and computational memory may outperform the Ordentlich-Cover universal portfolios. In Chapter Three, we extend the work of Tan and Lim (2011b) by introducing complementary pairs of additive universal portfolios. The performance of complementary pairs of additive universal portfolios is compared with the results of Tan and Lim (2011c). Some graphs are shown to illustrate the relationship between the performance wealths and the portfolio parameter. In Chapter Four, the universal portfolios generated by Toeplitz matrices are studied for three-stock data for 500 trading days and five-stock data for 1500 trading days.

1.1 Basic Definitions

Consider a market of m stocks described by a sequence of price-relative vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n, \dots$ where the price-relative vector $\boldsymbol{x}_n = (x_{ni})$ on the $n^{th.}$ trading day consists of the $i^{th.}$ price relative x_{ni} which is the ratio of the closing price of the $i^{th.}$ stock to its opening price on day n, for $i = 1,2, \dots, m$. We assume that $x_{ni} \geq 0$ for all $i = 1,2, \dots, m$ and all $n = 1,2, \dots$. Let $\boldsymbol{x}^n$ denote the sequence of n vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_n$. The portfolio vector $\boldsymbol{b}_n = (b_{ni})$ on day n is the investment strategy used on day n where b_{ni} is the proportion of the current wealth invested on stock i for $i = 1,2, \dots, m$, with $0 \leq b_{ni} \leq 1$ and $\sum_{i=1}^{m} b_{ni} = 1$. Let $\hat{S}_n(\boldsymbol{x}^n)$ denote the universal wealth at the end of day n, given $\boldsymbol{x}^n$ and assuming that the universal portfolios $\widehat{\boldsymbol{b}}_1, \widehat{\boldsymbol{b}}_2, \dots, \widehat{\boldsymbol{b}}_n$ are used in this period. Furthermore, assuming that the initial wealth $S_o = 1s$ unit, then

$$\hat{S}_n(\boldsymbol{x}^n) = \prod_{j=1}^{n} \widehat{\boldsymbol{b}}_j^t \boldsymbol{x}_j\,, \tag{1.1}$$

where $\widehat{\boldsymbol{b}}_j^t \boldsymbol{x}_j = \sum_{i=1}^{m} \boldsymbol{b}_{ji} \boldsymbol{x}_{ji}$.

For a constant rebalanced portfolio, $\widehat{\boldsymbol{b}}_j = \boldsymbol{b}$ for all $j = 1,2, \dots$, and for some constant portfolio $\boldsymbol{b}$. The constant-rebalanced-portfolio wealth $S_n(\boldsymbol{x}^n)$ at the end of day n is

$$S_n(\boldsymbol{x}^n) = \prod_{j=1}^{n} \boldsymbol{b}_j^t \boldsymbol{x}_j\,. \tag{1.2}$$

Given $\boldsymbol{x}^n$, if (1.2) is maximized over all possible portfolio vectors $\boldsymbol{b} = (b_i)$ in the simplex $\mathcal{B}$ of portfolio vectors, then $S_n^*(\boldsymbol{x}^n)$ is known as the BCRP wealth, where

$$S_n^*(\boldsymbol{x}^n) = \max_{\boldsymbol{b}\in\mathcal{B}} \left\{ \prod_{j=1}^{n} \boldsymbol{b}^t \boldsymbol{x}_j \right\} = \prod_{j=1}^{n} \boldsymbol{b}^{*t} \boldsymbol{x}_j \ , \tag{1.3}$$

$$\mathcal{B} = \left\{ \boldsymbol{b}: 0 \le b_i \le 1, i = 1,2, \dots, m \, , \sum_{i=1}^{m} b_i = 1 \right\} \tag{1.4}$$

and $\boldsymbol{b}^*$ in $\mathcal{B}$ achieves the maximum in (1.3). We shall use the abbreviation BCRP to denote the best constant rebalanced portfolio.

1.2 Literature Review

Kivinen and Warmuth (1997) introduced a common framework for deriving a learning algorithm based on the trade-off between the distance travelled from the current weight vector and a loss function by minimizing

$$M(\boldsymbol{b}_{n+1}) = d(\boldsymbol{b}_{n+1}, \boldsymbol{b}_n) + \eta_n L(y_n, \, \boldsymbol{b}_{n+1}^t \cdot \boldsymbol{x}_n) \tag{1.5}$$

where $d(\boldsymbol{b}_{n+1}, \boldsymbol{b}_n)$ is some measure of distance from the old vector $\boldsymbol{b}_n$ to the new weight vector $\boldsymbol{b}_{n+1}$, L is the loss function, the magnitude of the positive constant η_n represents the importance of correctiveness compared to the importance of conservativeness and the inner product $\boldsymbol{y}_n = \boldsymbol{b}_n^t \cdot \boldsymbol{x}_n$. Relative entropy which also known as the Kullback-Leibler distance is used as the distance measure between two portfolio vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, which is defined as

$$d(\boldsymbol{u}, \boldsymbol{v}) = \sum_{i=1}^{m} u_i ln \frac{u_i}{v_i} \ . \tag{1.6}$$

Helmbold et al. (1998) adapt the Kivinen and Warmuth (1997) method to find a new portfolio vector $\mathbf{b}_{\mathrm{n+1}}$ given that $\boldsymbol{b}_n$and $\boldsymbol{x}_n$ that maximizes

$$F(\boldsymbol{b}_{n+1}) = \eta \log(\boldsymbol{b}_{n+1}^t \cdot \boldsymbol{x}_n) - d(\boldsymbol{b}_{n+1}, \boldsymbol{b}_n), \tag{1.7}$$

where $\eta > 0$ is a parameter called the learning rate and $d(.)$ is a distance measure that serves as a penalty term. This penalty term $d(\boldsymbol{b}_{n+1} \cdot \boldsymbol{b}_n)$ tends to keep $\boldsymbol{b}_{n+1}$ close to $\boldsymbol{b}_n$ while the learning rate η controls the relative importance between the two terms. Helmbold et al. (1998) emphasize that different distance functions lead to different update rules. For example, by using the squared distance measure

$$d_{EUC}\,(\boldsymbol{u} \parallel \boldsymbol{v}) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{i=1}^{m} (u_i - v_i)^2 = \frac{1}{2} \parallel \boldsymbol{u} - \boldsymbol{v} \parallel^2 , \tag{1.8}$$

it leads to the update rule of gradient projection algorithms; using the relative entropy distance gives us exponential gradient update ($\mathrm{EG_n}$) algorithm, where

$$d_{RE}(\boldsymbol{u} \parallel \boldsymbol{v}) \stackrel{\text{def}}{=} \sum_{i=1}^{m} u_i \ln \frac{u_i}{v_i} \tag{1.9}$$

with $\boldsymbol{u} = \boldsymbol{b}_{n+1}$ and $\boldsymbol{v} = \boldsymbol{b}_n$.

Furthermore, using the second order Taylor series expansion of the relative entropy reduces (1.9) to the χ^2 distance function:

$$d_{\chi^2}(\boldsymbol{u} \parallel \boldsymbol{v}) \stackrel{\text{def}}{=} \frac{1}{2} \sum_{i=1}^{m} \frac{(u_i - v_i)^2}{v_i}. \tag{1.10}$$

In Cover (1991), the computation of the uniform universal portfolio requires computer memory storage that grows exponentially with the number of stocks. However, it was shown that the uniform universal portfolios can achieve investment returns with a doubling rate close to that of the best constant rebalanced portfolio and can outperform the best buy-and-hold strategy over a longer period of trading.

Cover and Ordentlich (1996) introduced the parametric family of Dirichlet $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ universal portfolios and studied the influence of side information on the wealth achieved. The Dirichlet-weighted universal portfolios can achieve investment returns higher than the best stock strategy but it requires computer memory capacity growing exponentially with the number of stocks invested in the market. They define the Dirichlet-weighted $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ universal portfolios as

$$\hat{b}_{n+1,i} = \frac{\int_B b_i \prod_{j=1}^{n} \boldsymbol{b}^t \boldsymbol{x}_n \, d\mu(\boldsymbol{b})}{\int_B \prod_{j=1}^{n} \boldsymbol{b}^t \boldsymbol{x}_n \, d\mu(\boldsymbol{b})} \; for \; i = 1,2, \dots, m, \tag{1.11}$$

where

$$d\mu(b) = \frac{\Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_m)}{\Gamma(\alpha_1)\Gamma(\alpha_2) \dots \Gamma(\alpha_m)} \prod_{j=1}^{m} b_j^{\alpha_j - 1} d\boldsymbol{b}.$$

Cover and Ordentlich also studied the ratio of the universal capital $\hat{S}_n(\boldsymbol{x}^n)$ achieved on day n to the best constant rebalanced capital $S_n^*(\boldsymbol{x}^n)$ by showing that for the μ-weighted universal portfolio,

$$\frac{S_n^*(\boldsymbol{x}^n)}{\hat{S}_n(\boldsymbol{x}^n)} \leq \max_{j^n} \frac{\prod_{i=1}^{n} b_{j_i}^*}{\int_B \prod_{i=1}^{n} b_{j_i} \, d\mu(\boldsymbol{b})}, \tag{1.12}$$

where the maximum is over the set of indices $j^n = (j_1, j_2, \dots, j_n) \epsilon \{1, \dots, m\}^n$ and $b^* = (b_1^*, \dots, b_m^*)^t$ is the best constant rebalanced portfolio for the sequence of the price relatives $\boldsymbol{x}^n$. For the Dirichlet (1, 1,…, 1) universal portfolio, it was shown that

$$\frac{S_n^*(\boldsymbol{x}^n)}{\hat{S}_n(\boldsymbol{x}^n)} \leq (n+1)^{m-1} \tag{1.13}$$

and for the Dirichlet $(1/2\,, 1/2\,, \dots, 1/2)$ universal portfolio,

$$\frac{S_n^*(\boldsymbol{x}^n)}{\hat{S}_n(\boldsymbol{x}^n)} \leq \ 2(n+1)^{\frac{(m-1)}{2}}. \quad (1.14)$$

The exponentiated-gradient (or HSSW) universal portfolios of Hembold et al. (1998) are implemented by day to day multiplicative updates of the current portfolio with require very much lesser computer memory capacity and may perform better than the uniform universal portfolio. It was shown that the HSSW portfolios can perform better than the Dirichlet(1,1,..,1) portfolios on some stock data sets from the New York Stock Exchange (NYSE).

Tan and Tang (2003) have shown that there are Dirichlet-weighted universal portfolios that can perform better than HSSW universal portfolios. Tan and Tang compare the performance of the HWWS universal portfolios and the Dirichlet-weighted universal portfolios by running the portfolios on some stock data sets from the Kuala Lumpur Stock Exchange (KLSE). They have shown that there are Dirichlet-weighted universal portfolios that can outperform the HSSW universal portfolios. Tan and Tang (2003) have also shown that the Helmbold or HSSW portfolio is sensitive to the initial starting portfolio of the algorithm and has the approximate behaviour of a constant rebalanced portfolio.

In a recent work, Tan and Lim (2011b) has shown that is not necessary to restrict the parameter η of the HSSW universal portfolio to $0 < \eta < 2\sqrt{\frac{2\log m}{N}} < 1$. They also illustrate that higher investment returns can be obtained for large positive values of η or negative η and choosing a proper initial starting portfolio can lead to a higher investment return.

Tan and Lim (2011a) introduces an additive update universal portfolio which uses the chi-square divergence distance measure defined as

$$D(\boldsymbol{b}_{n+1}||\boldsymbol{b}_n) = \sum_{i=1}^{m} \frac{\left(b_{n+1,i} - b_{ni}\right)^2}{b_{ni}} \ . \tag{1.15}$$

This universal portfolio is known as the chi-square divergence (CSD) universal portfolio. The CSD universal portfolio is generated by the following algorithm

$$b_{n+1,i} = b_{ni}\left[\frac{\xi(x_{ni} - \boldsymbol{b}_n^t\boldsymbol{x}_n)}{\boldsymbol{b}_n^t\boldsymbol{x}_n} + 1\right], \tag{1.16}$$

where ξ is any chosen real number such that $\boldsymbol{b}_{n+1}$ is a valid portfolio vector. Tan and Lim demonstrate the performance of this universal portfolio by running it on some stock data from the KLSE and show that the CSD universal portfolios can outperform HSSW universal portfolios for these data sets.

Another universal portfolio studied by Tan and Lim (2011c) is the Mahalanobis universal portfolio. The distance measure used is the Mahalanobis squared divergence defined as:

$$d_A(\boldsymbol{b}_k, \boldsymbol{b}_n) = [\boldsymbol{b}_k - \boldsymbol{b}_n]^t A[\boldsymbol{b}_k - \boldsymbol{b}_n] \tag{1.17}$$

where $\boldsymbol{b}_k$ and $\boldsymbol{b}_n$ are any two portfolio vectors and $A = (a_{ij})$ is a symmetric, positive definite matrix. This distance measure also can be written as

$$d_A(\boldsymbol{b}_k, \boldsymbol{b}_n) = \sum_{i=1}^{m} a_{ii}[b_{ki} - b_{ni}]^2 + \sum_{i<j} 2a_{ij}[b_{ki} - b_{ni}][b_{kj} - b_{nj}] \,. \tag{1.18}$$

They have shown that the Mahalanobis universal portfolios generated by certain diagonal matrices can outperform the CSD and HWWS universal portfolios on certain data sets. The CSD universal portfolio is in fact a special Mahalanobis universal portfolio generated by a special diagonal matrix.

It is the objective of this thesis to study the performance of different parametric families of universal portfolios generated by different methods. Many researches have tried to find universal portfolios that can achieve wealths close to or exceeding the BCRP wealth. We show empirically in this thesis that there are universal portfolio that can outperform the BCRP. The finite-order universal portfolios and some special Mahalanobis universal portfolios introduced in this thesis can be used by trust fund managers to increase the wealth of their clients.

CHAPTER TWO

FINITE-ORDER UNIVERSAL PORTFOLIOS GENERATED BY PROBABILITY MASS FUNCTIONS

Cover and Ordentlich (1996) studies the moving-order universal portfolios generated by moments of the Dirichlet probability distribution. Noting that the joint nth. moments of the Dirichlet distribution for a fixed integer n form a probability mass function, Cover and Ordentlich (1996) introduces the concept of a moving-order universal portfolio generated by a sequence of probability mass functions. A mini-max theorem for the ratio of the universal wealth to the best constant-rebalanced-portfolio wealth is derived and an asymptotic upper bound on the reciprocal of this ratio of wealths behaving as a polynomial in the number of trading days is exhibited. In practice, the computation of the moving-order Ordentlich-Cover universal portfolio is time-consuming and memory-extensive. With this in mind, the time and memory-saving finite-order universal portfolios generated by probability mass functions will be introduced and studied in this thesis. Special focus will be directed on the class of finite-order universal portfolios generated by the Dirichlet distribution.

2.1 The moving-order universal portfolio

For sequences $j^{\nu+1} = (j_1, \dots, j_\nu, j_{\nu+1}) \in \{1, \dots, m\}^{\nu+1}$, define a probability mass function $p_{\nu+1}(j^{\nu+1})$, where $0 \le p_{\nu+1}(j^{\nu+1}) \le 1$ for all $j^{\nu+1}$ and $\sum_{j_1,\dots,j_{\nu+1}} p_{\nu+1}(j^{\nu+1}) = 1$. It is clear that for $l < \nu + 1$,

$$p_{\nu+1}(j^l) = \sum_{j_{l+1},\dots,j_{\nu+1}} p_{\nu+1}(j^l, j_{l+1}, \dots, j_{\nu+1}) \tag{2.1}$$

is the marginal probability mass of $j^l = (j_1, \dots, j_l)$. The notation $p_{\nu+1}(j^{r+s}) = p_{\nu+1}(j^r, j^s)$ is also adopted for $1 \le r + s \le \nu + 1$. Let ν be a fixed positive integer and the probability mass function $p_{\nu+1}(j^{\nu+1})$ is given. The sequence of universal portfolios $\{\hat{\boldsymbol{b}}_{n+1}\}$ generated by $p_{\nu+1}(j^{\nu+1})$ is defined as:

$$\hat{b}_{n+1,k} = \frac{\sum_{j_1=1}^{m}\sum_{j_2=1}^{m}\cdots\sum_{j_\nu=1}^{m} p_{\nu+1}(j_1, j_2, \dots, j_\nu, k)\, x_{nj_1} x_{n-1,j_2} \cdots x_{n-(\nu-1),j_\nu}}{\sum_{j_1=1}^{m}\sum_{j_2=1}^{m}\cdots\sum_{j_\nu=1}^{m} p_{\nu+1}(j_1, j_2, \dots, j_\nu)\, x_{nj_1} x_{n-1,j_2} \cdots x_{n-(\nu-1),j_\nu}} \tag{2.2}$$

for $k = 1, \dots, m$ and $n = \nu, \nu + 1, \dots$. The portfolios $\hat{\boldsymbol{b}}_{n+1}$ for $n < \nu$ can be arbitrary. When $\nu = n$, the portfolio is known as a *moving-order universal portfolio*.

2.2 The finite-order universal portfolios generated by probability mass function

Let $Y_1, \dots, Y_{\nu+1}$ be $\nu + 1$ finite-valued, mutually independent random variables, where $P(Y_i = j) = q_i(j)$ for $j \in \{1, \dots, m\}$ and $i = 1, \dots, \nu + 1$.

Proposition 1. *Let* $Y_1, \dots, Y_{\nu+1}$ *be* $\nu + 1$ *finite-valued, mutually independent random variables with joint probability function* $p_{\nu+1}(j^{\nu+1})$ *. Then the sequence of universal portfolios* $\{\hat{\boldsymbol{b}}_{n+1}\}$ *generated by* $p_{\nu+1}(j^{\nu+1})$ *is a constant rebalanced portfolio given by*:

$$\hat{b}_{n+1,k} = q_{\nu+1}(k), for\ k = 1, \dots, m\ ; n = 1,2, \dots, \qquad (2.3)$$

where $q_{\nu+1}(k) = P(Y_{\nu+1} = k)$ *is the probability mass function of* Y_k.

Proof. If $q_i(j) = P(Y_i = j)$ is the probability mass function of Y_i for $i = 1, \dots, \nu + 1$, then $p_{\nu+1}(j^{\nu+1}) = \prod_{i=1}^{\nu+1} q_i(j_i)$ is the joint probability function of $Y_1, \dots, Y_{\nu+1}$. From (2.2), the result (2.3) is obtained. ■

2.3 The finite-order universal portfolios generated by the Dirichlet($\propto_1, \dots, \propto_m$) distribution

Let $\boldsymbol{Y} = (Y_1, \dots, Y_m)$ be m dependent random variables having a joint Dirichlet $(\propto_1, \dots, \propto_m)$ distribution with joint probability density function:

$$f(y_1, \dots, y_m) = \frac{\Gamma(\alpha_1 + \cdots + \alpha_m)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_m)} y_1^{\alpha_1 - 1} \dots y_m^{\alpha_m - 1}, \qquad (2.4)$$

where $0 \leq y_i \leq 1$ for $i = 1, \dots, m$ and $\sum_{i=1}^{m} y_i = 1$. The parameters $\propto_i$ are all positive, $i = 1, \dots, m$. Let B denote the simplex of portfolio vectors defined as:

$$B = \left\{ (y_1, \dots, y_m) \colon 0 \leq y_i \leq 1, i = 1, \dots, m; \sum_{i=1}^{m} y_i = 1 \right\}. \qquad (2.5)$$

Then from Tan (2013), the Dirichlet$(\propto_1, \dots, \propto_m; \upsilon)$ universal portfolio of order ν is defined as the sequence $\{\widehat{\boldsymbol{b}}_{n+1}\}$ of universal portfolios for a fixed integer ν given by:

$$\hat{b}_{n+1,k} = \frac{\int_B y_k (\boldsymbol{y}^t \boldsymbol{x}_n)(\boldsymbol{y}^t \boldsymbol{x}_{n-1}) \dots \left(\boldsymbol{y}^t \boldsymbol{x}_{n-(\nu-1)}\right) f(y_1, \dots, y_m) d\boldsymbol{y}}{\int_B (\boldsymbol{y}^t \boldsymbol{x}_n)(\boldsymbol{y}^t \boldsymbol{x}_{n-1}) \dots \left(\boldsymbol{y}^t \boldsymbol{x}_{n-(\nu-1)}\right) f(y_1, \dots, y_m) d\boldsymbol{y}} \qquad (2.6)$$

for $k = 1, \dots, m$ and $n = \nu, \nu + 1, \dots$. The portfolios $\widehat{\boldsymbol{b}}_{n+1}$ for $n < \nu$ can be arbitrary.

Note that the numerator of (2.6) can be written as:

$$\int_B y_k \left(\sum_{j_1=1}^{m} y_{j_1} x_{n j_1} \right) \left(\sum_{j_2=1}^{m} y_{j_2} x_{n-1, j_2} \right) \cdots \left(\sum_{j_\nu=1}^{m} y_{j_\nu} x_{n-\nu+1, j_\nu} \right) f(y_1, \dots, y_m) d\boldsymbol{y}$$

$$= \int_B y_k \left(\sum_{j_1=1}^{m} \sum_{j_2=1}^{m} \dots \sum_{j_\nu=1}^{m} (y_{j_1} y_{j_2} \dots y_{j_\nu})(x_{nj_1} x_{n-1,j_2} \dots x_{n-\nu+1,j_\nu}) \right) f(y_1, \dots, y_m) d\boldsymbol{y}$$

(2.7)

$$\sum_{j_1=1}^{m} \sum_{j_2=1}^{m} \dots \sum_{j_\nu=1}^{m} (x_{nj_1} x_{n-1,j_2} \dots x_{n-\nu+1,j_\nu}) E\left[Y_1^{n_1(k;\boldsymbol{j})} Y_2^{n_2(k;\boldsymbol{j})} \dots Y_m^{n_m(k;\boldsymbol{j})} \right], \quad (2.8)$$

where $n_r(k;\boldsymbol{j})$ is the number of y_r's in the product $(y_k y_{j_1} y_{j_2} \dots y_{j_\nu})$, $\boldsymbol{j} = (j_1, j_2, \dots, j_m)$ for $1 \le j_r \le m, r = 1,2, \dots, m;$ $0 \le n_r(k;\boldsymbol{j}) \le \nu + 1$ and $\sum_{r=1}^{m} n_r(k;\boldsymbol{j}) = \nu + 1$.Note that $n_r(k;\boldsymbol{j}) = n_k(\boldsymbol{j}) + 1$ if $r = k$ and $n_r(k;\boldsymbol{j}) = n_r(\boldsymbol{j})$ if $r \neq k$, where $n_r(\boldsymbol{j})$ is the number of y_r's in the product $(y_{j_1} y_{j_2} \dots y_{j_\nu})$.

Proposition 2. *The Dirichlet* $(\propto_1, \dots, \propto_m; \nu)$*universal portfolio of order* ν *is generated by the probability mass function*

$$p_{\nu+1}(j_1, \dots, j_\nu, k) = E\left[Y_{j_1} \dots Y_{j_\nu} Y_k\right], \quad (2.9)$$

for $(j_1, \dots, j_\nu, k) \in \{1, \dots, m\}^{\nu+1}$, *where the random variables* $Y_1, \dots, Y_m$ *have the joint Dirichlet* $(\propto_1, \dots, \propto_m)$ *distribution given by (2.4).*

Proof. For the probability mass function given by (2.9), the portfolio components (2.2) and (2.6) are the same as a consequence of (2.7). Since $0 \le Y_i \le 1$ for $i = 1, \dots, m$, it is clear that $0 \le E[Y_{j_1} \dots Y_{j_\nu} Y_k] \le 1$.

Furthermore,

$$\sum_{j_1,\dots,j_\nu}\sum_{k} E\left[Y_{j_1}\dots\ Y_{j_\nu}Y_k\right] = \sum_{j_1,\dots,j_\nu} E\left[(Y_1+\cdots+Y_m)Y_{j_1}\dots\ Y_{j_\nu}\right]$$

$$= \sum_{j_1,\dots,j_{\nu-1}} E\left[(Y_1+\cdots+Y_m)Y_{j_1}\dots\ Y_{j_{\nu-1}}\right]$$

$$= \vdots$$

$$= 1$$

by repeated application of the fact that $Y_1+\cdots+Y_m=1$. Thus $p_{\nu+1}(j_1,\dots,j_\nu,k)$ given by (2.9) is a probability mass function. ■

2.4 The Dirichlet($\propto_1,\dots,\propto_m;\ \nu$) universal portfolios for $\nu = 1, 2, 3$

In the sequel, the first four joint moments of Dirichlet $(\propto_1,\dots,\propto_m)$ distribution given by (2.4) are evaluated. They are required in the calculation of the portfolio components (2.6) in the Dirichlet $(\propto_1,\dots,\propto_m;\ \nu)$ universal portfolios for $\nu = 1, 2, 3$.

To evaluate the joint moments of (2.4), the gamma function property $\Gamma(u) = (u-1)\Gamma(u-1)$ for $u > 1$ is used.

For the first moment,

$$E(Y_i) = \frac{\Gamma(\alpha_1+\cdots+\alpha_m)}{\Gamma(\alpha_1)\dots\Gamma(\alpha_m)}\frac{\Gamma(\alpha_1)\dots\Gamma(\alpha_i+1)\dots\Gamma(\alpha_m)}{\Gamma(\alpha_1)\dots\Gamma(\alpha_m)}$$

$$= \frac{\alpha_i}{(\alpha_1+\cdots+\alpha_m)}. \tag{2.10}$$

For the second moment of (2.4) and $i \neq j$,

$$E\left(Y_iY_j\right) = \frac{\Gamma(\alpha_1 + \cdots + \alpha_m)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_m)} \frac{\Gamma(\alpha_1) \dots \Gamma(\alpha_i + 1) \dots \Gamma\left(\alpha_j + 1\right) \dots \Gamma(\alpha_m)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_m)}$$

$$= \frac{\alpha_i \alpha_j}{(\alpha_1 + \cdots + \alpha_m + 1)(\alpha_1 + \cdots + \alpha_m)}. \qquad (2.11)$$

If $i = j$,

$$E\left(Y_iY_j\right) = \frac{(\alpha_i + 1)\alpha_i}{(\alpha_1 + \cdots + \alpha_m + 1)(\alpha_1 + \cdots + \alpha_m)}. \qquad (2.12)$$

The third and fourth moments of (2.4) are given by:

$$E\left(Y_iY_jY_k\right) =$$

$$\begin{cases} \frac{(\alpha_i+2)(\alpha_i+1)\alpha_i}{(\alpha_1+\cdots+\alpha_m+2)(\alpha_1+\cdots+\alpha_m+1)(\alpha_1+\cdots+\alpha_m)} & \text{for } i = j = k, \\ \frac{\alpha_i(\alpha_j+1)\alpha_j}{(\alpha_1+\cdots+\alpha_m+2)(\alpha_1+\cdots+\alpha_m+1)(\alpha_1+\cdots+\alpha_m)} & \text{for } i \neq j, j = k, \\ \frac{\alpha_i\alpha_j\alpha_k}{(\alpha_1+\cdots+\alpha_m+2)(\alpha_1+\cdots+\alpha_m+1)(\alpha_1+\cdots+\alpha_m)} & \text{for } i \neq j, i \neq k, j \neq k, \end{cases} \qquad (2.13)$$

$$E\left(Y_iY_jY_kY_l\right) =$$

$$\begin{cases} \frac{(\alpha_i+3)(\alpha_i+2)(\alpha_i+1)\alpha_i}{(\alpha_1+\cdots+\alpha_m+3)(\alpha_1+\cdots+\alpha_m+2)(\alpha_1+\cdots+\alpha_m+1)(\alpha_1+\cdots+\alpha_m)} & \text{for } i = j = k = l, \\ \frac{\alpha_i\left(\alpha_j+2\right)\left(\alpha_j+1\right)\alpha_j}{(\alpha_1+\cdots+\alpha_m+3)(\alpha_1+\cdots+\alpha_m+2)(\alpha_1+\cdots+\alpha_m+1)(\alpha_1+\cdots+\alpha_m)} & \text{for } i \neq j, j = k = l, \\ \frac{(\alpha_i+1)\alpha_i(\alpha_k+1)\alpha_k}{(\alpha_1+\cdots+\alpha_m+3)(\alpha_1+\cdots+\alpha_m+2)(\alpha_1+\cdots+\alpha_m+1)(\alpha_1+\cdots+\alpha_m)} & \text{for } i = j, k = l\,, \\ \frac{\alpha_i\alpha_j(\alpha_k+1)\alpha_k}{(\alpha_1+\cdots+\alpha_m+3)(\alpha_1+\cdots+\alpha_m+2)(\alpha_1+\cdots+\alpha_m+1)(\alpha_1+\cdots+\alpha_m)} & \text{for } i \neq j, i \neq k, j \neq k, k = l, \\ \frac{\alpha_i\alpha_j\alpha_k\alpha_l}{(\alpha_1+\cdots+\alpha_m+3)(\alpha_1+\cdots+\alpha_m+2)(\alpha_1+\cdots+\alpha_m+1)(\alpha_1+\cdots+\alpha_m)} & \text{for } (i, j, k, l)\ distinct. \end{cases}$$

(2.14)

2.4.1 The Dirichlet$(\propto_1, \ldots, \propto_m;\ 1)$ universal portfolio (order-1 universal portfolio)

By substituting (2.10)-(2.12) into (2.8), the Dirichlet$(\propto_1, \ldots, \propto_m;\ 1)$ universal portfolio is given by:

$$\hat{b}_{n+1,k} = \frac{1}{\sum_{j=1}^{m} \alpha_j + 1}\left[\alpha_k + \frac{\alpha_k x_{nk}}{\left(\sum_{j=1}^{m} \alpha_j x_{nj}\right)}\right], \tag{2.15}$$

$for\ k = 1, \ldots., m\ , n = 1, 2, \ldots.$

There is some relationship between the Dirichlet$(\propto_1, \ldots, \propto_m;\ 1)$ universal portfolios and the Chi-Square Divergence (CSD) universal portfolios.

Consider the CSD universal portfolio with parameter ξ ,

$$b_{n+1,k} = \xi \left(\frac{b_{nk} x_{nk}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right) + (1 - \xi) b_{nk} \quad \text{for } k = 1,2, \ldots, m.$$

When $\alpha_k = b_{nk}\ for\ k = 1, 2, \ldots m$, the Dirichlet $(\propto_1, \ldots, \propto_m;\ 1)$ universal portfolio becomes the CSD universal portfolio with parameter $\xi = \frac{1}{2}$.

$$\hat{b}_{n+1,k} = \frac{1}{\sum_{l=1}^{m} b_{nl} + 1}\left[b_{nk} + \frac{b_{nk} x_{nk}}{\sum_{l=1}^{m} b_{nl} x_{nl}}\right]$$

$$= \frac{1}{2}\left[b_{nk} + \frac{b_{nl} x_{nl}}{\sum_{l=1}^{m} b_{nl} x_{nl}}\right]$$

$$= (1 - \xi) b_{nk} + \xi \left[\frac{b_{nk} x_{nk}}{\sum_{l=1}^{m} b_{nl} x_{nl}}\right]\ where\ \xi = \frac{1}{2}.$$

We run the order-1 universal portfolio on three 3-stock data sets A, B and C. The Malaysian companies which are in the 3-stock portfolios are shown in Table 2.1. There is a total of 500 trading days of the stocks of the companies displayed in Table 2.1, where the period of trading is from January 1, 2003 until November 30, 2004.

Table 2.1: The three Malaysian companies in the three-stock portfolios A, B and C

Data Set	Malaysian Companies in Each Portfolio
A	Malayan Banking, Genting, Amway (M) Holdings
B	Public Bank, Sunrise, YTL Corporation
C	Hong Leong Bank, RHB Capital, YTL Corporation

For selected values of the parameters α_1, α_2 and α_3, the wealths S_{500} achieved after 500 trading days for data sets A, B and C are displayed in Table 2.2. It is observed that the best wealths achieved by the order-1 universal portfolios are 1.8533, 4.2962 and 4.2960 units for data sets A, B and C respectively, assuming an initial wealth of 1 unit. Set A achieved 1.8533 when we put more weight on α_1 while Set B and C achieved best wealths of 4.2970 when we put more weight on α_3. Higher weight in α_2 does not effect the portfolio performance for this data set. The BCRP wealths for data set A, B and C are 1.8534, 4.2970and 4.2970 respectively. Compared with the wealths achieved by the order-1 universal portfolio, we observe that best wealths achieved by set A, B and C are very close to the BCRP wealth and that the wealth difference is only 0.0001 for set A and 0.0008 for set B and C.

Table 2.2: The best wealth S_{500} and the final portfolio (b_1, b_2, b_3) achieved by the Dirichlet$(\propto_1, \ldots, \propto_m;\ 1)$ universal portfolio with selected values of $(\alpha_1, \alpha_2, \alpha_3)$.

Set	α_1	α_2	α_3	S_{500}	b_1	b_2	b_3
A	10000	1	1	1.8533	0.9998	0.0001	0.0001
	1000	1	1	1.8526	0.9980	0.0010	0.0010
	100	1	1	1.8453	0.9804	0.0098	0.0098
	50	1	1	1.8375	0.9615	0.0192	0.0192
	10	1	1	1.7859	0.8333	0.0833	0.0834
	5	1	1	1.7387	0.7143	0.1428	0.1430
	3	1	1	1.6933	0.6000	0.1998	0.2002
	1	1	10000	1.6539	0.0001	0.0001	0.9998
	1	1	1000	1.6538	0.0010	0.0010	0.9980
	2	1	1	1.6529	0.4999	0.2497	0.2503
	1	1	100	1.6523	0.0098	0.0098	0.9804
	1	1	50	1.6507	0.0192	0.0192	0.9615
	1	1	10	1.6402	0.0833	0.0833	0.8334
	1	1	5	1.6298	0.1428	0.1427	0.7145
	1	1	3	1.6185	0.1999	0.1997	0.6004
	1	1	2	1.6070	0.2499	0.2496	0.5005
	1	1	1	1.5838	0.3333	0.3328	0.3339
	1	2	1	1.4757	0.2500	0.4995	0.2504
	1	3	1	1.4107	0.2001	0.5996	0.2003
	1	5	1	1.3373	0.1429	0.7141	0.1430
	1	10	1	1.2628	0.0834	0.8332	0.0834
	1	50	1	1.1859	0.0192	0.9615	0.0192
	1	100	1	1.1749	0.0098	0.9804	0.0098
	1	1000	1	1.1648	0.0010	0.9980	0.0010
A	1	10000	1	1.1638	0.0001	0.9998	0.0001
B	1	1	10000	4.2962	0.0001	0.0001	0.9998
	1	1	1000	4.2886	0.0010	0.0010	0.9980
	1	1	100	4.2148	0.0098	0.0098	0.9804
	1	1	50	4.1374	0.0192	0.0192	0.9616
	1	1	10	3.6470	0.0832	0.0832	0.8336
	1	1	5	3.2416	0.1424	0.1427	0.7149
	1	1	3	2.8910	0.1993	0.1997	0.6010
	1	1	2	2.6113	0.2491	0.2497	0.5012
	1	1	1	2.1949	0.3321	0.3332	0.3347
	1	2	1	2.0320	0.2493	0.4999	0.2508
	2	1	1	1.9571	0.4989	0.2501	0.2510
	1	3	1	1.9331	0.1995	0.5999	0.2006
	3	1	1	1.8240	0.5991	0.2001	0.2007
	1	5	1	1.8211	0.1426	0.7142	0.1432
	1	10	1	1.7074	0.0832	0.8333	0.0834
	5	1	1	1.6810	0.7137	0.1430	0.1433

	1	50	1	1.5906	0.0192	0.9615	0.0192
	1	100	1	1.5740	0.0098	0.9804	0.0098
	1	1000	1	1.5587	0.0010	0.9980	0.0010
	1	10000	1	1.5572	0.0001	0.9998	0.0001
	10	1	1	1.5427	0.8331	0.0834	0.0835
	50	1	1	1.4061	0.9615	0.0192	0.0192
	100	1	1	1.3872	0.9804	0.0098	0.0098
	1000	1	1	1.3697	0.9980	0.0010	0.0010
B	10000	1	1	1.3679	0.9998	0.0001	0.0001
C	1	1	10000	4.2960	0.0001	0.0001	0.9998
	1	1	1000	4.2864	0.0010	0.0010	0.9980
	1	1	100	4.1942	0.0098	0.0098	0.9804
	1	1	50	4.0976	0.0192	0.0192	0.9615
	1	1	10	3.4978	0.0832	0.0833	0.8334
	1	1	5	3.0182	0.1426	0.1429	0.7145
	1	1	3	2.6167	0.1996	0.2000	0.6004
	1	1	2	2.3063	0.2495	0.2501	0.5004
	1	1	1	1.8621	0.3326	0.3336	0.3338
	2	1	1	1.7320	0.4993	0.2503	0.2504
	3	1	1	1.6548	0.5995	0.2002	0.2003
	1	2	1	1.5890	0.2495	0.5002	0.2503
	5	1	1	1.5684	0.7139	0.1430	0.1431
	10	1	1	1.4813	0.8332	0.0834	0.0834
	1	3	1	1.4405	0.1997	0.6002	0.2002
	50	1	1	1.3921	0.9615	0.0192	0.0192
	100	1	1	1.3794	0.9804	0.0098	0.0098
	1000	1	1	1.3678	0.9980	0.0010	0.0010
	10000	1	1	1.3666	0.9998	0.0001	0.0001
	1	5	1	1.2848	0.1427	0.7144	0.1429
	1	10	1	1.1384	0.0833	0.8334	0.0834
	1	50	1	0.9982	0.0192	0.9615	0.0192
	1	100	1	0.9790	0.0098	0.9804	0.0098
	1	1000	1	0.9615	0.0010	0.9980	0.0010
C	1	10000	1	0.9597	0.0001	0.9998	0.0001

The empirical results for the parameters α_k equal to b_{nk} for $k = 1, 2, \ldots, m$ is shown in Table 2.3 where the order-1 universal portfolio becomes CSD universal portfolio with $\xi = \frac{1}{2}$. Set A achieves a wealth 1.7892 when $\boldsymbol{\alpha} = (0.8, 0.1, 0.1)$.Set B and C achieve wealth of 3.5759 and 3.3974 respectively when $\boldsymbol{\alpha} = (0.1, 0.1, 0.8)$. The portfolios performs better on sets B and C than on set A but not as well as the performance of order-1 universal portfolios for $\alpha_k \geq 1$.

Table 2.3: The best wealth S500 and the final portfolio (b_1, b_2, b_3) achieved by the Dirichlet$(\propto_1, \dots, \propto_m;\ 1)$ universal portfolio with selected values of $(\alpha_1, \alpha_2, \alpha_3)$ when $\alpha_k = b_{nk}$.

Set		initial b		s500	b1	b2	b3
A	0.8	0.1	0.1	1.7892	0.7999	0.0997	0.1004
	0.7	0.1	0.2	1.7798	0.6997	0.0997	0.2007
	0.5	0.1	0.4	1.7507	0.4994	0.0996	0.4010
	0.4	0.1	0.5	1.7310	0.3994	0.0996	0.5011
	0.2	0.1	0.7	1.6820	0.1995	0.0995	0.7010
	0.1	0.1	0.8	1.6530	0.0997	0.0995	0.8008
	0.1	0.2	0.7	1.6130	0.0998	0.1991	0.7012
	0.1	0.5	0.4	1.4690	0.1000	0.4986	0.4014
A	0.1	0.8	0.1	1.2979	0.1002	0.7993	0.1006
B	0.1	0.1	0.8	3.5759	0.0987	0.0994	0.8019
	0.1	0.2	0.7	3.2817	0.0988	0.1989	0.7023
	0.2	0.1	0.7	3.2125	0.1977	0.0995	0.7028
	0.4	0.1	0.5	2.5728	0.3967	0.0998	0.5035
	0.1	0.5	0.4	2.4618	0.0991	0.4985	0.4024
	0.5	0.1	0.4	2.2934	0.4966	0.1000	0.4035
	0.7	0.1	0.2	1.8082	0.6974	0.1003	0.2024
	0.1	0.8	0.1	1.7654	0.0993	0.7998	0.1009
B	0.8	0.1	0.1	1.5992	0.7982	0.1004	0.1013
C	0.1	0.1	0.8	3.3974	0.0993	0.0999	0.8007
	0.2	0.1	0.7	3.0567	0.1988	0.1000	0.7011
	0.1	0.2	0.7	2.9609	0.0994	0.1999	0.7007
	0.4	0.1	0.5	2.4527	0.3983	0.1002	0.5016
	0.5	0.1	0.4	2.1873	0.4982	0.1002	0.4015
	0.1	0.5	0.4	1.9142	0.0994	0.5000	0.4006
	0.7	0.1	0.2	1.7241	0.6985	0.1004	0.2011
	0.8	0.1	0.1	1.5238	0.7989	0.1005	0.1006
C	0.1	0.8	0.1	1.1939	0.0994	0.8004	0.1002

2.4.2 The Dirichlet$(\propto_1, \ldots, \propto_m; 2)$ universal portfolio (order-2 universal portfolio)

The order-2 Dirichlet $(\propto_1, \ldots, \propto_m)$ universal portfolios are given by substituting into (2.16) the respective moments (2.11), (2.12) and (2.13):

$$\hat{b}_{n+1,k} = \frac{\sum_{j_1=1}^{m} \sum_{j_2=1}^{m} x_{nj_1} x_{n-1,j_2} \, E\left[Y_{j_1} Y_{j_2} Y_k\right]}{\sum_{j_1=1}^{m} \sum_{j_2=1}^{m} x_{nj_1} x_{n-1,j_2} \, E\left[Y_{j_1} Y_{j_2}\right]} \qquad (2.16)$$

$for\ k = 1, \ldots, m\ , n = 2,3, \ldots.,$

For m=3 stocks and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, define the normalizing constant

$$\zeta_{n-1,n} = [(\alpha_1 + \alpha_2 + \alpha_3 + 2)\{\, (\boldsymbol{\alpha}^t \boldsymbol{x}_n)(\boldsymbol{\alpha}^t \boldsymbol{x}_{n-1}) + 1^t(\boldsymbol{x}_{n-1} \times \boldsymbol{x}_n \times \boldsymbol{\alpha})\}]^{-1}$$

and the price relative matrices

$$U_n = \begin{bmatrix} x_{n1} & x_{n1} & x_{n1} \\ x_{n2} & x_{n2} & x_{n2} \\ x_{n3} & x_{n3} & x_{n3} \end{bmatrix} = (\boldsymbol{x}_n, \boldsymbol{x}_n, \boldsymbol{x}_n) = \ \boldsymbol{x}_n \mathbf{1}^t$$

$$U_{n-1} = \begin{bmatrix} x_{n-1,1} & x_{n-1,1} & x_{n-1,1} \\ x_{n-1,2} & x_{n-1,2} & x_{n-1,2} \\ x_{n-1,3} & x_{n-1,3} & x_{n-1,3} \end{bmatrix} = (\boldsymbol{x}_{n-1}, \boldsymbol{x}_{n-1}, \boldsymbol{x}_{n-1}) = \ \boldsymbol{x}_{n-1} \mathbf{1}^t \, .$$

where $1 = (1, 1, \ldots, 1)$.

Hence,

$$U_{n-1}^t = \begin{bmatrix} x_{n-1,1} & x_{n-1,1} & x_{n-1,1} \\ x_{n-1,2} & x_{n-1,2} & x_{n-1,2} \\ x_{n-1,3} & x_{n-1,3} & x_{n-1,3} \end{bmatrix}^t = (\boldsymbol{x}_{n-1}, \boldsymbol{x}_{n-1}, \boldsymbol{x}_{n-1})^t = \ \mathbf{1} \boldsymbol{x}_{n-1}^t \, .$$

The Schur product of two matrices is known as the entrywise product or Hadamard product. Let A and B be two rectangular matrices of the same dimension $m \times n$. The Schur product of A and B, denoted by $A \times B$, is defined as

$$A \times B = \left(a_{ij}b_{ij}\right) \, where \, A = \left(a_{ij}\right) and \, B = \left(b_{ij}\right).$$

Then for n=2, 3, 4,…

$$\hat{b}_{n+1,1} = \zeta_{n-1,n}\mathbf{1}^{\mathrm{t}}(U_n \times U_{n-1}^t \times W(2,0))\mathbf{1}, \tag{2.17}$$

$$\hat{b}_{n+1,2} = \zeta_{n-1,n}\mathbf{1}^{\mathrm{t}}(U_n \times U_{n-1}^t \times W(2,1))\mathbf{1}, \tag{2.18}$$

$$\hat{b}_{n+1,3} = \zeta_{n-1,n}\mathbf{1}^{\mathrm{t}}(U_n \times U_{n-1}^t \times W(2,2))\mathbf{1}, \tag{2.19}$$

where the operation "$\times$ " denotes the Schur product of two matrices, the weight matrices $W(\nu, k)$ depend on $\boldsymbol{\alpha}$ only and are independent of n which are given by:

$$W(2,0) = \begin{bmatrix} (\alpha_1+2)(\alpha_1+1)\alpha_1 & \alpha_2(\alpha_1+1)\alpha_1 & \alpha_3(\alpha_1+1)\alpha_1 \\ \alpha_2(\alpha_1+1)\alpha_1 & \alpha_1(\alpha_2+1)\alpha_2 & \alpha_1\alpha_2\alpha_3 \\ \alpha_3(\alpha_1+1)\alpha_1 & \alpha_1\alpha_2\alpha_3 & \alpha_1(\alpha_3+1)\alpha_3 \end{bmatrix}, \tag{2.20}$$

$$W(2,1) = \begin{bmatrix} \alpha_2(\alpha_1+1)\alpha_1 & \alpha_1(\alpha_2+1)\alpha_2 & \alpha_1\alpha_2\alpha_3 \\ \alpha_1(\alpha_2+1)\alpha_2 & (\alpha_2+2)(\alpha_2+1)\alpha_2 & \alpha_3(\alpha_2+1)\alpha_2 \\ \alpha_1\alpha_2\alpha_3 & \alpha_3(\alpha_2+1)\alpha_2 & \alpha_2(\alpha_3+1)\alpha_3 \end{bmatrix}, \tag{2.21}$$

$$W(2,2)=\begin{bmatrix}\alpha_3(\alpha_1+1)\alpha_1 & \alpha_1\alpha_2\alpha_3 & \alpha_1(\alpha_3+1)\alpha_3\\ \alpha_1\alpha_2\alpha_3 & \alpha_3(\alpha_2+1)\alpha_2 & \alpha_2(\alpha_3+1)\alpha_3\\ \alpha_1(\alpha_3+1)\alpha_3 & \alpha_2(\alpha_3+1)\alpha_3 & (\alpha_3+2)(\alpha_3+1)\alpha_3\end{bmatrix}, \qquad (2.22)$$

Therefore,

$$\hat{b}_{n+1,1}=\zeta_{n-1,n}\begin{bmatrix}x_{n1} & x_{n1} & x_{n1}\\ x_{n2} & x_{n2} & x_{n2}\\ x_{n3} & x_{n3} & x_{n3}\end{bmatrix}\times\left[\begin{bmatrix}x_{n-1,1} & x_{n-1,1} & x_{n-1,1}\\ x_{n-1,2} & x_{n-1,2} & x_{n-1,2}\\ x_{n-1,3} & x_{n-1,3} & x_{n-1,3}\end{bmatrix}\right]\times$$

$$\begin{bmatrix}(\alpha_1+2)(\alpha_1+1)\alpha_1 & \alpha_2(\alpha_1+1)\alpha_1 & \alpha_3(\alpha_1+1)\alpha_1\\ \alpha_2(\alpha_1+1)\alpha_1 & \alpha_1(\alpha_2+1)\alpha_2 & \alpha_1\alpha_2\alpha_3\\ \alpha_3(\alpha_1+1)\alpha_1 & \alpha_1\alpha_2\alpha_3 & \alpha_1(\alpha_3+1)\alpha_3\end{bmatrix}, \qquad (2.23)$$

$$\hat{b}_{n+1,2}=\zeta_{n-1,n}\begin{bmatrix}x_{n1} & x_{n1} & x_{n1}\\ x_{n2} & x_{n2} & x_{n2}\\ x_{n3} & x_{n3} & x_{n3}\end{bmatrix}\times\left[\begin{bmatrix}x_{n-1,1} & x_{n-1,1} & x_{n-1,1}\\ x_{n-1,2} & x_{n-1,2} & x_{n-1,2}\\ x_{n-1,3} & x_{n-1,3} & x_{n-1,3}\end{bmatrix}\right]$$

$$\times\begin{bmatrix}\alpha_2(\alpha_1+1)\alpha_1 & \alpha_1(\alpha_2+1)\alpha_2 & \alpha_1\alpha_2\alpha_3\\ \alpha_1(\alpha_2+1)\alpha_2 & (\alpha_2+2)(\alpha_2+1)\alpha_2 & \alpha_3(\alpha_2+1)\alpha_2\\ \alpha_1\alpha_2\alpha_3 & \alpha_3(\alpha_2+1)\alpha_2 & \alpha_2(\alpha_3+1)\alpha_3\end{bmatrix}, \qquad (2.24)$$

$$\hat{b}_{n+1,3}$$

$$=\zeta_{n-1,n}\begin{bmatrix}x_{n1} & x_{n1} & x_{n1}\\ x_{n2} & x_{n2} & x_{n2}\\ x_{n3} & x_{n3} & x_{n3}\end{bmatrix}\times\left[\begin{bmatrix}x_{n-1,1} & x_{n-1,1} & x_{n-1,1}\\ x_{n-1,2} & x_{n-1,2} & x_{n-1,2}\\ x_{n-1,3} & x_{n-1,3} & x_{n-1,3}\end{bmatrix}\right] \qquad (2.25)$$

$$\times\begin{bmatrix}\alpha_3(\alpha_1+1)\alpha_1 & \alpha_1\alpha_2\alpha_3 & \alpha_1(\alpha_3+1)\alpha_3\\ \alpha_1\alpha_2\alpha_3 & \alpha_3(\alpha_2+1)\alpha_2 & \alpha_2(\alpha_3+1)\alpha_3\\ \alpha_1(\alpha_3+1)\alpha_3 & \alpha_2(\alpha_3+1)\alpha_3 & (\alpha_3+2)(\alpha_3+1)\alpha_3\end{bmatrix},$$

where

$$\zeta_{n-1,n} = \left[(\alpha_1 + \alpha_2 + \alpha_3 + 2)\left\{ \left(\sum_{i=1}^{3} \alpha_i x_{ni}\right)\left(\sum_{i=1}^{3} \alpha_i x_{n-1,i}\right) + \sum_{j=1}^{3} x_{nj} x_{n-1,j} \alpha_j \right\}\right]^{-1} \tag{2.26}$$

The empirical results of order-2 universal portfolios are presented in Table 2.4 for selected values of α_1, $\alpha_2$and α_3. From Table 2.4, we observe that the best wealths for Sets A, B and C have no significant change when the parametric value is increased. For order-2 universal portfolios, the best wealths achieved by A, B and C are 1.8533, 4.2962 and 4.2960 units respectively. The weighting on each α_k to get higher wealths is similar to that of order-1 universal portfolios. The performance of this universal portfolio is very close to BCRP wealth. It is shown in Table 2.6, that by putting more weight on α_3, we can get higher returns namely 4.2970 for sets B and C which is the BCRP wealth.

Table 2.4: The best wealth S_{500} and the final portfolio $(\boldsymbol{b_1}, \boldsymbol{b_2}, \boldsymbol{b_3})$ achieved by the Dirichlet$(\propto_1, \ldots, \propto_m;\ 2)$ universal portfolio with selected values of $(\boldsymbol{\alpha_1}, \boldsymbol{\alpha_2}, \boldsymbol{\alpha_3})$.

Set	$\propto_1$	$\propto_2$	$\propto_3$	S_n	b_1	b_2	b_3
A	10000	1	1	1.8533	0.9998	0.0001	0.0001
	1000	1	1	1.8526	0.9980	0.0010	0.0010
	100	1	1	1.8453	0.9804	0.0098	0.0098
	50	1	1	1.8375	0.9616	0.0192	0.0192
	10	1	1	1.7860	0.8337	0.0831	0.0832
	5	1	1	1.7389	0.7152	0.1422	0.1426
	3	1	1	1.6936	0.6014	0.1990	0.1996
	1	1	10000	1.6539	0.0001	0.0001	0.9998
	1	1	1000	1.6538	0.0010	0.0010	0.9980
	2	1	1	1.6534	0.5018	0.2487	0.2496
	1	1	100	1.6523	0.0098	0.0098	0.9804

	1	1	50	1.6507	0.0192	0.0192	0.9615
	1	1	10	1.6403	0.0835	0.0832	0.8333
	1	1	5	1.6301	0.1433	0.1425	0.7142
	1	1	3	1.6190	0.2008	0.1993	0.5998
	1	1	2	1.6077	0.2512	0.2490	0.4998
	1	1	1	1.5846	0.3353	0.3316	0.3331
	1	2	1	1.4764	0.2515	0.4984	0.2501
	1	3	1	1.4112	0.2011	0.5987	0.2002
	1	5	1	1.3376	0.1435	0.7135	0.1430
	1	10	1	1.2629	0.0836	0.8330	0.0834
	1	50	1	1.1859	0.0192	0.9615	0.0192
	1	100	1	1.1749	0.0098	0.9804	0.0098
	1	1000	1	1.1648	0.0010	0.9980	0.0010
A	1	10000	1	1.1638	0.0001	0.9998	0.0001
B	1	1	10000	4.2962	0.0001	0.0001	0.9998
	1	1	1000	4.2886	0.0010	0.0010	0.9980
	1	1	100	4.2149	0.0098	0.0098	0.9804
	1	1	50	4.1374	0.0192	0.0192	0.9616
	1	1	10	3.6476	0.0831	0.0830	0.8338
	1	1	5	3.2429	0.1424	0.1421	0.7155
	1	1	3	2.8929	0.1994	0.1988	0.6018
	1	1	2	2.6137	0.2493	0.2484	0.5023
	1	1	1	2.1977	0.3328	0.3328	0.3359
	1	2	1	2.0342	0.2500	0.4981	0.2519
	2	1	1	1.9586	0.4995	0.2488	0.2516
	1	3	1	1.9348	0.2001	0.5985	0.2014
	3	1	1	1.8250	0.5996	0.1992	0.2011
	1	5	1	1.8221	0.1430	0.7133	0.1437
	1	10	1	1.7078	0.0834	0.8329	0.0837
	5	1	1	1.6815	0.7140	0.1425	0.1435
	1	50	1	1.5906	0.0192	0.9615	0.0193
	1	100	1	1.5740	0.0098	0.9804	0.0098
	1	1000	1	1.5587	0.0010	0.9980	0.0010
	1	10000	1	1.5572	0.0001	0.9998	0.0001
	10	1	1	1.5429	0.8332	0.0832	0.0836
	50	1	1	1.4062	0.9615	0.0192	0.0192
	100	1	1	1.3872	0.9804	0.0098	0.0098
	1000	1	1	1.3697	0.9980	0.0010	0.0010
B	10000	1	1	1.3679	0.9998	0.0001	0.0001
C	1	1	10000	4.2960	0.0001	0.0001	0.9998
	1	1	1000	4.2864	0.0010	0.0010	0.9980
	1	1	100	4.1942	0.0098	0.0098	0.9804
	1	1	50	4.0976	0.0192	0.0192	0.9616
	1	1	10	3.4979	0.0831	0.0832	0.8337
	1	1	5	3.0183	0.1423	0.1426	0.7152
	1	1	3	2.6169	0.1990	0.1996	0.6014

	1	1	2	2.3066	0.2487	0.2495	0.5018
	1	1	1	1.8623	0.3316	0.3330	0.3353
	2	1	1	1.7319	0.4985	0.2501	0.2515
	3	1	1	1.6547	0.5988	0.2002	0.2011
	1	2	1	1.5893	0.2490	0.4997	0.2512
	5	1	1	1.5683	0.7135	0.1430	0.1435
	10	1	1	1.4813	0.8330	0.0834	0.0836
	1	3	1	1.4407	0.1994	0.5998	0.2008
	50	1	1	1.3921	0.9615	0.0192	0.0192
	100	1	1	1.3794	0.9804	0.0098	0.0098
	1000	1	1	1.3678	0.9980	0.0010	0.0010
	10000	1	1	1.3666	0.9998	0.0001	0.0001
	1	5	1	1.2849	0.1425	0.7141	0.1433
	1	10	1	1.1384	0.0832	0.8333	0.0835
	1	50	1	0.9982	0.0192	0.9615	0.0192
	1	100	1	0.9790	0.0098	0.9804	0.0098
	1	1000	1	0.9615	0.0010	0.9980	0.0010
C	1	10000	1	0.9597	0.0001	0.9998	0.0001

2.4.3 The Dirichlet$(\propto_1, \ldots, \propto_m;\ 3)$ universal portfolio (order-3 universal portfolio)

From (2.8), (2.13) and (2.14), the order-3 universal portfolio is given by:

$$\hat{b}_{n+1,k} = \frac{\sum_{j_1=1}^{m}\sum_{j_2=1}^{m}\sum_{j_3=1}^{m} x_{nj_1}x_{n-1,j_2}x_{n-2,j_3}\, E\left[Y_{j_1}Y_{j_2}Y_{j_3}Y_k\right]}{\sum_{j_1=1}^{m}\sum_{j_2=1}^{m}\sum_{j_3=1}^{m} x_{nj_1}x_{n-1,j_2}\, x_{n-2,j_3}\, E\left[Y_{j_1}Y_{j_2}Y_{j_3}\right]} \qquad (2.27)$$

$for\ k = 1, 2, 3\ , n = 3, 4, \ldots.$

Define the price-relative matrices

$$U_n = \begin{bmatrix} x_{n1} & x_{n1} & x_{n1} \\ x_{n2} & x_{n2} & x_{n2} \\ x_{n3} & x_{n3} & x_{n3} \end{bmatrix} = (\boldsymbol{x}_n, \boldsymbol{x}_n, \boldsymbol{x}_n)$$

Similarly, $U_{n-1} = (\boldsymbol{x}_{n-1}, \boldsymbol{x}_{n-1}, \boldsymbol{x}_{n-1})$. Hence

$$U_{n-1}^t = \begin{bmatrix} x_{n-1,1} & x_{n-1,2} & x_{n-1,3} \\ x_{n-1,1} & x_{n-1,2} & x_{n-1,3} \\ x_{n-1,1} & x_{n-1,2} & x_{n-1,3} \end{bmatrix}$$

The Schur product of U_n and U_{n-1}^t is

$$U_n \times U_{n-1}^t = \begin{bmatrix} x_{n1}x_{n-1,1} & x_{n1}x_{n-1,2} & x_{n1}x_{n-1,3} \\ x_{n2}x_{n-1,1} & x_{n2}x_{n-1,2} & x_{n2}x_{n-1,3} \\ x_{n3}x_{n-1,1} & x_{n3}x_{n-1,2} & x_{n3}x_{n-1,3} \end{bmatrix}$$

Let A be a rectangular matrix with columns $\boldsymbol{a}_1,\ \boldsymbol{a}_2, \dots, \boldsymbol{a}_k$, where $\boldsymbol{a}_i$ is $m \times 1$ and A is $m \times k$. Then $vec(A)$ is the column vector of dimension $mk \times 1$, $vec(A) = \begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \boldsymbol{a}_k \end{bmatrix}$.

Let $V_n = U_n \times U_{n-1}^t$. Then $vec(V_n)$ is the column vector

$$vec(V_n) = \begin{bmatrix} x_{n1}x_{n-1,1} \\ x_{n2}x_{n-1,1} \\ x_{n3}x_{n-1,1} \\ x_{n1}x_{n-1,2} \\ x_{n2}x_{n-1,2} \\ x_{n3}x_{n-1,2} \\ x_{n1}x_{n-1,3} \\ x_{n2}x_{n-1,3} \\ x_{n3}x_{n-1,3} \end{bmatrix}$$

We form the 9×3 augmented matrix $(vec(V_n), vec(V_n), vec(V_n))$.

Similarly, we form the 3×9 augmented matrix $(\ U_{n-2},\ U_{n-2},\ U_{n-2})$ given by:

$$\begin{bmatrix} x_{n-2,1} & x_{n-2,1} & x_{n-2,1} & x_{n-2,1} & x_{n-2,1} & x_{n-2,1} & x_{n-2,1} & x_{n-2,1} & x_{n-2,1} \\ x_{n-2,2} & x_{n-2,2} & x_{n-2,2} & x_{n-2,2} & x_{n-2,2} & x_{n-2,2} & x_{n-2,2} & x_{n-2,2} & x_{n-2,2} \\ x_{n-2,3} & x_{n-2,3} & x_{n-2,3} & x_{n-2,3} & x_{n-2,3} & x_{n-2,3} & x_{n-2,3} & x_{n-2,3} & x_{n-2,3} \end{bmatrix}$$

Define the normalizing constant

$$\zeta_{n-2,n} = [(\alpha_1 + \alpha_2 + \alpha_3 + 3)\left\{ \mathbf{1}^t \left(\left(vec(V_n), vec(V_n), vec(V_n)\right) \times (U_{n-2}, U_{n-2}, U_{n-2})^t \times W(3,0) \right) \mathbf{1} \right\}]^{-1}. \quad (2.28)$$

Then for n= 3, 4,…, the portfolio components are given by:

$$\hat{b}_{n+1,1} = \zeta_{n-2,n} \mathbf{1}^{\mathrm{t}} \left(\left(vec(V_n), vec(V_n), vec(V_n)\right) \times (U_{n-2}, U_{n-2}, U_{n-2})^t \times W(3,1) \right) \mathbf{1}, \quad (2.29)$$

$$\hat{b}_{n+1,2} = \zeta_{n-2,n} \mathbf{1}^{\mathrm{t}} \left(\left(vec(V_n), vec(V_n), vec(V_n)\right) \times (U_{n-2}, U_{n-2}, U_{n-2})^t \times W(3,2) \right) \mathbf{1}, \quad (2.30)$$

$$\hat{b}_{n+1,3} = \zeta_{n-2,n} \mathbf{1}^{\mathrm{t}} \left(\left(vec(V_n), vec(V_n), vec(V_n)\right) \times (U_{n-2}, U_{n-2}, U_{n-2})^t \times W(3,3) \right) \mathbf{1}, \quad (2.31)$$

where the operation “$\times$" denotes the Schur product of two matrices, the weight matrices $W(\nu, k)$ depend on $\boldsymbol{\alpha}$ only and are independent of n are given by:

$$W(3,0) =$$

$$\begin{bmatrix} (\alpha_1+2)(\alpha_1+1)\alpha_1 & (\alpha_1+1)\alpha_1\alpha_2 & (\alpha_1+1)\alpha_1\alpha_3 \\ (\alpha_1+1)\alpha_1\alpha_2 & (\alpha_2+1)\alpha_2\alpha_1 & \alpha_1\alpha_2\alpha_3 \\ (\alpha_1+1)\alpha_1\alpha_3 & \alpha_1\alpha_2\alpha_3 & (\alpha_3+1)\alpha_1\alpha_3 \\ (\alpha_1+1)\alpha_1\alpha_2 & (\alpha_2+1)\alpha_1\alpha_2 & \alpha_1\alpha_2\alpha_3 \\ (\alpha_2+1)\alpha_1\alpha_2 & (\alpha_2+2)(\alpha_2+1)\alpha_1 & (\alpha_2+1)\alpha_2\alpha_3 \\ \alpha_1\alpha_2\alpha_3 & (\alpha_2+1)\alpha_2\alpha_3 & (\alpha_3+1)\alpha_2\alpha_3 \\ (\alpha_1+1)\alpha_1\alpha_3 & \alpha_1\alpha_2\alpha_3 & (\alpha_3+1)\alpha_1\alpha_3 \\ \alpha_1\alpha_2\alpha_3 & (\alpha_2+1)\alpha_2\alpha_3 & (\alpha_3+1)\alpha_2\alpha_3 \\ (\alpha_3+1)\alpha_1\alpha_3 & (\alpha_3+1)\alpha_2\alpha_3 & (\alpha_3+2)(\alpha_3+1)\alpha_3 \end{bmatrix}, \quad (2.32)$$

$$W(3,1) =$$

$$\begin{bmatrix} (\alpha_1+3)(\alpha_1+2)(\alpha_1+1)\alpha_1 & (\alpha_1+2)(\alpha_1+1)\alpha_1\alpha_2 & (\alpha_1+2)(\alpha_1+1)\alpha_1\alpha_3 \\ (\alpha_1+2)(\alpha_1+1)\alpha_1\alpha_2 & (\alpha_1+1)(\alpha_2+1)\alpha_2\alpha_1 & (\alpha_1+1)\alpha_1\alpha_2\alpha_3 \\ (\alpha_1+2)(\alpha_1+1)\alpha_1\alpha_3 & (\alpha_1+1)\alpha_1\alpha_2\alpha_3 & (\alpha_1+1)(\alpha_3+1)\alpha_1\alpha_3 \\ (\alpha_1+2)(\alpha_1+1)\alpha_1\alpha_2 & (\alpha_1+1)(\alpha_2+1)\alpha_1\alpha_2 & (\alpha_1+1)\alpha_1\alpha_2\alpha_3 \\ (\alpha_1+1)(\alpha_2+1)\alpha_1\alpha_2 & (\alpha_2+2)(\alpha_2+1)\alpha_1\alpha_2 & (\alpha_2+1)\alpha_1\alpha_2\alpha_3 \\ (\alpha_1+1)\alpha_1\alpha_2\alpha_3 & (\alpha_2+1)\alpha_1\alpha_2\alpha_3 & (\alpha_3+1)\alpha_1\alpha_2\alpha_3 \\ (\alpha_1+2)(\alpha_1+1)\alpha_1\alpha_3 & (\alpha_1+1)\alpha_1\alpha_2\alpha_3 & (\alpha_1+1)(\alpha_3+1)\alpha_1\alpha_3 \\ (\alpha_1+1)\alpha_1\alpha_2\alpha_3 & (\alpha_2+1)\alpha_1\alpha_2\alpha_3 & (\alpha_3+1)\alpha_1\alpha_2\alpha_3 \\ (\alpha_1+1)(\alpha_3+1)\alpha_1\alpha_3 & (\alpha_3+1)\alpha_1\alpha_2\alpha_3 & (\alpha_3+2)(\alpha_3+1)\alpha_1\alpha_3 \end{bmatrix},$$

(2.33)

$W(3,2) =$

$$\begin{bmatrix} (\alpha_1+2)(\alpha_1+1)\alpha_1\alpha_2 & (\alpha_1+1)(\alpha_2+1)\alpha_1\alpha_2 & (\alpha_1+1)\alpha_1\alpha_2\alpha_3 \\ (\alpha_1+1)(\alpha_2+1)\alpha_1\alpha_2 & (\alpha_2+2)(\alpha_2+1)\alpha_2\alpha_1 & (\alpha_2+1)\alpha_1\alpha_2\alpha_3 \\ (\alpha_1+1)\alpha_1\alpha_2\alpha_3 & (\alpha_2+1)\alpha_1\alpha_2\alpha_3 & (\alpha_3+1)\alpha_1\alpha_2\alpha_3 \\ (\alpha_1+1)(\alpha_2+1)\alpha_1\alpha_2 & (\alpha_2+2)(\alpha_2+1)\alpha_1\alpha_2 & (\alpha_2+1)\alpha_1\alpha_2\alpha_3 \\ (\alpha_2+2)(\alpha_2+1)\alpha_1\alpha_2 & (\alpha_2+3)(\alpha_2+2)(\alpha_2+1)\alpha_1 & (\alpha_2+2)(\alpha_2+1)\alpha_2\alpha_3 \\ (\alpha_2+1)\alpha_1\alpha_2\alpha_3 & (\alpha_2+2)(\alpha_2+1)\alpha_2\alpha_3 & (\alpha_2+1)(\alpha_3+1)\alpha_2\alpha_3 \\ (\alpha_1+1)\alpha_1\alpha_2\alpha_3 & (\alpha_2+1)\alpha_1\alpha_2\alpha_3 & (\alpha_3+1)\alpha_1\alpha_2\alpha_3 \\ (\alpha_2+1)\alpha_1\alpha_2\alpha_3 & (\alpha_2+2)(\alpha_2+1)\alpha_2\alpha_3 & (\alpha_2+1)(\alpha_3+1)\alpha_2\alpha_3 \\ (\alpha_3+1)\alpha_1\alpha_2\alpha_3 & (\alpha_2+1)(\alpha_3+1)\alpha_2\alpha_3 & (\alpha_3+2)(\alpha_3+1)\alpha_2\alpha_3 \end{bmatrix}, \tag{2.34}$$

$W(3,3) =$

$$\begin{bmatrix} (\alpha_1+2)(\alpha_1+1)\alpha_1\alpha_3 & (\alpha_1+1)\alpha_1\alpha_2\alpha_3 & (\alpha_1+1)(\alpha_3+1)\alpha_1\alpha_3 \\ (\alpha_1+1)\alpha_1\alpha_2\alpha_3 & (\alpha_2+1)\alpha_2\alpha_1\alpha_3 & (\alpha_3+1)\alpha_1\alpha_2\alpha_3 \\ (\alpha_1+1)(\alpha_3+1)\alpha_1\alpha_3 & (\alpha_3+1)\alpha_1\alpha_2\alpha_3 & (\alpha_3+2)(\alpha_3+1)\alpha_1\alpha_3 \\ (\alpha_1+1)\alpha_1\alpha_2\alpha_3 & (\alpha_2+1)\alpha_1\alpha_2\alpha_3 & (\alpha_3+1)\alpha_1\alpha_2\alpha_3 \\ (\alpha_2+1)\alpha_1\alpha_2\alpha_3 & (\alpha_2+2)(\alpha_2+1)\alpha_1\alpha_3 & (\alpha_2+1)(\alpha_3+1)\alpha_2\alpha_3 \\ (\alpha_3+1)\alpha_1\alpha_2\alpha_3 & (\alpha_2+1)(\alpha_3+1)\alpha_2\alpha_3 & (\alpha_3+2)(\alpha_3+1)\alpha_2\alpha_3 \\ (\alpha_1+1)(\alpha_3+1)\alpha_1\alpha_3 & (\alpha_3+1)\alpha_1\alpha_2\alpha_3 & (\alpha_3+2)(\alpha_3+1)\alpha_1\alpha_3 \\ (\alpha_3+1)\alpha_1\alpha_2\alpha_3 & (\alpha_2+1)(\alpha_3+1)\alpha_2\alpha_3 & (\alpha_3+2)(\alpha_3+1)\alpha_2\alpha_3 \\ (\alpha_3+2)(\alpha_3+1)\alpha_1\alpha_3 & (\alpha_3+2)(\alpha_3+1)\alpha_2\alpha_3 & (\alpha_3+3)(\alpha_3+2)(\alpha_3+1)\alpha_3 \end{bmatrix}. \tag{2.35}$$

Therefore,

$$\hat{b}_{n+1,1} = \zeta_{n-2,n} \begin{bmatrix} x_{n1}x_{n-1,1} & x_{n1}x_{n-1,1} & x_{n1}x_{n-1,1} \\ x_{n2}x_{n-1,1} & x_{n2}x_{n-1,1} & x_{n2}x_{n-1,1} \\ x_{n3}x_{n-1,1} & x_{n3}x_{n-1,1} & x_{n3}x_{n-1,1} \\ x_{n1}x_{n-1,2} & x_{n1}x_{n-1,2} & x_{n1}x_{n-1,2} \\ x_{n2}x_{n-1,2} & x_{n2}x_{n-1,2} & x_{n2}x_{n-1,2} \\ x_{n3}x_{n-1,2} & x_{n3}x_{n-1,2} & x_{n3}x_{n-1,2} \\ x_{n1}x_{n-1,3} & x_{n1}x_{n-1,3} & x_{n1}x_{n-1,3} \\ x_{n2}x_{n-1,3} & x_{n2}x_{n-1,3} & x_{n2}x_{n-1,3} \\ x_{n3}x_{n-1,3} & x_{n3}x_{n-1,3} & x_{n3}x_{n-1,3} \end{bmatrix} \times$$

$$\begin{bmatrix} x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \end{bmatrix} \times$$

$$\begin{bmatrix} (\alpha_1+3)(\alpha_1+2)(\alpha_1+1)\alpha_1 & (\alpha_1+2)(\alpha_1+1)\alpha_1\alpha_2 & (\alpha_1+2)(\alpha_1+1)\alpha_1\alpha_3 \\ (\alpha_1+2)(\alpha_1+1)\alpha_1\alpha_2 & (\alpha_1+1)(\alpha_2+1)\alpha_2\alpha_1 & (\alpha_1+1)\alpha_1\alpha_2\alpha_3 \\ (\alpha_1+2)(\alpha_1+1)\alpha_1\alpha_3 & (\alpha_1+1)\alpha_1\alpha_2\alpha_3 & (\alpha_1+1)(\alpha_3+1)\alpha_1\alpha_3 \\ (\alpha_1+2)(\alpha_1+1)\alpha_1\alpha_2 & (\alpha_1+1)(\alpha_2+1)\alpha_1\alpha_2 & (\alpha_1+1)\alpha_1\alpha_2\alpha_3 \\ (\alpha_1+1)(\alpha_2+1)\alpha_1\alpha_2 & (\alpha_2+2)(\alpha_2+1)\alpha_1\alpha_2 & (\alpha_2+1)\alpha_1\alpha_2\alpha_3 \\ (\alpha_1+1)\alpha_1\alpha_2\alpha_3 & (\alpha_2+1)\alpha_1\alpha_2\alpha_3 & (\alpha_3+1)\alpha_1\alpha_2\alpha_3 \\ (\alpha_1+2)(\alpha_1+1)\alpha_1\alpha_3 & (\alpha_1+1)\alpha_1\alpha_2\alpha_3 & (\alpha_1+1)(\alpha_3+1)\alpha_1\alpha_3 \\ (\alpha_1+1)\alpha_1\alpha_2\alpha_3 & (\alpha_2+1)\alpha_1\alpha_2\alpha_3 & (\alpha_3+1)\alpha_1\alpha_2\alpha_3 \\ (\alpha_1+1)(\alpha_3+1)\alpha_1\alpha_3 & (\alpha_3+1)\alpha_1\alpha_2\alpha_3 & (\alpha_3+2)(\alpha_3+1)\alpha_1\alpha_3 \end{bmatrix}, \quad (2.36)$$

$$\hat{b}_{n+1,2}$$

$$= \zeta_{n-2,n} \begin{bmatrix} x_{n1}x_{n-1,1} & x_{n1}x_{n-1,1} & x_{n1}x_{n-1,1} \\ x_{n2}x_{n-1,1} & x_{n2}x_{n-1,1} & x_{n2}x_{n-1,1} \\ x_{n3}x_{n-1,1} & x_{n3}x_{n-1,1} & x_{n3}x_{n-1,1} \\ x_{n1}x_{n-1,2} & x_{n1}x_{n-1,2} & x_{n1}x_{n-1,2} \\ x_{n2}x_{n-1,2} & x_{n2}x_{n-1,2} & x_{n2}x_{n-1,2} \\ x_{n3}x_{n-1,2} & x_{n3}x_{n-1,2} & x_{n3}x_{n-1,2} \\ x_{n1}x_{n-1,3} & x_{n1}x_{n-1,3} & x_{n1}x_{n-1,3} \\ x_{n2}x_{n-1,3} & x_{n2}x_{n-1,3} & x_{n2}x_{n-1,3} \\ x_{n3}x_{n-1,3} & x_{n3}x_{n-1,3} & x_{n3}x_{n-1,3} \end{bmatrix} \times \begin{bmatrix} x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \end{bmatrix}$$

$$\times \begin{bmatrix} (\alpha_1+2)(\alpha_1+1)\alpha_1\alpha_2 & (\alpha_1+1)(\alpha_2+1)\alpha_1\alpha_2 & (\alpha_1+1)\alpha_1\alpha_2\alpha_3 \\ (\alpha_1+1)(\alpha_2+1)\alpha_1\alpha_2 & (\alpha_2+2)(\alpha_2+1)\alpha_2\alpha_1 & (\alpha_2+1)\alpha_1\alpha_2\alpha_3 \\ (\alpha_1+1)\alpha_1\alpha_2\alpha_3 & (\alpha_2+1)\alpha_1\alpha_2\alpha_3 & (\alpha_3+1)\alpha_1\alpha_2\alpha_3 \\ (\alpha_1+1)(\alpha_2+1)\alpha_1\alpha_2 & (\alpha_2+2)(\alpha_2+1)\alpha_1\alpha_2 & (\alpha_2+1)\alpha_1\alpha_2\alpha_3 \\ (\alpha_2+2)(\alpha_2+1)\alpha_1\alpha_2 & (\alpha_2+3)(\alpha_2+2)(\alpha_2+1)\alpha_1 & (\alpha_2+2)(\alpha_2+1)\alpha_2\alpha_3 \\ (\alpha_2+1)\alpha_1\alpha_2\alpha_3 & (\alpha_2+2)(\alpha_2+1)\alpha_2\alpha_3 & (\alpha_2+1)(\alpha_3+1)\alpha_2\alpha_3 \\ (\alpha_1+1)\alpha_1\alpha_2\alpha_3 & (\alpha_2+1)\alpha_1\alpha_2\alpha_3 & (\alpha_3+1)\alpha_1\alpha_2\alpha_3 \\ (\alpha_2+1)\alpha_1\alpha_2\alpha_3 & (\alpha_2+2)(\alpha_2+1)\alpha_2\alpha_3 & (\alpha_2+1)(\alpha_3+1)\alpha_2\alpha_3 \\ (\alpha_3+1)\alpha_1\alpha_2\alpha_3 & (\alpha_2+1)(\alpha_3+1)\alpha_2\alpha_3 & (\alpha_3+2)(\alpha_3+1)\alpha_2\alpha_3 \end{bmatrix}, \tag{2.37}$$

and

$$\hat{b}_{n+1,3}$$

$$= \zeta_{n-2,n} \begin{bmatrix} x_{n1}x_{n-1,1} & x_{n1}x_{n-1,1} & x_{n1}x_{n-1,1} \\ x_{n2}x_{n-1,1} & x_{n2}x_{n-1,1} & x_{n2}x_{n-1,1} \\ x_{n3}x_{n-1,1} & x_{n3}x_{n-1,1} & x_{n3}x_{n-1,1} \\ x_{n1}x_{n-1,2} & x_{n1}x_{n-1,2} & x_{n1}x_{n-1,2} \\ x_{n2}x_{n-1,2} & x_{n2}x_{n-1,2} & x_{n2}x_{n-1,2} \\ x_{n3}x_{n-1,2} & x_{n3}x_{n-1,2} & x_{n3}x_{n-1,2} \\ x_{n1}x_{n-1,3} & x_{n1}x_{n-1,3} & x_{n1}x_{n-1,3} \\ x_{n2}x_{n-1,3} & x_{n2}x_{n-1,3} & x_{n2}x_{n-1,3} \\ x_{n3}x_{n-1,3} & x_{n3}x_{n-1,3} & x_{n3}x_{n-1,3} \end{bmatrix} \times \begin{bmatrix} x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \end{bmatrix}$$

$$\times \begin{bmatrix} (\alpha_1+2)(\alpha_1+1)\alpha_1\alpha_3 & (\alpha_1+1)\alpha_1\alpha_2\alpha_3 & (\alpha_1+1)(\alpha_3+1)\alpha_1\alpha_3 \\ (\alpha_1+1)\alpha_1\alpha_2\alpha_3 & (\alpha_2+1)\alpha_2\alpha_1\alpha_3 & (\alpha_3+1)\alpha_1\alpha_2\alpha_3 \\ (\alpha_1+1)(\alpha_3+1)\alpha_1\alpha_3 & (\alpha_3+1)\alpha_1\alpha_2\alpha_3 & (\alpha_3+2)(\alpha_3+1)\alpha_1\alpha_3 \\ (\alpha_1+1)\alpha_1\alpha_2\alpha_3 & (\alpha_2+1)\alpha_1\alpha_2\alpha_3 & (\alpha_3+1)\alpha_1\alpha_2\alpha_3 \\ (\alpha_2+1)\alpha_1\alpha_2\alpha_3 & (\alpha_2+2)(\alpha_2+1)\alpha_1\alpha_3 & (\alpha_2+1)(\alpha_3+1)\alpha_2\alpha_3 \\ (\alpha_3+1)\alpha_1\alpha_2\alpha_3 & (\alpha_2+1)(\alpha_3+1)\alpha_2\alpha_3 & (\alpha_3+2)(\alpha_3+1)\alpha_2\alpha_3 \\ (\alpha_1+1)(\alpha_3+1)\alpha_1\alpha_3 & (\alpha_3+1)\alpha_1\alpha_2\alpha_3 & (\alpha_3+2)(\alpha_3+1)\alpha_1\alpha_3 \\ (\alpha_3+1)\alpha_1\alpha_2\alpha_3 & (\alpha_2+1)(\alpha_3+1)\alpha_2\alpha_3 & (\alpha_3+2)(\alpha_3+1)\alpha_2\alpha_3 \\ (\alpha_3+2)(\alpha_3+1)\alpha_1\alpha_3 & (\alpha_3+2)(\alpha_3+1)\alpha_2\alpha_3 & (\alpha_3+3)(\alpha_3+2)(\alpha_3+1)\alpha_3 \end{bmatrix}.$$

(2.38)

where

$$\zeta_{n-1,n}$$

$$= (\alpha_1+\alpha_2+\alpha_3+3)^{-1}\left[\mathbf{1}^t\begin{bmatrix} x_{n1}x_{n-1,1} & x_{n1}x_{n-1,1} & x_{n1}x_{n-1,1} \\ x_{n2}x_{n-1,1} & x_{n2}x_{n-1,1} & x_{n2}x_{n-1,1} \\ x_{n3}x_{n-1,1} & x_{n3}x_{n-1,1} & x_{n3}x_{n-1,1} \\ x_{n1}x_{n-1,2} & x_{n1}x_{n-1,2} & x_{n1}x_{n-1,2} \\ x_{n2}x_{n-1,2} & x_{n2}x_{n-1,2} & x_{n2}x_{n-1,2} \\ x_{n3}x_{n-1,2} & x_{n3}x_{n-1,2} & x_{n3}x_{n-1,2} \\ x_{n1}x_{n-1,3} & x_{n1}x_{n-1,3} & x_{n1}x_{n-1,3} \\ x_{n2}x_{n-1,3} & x_{n2}x_{n-1,3} & x_{n2}x_{n-1,3} \\ x_{n3}x_{n-1,3} & x_{n3}x_{n-1,3} & x_{n3}x_{n-1,3} \end{bmatrix}\right.$$

$$\times\begin{bmatrix} x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \\ x_{n-2,1} & x_{n-2,2} & x_{n-2,3} \end{bmatrix}$$

$$\left.\times\begin{bmatrix} (\alpha_1+2)(\alpha_1+1)\alpha_1 & (\alpha_1+1)\alpha_1\alpha_2 & (\alpha_1+1)\alpha_1\alpha_3 \\ (\alpha_1+1)\alpha_1\alpha_2 & (\alpha_2+1)\alpha_2\alpha_1 & \alpha_1\alpha_2\alpha_3 \\ (\alpha_1+1)\alpha_1\alpha_3 & \alpha_1\alpha_2\alpha_3 & (\alpha_3+1)\alpha_1\alpha_3 \\ (\alpha_1+1)\alpha_1\alpha_2 & (\alpha_2+1)\alpha_1\alpha_2 & \alpha_1\alpha_2\alpha_3 \\ (\alpha_2+1)\alpha_1\alpha_2 & (\alpha_2+2)(\alpha_2+1)\alpha_1 & (\alpha_2+1)\alpha_2\alpha_3 \\ \alpha_1\alpha_2\alpha_3 & (\alpha_2+1)\alpha_2\alpha_3 & (\alpha_3+1)\alpha_2\alpha_3 \\ (\alpha_1+1)\alpha_1\alpha_3 & \alpha_1\alpha_2\alpha_3 & (\alpha_3+1)\alpha_1\alpha_3 \\ \alpha_1\alpha_2\alpha_3 & (\alpha_2+1)\alpha_2\alpha_3 & (\alpha_3+1)\alpha_2\alpha_3 \\ (\alpha_3+1)\alpha_1\alpha_3 & (\alpha_3+1)\alpha_2\alpha_3 & (\alpha_3+2)(\alpha_3+1)\alpha_3 \end{bmatrix}\mathbf{1}\right]^{-1}.$$

(2.39)

The empirical results of order-3 universal portfolios are presented in Table 2.5 for selected values of α_1, α_2 and α_3 for data set A, B and C. When the order is increased to 3, there is a slight increase in wealth from 1.6539 to 1.8782 units for set A by increasing the value of α_1 to 10000 and decreasing the value of α_3 to 1. There is no significant change in the best wealths of

4.2962 and 4.2960 units achieved for sets B and C respectively when the order is increased from 2 to 3 with an increase in computational memory.

Table 2.5: The best wealth S_{500} and the final portfolio (b_1, b_2, b_3) achieved by the Dirichlet$(\propto_1, \dots, \propto_m;\ 3)$ universal portfolio with selected values of $(\alpha_1, \alpha_2, \alpha_3)$.

Set	α1	α2	α3	sn	b1	b2	b3
A	10000	1	1	1.8782	0.9998	0.0001	0.0001
	1000	1	1	1.8774	0.9980	0.0010	0.0010
	100	1	1	1.8694	0.9804	0.0098	0.0098
	50	1	1	1.8609	0.9616	0.0192	0.0192
	10	1	1	1.8044	0.8341	0.0828	0.0831
	5	1	1	1.7529	0.7161	0.1415	0.1425
	3	1	1	1.7034	0.6028	0.1976	0.1995
	2	1	1	1.6598	0.5036	0.2468	0.2496
	1	1	10000	1.6394	0.0001	0.0001	0.9998
	1	1	1000	1.6393	0.0010	0.0010	0.9980
	1	1	100	1.6383	0.0098	0.0098	0.9804
	1	1	50	1.6372	0.0192	0.0192	0.9615
	1	1	10	1.6299	0.0836	0.0830	0.8334
	1	1	5	1.6224	0.1437	0.1419	0.7145
	1	1	3	1.6139	0.2015	0.1982	0.6003
	1	1	2	1.6048	0.2523	0.2473	0.5003
	1	1	1	1.5856	0.3373	0.3290	0.3337
	1	2	1	1.4771	0.2531	0.4960	0.2509
	1	3	1	1.4118	0.2023	0.5968	0.2009
	1	5	1	1.3380	0.1443	0.7122	0.1435
	1	10	1	1.2632	0.0839	0.8325	0.0836
	1	50	1	1.1860	0.0193	0.9615	0.0193
	1	100	1	1.1750	0.0098	0.9804	0.0098
	1	1000	1	1.1648	0.0010	0.9980	0.0010
A	1	10000	1	1.1638	0.0001	0.9998	0.0001
B	1	1	10000	4.2962	0.0001	0.0001	0.9998
	1	1	1000	4.2887	0.0010	0.0010	0.9980
	1	1	100	4.2161	0.0098	0.0098	0.9804
	1	1	50	4.1398	0.0192	0.0192	0.9616
	1	1	10	3.6570	0.0832	0.0829	0.8340
	1	1	5	3.2574	0.1425	0.1417	0.7158
	1	1	3	2.9111	0.1995	0.1980	0.6024
	1	1	2	2.6344	0.2496	0.2474	0.5031
	1	1	1	2.2210	0.3335	0.3298	0.3368
	1	2	1	2.0642	0.2506	0.4968	0.2526
	2	1	1	1.9762	0.5001	0.2478	0.2520
	1	3	1	1.9680	0.2006	0.5974	0.2020

	1	5	1	1.8583	0.1433	0.7126	0.1440
	3	1	1	1.8396	0.6001	0.1985	0.2013
	1	10	1	1.7466	0.0836	0.8326	0.0838
	5	1	1	1.6932	0.7144	0.1421	0.1436
	1	50	1	1.6317	0.0192	0.9615	0.0193
	1	100	1	1.6154	0.0098	0.9804	0.0098
	1	1000	1	1.6004	0.0010	0.9980	0.0010
	1	10000	1	1.5989	0.0001	0.9998	0.0001
	10	1	1	1.5520	0.8334	0.0831	0.0836
	50	1	1	1.4129	0.9615	0.0192	0.0192
	100	1	1	1.3936	0.9804	0.0098	0.0098
	1000	1	1	1.3758	0.9980	0.0010	0.0010
B	10000	1	1	1.3740	0.9998	0.0001	0.0001
C	1	1	10000	4.2960	0.0001	0.0001	0.9998
	1	1	1000	4.2865	0.0010	0.0010	0.9980
	1	1	100	4.1952	0.0098	0.0098	0.9804
	1	1	50	4.0997	0.0192	0.0192	0.9616
	1	1	10	3.5057	0.0831	0.0832	0.8337
	1	1	5	3.0300	0.1423	0.1426	0.7151
	1	1	3	2.6312	0.1990	0.1997	0.6013
	1	1	2	2.3223	0.2487	0.2497	0.5016
	1	1	1	1.8790	0.3316	0.3333	0.3351
	2	1	1	1.7453	0.4984	0.2502	0.2513
	3	1	1	1.6664	0.5987	0.2003	0.2010
	1	2	1	1.6084	0.2490	0.4999	0.2511
	5	1	1	1.5782	0.7135	0.1431	0.1435
	10	1	1	1.4896	0.8330	0.0834	0.0836
	1	3	1	1.4607	0.1993	0.5999	0.2007
	50	1	1	1.3989	0.9615	0.0192	0.0192
	100	1	1	1.3860	0.9804	0.0098	0.0098
	1000	1	1	1.3742	0.9980	0.0010	0.0010
	10000	1	1	1.3730	0.9998	0.0001	0.0001
	1	5	1	1.3054	0.1425	0.7143	0.1433
	1	10	1	1.1591	0.0832	0.8333	0.0835
	1	50	1	1.0187	0.0192	0.9615	0.0192
	1	100	1	0.9995	0.0098	0.9804	0.0098
	1	1000	1	0.9819	0.0010	0.9980	0.0010
C	1	10000	1	0.9801	0.0001	0.9998	0.0001

2.4.4 Comparison of moving-order Cover-Ordentlich universal portfolio and the finite-order Dirichlet universal portfolios

It is observed that the finite-order Dirichlet universal portfolios perform well if heavier weights are placed on the parameter α_3. The performance of the finite-order Dirichlet universal portfolios is compared with that of the moving-order Cover-Ordentlich (1996) universal portfolio in Table 2.6. The moving-order universal portfolio achieves a slightly higher wealth of 1.8750 units compared with the finite-order wealths of 1.6394 - 1.6539 units for data set A. On the contrary, all the finite-order universal portfolios perform better than the moving-order universal portfolio for data sets B and C by achieving wealths in the range of 4.2764 - 4.2970 units compared with the moving-order wealths in the range of 4.0594 - 4.0761 units. A lot of implementation time and memory is saved by using the finite-order universal portfolios and at the same time performance is not sacrificed.

Table 2.6: Comparison of the wealths S_{500} achieved by the finite-order Dirichlet$(\propto_1, \propto_2, \propto_3;\ \nu)$ universal portfolios for with that of the moving-order $(\nu = n)$ Dirichlet$(\propto_1, \propto_2, \propto_3)$ universal portfolios for data sets A, B and C and selected values of $\propto_1$, $\propto_2$, $\propto_3$.

Data Set	α1	α2	α3	order n	order 1	order 2	order 3
A	0.01	0.01	5	1.874029	1.653751	1.653758	1.639339
	0.01	0.01	10	1.874523	1.653814	1.653816	1.639356
	0.01	0.01	50	1.874910	1.653912	1.653912	1.639414
	0.01	0.01	100	1.874958	1.653927	1.653927	1.639424
	0.01	0.01	1000	1.875001	1.653942	1.653942	1.639434

A	0.01	0.01	8000	1.875005	1.653944	1.653944	1.639435
B	0.01	0.01	5	4.062740	4.280718	4.280757	4.281026
	0.01	0.01	10	4.069261	4.288727	4.288738	4.288871
	0.01	0.01	50	4.074736	4.295338	4.295338	4.295364
	0.01	0.01	100	4.075440	4.296179	4.296179	4.296192
	0.01	0.01	1000	4.076078	4.296939	4.296939	4.296940
B	0.01	0.01	8000	4.076140	4.297013	4.297013	4.297013
C	0.01	0.01	5	4.059422	4.276393	4.276400	4.276642
	0.01	0.01	10	4.067466	4.286567	4.286568	4.286684
	0.01	0.01	50	4.074349	4.294906	4.294906	4.294928
	0.01	0.01	100	4.075244	4.295963	4.295963	4.295974
	0.01	0.01	1000	4.076058	4.296918	4.296918	4.296919
C	0.01	0.01	8000	4.076137	4.297010	4.297010	4.297011

The maximum wealths achieved by the finite-order Dirichlet universal portfolios and the BCRP wealths have been compared in Table 2.7 for data sets A, B and C. Order-3 universal portfolio achieve a wealth of 1.8782 which outperform the BCRP wealth 1.8534 in set A. The maximum wealths achieved by set B and C are very close to the wealths achieved by BCRP with a similar situation for order-2 and order-3 portfolios.

Table 2.7: Comparison of the maximum wealths S_{500} achieved by the BCRP and the finite-order Dirichlet $(\propto_1, \propto_2, \propto_3;\ \nu)$ universal portfolios for data sets A, B and C and selected values of $\propto_1$, $\propto_2$, $\propto_3$.

Set	BCRP	order 1	order 2	order 3
A	1.8534	1.8533	1.8533	1.8782
B	4.297	4.2962	4.29701	4.29701
C	4.297	4.296	4.29701	4.29701

In conclusion, the time and memory saving finite-order Dirichlet universal portfolio can outperform the BCRP for certain parametric values of $\propto_1, \propto_2, \propto_3$. The results of this chapter are reported in Tan, Chu and Pan (2014).

CHAPTER THREE

COMPLEMENTARY PAIRS OF ADDITIVE UNIVERSAL PORTFOLIOS

A universal portfolio does not depend on the stochastic model of the stock prices in an investment setting. Some recent studies on universal portfolios are Cover (1991), Cover and Ordentlich (1996), Hembold et al. (1998) and Tan and Lim (2011d). In Tan and Lim (2011c), the performance of some 3-stock universal portfolios generated by the quadratic divergence with respect to two special symmetric, positive definite matrices is studied. In this companion study, the performances of the complementary pairs of universal portfolios generated by the inverses of the matrices in Tan and Lim (2011c) are compared. Some sample graphs of the wealth achieved in our data sets will be plotted against the parameter for comparison of the performance of complementary pairs of universal portfolios.

3.1 The Mahalanobis Universal Portfolio

The Mahalanobis squared divergence with respect to a symmetric, positive definite matrix $A = (a_{ij})$ is defined in (1.17). The Mahalanobis universal portfolio is given by:

$$\boldsymbol{b}_{n+1} = \boldsymbol{b}_n + \frac{\xi}{(\boldsymbol{b}_n^t \boldsymbol{x}_n)}\left[A^{-1}\boldsymbol{x}_n - \left(\frac{\mathbf{1}^t A^{-1}\boldsymbol{x}_n}{\mathbf{1}^t A^{-1}\mathbf{1}}\right)A^{-1}\mathbf{1}\right], \tag{3.1}$$

where b_1 is given, ξ is any real number such that $\boldsymbol{b}_{n+1} \geq 0$ for $n = 1,2,3,\ldots$ The generating matrix A is assumed to be symmetric and positive definite.

Define the following objective function:

$$F(\boldsymbol{b}_{n+1}) = 2\xi\, log(\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n) - D(\boldsymbol{b}_{n+1}||\boldsymbol{b}_n)\,, \tag{3.2}$$

where

$$D_A(\boldsymbol{b}_{n+1}||\boldsymbol{b}_n) = (\boldsymbol{b}_{n+1} - \boldsymbol{b}_n)^t A(\boldsymbol{b}_{n+1} - \boldsymbol{b}_n)$$

is the quadratic or Mahalanobis squared divergence of $\boldsymbol{b}_{n+1}$ and $\boldsymbol{b}_n$ with respect to a symmetric, positive definite matrix A. The universal portfolio $\boldsymbol{b}_{n+1}(\xi)$ is said to generated by the quadratic divergence with respect to A if $\boldsymbol{b}_{n+1}(\xi)$ maximizes $F(\boldsymbol{b}_{n+1})$ and $\boldsymbol{b}_{n+1}(-\xi)$ maximizes $G(\boldsymbol{b}_{n+1})$for $\xi > 0$.

Proposition 1. The portfolio (3.1) maximizes the approximate of $F(\boldsymbol{b}_{n+1})$ given by $\hat{F}(\boldsymbol{b}_{n+1})$ in (3.3). We have the following results from Tan and Lim (2011). Let γ be the lagrange multiplier for the constraint $\sum_{i=1}^{m} b_{n+1,i} = 1$.

Consider maximizing the objective function

$$\hat{F}(\boldsymbol{b}_{n+1}, \gamma) = 2\xi \left[\log(\boldsymbol{b}_n^t \boldsymbol{x}_n) + \frac{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t x_n} - 1\right]$$

$$-\sum_{i=1}^{m} a_{ii}\left[b_{n+1,i} - b_{ni}\right]^2$$

$$-2\sum_{i<j} a_{ij}\left[b_{n+1,i} - b_{ni}\right]\left[b_{n+1,j} - b_{nj}\right] \qquad (3.3)$$

$$+\gamma\left(\sum_{i=1}^{m} b_{n+1,i} - 1\right).$$

.

Differentiating (3.3), we have

$$\frac{\partial \hat{F}(\boldsymbol{b}_{n+1}, \gamma)}{\partial \boldsymbol{b}_{n+1,i}} = 2\xi \frac{\boldsymbol{x}_n}{\boldsymbol{b}_n^t x_n} - 2a_{ii}\left[b_{n+1,i} - b_{ni}\right] - \sum_{k=1}^{m-i} 2a_{i,i+k}\left[b_{n+1,i+k} - b_{n,i+k}\right]$$

$$-\sum_{k=1}^{i-1} 2a_{i-k,i}\left[b_{n+1,i-k} - b_{n,i-k}\right] + \gamma$$

$$= 2\xi \frac{\boldsymbol{x}_n}{\boldsymbol{b}_n^t x_n} - 2\sum_{k=1}^{m} a_{ij}\left[b_{n+1,j} - b_{nj}\right] + \gamma.$$

Setting $\frac{\partial \hat{F}(\boldsymbol{b}_{n+1},\gamma)}{\partial \boldsymbol{b}_{n+1,i}} = 0 \; for \; i = 1,2, \ldots, m$, we obtain

$$2\xi \frac{\boldsymbol{x}_n}{\boldsymbol{b}_n^t x_n} + \gamma \mathbf{1} = 2A[\boldsymbol{b}_{n+1} - \boldsymbol{b}_n].$$

Hence,

$$\boldsymbol{b}_{n+1} = \boldsymbol{b}_n + \frac{\xi}{(\boldsymbol{b}_n^t \boldsymbol{x}_n)} A^{-1} \boldsymbol{x}_n + \frac{\gamma}{2} A^{-1} \mathbf{1}. \qquad (3.4)$$

To evaluate γ, pre-multiply (3.4) by $\mathbf{1}^t$ to obtain

$$1 = 1 + \frac{\xi}{(\boldsymbol{b}_n^t \boldsymbol{x}_n)} \mathbf{1}^t A^{-1} \boldsymbol{x}_n + \frac{\gamma}{2} \mathbf{1}^t A^{-1} \mathbf{1}.$$

Then substituting

$$\frac{\gamma}{2} = \frac{-\xi}{(\boldsymbol{b}_n^t \boldsymbol{x}_n)} \frac{\mathbf{1}^t A^{-1} \boldsymbol{x}_n}{\mathbf{1}^t A^{-1} \mathbf{1}}$$

into (3.4), we obtain the portfolio (3.1).

3.2 The complementary pairs of Mahalanobis universal portfolio generated by special symmetric matrices

Consider the following two special symmetric, positive definite matrices $A_1(r)$ and $A_2(r,t)$ given by:

$$A_1(r) = \frac{1}{(1-r^2)} \begin{bmatrix} 1 & -r & 0 \\ -r & 1+r^2 & -r \\ 0 & -r & 1 \end{bmatrix}, \qquad (3.5)$$

$where\ 0 < r < 1$

$$A_2(r,t) = \frac{1}{r(r^2-t^2)} \begin{bmatrix} r^2 & 0 & -rt \\ 0 & r^2-t^2 & 0 \\ -rt & 0 & r^2 \end{bmatrix}, \qquad (3.6)$$

$where\ r > t > 0$

Consider universal portfolios generated by the quadratic divergence with respect to $A_1(r)$ and $A_2(r,t)$ respectively. The $A_1(r)$ and $A_2(r,t)$ universal portfolios are studied in Tan and Lim (2011c). In this companion study, we study the performances of the $A_1^{-1}(r)$ and $A_2^{-1}(r,t)$ universal portfolios where the inverses of $A_1(r)$and $A_2(r,t)$are given by:

$$A_1^{-1}(r) = \begin{bmatrix} 1 & r & r^2 \\ r & 1 & r \\ r^2 & r & 1 \end{bmatrix}, \tag{3.7}$$

$$A_2^{-1}(r,t) = \begin{bmatrix} r & 0 & t \\ 0 & r & 0 \\ t & 0 & r \end{bmatrix}. \tag{3.8}$$

We shall simplify the formula (3.1) for m=3 stocks. Let

$$A^{-1} = E = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix},$$

where E is symmetric. Then

$$A^{-1}\boldsymbol{x}_n = \begin{bmatrix} e_{11}x_{n1} & e_{12}x_{n2} & e_{13}x_{n3} \\ e_{21}x_{n1} & e_{22}x_{n2} & e_{23}x_{n3} \\ e_{31}x_{n1} & e_{32}x_{n2} & e_{33}x_{n3} \end{bmatrix},$$

$$\mathbf{1}^t A^{-1}\boldsymbol{x}_n = (e_{11} + e_{21} + e_{31})x_{n1} + (e_{12} + e_{22} + e_{32})x_{n2}$$
$$+ (e_{13} + e_{23} + e_{33})x_{n3}$$

and

$$\mathbf{1}^t A^{-1}\mathbf{1} = \sum_i \sum_j e_{ij}.$$

Now

$$\left(\frac{\mathbf{1}^t A^{-1}\boldsymbol{x}_n}{\mathbf{1}^t A^{-1}\mathbf{1}}\right)A^{-1}\mathbf{1} = \left[\left(\frac{\sum_{\boldsymbol{i}} e_{i1}}{\sum_{\boldsymbol{i,j}} e_{ij}}\right)x_{n1} + \left(\frac{\sum_{\boldsymbol{i}} e_{i2}}{\sum_{\boldsymbol{i,j}} e_{ij}}\right)x_{n2} + \left(\frac{\sum_{\boldsymbol{i}} e_{i3}}{\sum_{\boldsymbol{i,j}} e_{ij}}\right)x_{n3}\right]A^{-1}\mathbf{1}$$

$$= \left[\left(\frac{\sum_{\boldsymbol{i}} e_{i1}}{\sum_{\boldsymbol{i,j}} e_{ij}}\right)x_{n1} + \left(\frac{\sum_{\boldsymbol{i}} e_{i2}}{\sum_{\boldsymbol{i,j}} e_{ij}}\right)x_{n2} + \left(\frac{\sum_{\boldsymbol{i}} e_{i3}}{\sum_{\boldsymbol{i,j}} e_{ij}}\right)x_{n3}\right]\begin{bmatrix}\sum_{\boldsymbol{j}} e_{1j} \\ \sum_{\boldsymbol{j}} e_{2j} \\ \sum_{\boldsymbol{j}} e_{3j}\end{bmatrix}$$

and hence

$$\left[A^{-1}\boldsymbol{x}_n - \left(\frac{\mathbf{1}^t A^{-1}\boldsymbol{x}_n}{\mathbf{1}^t A^{-1}\mathbf{1}}\right)A^{-1}\mathbf{1}\right]$$

$$= \left\{\left[e_{11} - \frac{(\sum_{\boldsymbol{i}} e_{i1})(\sum_{\boldsymbol{j}} e_{1j})}{\sum_{\boldsymbol{i,j}} e_{ij}}\right]x_{n1} + \left[e_{12} - \frac{(\sum_{\boldsymbol{i}} e_{i2})(\sum_{\boldsymbol{j}} e_{1j})}{\sum_{\boldsymbol{i,j}} e_{ij}}\right]x_{n2}\right.$$

$$+ \left[e_{13} - \frac{(\sum_{\boldsymbol{i}} e_{i3})(\sum_{\boldsymbol{j}} e_{1j})}{\sum_{\boldsymbol{i,j}} e_{ij}}\right]x_{n3}\,,$$

$$\left[e_{21} - \frac{(\sum_{\boldsymbol{i}} e_{i1})(\sum_{\boldsymbol{j}} e_{2j})}{\sum_{\boldsymbol{i,j}} e_{ij}}\right]x_{n1} + \left[e_{22} - \frac{(\sum_{\boldsymbol{i}} e_{i2})(\sum_{\boldsymbol{j}} e_{2j})}{\sum_{\boldsymbol{i,j}} e_{ij}}\right]x_{n2}$$

$$+ \left[e_{23} - \frac{(\sum_{\boldsymbol{i}} e_{i3})(\sum_{\boldsymbol{j}} e_{2j})}{\sum_{\boldsymbol{i,j}} e_{ij}}\right]x_{n3}\,,$$

$$\left[e_{31} - \frac{(\sum_{\boldsymbol{i}} e_{i1})(\sum_{\boldsymbol{j}} e_{3j})}{\sum_{\boldsymbol{i,j}} e_{ij}}\right]x_{n1} + \left[e_{32} - \frac{(\sum_{\boldsymbol{i}} e_{i2})(\sum_{\boldsymbol{j}} e_{3j})}{\sum_{\boldsymbol{i,j}} e_{ij}}\right]x_{n2}$$

$$\left. + \left[e_{33} - \frac{(\sum_{\boldsymbol{i}} e_{i3})(\sum_{\boldsymbol{j}} e_{3j})}{\sum_{\boldsymbol{i,j}} e_{ij}}\right]x_{n3}\right\}^T.$$

Consider

$$(A_1^{-1}(r))^{-1} = E = \frac{1}{(1-r^2)}\begin{bmatrix} 1 & -r & 0 \\ -r & 1+r^2 & -r \\ 0 & -r & 1 \end{bmatrix} \; where \; 0 < r < 1$$

and

$$\left(\sum_i e_{i1}\right) = \left(\sum_j e_{1j}\right) = \left(\frac{1-r}{1-r^2}\right) = \frac{1}{1+r} \;\; ,$$

$$\left(\sum_i e_{i2}\right) = \left(\sum_j e_{2j}\right) = \left(\frac{1-2r+r^2}{1-r^2}\right) = \frac{(1-r)^2}{(1+r)(1-r)} = \frac{1-r}{1+r} \; ,$$

$$\left(\sum_i e_{i3}\right) = \left(\sum_j e_{3j}\right) = \frac{1-r}{1-r^2} = \frac{1}{1+r} \;\; ,$$

and

$$\sum_i \sum_j e_{ij} = \frac{2}{1+r} + \frac{1-r}{1+r} = \frac{3-r}{1+r} \;\; .$$

For $A^{-1}(r)$ given by (3.7), the first element of $\left[A^{-1}\boldsymbol{x}_n + \left(\frac{\boldsymbol{1}^t A^{-1}\boldsymbol{x}_n}{\boldsymbol{1}^t A^{-1}\boldsymbol{1}}\right)A^{-1}\boldsymbol{1}\right]$ is

$$\left[\frac{1}{(1-r^2)} - \frac{\left(\frac{1}{1+r}\right)\left(\frac{1}{1+r}\right)}{\frac{3-r}{1+r}}\right]x_{n1} + \left[\frac{-r}{(1-r^2)} - \frac{\left(\frac{1-r}{1+r}\right)\left(\frac{1}{1+r}\right)}{\frac{3-r}{1+r}}\right]x_{n2}$$

$$+ \left[\frac{-r}{(1-r^2)} - \frac{\left(\frac{1}{1+r}\right)\left(\frac{1}{1+r}\right)}{\frac{3-r}{1+r}}\right]x_{n3}$$

$$= \left[\frac{2}{(1-r^2)(3-r)}\right]x_{n1} - \left[\frac{1}{(3-r)(1-r)}\right]x_{n2}$$

$$- \left[\frac{1}{(3-r)(1+r)}\right]x_{n3}$$

$$= \left[\frac{2}{(1-r^2)(3-r)}\right]x_{n1} - \left[\frac{1}{(3-r)(1-r)}\right]x_{n2}$$

$$- \left[\frac{1}{(3-r)(1+r)}\right]x_{n3}.$$

Second element of $\left[A^{-1}\boldsymbol{x}_n + \left(\frac{\mathbf{1}^t A^{-1}\boldsymbol{x}_n}{\mathbf{1}^t A^{-1}\mathbf{1}}\right) A^{-1}\mathbf{1}\right]$ is

$$\left[\frac{-r}{1-r^2} - \frac{\left(\frac{1}{1+r}\right)\left(\frac{1-r}{1+r}\right)}{\left(\frac{3-r}{1+r}\right)}\right] x_{n1} + \left[\frac{1+r^2}{1-r^2} - \frac{\left(\frac{1-r}{1+r}\right)\left(\frac{1-r}{1+r}\right)}{\left(\frac{3-r}{1+r}\right)}\right] x_{n2}$$

$$+ \left[\frac{-r}{1-r^2} - \frac{\left(\frac{1}{1+r}\right)\left(\frac{1-r}{1+r}\right)}{\left(\frac{3-r}{1+r}\right)}\right] x_{n3}$$

$$= \left[\frac{-1}{(3-r)(1-r)}\right] x_{n1} + \left[\frac{2}{(3-r)(1-r)}\right] x_{n2}$$

$$+ \left[\frac{-1}{(3-r)(1-r)}\right] x_{n3} .$$

Third element of $\left[A^{-1}\boldsymbol{x}_n + \left(\frac{\mathbf{1}^t A^{-1}\boldsymbol{x}_n}{\mathbf{1}^t A^{-1}\mathbf{1}}\right) A^{-1}\mathbf{1}\right]$ is

$$\left[- \frac{\left(\frac{1}{1+r}\right)\left(\frac{1}{1+r}\right)}{\left(\frac{3-r}{1+r}\right)}\right] x_{n1} + \left[\frac{-r}{1-r^2} - \frac{\left(\frac{1}{1+r}\right)\left(\frac{1-r}{1+r}\right)}{\left(\frac{3-r}{1+r}\right)}\right] x_{n2}$$

$$+ \left[\frac{1}{1-r^2} - \frac{\left(\frac{1}{1+r}\right)\left(\frac{1}{1+r}\right)}{\left(\frac{3-r}{1+r}\right)}\right] x_{n3}$$

$$= \left[-\frac{1}{(3-r)(1+r)}\right] x_{n1} - \left[\frac{1}{(3-r)(1-r)}\right] x_{n2}$$

$$+ \left[\frac{2}{(1-r^2)(3-r)}\right] x_{n3} .$$

The $A_1^{-1}(r)$ universal portfolio is given by:

$$b_{n+1,1} = b_{n1} + \frac{\xi}{(\boldsymbol{b}_n^t \boldsymbol{x}_n)}[c_1' x_{n1} + c_2' x_{n2} + c_3' x_{n3}], \tag{3.9}$$

$$b_{n+1,2} = b_{n2} + \frac{\xi}{(\boldsymbol{b}_n^t \boldsymbol{x}_n)}[c_2' x_{n1} - 2c_2' x_{n2} + c_2' x_{n3}], \tag{3.10}$$

$$b_{n+1,3} = b_{n3} + \frac{\xi}{(\boldsymbol{b}_n^t \boldsymbol{x}_n)}[c_3' x_{n1} + c_2' x_{n2} + c_1' x_{n3}], \tag{3.11}$$

where

$$c_1' = \frac{2}{(1-r^2)(3-r)} = \frac{2}{(3-r-3r^2+r^3)}, \tag{3.12}$$

$$c_2' = \frac{-1}{(3-r)(1-r)} = \frac{-1}{(3-4r+r^2)}, \tag{3.13}$$

$$c_3' = -(c_1' + c_2') = \frac{-1}{(3-r)(1+r)} = \frac{-1}{(3+2r-r^2)}, \tag{3.14}$$

and $0 < r < 1$. The parameter ξ is valid provided $b_{n+1,i} \geq 0$ for all n=1, 2, 3, … and $i = 1, 2, 3$, given $\boldsymbol{b}_1$.

Consider

$$\left(A_1^{-1}(r)\right)^{-1} = E = A_2(r,t)$$

$$= \frac{1}{r(r^2-t^2)} \begin{bmatrix} r^2 & 0 & -rt \\ 0 & r^2-t^2 & 0 \\ -rt & 0 & r^2 \end{bmatrix} where\ r > t > 0\,.$$

Now

$$\left(\sum_i e_{i1}\right) = \left(\sum_j e_{1j}\right) = \frac{r^2 - rt}{r(r^2-t^2)} = \frac{1}{r+t} \quad ,$$

$$\left(\sum_i e_{i2}\right) = \left(\sum_j e_{2j}\right) = \frac{r^2 - t^2}{r(r^2 - t^2)} = \frac{1}{r} \quad ,$$

$$\left(\sum_i e_{i3}\right) = \left(\sum_j e_{3j}\right) = \frac{r^2 - rt}{r(r^2 - t^2)} = \frac{1}{r + t} \quad ,$$

$$and \sum_i \sum_j e_{ij} = \frac{1}{r} + \frac{2}{r+t} = \frac{3r+t}{r(r+t)} \ .$$

For $A^{-1}(r,t)$ given by (3.8), the first element of $\left[A^{-1}\boldsymbol{x}_n + \left(\frac{\mathbf{1}^t A^{-1}\boldsymbol{x}_n}{\mathbf{1}^t A^{-1}\mathbf{1}}\right) A^{-1}\mathbf{1}\right]$ is

$$= \left[\frac{r^2}{r(r^2-t^2)} - \frac{\left(\frac{1}{r+t}\right)^2}{\frac{3r+t}{r(r+t)}}\right] x_{n1} + \left[0 - \frac{\left(\frac{1}{r}\right)\left(\frac{1}{r+t}\right)}{\frac{3r+t}{r(r+t)}}\right] x_{n2}$$

$$+ \left[\frac{-rt}{r(r^2-t^2)} - \frac{\left(\frac{1}{r+t}\right)^2}{\frac{3r+t}{r(r+t)}}\right] x_{n3}$$

$$= \left[\frac{2r}{(r-t)(3r+t)}\right] x_{n1} - \left[\frac{1}{(3r+t)}\right] x_{n2}$$

$$+ \left[\frac{-(r+t)}{(r-t)(3r+t)}\right] x_{n3} \ .$$

Second element of $\left[A^{-1}\boldsymbol{x}_n + \left(\frac{\mathbf{1}^t A^{-1}\boldsymbol{x}_n}{\mathbf{1}^t A^{-1}\mathbf{1}}\right) A^{-1}\mathbf{1}\right]$ is

$$= \left[0 - \frac{\left(\frac{1}{r}\right)\left(\frac{1}{r+t}\right)}{\frac{3r+t}{r(r+t)}}\right] x_{n1} + \left[\frac{r^2-t^2}{r(r^2-t^2)} - \frac{\left(\frac{1}{r}\right)^2}{\frac{3r+t}{r(r+t)}}\right] x_{n2}$$

$$+ \left[0 - \frac{\left(\frac{1}{r}\right)\left(\frac{1}{r+t}\right)}{\frac{3r+t}{r(r+t)}}\right] x_{n3}$$

$$= \left[-\frac{1}{(3r+t)}\right] x_{n1} + \left[\frac{2}{3r+t}\right] x_{n2} - \left[\frac{1}{3r+t}\right] x_{n3}$$

Third element of $\left[A^{-1}\boldsymbol{x}_n + \left(\frac{\mathbf{1}^t A^{-1} x_n}{\mathbf{1}^t A^{-1}\mathbf{1}}\right) A^{-1}\mathbf{1}\right]$ is

$$= \left[\frac{-rt}{r(r^2-t^2)} - \frac{\left(\frac{1}{r+t}\right)^2}{\frac{3r+t}{r(r+t)}}\right] x_{n1} + \left[0 - \frac{\left(\frac{1}{r}\right)\left(\frac{1}{r+t}\right)}{\frac{3r+t}{r(r+t)}}\right] x_{n2}$$

$$+ \left[\frac{r^2}{r(r^2-t^2)} - \frac{\left(\frac{1}{r+t}\right)^2}{\frac{3r+t}{r(r+t)}}\right] x_{n3}$$

$$= \left[\frac{-(r+t)}{(r-t)(3r+t)}\right] x_{n1} - \left[\frac{1}{(3r+t)}\right] x_{n2}$$

$$+ \left[\frac{2r}{(r-t)(3r+t)}\right] x_{n3}$$

The $A_2^{-1}(r,t)$ universal portfolio is given by:

$$b_{n+1,1} = b_{n1} + \frac{\xi}{(\boldsymbol{b}_n^t \boldsymbol{x}_n)}[p' x_{n1} + q' x_{n2} + w' x_{n3}], \tag{3.15}$$

$$b_{n+1,2} = b_{n2} + \frac{\xi}{(\boldsymbol{b}_n^t \boldsymbol{x}_n)}[q' x_{n1} - 2q' x_{n2} + q' x_{n3}], \tag{3.16}$$

$$b_{n+1,3} = b_{n3} + \frac{\xi}{(\boldsymbol{b}_n^t \boldsymbol{x}_n)}[w' x_{n1} + q' x_{n2} + p' x_{n3}], \tag{3.17}$$

where

$$p' = \frac{2r}{(r-t)(3r+t)}, \tag{3.18}$$

$$q' = \frac{-1}{(3r+t)}, \tag{3.19}$$

$$w' = -(p'+q') = \frac{-(r+t)}{(r-t)(3r+t)}, \tag{3.20}$$

and $r > t > 0$. Again ξ is a valid parameter if $b_{n+1,i} \geq 0$ for all $n = 1, 2, 3, \ldots$, and $i = 1,2,3$ given $\boldsymbol{b}_1$. We say that $(A_1(r), A_1^{-1}(r))$ and $(A_2(r), A_2^{-1}(r))$ are two complementary pairs of universal portfolios.

3.2.1 Empirical Results

The three data sets A, B, C used in this study are the same as those of Tan and Lim (2011c) for ease of comparison. There are a total of 500 trading days in each data set covering the period of January 1, 2003 until November 30, 2004. Each set consists of three Malaysian companies listed in the Kuala Lumpur Stock Exchange. Set A consists of the stocks of Malayan Banking, Genting and Amway (M) Holdings. Set B consists of the stocks of Public Bank, Sunrise and YTL Corporation. The last set C consists of the stocks of Hong Leong Bank, RHB Capital and YTL Corporation.

We run the $\mathrm{A}_1^{-1}(\mathrm{r})$ and $\mathrm{A}_2^{-1}(\mathrm{r}, \mathrm{t})$ universal portfolios on the three data sets for the selected values of r and t. The wealth S_{500} after 500 trading for the best parameter ξ that maximizes $\mathrm{S}_{500}(\xi)$ are displayed in Tables 3.1, 3.2 and 3.3 for the three data sets A, B and C, together with the resulting portfolio $\mathbf{b}_{501}$. The corresponding tables for wealth S_{500} achieved for the best parameter ξ for the $\mathrm{A}_1(\mathrm{r})$ and $\mathrm{A}_2(\mathrm{r}, \mathrm{t})$ universal portfolios are given in Tan and Lim (2011c). To compare the performance of the complementary pair of universal portfolios $(A_1(r), A_1^{-1}(r))$ and $(A_2(r,t), A_2^{-1}(r,t))$, we select the best wealth for fixed values of r or (r,t) and compare the pair of best wealths. The pairs of the best wealths for Tables I and II are listed in Tables 3.4 and 3.5 respectively. The first component of each pair is taken from the tables in Tan and Lim (2011c).

In Table 3.4, the $A_1(r)$ universal portfolios perform better than the $A_1^{-1}(r)$ universal portfolios except for the special cases $\mathrm{r} = 0.1, 0.2, 0.3, 0.4, 0.5$ and data set B, the $A_1^{-1}(r)$ universal portfolios perform better than $\mathrm{A}_1(\mathrm{r})$ universal portfolios. In Table 3.5, the $A_1^{-1}(r)$ universal portfolios achieve higher wealths than the $\mathrm{A}_1(\mathrm{r})$ universal portfolio except for the special cases $(2b_{n1}, b_{n1})$ for data set B and $(2b_{n2}, b_{n2})$ for all data sets. It is evident that the use of the $A_1^{-1}(r)$ or $A_2^{-1}(r,t)$ universal portfolios may help to improve the investment returns.

Some sample graphs of the wealth $S_{500}(\xi)$ against ξ are plotted in the Figures 1-8 for the pairs $(A_2(r,t), A_2^{-1}(r,t))$ of universal portfolios and (r, t) = (0.5, 0.25), $(2\mathrm{b}_{\mathrm{n}1}, \mathrm{b}_{\mathrm{n}1})$,$(2\mathrm{b}_{\mathrm{n}2}, \mathrm{b}_{\mathrm{n}2})$ and $(2\mathrm{b}_{\mathrm{n}3}, \mathrm{b}_{\mathrm{n}3})$ for data set A in Table 3.2. There is a small range of variation between $S_{500}(\xi)$ for $\mathrm{A}_2(\mathrm{r},\mathrm{t})$ and $\mathrm{A}_2^{-1}(\mathrm{r},\mathrm{t})$.

Sets A, B and C achieve the highest wealth $S_{500}(A_2)$ which are 1.6885, 2.7053 and 2.4971 respectively for (r, t) equal to $(2b_{n_2}, b_{n_2})$. We compare these performances with the BCRP wealths which are 1.8534, 4.297 and 4.297 respectively for Set A, B and C. We notice that the wealth for set A is close to the BCRP wealth whereas the wealth of B and C are far from the BCRP wealth. Different types of generating matrices of the Mahalanobis universal portfolio will lead to different performance. The results of this chapter are reported in Tan and Chu (2012).

Table 3.1: Values of $S_{500}(\max)$ achieved within the range if ξ listed in columns 3 and 4, and the best ξ values (column 5) achieving $S_{500}(max)$ together with b_{501} where $b_1 = (0.3333, 0.3333, 0.3334)$ for the $A_1^{-1}(r)$ universal portfolios and nine selected values of r.

Set	r	Smallest ξ	Largest ξ	Best ξ	S_{500}	b1	b2	b3
A	0.1	-1.2589	0.8935	-1.2589	1.5816	0.0530	0.7522	0.1948
	0.2	-1.1110	0.7669	-0.9666	1.5766	0.0914	0.7046	0.2040
	0.3	-0.9603	0.6471	-0.6848	1.5730	0.1373	0.6412	0.2216
	0.4	-0.8096	0.5341	-0.4701	1.5704	0.1753	0.5865	0.2381
	0.5	-0.6608	0.4280	-0.3091	1.5686	0.2073	0.5391	0.2536
	0.6	-0.5145	0.3287	-0.1909	1.5673	0.2345	0.4973	0.2681
	0.7	-0.3744	0.2362	-0.1071	1.5664	0.2580	0.4603	0.2817
	0.8	-0.2414	0.1506	-0.0510	1.5657	0.2782	0.4275	0.2943
A	0.9	-0.1163	0.0719	-0.0168	1.5653	0.2960	0.3979	0.3061
B	0.1	-0.4393	0.7127	0.7127	2.6470	0.0385	0.1103	0.8512
	0.2	-0.4139	0.7079	0.7079	2.6564	0.0457	0.0766	0.8777
	0.3	-0.3808	0.6886	0.6886	2.6663	0.0539	0.0391	0.9070
	0.4	-0.3406	0.6480	0.6480	2.6722	0.0654	0.0000	0.9346
	0.5	-0.2942	0.5186	0.5186	2.6132	0.1096	0.0000	0.8904
	0.6	-0.2425	0.3978	0.3978	2.5624	0.1484	0.0000	0.8516
	0.7	-0.1863	0.2856	0.2856	2.5182	0.1826	0.0001	0.8173
	0.8	-0.1265	0.1820	0.1820	2.4795	0.2131	0.0000	0.7869
B	0.9	-0.0641	0.0867	0.0867	2.4450	0.2405	0.0004	0.7592
C	0.1	-0.3452	0.4570	0.4570	2.3504	0.2304	0.0171	0.7524
	0.2	-0.3189	0.3922	0.3922	2.3157	0.2601	0.0171	0.7228
	0.3	-0.2877	0.3309	0.3309	2.2866	0.2852	0.0172	0.6976
	0.4	-0.2524	0.2731	0.2731	2.2619	0.3068	0.0172	0.6760
	0.5	-0.2139	0.2188	0.2188	2.2405	0.3256	0.0172	0.6572
	0.6	-0.1729	0.1681	0.1681	2.2222	0.3421	0.0171	0.6408
	0.7	-0.1304	0.1208	0.1208	2.2059	0.3566	0.0172	0.6262
	0.8	-0.0869	0.0770	0.0770	2.1914	0.3695	0.0173	0.6132
C	0.9	-0.0432	0.0367	0.0367	2.1781	0.3810	0.0177	0.6012

Table 3.2: Values of $S_{500}(\max)$ achieved within the range if ξ listed in columns 4 and 5, and the best ξ values (column 6) achieving $S_{500}(max)$ together with b_{501} where $b_1 = (0.3333, 0.3333, 0.3334)$ for the $A_2^{-1}(r,t)$ universal portfolios and eleven pairs of (r,t) where $r = 2t$.

Set	r	t	Smallest ξ	Largest ξ	Best ξ	S_{500}	b1	b2	b3
A	0.1	0.05	-0.12	0.12	-0.12	1.63	0.05	0.62	0.32
	0.2	0.1	-0.24	0.24	-0.24	1.63	0.05	0.62	0.32
	0.3	0.15	-0.36	0.36	-0.36	1.63	0.05	0.62	0.32
	0.5	0.25	-0.60	0.60	-0.60	1.63	0.05	0.62	0.32
	1	0.5	-1.20	1.20	-1.20	1.63	0.05	0.62	0.32
	5	2.5	-6.02	5.99	-6.02	1.63	0.05	0.62	0.32
	10	5	-12.04	11.97	-12.04	1.63	0.05	0.62	0.32
	20	10	-24.08	23.95	-24.08	1.63	0.05	0.62	0.32
	$2b_{n1}$	b_{n1}	-0.60	0.96	-0.60	1.65	0.27	0.53	0.20
	$2b_{n2}$	b_{n2}	-1.13	0.80	-1.13	1.60	0.11	0.66	0.23
A	$2b_{n3}$	b_{n3}	-0.67	1.12	-0.44	1.59	0.12	0.46	0.41
B	0.1	0.05	-0.03	0.03	0.03	2.60	0.01	0.26	0.73
	0.2	0.1	-0.05	0.06	0.06	2.60	0.01	0.26	0.73
	0.3	0.15	-0.08	0.10	0.10	2.60	0.01	0.26	0.73
	0.5	0.25	-0.13	0.16	0.16	2.60	0.01	0.26	0.73
	1	0.5	-0.26	0.32	0.32	2.60	0.01	0.26	0.73
	5	2.5	-1.29	1.59	1.59	2.60	0.01	0.26	0.73
	10	5	-2.57	3.18	3.18	2.60	0.01	0.26	0.73
	20	10	-5.15	6.35	6.35	2.60	0.01	0.26	0.73
	$2b_{n1}$	b_{n1}	-0.24	0.12	0.12	2.54	0.10	0.06	0.84
	$2b_{n2}$	b_{n2}	-0.18	0.20	0.20	2.58	0.01	0.26	0.72
			-0.10	0.32	0.32	2.67	0.01	0.26	0.73
B	$2b_{n3}$	b_{n3}	-0.10	-0.10	-0.10	1.93	0.26	0.44	0.31
C	0.1	0.05	-0.02	0.04	0.04	2.43	0.01	0.14	0.84
	0.2	0.1	-0.05	0.07	0.07	2.43	0.01	0.14	0.85
	0.3	0.15	-0.07	0.11	0.11	2.43	0.01	0.14	0.85
	0.5	0.25	-0.11	0.19	0.19	2.43	0.01	0.14	0.85

	1	0.5	-0.23	0.37	0.37	2.43	0.01	0.14	0.85
	5	2.5	-1.14	1.85	1.85	2.43	0.01	0.14	0.85
	10	5	-2.28	3.71	3.71	2.43	0.01	0.14	0.85
	20	10	-4.57	7.42	7.42	2.43	0.01	0.14	0.85
	$2b_{n1}$	b_{n1}	-0.20	0.16	0.16	2.33	0.11	0.02	0.87
	$2b_{n2}$	b_{n2}	-0.18	0.17	0.17	2.30	0.02	0.16	0.82
			-0.11	-0.11	-0.11	1.63	0.24	0.51	0.24
C	$2b_{n3}$	b_{n3}	-0.11	0.41	0.41	2.52	0.01	0.16	0.83

Table 3.3: Values of $S_{500}(\max)$ achieved within the range if ξ listed in columns 4 and 5, and the best ξ values (column 6) achieving $S_{500}(max)$ together with b_{501} where $b_1 = (0.3333, 0.3333, 0.3334)$ for the $A_2^{-1}(r,t)$ universal portfolios and eleven pairs of (r,t) where $r = t + 1$.

Set	r	t	Smallest ξ	Largest ξ	Best ξ	S_{500}	b1	b2	b3
A	1.05	0.05	-1.47	1.10	-1.47	1.59	0.05	0.73	0.22
	1.1	0.1	-1.53	1.16	-1.53	1.59	0.05	0.72	0.22
	1.15	0.15	-1.60	1.23	-1.60	1.60	0.05	0.72	0.23
	1.25	0.25	-1.72	1.37	-1.72	1.60	0.05	0.70	0.24
	1.5	0.5	-1.98	1.71	-1.98	1.61	0.05	0.67	0.27
	3.5	2.5	-3.19	4.44	-3.19	1.67	0.05	0.54	0.41
	6	5	-3.67	4.64	-3.67	1.69	0.06	0.46	0.47
	11	10	-3.89	4.64	-3.89	1.70	0.08	0.41	0.51
	$b_{n1}+1$	b_{n1}	-1.76	1.53	-1.76	1.62	0.06	0.69	0.25
	$b_{n2}+1$	b_{n2}	-1.93	1.40	-1.93	1.60	0.05	0.69	0.26
A	$b_{n3}+1$	b_{n3}	-1.75	1.55	-1.75	1.61	0.07	0.67	0.26
B	1.05	0.05	-0.02	0.02	-0.02	2.38	0.00	0.62	0.38
	1.1	0.1	-0.03	0.04	-0.03	2.38	0.00	0.62	0.38
	1.15	0.15	-0.05	0.06	-0.05	2.38	0.00	0.62	0.38
	1.25	0.25	-0.08	0.10	-0.08	2.38	0.00	0.62	0.38
	1.5	0.5	-0.16	0.20	-0.16	2.38	0.00	0.62	0.38
	3.5	2.5	-0.82	1.02	-0.82	2.38	0.00	0.62	0.38

	6	5	-1.64	2.04	-1.64	2.38	0.00	0.62	0.38
	11	10	-3.28	4.07	-3.28	2.38	0.00	0.62	0.38
	$b_{n1}+1$	b_{n1}	-0.50	0.69	0.69	2.64	0.03	0.16	0.81
	$b_{n2}+1$	b_{n2}	-0.49	0.66	0.66	2.60	0.02	0.21	0.77
B	$b_{n3}+1$	b_{n3}	-0.47	0.66	0.66	2.61	0.01	0.23	0.76
C	1.05	0.05	-0.37	0.56	0.56	2.42	0.18	0.02	0.81
	1.1	0.1	-0.38	0.60	0.60	2.44	0.16	0.02	0.83
	1.15	0.15	-0.39	0.63	0.63	2.46	0.14	0.02	0.85
	1.25	0.25	-0.40	0.70	0.70	2.51	0.10	0.02	0.88
	1.5	0.5	-0.43	0.84	0.84	2.60	0.02	0.03	0.95
	3.5	2.5	-0.50	0.65	0.65	2.28	0.01	0.24	0.75
	6	5	-0.52	0.62	0.62	2.22	0.01	0.28	0.70
	11	10	-0.54	0.60	0.60	2.19	0.01	0.31	0.68
	$b_{n1}+1$	b_{n1}	-0.42	0.66	0.66	2.48	0.12	0.02	0.87
	$b_{n2}+1$	b_{n2}	-0.42	0.66	0.66	2.46	0.12	0.02	0.86
C	$b_{n3}+1$	b_{n3}	-0.39	0.81	0.81	2.56	0.02	0.06	0.92

Table 3.4: The pairs of best wealths $(\boldsymbol{S_{500}(A_1)}, \boldsymbol{S_{500}(A_1^{-1})})$ acheived by the complementary pairs $(\boldsymbol{A_1(r)}, \boldsymbol{A_1^{-1}(r)})$ of universal portfolios for Table 3.1

	r	$S_{500}(A_1)$	$S_{500}(A_1^{-1})$
A	0.1	1.594	1.5816
	0.2	1.6005	1.5766
	0.3	1.607	1.573
	0.4	1.6135	1.5704
	0.5	1.6199	1.5686
	0.6	1.6261	1.5673
	0.7	1.6321	1.5664
	0.8	1.6379	1.5657
A	0.9	1.6435	1.5653
B	0.1	2.6304	2.647
	0.2	2.6237	2.6564
	0.3	2.6181	2.6663

	0.4	2.612	2.6722
	0.5	2.6063	2.6132
	0.6	2.6015	2.5624
	0.7	2.5974	2.5182
	0.8	2.5939	2.4795
B	0.9	2.5909	2.445
C	0.1	2.4417	2.3504
	0.2	2.4964	2.3157
	0.3	2.5568	2.2866
	0.4	2.607	2.2619
	0.5	2.5366	2.2405
	0.6	2.4821	2.2222
	0.7	2.4391	2.2059
	0.8	2.4043	2.1914
C	0.9	2.3758	2.1781

Table 3.5: The pairs of best wealths $(S_{500}(A_2), S_{500}(A_2^{-1}))$ acheived by the complementary pairs $(A_2(r), A_2^{-1}(r))$ of universal portfolios for Table 3.2

Set	(r, t)	$S_{500}(A_2)$	$S_{500}(A_2^{-1})$
A	(0.1, 0.05)	1.5663	1.634
	(0.2, 0.1)	1.5663	1.634
	(0.3, 0.15)	1.5663	1.634
	(0.5, 0.25)	1.5663	1.634
	(1, 0.5)	1.5663	1.634
	(5, 2.5)	1.5663	1.634
	(10, 5)	1.5663	1.634
	(20, 10)	1.5663	1.634
	$(2b_{n1}, b_{n1})$	1.5652	1.6513
	$(2b_{n2}, b_{n2})$	1.6885	1.5972
A	$(2b_{n3}, b_{n3})$	1.5653	1.5923
B	(0.1, 0.05)	2.5124	2.5954

	(0.2, 0.1)	2.5124	2.5961
	(0.3, 0.15)	2.5124	2.5959
	(0.5, 0.25)	2.5124	2.5962
	(1, 0.5)	2.5124	2.5962
	(5, 2.5)	2.5123	2.5962
	(10, 5)	2.5121	2.5962
	(20, 10)	2.5118	2.5962
	$(2b_{n1}, b_{n1})$	2.5845	2.5426
	$(2b_{n2}, b_{n2})$	2.7053	2.5754
B	$(2b_{n3}, b_{n3})$	2.4134	2.6711
C	(0.1, 0.05)	2.2038	2.4254
	(0.2, 0.1)	2.2038	2.4263
	(0.3, 0.15)	2.2038	2.4266
	(0.5, 0.25)	2.2038	2.4268
	(1, 0.5)	2.2038	2.427
	(5, 2.5)	2.2034	2.4271
	(10, 5)	2.203	2.4271
	(20, 10)	2.203	2.4271
	$(2b_{n1}, b_{n1})$	2.1924	2.3314
	$(2b_{n2}, b_{n2})$	2.4972	2.2985
C	$(2b_{n3}, b_{n3})$	2.1519	2.5221

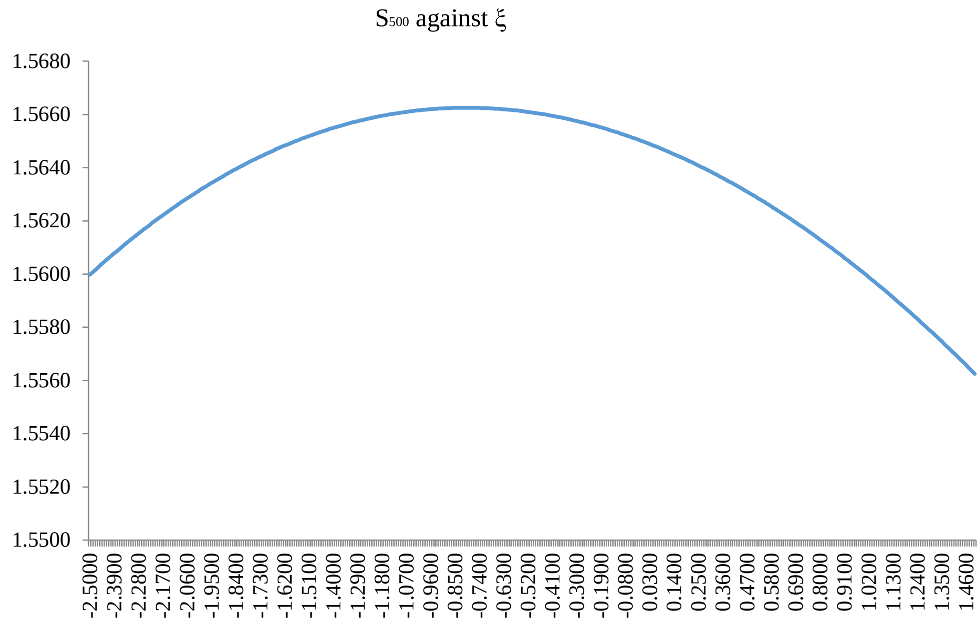

Figure 3.1: The graph of S_{500} against ξ for the $A_2(0.5, 0.25)$ universal portfolio and data set A in Table 3.2.

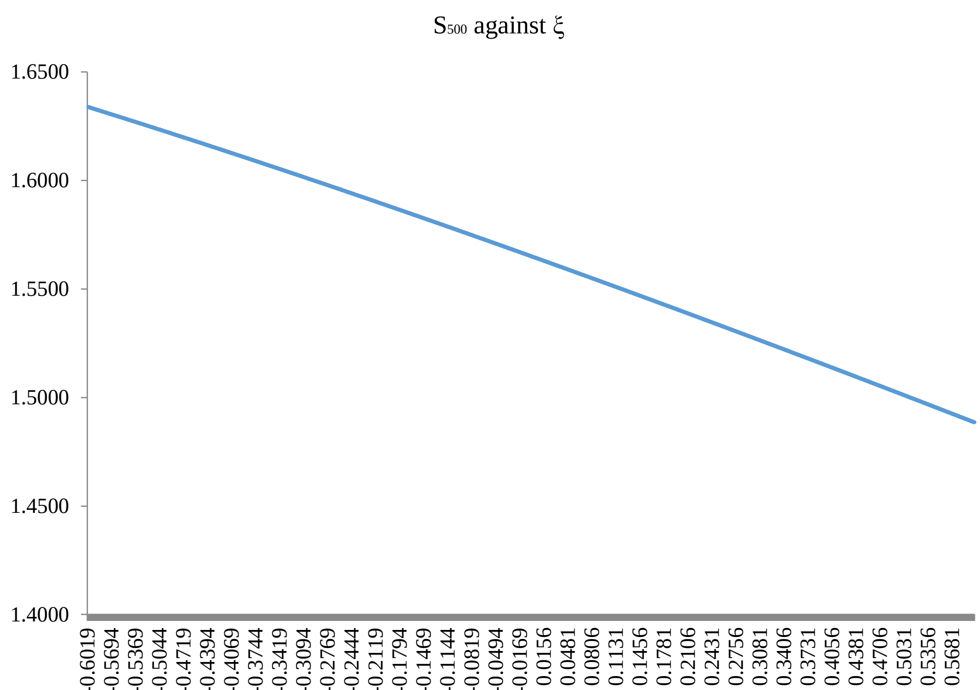

Figure 3.2: The graph of S_{500} against ξ for the $A_2^{-1}(0.5, 0.25)$ universal portfolio and data set A in Table 3.2.

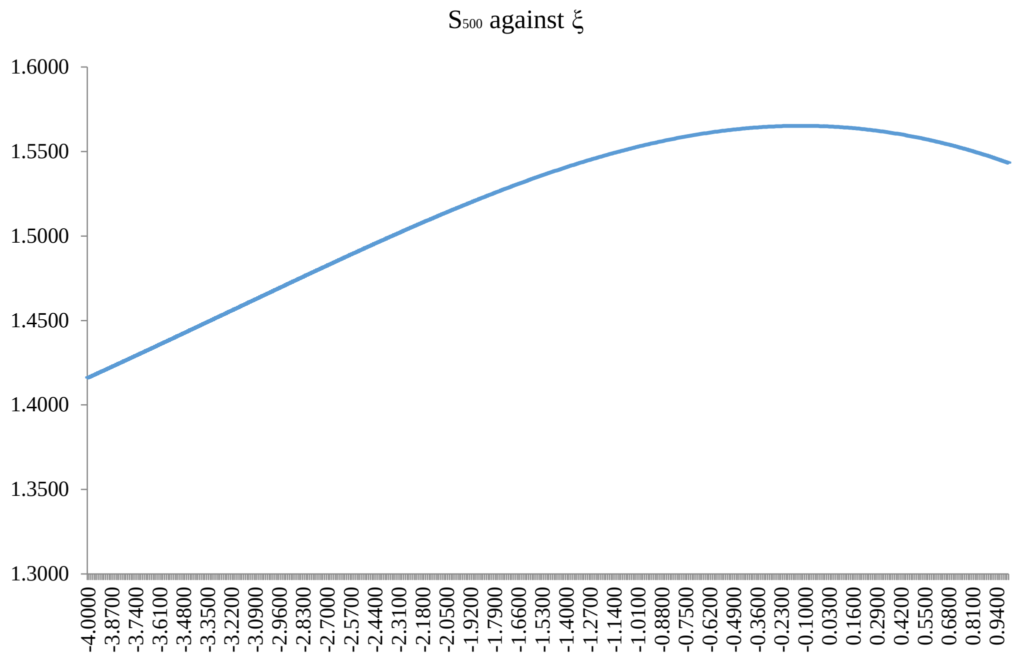

Figure 3.3: The graph of S_{500} against ξ for the $A_2(2b_{n1}, b_{n1})$ universal portfolio and data set A in Table 3.2.

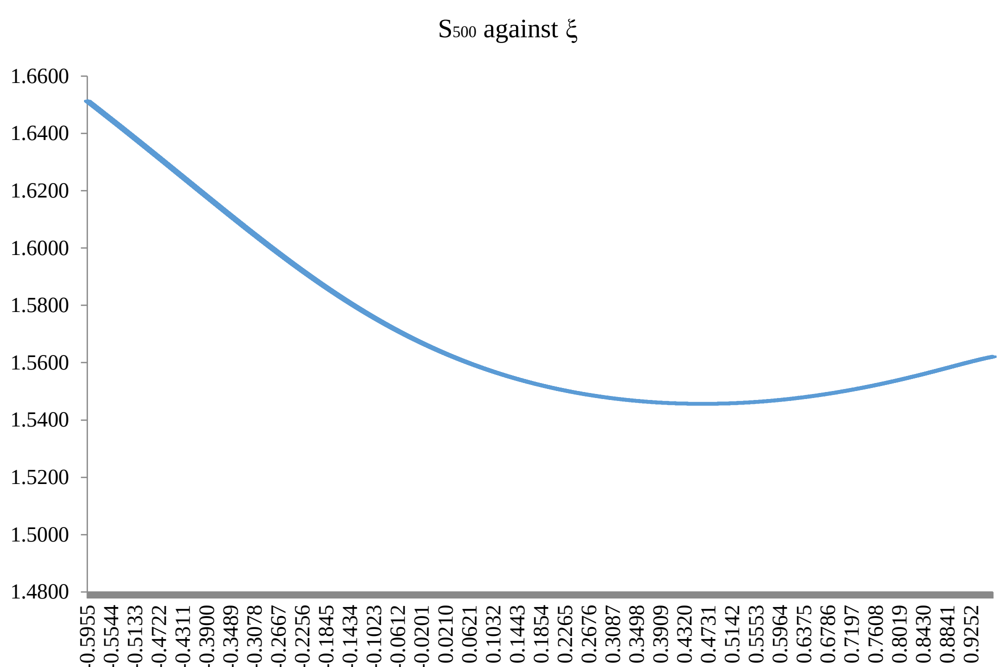

Figure 3.4: The graph of S_{500} against ξ for the $A_2^{-1}(2b_{n1}, b_{n1})$ universal portfolio and data set A in Table 3.2.

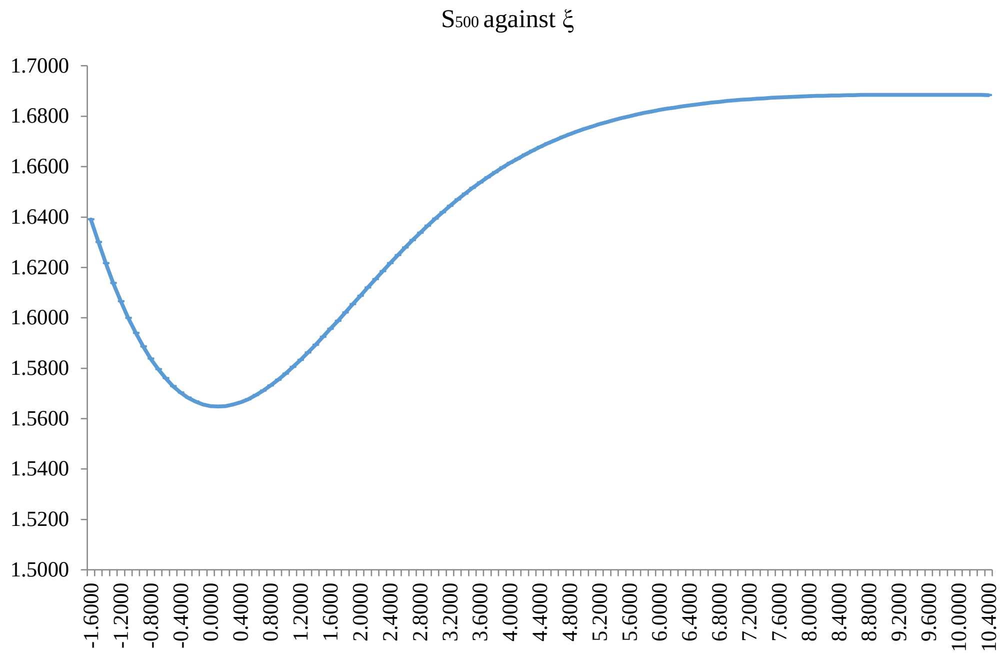

Figure 3.5: The graph of S_{500} against ξ for the $A_2(2b_{n2}, b_{n2})$ universal portfolio and data set A in Table 3.2

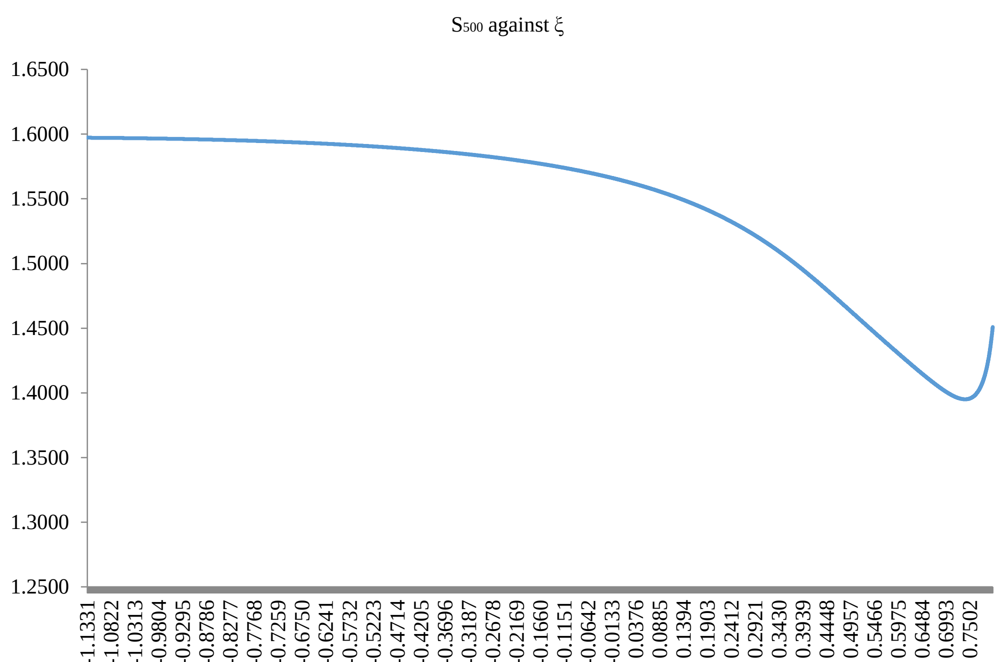

Figure 3.6: The graph of S_{500} against ξ for the $A_2^{-1}(2b_{n2}, b_{n2})$ universal portfolio and data set A in Table 3.2.

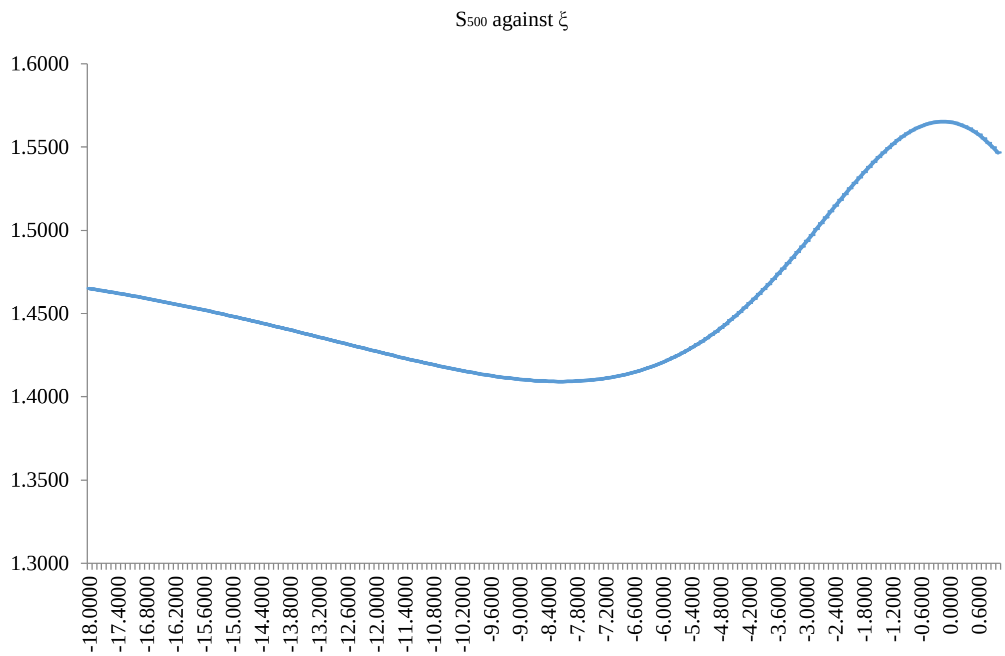

Figure 3.7: The graph of S_{500} against ξ for the $A_2(2b_{n3}, b_{n3})$ universal portfolio and data set A in Table 3.2.

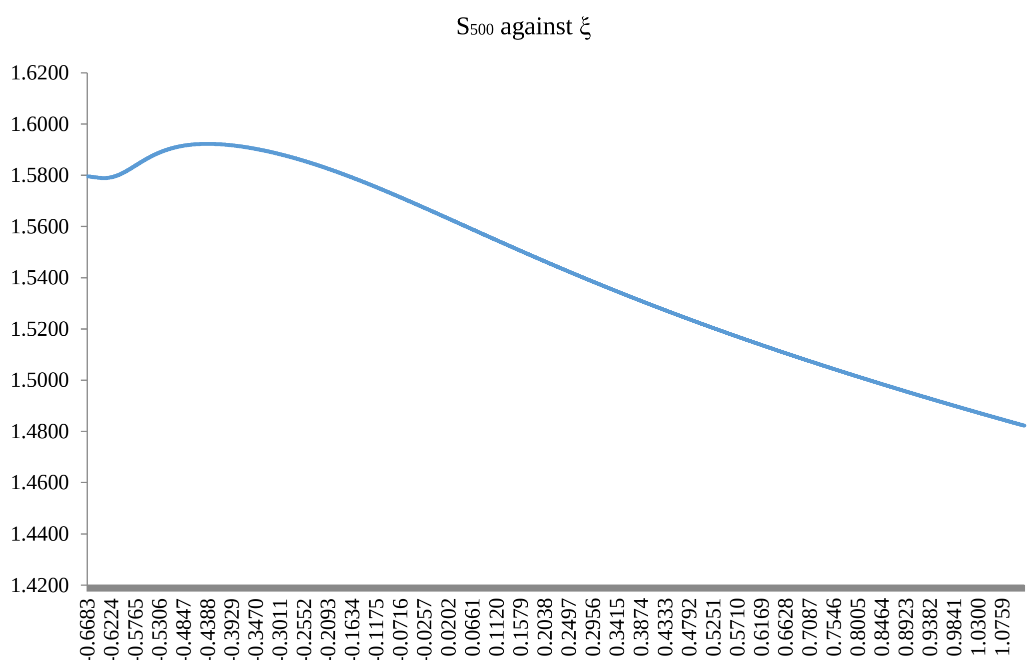

Figure 3.8: The graph of S_{500} against ξ for the $A_2^{-1}(2b_{n3}, b_{n3})$ universal portfolio and data set A in Table 3.2.

CHAPTER FOUR

THE TOEPLITZ UNIVERSAL PORTFOLIO

Multiplicative-update universal portfolios were introduced in Hembold et al. (1998). The generalization to additive-update universal portfolios is given by Tan and Lim (2011d). The symmetric, positive definite matrix generating the Mahalanobis universal portfolio has some nice property. The performance of the Mahalanobis universal portfolios generated by diagonal matrices is studied in Tan and Lim (2011c). A natural extension to Mahalanobis universal portfolios generated by Toeplitz matrices is studied in this chapter.

4.1 The Mahalanobis Universal Portfolio generated by Toeplitz matrices

Let $A = (a_{ij})$ be an $m \times m$ symmetric, positive definite matrix. The Mahalanobis universal portfolio $(A, \boldsymbol{b}_1, \xi)$ generated by A and $\boldsymbol{b}_1$ is the sequence of universal portfolios $\{\boldsymbol{b}_{n+1}\}$ given by

$$\boldsymbol{b}_{n+1} = \boldsymbol{b}_n + \frac{\xi}{(\boldsymbol{b}_n^t x_n)} C\boldsymbol{x}_n, \tag{4.1}$$

where $\boldsymbol{b}_1$ is given, ξ is any real number such that $\boldsymbol{b}_{n+1} \geq \boldsymbol{0}$ for $n = 0,1,2,\ldots;$ $C = (c_{ij})$ is the companion matrix of A defined as:

$$c_{ij} = a_{ij} - \frac{a_{i\cdot}a_{\cdot j}}{a_{\cdot\cdot}} \ for\ i, j = 1,2, \dots, m, \qquad (4.2)$$

where $a_{i\cdot} = \sum_{j=1}^{m} a_{ij}$, $a_{\cdot j} = \sum_{i=1}^{m} a_{ij}$ and $a_{\cdot\cdot} = \sum_{i,j=1}^{m} a_{ij}$ denote the ith row sum, jth column sum and total sum of entries of A respectively (see Tan and Lim (2011c)). Note that (4.1) is the simplified form of (3.1) where A^{-1} is replaced by A. The parametric family of universal portfolios generated by A and $\boldsymbol{b}_1$ is the sequence $\{\boldsymbol{b}_{n+1}\}$ given by (4.1) for any real number ξ such that $\boldsymbol{b}_{n+1} \geq \mathbf{0}$ for $n = 0,1,2, \dots$. It is clear that the parametric family of universal portfolios generated by A and $\boldsymbol{b}_1$ is the same as the parametric family of universal portfolios generated by cA and $\boldsymbol{b}_1$ for any positive constant c because the companion matrix of cA is cC. Sometimes it is more convenient to deal with the generating matrix cA by choosing an appropriate constant $c > 0$. If A is symmetric but not positive definite, the sequence $\{\boldsymbol{b}_{n+1}\}$ generated by (4.1) is said to be *pseudo-Mahalanobis*. If A is positive definite, the inverse A^{-1} is positive definite and the Mahalanobis squared divergence for any two portfolio vectors $\boldsymbol{b}_k$ and $\boldsymbol{b}_k$ is well-defined as follows:

$$d_{A^{-1}}(\boldsymbol{b}_k, \boldsymbol{b}_n) = [\boldsymbol{b}_k - \boldsymbol{b}_n]^t A^{-1} [\boldsymbol{b}_k - \boldsymbol{b}_n]. \qquad (4.3)$$

Rewriting (3.1) where A^{-1} replaced by A, we have:

$$\boldsymbol{b}_{n+1} = \boldsymbol{b}_n + \frac{\xi}{(\boldsymbol{b}_n^t \boldsymbol{x}_n)} \left[A\boldsymbol{x}_n - \left(\frac{\mathbf{1}^t A \boldsymbol{x}_n}{\mathbf{1}^t A \mathbf{1}} \right) A\mathbf{1} \right] \qquad (4.4)$$

where $\mathbf{1} = (1,1,\ldots,1)$. The computational formula for (4.4) is (4.1). If A is positive definite, the portfolio (4.1) or (4.4) maximizes the following objective function

$$F(\boldsymbol{b}_{n+1}) = 2\xi\left[\log(\boldsymbol{b}_n^t\boldsymbol{x}_n) + \frac{\boldsymbol{b}_{n+1}^t\boldsymbol{x}_n}{\boldsymbol{b}_n^t\boldsymbol{x}_n} - 1\right] - d_{A^{-1}}(\boldsymbol{b}_{n+1}, \boldsymbol{b}_n) \tag{4.5}$$

over all portfolio vectors $\boldsymbol{b}_{n+1}$ where ξ is fixed. Note that $\left[\log(\boldsymbol{b}_n^t\boldsymbol{x}_n) + \frac{\boldsymbol{b}_{n+1}^t\boldsymbol{x}_n}{\boldsymbol{b}_n^t\boldsymbol{x}_n} - 1\right]$ in (4.5) is a first-order approximation of $\log(\boldsymbol{b}_{n+1}^t\boldsymbol{x}_n)$.

4.2 Structure of the companion matrix of the Toeplitz universal portfolio

Let $A = (a_{ij})$ be an $m \times m$ symmetric matrix. The matrix A is said to be a *symmetric Toeplitz matrix* of bandwidth $2k+1$ for a positive integer k if $a_{ij} = a_{|i-j|}$ for $|i-j| \leq k$ and $a_{|i-j|} = 0$ for $|i-j| > k$. For simplicity, this matrix is referred to as a $(2k+1)$-band symmetric Toeplitz matrix.

Proposition. Let $A = (a_{ij})$ be a $(2k+1)$-band symmetric Toeplitz matrix with companion matrix C given by (4.2) where $a_{\cdot\cdot} > 0$. Then $\phi_{ij} = -(a_{i\cdot})(a_{\cdot j})\ for\ |i-j| > k$ and $\phi_{ij} = (a_{\cdot\cdot})(a_{|i-j|}) - (a_{i\cdot})(a_{\cdot j})$ for $|i-j| \leq k$ are the distinct entries of the companion matrix $a_{\cdot\cdot}C$ of $a_{\cdot\cdot}A$.

Proof. Since $a_{ij} = a_{|i-j|}\ for\ |i-j| \leq k$, the distinct entries of A are 0 and $a_{|i-j|}$ for $|i-j| \leq k$. From (4.2), $a_{..}c_{ij} = (a_{..})(a_{ij}) - (a_{i.})(a_{.j})$ is the (i,j) entry of the companion matrix $a_{..}C$ of $a_{..}A$ and hence $\phi_{ij} = (a_{..})(a_{|i-j|}) - (a_{i.})(a_{.j})\ for\ |i-j| \leq k$ is the (i,j) entry of $a_{..}C$. Similarly for $|i-j| > k, a_{ij} = 0$ and hence $\phi_{ij} = -(a_{i.})(a_{.j})$ is the (i,j) entry of $a_{..}C$. ■

4.3 Mahalanobis Universal Portfolio generated by 3-band m×m Teoplitz Matrix

For $k = 1$, the 3-band $m \times m$ symmetric Toeplitz matrix A can be written as:

$$A = a_0 \begin{bmatrix} 1 & r & 0 & 0 & & 0 & 0 & 0 & 0 \\ r & 1 & r & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & r & 1 & r & & 0 & 0 & 0 & 0 \\ & \vdots & & & \ddots & & & \vdots & \\ 0 & 0 & 0 & 0 & & 1 & r & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & r & 1 & r & 0 \\ 0 & 0 & 0 & 0 & & 0 & r & 1 & r \end{bmatrix} \tag{4.6}$$

where $a_0 > 0$ and $r = {}^{a_1}/_{a_0}$. It is well-known (see Graybill (1983)) that A is positive definite if and only if the m eigenvalues of A , $\lambda_l = a_0\left[1 + 2r\cos\left(\frac{l\pi}{m+1}\right)\right]$ for $l = 1,2,\ldots,m$ are positive. A sufficient condition for A to be positive definite is that $a_0 > 0$ and $|r| \leq \frac{1}{2}$. Note that the sum of all entries of A is $a_{..} = a_0[m + 2(m-1)r]$. Let C be the companion matrix of A. Then for $m \geq 5$, the companion matrix $[m+2(m-1)r]C$ of $[m+2(m-1)r]A$ is of the form $a_0[m+2(m-1)r]C'$ where

$$C' = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_3 & \cdots & \phi_3 & \phi_3 & \phi_3 & \phi_4 \\ \phi_2 & \phi_5 & \phi_6 & \phi_7 & \cdots & \phi_7 & \phi_7 & \phi_7 & \phi_3 \\ \phi_3 & \phi_6 & \phi_5 & \phi_6 & \cdots & \phi_7 & \phi_7 & \phi_7 & \phi_3 \\ \phi_3 & \phi_7 & \phi_6 & \phi_5 & \cdots & \phi_7 & \phi_7 & \phi_7 & \phi_3 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ \phi_3 & \phi_7 & \phi_7 & \phi_7 & \cdots & \phi_5 & \phi_6 & \phi_7 & \phi_3 \\ \phi_3 & \phi_7 & \phi_7 & \phi_7 & \cdots & \phi_6 & \phi_5 & \phi_6 & \phi_3 \\ \phi_3 & \phi_7 & \phi_7 & \phi_7 & \cdots & \phi_7 & \phi_6 & \phi_5 & \phi_2 \\ \phi_4 & \phi_3 & \phi_3 & \phi_3 & \cdots & \phi_3 & \phi_3 & \phi_2 & \phi_1 \end{bmatrix}, \tag{4.7}$$

and the 7 possible distinct entries are:

$$\begin{aligned} \phi_1 &= (m-1)+(2m-4)r-r^2, \\ \phi_2 &= -1+(m-3)r+(2m-4)r^2, \\ \phi_3 &= -(1+r)(1+2r), \\ \phi_4 &= -(1+r)^2, \\ \phi_5 &= (m-1)+2(m-3)r-4r^2, \\ \phi_6 &= -1+(m-4)r+(2m-6)r^2, \\ \phi_7 &= -(1+2r)^2, \end{aligned} \tag{4.8}$$

The parametric family of universal portfolios generated by $[m+2(m-r)]a_0^{-1}A$ with companion matrix C' is the same as the parametric family of universal portfolios generated by A with companion matrix C.

4.3.1 Running the Mahalanobis universal portfolio on 3 stocks data

For m=3, the companion matrix (4.7) becomes

$$C' = \begin{bmatrix} \phi_1 & \phi_2 & \phi_4 \\ \phi_2 & \phi_5 & \phi_2 \\ \phi_4 & \phi_2 & \phi_1 \end{bmatrix} \quad (4.9)$$

$$\begin{aligned} \phi_1 &= 2 + 2r - r^2 \\ \phi_2 &= -1 + 2r^2 \\ \phi_4 &= -(1 + r^2) \\ \phi_5 &= 2 - 4r^2 \end{aligned} \quad (4.10)$$

For A to be positive definite, the sufficient condition $-\frac{1}{2} \leq r \leq \frac{1}{2}$ is imposed.

We run the 3-band 3×3 Toeplitz universal portfolios for data sets A, B, C and selected values of r from -0.5 to 0.5 using $\boldsymbol{b_1} = (0.3333, 0.3333, 0.3333)$. The proportions to invest on day 501 and the maximum wealths achieved by the universal portfolios generated for different values of r are listed in Table 4.1. For m=3, the Toeplitz universal portfolios generated by $(3 + 4r)A$ with companion matrix $a_0[3 + 4r]C'$ where A and C' are given (4.6) and (4.9) respectively are run on the data sets A, B and C for selected values of the parameter r. The Toeplitz universal portfolios have the best wealths when r=0.5, r=-0.5 and r=0.3 respectively for sets A, B and C. Set B with $S_{500} = 2.6672$ performs slightly better than set A ($S_{500} = 1.6549$) and C ($S_{500} = 2.5789$).

Table 4.1: The best wealth S_{500} and the final portfolio (b_1, b_2, b_3) achieved by the 3-band 3×3 Toeplitz universal portfolios generated with $m = 3$, for data sets A, B, C and selected values of r, the valid ξ intervals and the best ξ.

Set	r	Largest ξ	Smallest ξ	best ξ	S_{500}	b_1	b_2	b_3
A	0.5	3.41	-2.86	-2.86	1.6549	0.0546	0.5728	0.3726
	0.4	2.31	-2.36	-2.36	1.6319	0.0542	0.6289	0.3169
	0.3	1.75	-2.01	-2.01	1.6157	0.0538	0.6685	0.2777
	0.2	1.41	-1.75	-1.75	1.6038	0.0538	0.6973	0.2489
	0.1	1.18	-1.55	-1.55	1.5946	0.0542	0.7187	0.2271
	0.0	1.02	-1.40	-1.40	1.5875	0.0532	0.7374	0.2094
	-0.1	0.90	-1.27	-1.27	1.5818	0.0548	0.7488	0.1965
	-0.2	0.81	-1.17	-1.07	1.5775	0.0805	0.7194	0.2001
	-0.3	0.75	-1.10	-0.87	1.5745	0.1152	0.6722	0.2126
	-0.4	0.70	-1.04	-0.74	1.5727	0.1393	0.6386	0.2222
A	-0.5	0.68	-1.02	-0.68	1.5720	0.1514	0.6210	0.2276
B	-0.5	0.76	-0.41	0.76	2.6672	0.0581	0.0271	0.9148
	-0.4	0.75	-0.41	0.75	2.6613	0.0576	0.0392	0.9033
	-0.3	0.74	-0.42	0.74	2.6564	0.0525	0.0602	0.8873
	-0.2	0.73	-0.43	0.73	2.6518	0.0456	0.0850	0.8694
	-0.1	0.71	-0.44	0.71	2.6401	0.0418	0.1145	0.8436
	0.0	0.70	-0.45	0.70	2.6353	0.0337	0.1417	0.8246
	0.1	0.68	-0.46	0.68	2.6231	0.0298	0.1716	0.7986
	0.2	0.67	-0.48	0.67	2.6179	0.0216	0.1989	0.7795
	0.3	0.65	-0.49	0.65	2.6053	0.0183	0.2277	0.7541
	0.4	0.63	-0.51	0.63	2.5923	0.0155	0.2554	0.7291
B	0.5	0.61	-0.53	0.61	2.5791	0.0132	0.2821	0.7047
C	0.3	0.83	-0.42	0.83	2.5789	0.0157	0.0403	0.9440
	0.2	0.72	-0.40	0.72	2.5230	0.0868	0.0185	0.8947
	0.1	0.60	-0.38	0.60	2.4391	0.1517	0.0210	0.8274
	0.4	0.74	-0.45	0.74	2.4313	0.0178	0.1345	0.8476
	0.0	0.52	-0.36	0.52	2.3862	0.1964	0.0203	0.7834
	-0.1	0.46	-0.34	0.46	2.3467	0.2298	0.0202	0.7500
	0.5	0.68	-0.48	0.68	2.3269	0.0135	0.2093	0.7773
	-0.2	0.41	-0.33	0.41	2.3112	0.2560	0.0236	0.7204
	-0.3	0.38	-0.32	0.38	2.2943	0.2743	0.0207	0.7050
	-0.4	0.36	-0.31	0.36	2.2839	0.2871	0.0176	0.6954
C	-0.5	0.35	-0.31	0.35	2.2777	0.2928	0.0173	0.6899

For $m = 5$, the companion matrix (4.7) becomes

$$C' = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_3 & \phi_4 \\ \phi_2 & \phi_5 & \phi_6 & \phi_7 & \phi_3 \\ \phi_3 & \phi_6 & \phi_5 & \phi_6 & \phi_3 \\ \phi_3 & \phi_7 & \phi_6 & \phi_5 & \phi_2 \\ \phi_4 & \phi_3 & \phi_3 & \phi_2 & \phi_1 \end{bmatrix}, \tag{4.11}$$

$$\begin{aligned} \phi_1 &= 4 + 6r - r^2, \\ \phi_2 &= -1 + 2r + 6r^2, \\ \phi_3 &= -(1 + r)(1 + 2r), \\ \phi_4 &= -(1 + r)^2, \\ \phi_5 &= 4 + 4r - 4r^2, \\ \phi_6 &= -1 + r + 4r^2, \\ \phi_7 &= -(1 + 2r)^2. \end{aligned} \tag{4.12}$$

Five stock-price data sets D, E, F, G and H are selected where each data set consists of five Malaysian companies listed on Table 4.2. The period of trading is from 1st March 2006 untill 2nd August 2012, consisting of a total of 1500 trading days. To make sure that companies in data sets D, E, F, G and H are liquid and active enough to trade, we choose the companies from the largest 30 companies listed on the Kuala Lumpur Stocks Exchange by full market capitalization. Different sectors of trading are selected in each data set to diversify risk of investment.

For m=5, the Toeplitz universal portfolios generated by $(5+8r)A$ with companion matrix $a_0[5+8r]C'$ where A and C' are given (4.6) and (4.9) respectively, are run on the data sets D, E, F, G and H for selected values of the parameter r. The best wealth S_{1500} achieved at the end of 1500 days together with the final portfolio is displayed in Table 4.3. The valid ξ interval for the Mahalanobis universal portfolio used and the best ξ achieving the highest wealth for a given r are also shown in the table. The Toeplitz universal portfolio performs poorly on data sets D and F with the ranges of wealths 2.4265-2.6839 and 1.3639-1.5199 units respectively. A good performance is observed for data set G with the range of wealths 4.4211-5.0088 achieved. Higher wealths in the range 8.7177-9.3208 are achieved for data set E. In conclusion, the Toeplitz universal portfolios performs well for data set E where the highest wealth of 9.3208 units is achieved for r = -0.5.

Table 4.2: List of companies in the data set D, E, F, G and H

Data Set	Malaysian Companies in Each Portfolio
D	IOI Corporation, Carlsberg Brewery Malaysia, British American Tobacco, Nestle, Digi.com
E	Public Bank, Kulim, KLCC Property Holdings, AEON Corporation, Kuala Lumpur Kepong
F	AMMB Holdings, Berjaya Sports TOTO, Airasia, Gamuda, Genting
G	AEON Corporation, British American Tobacco, Kulim, Nestle, Digi.com
H	DIGI.com , Public Bank, KLCC Property Holdings, Carlsberg Brewery Malaysia, Kuala Lumpur Kepong

Table 4.3: The best wealth S_{1500} and the final portfolio $(b_1, b_2, b_3, b_4, b_5)$ achieved by the 3-band 5×5 Toeplitz universal portfolios generated with $m = 5$, for data sets D,E,F,G,H and selected values of r, the valid ξ intervals and the best ξ.

Set	r	Largest ξ	Smallest ξ	best ξ	s1500	b1	b2	b3	b4	b5
D	-0.5	0.12	-0.11	0.12	2.6839	0.1927	0.2470	0.2136	0.2146	0.1321
	-0.1	0.05	-0.03	0.05	2.6470	0.1902	0.2491	0.2224	0.2110	0.1273
	-0.4	0.07	-0.06	0.07	2.6258	0.1932	0.2506	0.2270	0.2050	0.1242
	-0.2	0.05	-0.04	0.05	2.6086	0.1975	0.2522	0.2305	0.1983	0.1214
	-0.3	0.05	-0.05	0.05	2.6037	0.2024	0.2566	0.2348	0.1911	0.1150
	0	0.04	-0.03	0.04	2.5788	0.2076	0.2562	0.2360	0.1843	0.1160
	0.1	0.04	-0.03	0.04	2.5376	0.2110	0.2505	0.2334	0.1802	0.1248
	0.2	0.04	-0.03	0.04	2.4967	0.2233	0.2551	0.2363	0.1640	0.1213
	0.5	0.04	-0.04	-0.04	2.4914	0.2168	0.2468	0.2313	0.1731	0.1320
	0.3	0.05	-0.03	0.05	2.4842	0.2114	0.2394	0.2266	0.1807	0.1419
D	0.4	0.04	-0.03	-0.03	2.4265	0.2137	0.2304	0.2204	0.1784	0.1570
E	-0.5	0.33	-0.30	-0.30	9.3208	0.3563	0.0746	0.1847	0.1742	0.2101
	-0.4	0.20	-0.17	-0.17	9.2477	0.3755	0.0716	0.1618	0.1593	0.2318
	-0.2	0.10	-0.09	-0.09	9.0895	0.3761	0.0920	0.1370	0.1520	0.2429
	-0.3	0.13	-0.11	-0.11	9.0748	0.3657	0.0891	0.1515	0.1567	0.2371
	-0.1	0.08	-0.07	-0.07	8.9698	0.3658	0.1087	0.1299	0.1528	0.2427
	0.0	0.07	-0.06	-0.06	8.9256	0.3652	0.1193	0.1193	0.1519	0.2442
	0.2	0.05	-0.05	-0.05	8.8927	0.3716	0.1388	0.0917	0.1491	0.2488
	0.1	0.06	-0.05	-0.05	8.8242	0.3553	0.1349	0.1136	0.1537	0.2426
	0.5	0.03	-0.04	-0.04	8.7641	0.3682	0.1789	0.0510	0.1502	0.2517
	0.4	0.04	-0.04	-0.04	8.7466	0.3594	0.1677	0.0739	0.1520	0.2470
E	0.3	0.04	-0.04	-0.04	8.7177	0.3490	0.1591	0.0949	0.1547	0.2424
F	-0.5	0.09	-0.11	0.09	1.5199	0.0924	0.1550	0.3688	0.1331	0.2507
	-0.4	0.05	-0.07	0.05	1.5144	0.0794	0.1759	0.3249	0.1940	0.2257
	-0.3	0.04	-0.05	0.04	1.4918	0.0977	0.1691	0.3293	0.1690	0.2349
	-0.2	0.03	-0.04	0.03	1.4661	0.0962	0.1864	0.2846	0.2186	0.2143
	-0.1	0.02	-0.04	0.02	1.4457	0.1304	0.1605	0.3274	0.1354	0.2463
	0	0.02	-0.03	0.02	1.4302	0.1463	0.1495	0.3418	0.1031	0.2593
	0.5	0.02	-0.02	-0.02	1.4225	0.1445	0.1594	0.3206	0.1274	0.2481
	0.1	0.02	-0.03	0.02	1.4008	0.1604	0.1539	0.3227	0.1084	0.2546
	0.2	0.02	-0.03	0.02	1.3899	0.1672	0.1524	0.3217	0.1023	0.2564
	0.4	0.02	-0.02	-0.02	1.3794	0.1730	0.1512	0.3207	0.0972	0.2579
F	0.3	0.02	-0.03	0.02	1.3639	0.1787	0.1516	0.3160	0.0962	0.2575
G	-0.5	0.12	-0.11	0.12	5.0088	0.3207	0.0164	0.3704	0.1062	0.1862
	-0.1	0.05	-0.03	0.05	4.9690	0.3712	0.0064	0.3871	0.0792	0.1561

	-0.4	0.07	-0.06	0.07	4.9634	0.3245	0.0235	0.3605	0.1052	0.1864
	-0.2	0.05	-0.04	0.05	4.9600	0.3493	0.0153	0.3728	0.0915	0.1712
	-0.3	0.05	-0.05	0.05	4.9066	0.3220	0.0373	0.3494	0.1086	0.1827
	0	0.04	-0.03	0.04	4.8344	0.3502	0.0469	0.3567	0.0959	0.1503
	0.1	0.04	-0.03	0.04	4.7742	0.3591	0.0604	0.3544	0.0936	0.1324
	0.2	0.04	-0.03	0.04	4.6815	0.3637	0.0846	0.3451	0.0951	0.1115
	0.5	0.04	-0.04	-0.04	4.6618	0.0684	0.1859	0.1130	0.2632	0.3695
	0.3	0.05	-0.03	0.05	4.5936	0.4063	0.1008	0.3579	0.0758	0.0593
G	0.4	0.04	-0.03	-0.03	4.4211	0.0899	0.2314	0.1104	0.2632	0.3051
H	-0.5	0.09	-0.11	0.09	4.9115	0.1724	0.1777	0.3095	0.0685	0.2719
	-0.4	0.05	-0.07	0.05	4.8613	0.1734	0.1816	0.3015	0.0783	0.2653
	-0.3	0.04	-0.05	0.04	4.8559	0.1696	0.1839	0.3080	0.0717	0.2668
	-0.2	0.03	-0.04	0.03	4.8068	0.1704	0.1892	0.2974	0.0857	0.2573
	-0.1	0.02	-0.04	0.02	4.7324	0.1758	0.1952	0.2733	0.1152	0.2405
	0	0.02	-0.03	0.02	4.7192	0.1712	0.1993	0.2785	0.1113	0.2398
	0.5	0.02	-0.02	-0.02	4.7180	0.2448	0.1477	0.1363	0.2584	0.2127
	0.1	0.02	-0.03	0.02	4.6936	0.1664	0.2050	0.2810	0.1112	0.2364
	0.2	0.02	-0.03	0.02	4.6558	0.1615	0.2122	0.2807	0.1151	0.2306
	0.4	0.02	-0.02	-0.02	4.6313	0.2422	0.1625	0.1272	0.2728	0.1953
H	0.3	0.02	-0.03	0.02	4.6060	0.1564	0.2208	0.2776	0.1230	0.2222

4.4 Running the pseudo-Mahalanobis universal portfolio generated by the Teoplitz matrix

Consider the following 3×3 Toeplitz matrix which is 5-band:

$$A = \begin{bmatrix} a_0 & a_1 & a_2 \\ a_1 & a_0 & a_1 \\ a_2 & a_1 & a_0 \end{bmatrix} \tag{4.13}$$

The universal portfolio generated by A is given by (4.4). The portfolio is said to be Mahalanobis if A is positive-definite and pseudo-Mahalanobis if A is non-positive-definite. The criterion that A is positive definite if and only if all the eigenvalues of A are positive cans be used to determine whether a portfolio is Mahalanobis or pseudo-Mahalanobis.

We use the initial starting portfolio $\boldsymbol{b}_1 = (0.3333, 0.3333, 0.3334)$ for all the three data sets. Table 4.4 shows the performance of Mahalanobis universal portfolios generated by the matrix A in (4.13) for data sets A, B and C the maximum wealths after 500 trading days. Set A achieve the highest wealth ($S_{500} = 1.7384$) at $(a_0, a_1, a_2) = (1, 1{,}5)$ when ξ=1.14 whereas sets B and C achieve highest wealths at (a_0, a_1, a_2) =(3, 1, 1) when $\xi = 0.35$ for set B and $\xi = 0.26$ for set C.

Table 4.4: The best wealth S500 and the final portfolio $(\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3)$ achieved by the 5-band 3×3 Toeplitz universal portfolios generated with $\boldsymbol{m} = 3$, for data sets A, B, C and selected values of $(\boldsymbol{a}_0,\ \boldsymbol{a}_1,\ \boldsymbol{a}_2)$, the valid $\boldsymbol{\xi}$ intervals and the best $\boldsymbol{\xi}$.

Set	a_0	a_1	a_2	Largest ξ	Smallest ξ	Best ξ	S_{500}	b1	b2	b3
A	1	1	5	1.14	-0.88	1.14	1.7384	0.1815	0.1301	0.6884
	1	1	4	1.52	-1.13	1.52	1.7384	0.1942	0.1045	0.7014
	1	1	10	0.50	-0.42	0.50	1.7354	0.1478	0.2047	0.6475
	1	1	3	2.23	-1.60	2.23	1.7342	0.2133	0.0768	0.7099
	1	1	100	0.04	-0.04	0.04	1.7112	0.1209	0.3181	0.5609
	1	1	2	3.77	-3.00	3.77	1.7073	0.2509	0.0771	0.6721
	2	1	1	1.02	-1.40	-1.40	1.5875	0.0532	0.7374	0.2094
	3	1	1	0.51	-0.70	-0.70	1.5875	0.0532	0.7374	0.2094
	5	1	1	0.25	-0.35	-0.35	1.5875	0.0532	0.7374	0.2094
	4	1	1	0.34	-0.46	-0.46	1.5874	0.0574	0.7312	0.2114
	10	1	1	0.11	-0.15	-0.15	1.5872	0.0638	0.7218	0.2144
	100	1	1	0.01	-0.01	-0.01	1.5837	0.1386	0.6120	0.2494
	1	0	1	0.85	-1.52	0.73	1.5692	0.4436	0.1128	0.4437

	0	1	0	1.22	-0.68	-0.59	1.5692	0.4446	0.1106	0.4447
	1	3	1	0.64	-0.36	-0.31	1.5692	0.4435	0.1130	0.4436
	1	2	1	1.32	-0.74	-0.64	1.5692	0.4448	0.1104	0.4449
	1	5	1	0.31	-0.17	-0.15	1.5692	0.4421	0.1157	0.4422
	1	4	1	0.42	-0.23	-0.21	1.5692	0.4464	0.1071	0.4465
	1	10	1	0.13	-0.07	-0.07	1.5692	0.4493	0.1014	0.4494
A	1	100	1	0.01	0.00	0.00	1.5650	0.3333	0.3333	0.3334
B	3	1	1	0.35	-0.22	0.35	2.6353	0.0337	0.1417	0.8246
	2	1	1	0.70	-0.45	0.70	2.6353	0.0337	0.1417	0.8246
	4	1	1	0.23	-0.15	0.23	2.6282	0.0379	0.1444	0.8177
	5	1	1	0.17	-0.11	0.17	2.6210	0.0421	0.1470	0.8109
	10	1	1	0.07	-0.05	0.07	2.5854	0.0633	0.1602	0.7765
	1	1	5	0.14	-0.14	-0.14	2.5458	0.0013	0.3613	0.6373
	1	1	2	0.59	-0.54	-0.54	2.5317	0.0056	0.3752	0.6193
	1	1	3	0.29	-0.27	-0.27	2.5315	0.0088	0.3687	0.6225
	1	1	4	0.19	-0.18	-0.18	2.5314	0.0112	0.3639	0.6248
	1	1	10	0.06	-0.06	-0.06	2.5309	0.0181	0.3503	0.6317
	1	0	1	1.02	-1.77	-0.31	2.1612	0.2795	0.4410	0.2796
	1	2	1	1.53	-0.88	0.27	2.1612	0.2792	0.4415	0.2793
	0	1	0	1.41	-0.82	0.25	2.1612	0.2790	0.4418	0.2791
	1	3	1	0.75	-0.43	0.13	2.1612	0.2802	0.4395	0.2803
	1	4	1	0.49	-0.28	0.09	2.1612	0.2775	0.4450	0.2776
	1	10	1	0.16	-0.09	0.03	2.1612	0.2759	0.4480	0.2760
	1	5	1	0.36	-0.21	0.06	2.1612	0.2834	0.4332	0.2835
	1	1	100	0.00	0.00	0.00	2.1600	0.3333	0.3333	0.3334
	1	100	1	0.01	0.00	0.00	2.1600	0.3333	0.3333	0.3334
B	100	1	1	0.00	0.00	0.00	2.1600	0.3333	0.3333	0.3334
C	3	1	1	0.26	-0.18	0.26	2.3862	0.1964	0.0203	0.7834
	5	1	1	0.13	-0.09	0.13	2.3862	0.1964	0.0203	0.7834
	2	1	1	0.52	-0.36	0.52	2.3862	0.1964	0.0203	0.7834
	4	1	1	0.17	-0.12	0.17	2.3745	0.1990	0.0262	0.7748
	10	1	1	0.05	-0.04	0.05	2.3049	0.2146	0.0618	0.7236
	1	1	10	0.06	-0.06	-0.06	2.1093	0.0105	0.3701	0.6194
	1	1	5	0.15	-0.13	-0.13	2.0824	0.0118	0.3896	0.5986
	1	1	4	0.20	-0.17	-0.17	2.0718	0.0142	0.3959	0.5899
	0	1	0	0.67	-0.35	-0.35	2.0603	0.4913	0.0173	0.4914
	1	1	3	0.31	-0.25	-0.25	2.0597	0.0157	0.4042	0.5802
	1	5	1	0.17	-0.09	-0.09	2.0574	0.4893	0.0212	0.4894
	1	0	1	0.43	-0.84	0.43	2.0563	0.4886	0.0226	0.4887
	1	4	1	0.23	-0.12	-0.12	2.0555	0.4881	0.0236	0.4882
	1	2	1	0.73	-0.37	-0.37	2.0547	0.4875	0.0248	0.4876
	1	3	1	0.35	-0.18	-0.18	2.0528	0.4863	0.0272	0.4864
	1	1	2	0.64	-0.49	-0.49	2.0454	0.0156	0.4154	0.5690

	1	10	1	0.07	-0.03	-0.03	2.0034	0.4530	0.0940	0.4531
	1	100	1	0.00	0.00	0.00	1.8343	0.3333	0.3333	0.3334
	1	1	100	0.00	0.00	0.00	1.8343	0.3333	0.3333	0.3334
C	100	1	1	0.00	0.00	0.00	1.8343	0.3333	0.3333	0.3334

The comparison of the BCRP wealth and the highest wealths achieved by the Mahalanobis universal portfolios generated by 3-band and 5-band Toeplitz matrices is shown in Table 4.5. Performance of the 3-band and 5-band Toeplitz universal portfolios on in sets A, B and C cannot outperform the BCRP. However the maximum wealths achieved by the 5-band Toeplitz universal portfolio on set A is very close to the BCRP wealth. Set A achieves a maximum wealth of 1.7384 for a non-positives definite A. This wealth is slightly higher than the wealth of 1.6549 achieved for a positives definite A.

Table 4.5: Comparison of the BCRP with the maximum wealth S_{500} achieved by the 3-band and 5-band Toeplitz universal portfolios generated with $m = 3$, for data sets A, B, C.

	BCRP	3-band Toeplitz Portfolio	5-band Toeplitz Portfolio
Set A	1.8534	1.6549	1.7384
Set B	4.2970	2.6672	2.6353
Set C	4.2970	2.5789	2.3862

Next, consider the following 5×5 Toeplitz matrix which is 9-band:

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 \\ a_1 & a_0 & a_1 & a_2 & a_3 \\ a_2 & a_1 & a_0 & a_1 & a_2 \\ a_3 & a_2 & a_1 & a_0 & a_1 \\ a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix}. \qquad (4.14)$$

The empirical results for the best wealth S_{1500} achieved by the 9-band Toeplitz universal portfolios generated by A together with the final portfolio is displayed in Table 4.6 for selected values of the parametric vector $(a_0, a_1, a_2, a_3, a_4)$. The best wealth S_{1500} refers to the highest wealth achieved for ξ in the valid ξ interval, given $(a_0, a_1, a_2, a_3, a_4)$. The first 5 rows of the table correspond to 5 non-positives definite A and the remaining rows correspond to positive definite A. Set G and H in Table 4.6 indicates that the pseudo-Mahalanobis portfolios can perform better than the Mahalanobis portfolios, with the former achieving wealths in the range 4.9225-5.4003 and the latter achieving wealths around 5.2600 units.

Table 4.6: The best wealth S_{1500} and the final portfolio $(\boldsymbol{b_1}, \boldsymbol{b_2}, \boldsymbol{b_3}, \boldsymbol{b_4}, \boldsymbol{b_5})$ achieved by the 9-band 5×5 Toeplitz universal portfolios generated with $\boldsymbol{m} = 5$, for data sets D, E, F, G and H for selected values of $(\boldsymbol{a_0}, \boldsymbol{a_1}, \boldsymbol{a_2}, \boldsymbol{a_3}, \boldsymbol{a_4})$, the valid $\boldsymbol{\xi}$ intervals and the best $\boldsymbol{\xi}$.

Set	a0	a1	a2	a3	a4	best	s1500	b1	b2	b3	b4	b5
D	2	1	1	1	1	-0.22	2.60	0.21	0.26	0.24	0.18	0.11
	3	1	1	1	1	-0.11	2.60	0.21	0.26	0.24	0.18	0.11
	4	1	1	1	1	-0.07	2.59	0.21	0.26	0.24	0.18	0.11
	5	1	1	1	1	-0.05	2.58	0.21	0.26	0.24	0.18	0.12
	10	1	1	1	1	-0.02	2.56	0.21	0.25	0.23	0.19	0.13
	1	0	1	0	1	-0.23	2.52	0.15	0.27	0.15	0.27	0.15
	1	2	1	1	1	0.20	2.48	0.16	0.19	0.17	0.26	0.22
	1	10	1	1	1	0.02	2.47	0.16	0.19	0.18	0.25	0.22
	1	4	1	1	1	0.06	2.47	0.16	0.19	0.18	0.25	0.22
	1	3	1	1	1	0.09	2.47	0.17	0.19	0.18	0.25	0.22
	1	5	1	1	1	0.04	2.45	0.17	0.19	0.18	0.25	0.22
	1	1	1	1	5	0.08	2.44	0.30	0.19	0.19	0.19	0.14
	1	1	1	1	4	0.11	2.44	0.31	0.18	0.18	0.18	0.14
	1	1	1	1	3	0.17	2.44	0.31	0.18	0.18	0.18	0.14
	1	1	1	1	2	0.34	2.43	0.32	0.18	0.18	0.18	0.14
	1	1	1	1	10	0.03	2.43	0.28	0.19	0.19	0.19	0.15
	0	1	0	1	0	-0.15	2.41	0.23	0.21	0.24	0.16	0.16
	1	1	2	1	1	0.17	2.39	0.17	0.22	0.27	0.16	0.17

	1	1	5	1	1	0.04	2.39	0.17	0.22	0.26	0.16	0.17
	1	1	3	1	1	0.08	2.39	0.17	0.22	0.26	0.16	0.17
	1	1	4	1	1	0.05	2.38	0.18	0.22	0.26	0.17	0.18
	1	1	1	5	1	0.04	2.38	0.21	0.26	0.20	0.19	0.15
	1	1	1	3	1	0.08	2.38	0.21	0.26	0.20	0.19	0.15
	1	1	1	2	1	0.16	2.38	0.21	0.26	0.20	0.19	0.15
	1	1	1	4	1	0.05	2.38	0.21	0.26	0.20	0.19	0.15
	1	1	10	1	1	0.01	2.37	0.19	0.21	0.24	0.18	0.19
D	1	1	1	10	1	0.01	2.37	0.20	0.23	0.20	0.19	0.17
E	3	1	1	1	1	-0.16	9.00	0.38	0.11	0.11	0.15	0.25
	2	1	1	1	1	-0.32	9.00	0.38	0.11	0.11	0.15	0.25
	5	1	1	1	1	-0.08	9.00	0.38	0.11	0.11	0.15	0.25
	4	1	1	1	1	-0.10	8.93	0.37	0.12	0.12	0.15	0.24
	10	1	1	1	1	-0.03	8.81	0.35	0.13	0.13	0.16	0.24
	1	2	1	1	1	0.34	8.62	0.24	0.05	0.29	0.20	0.22
	1	4	1	1	1	0.11	8.59	0.24	0.05	0.29	0.20	0.22
	1	3	1	1	1	0.16	8.57	0.24	0.06	0.29	0.20	0.22
	1	5	1	1	1	0.08	8.56	0.24	0.05	0.28	0.20	0.22
	1	10	1	1	1	0.03	8.45	0.24	0.08	0.27	0.19	0.22
	1	1	1	1	10	0.06	8.38	0.25	0.24	0.24	0.24	0.02
	1	1	1	1	3	0.24	8.35	0.22	0.25	0.25	0.25	0.02
	1	1	1	1	2	0.47	8.35	0.21	0.26	0.26	0.26	0.01
	1	1	1	1	4	0.16	8.35	0.22	0.25	0.25	0.25	0.02
	1	1	1	1	5	0.12	8.34	0.23	0.25	0.25	0.25	0.03
	1	0	1	0	1	-0.24	8.25	0.28	0.08	0.28	0.08	0.28
	0	1	0	1	0	0.22	8.25	0.28	0.08	0.28	0.08	0.28
	1	1	10	1	1	0.02	8.00	0.24	0.22	0.06	0.24	0.24
	1	1	2	1	1	0.19	8.00	0.24	0.22	0.06	0.24	0.24
	1	1	4	1	1	0.06	7.99	0.24	0.22	0.06	0.24	0.24
	1	1	3	1	1	0.09	7.99	0.24	0.22	0.06	0.24	0.24
	1	1	1	2	1	-0.30	7.99	0.13	0.22	0.19	0.35	0.11
	1	1	5	1	1	0.04	7.98	0.24	0.22	0.08	0.23	0.24
	1	1	1	3	1	-0.15	7.98	0.13	0.21	0.19	0.35	0.11
	1	1	1	4	1	-0.10	7.97	0.13	0.21	0.19	0.35	0.11
	1	1	1	5	1	-0.07	7.96	0.14	0.21	0.19	0.34	0.12
E	1	1	1	10	1	-0.03	7.95	0.14	0.21	0.19	0.33	0.12
F	2	1	1	1	1	-0.24	1.45	0.13	0.15	0.35	0.11	0.26
	3	1	1	1	1	-0.12	1.45	0.13	0.15	0.35	0.11	0.26
	4	1	1	1	1	-0.08	1.45	0.13	0.15	0.35	0.11	0.26
	5	1	1	1	1	-0.06	1.45	0.13	0.15	0.35	0.11	0.26
	10	1	1	1	1	-0.02	1.41	0.15	0.16	0.31	0.14	0.24
	1	1	3	1	1	0.11	1.36	0.10	0.28	0.25	0.26	0.10
	1	1	2	1	1	0.22	1.36	0.10	0.28	0.25	0.26	0.10

	1	1	4	1	1	0.07	1.36	0.11	0.28	0.25	0.26	0.11
	1	1	5	1	1	0.05	1.36	0.11	0.27	0.25	0.25	0.11
	1	1	10	1	1	0.02	1.35	0.12	0.26	0.25	0.25	0.12
	1	1	1	2	1	0.30	1.34	0.27	0.09	0.17	0.24	0.23
	1	1	1	5	1	0.07	1.34	0.26	0.10	0.18	0.24	0.22
	1	1	1	10	1	0.03	1.34	0.26	0.10	0.19	0.23	0.22
	1	1	1	3	1	0.14	1.34	0.26	0.10	0.18	0.24	0.22
	1	1	1	4	1	0.09	1.34	0.26	0.10	0.18	0.24	0.22
	1	0	1	0	1	-0.12	1.31	0.25	0.12	0.25	0.12	0.25
	0	1	0	1	0	0.11	1.31	0.25	0.12	0.25	0.12	0.25
	1	1	1	1	10	0.05	1.31	0.09	0.20	0.20	0.20	0.32
	1	1	1	1	5	0.13	1.30	0.07	0.20	0.20	0.20	0.34
	1	1	1	1	4	0.17	1.30	0.07	0.20	0.20	0.20	0.34
	1	1	1	1	3	0.26	1.29	0.07	0.20	0.20	0.20	0.34
	1	1	1	1	2	0.53	1.29	0.07	0.20	0.20	0.20	0.34
	1	3	1	1	1	0.07	1.29	0.23	0.16	0.28	0.08	0.25
	1	5	1	1	1	0.03	1.29	0.22	0.16	0.27	0.10	0.24
	1	4	1	1	1	0.04	1.29	0.22	0.16	0.27	0.10	0.24
	1	2	1	1	1	0.14	1.29	0.23	0.16	0.28	0.08	0.25
F	1	10	1	1	1	0.01	1.29	0.22	0.17	0.25	0.13	0.23
G	1	1	2	1	1	0.20	5.40	0.33	0.07	0.26	0.01	0.33
	1	1	3	1	1	0.10	5.40	0.33	0.07	0.25	0.01	0.33
	1	1	5	1	1	0.05	5.39	0.33	0.07	0.25	0.01	0.33
	1	1	4	1	1	0.06	5.29	0.32	0.08	0.25	0.03	0.32
	1	1	10	1	1	0.02	5.29	0.32	0.08	0.24	0.03	0.32
	1	0	1	0	1	0.12	5.14	0.32	0.02	0.32	0.02	0.32
	0	1	0	1	0	-0.11	5.13	0.32	0.02	0.32	0.02	0.32
	1	2	1	1	1	-0.11	5.04	0.28	0.02	0.33	0.13	0.25
	1	3	1	1	1	-0.05	4.98	0.27	0.04	0.32	0.13	0.24
	1	4	1	1	1	-0.03	4.92	0.26	0.05	0.31	0.14	0.24
	1	10	1	1	1	-0.01	4.92	0.26	0.05	0.31	0.14	0.24
	4	1	1	1	1	0.08	4.92	0.38	0.02	0.39	0.08	0.14
	5	1	1	1	1	0.06	4.92	0.38	0.02	0.39	0.08	0.14
	2	1	1	1	1	0.24	4.92	0.38	0.02	0.39	0.08	0.14
	3	1	1	1	1	0.12	4.92	0.38	0.02	0.39	0.08	0.14
	1	5	1	1	1	-0.02	4.87	0.26	0.07	0.30	0.15	0.24
	10	1	1	1	1	0.02	4.79	0.34	0.06	0.34	0.11	0.16
	1	1	1	1	10	-0.03	4.56	0.31	0.22	0.22	0.22	0.04
	1	1	1	1	3	-0.13	4.56	0.29	0.22	0.22	0.22	0.04
	1	1	1	1	2	-0.25	4.55	0.29	0.22	0.22	0.22	0.04
	1	1	1	1	4	-0.08	4.54	0.29	0.22	0.22	0.22	0.05
	1	1	1	1	5	-0.06	4.54	0.29	0.22	0.22	0.22	0.05
	1	1	1	10	1	-0.02	4.54	0.26	0.21	0.19	0.03	0.31

	1	1	1	4	1	-0.06	4.53	0.26	0.21	0.18	0.03	0.31
	1	1	1	3	1	-0.09	4.53	0.26	0.21	0.18	0.03	0.31
	1	1	1	2	1	-0.18	4.52	0.27	0.22	0.17	0.03	0.31
G	1	1	1	5	1	-0.04	4.52	0.26	0.21	0.18	0.05	0.30
H	1	2	1	1	1	-0.16	5.26	0.19	0.12	0.33	0.02	0.34
	1	3	1	1	1	-0.08	5.26	0.19	0.12	0.33	0.02	0.34
	1	5	1	1	1	-0.04	5.26	0.19	0.12	0.33	0.02	0.34
	1	4	1	1	1	-0.05	5.21	0.20	0.12	0.33	0.03	0.33
	1	10	1	1	1	-0.01	4.96	0.20	0.15	0.28	0.09	0.28
	1	0	1	0	1	0.12	4.86	0.28	0.08	0.28	0.08	0.28
	0	1	0	1	0	-0.11	4.86	0.28	0.08	0.28	0.08	0.28
	1	1	1	2	1	0.14	4.81	0.09	0.28	0.22	0.19	0.22
	1	1	1	3	1	0.07	4.81	0.09	0.28	0.22	0.19	0.22
	1	1	1	4	1	0.04	4.77	0.11	0.27	0.21	0.19	0.22
	1	1	1	5	1	0.03	4.77	0.11	0.27	0.21	0.19	0.22
	2	1	1	1	1	0.13	4.76	0.16	0.20	0.30	0.09	0.25
	5	1	1	1	1	0.03	4.75	0.17	0.20	0.29	0.09	0.25
	4	1	1	1	1	0.04	4.75	0.17	0.20	0.29	0.09	0.25
	3	1	1	1	1	0.06	4.75	0.17	0.20	0.29	0.09	0.25
	1	1	1	10	1	0.01	4.72	0.13	0.25	0.21	0.19	0.22
	1	1	2	1	1	0.11	4.71	0.27	0.08	0.19	0.18	0.27
	10	1	1	1	1	0.01	4.71	0.17	0.20	0.27	0.12	0.24
	1	1	3	1	1	0.05	4.70	0.26	0.09	0.19	0.19	0.26
	1	1	4	1	1	0.03	4.69	0.26	0.11	0.19	0.19	0.26
	1	1	10	1	1	0.01	4.68	0.26	0.11	0.19	0.19	0.26
	1	1	5	1	1	0.02	4.67	0.25	0.12	0.19	0.19	0.25
	1	1	1	1	2	-0.46	4.65	0.02	0.21	0.21	0.21	0.34
	1	1	1	1	3	-0.23	4.63	0.02	0.21	0.21	0.21	0.34
	1	1	1	1	10	0.06	4.62	0.39	0.19	0.19	0.19	0.03
	1	1	1	1	4	-0.15	4.62	0.03	0.21	0.21	0.21	0.34
H	1	1	1	1	5	-0.11	4.61	0.03	0.21	0.21	0.21	0.34

Table 4.7: Comparison of the BCRP wealth with the maximum wealth S_{1500} achieved by the 3-band and 5-band Toeplitz universal portfolios generated with $m = 5$, for data sets D, E, F, G and H.

	BCRP	3-band Toeplitz Portfolio	9-band Toeplitz Portfolio
Set D	3.9315	2.6839	2.6025
Set E	10.4317	9.3208	9.0007
Set F	2.2428	1.5199	1.4523
Set G	10.2710	5.0088	5.4003
Set H	9.7598	4.9115	5.2622

Table 4.7 shows the comparison of the BCRP wealth with the maximum wealth S_{1500} achieved by the 3-band and 5-band Toeplitz universal portfolios generated with $m = 5$, for data sets D, E, F, G and H. We notice that the pseudo-Mahalanobis universal portfolios generated by the 9-band Toeplitz matrices achieving a maximum wealth of 5.4003 that exceeds the Mahalanobis universal portfolio wealths generated by the 3-band Toeplitz matrices for data set G, with the latter achieving a highest wealth of 5.0088. This is also observed for Set H where the pseudo-Mahalanobis universal portfolios generated by the 9-band Toeplitz matrices achieved capital return of 5.2622 after 1500 trading days which is higher than wealth of the Mahalanobis universal portfolios generated by the 3-band Toeplitz matrix which is 4.9115. The BCRP wealths for sets D, E, F, G and H are 3.9315, 10.4317, 2.2428, 10.2710, and 9.7598 respectively. The closest value to the BCRP wealth is 9.3208 which is the maximum wealth achieved by the 3-band Toeplitz universal portfolio on set E. The results of this chapter are reported in Tan, Chu and Pan (2013).

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APPENDIX A
Matlab Coding for Order-2 Universal Portfolios

```
function order2setA()
xn1=xlsread('malayan banking bhd(1155.kl).xls',1,'I2:I501');
xn2=xlsread('genting bhd(3182.kl).xls',1,'I2:I501');
xn3=xlsread('AMWAY (M) holdings bhd(6351.kl).xls',1,'I2:I501');
xn=[xn1 xn2 xn3];
a1 = input('Please enter a value for a1:');
a2 = input('Please enter a value for a2:');
a3 = input('Please enter a value for a3:');
a=[a1 a2 a3];
bn=[0.3333 0.3333 0.3334];
v1=[1;1;1];
sn=1;

for i= 2:1:500
   A=(sum(a(1,:))+2)*(((sum(a(1,:).*xn(i,:)))*(sum(a(1,:).*xn(i-1,:))))
+(sum(a(1,:).*xn(i,:).*xn(i-1,:))));
    zeta=A^(-1);
   for j=1:1:3
       un=[xn(i,:); xn(i,:); xn(i,:)]';
       umt=[xn(i-1,:);xn(i-1,:);xn(i-1,:)];
    for k=1:1:3
        for l=1:1:3
            if j==k&&j==l&&k==j
                w(k,l)=(a(1,j)+2)*(a(1,j)+1)*a(1,j);
            elseif j~=k&&j~=l&&k~=l
                w(k,l)=a(1,j)*a(1,k)*a(1,l);
            elseif j~=k&&k==l
                w(k,l)=a(1,j)*((a(1,l))+1)*a(1,l);
            else
                w(k,l)=a(1,k)*((a(1,j))+1)*a(1,l);
            end
        end
    end
          bn(i,j)=zeta*v1'*(un.*umt.*w)*v1;
   end
     sn=sum(bn(i,:).*xn(i,:))*sn;
end

%bn(i,:)
%sn

fprintf ( 1, 'The b is %.9f %.9f %.9f\n',bn(i,:) );
fprintf ( 1, 'The sn is %.9f\n\n', sn );
end
```