PRICING OF ANNUITIES WITH GUARANTEED MINIMUM WITHDRAWAL BENEFITS UNDER STOCHASTIC INTEREST RATES

By

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ABSTRACT

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Variable annuity with Guaranteed Minimum Withdrawal Benefits (GMWB) has become a popular tool in retirement planning as it provides downside protection against investment risk while policyholder may enjoy the potential upside gain when the market performance is good. GMWB is purchased by paying a fee proportionate to the policyholder's fund value.

Our main objective of this research is to explore the pricing model of variable annuities with GMWB, to determine the insurance fee that needs to be charged for providing the guarantee under a deterministic withdrawal rate. Besides, we estimate the model parameters based on Malaysia market.

We first examine pricing models of GMWB under constant interest rate and extend to Vasicek and CIR interest rate model with parameters estimated from bond market using least square method. As the model becomes complex when stochastic interest rate is incorporated, a change of numéraire is performed to simplify the computational work. Furthermore, sensitivity tests

are also performed to examine the GMWB pricing behavior when different parameters of the model are changed, as well as under different interest rate models. We find out that the pricing of GMWB is sensitive to the volatility of interest rate and the correlation between underlying asset and interest rate.

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APPROVAL SHEET

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I understand that University will upload softcopy of my dissertation in pdf format into UTAR Institutional Repository, which may be made accessible to UTAR community and public.

Yours truly,

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(Khow Weai Chiet)

DECLARATION

I, Khow Weai Chiet, hereby declare that the dissertation is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UTAR or other institutions.

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Date ________________________

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CHAPTER 1

INTRODUCTION

Variable annuity has been widely used as a retirement planning instrument. Recently variable annuity with Guaranteed Minimum Withdrawal Benefits (GMWB) rider is widely used as a retirement planning instrument in the US as it provides downside protection against investment risk while policyholder may enjoy the potential upside gain when the market performance is good. To enjoy this benefit, the policyholder is required to pay a fee proportionate to the policyholder's fund value.

In this research, we will explore the pricing of variable annuity with GMWB rider by using the Monte Carlo simulation approach. As the GMWB could be decomposed into an annuity certain and an Asian put option (Milevsky and Salisbury, 2006), its value could be approximated by quantifying the Asian put option's value. To achieve this, we simulate the performance of the subaccount. Assumption is made that the underlying asset follows a Geometric Brownian Motion. Using the risk neutral valuation approach, the option value is determined by discounting the terminal payoff using risk free rate.

We begin with a simpler model by assuming a constant risk free rate to serve as a benchmark. This will be compared against the results produced by the proposed models. By leveraging on a similar methodology, we further expand the pricing model to allow for stochastic interest rate models, where the model's parameters are calibrated to Malaysia's historical data using least square method. Using the calibrated parameters, future interest rates are simulated to determine the discount factor that will be used to find the present value of the terminal payoff. The process is then repeated for Vasicek model and the Cox-Ingersoll Ross (CIR) model. Lower bound of the GMWB pricing model under Vasicek model is deduced as an approximation of the option value. Furthermore, sensitivity tests are also performed to examine the GMWB pricing behavior when different parameters are used in the model, as well as under different interest rate models.

1.1 Variable Annuity

Variable annuity is an agreement between an insurer and a policyholder where policyholder will make a purchase payment to the insurer and the insurer will make a serial of payments to the policyholder. The purchase payment can be a single payment or a series of payments begins at the inception or sometime in the future. In this thesis, we will assume single purchase payment and the policyholder will receive the payment immediately.

When a variable annuity is purchased, the premium paid is allocated in a subaccount where the policyholder is offered to invest the purchase payment in a selection of investment fund comprising of equities, bonds, money market or a combination of these three classes of asset. At any time t, the value of the policyholder's account is depending on the performance of the investment fund. In other words, the investment risk is borne by the policyholder. Thus, by purchasing a variable annuity there is a possibility to lose all of the money. As variable annuity is commonly used as a retirement planning instrument, the washed-out of the account value could be a nightmare to a retiree.

1.2 Types of Guaranteed Minimum Benefit in Variable Annuity

Optional guarantees are often offered to variable annuity for additional charges. Guaranteed Minimum Benefits (GMBs) that are currently available in the market are Guaranteed Minimum Death Benefit (GMDB), Guaranteed Minimum Accumulation Benefit (GMAB), Guaranteed Minimum Income Benefit (GMIB) and Guaranteed Minimum Withdrawal Benefit (GMWB). Generally GMAB, GMIB and GMWB are categorized as Guaranteed Minimum Living Benefit (GMLB).

Traditionally, a variable annuity with GMDB will guarantee the policyholder a lump sum payment of the higher of account value or the return of purchase payment upon death. Nowadays, a great variety of GMWB is offered. For example, the policyholder is guaranteed the death benefit equal to the minimum of the roll up benefit base, or death benefit equal to the higher of the annual ratchet benefit base or the account value.

With GMAB, the return of the purchase payment is guaranteed at the end of the specified period, typically five to ten years from issue, regardless of the investment performance. In other words, if the subaccount value is less than the guaranteed amount at the specified period, the insurer has to top up to the guaranteed amount.

GMIB guarantees the policyholder to receive periodical payment for life when the GMIB amount is annuitized. The income amount that is received by the policyholder is calculated based on the greater of the subaccount value or the guaranteed amount.

Basically GMWB guarantees a return of purchase payment over a specified time through a series of periodic withdrawal. We will look into this further in the next section.

Among these GMBs, GMWB has attracted significant attentions and become more popular.

1.3 Overview of GMWB

GMWB was first introduced in Canada in the last decade. The key reason of the emerging of GMWB was due to the baby boomer who has accumulate a large amount of saving will be retiring in the next 10 to 15 years. A conventional annuities is relatively not exciting as people are more willing to have a control over their saving.

GMWB manages to attract more than $2 billion asset for Canadian insurance companies. But during the global financial crisis, the insurers faced a difficult time to set up reserve for the guaranteed provided. Thus, fees were increased or the features were cut back and some even discontinued the product. Nowadays, GMWB is still popular among those who want to have a peace of mind in their retirement years.

GMWB is a rider attached to variable annuity. With GMWB rider, the insurer guarantees the policyholder to have periodic withdrawal up to the value of the initial capital. When the GMWB is purchased, initial capital is paid to the insurance company. This capital can be invested in a series of investment funds at the policyholder's discretion. At maturity, any balance in the account will be given back to the policyholder. Thus, the policyholder can enjoy the upside equity gain while downside risk is protected. However, the GMWB does not provide protection against inflation as the initial capital can only be recovered after a long period.

To have a better view, we will illustrate the calculation of a variable annuity with GMWB rider under two scenarios using the following example:-

- Single premium (initial investment) = RM100
- Annual withdrawal amount = RM 5
- Contract term = 20 years
- Assuming no insurance fee

In this example we assume no insurance fee but in reality, insurer charges a proportional fee, α ranging from 35bps to 75bps per annum.

Table 1.1 shows the scenario under bullish market while Table 1.2 describes the scenario under bearish market. From table 1.1, in the bullish market, investors of the GMWB enjoy the potential gains from the equity market. Throughout the 20 years, total amount that will be received by the investors are the guaranteed periodical withdrawal (RM5 for 20 years) and the sub-account value at maturity (RM 60.95), a total of RM160.95. Table 1.2 shows that in year 15, the subaccount value at the end of the year is RM 0.63. Under the regular plan without GMWB rider, the policyholder can no longer withdraw from the policy as there is not enough money to be withdrawn. But under GMWB, the policyholder can withdraw RM 5 until the total withdrawal is equal to the initial investment (RM100), which means he can continue to withdraw until year 20, even though the subaccount value is negative. Thus, the policyholder is protected against downside risk during bad performance year.

Table 1.1 Example cash flow of GMWB under Bullish Market

Time	Beginning of Year	Investment Return (%)	Withdrawal (GMWB)	Balance (EOY)
1	100.00	4.00	5.00	99.00
2	99.00	10.00	5.00	103.90
3	103.90	(1.33)	5.00	97.52
4	97.52	4.89	5.00	97.29
5	97.29	6.00	5.00	98.12
6	98.12	8.20	5.00	101.17
7	101.17	8.05	5.00	104.31
8	104.31	13.93	5.00	113.85
9	113.85	(6.00)	5.00	102.01
10	102.01	(1.23)	5.00	95.76
11	95.76	4.45	5.00	95.02
12	95.02	(13.64)	5.00	77.06
13	77.06	(3.08)	5.00	69.69
14	69.69	(10.00)	5.00	57.72
15	57.72	7.36	5.00	56.97
16	56.97	13.04	5.00	59.39
17	59.39	9.98	5.00	60.32
18	60.32	19.79	5.00	67.26
19	67.26	3.04	5.00	64.30
20	64.30	2.56	5.00	60.95

Table 1.2 Example cash flow of GMWB under Bearish Market

Time	Beginning of Year	Investment Return (%)	Withdrawal (GMWB)	Balance (EOY)
1	100.00	0.40	5.00	95.40
2	95.40	(3.57)	5.00	86.99
3	86.99	(1.33)	5.00	80.84
4	80.84	4.89	5.00	79.79
5	79.79	(20.38)	5.00	58.53
6	58.53	(26.71)	5.00	37.90
7	37.90	8.05	5.00	35.95
8	35.95	13.93	5.00	35.95
9	35.95	(17.47)	5.00	24.67
10	24.67	25.54	5.00	25.97
11	25.97	4.45	5.00	22.13
12	22.13	(13.64)	5.00	14.11
13	14.11	(3.08)	5.00	8.68
14	8.68	18.09	5.00	5.25
15	5.25	7.36	5.00	0.63
16	0.63	r	5.00	0.00
17	0.00	r	5.00	0.00
18	0.00	r	5.00	0.00
19	0.00	r	5.00	0.00
20	0.00	r	5.00	0.00

1.4 Literature Review

Milevsky and Salisbury (2006) presented a model to evaluate the price of variable annuities with GMWB rider from financial economic perspective. They considered two different withdrawal behaviors in the pricing model, which are static and dynamic withdrawal policy. The static approach assumes an individual periodically withdraws a fixed amount from their account while the dynamic approach assumes individual can withdraw a flexible amount from their account in accordant to economic advantages. Assuming that the underlying assets follow a Geometric Brownian Motion (GBM) with constant interest rate, they have shown that under static withdrawal policy, variable annuity with GMWB rider can be decomposed into a type of Quanto Asian put option plus a generic annuity certain. While under dynamic withdrawal policy, assuming that investors were to maximize the embedded options, the evaluation is similar to the pricing of an American put option. Hence the existing pricing techniques can readily be used for the evaluation of a GMWB. Besides, it provides an alternative for the insurance company to use those options as a hedging tool as both Asian option and American put option markets are well established. They also concluded that the insurance fee charged in the market (as of mid-2004) is too low to cover the hedging cost for providing the guarantee under two extreme withdrawal approach (static and dynamic withdrawal).

Since policyholder is assumed to be rational, under dynamic withdrawal approach, they will lapse the product at an optimal time to maximize the annuity value. Thus, Dai et al. (2008) explored the optimal withdrawal strategy for the pricing of variable annuities with GMWB and constructed a singular stochastic control model. In the model, withdrawal is used as the control variable under both continuous and discrete withdrawal framework.

Dai et al. (2008) used Hamilton-Jacobi-Bellman (HJB) approach to evaluate the value of a variable annuity with GMWB since the value function can be shown as the generalized solution to the HJB equation. Using penalty approximation approach, Dai et al. (2008) proposed an efficient finite difference scheme to obtain the fair value of a GMWB. For both continuous and discrete withdrawal policy, a penalty fee is imposed for the withdrawal greater than the contractual withdrawal. Their pricing model is performed under risk neutral framework assuming deterministic interest rate.

Based on the idealized contract discussed by Dai et al. (2008), Chen et al. (2008) examined the assumptions of the model in detail. Besides, the effect of different parameters on the fair fee is examined. They have included mutual fund fees in the model with the argument that mutual fund charges a separate management fees on top of the hedging fees. Furthermore, both optimal investor behavior and sub-optimal behavior are studied. Besides assuming underlying asset follows a Geometric Brownian Motion, jump diffusion processes is incorporated in the model. Incorporating jumps in the model is believed to be more realistic especially for the pricing of a long term product.

Ignoring the interest rate risk, their model shows that insurances companies are not charging enough to cover the cost of the contract.

Huang and Kwok (2014) extended the singular stochastic control model proposed by Dai et al. (2008). In the pricing model, optimal dynamic withdrawal policies are analyzed. The analytical approximation of the pricing model is derived under numerous limiting conditions. In this paper (Huang and Kwok, 2014), they showed that the pricing formulation can be simplified to an optimal stopping problem if the penalty charge is imposed on the withdrawal amount.

Bauer et al. (2008) proposed a general framework that fits all types of guarantees offered within a variable annuity contract in the US, namely Guaranteed Minimum Death Benefits (GMDB), Guaranteed Minimum Accumulation Benefits (GMAB), Guaranteed Minimum Income Benefits (GMIB) and Guaranteed Minimum Withdrawal Benefits (GMWB). The pricing model is developed under risk neutral measure assuming underlying fund follows a Geometric Brownian Motion (GBM). In the framework, both deterministic withdrawal strategies and optimal withdrawal strategies are considered. As there is no closed form solution for the complex valuation, Monte Carlo simulation and generalized finite mesh discretization approach are used to determine the cost for the guarantee provided. According to the paper, Monte Carlo methods are not preferable for the evaluation under optimal withdrawal strategies, thus discretization approach is applied. The numerical results showed that some of the guarantees provided in the market are overpriced whereas variable annuities with GMIB options are underpriced. As their model does not include charges other than insurance fees, it tends to underestimate the option values. Thus, they concluded that the insurer should reexamine their pricing schemes if the guaranteed provided is underpriced under their model.

The pricing model assumption is usually simplified to make the model tractable. However over simplification of the model might lead to mispricing. Extending the existing literature on GMWB, Yang and Dai (2013) proposed a tree model to evaluate the values of GMWB with complex provision. The tree model allowed the flexibility to include deferred withdrawal, penalty for surrender, mortality risk, rollup interest rate guarantee as well as discrete withdrawal behavior in the structure. To verify the accuracy of the tree model, Monte Carlo simulation is used as the benchmark. From the numerical analysis and sensitivity analysis, they conclude that developing a pricing model that incorporates provision of GMWB contract is important for risk management. As this tree model employs a deterministic mortality model, it will be more capable to capture the mortality risk if stochastic mortality was being used. Furthermore, construct a tree model assuming stochastic interest rate will be more sensible especially for the pricing of a long term product.

Hyndman and Wenger (2014) considered the fair pricing of variable annuities with GMWB from financial economic perspective. As numerical method is used to solve for the insurance fees, it raises a theoretical question of existence and uniqueness of the insurance fee. In this paper, they have proven that the insurance fee exist and is unique. They further extend the work of Peng et al. (2012) to decompose the contract value when lapses are allowed. However their model only account for equity risk, interest rate risk is not considered in the model.

All of the paper mentioned assumed constant interest rate in their pricing model. As GMWB is a long term product, it will be more realistic to assume stochastic interest rate.

Benhamou and Gauthier (2009) considered both stochastic volatility and stochastic interest rate model in the pricing of GMB. Combining Heston model and Hull-White interest rate model in the pricing model, they found out that there is an impact of stochastic volatility and stochastic interest rate on the insurance fees charge by insurer. The impact is even more significant with longer maturity period.

Peng et al. (2012) considered the pricing of a GMWB under the Vasicek stochastic interest rate framework, assuming deterministic withdrawal rates. They use the lower and upper bound approximation method to estimate the option value. Furthermore, the sensitivity of the pricing model is examined using different parameter values. But the drawback is, Vasicek interest rate model has positive possibility of having negative interest rate, a situation that is not usually seen in a normal market condition. Thus, we believe that CIR interest rate model with zero probability of negative rate under certain restriction of its parameters should be a more realistic model. However, recently a number of major central banks in Europe have reduced the interest rate below zero in order to fight the growing trend of deflation. This has challenged the conventional economic theory saying that the rates cannot stay in negative territory, as investor will withdraw their cash, emptying the banks and crash the financial system. It has sparked a debate whether negative interest rate is a temporary phenomenon or a new normal.

A summary of literature review on GMWB is shown in table below:-

Paper	Withdrawal Assumption	Withdrawal Process	Underlying Asset	Interest Rate	Method
Milevsky and Salisbury (2006)	Static and dynamic	Continuous	GBM	Constant	PDE
Dai et al. (2008)	Dynamic	Continuous and discrete	GBM	Constant	PDE - singular stochastic control
Chen et al. (2008)	Dynamic	Continuous and discrete	GBM with jump diffusion	Constant	PDE
Bauer et al. (2008)	Static and dynamic	Discrete	GBM	Constant	Monte Carlo and generalized finite mesh discretization approach
Bacinello et al. (2011)	Static, dynamic and mixed	Discrete	Stochastic volatility Heston Model	Stochastic (CIR)	Least square Monte Carlo
Peng et al. (2012)	Static	Continuous	GBM	Stochastic (Vasicek)	Analytical approximation (lower bound and upper bound)

Paper	Withdrawal Assumption	Withdrawal Process	Underlying Asset	Interest Rate	Method
Yang and Dai (2013)	Static	Discrete	GBM	Constant	Tree model and Monte Carlo
Bacinello et al. (2013)	Dynamic	Discrete	Levy Process	Constant	Dynamic Programming
Huang and Kwok (2014)	Dynamic	Continuous	GBM	Constant	Analytical approximation
Hyndman and Wenger (2014)	Static	Continuous	GBM	Constant	Binomial model

CHAPTER 2

PRICING MODEL OF GMWB

A simple model with constant withdrawal rate and fixed total withdrawal amount is first considered for pricing GMWB. In addition, we assume constant interest rate and there are no lapses or mortalities throughout the period. The pricing formulation considered here follows the method introduced by Milevsky and Salisbury (2006). One of the advantages of using simple model is that it allows us to have a better visualization of the annuity problem. Thus, it can be served as a benchmark for future analysis. Then, in section 2.2, we will extend the pricing model to incorporate stochastic interest rate. As the interest rate in the long run tends to fluctuate, we believe that pricing of a long term product will be more realistic under stochastic interest rate.

2.1 Pricing under Constant Interest Rate

Pricing model with constant interest rate will be discussed in this section. From the policyholder's perspective, there are two cash inflows, that is periodical withdrawal and the balance in the subaccount at maturity. Let T be the maturity time and W_T be the subaccount value at maturity. To evaluate W_T we need to understand the dynamics of the subaccount. As GMWB allows policyholders to invest their initial capital in a selected fund, the value of the subaccount is affected by the performance of the underlying fund. We assume the value of underlying fund follows a Geometric Brownian motion which is defined by the following stochastic differential equation (SDE) under risk neutral measure:

$$dS_t = rS_t dt + \sigma_S S_t dB_t \tag{1}$$

where

S_t : Underlying asset value before deduction of proportionate insurance fees

r : Risk free interest rate

σ_S : Volatility of the underlying asset

B_t: Standard Brownian process

In our model we ignore lapses, partial withdrawals and mortalities to simplify the model as our main focus is on interest rate.

Under risk neutral valuation framework, the value of the subaccount is accumulated at the risk free rate of return, r and deducted for periodical withdrawal at a constant rate G and proportionate insurance fees, α. We assume policyholder's initial investment amount is w_0. We let $W_0 = w_0$ and $G = \frac{W_0}{T}$**.** Thus the dynamic of the subaccount value could be shown to satisfy the following SDE:-

$$dW_t = (r-\alpha)W_t dt - G dt + \sigma_S W_t dB_t \text{ for } W_t > 0 \tag{2}$$

This dynamic holds when W_t is positive. Once the subaccount value hits zero, it will remain at zero. This is the time the insurer has to fund the guaranteed withdrawal from its own account rather than deducted from the policyholder's subaccount.

Let τ_0 be the first time W_t hits zero. Then W_t satisfies

$$dW_t = (r-\alpha)W_t dt - Gdt + \sigma_S W_t dB_t,\ \ 0 \le t < \tau_0 \qquad (3)$$

$$W_t = 0 \qquad\qquad t \ge \tau_0$$

As W_t has an absorbing barrier at zero, we let $\widetilde{W}_t$be the unrestricted process of W_t, then the dynamic of $\widetilde{W}_t$ is:-

$$d\widetilde{W}_t = (r-\alpha)\widetilde{W}_t dt - Gdt + \sigma_S \widetilde{W}_t dB_t \qquad (4)$$

where W_t and $\widetilde{W}_t$ is linked by $W_t = \left(\widetilde{W}_t\right)^+$(Karatzas, 1991)

We find W_t by first solving SDE (4) for the unrestricted process $\widetilde{W}_t$. We use method of integrating factor to solve for $\widetilde{W_t}$. The integrating factor is

$$I(t) = e^{-\left(r-\alpha-\frac{1}{2}\sigma_S^2\right)t-\sigma_S B_t} \qquad (5)$$

Differentiate the integrating factor yields

$$dI(t) = [-(r-\alpha-\sigma_s{}^2)dt - \sigma_s dB_t]e^{-\left(r-\alpha-\frac{1}{2}\sigma_S^2\right)t-\sigma_S B_t}$$
$$= [-(r-\alpha-\sigma_s{}^2)dt - \sigma_s dB_t]I(t)$$

Rearrange (4) as

$$d\widetilde{W}_t - (r-\alpha)\widetilde{W}_t dt - \sigma_S \widetilde{W}_t dB_t = -Gdt \qquad (6)$$

Multiply both side of equation (6) by $I(t)$, we get

$$I(t)d\widetilde{W}_t - I(t)(r-\alpha)\widetilde{W}_t dt - I(t)\sigma_S \widetilde{W}_t dB_t = -I(t)Gdt$$

$$I(t)d\widetilde{W}_t - \widetilde{W}_t I(t)[(r-\alpha)dt + \sigma_S dB_t] = -I(t)Gdt$$

$$I(t)d\widetilde{W}_t + \widetilde{W}_t I(t)[-(r-\alpha-\sigma_s{}^2)dt - \sigma_S dB_t] - \sigma_s{}^2 \widetilde{W}_t I(t)dt = -I(t)Gdt$$

$$I(t)d\widetilde{W}_t + \widetilde{W}_t dI(t) + d\widetilde{W}_t dI(t) = -I(t)Gdt$$

$$d\big(I(t)\widetilde{W}_t\big) = -I(t)Gdt$$

Integrate both sides of the equation yields

$$I(t)\widetilde{W}_t - w_0 = -\int_0^t I(u)Gdu$$

Denote $I(t) = \frac{1}{X_t}$, then

$$\frac{\widetilde{W}_t}{X_t} - w_0 = -\int_0^t \frac{G}{X_u} du$$

$$\widetilde{W}_t = X_t \left(w_0 - \int_0^t \frac{G}{X_u} du \right) \tag{7}$$

where $X_t = e^{\left(r-\alpha-\frac{1}{2}\sigma_S^2\right)t + \sigma_S B_t}$

As W_t and $\widetilde{W}_t$ is linked by $W_t = \left(\widetilde{W}_t\right)^+$, hence

$$W_t = X_t \left(w_0 - \int_0^t \frac{G}{X_u} du \right)^+ \tag{8}$$

By purchasing a GMWB rider, the policyholder is eligible to receive periodical withdrawal and the remaining subaccount value at maturity if there is still a positive balance. Thus, from a policyholder’s perspective, the value of a variable annuity with GMWB rider at time t can be articulated mathematically as

$$
\begin{aligned}
V(W,r,t) &= E_Q\left[e^{-\int_t^T r ds} W_T \middle| \mathcal{F}_\mathrm{t} + \int_t^T e^{-\int_t^u r ds} G\, du\right] \\
&= E_Q\left[e^{-r(T-t)} W_T \middle| \mathcal{F}_\mathrm{t} + \int_t^T e^{-r(u-t)} G\, du\right] \qquad (9) \\
&= E_Q\left[e^{-r(T-t)} W_T \middle| \mathcal{F}_\mathrm{t}\right] + \frac{G}{r}\left[1 - e^{-r(T-t)}\right]
\end{aligned}
$$

Where $\mathcal{F}_\mathrm{t}$ is the filtration generated by the Brownian motion up to time t and E_Q denotes the expectation under risk neutral measure Q. In other words, the first term can be viewed as a present value of sub account value at maturity, while second term is present value of an annuity certain.

Present value of the cash flow of GMWB at time $t = 0$ is therefore:

$$V(W,r,0) = e^{-rT} E_Q[W_T] + \frac{G}{r}[1 - e^{-rT}] \qquad (10)$$

By substituting W_T into the equation, we can view the value of a variable annuity with GMWB rider as an Asian option added to a term-certain annuity,

$$V(W,r,0) = e^{-rT} E_Q\left[X_T\left(w_0 - \int_0^T \frac{G}{X_u} du\right)^+\right] + \frac{G}{r}[1 - e^{-rT}] \qquad (11)$$

The difference between a GMWB rider and most of the normal insurance products is that the insurance fee is charged based on the asset value instead of an upfront charge. The insurance fee is determined based on equivalence principle. According to the equivalence principle, the net premiums are chosen so that the actuarial present value of the benefits equals the actuarial present value of the net premiums. To fairly price a GMWB, proportional fee is determined so that the amount invested by the policyholder at inception, w_0 is equal to the present value of the total amount of money being received. Therefore,

$$
\begin{aligned}
w_0 &= V(W, r, 0) \\
&= e^{-rT} E_Q[W_T] + \frac{G}{r}[1 - e^{-rT}]
\end{aligned}
\tag{12}
$$

The proportionate insurance fee, α can be determined by solving the equation above. Thus in order to determine the value of $V(W, r, 0)$, we separate equation (12) into two parts. We will first use Monte Carlo simulation to approximate the value of the first term of equation (12) and sum it up with the annuity certain, $\frac{G}{r}[1 - e^{-rT}]$. The insurance fee, α is charged as a percentage of the sub account value. Typically a GMWB carries a fee range from 35 to 75 basis points.

Simulation algorithm for GMWB model under constant interest rate

```
for i=1:PathNum
      At=0;
      Bt=0;
      ## To simulate subaccount value using Euler method
      for j=1:N
            t=(j-1)*dt;
            Bt=Bt+sqrt(dt)*randn;
             At=At+G*exp((alpha-r-
      0.5*sigmas^2)*t+sigmas*Bt)*dt;
      end;

      W(i)=max(0,1-At);
end;
## To find the expected value
M=exp(-alpha*T)*mean(W)
```

2.2 Pricing Under Stochastic Interest Rate

After presenting GMWB model with constant interest rate, we now incorporate stochastic interest rate model, namely, Vasicek model of Vasicek (1977) and CIR model (Cox et al., 1985) into the GMWB model.

Vasicek interest model can be categorized as a Gaussian interest rate model, in which the interest rate is normally distributed. It is commonly used because of its mean reverting property and its simplicity. However, the production of negative rate is a major drawback for this model. In real market, negative interest rate rarely occurred.

While CIR model shares the same mean reverting property as Vasicek model, it preserves the positivity of interest rate. However, derivative pricing under CIR model is much more complex as the model involves square root process which has a non-central Chi Square distribution.

2.2.1 Model Formulation

We assume the same set of pricing assumptions as stated in the constant interest rate pricing model. As we incorporate stochastic interest rate into the model, the underlying asset is assumed to satisfy the following SDE under risk neutral measure Q

$$dS_t = r_t S_t dt + \sqrt{1-\rho^2}\sigma_S S_t dB_{1,t} + \rho\sigma_S S_t dB_{2,t} \quad (13)$$

Under Vasicek Model, the stochastic interest rate r_t is governed by the following SDE:

$$dr_t = k(\theta - r_t)dt + \sigma_r dB_{2,t} \quad (14)$$

While under CIR Model, the rate r_t satisfies:

$$dr_t = k(\theta - r_t)dt + \sigma_r \sqrt{r_t} dB_{2,t} \quad (15)$$

where

S_t	: Underlying asset value before deduction of proportional insurance fees.
$B_{1,t}, B_{2,t}$	: Independent standard Q-Brownian process.
ρ	: Correlation coefficient between S_t and r_t.
k	: Speed of conversion to long term interest rate
θ	: Long term interest rate
σ_S	: Volatility of S_t
σ_r	: Volatility of r_t

It can be proven that (Cairns, 2004) under Vasicek interest rate model, the price of a zero coupon bond at time t with maturity $T \geq t$ is given by

$$D(t,T) = a(t,T)e^{-b(t,T)r_t} \quad (16)$$

where

$$b(t,T) = \frac{1}{k}[1 - e^{-k(T-t)}]$$

$$a(t,T) = \exp\left(\left(\theta - \frac{\sigma_r^2}{2k^2}\right)[b(t,T) - (T-t)] - \frac{\sigma_r^2}{4k}b(t,T)^2\right)$$

While bond price $D(t,T)$ for CIR interest rate model is

$$D(t,T) = a(t,T)e^{-b(t,T)r_t} \tag{17}$$

where

$$b(t,T) = \frac{2(\exp\{(T-t)h\} - 1)}{2h + (k+h)(\exp\{(T-t)h\} - 1)}$$

$$a(t,T) = \left[\frac{2h\exp\{(k+h)(T-t)/2\}}{2h + (k+h)(\exp\{(T-t)h\} - 1)}\right]^{2k\theta/\sigma_r^2}$$

$$h = \sqrt{k^2 + 2\sigma^2}$$

The dynamic of the bond price $D(t,T)$, satisfies the SDE below:-

$$dD(t,T) = D(t,T)[r_t dt + \sigma_D(t,T)dB_t] \tag{18}$$

where $\sigma_D(t,T)$ is the volatility term. It can be shown that the volatility term of Vasicek model is $-\sigma_r b(t,T)$. Thus, Vasicek model has a deterministic volatility term, while the volatility term of CIR model is $\sigma_r\sqrt{r_t}b(t,T)$, which is a stochastic variable.

As we introduce stochastic interest rate model into the pricing model, the dynamic of subaccount value follows:-

$$dW_t = [(r_t - \alpha)W_t - G]dt + W_t \boldsymbol{\sigma}_S d\boldsymbol{B}_t \tag{19}$$

$\boldsymbol{\sigma}_S \equiv (\sqrt{1-\rho^2}\sigma_s \quad \rho\sigma_s)$

$\boldsymbol{B}_t \equiv \begin{pmatrix} B_{1,t} \\ B_{2,t} \end{pmatrix}$

The equation above holds for $W_t > 0$, that is when $t < \tau_0$, where τ_0 is the first time W_t hits zero. Once the subaccount value hits zero, it is considered ruined. Hence, it will continue to remain zero, that is $W_t = 0$ for $t \geq \tau_0$. As W_t has an absorbing barrier at zero, we let $\widetilde{W}_t$ be the unrestricted process of W_t, then the dynamic of $\widetilde{W}_t$ is:-

$$d\widetilde{W}_t = \left[(r_t - \alpha)\widetilde{W}_t - G\right]dt + \widetilde{W}_t \boldsymbol{\sigma}_S d\boldsymbol{B}_t \tag{20}$$

We solve the SDE using method of integrating factor as shown in previous section with $I(t) = \exp\left[-\int_0^t \left(r_u - \alpha - \frac{1}{2}\boldsymbol{\sigma}_S\boldsymbol{\sigma}_S^T\right)du - \int_0^t \boldsymbol{\sigma}_S\, d\boldsymbol{B}_u\right]$.

By differentiating the integrating factor we have

$$\begin{aligned} dI(t) = \exp&\left[-\int_0^t \left(r_u - \alpha - \frac{1}{2}\boldsymbol{\sigma}_S\boldsymbol{\sigma}_S^T\right)du \right. \\ &\left. - \int_0^t \boldsymbol{\sigma}_S\, d\boldsymbol{B}_u\right][-(r_t - \alpha - \boldsymbol{\sigma}_S\boldsymbol{\sigma}_S^T)dt - \boldsymbol{\sigma}_s dB_t] \\ = -[(r_t - \alpha &- \boldsymbol{\sigma}_S\boldsymbol{\sigma}_S^T)dt + \boldsymbol{\sigma}_s d\boldsymbol{B}_t]I(t) \end{aligned}$$

Rearrange equation (20) as

$$d\widetilde{W}_t - (r_t - \alpha)\widetilde{W}_t dt - \widetilde{W}_t \boldsymbol{\sigma}_S d\boldsymbol{B}_t = -Gdt \tag{21}$$

Multiply both side of equation (21) by $I(t)$, we get

$$I(t)d\widetilde{W}_t - I(t)(r_t-\alpha)\widetilde{W}_t dt - I(t)\boldsymbol{\sigma}_S\widetilde{W}_t d\boldsymbol{B}_t = -I(t)Gdt$$

$$I(t)d\widetilde{W}_t - \widetilde{W}_t I(t)[(r_t-\alpha)dt + \boldsymbol{\sigma}_S d\boldsymbol{B}_t] = -I(t)Gdt$$

$$I(t)d\widetilde{W}_t + \widetilde{W}_t I(t)[-(r_t-\alpha-\boldsymbol{\sigma}_S\boldsymbol{\sigma}_S^T)dt - \boldsymbol{\sigma}_S d\boldsymbol{B}_t] - \boldsymbol{\sigma}_S\boldsymbol{\sigma}_S^T\widetilde{W}_t I(t)dt$$
$$= -I(t)Gdt$$

$$I(t)d\widetilde{W}_t + \widetilde{W}_t dI(t) + d\widetilde{W}_t dI(t) = -I(t)Gdt$$

$$d\left(I(t)\widetilde{W}_t\right) = -I(t)Gdt$$

Integrate both sides of the equation yields

$$I(t)\widetilde{W}_t - w_0 = -\int_0^t I(u)Gdu$$

Substitute $I(t) = \frac{1}{X_t}$

$$\frac{\widetilde{W}_t}{X_t} - w_0 = -\int_0^t \frac{G}{X_u}du$$

$$\widetilde{W}_t = X_t\left(w_0 - \int_0^t \frac{G}{X_u}du\right)$$

Where $X_t = \exp\left(\int_0^t\left(r_u - \alpha - \frac{1}{2}\|\boldsymbol{\sigma}_S\|^2\right)du + \int_0^t \boldsymbol{\sigma}_S\, d\boldsymbol{B}_u\right)$

As W_t and $\widetilde{W}_t$ is linked by $W_t = \left(\widetilde{W}_t\right)^+$, hence

$$W_t = X_t\left(w_0 - \int_0^t \frac{G}{X_u}du\right)^+ \tag{22}$$

Therefore by considering equity and interest risk, the fair value of a variable annuity with GMWB at time t can be shown as

$$V(W,r,t) = E_Q\left[\int_t^T e^{-\int_t^u r_s ds} G du + e^{-\int_t^T r_s ds} W_T \middle| \mathcal{F}_t\right] \tag{23}$$

where $\mathcal{F}_t$ is the filtration generated by the stochastic process and $E_Q[\cdot]$ is the expectation under risk measure Q.

At time $t = 0$ this value is reduced to

$$V(W,r,0) = E_Q\left[\int_0^T e^{-\int_0^u r_s ds} G du + e^{-\int_0^T r_s ds} W_T\right] \tag{24}$$

Note that $e^{-\int_t^T r_s ds}$ is a stochastic discount factor, hence the value of the first term in the equation above can be expressed as $\int_0^T D(0,u) G\, du$ where $D(0,u)$ is the bond price at time 0, with maturity at time u. Hence, the value of GMWB at 0 can be written as:-

$$V(W,r,0) = \int_0^T D(0,u) G\, du + E_Q\left[e^{-\int_0^T r_s ds} W_T\right] \tag{25}$$

Evaluation of the option value part or the second term of the value of GMWB is complex as it is an expectation of a product of three correlated random variable, X_t, r_t and $\int_0^t \frac{G}{X_u} du$. The calculation would become easier if we perform a change of measure from risk neutral measure. Thus, we will apply the technique of change of numéraire to simplify the calculation of the expectation.

Recall that in term of $\widetilde{W}_T$, the value of GMWB at time $t = 0$ is

$$V(W,r,0) = \int_0^T D(0,u) G\, du + E_Q\left[e^{-\int_0^T r_s ds} \left(\widetilde{W}_T\right)^+\right]$$

$$= \int_0^T D(0,u)G\,du + w_0 E_Q\left[e^{-\int_0^T r_s ds} X_T\left(1 - \int_0^T \frac{G}{w_0 X_u} du\right)^+\right]$$

Define $A_t = \int_0^t \frac{G}{w_0 X_u} du$, then

$$V(W, r, 0) = \int_0^T D(0,u)G_u\,du + w_0 E_Q\left[e^{-\int_0^T r_s ds} X_T (1 - A_T)^+\right] \tag{26}$$

Evaluation of the second term of GMWB is then:

$$E_Q\left[e^{-\int_0^T r_s ds} X_T (1 - A_T)^+\right]$$

$$= \int_\Omega e^{-\int_0^T r_s ds} X_T (1 - A_T)^+\, dQ$$

$$= \int_\Omega e^{-\int_0^T r_s ds} X_T (1 - A_T)^+ \frac{dQ}{dQ_S} dQ_S$$

$$= \int_\Omega e^{-\int_0^T r_s ds} \frac{S_T}{S_0} e^{-\alpha T} (1 - A_T)^+ \frac{dQ}{dQ_S} dQ_S$$

where Q_S is the measure we wish to choose so that the integral become tractable. Hence, let $e^{-\int_0^T r_s ds} \frac{S_T}{S_0} \frac{dQ}{dQ_S} = 1$, or $\frac{dQ_S}{dQ} = \frac{e^{-\int_0^T r_s ds} S_T}{S_0} = \frac{e^{-\int_0^T r_s ds} S_T}{S_0}$, thus we have

$$\int_\Omega e^{-\int_0^T r_s ds} \frac{S_T}{S_0} e^{-\alpha T} (1 - A_T)^+ \frac{dQ}{dQ_S} dQ_S = \int_\Omega e^{-\alpha T} (1 - A_T)^+\, dQ_S$$

$$= e^{-\alpha T} E_{Q_S}[(1 - A_T)^+].$$

It can be shown that $Q_S(A) = \frac{1}{S_0} \int_A D(T) S_T\, dQ$ is a measure on Ω, the set of all possible outcome and $\frac{dQ_S}{dQ}$ is the Radon Nikodym derivative of Q_S with respect to Q.

Therefore if we know the distribution law of $\frac{1}{X_u}$ under Q_S probability measure, we can evaluate

$$e^{-\alpha T} E_{Q_S}[(1-A_T)^+] = e^{-\alpha T} E_{Q_S}\left[\left(1-\int_0^T \frac{G}{w_0 X_u} du\right)^+\right]$$

To determine the distribution law of $\frac{1}{X_u}$, recalled that SDE of the underlying asset under Q_S measure is

$$dS_t = r_t S_t dt + \sqrt{1-\rho^2}\sigma_S S_t dB_{1,t} + \rho\sigma_S S_t dB_{2,t} \quad \text{or}$$

$$dS_t = S_t[r_t dt + \boldsymbol{\sigma}_S d\boldsymbol{B}_t],$$

Where

$\boldsymbol{\sigma}_S \equiv (\sqrt{1-\rho^2}\sigma_s \qquad \rho\sigma_s)$

$\boldsymbol{B}_t \equiv \begin{pmatrix} B_{1,t} \\ B_{2,t} \end{pmatrix}$

Solving the SDE yield

$$S_t = S_0 \exp\left(\int_0^t \left(r_u - \frac{1}{2}\|\boldsymbol{\sigma}_S\|^2\right) du + \int_0^t \boldsymbol{\sigma}_S d\boldsymbol{B}_u\right) \tag{27}$$

Suppose $D(t,T)$ and S_t are denominated in the same currency. If we take the asset price S_t as the numéraire, in terms of this numéraire, the asset price is equal to 1. The measure defined by this numéraire is given by

$Q_S(A) = \frac{1}{S_0}\int_A D(T)S_T \, dQ$ for all $A \in \mathcal{F}$, or

$\left.\frac{dQ_S}{dQ}\right| \mathcal{F}(T) = \frac{1}{S_0} D(T) S_T$

The price of $D(t,T)$ denominated in units of shares of S_t is $\frac{D(t,T)}{S_t}$. Differentiate by using Ito's lemma find that

$$d\left(\frac{D(t,T)}{S_t}\right) = dD(t,T)\left(\frac{1}{S_t}\right) + d\left(\frac{1}{S_t}\right)D(t,T) + dD(t,T)d\left(\frac{1}{S_t}\right)$$

$$= \frac{D(t,T)}{S_t}[r_t dt + \boldsymbol{\sigma}_D d\boldsymbol{B}_t] - \frac{D(t,T)}{S_t}(r_t dt + \boldsymbol{\sigma}_S d\boldsymbol{B}_t - \boldsymbol{\sigma}_S^2 dt)$$

$$+ D(t,T)(r_t dt + \boldsymbol{\sigma}_D d\boldsymbol{B}_t)\left(-\frac{1}{S_t}\right)(r_t dt + \boldsymbol{\sigma}_S d\boldsymbol{B}_t - \boldsymbol{\sigma}_S^2 dt)$$

$$= \frac{D(t,T)}{S_t}[\boldsymbol{\sigma}_D d\boldsymbol{B_t} - \boldsymbol{\sigma}_S d\boldsymbol{B}_t + \boldsymbol{\sigma}_S^2 dt - \boldsymbol{\sigma}_D\boldsymbol{\sigma}_S dt]$$

$$= \frac{D(t,T)}{S_t}[(\boldsymbol{\sigma}_D - \boldsymbol{\sigma}_S)(-\boldsymbol{\sigma}_S dt + d\boldsymbol{B}_t)]$$

$$\equiv \frac{D(t,T)}{S_t}\left[(\boldsymbol{\sigma}_D - \boldsymbol{\sigma}_S)d\boldsymbol{B}_t^{Qs}\right] \tag{28}$$

By Girsanov theorem, $\boldsymbol{B}_t^{Qs} = \boldsymbol{B}_t - \int_0^t \boldsymbol{\sigma}_S\, ds$ is a 2-dimensional Brownian motion under Q_S measure which is equivalent to the Q measure. We can see that the process $\frac{D(t,T)}{S(t)}$ is a Q_S-martingale as there is no drift term in the equation and $\boldsymbol{B}_t$ and $\boldsymbol{B}_t^{Qs}$ are related by $d\boldsymbol{B}_t^{Qs} = d\boldsymbol{B}_t - \boldsymbol{\sigma}_S$.

Replace t by u, and T by t,

$$d\left(\frac{D(u,t)}{S_u}\right) = \frac{D(u,t)}{S_u}\left[(\boldsymbol{\sigma}_D(u,t) - \boldsymbol{\sigma}_S)\cdot d\boldsymbol{B}_u^{Qs}\right]$$

$$= \frac{D(u,t)}{S_u}\boldsymbol{\sigma}_{Q_S}(u,t)\cdot d\boldsymbol{B}_u^{Qs}, \text{for } u \le t$$

$$d\log(D(u,t)/S_u) = -\frac{1}{2}\left\|\boldsymbol{\sigma}_{Q_S}(u,t)\right\|^2 du + \boldsymbol{\sigma}_{Q_S}(u,t)\cdot d\boldsymbol{B}_u^{Qs}$$

Integrate with respect to u from 0 to t and observe that $D(t,t) = 1$, we have

$$\frac{D(t,t)}{S_t} = \frac{D(0,t)}{S_0}\exp\left(-\frac{1}{2}\int_0^t \left\|\boldsymbol{\sigma}_{Q_S}(u,t)\right\|^2 du + \int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\cdot d\boldsymbol{B}_u^{Qs}\right)$$

$$\frac{1}{S_t} = \frac{D(0,t)}{S_0}\exp\left(-\frac{1}{2}\int_0^t \left\|\boldsymbol{\sigma}_{Q_S}(u,t)\right\|^2 du + \int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\cdot d\boldsymbol{B}_u^{Qs}\right)$$

$$\frac{S_0}{S_t} = D(0,t)\exp\left(-\frac{1}{2}\int_0^t \left\|\boldsymbol{\sigma}_{Q_S}(u,t)\right\|^2 du + \int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\cdot d\boldsymbol{B}_u^{Qs}\right) \quad (29)$$

where

$\boldsymbol{\sigma}_{Q_S} = \boldsymbol{\sigma}_D - \boldsymbol{\sigma}_S$

Note that $X_t = e^{-\alpha t}\frac{S_t}{S_0}$. Hence the distribution law of $\frac{1}{X_t}$ under probability measure Q_S is,

$$\frac{1}{X_t} = e^{\alpha t}\frac{S_0}{S_t} = e^{\alpha t}D(0,t)\exp\left(-\frac{1}{2}\int_0^t \left\|\boldsymbol{\sigma}_{Q_S}(u,t)\right\|^2 du + \int_0^t \boldsymbol{\sigma}_{Q_S}(u,t) d\boldsymbol{B}_u^{Qs}\right)$$

Replace u by s and then t by u to obtain

$$\frac{1}{X_u} = e^{\alpha u}D(0,u)\exp\left(-\frac{1}{2}\int_0^u \left\|\boldsymbol{\sigma}_{Q_S}(s,u)\right\|^2 ds + \int_0^u \boldsymbol{\sigma}_{Q_S}(s,u) d\boldsymbol{B}_s^{Qs}\right) \quad (30)$$

Observe from the equation above that the computational works can be simplified if $\boldsymbol{\sigma}_{Q_S}$ is a deterministic function. As $\boldsymbol{\sigma}_S$ is a deterministic function, $\boldsymbol{\sigma}_{Q_S}$ will be a deterministic function if $\boldsymbol{\sigma}_D$ is a deterministic function. By changing from risk neutral measure Q to a new measure Qs, the evaluation of option value can be simplified if the volatility term of bond price, $\boldsymbol{\sigma}_D$ is deterministic. Therefore change of numéraire technique can reduce the calculation work for the pricing model under Vasicek interest rate model but it is not the same for CIR interest rate model which has a stochastic volatility term. Thus under Gaussian interest rate model where the volatility term is deterministic, the value of variable annuity with GMWB is simplified to

$$
\begin{aligned}
&V(W,r,0) \\
&= \int_0^T D(0,u)G du + E_Q\left[e^{-\int_0^T r_s ds} W_T\right] \\
&= \int_0^T D(0,u)G du + e^{-\alpha T} E_{Q_S}\left[\left(1 - \int_0^T \frac{G}{w_0 X_u} du\right)^+\right] \\
&= \int_0^T D(0,u)G du \; + e^{-\alpha T} E_{Q_S}\Bigg[\Bigg(1 \\
&\qquad - \frac{G}{w_0}\int_0^T e^{\alpha t} D(0,t) \exp\left(-\frac{1}{2}\int_0^t \left\|\boldsymbol{\sigma}_{Q_S}(u,t)\right\|^2 du \right. \qquad (31) \\
&\qquad \left. + \int_0^t \boldsymbol{\sigma}_{Q_S}(u,t) d\boldsymbol{B}_u^{Q_S}\right) dt\Bigg)^+\Bigg]
\end{aligned}
$$

In order to perform simulation to approximate the option value we will need to evaluate $D(0,t)$, $\int_0^t \left\|\boldsymbol{\sigma}_{Q_S}(u,t)\right\|^2 du$ and $\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t) d\boldsymbol{B}_u^{Q_S}$.

As there are closed form solutions for bond price under both Vasicek and CIR interest rate model, $D(0,t)$ is a deterministic function that readily available. As for $\int_0^t \left\|\boldsymbol{\sigma}_{Q_S}(u,t)\right\|^2 du$, under Vasicek interest rate model it is

$$
\begin{aligned}
&\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\boldsymbol{\sigma}_{Q_S}(u,t)^T du \\
&= \int_0^t \left(-\sqrt{1-\rho^2}\sigma_S \qquad -(\sigma_r b(u,t) + \rho\sigma_S)\right) \\
&\qquad \times \left(-\sqrt{1-\rho^2}\sigma_S \qquad -(\sigma_r b(u,t) + \rho\sigma_S)\right)^T du
\end{aligned}
$$

$$= \int_0^t \sigma_S^2 + \sigma_r^2 b^2(u,t) + 2\sigma_r\sigma_S\rho b(u,t)\, du$$

$b(t,T) = \frac{1}{k}[1 - e^{-k(T-t)}]$

$\sigma_{Q_S}(t,T) = \sigma_S^2 + \sigma_r^2 b^2(t,T) + 2\rho\sigma_r\sigma_s b(t,T)$

$\sigma_{Q_S}(u,t) = \sigma_S^2 + \sigma_r^2 b^2(u,t) + 2\rho\sigma_r\sigma_s b(u,t)$

For the stochastic integral $\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t) d\boldsymbol{B}_u^{Qs}$, since $\boldsymbol{\sigma}_{Q_S}$ is a deterministic function, it has a centered Gaussian distribution. So it has mean zero and variance

$$\mathrm{Var}\left[\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t) d\boldsymbol{B}_u^{Qs}\right]$$

$$= E\left[\left(\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t) d\boldsymbol{B}_u^{Qs}\right)^2\right] - \left(E\left[\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t) d\boldsymbol{B}_u^{Qs}\right]\right)^2$$

$$= E\left[\left(\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t) d\boldsymbol{B}_u^{Qs}\right)^2\right]$$

By Ito Isometry,

$$E\left[\left(\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t) d\boldsymbol{B}_u^{Qs}\right)^2\right] = \int_0^t \left\|\boldsymbol{\sigma}_{Q_S}(u,t)\right\|^2 du$$

$$= \int_0^t \sigma_S^2 + \sigma_r^2 b^2(u,t) + 2\sigma_r\sigma_S\rho b(u,t) du$$

Let $f(t) = \int_0^t \sigma_S^2 + \sigma_r^2 b^2(u,t) + 2\sigma_r\sigma_S\rho b(u,t) du$

Hence, $\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t) d\boldsymbol{B}_u^{Qs}$ has a centered Gaussian distribution with variance $f(t)$.

$$\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)d\boldsymbol{B}_u^{Qs} \sim N\big(0, f(t)\big) \sim \sqrt{f(t)}N(0,1)$$

Thus, simulation can be performed based on this information.

Algorithm to simulate $e^{-\alpha T}E_{Q_S}[W_T]$ under Vasicek model:-

```
% Define the variable

r0=0.03025;       % initial interest rate
sigmas=0.1280;    % $\sigma_S$, volatility of stock market
T=10;             % Time to maturity
alpha=0.006;      % $\alpha$, the insurance fee
w0=1;             % subaccount value at t=0
G=w0/T;           % periodical withdrawal rate
rho=-0.05;        % $\rho$ , correlation coefficient between
                  equity and interest rate
theta=0.0291;     % $\theta$, long term interest rate
k=0.2718;         % $k$ , Speed of conversion to long term
sigmar=0.0036;    % $\sigma_r$,interest rate volatility
N=10000;          % No of partition
PathNum=100000;   % No of simulation
dt=T/N;           % considering discretized process

% Repeat the simulation to find the expected value
For i=1:PathNum
      At=0;
      Bt=0;
      % Simulate the subaccount value using Euler method
      for j=1:N
            t=(j-1)*dt;
            Bt=Bt+sqrt(F1_Func(sigmas,k,sigmar,rho,dt))*ra
            ndn;
            At=At+G*D(r0,theta,k,sigmar,0,t)*exp(alpha*t-
            1/2*F1_Func(sigmas,k,sigmar,rho,t)+Bt)*dt;
      end;
      % Bt for $\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)d\boldsymbol{B}_u^{Qs}$
      % At for $\frac{G}{w_0}\int_0^T e^{\alpha t}D(0,t)\exp\left(-\frac{1}{2}\int_0^t \left\|\boldsymbol{\sigma}_{Q_S}(u,t)\right\|^2 du +$
      $\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)d\boldsymbol{B}_u^{Qs}\right) dt$
      W(i)=max ((1-At),0);
end;

% To find the expected value
y= exp(-alpha*T)*mean(W);
```

Listing 1

Listing 1 is used to perform the simulation of the option value, $e^{-\alpha T}E_{Q_S}[W_T]$. The random number generator, `randn` is used. Thus, each call to `randn` will produce a random number generated from the standard normal distribution. `F1_Func()` is a function created for $\int_0^t \sigma_S^2 + \sigma_r^2 b^2(u,t) + 2\sigma_r\sigma_S\rho b(u,t)du$ while `D()`is for $a(t,s)e^{-b(t,s)r_t}$. As for Vasicek model, $b(t,s) = \frac{1}{k}[1 - e^{-k(s-t)}]$ and $a(t,s) = \exp\left(\left(\theta - \frac{\sigma_r^2}{2k^2}\right)[b(t,s) - (s-t)] - \frac{\sigma_r^2}{4k}b(t,s)^2\right)$. The function for `F1_Func()` and `D()` is shown in Listing 2 and Listing 3.

`F1_Func(sigmas,k,sigmar,rho,dt)`

```
y =sigmas.^2.*t + (sigmar.^2.*(t - (1./(2.*exp(2.*k.*t))
- 2./exp(k.*t) + 3./2)./k))./k.^2 +
(2.*rho.*sigmar.*sigmas.*(1./exp(k.*t) - 1))./k.^2 +
(2.*rho.*sigmar.*sigmas.*t)./k;
```

Listing 2

`D(r0,theta,k,sigmar,0,t)`

```
bts=(1-exp(-k.*(s-t)))./k;
ats=exp((theta-sigmar.^2./(2.*k.^2)).*(bts-s+t)-
sigmar.^2.*bts.*bts./(4.*k));

v=ats.*exp(-bts.*r0);
```

Listing 3

2.2.2 Simulation of GMWB under CIR Model

As CIR interest rate model has a non-central Chi square distribution, the change of numéraire technique does not help to reduce the computational work. Thus, to approximate the price of GMWB under CIR interest rate model, we will only consider Monte Carlo simulation under probability measure Q.

Under probability measure Q, the value of a GMWB is

$$V(W, r, 0) = \int_0^T D(0, u) G \, du + E_Q \left[e^{-\int_0^T r_s ds} \left(\widetilde{W}_T \right)^+ \right]$$

Thus, we will approximate the value of $E_Q \left[e^{-\int_0^T r_s ds} \left(\widetilde{W}_T \right)^+ \right]$ based on Monte Carlo simulation. We divide the interval [0 T] into N subintervals, with each time step $\Delta t = T/N$ and approximate the value $\widetilde{W}_T$ based on the discretized equation

$$\widetilde{W}_{t+\Delta t} = \widetilde{W}_t + \widetilde{W}_t \left[(r_t - \alpha)\Delta t + \sqrt{(1-\rho^2)\Delta t}\,\sigma_S z_t + \rho \sigma_S \sqrt{\Delta t}\,\varepsilon_t \right] - G\Delta t \quad (32)$$

Where $z_t \sim N(0,1)$ and $\varepsilon_t \sim N(0,1)$, a standard normal distribution. Thus we calculate $\widetilde{W}_{\Delta t}$ from $\widetilde{W}_0$, where $\widetilde{W}_0$ is the initial subaccount value which is denoted as w_0. $\widetilde{W}_{2\Delta t}$ is calculated from $\widetilde{W}_{\Delta t}$ and so on until we reach $\widetilde{W}_{N\Delta t} = \widetilde{W}_T$.

As the simulation of CIR model involves square root process, it raises some problems in generating the Monte Carlo paths on discrete timeline. The classical discretization is using Euler Method, where the discretization process could be written as

$$r_{t+\Delta t} = r_t + k(\theta - r_t)\Delta t + \sigma\sqrt{r_t \Delta t}\, \varepsilon_t$$

But we notice that under this method, the discretization process for r_t can become negative when the last term is comparatively a large negative number. For instance, when ε_t is a large negative number, then the last term may become a large negative number, hence r_t can become negative. If r_t can become negative, it is impossible for us to compute $\sqrt{r_t}$ that make the simulation fail. The problem can be fixed by replacing r_t with $(r_t)^+$ which mean whenever r_t fall negative, it will be considered as 0.

$$r_{t+\Delta t} = r_t + k(\theta - (r_t)^+)\Delta t + \sigma\sqrt{(r_t)^+ \Delta t}\, \varepsilon_t$$

Another drawback of this method is that it has first order weak convergence which can be further improved by Milstein approximation. The Milstein approximation increases the accuracy of discretization by applying Ito-Taylor expansion. This is a higher order scheme which adds an additional correction term to Euler Approximation where the discretization process is as follow:

$$r_{t+\Delta t} = r_t + k(\theta - r_t)\Delta t + \sigma\sqrt{r_t \Delta t}\varepsilon_t + \frac{1}{4}\sigma^2 \Delta t({\varepsilon_t}^2 - 1)$$

Although this method can improve the accuracy of discretization, it failed to preserve the positivity of r_t. Therefore it cannot be used without suitable modification. As discussed by Andersen *et al*, one of the Milstein-type approximations that control the probability of getting negative value is implicit Milstein scheme which is defined as:-

$$r_{t+\Delta t} = \frac{r_t + k\theta\Delta t + \sigma\sqrt{r_t\Delta t}\varepsilon_t + \frac{1}{4}\sigma^2\Delta t({\varepsilon_t}^2 - 1)}{1 + k\Delta}$$

This discretization will ensure positive paths for r_t if $4k\theta > \sigma^2$. Thus in our simulation, we will use implicit Milstein scheme for the cases where $4k\theta > \sigma^2$ and Euler discretization with adjustment when the bound is not hold to prevent the problem of generating negative r_t.

Simulation of option value under CIR interest rate

```
% Initialize the variable
r0=0.03025;           % initial interest rate
sigmas=0.1280;        % σ_S, volatility of stock market
T=10;                 % Time to maturity
alpha=0.006;          % α, the insurance fee
w0=1;                 % subaccount value at t=0
G=w0/T;               % periodical withdrawal rate
rho=-0.05;            %ρ , correlation coefficient between
                      equity and interest rate
theta=0.0291;         % θ, long term interest rate
k=0.2718;             % k , Speed of conversion to long
sigmar=0.0036;        % σ_r,interest rate volatility
N=10000;              % No of partition
PathNum=100000;       % No of simulation
randn('seed');        % Generate random number from
                      standard normal distribution
r(N,PathNum)=0;       % Preallocate arrays for efficiency
normV(N)=0;
exprt(PathNum)=0;
Payoff(PathNum)=0;
Annuity(PathNum)=0;
SubAcc(PathNum)=0;
w(1)=w0;
GSum=0;
GSumDelta=G*dt;
dt=T/N;               % considering discretized process
```

```
% Repeat the process to find the mean
for path=1:PathNum
r(1,path)=r0;
w(N)=0;w1(N)=0;
tmp = r(1,path);
GSum=GSum+exp(-tmp*dt)*GSumDelta;

% Simulate the interest rate under CIR model based on
equation 33
if 4*k*theta > sigmar^2
      for j=2:N
      normV(j)=randn;
      r(j,path)=(r(j-1,path)+k*theta*dt+sigmar*sqrt(r(j-
      1,path))*normV(j)*sqrt(dt)+0.25*sigmar^2*dt*(normV(
      j)^2-1))/(1+k*dt);
      tmp=tmp+r(j,path);
      GSum=GSum+exp(-tmp*dt)*GSumDelta;
      end;
else
      for j=2:N
      normV(j)=randn;
      if r(j-1,path)<0
            r(j,path)=r(j-1,path)+k*(theta-max(r(j-
            1,path),0))*dt+sigmar*sqrt(max(r(j-
            1,path),0))*sqrt(dt)*normV(j);
      else
            r(j,path)=(r(j-1,path)+k*theta*dt
      +sigmar*sqrt(r(j-1,path))*normV(j)*sqrt(dt)
      +0.25*sigmar ^2*dt*(normV(j) ^ 2- 1))/(1+k*dt);
      end
      tmp=tmp+r(j,path);
      GSum=GSum+exp(-tmp*dt)*GSumDelta;
      end;
end;

% Simulate the subaccount value using discretization
method based on equation 32
for j=2:N
      w(j)=w(j-1)*(1+r(j,path)-alpha)*dt+sqrt(1-
ro^2)*sigmas*sqrt(dt)*randn+ro*sigmas*sqrt(dt)*normV(j))-
G*dt;
end;

% To get the present value of the subaccount value
Payoff(path)=exp(-tmp*dt)*w(N);
end;

% To get the mean of present value of subaccount value
M=mean(Payoff);
```

CHAPTER 3

LOWER BOUND UNDER VASICEK MODEL

As shown in previous chapter, the pricing of an annuity with GMWB can be shown as an option plus an annuity certain. The option term $E_{Q_S}[(1-A_T)^+]$ can be evaluated as an Asian option. Although there is no closed form formula for pricing an Asian option, its analytic approximation does exist. Following the method introduced by Rogers and Shi (1995), Peng et al. (2012) deduced a lower bound for pricing of GMWB. It turns out that, this lower bound itself is a good approximation for Asian Option pricing. In this chapter, we compute the lower bound by modifying slightly the method proposed by Peng et al. (2012).

The lower bound is based on the concept of conditional expectation. As x^+ is a convex function, by Jensen Inequality,

$$\begin{aligned}
&E_{Q_S}[(1-A_T)^+] \\
&= E_{Q_S}\left[E_{Q_S}[(1-A_T)^+|Z]\right] \\
&\geq E_{Q_S}\left[\left(E_{Q_S}[1-A_T|Z]\right)^+\right] \\
&= \ell_Z
\end{aligned}$$

where Z is a conditional variable. Rogers and Shi (1995) found out that it will give a good lower bound approximation for fixed strike Asian Option if $Z=\int_0^T \boldsymbol{B}_u du$ is chosen.

The error of lower bound approximation can be deduced based on the idea below.

For any random variable U, it can be stated that $U = U^{+} - U^{-}$ and $|U| = U^{+} + U^{-}$. Substitute $U = U^{+} - U^{-}$, we have

$$E(U^{+}) - [E(U)]^{+} = E(U^{+}) - [E(U^{+}) - E(U^{-})]^{+}$$

$$= \begin{cases} E(U^{-}) \text{ if } E(U^{+}) \geq E(U^{-}) \\ E(U^{+}) \text{ if } E(U^{+}) < E(U^{-}) \end{cases}$$

On the other hand,

$$\frac{(E(|U|) - |E(U)|)}{2} = \frac{[E(U^{+}) + E(U^{-})] - |E(U^{+}) - E(U^{-})|}{2}$$

$$= \begin{cases} E(U^{-}) \text{ if } E(U^{+}) \geq E(U^{-}) \\ E(U^{+}) \text{ if } E(U^{+}) < E(U^{-}) \end{cases}$$

Hence, $E(U^{+}) - [E(U)]^{+} = \frac{1}{2}(E(|U|) - |E(U)|)$

Write $U = U - E(U) + E(U)$. By triangle inequality, we obtain

$$|U| \leq |U - E(U)| + |E(U)|$$

$$E(|U|) - |E(U)| \leq E|U - E(U)|$$

Combining the equation we get

$$E(U^{+}) - [E(U)]^{+} = \frac{1}{2}(E(|U|) - |E(U)|) \leq \frac{1}{2}E|U - E(U)| \leq \frac{1}{2}\sqrt{\mathrm{var}(U)}$$

Therefore, substitute $U = 1 - A_T$ and the inner expectation by conditional expectation accordingly, we find that the error of lower bound is

$$0 \le E_{Q_S}\left[E_{Q_S}[(1-A_T)^+|Z]\right] - E_{Q_S}\left[\left(E_{Q_S}[1-A_T|Z]\right)^+\right]$$
$$\le \frac{1}{2}E_{Q_S}\left[\sqrt{\mathrm{var}(A_T|Z)}\right]$$

Thus it can be seen that the quality of the lower bound is highly dependent on the choice of Z. It is the best choice if $E_{Q_S}\left[\sqrt{\mathrm{var}(A_T|Z)}\right]$ is minimized. Here we follow the choice as shown in Peng et al. (2012) and consider

$$Z = \frac{1}{\Sigma}\int_0^T \left(\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\, d\boldsymbol{B}_u^{Q_S}\right) dt$$

where

$$\Sigma^2 = \mathrm{var}_{\mathrm{Q_S}} \int_0^T \left(\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\, d\boldsymbol{B}_u^{Q_S}\right) dt$$

Under $\mathrm{Q_S}$ measure, Z is normally distributed with mean zero and standard deviation of one.

We define

$$\xi_t = -\frac{1}{2}\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\boldsymbol{\sigma}_{Q_S}(u,t)^T\, du + \int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\, d\boldsymbol{B}_u^{Q_S}$$

a normal variable which appears in the formula of $\frac{1}{X_t}$ under Q_S measure. Thus

$$\frac{1}{X_t} = D(0,t)e^{\alpha t}e^{-\frac{1}{2}\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\boldsymbol{\sigma}_{Q_S}(u,t)^T du + \int_0^t \boldsymbol{\sigma}_{Q_S}(u,t) d\boldsymbol{B}_u^{Q_S}}$$
$$= D(0,t)e^{\alpha t}e^{\xi_t}$$

Based on the properties of jointly normal random variable

$$E[Z_1|Z_2] = E[Z_1] + \frac{\mathrm{Cov}(Z_1, Z_2)}{\mathrm{var}(Z_2)}(Z_2 - E[Z_2])$$

$$\mathrm{var}[Z_1|Z_2] = \mathrm{var}[Z_1] - \frac{\mathrm{Cov}\left(Z_{1_t}, Z_2\right)^2}{\mathrm{var}(Z_2)}$$

Thus, by substituting $Z_1 = \xi_t$, $Z_2 = Z = \frac{1}{\Sigma}\int_0^T \left(\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\, d\boldsymbol{B}_u^{Qs}\right) dt$ and note that $\mathrm{Cov}(\xi_t, Z) = E[(\xi_t - \mu_1)Z] = E[\xi_t Z] \equiv m_t$ we obtain

$E_{Q_S}[\xi_t|Z]$

$= E_{Q_S}[\xi_t] + \frac{\mathrm{Cov}(\xi_t,Z)}{\mathrm{var}(Z)}\left(Z - E_{Q_S}[Z]\right)$

$= E_{Q_S}\left[-\frac{1}{2}\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\boldsymbol{\sigma}_{Q_S}(u,t)^T\, du + \int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\, d\boldsymbol{B}_u^{Qs}\right] + \frac{\mathrm{Cov}(\xi_t,Z)}{\mathrm{var}(Z)}\left(Z - E_{Q_S}[Z]\right)$

$$= -\frac{1}{2}\int_0^t \boldsymbol{\sigma}_{\boldsymbol{Q_S}}(u,t)\boldsymbol{\sigma}_{\boldsymbol{Q_S}}(u,t)^T\, du + E_{Q_S}\left[\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\, d\boldsymbol{B}_u^{Qs}\right]$$

$+\frac{m_t}{1}\left(Z - E_{Q_S}[Z]\right)$

$= -\frac{1}{2}\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\boldsymbol{\sigma}_{Q_S}(u,t)^T\, du + m_t Z$

$= -\frac{1}{2}f(t) + m_t Z$

Note: $E_{Q_S}\left[\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)\, d\boldsymbol{B}_u^{Qs}\right] = 0$ and $E_{Q_S}[Z] = 0$

By Ito Isometry,

$$\text{var}\left(\int_0^t \boldsymbol{\sigma}_{Q_S}(u,t)d\boldsymbol{B}_u^{Q_S}\right)$$

$$= \int_0^t \left\|\boldsymbol{\sigma}_{Q_S}(u,t)\right\|^2 du$$

$$= \int_0^t \sigma_S^2 + \sigma_r^2 b^2(u,t) + 2\sigma_r\sigma_S\rho b(u,t)du$$

$$= f(t)$$

$$\frac{Cov(\xi_t, Z)}{var(Z)} = E[(\xi_t - E\xi_t)Z] = E[\xi_t Z] \equiv m_t$$

$$\text{var}[Z_1|Z_2] = \text{var}[Z_1] - \frac{\text{Cov}\left(Z_{1_t}, Z_2\right)^2}{\text{var}(Z_2)}$$

$$\text{Var}(\xi_t|Z) = var(\xi_t) - \frac{m_t^2}{1}$$

$$= f(t) - m_t^2$$

Hence $\xi_t|Z$ is normal with mean $-\frac{1}{2}f(t) + m_t Z$ and variance $f(t) - m_t^2$.

$$E_{Q_S}[1/X_t|Z] = D(0,t)e^{\alpha t}E_{Q_S}\left[e^{\xi_t}\middle|Z\right]$$

$$= D(0,t)e^{\alpha t}E_{Q_S}\left[e^{\xi_t|Z}\right]$$

$$= D(0,t)e^{\alpha t}e^{-\frac{1}{2}f(t)+m_t Z+\frac{1}{2}\left(f(t)-m_t^2\right)}$$

$$= D(0,t)e^{\alpha t}e^{m_t Z-\frac{1}{2}m_t^2}$$

As a result, we obtain

$$\ell_Z = E_{Q_S}\left[\left(E_{Q_S}[1 - A_T|Z]\right)^+\right]$$

$$= E_{Q_S}\left[\left(E_{Q_S}\left[1 - \int_0^T \frac{dt}{X_t}\middle| Z\right]\right)^+\right]$$

$$= E_{Q_S}\left[\left(1 - \int_0^T E_{Q_S}[1/X_t|Z]dt\right)^+\right]$$

$$= E_{Q_S}\left[\left(1 - \int_0^T G_t D(0,t) e^{\alpha t} e^{m_t Z - \frac{m_t^2}{2}} dt\right)^+\right]$$

$$= E_{Q_S}\left[\left(1 - \int_0^T G_t D(0,t) e^{\alpha t} g(Z)\, dt\right)^+\right]$$

where $g(z) = e^{m_t z - \frac{m_t^2}{2}}$.

To derive the analytic formula for m_t recall:

$$\xi_t = -\frac{1}{2}\int_0^t \boldsymbol{\sigma}_{Q_S}(v,t)\boldsymbol{\sigma}_{Q_S}(v,t)^T\, dv + \int_0^t \boldsymbol{\sigma}_{Q_S}(v,t)\, d\boldsymbol{B}_v^{Q_S}$$

$$= -\frac{1}{2}f(t) + \int_0^t \boldsymbol{\sigma}_{Q_S}(v,t)\, d\boldsymbol{B}_v^{Q_S}$$

$$\mathrm{var}(\xi_t) = \mathrm{var}\int_0^t \boldsymbol{\sigma}_{Q_S}(v,t)\, d\boldsymbol{B}_v^{Q_S}$$

$$= \int_0^t \left\|\boldsymbol{\sigma}_{Q_S}(v,t)\right\|^2 dv = f(t)$$

$$Z = \frac{1}{\Sigma}\int_0^T \left(\int_0^s \boldsymbol{\sigma}_{Q_S}(u,s)\, d\boldsymbol{B}_u^{Q_S}\right) ds \sim N(0,1)$$

As we define $m_t \equiv E_{Q_S}[\xi_t Z]$

$$= E_{Q_S}\left[\left(-\frac{1}{2}f(t) + \int_0^t \boldsymbol{\sigma}_{Q_S}(v,t)\, d\boldsymbol{B}_v^{Q_S}\right) Z\right]$$

$$= E_{Q_S}\left[-\frac{1}{2}f(t)Z\right] + E_{Q_S}\left[\left(\int_0^t \boldsymbol{\sigma}_{Q_S}(v,t)\, d\boldsymbol{B}_v^{Q_S}\right) Z\right]$$

$$= 0 + E_{Q_S}\left[\left(\int_0^t \boldsymbol{\sigma}_{Q_S}(v,t)\, d\boldsymbol{B}_v^{Q_S}\right)\frac{1}{\Sigma}\int_0^T \left(\int_0^s \boldsymbol{\sigma}_{Q_S}(u,s)\, d\boldsymbol{B}_u^{Q_S}\right) ds\right]$$

$$= \frac{1}{\Sigma} E_{Q_S}\left[\int_0^t \boldsymbol{\sigma}_{Q_S}(v,t)\, d\boldsymbol{B}_v^{Q_S} \times \int_0^T \left(\int_0^s \boldsymbol{\sigma}_{Q_S}(u,s)\, d\boldsymbol{B}_u^{Q_S}\right) ds\right]$$

As $v < t$, $d\boldsymbol{B}_v^{Q_S}.d\boldsymbol{B}_u^{Q_S} = 0$ for $u > t$, we change the integration limit so that $\int_0^T \left(\int_0^s \boldsymbol{\sigma}_{Q_S}(u,s)\, d\boldsymbol{B}_u^{Q_S}\right) ds$ become $\int_0^t \left(\int_0^s \boldsymbol{\sigma}_{Q_S}(u,s)\, d\boldsymbol{B}_u^{Q_S}\right) ds$. Thus $E_{Q_S}[\xi_t Z] = \frac{1}{\Sigma} E_{Q_S}\left[\int_0^t \boldsymbol{\sigma}_{Q_S}(v,t)\, d\boldsymbol{B}_v^{Q_S} \times \int_0^t \left(\int_0^s \boldsymbol{\sigma}_{Q_S}(u,s)\, d\boldsymbol{B}_u^{Q_S}\right) ds\right]$

Changing the integration order yields,

$$= \frac{1}{\Sigma} E_{Q_S}\left[\left(\int_0^t \boldsymbol{\sigma}_{Q_S}(v,t)\, d\boldsymbol{B}_v^{Q_S}\right) \times \left(\int_0^t \left(\int_u^t \boldsymbol{\sigma}_{Q_S}(u,s) ds\right) d\boldsymbol{B}_u^{Q_S}\right)\right]$$

$$= \frac{1}{\Sigma} \int_0^t \int_v^t \boldsymbol{\sigma}_{Q_S}(v,t) \boldsymbol{\sigma}_{Q_S}(v,s) ds dv$$

Let $F(z) = 1 - \int_0^T G_t D(0,t) e^{\alpha t} g(z)\, dt$ where $g(z) = e^{m_t z - \frac{m_t^2}{2}}$. Plot of $F(z)$ in Figure 3.1 shows that there is a unique root.

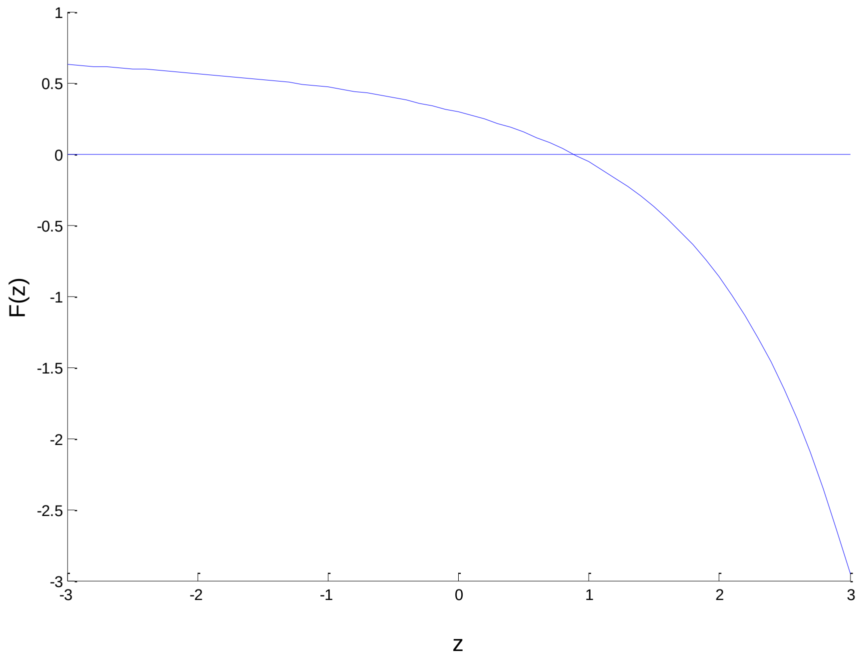

Figure 3.1 Plot of $\boldsymbol{F(z)}$

Suppose $F(z)$ has a root z_2. We observe that $F(z) > 0$ for $-\infty < z < z_2$.

Hence

$$\ell_Z = E_{Q_S}\left[\left(1 - \int_0^T G_t D(0,t) e^{\alpha t} g(Z)\, dt\right)^+\right]$$

$$= \left(1 - \int_0^T G_t D(0,t) e^{\alpha t} g(z)\, dt\right) \int_{-\infty}^{z_2} f(z) dz$$

$$= N(z_2) - \int_0^T G_t D(0,t) e^{\alpha t} \int_{-\infty}^{z_2} g(z) f(z) dz\, dt,$$

where $N(.)$ is the standard normal distribution function.

Since $f(z) = \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}}$, thus

$$\ell_Z = N(z_2) - \int_0^T G_t D(0,t) e^{\alpha t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_2} e^{-\frac{(z-m_t)^2}{2}} dz\, dt$$

$$= N(z_2) - \int_0^T G_t D(0,t) e^{\alpha t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_2 - m_t} e^{-\frac{y^2}{2}} dy\, dt$$

$$= N(z_2) - \int_0^T G_t D(0,t) e^{\alpha t} N(z_2 - m_t)\, dt$$

Mathlab programs were written to implement the put option value, $w_0 e^{-\alpha T} \ell_Z$.

It is found that this value is very close to the value reported in table 5.1.

CHAPTER 4

EMPIRICAL STUDY IN MALAYSIAN MARKET

In order to use the pricing model in practice, it is essential to estimate the model parameters. Vasicek and CIR model will be fitted to the data to estimate the interest rate model parameters. As we would like to explore the pricing of GMWB in Malaysia, Malaysian Government Securities (MGS) with one year maturity will be used as a proxy to risk neutral rate while Kuala Lumpur Composite Index (KLCI) will be used to estimate the parameters for equities portfolio. The parameters will be estimated using historical data instead of calibrating using option price as options data are not easily available. For parameter estimation of an interest rate model, different methods can be used since there are three dimensions that can be considered: time to maturity, yield rate and time. One method is to do a cross-sectional estimation where different maturity times are considered at a fixed moment of time. Another method is using time series estimation. This method considers the evolution of yield rate over a time period with fixed maturity. In this research, time series estimation method will be used for both Vasicek and CIR model.

4.1 Data

MGS are long term bonds issued by Malaysian government and they are the most actively traded bonds. Thus MGS will be used to estimate the parameter for interest rate model. Since we are considering the pricing of a long term product, 8 years of data will be used in our study covering the period from Jan 2006 to Dec 2013, a total of 1973 trade days. Table 4.1 shows the fraction of the data used while Figure 4.2 shows the trend of KLCI and one year MGS for the past 8 years. Figure 4.1 shows the daily yield curve for the MGS from 2006 to 2013. The bold line highlights one year MGS rate which is the yield rate being used in this research. From Figure 4.1, it can be seen that in general the interest rate has a great drop for all maturity time around year 2008 to 2009 and its movement is seen to be stable after year 2009. The decrease in interest rate was consistent with the action of central bank of Malaysia to reduce its Overnight Policy Rate (OPR). From November 2008 to February 2009, Bank Negara Malaysia has cut the OPR by a total of 150 basis points. During that time, Malaysian economy experienced a severe fundamental slowdown as an impact of global financial crisis. The interest rate cut was to prevent Malaysian economy from entering a deep and prolonged recession.

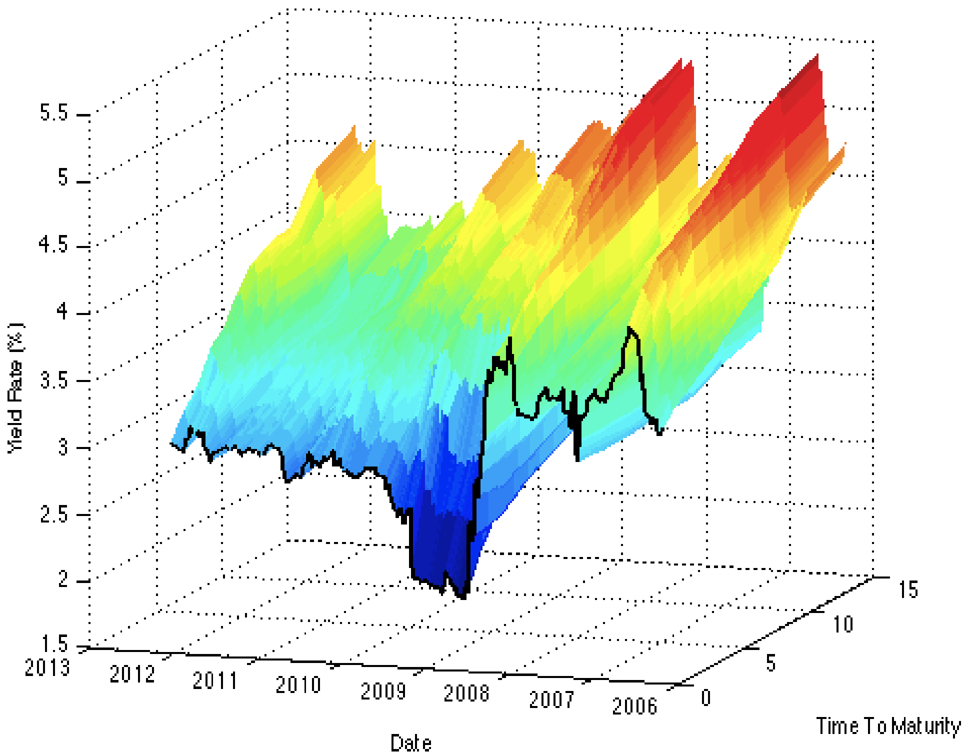

Figure 4.1 3D-plot of Malaysian Government Securities over 8 year period

Table 4.1 Sample of data used for parameters estimation

DATE	MGS1Y	KLCI
20060103	3.347	892.85
20060104	3.350	897.13
20060105	3.333	906.66
20060106	3.325	911.67
20060109	3.323	913.80
20060111	3.327	909.59
20060112	3.321	911.16
20060113	3.307	911.90
20060116	3.303	906.53
20060117	3.297	907.52
20060118	3.300	901.32

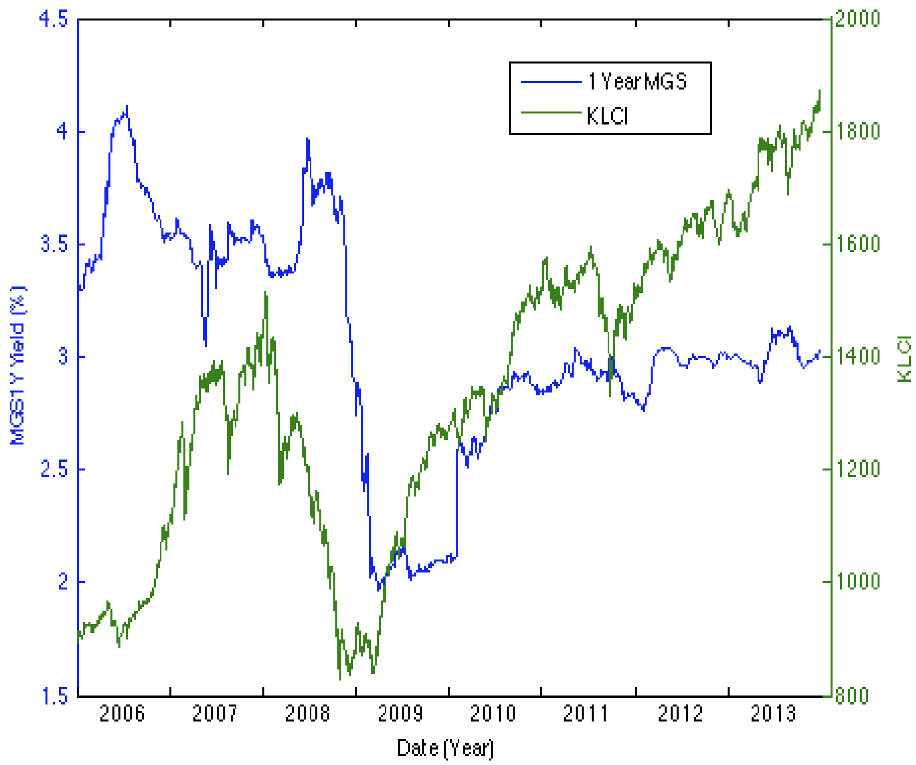

Figure 4.2 Trend of KLCI and 1 year MGS over 8 year period.

4.2 Estimation of parameters

For the parameter estimation of both the short rate model and stock price model, we will first obtain the discretize model using Euler discretization method and then least square method is used to estimate the parameter. The least square method estimates the coefficient by minimizing the sum of square of the deviation. The SDE of the short rate model consists of drift term and volatility term. The drift term is mainly driven by different legislative, regulatory, and economic environment; while the volatility term can be seen as the noise that is affected by the market supply and demand. Different interest rate data used will imply different parameter estimates. In this research, one year maturity yield is used. As we are considering the parameter estimation for a short rate model, the time to maturity should not be too long. Since daily data is used, each time step Δt is equal to 1/247 where we assume there are 247 trading days in a year.

4.2.1 Stock Price Model

Volatility of the return of the equity market σ_S is estimated using least square method by considering the discretized stock price model as follow:

$$ln\left(\frac{S_{t+\Delta t}}{S_t}\right) = \left(r - \frac{1}{2}\sigma_S^2\right)\Delta t + \sigma_S\sqrt{\Delta t}\varepsilon_{1,t}$$

where $\varepsilon_{1,t} \sim N(0,1)$

In this study, we will not estimate the drift tern from the historical data as the pricing model is built under risk neutral measure. Under risk neutral measure, the drift term is replaced by the risk free rate.

Since our main purpose is to investigate the pricing of a GMWB under stochastic interest rate which can be viewed as an option, it is needed to estimate the correlation coefficient between interest rate and stock return. The correlation coefficient ρ is estimated from the same set of historical data.

4.2.2 Vasicek Model

To estimate the parameters via time series data, we make use of the Ordinary Least Square method. A discrete time version model is needed for the parameter estimation. The discretization scheme of Vasicek model is written as:-

$$r_{t+\Delta t} - r_t = k\theta\Delta t - k\Delta t r_t + \sigma_r\sqrt{\Delta t}\varepsilon_{2,t}$$

where $\varepsilon_{2,t} \sim N(0,1)$

Parameter k and θ is estimated by solving the following equation.

$$(k,\theta) = arg\min_{(k,\theta)} \sum_{i=1}^{n-1} \left(r_{(i+1)\Delta t} - r_{i\Delta t} - k\theta\Delta t + k\Delta t r_{i\Delta t}\right)^2$$

In this research we use Microsoft Excel Solver to solve the equation above. Solver is an optimization tool that helps to find an optimal value in the target cells by changing a group of cells linked to the target cell. In our case, we set the parameter k and θ as the target cells by minimizing the sum of square of the residual. As for σ_r , the standard deviation of the residual will be used as an estimator.

4.2.3 CIR Model

Same estimation method is applied to the CIR model. We will first discretized the CIR model using Euler discretization method which is:

$$r_{t+\Delta t} - r_t = k\theta\Delta t - kr_t\Delta t + \sigma_r\sqrt{r_t\Delta t}\varepsilon_{2,t}$$

Then, Solver will be used to estimate the parameter k and θ by solving this equation.

$$(k,\theta) = arg\,\min_{(k,\theta)} \sum_{i=1}^{n-1} \left(\frac{r_{(i+1)\Delta t} - r_{i\Delta t} - k\theta\Delta t + k\Delta t r_{i\Delta t}}{\sqrt{r_{i\Delta t}\Delta t}} \right)^2$$

Where $\varepsilon_{2,t} \sim N(0,1)$

After we have obtained k and θ, σ_r is estimated by calculating the standard deviation of the sum of square of the residual.

The parameters estimated based on 8 years historical data will be used as the input to our pricing model to determine the insurance cost that need to be charged. The results are shown in Table 4.2. We find out that all parameters estimated are almost the same for both model except σ_r is higher for Vasicek model.

Besides, we are also interested to examine the movement of the parameters for both interest rate models. A total of 8 years data is used to examine the movement of the parameter over 2 years. We will first obtain the parameters for the first 6 years (1 Jan 2006 - 31 Dec 2011). Then we will shift the time frame by one day (2 Jan 2006 - 1 Jan 2012) to get the second set of parameters. This process is performed for a total of 493 different time frames, a period of two years. The results of the movement of parameters are shown in Figure 4.3 to Figure 4.5. Figure 4.3 and Figure 4.4 show that parameter k and θ estimated for both Vasicek and CIR model are close to each other. Although σ_r has the same trend for both models, the level is different. As predicted, CIR model has a higher σ_r as compared to Vasicek model. The higher σ_r for CIR model is due to the $\sqrt{r_t}$ in the volatility term of CIR model. As the volatility of CIR model depends on current interest rate, the higher the interest rate, the higher the volatility.

As a conclusion, the parameters estimated based on two different models give similar results for the data set we considered. Thus, we are unable to determine which model is better to be used in modeling interest rate in Malaysia.

Table 4.2 Parameter estimated for Vasicek and CIR model (Jan 2006-Dec 2013)

Parameter	Vasicek	CIR
$\boldsymbol{k}$	0.2718	0.2963
$\boldsymbol{\theta}$	0.0291	0.0292
$\boldsymbol{\sigma_r}$	0.0036	0.0021

As there is a major change in interest rate for the period of Nov 2008 to Feb 2009, we split the data into two partition: before Feb 2009 and after Feb 2009 to examine the impact of it. We find out that there is a huge different in the parameters calibrated. The estimation of the parameters is shown in Table 4.3.

Table 4.3 Parameter estimated for Vasicek and CIR model (before 2009 and after 2009)

	Before Feb 2009		After Feb 2009	
	Vasicek	**CIR**	**Vasicek**	**CIR**
k	-0.7320	-0.7685	0.5345	0.4683
θ	0.0388	0.0386	0.0312	0.0318
σ_r	0.0042	0.0231	0.0022	0.0137

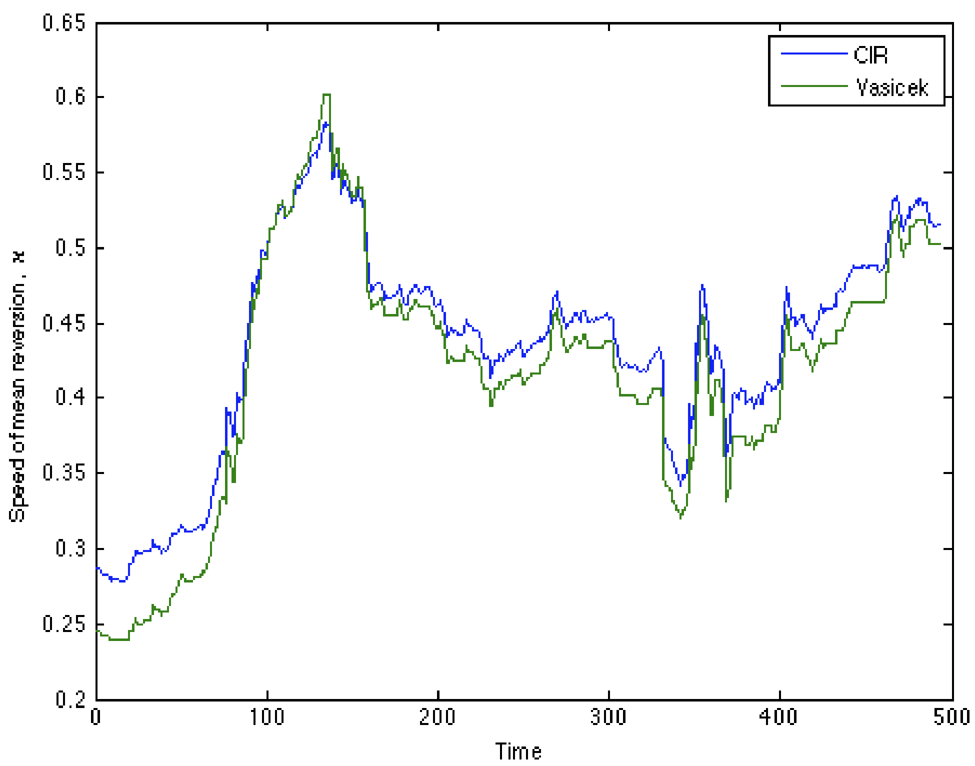

Figure 4.3 Estimation of Parameter *k* for Vasicek and CIR model from Jan 2012 to Dec 2013

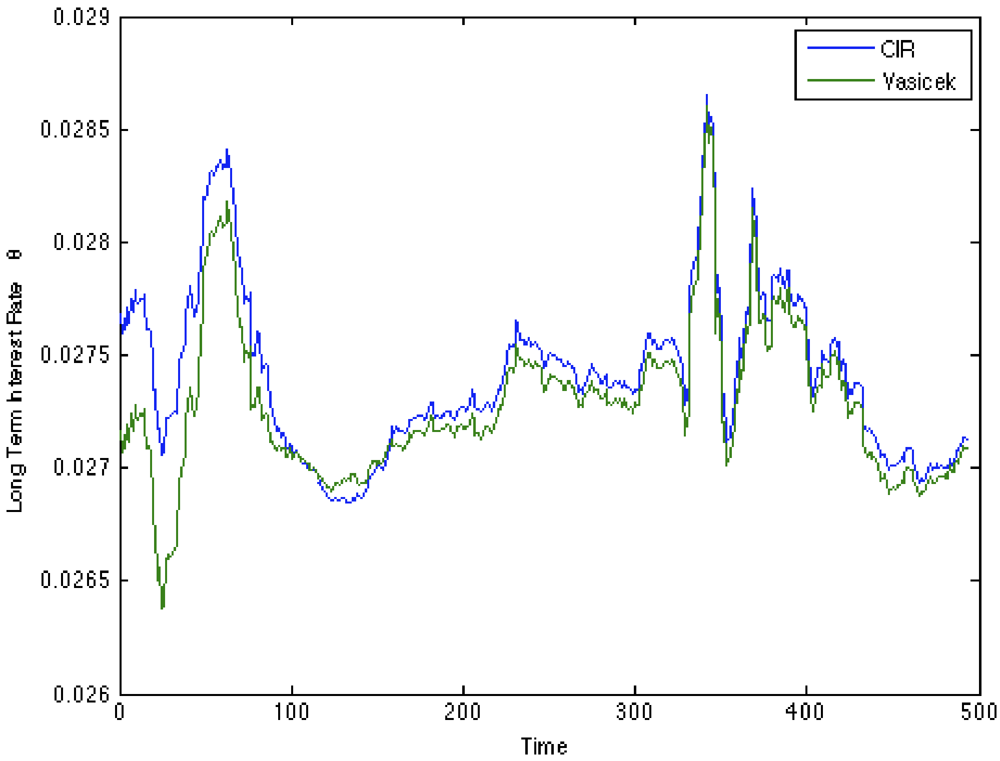

Figure 4.4 Estimation of Parameter θ for Vasicek and CIR model from Jan 2012 to Dec 2013

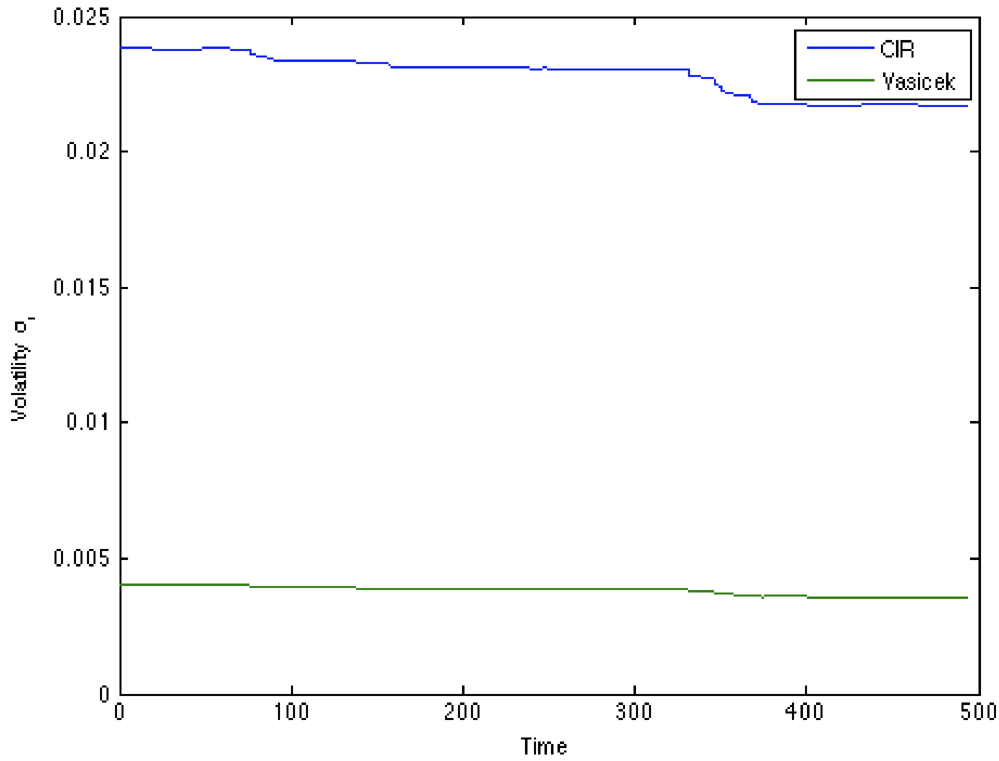

Figure 4.5 Estimation of Parameter σ_r for Vasicek and CIR model from Jan 2012 to Dec 2013

Based on the data set, the volatility of stock return is 0.1279. We notice that the stock return and interest rate return in Malaysia is negatively correlated with $\rho = -0.050$. That is, when interest rate increases, the stock market will not be performing. This is consistent with conventional economic argument that the raise of interest rate will negatively affect the stock market. When the interest rate increased, it will shift the investor behavior to invest in government securities instead of invest in stock market. Besides, the company profit is seen to be lower with increasing debt expenses. Thus all else being equal, it will lower the company stock price as expected future cash flow will drop.

CHAPTER 5

NUMERICAL RESULTS

5.1 Comparison with Benchmark

The simulation result of $E_{Q_S}[(1 - A_T)^+]$ under Vasicek interest rate model has been verified with the result obtained from Peng et al. (2012). Table 5.1 shows the comparison of the results. The simulation is performed with 100,000 trials for 10 years maturity and 15 years maturity. The parameter that is used in the model is $\theta = 0.05$, $k = 0.0349$, and $\alpha = 0.006$. The results that we obtained is quite close to the result obtained from Peng et al. (2012) The maximum difference is 1.41% observed from 15 years maturity with $\sigma_r = 0.03$. We noticed that the gap is smaller for lower σ_r. When the time to maturity is increased from 10 years to 15 years, the difference is increased as well.

Table 5.1 Comparison of the result

T	σ_r	σ_S	ρ	Peng *et al.* Result	Our Results	% Diff
10	0.01	0.2	-0.2	0.2404	0.2412	0.35
		0.2	0	0.2440	0.2444	0.18
		0.2	0.2	0.2476	0.2476	0.01
		0.3	-0.2	0.2942	0.2942	0.01
		0.4	-0.2	0.3492	0.3495	0.07
	0.02	0.2	-0.2	0.2370	0.2385	0.63
		0.3	-0.2	0.2902	0.2908	0.22
	0.03	0.2	-0.2	0.2340	0.2356	0.69
		0.3	-0.2	0.2851	0.2873	0.78
15	0.01	0.2	-0.2	0.3102	0.3102	0.00
		0.2	0	0.3154	0.3156	0.05
		0.2	0.2	0.3203	0.3217	0.43
		0.3	-0.2	0.3667	0.3680	0.35
		0.4	-0.2	0.4261	0.4271	0.22
	0.02	0.2	-0.2	0.3031	0.3055	0.79
		0.3	-0.2	0.3588	0.3604	0.45
	0.03	0.2	-0.2	0.2952	0.2994	1.41
		0.3	-0.2	0.3501	0.3539	1.07

5.2 Sensitivity Analysis

We explore the sensitivity of the pricing of an annuity with GMWB against different parameter values. In particular, we would like to study the behavior of option value, $E_{Q_S}[(1 - A_T)^+]$ against the interest rate volatility, σ_r and the instantaneous correlation coefficient, ρ. Table 5.2 shows the sensitivity of the option price against interest rate volatility and correlation coefficient respectively under Vasicek model and CIR model. From Table 5.2, it can be seen that option value is increasing with interest rate volatility except for postive correlation coefficient and decreasing for negative correlation coefficient. While under CIR interest model, option price is less sensitive to interest rate volatility where the gap of the option price is narrower compare to the option value under Vasicek model. We notice that the sensitivity of option value against correlation coefficient has the same trend under two different interest rate models, which is the sensitivity of option value to correlation coefficient increases when interest rate volatility increases. However, option value under CIR model is less sensitive compared to Vasicek model.

Table 5.2 Sensitivity analysis of option value under Vasicek and CIR model

T	σ_r	σ_S	ρ	Vasicek	CIR
10	0	0.2	-0.2	0.1767	0.1754
			0	0.1767	0.1753
			0.2	0.1767	0.1769
		0.3	-0.2	0.2359	0.2334
			0	0.2342	0.2325
			0.2	0.2351	0.2315
		0.4	-0.2	0.2921	0.2913
			0	0.2907	0.2834
			0.2	0.2928	0.2912
	0.004	0.2	-0.2	0.1761	0.1752
			0	0.1767	0.1766
			0.2	0.1775	0.1744
		0.3	-0.2	0.2350	0.2354
			0	0.2351	0.2377
			0.2	0.2359	0.2319
		0.4	-0.2	0.2916	0.2927
			0	0.2924	0.2886
			0.2	0.2942	0.2899
	0.008	0.2	-0.2	0.1757	0.1774
			0	0.1769	0.1764
			0.2	0.1778	0.1757
		0.3	-0.2	0.2335	0.2377
			0	0.2351	0.2324
			0.2	0.2375	0.2320
		0.4	-0.2	0.2904	0.2875
			0	0.2921	0.2873
			0.2	0.2926	0.2951
	0.012	0.2	-0.2	0.1742	0.176
			0	0.176	0.1769
			0.2	0.1807	0.1787
		0.3	-0.2	0.232	0.2357
			0	0.2342	0.2339
			0.2	0.2375	0.2336
		0.4	-0.2	0.2896	0.2866
			0	0.2921	0.2851
			0.2	0.296	0.2955

5.3 Insurance Fees

The proportionate insurance fee is determined based on the parameters estimated from Malaysian market data. Table 5.3 shows the parameters estimated from one year MGS and KLCI for the period from Jan 2006 to Dec 2013. This set of parameters is used as an input to our pricing model. For the pricing model under constant interest rate model, initial interest rate is assumed throughout the period.

Table 5.3 Parameter used for the pricing model

Parameter	Constant	Vasicek	CIR
$\boldsymbol{\sigma_s}$	0.1279	0.1279	0.1279
$\boldsymbol{r_0}$	0.03025	0.03025	0.03025
$\boldsymbol{\rho}$	-	-0.050	-0.050
$\boldsymbol{k}$	-	0.2718	0.2963
$\boldsymbol{\theta}$	-	0.0291	0.0292
$\boldsymbol{\sigma_r}$	-	0.0036	0.0021

The proportionate insurance fee is determinded such that the amount invested at inception is equal to the present value of the total amount of money received. The insurance charges computed based on different interest rate model as shown in Table 5.4. It can be seen that the insurance charges under Vasicek and CIR model is indifferent at 0.0081 while insurance charges under constant interest rate is lower at 0.0079. Since the difference in the insurance fees is insignificant, pricing under constant interest rate will be more practical due to its feasibility.

Table 5.4 Insurance charges based on different interest rate model

Interest Rate Model	**Insurance Cost, α**
Constant Rate	0.0079
Vasicek	0.0081
CIR	0.0081

CHAPTER 6

CONCLUSION AND FUTURE WORK

6.1 Conclusion

We have estimated the parameters for Vasicek and CIR model based on 1 year MGS. The results shown that parameters estimated for both model have a quite similar trend over two year period which implies the two models were indifferent based on the data set.

In answering which interest rate model to be chosen, it depends on the derivative product. In general, if the interest rates are not close to zero, or far-off from zero, Vasicek model could be a better choice. Vasicek model is more tractable and the closed form solution for more complex financial product is also available. On the other hand, if the interest rates are approaching zero, the chances of getting negative interest rate is higher especially when volatility is high. In that case, CIR model would be a better choice as compare to Vasicek model as working with Vasicek model has the possibility to get negative interest rate which yields illogical results and prices.

We have considered the pricing of an annuity with GMWB rider under stochastic interest rate. We compared the insurance charges assuming constant rate, Vasicek model and CIR model. Our results show that pricing model under Vasicek and CIR interest rate model is indifferent. While the insurance charges based on the pricing model assuming constant interest rate would be slightly lower based on the same scenario.

The sensitivity analysis shows that the stochastic interest rate volatility could have a significant impact on the option price, especially under Vasicek interest rate model. As a consequent, insurance charges could be affected. Thus, under high volatility regime, this model needs to be used cautiously.

The results obtained in this research are noteworthy in implementing the model in practice.

6.2 Future Work

Several European Central Banks have reduced interest rate to below zero to fight deflation and the interest rates stay negative for more than a year. This has challenged the saying of not possible for the interest rates to stay at negative territory. Thus, future research could be extended to study the interest rate model under negative interest rate environment. Under negative interest era, the interest rate model that prevents generating negative interest rate, such as CIR model and Black-Derman-Toy model will break down. Frankena (2016) discussed three solution methods to cope with negative interest rate. First method is to use the normal models where negative rates are allowed naturally. Second method is shifting the boundary condition by a positive constant s, such that rates larger than −s are allowed. The third method is removing the boundary condition. The disadvantage of the normal models is that there is a possibility to get extreme negative values. Shifted model will need additional shift parameter, where choosing the shift parameter is more of an art than a science. Although no extra parameter is introduced in free boundary models, there is no analytical solution and the approximation is inaccurate.

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