BOND PORTFOLIO OPTIMISATION USING STOCHASTIC CONTROL UNDER LIBOR MARKET MODEL

By

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ABSTRACT

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This study aims to find the optimum positions of each bond in a bond portfolio that is available in the market that would maximize the utility of the investor. This help investors to strategically allocate their limited funds into the bonds to suit their investment needs. The study consists of two parts: the interest rate model and the optimisation process. Different interest rate models would affect investors' bond portfolio value and ultimately affect the utility of the investors. Using the Vasicek model, a closed form formula for the optimal weight has been found in Puhle 2008. Here we propose using the LIBOR Market Model (LMM), which describes the interest rate dynamic that would affect the bond prices in the market. The LMM parameters were calibrated using cap volatilities and swaption volatilities. The optimisation process is viewed as a maximisation problem of investors' utility. The dynamic programming method used to optimise the utility functional is the Hamilton-Jacobi-Bellman (HJM) equation. A closed form formula for the optimal weight under LMM is determined. Based on the optimal weights for both LMM and Vasicek model, the actual portfolio performance and simulated portfolio performance are computed. The results of both models are then being analysed and compared.

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APPROVAL SHEET

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I understand that University will upload softcopy of my thesis in pdf format into UTAR Institutional Repository, which may be made accessible to UTAR community and public.

Yours truly,

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(*Foong Shee Heng, Dennis*)

DECLARATION

I hereby declare that the dissertation is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UTAR or other institutions.

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CHAPTER 1

INTRODUCTION

Asset allocation is one of the common topics in finance and investment which deals with selecting the best investment vehicles under certain constraints. Investment vehicles may include bonds, equities and alternative investments; constraints including limited initial capital, current and forecasted future economic conditions as well as investor specific requirements. There are two types of asset allocations: (i) static asset allocation and (ii) dynamic asset allocation.

Static asset allocation is a strategy that involves a combination of buying and short selling some assets over a period of time. In other words, the portfolio selected would remain the same throughout the investment period. The research done by Markowitz (1952) tackles this static asset allocation problem using the mean variance approach. This approach works with the expectation on investment vehicle future return. Different portfolios, i.e. different combinations of investment vehicles, would give different means and variances on future expected returns. The paper explains that it is possible, with the appropriate diversification, that some portfolios are superior to others, i.e. portfolios that give the highest expected return for a given risk or portfolios that give the smallest risks for an expected return. The set of portfolios which satisfy the above conditions forms the efficient frontier.

The research done by Tobin (1958) considered an riskless investment vehicle, i.e. money market account, on top of the portfolio of investment vehicles considered by Markowitz (1952) and the economic conditions which will affect the allocation between the money market account and the portfolio. In other words, how would investors adjust their allocation into riskless investment vehicle and the risky portfolio when certain economic conditions occur? This idea of investing a portion in the chosen portfolio and money market account is called the Tobin's Separation Theorem.

The static asset allocation, however, has some limitations. In practical sense, it is very time consuming in finding the optimal portfolios as there are many possible combination of investment vehicles to consider. Static asset allocation assumes that every investor has the same and only one investment period and would prefer the same risk and return profile. These assumptions are not true in reality. Furthermore, return and variance, which are used to determine the optimal portfolios and the market portfolio, are insufficient in capturing the extreme events. This is very dangerous to the economic and financial institutions as many historical financial disasters are tail risk events. However, if Value-at-Risk ("VaR") or the expected shortfall is used to measure risk, then the static asset allocation may cover tail risk events.

Dynamic asset allocation relaxs the one investment period assumption made in static asset allocation by breaking down one investment period into multiple shorter investment periods. The decision period to select investment vehicles span throughout the investment period. The early 70s marks the start of the dynamic asset allocation period. Research work such as Samuelson (1969), Merton (1969) and Merton (1971) are the pioneers of dynamic asset allocation. They employed the calculus of variation in solving the optimsation problem. The dynamic asset allocation assumes that the portfolio could change even in an instantaneous time. Obviously, this is one of the drawbacks of this method, as in reality, it is hard to find an investor who constantly changes his portfolio instantaneously. Another difference between dynamic asset allocation and static asset allocation is the variables to be optimised. In static asset allocation, the objective is to either minimise risk or maximise the return while in dynamic asset allocation, the objective is to maximise the utility function as well as to maximise the consumption of an investor. The variables for such utility functions would normally be the wealth of the investor in the future. Li and Ng (2000) has expanded the work of Markowitz (1952) from 1 investment period to multiple investment period.

Both static and dynamic asset allocation do not specify the investment vehicle, i.e. whether the investment vehicle are stocks, bonds, derivatives or alternative investments. Research has been done by Korn and Kraft (2002) which narrows the scope of investment vehicles to only bonds in the dynamic asset allocation problem. In view that dynamic asset allocation often take the utility of terminal wealth of the investor as one of the variables and with the investment vehicle in consideration is bonds, it is logical to find out what would highly impact the bond prices. Litterman and Scheinkman (1991) states that the prices of bonds are greatly affected by the interest rate level, slope and curvature. Hence, a model is needed to describe the future interest rate levels. Therefore, interest rate models are introduced into the dynamic asset allocation problem. Some of the interest rate models considered by Korn and Korn (2001) are Ho and Lee (1986), Vasicek (1977), Dothan (1978), Black and Karasinksi (1991) and Cox et al. (1985) models. Note that the interest rate models considered by Korn and Korn (2001) are short rate models which only models the interest rate over an infinitesimal period of time. The short rate model is a stochastic process which can be described by a stochastic differential equation with certain drift and diffusion functions. Some drift function exhibits mean reverting properties like those proposed by Vasicek (1977) and Cox et al. (1985).

However, there is another type of interest rate models which differs from the short rate interest rate models. This dissertation will consider to use one of the market models, particularly, the LIBOR Market Model (LMM) as the model to describe the interest rate level in this dynamic asset allocation problem. The LMM was first introduced by Brace et al. (1997). LMM is also known as Brace-Gaterak-Musiela (BGM) model. LMM enjoys all properties of the market models such as modeling of forward rates instead of instantaneous interest rates. LMM is popular among practitioners as it is consistent with the standard practice of using Black (1976) to price cap. This means that the error between the market price of cap and the price implied by LMM is smaller than the error using short rate models.

The natural question to ask is where LMM could price bond as effective as pricing interest rate related derivatives. This dissertation will use LMM to price bond and also determine the optimal bond weights. The parameters of LMM is calibrated using cap volatilities and swaption volatilities. These parameters will be used to determine the optimal bond weights.

The structure of this dissertation is as follow: Chapter 2 contains the literature review. Chapter 3 will show the derivation of the optimal bond weights. The numerical illustration of the optimisation process will be covered in Chapter 4. Finally, conclusion and future work will be discussed in Chapter 5.

CHAPTER 2

LITERATURE REVIEW

This chapter covers some basic knowledge on mathematics and financial products that will be used in this dissertation. Section 2.1 starts by introducing some of the financial products which are commonly found in the market. The financial products which will be highlighted in this dissertation would be bonds. Certain properties of these financial products will be discussed. Assuming that the interest rate remain constant throughout the investment period, the discount formula and accumulation formula for fixed periodic payment will be shown here, however, these formula will not be proved. Basic concepts in mathematical finance such as Brownian Motion, Itô Integral and Itô Lemma will be discussed in Section 2.2. One of the dynamic programming used is the Bellman equation. The background of this equation will stated in Section 2.3.1 while its derivation, will be briefly discussed in Section 2.3.2. The Bellman equation caters for only discrete value functions. The continuous counterpart of the Bellman equation is the Hamilton-Jacobi-Bellman equation.

2.1 Financial products

This section covers the different financial products that will be used throughout this dissertation. Products that are covered are (i) zero coupon bond, (ii) cap and (iii) swaption.

2.1.1 Bond

We only discuss zero coupon bond in this dissertation as our optimisation process only consider zero coupon bond. The zero coupon bond is defined in Hull (2014) as follows:

Definition 2.1.1. A bond that provides no coupons

This means that the zero coupon bond, unlike the coupon paying bonds which pay coupons to the holder periodically as stated by Hull (2014).

2.1.2 Derivative

As defined by Hull (2014), derivative is a financial instrument whose value depends on the values of other, more basic, underlying variables. Some of the derivatives that will be used in this dissertation are cap and swaption. The underlying variable of cap and swaption is interest rate. Cap is also called as interest rate cap which is defined by Hull (2014) as follows:

Definition 2.1.2. An option that provides a payoff when a specified interest rate is above a certain level. The interest rate is a floating rate that is reset periodically

An interest rate cap consists of many interest rate caplets. The definition of interest rate caplet is one component of an interest rate cap as mentioned by Hull (2014).

The combined understanding of the two definition above implies that the valuation of a cap is the summation of each caplets' value. Each caplet is valued using the Black formula as found in Black (1976). There is a procedure of stripping the cap volatility into caplet volatilites in the later part of this dissertation. This procedure relies on these concepts.

Hull (2014) defines swaption as

Definition 2.1.3. Swaption is an option to enter into an interest rate swap where a specified fixed rate is exchanged for floating.

Hull (2014) suggests that swaption is to be valued using a Black formula as it was assumed that the underlying swap rate at the maturity of the option is lognormal.

2.2 Mathematical finance

Mathematical finance deals with problems that involve stochastic processes rather than deterministic variables. Therefore, the normal calculus could not be used and stochastic calculus is used instead. The difference between normal calculus and stochastic calculus is the introduction of randomness in the stochastic calculus. This randomness could arise from the introduction of the Brownian motion into the differential equation.

2.2.1 Brownian motion

Shreve (2004) define the Brownian motion as follows:

Definition 2.2.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. For each $\omega \in \Omega$, the continuous function $Z(t)$ of $t \geq 0$ that satisfies $Z(0) = 0$ and depends on ω is called a Brownian motion if it satisfies the following condition

- for all $0 = t_0 < t_1, \ldots, t_m$ the increments

$$Z(t_1) = Z(t_1) - Z(t_0), Z(t_2) - Z(t_1), \ldots, Z(t_m) - Z(t_{m-1})$$

  are independent normal distribution
- $\mathbb{E}^{\mathbb{P}}\left[Z(t_{i+1}) - Z(t_i)\right] = 0$
- $Var^{\mathbb{P}}\left[Z(t_{i+1}) - Z(t_i)\right] = t_{i+1} - t_i$

Shreve (2004) has also listed the following properties of the Brownian motion.

- $Z(t)$ is continuous for almost all ω
- The covariance of $Z(s)$ and $Z(t)$ is $\min(s, t)$
- the moment generating function of the random vector $(Z(t_1), Z(t_2), \ldots, Z(t_m))$ is

$$e^{\frac{1}{2}u_m^2(t_m - t_{m-1})} \cdot e^{\frac{1}{2}(u_{m-1}+u_m)^2(t_{m-1}-t_{m-2})} \cdot e^{\frac{1}{2}(u_1+u_2+\ldots+u_m)^2 t_1}$$

- Brownian motion is a martingale

- The quadratic variation of the Brownian motion, $[Z, Z](t) = t$ for all $t \geq 0$ almost surely.

Shreve (2004) has stated that the quadratic variation of the Brownian motion, $Z(t)$ is t, i.e

$$dZ(t)dZ(t) = t$$

In other words, Brownian motion accumulates quadratic variation at rate one per unit time. Other results that are stated are

$$\begin{aligned} dZ(t)dt &= 0 \\ dtdt &= 0 \end{aligned}$$

2.2.2 Itô integral

Itô (1944) has constructed the stochastic integral where the integrator is a normalised Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The Itô is defined as

$$I(t) = \int_0^t \Delta(u)dZ(u) \tag{2.1}$$

Note that the Itô integral does not have drift component. Shreve (2004) states the generalisation of the Itô integral with the definition of the Itô process which adds a drift component to the Itô integral. Let $X(t)$ be the Itô process. $X(t)$ is defined as

$$X(t) = \int_0^t \mu(u)du + \int_0^t \sigma(u)dZ(u)$$

where $\mu(u)$ is the drift coefficient of the Itô process.

There is a need to differentiate a function which has Itô process as it's variable, $f(t, X(t))$. This is where Itô comes in. Shreve (2004) that the Itô lemma is the result of applying the non zero quadratic variation of the Brownian motion to $df(t, X(t))$. The Itô lemma is stated as follows:

Theorem 2.2.1. Let $f(t,x)$ be any twice differentiable function of two (2) real variables t and x. Let the Itô process be $X(t) = \int_0^t \mu(u)du + \int_0^t \sigma(u)dZ(u)$. We have the following

$$df(t,X(t)) = \left(\frac{\partial f}{\partial t} + \mu(t)\frac{\partial f}{\partial X(t)} + \frac{1}{2}\sigma(t)^2\frac{\partial^2 f}{\partial X^2(t)}\right)dt + \sigma(t)\frac{\partial f}{\partial X(t)}dZ(t)$$

2.3 Dynamic programming

Dynamic programming involves breaking one large optimisation problem into many smaller subproblems. This is achieved by first obtaining a solution to the first optimisation subproblem as used this solution to be the starting point for the next optimisation subproblem. This procedure is iterated in a recursive manner for all subsequent subproblems.

There are many ways to solve these subproblems. One of the ways is to apply the greedy algorithm which chooses the local optimsed solution for each subproblem. However, this doesn't gurantee the best solution to the optimisation problem. For example, considering the problem of one driver in getting from point A to point B during rush hour as fast as possible. The application of the greedy algorithm may lead the driver to a traffic jam causing him to be slower in subsequent subproblems.

The other way of solving the optimisation problem is by solving it backward recursively. Olver and Sookne (1972) states that computing the optimised solution of a system of difference equation by forward recurrence is not practical due to instability where as backward recursive computation is a stable way.

Bellman (1957) proposed the Bellman equation which solves discrete optimisation problem in a backward recursive manner. This Bellman equation can be seen as an extension to the Hamilton-Jacobi-Bellman equation to solve optimisation problem with stochastic variables in the continuous setting.

2.3.1 Background of Bellman equation

Bellman (1957) has studied the structure of the dynamic programming process. He defined the multi stage optimisation problem based on the following:

- A physical system characterised at any stage, t, by a small set of parameters which are called state variables, X_t.
- The variables that would change the state variables are called the control variables, q.
- At each stage, there is a number of decisions which would cause transformation, T_q, of the state variables by changing the control variables.
- The set of all obtainable control variables is denoted by S.
- The determination of which decision to take would based on the objective function, f.
- The set that consists of all decisions made is called policy, p. The policy which optimise the objective function is said to be the optimal policy

2.3.2 Bellman equation

Next, the Bellman's Principle of Optimality is also defined in Bellman (1957) which states that an optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

He has mentioned in Bellman (1957) that the result from the definition above is the following basic recurrence relation: each stage of the multi stage optimisation problem will be related to the next stage of the multi stage optimisation problem. In other words, if the optimal policy is selected at the beginning stage, the policy will contain the decision which will optimise the next stage.

The Bellman equation is written as

$$f(X_t) = \max_{q \in S} f(T_q(X_t)) \tag{2.2}$$

This method is useful as the bond portfolio considered in this dissertation will be rebalanced at fixed interval. Therefore, it is similar to breaking one single long term bond portfolio optimisation problem into multiple short term bond portfolio optimisation problem.

Other optimisation involves determine the optimised outcome of the problem at the end of the optimisation process and then only determine the process to archieve that optimised outcome. This is not applicable to the bond portfolio optimisation as investors would naturally not limit themselves to the gain they could obtain from their portfolio.

2.4 LIBOR market model

2.4.1 Background of LMM

LIBOR stands for London Interbank Offer Rates. Banks use this interest rate as a reference for unsecured loans to other banks in the London wholesale money market. The valuation of interest rate derivatives such as Bermudan swaptions, caps and floors are commonly based on LIBOR.

LMM was introduced by Brace et al. (1997). LMM was built on the Heath et al. (1992) (HJM) framework. The HJM framework allows the pricing of bond under no arbitrage pricing condition. They have shown that, under the no arbitrage pricing condition, the market price of interest rate risk could be removed from the forward rate process.

LMM was made popular as it agrees with the well-established Black formula from Black (1976) in pricing interest rate derivatives such as caps, floors and swaptions unlike most short rate models. The lognormal forward-LIBOR model (LFM) coincide with Black's cap formula while the lognormal forward-swap model (LSM) coincide with the Black's swaption formula. Brace et al. (1997) has shown that pricing interest rate derivatives such as cap, floor and swaption through simulation assuming LMM describing the changes in the interest rate levels would produce small deviation from the Black formula. However, this dissertation will focus on bonds pricing under LMM rather than valuation of interest rate derivatives.

The idea that LMM is part of the HJM framework is discussed in Section 2.4.2. The definition of a market model is detailed in Section A.1. In Section A.2, the derivation of LMM process under different measures by using the change of numeraire technique. In Section 2.4.3, the calibration process of LMM process parameter such as standard deviation and correlation coefficient is discussed.

2.4.2 The connection between HJM and LMM

The Heath et al. (1992) has introduced a unifying theory for valuing default free zero coupon bonds and contingent claims under a stochastic term structure of interest rates as well as formulated the martingale representation of the forward rate under no-arbitrage pricing. The following equation is famous HJM equation.

$$df(0;t,T-t) = \left(\sigma(0;t,T)\int_t^T \sigma(0;t,s)ds\right)dt + \sigma(0;t,T)dZ(t)$$

where $\sigma(0;t,T)$ is the volatility for the forward rate from time t to time T observed at time 0 and $Z(t)$ is the standard Brownian motion. Note that the drift term of the forward rate process in the HJM framework consists of only the standard deviation of the forward rate. The standard deviation dictates the drift term

Table 2.1: Piecewise-constant function for volatility

Instant. Vols	Time: $t \in (0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	$\ldots$	$(T_{M-2}, T_{M-1}]$
Fwd Rate: $F_1(t)$	$\sigma_{1,1}$	Dead	Dead	$\ldots$	Dead
$F_2(t)$	$\sigma_{2,1}$	$\sigma_{2,2}$	Dead	$\ldots$	Dead
$\vdots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$F_M(t)$	$\sigma_{M,1}$	$\sigma_{M,2}$	$\sigma_{M,3}$	$\ldots$	$\sigma_{M,M}$

of the forward rate process and this is called the forward rate drift restriction.

2.4.3 Calibration

The discussion of the calibration process for the LMM's parameters will make reference to the following LMM process which is Equation (A.3). The LMM process contains parameters such as σ and ρ. These parameters are interpreted as the instantaneous volatility and instantaneous correlation respectively.

Brigo and Mercurio (2006) mentioned that the instantaneous volatility, σ could be assumed to follow a piecewise-constant or a parametric function while the instantaneous correlation, ρ could be assumed to follow either a full rank or a reduced rank parametric function. The next few subsections will describe these in more detail.

Volatility function

An example of a σ function that follows a piecewise-constant function would be as in Table 2.1. Consider a cap which consists of M caplets. Each caplets would have their respective σ, therefore, there would be M caplet volatilities spanning the M periods. The volatility for each period could assumed to be constant throughout the period. In this case, there would be $\frac{M(M-1)}{2}$ piecewise constant volatilities to calibrate from given market cap volatilities.

On the other hand, the cap volatiltiy could be described by a parametric function. Most parametric function used by practitioners would be less than $\frac{M(M-1)}{2}$.

Correlation function

Brigo and Mercurio (2006) states some properties that an instantaneous correlation matrix ρ used in LMM should have after observing the trends of the forward rates. The properties as as follows:

- positive correlation, $\rho_{ik} \geq 0$ for all i, j
- monotonic decreasing when moving away from the diagonal entry
- more correlated changes among adjacent forward rates when the tenor gets larger

There 2 types of correlation functions which are

- full rank
- reduced rank

The full rank correlation is similar to piecewise volatility structure in the sense that there are $\frac{M(M-1)}{2}$ entries in the correlation matrix. Each entry of the correlation matrix need to be estimated from the swaption volatility. There are times where the size of the correlation matrix is so huge that it takes a long time in order to calibrate all entries from the swaption volatilities. This leads to the introduction of reduced rank correlation matrix.

Reduced rank correlation matrix uses less parameters to describe the correlation matrix. One of the methods is to define a parametric equation which could describe the correlation matrix with a few parameters. Gatarek et al. (2006) suggested that the number of parameters could be as few as 8 parameters. Therefore, its calibration process would be faster than the calibration process of the full rank correlation matrix.

Brigo and Mercurio (2006) states that the size of the correlation matrix that is being considered in this bond optimisation problem depends on the number of forward rates. In general, the full rank correlation has more entries to be calibrated and it is not practical to do so. Therefore, the natural alternative is to describe the correlation matrix using reduced rank correlation matrix. This can be archieved by Rebonato's angles parameterisation.

CHAPTER 3

PORTFOLIO OPTIMISATION PROCESS

This dissertation seeks to optimise a portfolio which consists of a zero coupon bond and money market account using dynamic programming. The end result of this dissertation is the optimal weights in the zero coupon bond throughout the investment period.

The advantages of dynamic asset allocation over static asset allocation.

- The static asset allocation only cover 1 investment period while dynamic asset allocation covers multiple investment periods.
- The static asset allocation maximise the expected future returns or minimise the risk involved while the dynamic asset allocation maximise the utility function with the terminal wealth being the variable.

The next few subsections will compare the optimal bond weights which lead to a maximised utility function on the terminal wealth of a bond investor as a result of investing a fixed amount of initial capital in either a zero coupon bond or money market account under two different interest rate models which are (i) Vasicek model and (ii) LIBOR Market Model (”LMM”).

3.1 Assumptions

A few assumptions will be made in deriving the optimal bond weights for both models. The following list are the assumption being made:

- There is no transaction cost.
- There is no restriction in buying or selling any less than one unit of zero coupon bond.
- There is no restriction in depositing and borrowing of any amount of cash from the money market account.

- The interest rate for the money market account is fixed throughout the investment period.
- There is no consumption throughout the investment period. All interest gained and capital gain obtained will be either be invested in the zero coupon bond or deposited in the money market account.
- The maturity date of the zero coupon bond, T_B, is later than the end of the investment period, T, i.e. $T < T_B$.
- The market price of interest rate risk is assumed to be constant.
- The bond portfolio is rebalance at a fixed interval, Δt. This means that the portfolio (one zero coupon bond and money market account) will be readjusted to the weights suggested by the models after Δt time has passed. Δt taken to be (i) daily, (ii) weekly and (iii) monthly.

3.1.1 Optimal bond weights under Vasicek model

Puhle (2008) considered the same dynamic asset allocation problem, however, assuming interest rate dynamics follow Vasicek model, $dr(t) = a(b - r(t))dt + \sigma^r dZ(t)$. He has derived the optimal bond weights to be the following:

$$\pi^*(t) = \frac{1}{1-\gamma}\left(\frac{\lambda a}{\sigma^r(1-e^{-a(T_B-t)})} - \gamma\frac{1-e^{-a(T-t)}}{1-e^{-a(T_B-t)}}\right) \tag{3.1}$$

where

- λ is the market price of interest rate risk
- γ is the risk aversion coefficient of the investor where $0 < \gamma < 1$
- a is the mean reverting rate in the Vasicek model
- σ^r is the volatility of the short rate in the Vasicek model
- t is the current time
- T_B is the maturity date for the zero coupon bond
- T is the end of the investment period

The market price of interest rate risk, λ , as suggested by Ahmad and Wilmott (nd), is determined by letting the slope of the yield curve to be equal to $(\mu - \lambda\sigma^r)/2$, where μ is the drift term of the interest rate model. When Vasick model is considered as the interest rate model, the equation to describe the slope is as follows:

$$\text{Slope} = (a(b - r(t)) - \lambda\sigma^r)/2$$

where

- σ^r is the volatility of the Vasicek model
- Slope is the slope of the yield curve which is measured by the gradient of the shortest term rate and the longest term rate

Rearranging the above equation,

$$\lambda = \frac{a(b - r(t)) - 2 \times \text{Slope}}{\sigma^r}$$

The market price of interest rate risk, λ is assumed to be constant throughout this dissertation. The consideration of a non-constant λ is left for future work. The λ is calibrated and fixed at the start of the investment period.

3.2 Optimal bond weights under LMM

The derivation of the optimal bond weight is the similar as Korn and Kraft (2002) and Puhle (2008) however instead of using short rate model to model future interest rates, the LMM was used as the interest rate model.

3.2.1 Deriving the optimal bond weights under LMM

The following are the steps used in deriving the optimal bond weights under LMM:

- Derive the wealth process which is based on the money market account process and the bond process assuming that LMM describes the changes in the interest rate level

- Maximise the utility function which takes the terminal wealth as the variable using the HJB equation. The end product of this step would be the optimsed bond weights under LMM

Moving on to the first step in deriving the optimal bond weights under LMM which is to derive the wealth process which is based on the money market account process and the bond process under LMM. No one would not know anyone's wealth at the end of a future day. One of the workaround would be to describe the wealth process based on the investor's portfolio and how the value each assets in the portfolio would change in the future. Note that for this bond optimisation problem, the investor's portfolio could only consists of money market account and a zero coupon bond.

Let $P(t,T)$ be the price of the zero coupon bond at time t which matures at time T. Define the forward rate, $f(t,T)$ as follows:

$$P(t,T) = \frac{1}{1 + f(t,T)\tau}$$

where $\tau = T - t$ denote the time to maturity of the zero coupon bond.

Apply Itô lemma on the above equation, the following holds:

$$\frac{dP(t,T)}{P(t,T)} = \frac{f(t,T)}{1 + f(t,T)\tau}\left[(1 + \frac{\sigma^2(t,T)f(t,T)\tau}{1 + f(t,T)\tau})dt - \sigma(t,T)\tau dZ^{\mathbb{Q}}(t)\right]$$

where $\sigma(t,T)$ is the standard deviation of the forward rate for the period $[t,T]$.

The $\mathbb{Q}$-measure is suitable for derivative pricing as the payoff of the derivative can be replicated by a portfolio that grows at risk free rate which consists of the underlying asset and zero coupon bond. In portfolio optimisation, investors are interested in the real probability measure, $\mathbb{P}$ rather than the $\mathbb{Q}$-measure. Therefore, the zero coupon bond process that should be used has to be under $\mathbb{P}$-measure.

Girsanov (1960) has shown that it is possible to change the measure of the zero coupon bond process from the $\mathbb{Q}$-measure to $\mathbb{P}$-measure by applying the Girsanov theorem and the Radon-Nikodym derivative. The relationship between $\mathbb{Q}$-measure and $\mathbb{P}$-measure is described by the following equation:

$$dZ^{\mathbb{P}}(t) = dZ^{\mathbb{Q}}(t) + \lambda(t)dt$$

where $\lambda(t)$ is called the market price of risk. In view that the process is the zero coupon bond process, this $\lambda(t)$ is called the market price of interest rate risk. As for the Vasicek model, the market price of interest rate risk, $\lambda(t)$ for LMM is also assumed to be constant throughout time.

Therefore, the zero coupon bond process, under $\mathbb{P}$-measure is

$$\begin{aligned} & \frac{dP(t,T)}{P(t,T)} \\ = \; & \frac{f(t,T)}{1+f(t,T)\tau} \times \\ & \left[\left(1 + \frac{(\sigma^f(t,T))^2 f(t,T)\tau}{1+f(t,T)\tau} + \sigma^f(t,T)\tau\lambda\right) dt - \sigma^f(t,T)\tau dZ^{\mathbb{P}}(t)\right] \end{aligned}$$

Recall that $P(t,T)$ is the price at time t of the zero coupon bond which matures at time T. However, in this dissertation, the zero coupon bond considered is assumed to mature at time T_B. Therefore, rewriting the above equation as follows,

$$\begin{aligned} & \frac{dP(t,T_B)}{P(t,T_B)} \\ = \; & \frac{f(t,T_B)}{1+f(t,T_B)\tau_B} \times \\ & \left[\left(1 + \frac{(\sigma^f(t,T_B))^2 f(t,T_B)\tau_B}{1+f(t,T_B)\tau_B} + \sigma^f(t,T_B)\tau_B\lambda\right) dt - \sigma^f(t,T_B)\tau_B dZ^{\mathbb{P}}(t)\right] \end{aligned} \tag{3.2}$$

where τ_B is define as $\tau = T_B - t$.

The money market account, under HJM framework, follows a process called the money market account process, $M(t)$. It is defined as follows:

$$dM(t) = r(t)dt \tag{3.3}$$

where $r(t)$ is the interest rate generated by the money market account.

One additional assumption in deriving the optimal weight for LMM is that the money market account will generate interest at a fixed rate. Therefore the above equation would be written as follows:

$$dM(t) = rdt$$

Since the bond investor could only invest in either the zero coupon bond maturing at T_B or in the money market account, the wealth of the investor would be the sum of the values in the zero coupon bond and the money market account. The changes in the wealth of the investor would be reflected by the changes in the zero coupon bond value and the changes in the money market account. By denoting, $W(t)$ as the wealth of the investor at time t and π as the weights in the zero coupon bond, the changes in the investor's wealth could be written in an equation as:

$$dW(t) = W(t)(1 - \pi(t))dM(t) + \frac{W(t)\pi(t)}{P(t, T_B)}dP(t, T_B) \tag{3.4}$$

The above equation could be interpreted as the investor holding $W(t)(1-\pi(t))$ in the money market account which will grow at $dM(t)$ while at the same time holding $\frac{W(t)\pi(t)}{P(t,T_B)}$ units of the zero coupon bond which will grow at $dP(t, T_B)$.

Note that $\pi(t)$ is not restricted between 0 and 1 due to the assumption that there is no restriction in depositing and borrowing of any amount of cash from the money market account.

Substitute equations (3.2) and (3.3) into Equation (3.4),

$$
\begin{aligned}
& dW(t) \\
& = W(t)(1-\pi(t))r dt \\
& + \frac{W(t)\pi(t) f(t,T_B)}{1+f(t,T_B)\tau_B} \times \\
& \left[\left(1+\frac{(\sigma^f(t,T_B))^2 f(t,T_B)\tau_B}{1+f(t,T_B)\tau_B}+\sigma^f(t,T_B)\tau_B\lambda\right) dt-\sigma^f(t,T_B)\tau_B dZ^{\mathbb{P}}(t)\right]
\end{aligned}
$$

which is the wealth process that describes the investor's wealth over the period $[t,T]$. Note that $dW(t)$ is a controlled SDE with the optimal bond weights process, π being the control variable.

Moving on to the second step in deriving the optimal bond weights under LMM which is to maximise the utility function which takes the terminal wealth as the variable using the HJB equation.

The stochastic control method, particularly the HJB equation, is used in portfolio optimisation problem as seen by Merton (1969), Merton (1971) and Kamien and Schwartz (1991), while HJB equation is used in bond portfolio optimisation problem as seen by Korn and Kraft (2002) and Puhle (2008). Note that the bond weights are the control variable of this bond optimisation problem.

In this setting, the investor chooses the optimal bond weights process, π^* which maximise the utility function on the terminal wealth $u(W(T))$, considering potential changes in the future forward curves, $df(u,T_B)$ associated with the real measure, $\mathbb{P}$ and the potential changes in the future total wealth, $dW(u)$ where $t<u<T$. The problem could be written mathematically as follows:

$$\max_{\pi}\{\mathbb{E}^{\mathbb{P}}[u(W(T))]\}$$

subject to the initial wealth of the investor, $W(0) = W_0$ and the LMM dynamic, Equation (A.3).

Bond portfolio optimisation in terms of functional $J(t, f(t, T_B), W(t); \pi)$

Define the optimised functional, $J(t, f(t, T_B), W(t); \pi)$ as

$$J(t, f(t, T_B), W(t); \pi) = \max_{\pi}\{\mathbb{E}^{\mathbb{P}}[u(W(T))]\}$$

There are certain conditions in which the above functional should satisfy in order to for it to be bounded. Such conditions are as follows:

- Recall from Definition 2.3.1 that the control variable has to satisfy the definition admissible control variable.
- The function $u(W(T))$ is a continuous function that satisfy the polynomial growth conditions, $||u(W(T))|| \leq C(1 + ||W(T)||^k)$ where $C > 0$ and $k \in \mathbb{N}$.

Referring to LMM that is described in Equation (A.3). The LMM process would be different when different measure is considered. The above functional is assumed to be subjected to the following case of the LMM process

$$df(t, T_B) = \sigma^f(t, T_B) f(t, T_B) dZ^{\mathbb{Q}}(t)$$

The above form of the LMM process assumes forward measure is taken to be the measure. The forward measure is associated with a numeraire which is a zero coupon bond. The LMM process would coincide with the Black formula if the forward measure is chosen as stated in Section A.2.1, i.e. the interest rate follows a lognormal process. More information could be found in appendix A.2.1.

However, in real measure, $\mathbb{P}$, the LMM process would be

$$df(t, T_B) = -\sigma(t, T_B)f(t, T_B)\lambda + \sigma^f(t, T_B)f(t, T_B)dZ^{\mathbb{P}}(t)$$

As observed in Merton (1969) and Merton (1971), that the functional at terminal time T is

$$J(T, f(T, T_B), W(T); \pi) = u(W(T))$$

Note that the above equation only holds at time T, i.e. the end of the investment period. As for time between the start of the investment period and the end of the investment period, the HJB equation is applied to obtain the following equation:

$$J(t, f(t, T_B), W(t); \pi) = \max_{\pi}\{\mathbb{E}^{\mathbb{P}}[J(t+\Delta t, f(t+\Delta t, T_B), W(t+\Delta t)]\} \quad (3.5)$$

Next is to find $J(t+\Delta t, f(t+\Delta t, T_B)), W(t+\Delta t)$. Applying Taylor's theorem on optimsed functional J around $(t, f(t, T_B), W(t))$,

$$\begin{aligned} & J(t+\Delta t, f(t+\Delta t, T_B), W(t+\Delta t)) \\ = \;& J(t, f(t, T_B), W(t)) + J_t\Delta t + J_f\Delta f(t, T_B) + J_W\Delta W(t) \\ + \;& \frac{1}{2}J_{ff}(\Delta f(t, T_B))^2 + J_{Wf}\Delta W(t)\Delta f(t, T_B) + \frac{1}{2}J_{WW}(\Delta W(t))^2 \quad (3.6) \end{aligned}$$

Taking expectation at time t on Equation (3.6) with $\mathbb{E}^{\mathbb{P}}$ are short written as $\mathbb{E}$,

$$\begin{aligned}
& \mathbb{E}[J(t+\Delta t, f(t+\Delta t, T_B), W(t+\Delta t)] \\
= \;& J(t, f(t,T_B), W(t)) + J_t\Delta t + J_f\mathbb{E}[\Delta f(t,T_B)] + J_W\mathbb{E}[\Delta W(t)] \\
+ \;& \frac{1}{2}J_{ff}\mathbb{E}[(\Delta f(t,T_B))^2] + J_{Wf}\mathbb{E}[\Delta W(t)\Delta f(t,T_B)] \\
+ \;& \frac{1}{2}J_{WW}\mathbb{E}[(\Delta W(t))^2]
\end{aligned} \tag{3.7}$$

Insert Equation (3.7) into Equation (3.5) and simplify,

$$\begin{aligned}
0 \;=\;& \max_{\pi}\{J_t\Delta t + J_f\mathbb{E}[\Delta f(t,T_B)] + J_W\mathbb{E}[\Delta W(t)] \\
+ \;& \frac{1}{2}J_{ff}\mathbb{E}[(\Delta f(t,T_B))^2] + J_{Wf}\mathbb{E}[\Delta W(t)\Delta f(t,T_B)] \\
+ \;& \frac{1}{2}J_{WW}\mathbb{E}[(\Delta W(t))^2]\}
\end{aligned} \tag{3.8}$$

The expectations in the above equation have to be found before proceeding further.

Taking conditional expectation on the wealth process, conditioned that the current wealth is W, on the LMM process under the $\mathbb{P}$ measure, the following follows:

- $\mathbb{E}[\Delta f(t,T_B)] = -\sigma^f(t,T_B)f(t,T_B)\lambda\Delta t$
- $\mathbb{E}[(\Delta f(t,T_B))^2] = [\sigma^f(t,T_B)f(t,T_B)]^2\Delta t$
- $\mathbb{E}[\Delta W(t)] = W(t)(1-\pi(t))r\Delta t + \frac{W(t)\pi(t)f(t,T_B)}{1+f(t,T_B)\tau_B}[(1+\frac{(\sigma^f(t,T_B))^2 f(t,T_B)\tau_B}{1+f(t,T_B)\tau_B} + \sigma^f(t,T_B)\tau_B\lambda)\Delta t]$
- $\mathbb{E}[(\Delta W(t))^2] = -\frac{W^2(t)\pi^2(t)f^2(t,T_B)}{(1+f(t,T_B)\tau_B)^2}(\sigma^f(t,T_B))^2\tau_B^2\Delta t$
- $\mathbb{E}[\Delta f(t,T)dW(t)] = -\frac{W(t)\pi(t)f(t,T_B)}{1+f(t,T_B)\tau_B}(\sigma^f(t,T_B))^2 f(t,T_B)\tau_B\Delta t$

Substitute the above expectations to Equation (3.8), divide by Δt,

$$
\begin{aligned}
0 &= J_{\tau_B} - J_f \sigma^f(t,T_B) f(t,T_B)\lambda + \frac{1}{2} J_{ff} (\sigma^f(t,T_B))^2 f^2(t,T_B) \\
&\quad - \frac{J_{Wf} W(t)\pi(t)(\sigma^f(t,T_B))^2 f^2(t,T_B)\tau_B}{1+f(t,T_B)\tau_B} \\
&\quad + \frac{J_{WW} W^2(t)\pi^2(t)(\sigma^f(t,T_B))^2 f^2(t,T_B)\tau_B^2}{2(1+f(t,T_B)\tau_B)^2} \\
&\quad + J_W W(t)(1-\pi(t))r \\
&\quad + \frac{J_W W(t)\pi(t) f(t,T_B)}{1+f(t,T_B)\tau_B} \times \\
&\quad \left(1 + \frac{(\sigma^f(t,T_B))^2 f(t,T_B)\tau_B}{1+f(t,T_B)\tau_B} + \sigma^f(t,T_B)\tau_B\lambda\right) \qquad (3.9)
\end{aligned}
$$

In order to find the optimal bond weights, π^* which will maximise the above equation, differentiate the above equation with respect to π and rearrange the equation :

$$
\begin{aligned}
\pi^*(t) &= \frac{J_{Wf}(1+f(t,T_B)\tau_B)}{J_{WW} W(t)\tau_B} \\
&\quad + \frac{J_W r (1+f(t,T_B)\tau_B)^2}{J_{WW} W(t)(\sigma^f(t,T_B))^2 f^2(t,T_B)\tau_B^2} \\
&\quad - \frac{J_W(1+f(t,T_B)\tau_B)}{J_{WW} W(t)(\sigma^f(t,T_B)^2 f(t,T_B)\tau_B^2} \times \\
&\quad \left(1 + \frac{(\sigma^f(t,T_B))^2 f(t,T_B)\tau_B}{1+f(t,T_B)\tau_B} + \sigma^f(t)\tau_B\lambda\right) \qquad (3.10)
\end{aligned}
$$

There are unknown variables in Equation (3.10) such as J_W, J_{Wf} and J_{WW}. Additional assumptions are needed in order to remove these unknown variables from the optimal bond weight equation. The derivation of the optimal bond weights under LMM will continue from Equation (3.10). The partial differential equation, Equation (3.9) will only be brought to our attention when there is no way to remove the unknown variables from the optimal bond weight equation.

The form of the utility function

As proposed by Merton (1971) and Puhle (2008), the utility function is assumed to be the form yielding Constant Relative Risk Aversion (”CRRA”), particularly, $u(W(T)) = W^{\gamma}(T)$ where $0 < \gamma < 1$ is called the risk aversion coefficient of the investor. Other forms of utility functions could be considered as future work.

Note that the CRRA function satisfy the second condition in order for the functional to be bounded by ∞. The second condition states that the CRRA function should be a continuous function and satisfies the polynomial growth function.

Bond portfolio optimisation in terms of functional $G(t, f(t, T_B); \pi)$

Note that the optimal bond weights in Equation (3.10) depends on the optimised functional, $J(t, f(t, T_B), W(t); \pi)$ which is so far, unknown. In order to proceed further, Korn and Kraft (2002) recommends the following separation,

$$J(t, f(t, T_B), W(t); \pi) = G(t, f(t, T_B); \pi)W^{\gamma}(t)$$

where $0 < \gamma < 1$ and the functional $G(t, f(t, T_B))$ has a boundary condition $G(T, f(T, T_B)) = 1$.

The partial derivatives of $J(t, f(t, T_B), W(t); \pi)$ in terms of $G(t, f(t, T_B); \pi)$ and $W(t)$ would be as follows:

- $J_t = G_t W^{\gamma}(t)$
- $J_f = G_f W^{\gamma}(t)$
- $J_{ff} = G_{ff} W^{\gamma}(t)$
- $J_W = \gamma G W^{\gamma-1}(t)$
- $J_{WW} = \gamma(\gamma - 1) G W^{\gamma-2}(t)$
- $J_{Wf} = \gamma G_f W^{\gamma-1}(t)$

Substitute the above equations into Equation (3.10),

$$\begin{aligned}\pi^*(t) &= \frac{G_f(1+f(t,T_B)\tau_B)}{(\gamma-1)G\tau_B} \\ &+\frac{r(1+f(t,T_B)\tau_B)^2}{(\sigma^f(t,T_B))^2 f^2(t,T_B)\tau_B^2} \\ &-\frac{(1+f(t,T_B)\tau_B)}{(\gamma-1)(\sigma^f(t,T_B)^2 f(t,T_B)\tau_B^2} \times \\ &\left(1+\frac{(\sigma^f(t,T_B))^2 f(t,T_B)\tau_B}{1+f(t,T_B)\tau_B}+\sigma^f(t)\tau_B\lambda\right) \qquad (3.11)\end{aligned}$$

Bond portfolio optimisation in terms of functions $A(t)$ and $f(t,T_B)$

As per in the previous section, next is to further separate the interdependency of t and $f(t,T_B)$ in $G(t,f(t,T_B))$. Express $G(t,f(t,T_B))$ as follows:

$$G(t,f(t,T_B)) = A(t)f(t,T_B) \qquad (3.12)$$

such that $A(T)=\frac{1}{f(T,T_B)}$. The end point for $A(t)$ is fixed in this manner so that $G(T,f(T,T_B))=1$ is satisfied.

With the above expression of $G(t,f(t,T_B))$ in terms of $A(t)$, some of the partial derivative of $G(t,f(t,T_B))$ are as follows,

- $G_t = A'(t)f(t,T_B)$
- $G_f = A(t)$
- $G_{ff} = 0$

Substituting Equation (3.12) into Equation (3.11), the optimal bond weights would then be:

$$\begin{aligned}\pi^*(t) &= \frac{1+f(t,T_B)\tau_B}{(\gamma-1)f(t,T_B)\tau_B} \\ &+\frac{r(1+f(t,T_B)\tau_B)^2}{(\sigma^f(t,T_B))^2 f^2(t,T_B)\tau_B^2} \\ &-\frac{(1+f(t,T_B)\tau_B)}{(\gamma-1)(\sigma^f(t,T_B)^2 f(t,T_B)\tau_B^2} \times \\ &\left(1+\frac{(\sigma^f(t,T_B))^2 f(t,T_B)\tau_B}{1+f(t,T_B)\tau_B}+\sigma^f(t,T_B)\tau_B\lambda\right) \qquad (3.13)\end{aligned}$$

Note that $A(t)$ is not present in $\pi^*(t)$.

In the next chapter, this optimisation method will be presented using a numerical illustrated with real life data.

CHAPTER 4

NUMERICAL ILLUSTRATION

This chapter illustrates the proposed optimisation method mentioned in previous chapter using real life data. Calibration method is found in Section 4.1. Section 4.2 contains the calibration results, optimal bond weights and simulated portfolio return.

4.1 Calibration method

Calibration is the process where the model parameters are set in such a way so that the agreement between the proposed model and the data provided is maximised.

4.1.1 Calibration of Vasicek model

The Vasicek model is defined as follows:

$$dr(t) = a(b - r(t))dt + \sigma^r dZ^{\mathbb{Q}}(t)$$

where

- $r(t)$ is the short rate at time t
- a is the mean reverting rate
- b is the long run rate
- σ^r is the volatility of the short rate
- $Z^{\mathbb{Q}}(t)$ is the Brownian motion under $\mathbb{Q}$-measure

The calibration is done using maximum likelihood estimator method as suggested by Berg (2011). The formulas are attached in the appendix C.

4.1.2 Calibration of LMM

For this dissertation, $\sigma^f(t, T_B)$ is assumed to take the form of a parametric equation while $\rho(i,j)$ takes the form of a reduced rank parametric function. The specific parametric equation for $\sigma^f(t, T_B)$ and $\rho(i,j)$ are (4.1) and (4.2) respectively.

$$\sigma^f(t, T_i) = (\alpha_4 + \alpha_1(T_i - t))e^{-\alpha_2(T_i - t)} + \alpha_3 \quad (4.1)$$

where $-2 \leq \alpha_1, \alpha_4 \leq 2, 0 \leq \alpha_2, \alpha_3 \leq 2$ and

$$\begin{aligned} \rho(i,j) &= \cos\Phi(i) - \cos\Phi(j) \quad (4.2) \\ &- \sin\Phi(i)\sin\Phi(j)(1 - \cos\Theta(i) - \Theta(j)) \\ \Phi(t, T_i) &= \phi_1 + (\phi_2 + \phi_3(T_i - t))e^{-\phi_4(T_i - t)} \\ \Theta(t, T_i) &= \theta_1 + (\theta_2 + \theta_3(T_i - t))e^{-\theta_4(T_i - t)} \end{aligned}$$

where $\frac{-\pi}{2} \leq \phi_i, \theta_i \leq \frac{\pi}{2}$ for $i = 1, 2, 3, 4$ and $T_i, i = 1, 2, 3, \ldots$ are the reset dates.

Brigo and Mercurio (2006) suggested this parametric equation for $\sigma^f(t, T_B)$ because of the faster calibration process compared to piecewise-constant functions. This is because parametric equation has fewer paremeters compared to the number of parameters in piecewise-constant function. Fewer parameters would speed up the calibration process.

Gatarek et al. (2006) suggested this parametric equation for $\rho(i,j)$. They have noted that the correlation matrix produced by this parametric equation deviates further from to the historical correlations compared to piecewise-constant correlation function. However, this algorithm may need to be run many times unlike derivative pricing which run only once, therefore, calibration speed would become a deciding factor on which method to use. They have commented that though the result is worse for parametric equation compared to piecewise-constant function, the calibration results of the parametric equation is acceptable.

Calibration algorithm

The algorithm in Nelder and Mead (1965) is called the downhill simplex algorithm or the Nelder-Mead method which is used to calibrate the LMM parameters. This method is commonly used to find the minimum or maximum of an objective function in a many-dimensional space. The advantage of this method is its application to non-linear optimisation problem without using the gradient.

Calibration of LMM's $\sigma^f(t, T_B)$ to the cap volatilities

The market data provider would normally quote cap volatilities. This cap volatilities provided by the market data provider describe a constant volatility in the forward rate throughout the duration of the cap. As assumed in Section 4.1.2 that the volatilty, particularly, $\sigma^f(t, T_B)$ for LMM would follow Equation (4.1). The work around this rely on the understanding that a cap is a series of caplets. Therefore, it is possible to strip the cap volatilities into caplet volatilities. Thus, LMM's $\sigma^f(t, T_B)$ would be calibrated to the stripped caplet volatilities instead.

The following describes the procedure for stripping the caplet volatilities from the market quoted cap volatilities and then calibrate the parameters of the parametric volatility function to the stripped caplet volatilities. This algorithm is taken from Gatarek et al. (2006).

1. Determine the parameters such as forward rate, strike rate, starting date and maturity date (reset date) for each caplet.

2. Denote T_m as the time at m. The standard KLIBOR has a tenor of 3 months, therefore, the caplet volatility should reflect a 3 months volatility. Linear interpolate the market cap volatilities to obtain $\sigma(t, T_{0.25}), \sigma(t, T_{0.5})$, $\ldots, \sigma(t, T_N)$ where T_N is the maturity time for the longest duration cap volatility.
3. Set the first caplet volatility to the first linear interpolated cap volatility, $\sigma(t, T_{0.25})$.
4. Compute this cap value .
5. Compute the cap value for the next reset date using the next linear interpolated cap volatility.
6. The difference between these cap values would be the value of the caplet maturing in the next reset date. Use numerical root finding methods to find the caplet volatility.
7. Repeat step (4) to (6) with next reset date with the recently computed caplet volatiltiy and continue up to the last reset date.

These caplet volatilities will be then used to calibrate the parameters in the parametric function for $\sigma^f(t, T_i)$ as mentioned in Section 4.1.2. The parameters will be chosen such that the sum of all differences between the parametric volatilities and the caplet volatilities would be minimal.

The procedure is as follow

1. Initialise the volatility parameters, $\alpha_1, \alpha_2, \alpha_3$ and α_4.
2. Compute $\sigma^f(t, T_i) = (\alpha_4 + \alpha_1(T_i - t))e^{-\alpha_2(T_i - t)} + \alpha_3$.
3. Repeat step (1) to (2) with different initialisation of $\alpha_1, \alpha_2, \alpha_3$ and α_4 which are determined by simplex downhill algorithm until the difference between the market cap volatilities and $\sigma_i^f(t)$ are minimised.

Calibration of LMM's $\rho(i, j)$ to the swaption volatilities

This calibration process requires the calibrated parameters of the parametric volatility function from previous section. The following describes the procedure for calibrating the correlation, $\rho(i,j)$ of LMM to the market swaption volatilities. This algorithm is modified from Gatarek et al. (2006).

1. Initialise intermediate correlation parameters $\xi_1,\ldots,\xi_4$ and $\eta_1,\ldots,\eta_4$.
2. Compute $\theta(T_i,T_{i+1}) = \xi_1+(\xi_2+\xi_3 f(T_i,T_{i+1}))e^{-\xi_4 f(T_i,T_{i+1})}$, $\phi(T_i,T_{i+1}) = \eta_1+(\eta_2+\eta_3 f(T_i,T_{i+1}))e^{-\eta_4 f(T_i,T_{i+1})}$ and $\psi(T_i,T_{i+1}) = 0.25 f(T_i,T_{i+1})^2$ for $i=0,\ldots,N-1$ where N is the number of caplet determined from earlier section.
3. Compute the correlation using the following formula:

$$\rho(i,j) = \cos(\phi(i)-\phi(j)) - \sin(\phi(i))\sin(\phi(j))(1-\cos(\theta(i)-\theta(j)))$$

4. Compute $\Psi = \sum_{i=1}^{N}\psi(T_i,T_{i+1})$.
5. Compute

$$\begin{aligned}\sigma_{\text{swaption}}(i,j) &= \frac{\psi(T_i,T_{i+1})\psi(T_j,T_{j+1})}{\Psi^2}\times \\ &\quad \rho(i,j)\sum_{k=1}^{N}\sigma^f(t,T_{i-k})\sigma^f(t,T_{j-k})f(T_i,T_{i+1})\end{aligned}$$

6. Repeat step (2) to (5) with different initialisation of $\xi_1,\ldots,\xi_4,\eta_1,\ldots,\eta_4$ which are determined by simplex downhill algorithm until the difference between the market swaption volatilities and σ_{swaption} are minimised.

4.2 Numerical results

4.2.1 Investment strategy

The investment period selected is from 31st December 2012 to 31st December 2013. Considering a hypothetical 2 year zero coupon bond issued on 31st December 2012 and matures on 31st December 2014 that is available for investment at any amount without transaction cost. There are two cases: (i) the optimal weight algorithm will run at fixed rebalancing period throughout the investment period, i.e. new bond optimal weight is generated after every rebalancing period and (ii) the optimal weight algorithm will only run on the first day of the investment period.

4.2.2 Source of data

The 1 year KLIBOR from 31st December 2003 to 31st December 2013 were used to parameters calibration for the Vasicek model. These rates are taken from Bloomberg. For the case where the optimal weight algorithm that will be run at fixed rebalancing period, 2 years of data will be used in the calibration while for the case where running the optimal weight algorithm at the first day of the investment period, 10 years historical data is used during the calibration process. The term structure on 31st December 2012 will be used to determine the market price of interest rate risk, λ.

The σ and ρ found in the LMM process are calibrated using the cap volatilities and swaption volatilities. The cap volatilities and swaption volatilities chosen have KLIBOR as the underlying . The cap volatilities and swaption volatilities from 31st December 2012 to 31st December 2013 were taken from Bloomberg. A point to mention is that both cap volatilities and swaption volatilities are the volatility implied from the at the money ("ATM") cap and ATM swaption respectively. Volatility smile is not considered, i.e. a flat volatility surface was assumed in the parameter calibration process.

An example of the cap volatility and swaption ATM volatility are as per Table 4.1 and 4.2 respectively.

Table 4.1: ATM Cap Volatility (non percentage) on 31st December 2012 quoted by Bloomberg

Cap Maturity	Cap Volatility
1Y	0.4402
2Y	0.49
3Y	0.3546
4Y	0.2766
5Y	0.2297
7Y	0.1552
10Y	0.1338

Table 4.2: ATM Swaption Volatility (non percentage) on 31st December 2012 quoted by Bloomberg

Option Maturity	Swap Maturity	Swaption Volatility	Option Maturity	Swap Maturity	Swaption Volatility	Option Maturity	Swap Maturity	Swaption Volatility
3M	1Y	0.9627	1Y	1Y	0.2431	4Y	1Y	0.2235
3M	2Y	0.7037	1Y	2Y	0.2375	4Y	2Y	0.194
3M	3Y	0.5072	1Y	3Y	0.2374	4Y	3Y	0.1892
3M	4Y	0.3928	1Y	4Y	0.213	4Y	4Y	0.1929
3M	5Y	0.3356	1Y	5Y	0.2181	4Y	5Y	0.1818
3M	7Y	0.3505	1Y	7Y	0.2263	4Y	7Y	0.1694
3M	10Y	0.3763	1Y	10Y	0.2148	4Y	10Y	0.1509
6M	1Y	0.6888	2Y	1Y	0.215	5Y	1Y	0.2075
6M	2Y	0.5271	2Y	2Y	0.1905	5Y	2Y	0.2141
6M	3Y	0.4037	2Y	3Y	0.1963	5Y	3Y	0.1992
6M	4Y	0.3326	2Y	4Y	0.2115	5Y	4Y	0.2014
6M	5Y	0.3039	2Y	5Y	0.2269	5Y	5Y	0.1843
6M	7Y	0.3204	2Y	7Y	0.2296	5Y	7Y	0.1623
6M	10Y	0.3309	2Y	10Y	0.2186	5Y	10Y	0.1487
9M	1Y	0.3786	3Y	1Y	0.219	7Y	1Y	0.2157
9M	2Y	0.3284	3Y	2Y	0.2077	7Y	2Y	0.22
9M	3Y	0.287	3Y	3Y	0.2061	7Y	3Y	0.2107
9M	4Y	0.2473	3Y	4Y	0.2231	7Y	4Y	0.3821
9M	5Y	0.2338	3Y	5Y	0.235	7Y	5Y	0.1682
9M	7Y	0.242	3Y	7Y	0.2345	7Y	7Y	0.1539
9M	10Y	0.2416	3Y	10Y	0.2178	7Y	10Y	0.1586

4.2.3 Initial values for calibration

An example of the parameters' initial values for volatility function and correlation function are as per Table 4.3 and 4.4 respectively.

Table 4.3: Initial Volatility Parametric Parameters

	Initial Point 1	Initial Point 2	Initial Point 3
α_1	-0.542823434	0.086501002	-0.530016541
α_2	-0.883896589	0.289165258	0.082565546
α_3	0.448771119	0.246480107	-0.465263724
α_4	0.397423267	-0.446387291	0.18347311

	Initial Point 1	Initial Point 2	Initial Point 3
φ_1	-0.713724732	0.069376349	-0.184113145
φ_2	-0.858497381	-0.921667814	-0.843816519
φ_3	0.834487557	-0.834399581	-0.020101905
φ_4	0.854531765	0.436388493	-0.150059223
θ_1	0.872402072	-0.598754048	0.591253161
θ_2	0.459782362	0.050008535	-0.812005281
θ_3	-0.850332141	-0.151284099	-0.689266086
θ_4	-0.401067734	0.092608452	0.078624725

Table 4.4: Initial Correlation Parametric Parameters

Note that there are 3 initial values for all 4 parameters of the parametric volatility function. This is because the simplex downhill algorithm uses as little as 3 points located in the search region and iteratively find the next 3 points in the search region, using certain rules, until one of the points is the optimal point.

4.2.4 Calibration results

An example of the calibrated parameters for Vasicek model volatility function, LMM instantaneous volatility function and LMM instantaneous correlation function are as per Table 4.5, 4.6 and 4.7 respectively.

Table 4.5: Calibrated Vasicek Parameters

b (mean)	0.032841026
a (reverting)	45.53130754
σ^f (volatility)	0.00021014

Table 4.6: Calibrated Volatility Parametric Parameters

α_1	0.086501002
α_2	0.289165258
α_3	0.246480107
α_4	0.397423267

Table 4.7: Calibrated Correlation Parametric Parameters

Φ_1	-0.71372473
Φ_2	-0.85849738
Φ_3	0.834487557
Φ_4	0.854531765
Θ_1	0.872402072
Θ_2	0.459782362
Θ_3	-0.85033214
Θ_4	-0.40106773

4.2.5 Optimal bond weights

There are a few options when deriving the optimal bond weights under Vasicek. The options are:

- Moving window and non moving window.
- Risk averse, risk neutral and risk taking investors.
- Daily, weekly and monthly rebalancing periods.

The rationale for each options

The advantage for considering the moving window option is the incorporation of latest data during the parameters calibration process. The non moving window option has its merit with its calibrated parameters being stable.

Each investors will have different risk appetite, i.e. some investors are wiling to risk more than other investors. Therefore, this behavior is roughly captured in the parameter called risk aversion coefficient parameter.

In the dynamic asset allocation process, the portfolio weights are being rebalance (adjusted) after a period of time. This dissertation considers daily, weekly and monthly rebalancing periods as these are the common rebalancing periods considered by many fund and portfolio managers.

Explanation of each options

The moving window option is not considered in the optimal bond weights under LMM. This is because the data used to calibrate the LMM's parameter depends on one day's cap volatility and swaption volatility and not a period of historical cap volalitities and swaption volatilities. Therefore, only the non moving average window option is considered for LMM which uses the current day's cap volatility and swaption volatility in the parameters calibration process. The option of risk averse, risk neutral and risk taking investors option is still considered in the derivation of optimal bond weights under LMM.

The moving window option will use past 2 years data for the calibration of Vasicek model's parameters. Subsequently, these calibrated parameters will be used to forecast the changes in the interest rate. The data for next business day's calibration process will use the same past 2 years data however, removing the oldest data while incorporates the latest forecasted interest rate. This process is repeated until the end of the investment period.

The non moving window option will use past 10 years data for the calibration of Vasicek model's parameters. The same calibrated parameters will be used to forecast the interest rate throughtout the investment period.

The risk averse, risk neutral and risk taking investors are captured using the risk aversion coefficient, γ , which takes the values of $0.25, 0.5$ and 0.75 for risk averse, risk neutral and risk taking investor respectively.

The optimal bond weights are presented as per the following subsections:

- Vasicek model with moving window application of optimal bond weight algorithm.
- Vasicek model with non moving window application of optimal bond weight algorithm.
- LMM with non moving window application of optimal bond weight algorithm.

Vasicek model with moving window application of optimal bond weight algorithm

The bond weights for Vasicek model with risk aversion coefficient of 0.25, 0.5, 0.75 and daily, weekly and monthly rebalancing are as per Figures 4.1, 4.2 and 4.3 respectively.

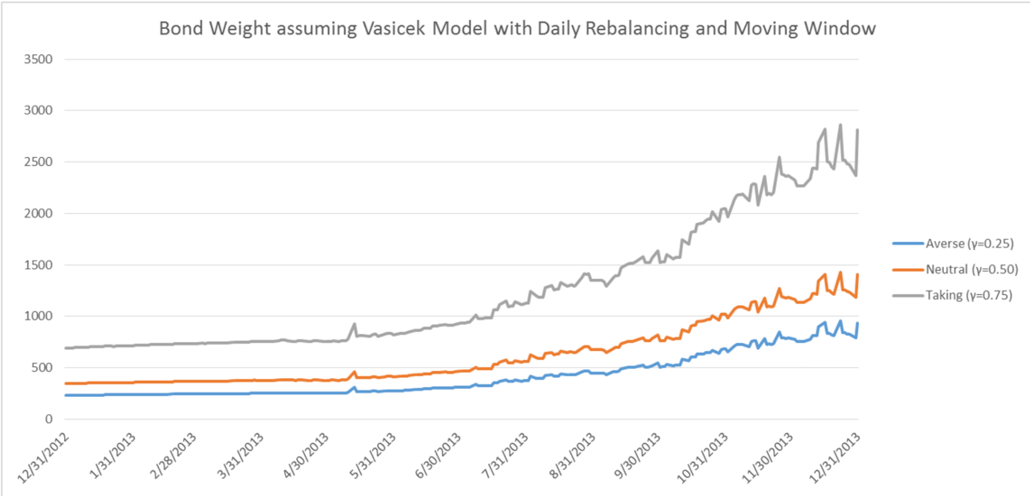

Figure 4.1: Bond Weights under Vasicek Model with Daily Rebalancing, Moving Window and Risk Aversion Coefficient of 0.25, 0.5 and 0.75

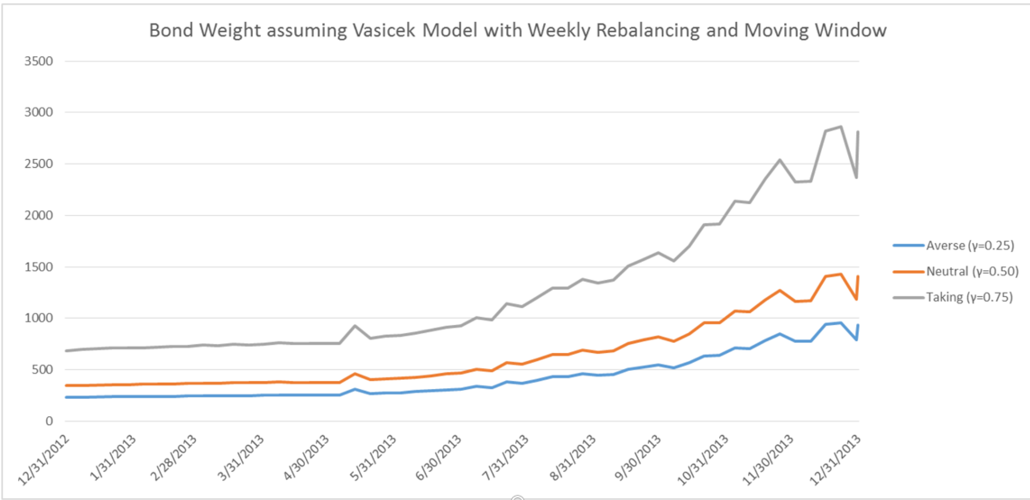

Figure 4.2: Bond Weights under Vasicek Model with Weekly Rebalancing, Moving Window and Risk Aversion Coefficient of 0.25, 0.5 and 0.75

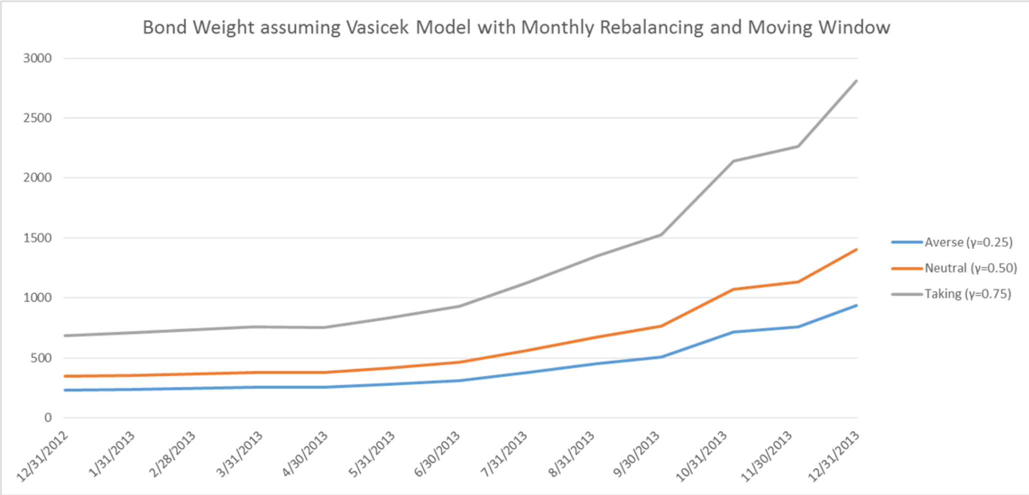

Figure 4.3: Bond Weights under Vasicek Model with Monthly Rebalancing, Moving Window and Risk Aversion Coefficient of 0.25, 0.5 and 0.75

Vasicek model with non moving window application of optimal bond weight algorithm

The bond weights for Vasicek model with risk aversion coefficient of 0.25, 0.5, 0.75 and daily, weekly and monthly rebalancing are as per Figure 4.4, 4.5 and 4.6 respectively.

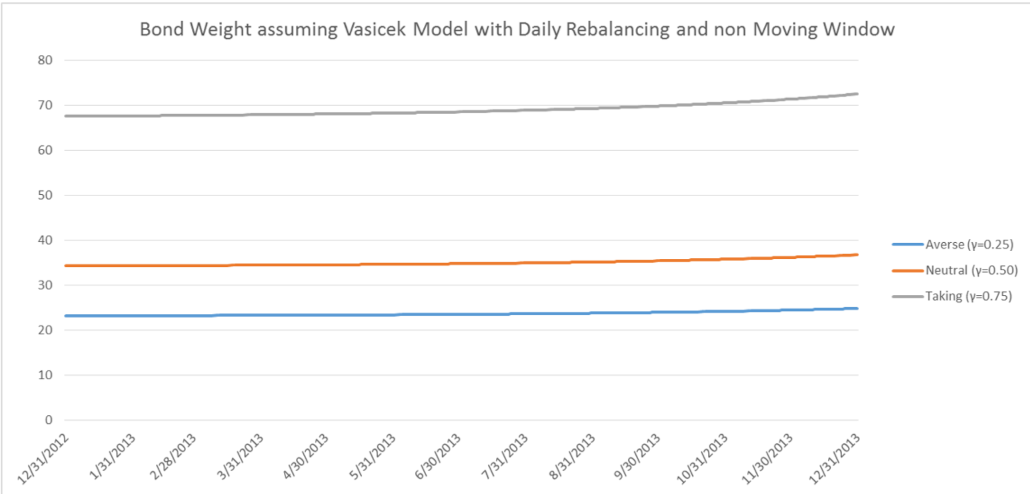

Figure 4.4: Bond Weights under Vasicek Model with Daily Rebalancing, non Moving Window and Risk Aversion Coefficient of 0.25, 0.5 and 0.75

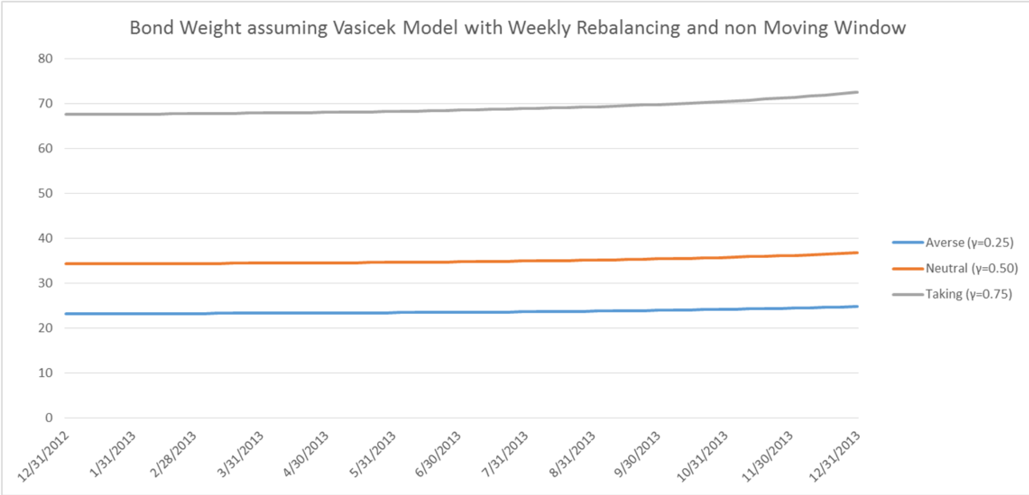

Figure 4.5: Bond Weights under Vasicek Model with Weekly Rebalancing, non Moving Window and Risk Aversion Coefficient of 0.25, 0.5 and 0.75

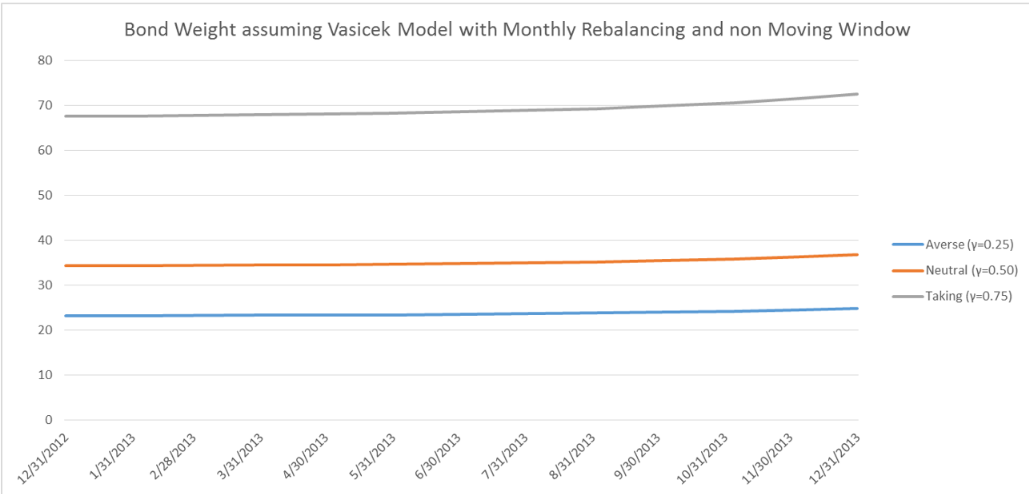

Figure 4.6: Bond Weights under Vasicek Model with Monthly Rebalancing, non Moving Window and Risk Aversion Coefficient of 0.25, 0.5 and 0.75

LMM with non moving window application of optimal bond weight algorithm

The bond weights for LMM with risk aversion coefficient of 0.25, 0.5, 0.75 and daily, weekly and monthly rebalancing are as per Figures 4.7, 4.8 and 4.9 respectively.

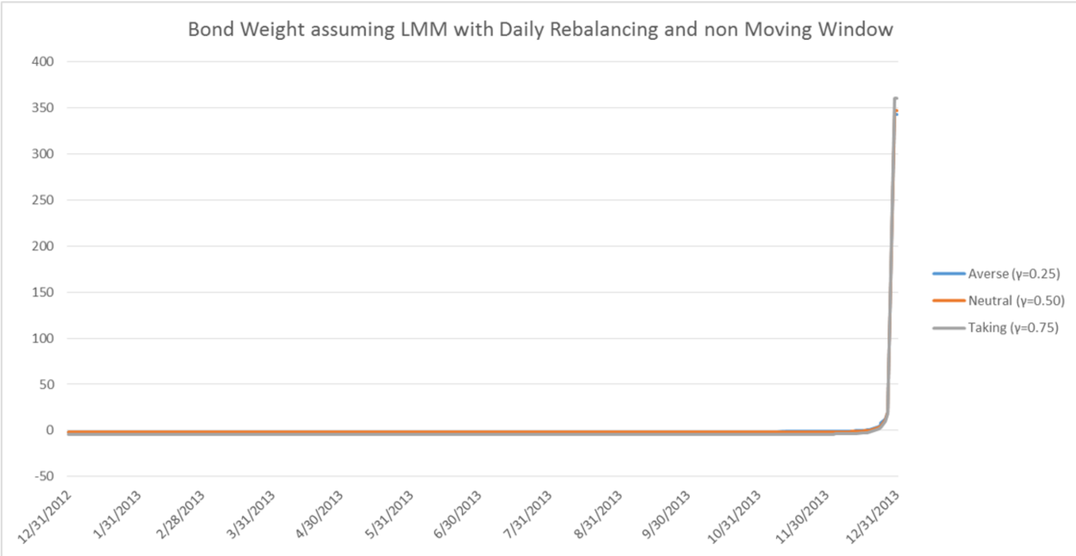

Figure 4.7: Bond Weights under LMM with Daily Rebalancing, non Moving Window and Risk Aversion Coefficient of 0.25, 0.5 and 0.75

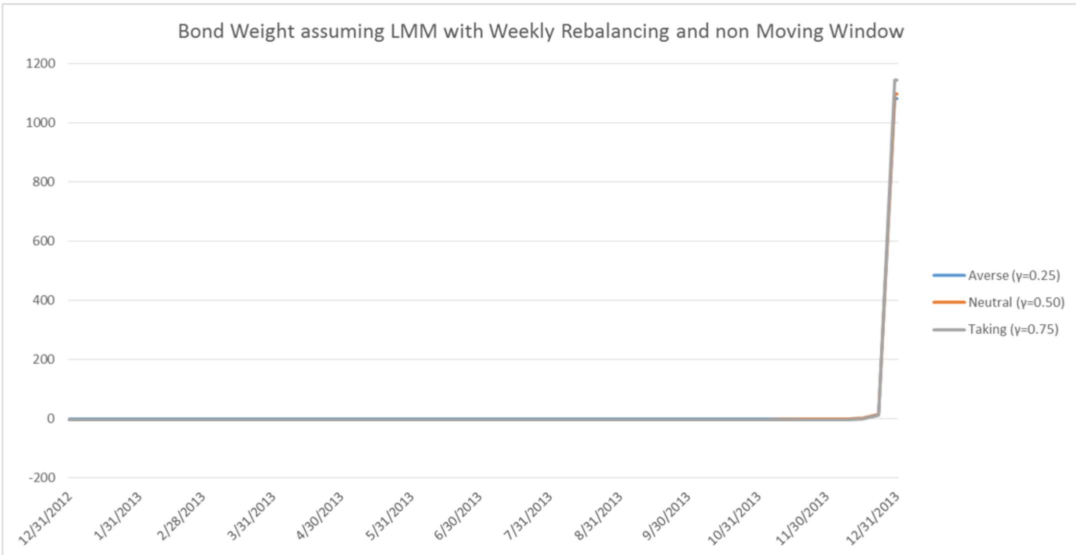

Figure 4.8: Bond Weights under LMM with Weekly Rebalancing, non Moving Window and Risk Aversion Coefficient of 0.25, 0.5 and 0.75

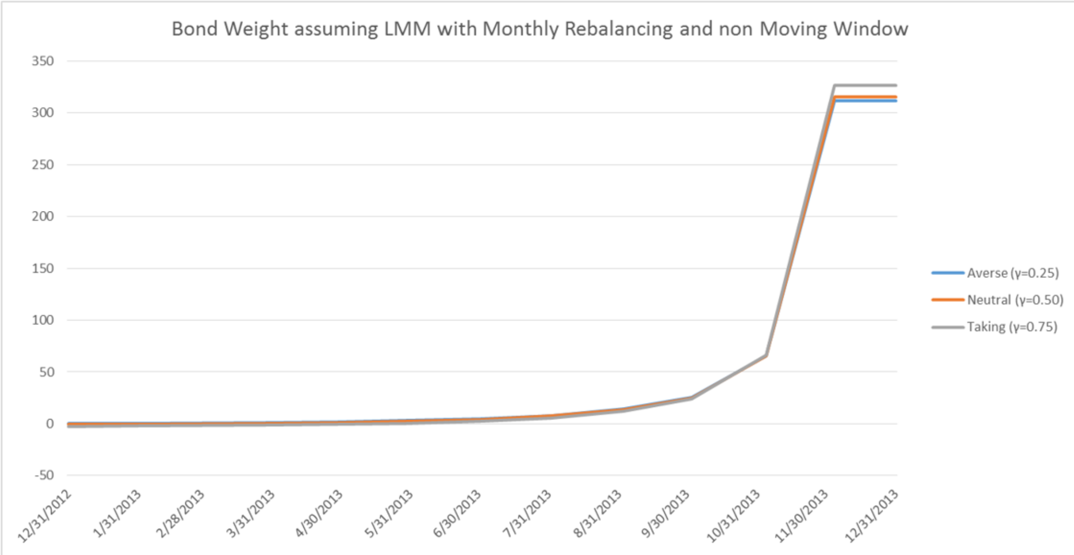

Figure 4.9: Bond Weights under LMM with Monthly Rebalancing, non Moving Window and Risk Aversion Coefficient of 0.25, 0.5 and 0.75

The y-axis of the above graphs should be interpreted as whole number rather than in percentage. Suppose the graph show that the bond weight is 100 and investor has an initial capital of MYR 1,000,000. The investor should borrow additional MYR 99,000,000 and invest MYR 100,000,000 into the zero coupon bond.

It is natural that the optimal bond weights, π would be in the $[0, 1]$ region. However, the assumption that there is no restriction in depositing and borrowing of any amount of cash from the money market account implicitly allow investors to borrow money for the purpose of investing the money into the zero coupon bond. Therefore, it is possible for the optimal bond weights to be outside of the $[0, 1]$ region.

4.2.6 Simulated invesment returns

The simulated investment returns are generated using the optimal bond weights computed from Section 4.2.5 and the interest models, particularly Vasicek model and LMM, which describe the changes in the interest rate. Firstly, the future possible interest rate paths are generated using the interest rate models. These interest rates will be used to value the zero coupon bond. Lastly, the investment returns are computed using the zero coupon bond prices on the last investment period date and the first investment period date.

This simulated returns are then compared to the actual investment return as the actual data is available during the time when this research is done. This step is used to gauge how far would the simulated investment returns deviate from the actual investment return.

The simulated investment returns are presented as per the following subsections:

- Vasicek model with moving window application of optimal bond weight algorithm
- Vasicek model with non moving window application of optimal bond weight algorithm
- LMM with non moving window application of optimal bond weight algorithm

Vasicek model with moving window application of optimal bond weight algorithm

The simulated investment returns assuming Vasicek model with risk aversion coefficient of 0.25, 0.5, 0.75 and with daily, weekly and monthly rebalancing are as per Tables 4.8, 4.9 and 4.10 respectively.

Table 4.8: Return Distribution under Vasicek model with Daily Rebalancing, Moving Window and Risk Aversion Coefficient of 0.25, 0.5 and 0.75

Daily, moving window	Averse (γ=0.25)	Neutral (γ=0.50)	Taking (γ=0.75)
Actual	74.9273%	127.3866%	397.6052%
Average	74.4175%	126.3999%	393.3930%
Max	74.4693%	126.5006%	393.8295%
Min	74.3794%	126.3259%	393.0707%
Standard Deviation	0.0003	0.0006	0.0025
Skewness	0.4899	0.4917	0.4991
Kurtosis	-1.0499	-1.0463	-1.0306

Table 4.9: Return Distribution under Vasicek model with Weekly Rebalancing, Moving Window and Risk Aversion Coefficient of 0.25, 0.5 and 0.75

Weekly, moving window	Averse (γ=0.25)	Neutral (γ=0.50)	Taking (γ=0.75)
Actual	79.0449%	134.9084%	423.2864%
Average	73.7372%	124.7739%	381.9404%
Max	74.2138%	125.6955%	385.8502%
Min	73.4120%	124.1451%	379.2719%
Standard Deviation	0.0013	0.0025	0.0107
Skewness	0.4071	0.4078	0.4096
Kurtosis	1.4133	1.4139	1.4155

Table 4.10: Return Distribution under Vasicek model with Monthly Rebalancing, Moving Window and Risk Aversion Coefficient of 0.25, 0.5 and 0.75

Monthly, moving window	Averse (γ=0.25)	Neutral (γ=0.50)	Taking (γ=0.75)
Actual	66.0614%	108.4283%	297.7207%
Average	71.4317%	118.9784%	342.2773%
Max	72.7742%	121.4943%	351.8398%
Min	69.7478%	115.8485%	330.6722%
Standard Deviation	0.0058	0.0109	0.0410
Skewness	0.1893	0.1992	0.2254
Kurtosis	-0.0116	-0.0200	-0.0413

Vasicek model with non moving window application of optimal bond weight algorithm

The simulated investment returns assuming Vasicek model with risk aversion coefficient of 0.25, 0.5, 0.75 and with daily, weekly and monthly rebalancing are as per Tables 4.11, 4.12 and 4.13 respectively.

Table 4.11: Return Distribution under Vasicek model with Daily Rebalancing, non Moving Window and Risk Aversion Coefficient of 0.25, 0.5 and 0.75

Daily, non moving window	Averse (γ=0.25)	Neutral (γ=0.50)	Taking (γ=0.75)
Actual	1.8840%	1.1683%	-0.9489%
Average	1.8989%	1.1902%	-0.9066%
Max	1.9016%	1.1942%	-0.8989%
Min	1.8963%	1.1864%	-0.9141%
Standard Deviation	0.0000	0.0000	0.0000
Skewness	0.4265	0.4263	0.4264
Kurtosis	0.4994	0.4995	0.4997

Table 4.12: Return Distribution under Vasicek model with Weekly Rebalancing, non Moving Window and Risk Aversion Coefficient of 0.25, 0.5 and 0.75

Weekly, non moving window	Averse (γ=0.25)	Neutral (γ=0.50)	Taking (γ=0.75)
Actual	1.9247%	1.2014%	-0.9388%
Average	1.9598%	1.2529%	-0.8393%
Max	1.9687%	1.2659%	-0.8141%
Min	1.9532%	1.2431%	-0.8581%
Standard Deviation	0.0000	0.0001	0.0001
Skewness	0.2651	0.2650	0.2652
Kurtosis	-0.9656	-0.9657	-0.9655

Table 4.13: Return Distribution under Vasicek model with Monthly Rebalancing, non Moving Window and Risk Aversion Coefficient of 0.25, 0.5 and 0.75

Monthly, non moving window	Averse (γ=0.25)	Neutral (γ=0.50)	Taking (γ=0.75)
Actual	1.8806%	1.1360%	-1.0689%
Average	2.0920%	1.4466%	-0.4677%
Max	2.1446%	1.5240%	-0.3178%
Min	2.0512%	1.3865%	-0.5841%
Standard Deviation	0.0003	0.0004	0.0008
Skewness	0.1418	0.1418	0.1418
Kurtosis	-1.0123	-1.0120	-1.0115

LMM with non moving window application of optimal bond weight algorithm

The simulated investment returns assuming LMM with risk aversion coefficient of 0.25, 0.5, 0.75 and with daily, weekly and monthly rebalancing are as per Tables 4.14, 4.15 and 4.16 respectively.

Table 4.14: Return Distribution under LMM with Daily Rebalancing, non Moving Window and Risk Aversion Coefficient of 0.25, 0.5 and 0.75

Daily, non moving window	Averse (γ=0.25)	Neutral (γ=0.50)	Taking (γ=0.75)
Actual	3.2147%	3.0889%	2.7285%
Average	1.8715%	1.9575%	2.5866%
Max	28.9209%	23.2143%	11.0391%
Min	-15.8418%	-37.1596%	-73.7855%
Standard Deviation	0.0392	0.0520	0.0969
Skewness	2.5779	-3.8117	-5.2821
Kurtosis	26.5936	34.7195	38.9459

Table 4.15: Return Distribution under LMM with Weekly Rebalancing, non Moving Window and Risk Aversion Coefficient of 0.25, 0.5 and 0.75

Weekly, non moving window	Averse (γ=0.25)	Neutral (γ=0.50)	Taking (γ=0.75)
Actual	7.9171%	7.8543%	7.6661%
Average	6.7877%	6.5479%	5.8308%
Max	164.9086%	161.9362%	153.0916%
Min	-44.4982%	-44.6584%	-45.1890%
Standard Deviation	0.3814	0.3768	0.3636
Skewness	2.0331	1.9897	1.8559
Kurtosis	5.3931	5.1878	4.5721

Table 4.16: Return Distribution under LMM with Monthly Rebalancing, non Moving Window and Risk Aversion Coefficient of 0.25, 0.5 and 0.75

Monthly, non moving window	Averse (γ=0.25)	Neutral (γ=0.50)	Taking (γ=0.75)
Actual	9.1844%	9.1631%	9.0991%
Average	9.1575%	9.1287%	9.0424%
Max	13.3418%	13.3261%	13.2784%
Min	5.0512%	5.0140%	4.9025%
Standard Deviation	0.0156	0.0156	0.0157
Skewness	-0.1535	-0.1518	-0.1470
Kurtosis	-0.2561	-0.2570	-0.2599

The main point of generating the investment return distribution is to know how much return could an investor expects from the initial wealth over the investment period. For example, an investor assuming Vasicek model to describe the changes in the interest rates, daily moving window and with a risk aversion coefficient of 0.25 would expect a return of 74.4175% at the end of the investment period.

The observations based on the bond weights and simulated results are,

- Risk taking investors would leverage more than risk averse investors
- The expected return for risk taking investors are higher than risk averse investors and likewise for the volatility
- Under Vasicek model, the optimal bond weights computed using parameters which are calibrated using moving window are more reactive com-

pared to those where its parameters are calibrated using non-moving window

- Under Vasicek model, more frequent rebalancing of the portfolio would have more volatile optimal bond weights.
- With the same risk aversion coefficient, Vasicek model suggests higher leverage compared to LMM. Under Vasicek model, the optimal bond weights, Equation (3.1), is high with high mean reverting rate, a and low volatility, σ^r. As shown in Figure 4.5, the mean reverting rate is high while the volatility is low. The values of these 2 parameters caused the optimal bond weights under Vasicek model to be high.
- Under both Vasicek model and LMM, as the time approaches the end of the investment period, there would be less volatility in either the short rates or the forward rates. This leads to a smaller volatility number and thus a larger optimal bond weights. Recall that the optimal bond weights for both models are inversely proportional to the volatility. This is natural, as there are less volatility among the interest rates, investors should be more confident in investing into interest rate related products such as the zero coupon bond.
- The higher simulated return based on Vasicek model is due to higher leverage than LMM with the exception for LMM with weekly rebalancing and non moving window. Close to 3000 times leverage for risk taking investors based on Vasicek model with moving window to as high as 300 times leverage for risk taking investors based on LMM other than weekly rebalancing and non moving window. Should the monthly rebalancing for LMM is as responsive as the daily and weekly rebalancing, I believe the optimal bond weights would be much higher than the optimal bond proposed by Vasicek model.

Caution should be taken for those who wish to invest based on Vasicek model on a moving window, daily rebalancing and weekly rebalancing basis, in view that the actual result is higher than the simulated results and avoid other strategies mentioned here. In hindsight, these results hold true, however, these results may not hold true in the future. Furthermore, the assumptions made earlier are not reflective of the actual financial market conditions.

CHAPTER 5

CONCLUSION AND FUTURE WORK

Consider the situation where an investor has set a fixed period to invest as well as a fixed amount of initial wealth. This investor would invest certain percentage of the wealth (called the bond weights) in a chosen zero coupon bond or deposit in a money market account. How would the investor allocate the wealth into these investment vehicles at the start of the investment period and how should the these bond weights would change, on a fixed interval basis, such that the investor would maximise his utility from the total wealth at the end of the investment period. In other words, what should be the bond weights in the zero coupon bond and the remaining in the money market account such that the investor's satisfaction will be the highest at the end of the investment period.

Assume that money market would generate interest at a fixed rate while most finance literature will state that the price of the zero coupon bond is inversely proportional to the interest rate level. This dissertation assume that the changes in the interest rate level follow certain models.

Korn and Kraft (2002) and Puhle (2008) has studied such problem assuming that the changes in the interest rate level follow some short rate models such as Vasicek (1977) model and Cox et al. (1985) model. This dissertation, however, studied such problem assuming that the changes in the interest rate level follow some market models such as the Brace et al. (1997) (BGM) model also known as LIBOR Market Model (LMM). A comparison was done on the bond weights process produced by assuming Vasicek model and LMM.

The potential future works mentioned throughout this dissertation are (i) optimsation methods other than stochastic control, (ii) other form of utility function for $u(W(T))$, (iii) taking volatility smile into consideration during the calibration of parameters for LMM, (iv) assume non constant market price of interest rate risk, λ, (v) considering more than one (1) bond to constitute the portfolio, (vi) adding stock into the portfolio and (vii) subject the optimsed portfolio to Value at Risk (VaR) constraint as a risk management tool.

Some papers that would be a leading point to some of the above potential future work mentioned above are as follows:

Chiu and Wong (2014) has applied the same optimisation method on a Markowitz portfolio efficient portfolio while considering the contingent claims arising from insurance products. In other words, this optimisation method is not only applicable for portfolios managed under asset approach but also for portfolios managed using asset liability approach. This dissertation can be expanded to optimising a bond portfolio such that it is sufficient in meeting all contingent claims of an insurance companies.

Dai and Yue (2016) developed an optimal control approach to recover the risk neutral drift term of the stochastic volatility. This paper shows the calibration of a stochastic volatility's drift term to the available option volatilities found in the market. This calibration process also consider the relationship between volatilty and option's strike, i.e. volatility smile.

The observations based on the bond weights and simulated results are (i) higher simulated return based on Vasicek model is due to higher leverage than LMM. Close to 3000 times leverage for risk taking investors based on Vasicek model to 4 times leverage for risk taking investors based on LMM, (ii) higher expected return and volatility for risk taking investors than risk averse investors and (iii) Vasicek model with moving window performed better than non moving window where slight improvement for LMM with moving window over non moving window.

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APPENDIX A

LIBOR MARKET MODEL

A.1 Definition of a market model

Prior to 1997 where the LMM process has yet to be introduced to the quantitative community, many traders will usually price interest rate derivatives using the standard Black formula. Back then, there was no SDE that would describe the LIBOR or the forward rate as most interest rate SDEs of those time describe short rate.

There is a need for a SDE that describes the forward rate especially LIBOR and that SDE would coinside with the market practice of using Black formula to price interest rate derivatives. There are many ways to define a SDE, such as Hunt and Kennedy (2004) has specified some steps to identify a SDE that fulfills the two requirements mentioned above. The steps are:

1. Specify a SDE for the forward LIBOR and determine the necessary relationship between the drift and diffusion terms for any corresponding term structure model to be arbitrage free
2. Prove that a solution exists and it is unique to the SDE defined in step 1
3. Check that, for this solution, there exists a numeraire pair $(N, \mathbb{N})$ for which all numeraire re-based bonds defined by the SDE are martingales under the measure $\mathbb{N}$
4. Demonstrate that this particularly defined model can be extended to a model for the whole term structure in such a way that the extended model also admits a numeraire pair

A.2 LMM process

The concept of martingale measure is important in the models for pricing financial derivatives. The valuation of financial derivatives are viewed as stochastic processes. There are some stochastic processes in financial derivatives that are supermartingale with an American style option being the standard example. However, most processes would be a martingale once these stochastic processes are adjusted using a numeraire.

A numeraire is defined to be a tradeable economic entity which all other prices are expressed. An example of numeraire is currency of a country. In other words, when a stock is said to be traded at \$1.20, this means that \$1.20 would be the tradeable economic entity where the price of a stock is expressed.

This paper would first assume that LMM is a no-arbitrage model then prove it mathematically later that it is indeed no-arbitrage. Heuristically, if LMM is not a no-arbitrage model i.e. the application of LMM model would lead to a situation where arbitrage might exist, it is common understanding that, on the buy side, investors will exploit this arbitrage. Eventually, the arbitrage opportunity would vanish. On the other hand, on the sell side, traders who are aware that LMM would lead to an arbitrage opportunity would adjust down their valuation, else no one would buy from them. This rejects the initial hypothesis that LMM is a no-arbitrage model.

Harrison and Kreps (1979) states that if a pricing model admits no simple free lunches i.e. no arbitrage exists then there is a one-to-one correspondence between equivalent martingale measure $\mathbb{Q}^*$ and the pricing functional. This means that each price or valuation of the interest rate derivative or bond would correspond to an equivalent martingale measure.

What should the equivalent martingale measure be? Before answering that question, some understanding of forward rates and the HJM framework have to be established.

Let $P(0,T)$ be the price of a zero coupon bond expiring at time T observed at time 0.

Definition A.2.1. The forward rate, $f(0;T,\Delta T)$ for the period $[T,T+\Delta T]$ observed at time 0, is defined as

$$P(0,T) = P(0,T+\Delta T)(1+\Delta T f(0;T,\Delta T))$$

Rearranging the above equation, we have

$$f(0;T,\Delta T) = \frac{1}{\Delta T}\left(\frac{P(0,T)}{P(0,T+\Delta T)} - 1\right)$$

Under the HJM framework, the forward rate process would be

$$df(0;T,\Delta T) = \left(\sigma(T,T+\Delta T)\int_0^{\Delta T}\sigma(T,T+s)ds\right)dt + \sigma(T,T+\Delta T)dZ(T)$$

A.2.1 LMM process under forward measure and forward tenor

Theorem A.2.1. The process of a forward rate, $f(t;T,\Delta T)$, is a martingale under the forward measure $\mathbb{Q}^{T+\Delta T}$.

Proof. With the following forward rate,

$$f(t;T,\Delta T) = \frac{1}{\Delta T}\left(\frac{P(t,T)}{P(t,T+\Delta T)} - 1\right)$$

Apply Itô's Lemma

$$\begin{aligned} d\left(\frac{P(t,T)}{P(t,T+\Delta T)}\right) &= \frac{P(t,T)}{P(t,T+\Delta T)}(\sigma(t,T)-\sigma(t,T+\Delta T)) \\ &\quad \times (dZ(t) - \sigma(t,T+\Delta T)dt) \end{aligned} \tag{A.1}$$

Since $\widehat{Z(t)} = Z(t) - \int_0^t \sigma(s,T+\Delta T)ds$ is a $\mathbb{Q}^{T+\Delta T}$ Brownian motion. So, $f(t,T,\Delta T)$ is a $\mathbb{Q}^{T+\Delta T}$ martingale. □

By Theorem A.2.1, we can write the forward rate process under the forward measure as

$$df(t,T,\Delta T) = \sigma(T,T+\Delta T)f(t,T,\Delta T)\widehat{dZ(t)} \qquad \text{(A.2)}$$

Note that this Equation (A.2) holds true only for the one particular forward rate that will be used to determine the interest cash flow at time $T+\Delta T$. In general, there will be a drift term whenever a different measure is considered. Equation (A.2) implies that the LMM process follows a lognormal distribution with mean 0 and standard deviation of $\sigma(T,T+\Delta T)$ under the forward measure.

Recall from Section A.1, Equation (A.2) fulfills the third step in defining the market model. However, this equation is restricted to the forward measure. Step one in defining the market model requires that the SDE for the forward LIBOR be able to describe the process of forward LIBOR under measures different from the forward measure. This leads to the next subsection.

A.2.2 LMM process under non forward measure

In Section A.2.1 we consider the LMM process under forward measure. How would the LMM process looks like if the measure is not the forward measure. This situation arises frequently especially in bonds with more than one coupon payment or swaps. There will be a lot of cash flow to considered and it would be hard to justify which period should be chosen as the forward period where the forward measure is defined on.

Under measure different from the forward measure, the LMM process under these measures make some adjustment in the drift term to account for the difference in measure.

Recall that the LMM process models the forward rate process, i.e. the LMM process describe how forward rates would be in the future with a certain probability.

In the following derivation of the LMM process, we will use slightly different notional for forward rate as to make equation neat. Consider the situation where there is a multiple coupon paying bond which pays coupons at time $T_1, T_2, \ldots, T_n$. The tenor, i.e. the time between two coupon paying times is denoted by τ. The instantaneous forward rate is the forward rate starting today to the maturity date observed date, i.e. $f(0, 0, \Delta T)$. The simplified notation for instantaneous forward rate for time T_k to time T_{k+1} observed at time t would be $f_k(t, T)$.

Theorem A.2.2. The process of $f_k(t, T, \Delta T)$ under the measure $\mathbb{Q}^i$ for the cases (i) $i < k$, (ii) $i = k$ and (iii) $i > k$ are

$$df_k(t,T_i) = \begin{cases} \sigma_k(t,T_i)f_k(t,T_i)\sum_{j=i+1}^{k}\frac{\rho(i,j)\tau_j\sigma_k(t,T_j)f_k(t,T_j)}{1+\tau_j f_k(t,T_j)}dt & \\ +\sigma_k(t,T_i)f_k(t,T_i)dZ_k(t), & i<k, t\le T_i \\ \sigma_k(t,T_i)f_k(t,T_i)dZ_k(t), & i=k, t\le T_i \\ -\sigma_k(t,T_i)f_k(t,T_i)\sum_{j=k+1}^{i}\frac{\rho(i,j)\tau_j\sigma_k(t,T_j)f_k(t,T_j)}{1+\tau_j f_k(t,T_j)}dt & \\ +\sigma_k(t,T_i)f_k(t,T_i)dZ_k(t), & i>k, t\le T_i \end{cases} \tag{A.3}$$

where $Z_k(t)$ is a Brownian motion under measure $\mathbb{Q}^i$.

Note that this satisfies the first specification of a market model stated in Section A.1. The proof could found in Brigo and Mercurio (2006). This satisfies the first step in defining a market model as mentioned in Section A.1.

The correlation, in the short rate models, is often interpreted as the correlation between Brownian shocks. The correlation, however, in LMM could be viewed as the correlation between the movement between different pairs of LIBOR that are of different reset period. For instance, $\rho_{3M,6M}$ describes the correlation between the 3 month LIBOR and the 6 month LIBOR. This is rather more intuitive and have a better economics interpretation than the correlation in the short models.

The next subsection describes the second step in defining a market model.

A.2.3 Existence and uniqueness of the LMM process

The Picard-Lindelof Theorem proves the existence and pathwise uniqueness for ODE. Itô (1946) has extend this theorem to prove the existence and pathwise uniqueness for SDE. This satisfies the second specification of a market model stated in Section A.1. The next subsection will describe the fourth step in defining the market model.

A.2.4 A martingale solution for the LMM process

As a result of the non-arbitrage of the HJM framework, there exist an equivalent martingale measure. Based on Harrison and Pliska (1981), the existence of an equivalent martingale measure implies that the market is complete. A market is said to be complete when every contingent claim is attainable.

The existence of the equivalent martingale measure implies that there is a numeraire $N(t)$ and also a probability measure $\mathbb{N}$ such that the price of every contingent claim $X(t, \omega)$ relative to $N(t)$ is a martingale under $\mathbb{N}$. This means that the following equation holds

$$\frac{X(t,\omega)}{N(t)} = \mathbb{E}^{\mathbb{N}}\left[\frac{X(T,\omega)}{N(T)}|\mathcal{F}_t\right]$$

where t is the current time and T is the time in the future where there is cash movement resulting from the contingent claim. Note that ω is to specify the particular state of the economy from all possible state of the economy, Ω.

Substitute $X(t, \omega)$ with $f_k(t)$ and $N(t)$ with $M(t)$, where $f_k(t)$ is the unique solution of the LMM process as shown in subsection A.2.3 and $M(t)$ is the money market account. Therefore, the forward rate process $f_k(t)$ relative to the money market account, $M(t)$ is a martingale process. This satisfies the third specification of a market model stated in Section A.1.

Combining the technique of change of measure and the existence of the equivalent martingale measure, we have another important equation

$$\mathbb{E}^{\mathbb{N}}\left[\frac{X(T,\omega)}{N(T)}|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\frac{X(T',\omega)}{Q(T')}|\mathcal{F}_t\right]$$

where $Q(t)$ is another numeraire with $\mathbb{Q}$ being its respective measure and T' is a time different from T. This satisfies the fourth specification of a market model stated in Section A.1.

APPENDIX B

DERIVATION OF OPTIMAL BOND WEIGHTS FOR VASICEK MODEL

From Puhle (2008), the zero coupon bond price process follows

$$\frac{dP(t,T)}{P(t,T)} = (r(t) + \sigma(t,T)\lambda)dt - \sigma(t,T)dZ(t) \tag{B.1}$$

where

- λ is the market price of interest rate risk
- $\sigma(t,T)$ is the volatility of the zero coupon bond process for the period $[t,T]$

.

The money market account process follows

$$dM(t) = M(t)r(t)dt \tag{B.2}$$

where $r(t)$ is the short rate.

Suppose that the $\pi(t)$ portion of the investor's portfolio consists of one zero coupon bond at time t and the remaining $1 - \pi(t)$ portion is held as some cash in the money market account and this constitute the investor's total wealth. Therefore the total wealth of the investor follows the following process

$$\frac{dW(t)}{W(t)} = (r(t) + \pi(t)\sigma(t,T)\lambda)dt - \pi(t)\sigma(t,T)dZ(t) \tag{B.3}$$

B.0.5 Deriving the optimal bond weights under Vasicek model

The goal of the investor is to invest in a certain way such that the utility gained from the wealth at the end of the investment period is maximised.

Mathematically,

$$\max_{\pi}\{\mathbb{E}_t[u(W(T))]\}$$

where $u(\cdot)$ is the utility function of the investor and this decision is made on time t.

Define the optimised functional at time t as

$$J(t, r(t), W(t)) = \max_{\pi}\{\mathbb{E}_t[u(W(T))]\}$$

Likewise, the optimsed functional at time T is

$$J(T, r(T), W(T)) = u(W(T))$$

Suppose the portfolio is rebalanced after a period of time say, Δt. The optimised functional at time $T - \Delta t$ is

$$\begin{aligned} J(T-\Delta t, r(T-\Delta t), W(T-\Delta t)) &= \max_{\pi}\{\mathbb{E}_{T-\Delta t}[u(W(T))]\} \\ &= \max_{\pi}\{\mathbb{E}_{T-\Delta t}[J(T, r(T), W(T))]\} \end{aligned}$$

Note that the optimised functional at time $T - \Delta t$ depends on time T. Recursively, the optimised functional at time t depends on time $t + \Delta t$

$$J(t, r(t), W(t)) = \max_{\pi}\{\mathbb{E}_t[J(t+\Delta t, r(t+\Delta t), W(t+\Delta t))]\} \tag{B.4}$$

Applying Taylor's theorem on the functional J around $(t, r(t), W(t))$

$$\begin{aligned} J(t+\Delta t, r(t+\Delta t), W(t+\Delta t)) &= J(t, r(t), W(t)) + J_t\Delta t + J_W\Delta W + J_r\Delta r \\ &+ J_{Wr}\Delta W\Delta r + \frac{1}{2}J_{WW}\Delta W^2 + \frac{1}{2}J_{rr}\Delta r^2 \end{aligned}$$

Taking expectation on the above equation as the variables such as $r(t)$

and $W(t)$ are stochastic processes

$$\begin{aligned}
& \mathbb{E}_t[J(t+\Delta t, r(t+\Delta t), W(t+\Delta t))] \quad \text{(B.5)}\\
= \ & J(t, r(t), W(t)) + J_t\Delta t + J_W\mathbb{E}_t[\Delta W] + J_r\mathbb{E}_t[\Delta r]\\
+ \ & J_{Wr}\mathbb{E}_t[\Delta W\Delta r] + \frac{1}{2}J_{WW}\mathbb{E}_t[\Delta W^2] + \frac{1}{2}J_{rr}\mathbb{E}_t[\Delta r^2]
\end{aligned}$$

Substitute Equation (B.5) into Equation (B.4)

$$\begin{aligned}
0 = \ & \max_\pi\{J_t\Delta t + J_W\mathbb{E}_t[\Delta W] + J_r\mathbb{E}_t[\Delta r] \quad \text{(B.6)}\\
+ \ & J_{Wr}\mathbb{E}_t[\Delta W\Delta r] + \frac{1}{2}J_{WW}\mathbb{E}_t[\Delta W^2] + \frac{1}{2}J_{rr}\mathbb{E}_t[\Delta r^2]\}
\end{aligned}$$

The above equation relies on the expectation of Equation (C.1) and (B.3) which are

- $\mathbb{E}[\Delta r] = a(b - r(t))\Delta t$
- $\mathbb{E}[\Delta W] = W(t)(r(t) + \pi(t)\sigma(t,T)\lambda)\Delta t$
- $\mathbb{E}[\Delta r^2] = (\sigma^r)^2\Delta t$
- $\mathbb{E}[(\Delta W)^2] = W^2(t)\pi^2(t)\sigma^2(t,T)\Delta t$
- $\mathbb{E}[\Delta W\Delta r] = -W(t)\pi(t)\sigma(t,T)\sigma^r\Delta t$

The expectations, $\mathbb{E}_t$ are short written as $\mathbb{E}$.

Inserting these expectations into Equation (B.6)

$$\begin{aligned}
0 = \ & \max_\pi\{J_t\Delta t + J_W W(t)(r(t) + \pi(t)\sigma(t,T)\lambda)\Delta t \quad \text{(B.7)}\\
+ \ & J_r a(b - r(t))\Delta t - J_{Wr}W(t)\pi(t)\sigma(t,T)\sigma^r\Delta t\\
+ \ & \frac{1}{2}J_{WW}W^2(t)\pi^2(t)\sigma^2(t,T)\Delta t + \frac{1}{2}J_{rr}(\sigma^r)^2\Delta t\}
\end{aligned}$$

Divided the above by Δt

$$\begin{aligned}
0 = \ & \max_\pi\{J_t + J_W W(t)(r(t) + \pi(t)\sigma(t,T)\lambda) + J_r a(b - r(t))\\
- \ & J_{Wr}W(t)\pi(t)\sigma(t,T)\sigma^r + \frac{1}{2}J_{WW}W^2(t)\pi^2(t)\sigma^2(t,T) + \frac{1}{2}J_{rr}(\sigma^r)^2\}
\end{aligned}$$

Differentiate the above equation with respect to $\pi(t)$ and set it to 0 in order to find the $\pi^*(t)$ which is the portfolio process that would maximise the utility gained from the total wealth at the end of the investment period.

$$\pi^*(t) = \frac{J_{Wr}s - J_W\lambda}{J_{WW}W(t)\sigma(t,T)} \tag{B.8}$$

Substitute the above equation into Equation (B.7) and simplify

$$\begin{aligned} 0 &= J_tJ_{WW} + J_WJ_{WW}W(t)r(t) + J_WJ_{Wr}\sigma^r\lambda \\ &+ J_rJ_{WW}a(b-r(t)) - \frac{J_{Wr}^2(\sigma^r)^2}{2} + \frac{J_{rr}J_{WW}(\sigma^r)^2}{2} \end{aligned} \tag{B.9}$$

with the boundary condition

$$J(T, r(T), W(T)) = \mathbb{E}[u(W(T))]$$

Assume that the utility function follows the Constant Relative Risk Aversion (CRRA) form which is $u(W(T)) = W^\gamma$ where γ is called the risk aversion coefficient of the investor which satisfy $0 < \gamma < 1$. Therefore, the boundary condition for Equation (B.9 is

$$J(T, r(T), W(T)) = W^\gamma$$

Note that $J(t, r(t), W(t))$ is a general function. Korn and Kraft (2002) suggested the following seperation

$$J(t, r(t), W(t)) = G(t, r(t))W^\gamma$$

The boundary condition for $G(t, r(t))$ is $G(T, r(T)) = 1$.

The partial derivatives for $J(t, r(t), W(t))$ in terms of $G(t, r(t))$ and W are

- $J_t = G_t W^\gamma$
- $J_r = G_r W^\gamma$
- $J_{rr} = G_{rr} W^\gamma$
- $J_W = \gamma G W^{\gamma-1}$
- $J_{WW} = \gamma(\gamma-1) G W^{\gamma-2}$
- $J_{Wr} = \gamma G_r W^{\gamma-1}$

Substitute the above partial deriviatives into Equation (B.9) and simplify

$$\begin{aligned} 0 &= (\gamma-1)GG_t + \gamma(\gamma-1)G^2 r(t) + \gamma G G_r \sigma^r \lambda \\ &+ (\gamma-1)GG_r a(b-r(t)) - \frac{\gamma G_r^2(\sigma^r)^2}{2} + \frac{(\gamma-1)GG_{rr}(\sigma^r)^2}{2} \end{aligned} \quad \text{(B.10)}$$

with the boundary condition $G(T, r(T)) = 1$.

The optimal bond weights would be

$$\begin{aligned} \pi^*(t) &= \frac{G_r\sigma^r - G\lambda}{(\gamma-1)G\sigma(t,T)} \\ &= \frac{1}{1-\gamma}\left(\frac{\lambda}{\sigma(t,T)} - \frac{\sigma^r}{\sigma(t,T)}\frac{G_r}{G}\right) \end{aligned} \quad \text{(B.11)}$$

Likewise, the following seperation on $G(t, r(t))$ can be made

$$G(t, r(t)) = A(t)e^{B(t)r}$$

with the boundary condition $B(T) = 0$.

The partial derivatives for $G(t, r(t))$ in terms of $A(t)$ and $B(t)$ are

- $G_t = A'(t)e^{B(t)r} + A(t)B'(t)re^{B(t)r}$
- $G_r = A(t)B(t)e^{B(t)r}$
- $G_{rr} = A(t)B^2(t)e^{B(t)r}$

Subsititute the above partial derivatives into Equation (B.10) and simplify

$$
\begin{aligned}
0 &= (\gamma-1)(A'(t)+A(t)B'(t)r)+\gamma(\gamma-1)A(t)r+\gamma A(t)B(t)\sigma^r\lambda \\
&+ (\gamma-1)A(t)B(t)a(b-r)-\frac{A(t)B^2(t)(\sigma^r)^2}{2}
\end{aligned}
$$

The optimal bond weights after simplification is then

$$
\pi^*(t)=\frac{1}{1-\gamma}\left(\frac{\lambda}{\sigma(t,T)}-\frac{\sigma^r}{\sigma(t,T)}B(t)\right)
$$

As mentioned by Korn and Kraft (2002), the seperation would be meaningful if the following holds

$$
0=(\gamma-1)A(t)r(B'(t)+\gamma-aB(t))
$$

i.e. the coefficient of the r term is zero.

This involves solving an inhomogeneous ODE

$$
0=B'(t)-aB(t)+\gamma
$$

such that $B(T)=0$ and the solution is as follows

$$
B(t)=\frac{1-e^{-a(T-t)}\gamma}{a} \tag{B.12}
$$

In Vasicek case, note that the zero coupon bond price dynamic is

$$
\begin{aligned}
\frac{dP(t,T)}{P(t,T)} &= r(t)dt-\sigma^r B^*(t,T)dZ(t) \\
&= r(t)dt-\sigma(t,T)dZ(t)
\end{aligned}
$$

where $B^*(t,T)=\frac{1-e^{-a(T-t)}}{a}$ and $\sigma(t,T)$ is the volatility for the zero coupon bond process for the period $[t,T]$ with $\sigma(t,T)=\sigma^r B^*(t,T)$.

Note the relation between $B(t)$ and $B^*(t,T)$

$$B(t) = \gamma \frac{1 - e^{-a(T-t)}}{a} = \gamma B^*(t,T)$$

Combine the above two equations

$$\sigma(t,T) = \frac{\sigma^r B(t)}{\gamma}$$

Substitute the above relationship and Equation (B.12) into (B.11) and simplify

$$\pi^*(t) = \frac{1}{1-\gamma} \left(\frac{\lambda a}{\sigma^r (1 - e^{-a(T_B - t)})} - \gamma \frac{1 - e^{-a(T-t)}}{1 - e^{-a(T_B - t)}} \right)$$

APPENDIX C

CALIBRATION OF VASICEK MODEL USING MAXIMUM LIKELIHOOD ESTIMATOR

The SDE for the Vasicek Model is

$$dr(t) = a(b - r(t))dt + \sigma dZ(t) \tag{C.1}$$

where

- $r(t)$ is the current interest rate level
- a is the mean reversion rate
- b is the long term interest rate level
- σ is the interest rate volatility
- $Z(t)$ is the Brownian motion

Let $\delta = \frac{1}{252}$ (one (1) day expressed in term of year) be the fixed time step. Define the following:

$$S_x = \sum_{i=1}^{N} r_{i-1}$$
$$S_y = \sum_{i=1}^{N} r_i$$
$$S_{xx} = \sum_{i=1}^{N} r_{i-1}^2$$
$$S_{xy} = \sum_{i=1}^{N} r_{i-1} r_i$$
$$S_{yy} = \sum_{i=1}^{N} r_i^2$$

where N is the number of historical interest rate used in the calibration.

Apply the following formula to obtain the calibrated parameters for Vasicek model using maximum likelihood estimator:

$$
\begin{aligned}
b &= \frac{S_y S_{xx} - S_x S_{xy}}{N(S_{xx} - S_{xy}) - (S_x^2 - S_x S_y)} \\
a &= -\frac{1}{\delta} \ln \frac{S_{xy} - bS_x - bS_y + Nb^2}{S_{xx} - 2bS_x + Nb^2} \\
\sigma &= \frac{2a}{N(1 - e^{-2a\delta})} \cdot \\
& \big(S_{yy} - 2e^{-a\delta} S_{xy} + e^{-2a\delta} S_{xx} - 2b(1 - e^{-a\delta})(S_y - e^{-a\delta} S_x) + Nb^2(1 - e^{-a\delta})^2\big)
\end{aligned}
$$