A STUDY ON UNIVALENT FUNCTIONS AND THEIR GEOMETRICAL PROPERTIES

By

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ABSTRACT

A STUDY ON UNIVALENT FUNCTIONS AND THEIR GEOMETRICAL PROPERTIES

Wei Dik Kai

A study of univalent functions was carried out in this dissertation. An introduction and some known results on univalent functions were given in the first two chapters.

In Chapter 3 of this dissertation, the mapping f_R from unit disk D to disk of specified radius E_R for which $f \in S$ was identified explicitly. Moreover, the mapping was studied as the radius R approaches to infinity. It is then found that when $R \to \infty$, the mapping f_R tends to the function $z(1-z)^{-1}$ analytically and geometrically.

In Chapter 4, functions from subclasses of S consist of special geometrical properties such as starlike and convex functions were defined geometrically and analytically. It is known that $f(z) = z + az^2$ is starlike or convex under a certain conditions on the complex constant a, we are able to obtain similar results for the more general function $f(z) = z + az^m$.

Furthermore, the Koebe function is generalized into $z(1-z)^{-\alpha}$ and we were able to show that it is starlike if and only if $0 \le \alpha \le 2$. The range of the generalized Koebe function was studied afterward and we found that the range contain the disk of radius $1/2^{\alpha}$. At the end of the dissertation, some well-known inequalities involve function of class S were improved to inequalities involving convex functions.

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APPROVAL SHEET

The dissertation entitled " **A STUDY ON UNIVALENT FUNCTIONS AND THEIR GEOMETRICAL PROPERTIES**" was prepared by WEI DIK KAI and submitted as partial fulfillment for the requirements for the degree of Master of Science at University Tunku Abdul Rahman.

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SUBMISSION OF DISSERTATION

It is hereby certified that **WEI DIK KAI** (ID No: **13UEB07929**) has completed this dissertation entitled "**A STUDY ON UNIVALENT FUNCTIONS AND THEIR GEOMETRICAL PROPERTIES**" under the supervision of Dr. Tan Sin Leng (Supervisor) from the Department of Mathematical and Actuarial Sciences, Lee Kong Chian Faculty of Engineering and Science, and Dr. Wong Wing Yue (Co-Supervisor) from the Department of Mathematical and Actuarial Sciences, Lee Kong Chian Faculty of Engineering and Science.

I understand that University will upload softcopy of my dissertation in pdf format into UTAR Institutional Repository, which may be made accessible to UTAR community and public.

Yours truly,

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Wei Dik Kai

DECLARATION

I hereby declare that the dissertation is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UTAR or other institutions.

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CHAPTER 1

HOLOMORPHIC FUNCTIONS

The main purpose of this chapter is to introduce the holomorphic functions and some of their properties that will be used throughout the dissertation. We are interested in studying the analyticity of complex differentiable function and some of its properties.

1.1 Real Differentiable Functions

First of all, we begin our study on real differentiable functions. A real function $f(x)$ is said to be differentiable at point x_0 if the quotient

$$\frac{f(x_0+h)-f(x_0)}{h}$$

converges to a limit as $h \to 0$. If the limit exists, it is denoted as $f'(x_0)$, and called as the derivative of f at x_0. If the limit doesn't exists, then $f(x)$ is not differentiable at x_0.

If f is differentiable at x_0, then $f(x)$ is continuous at x_0. The converse is not true. At certain points in its domain, a function can be continuous but not differentiable. For example, $f(x)=|x|$ is continuous on the real line but not differentiable at $x=0$.

The term analytic function is often used interchangeably with holomorphic function in complex analysis. However, this is not true in general for real functions. Analytic is different from real differentiable. Analyticity is used in describing whether the function value near a fixed point can be obtained from its Taylor's series expansion at that point. More precisely, a real-valued function f on a nonempty, open interval (a,b) is said to be analytic at $x_0 \in (a,b)$ if there exist coefficient $\{a_n\}_{n=0}^{\infty}$ and $\delta > 0$ such that

$$f(x)=\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

for all $x \in (x_0-\delta, x_0+\delta) \subset (a,b)$. The function f is said to be analytic on $(r,s) \subset (a,b)$, if it is analytic at every point in (r,s). If $f(x)$ is analytic at x_0, then from Taylor's Theorem, $a_n = f^{(n)}(x_0)/n!$ and $f(x)$ is infinitely differentiable at x_0. However, the converse is not true. For example,

$$h(x)=\begin{cases} e^{-1/x} & ,x>0 \\ 0 & ,x\le 0 \end{cases}$$

The derivative of h of all orders at $x=0$ is equal to 0. Therefore, the Taylor's series of h at the origin is $0+0x+0x^2+\cdots$ which converges everywhere to the zero function. Hence the Taylor's series does not converge to h for $x>0$. Consequently, h is not analytic at the origin.

1.2 Complex Differentiable Functions

Let Ω be an open set in the complex plane $\mathbb{C}$, and f be a complex-valued function on Ω. The function f is said to be differentiable at the point $z_0 \in \Omega$ if the quotient

$$\frac{f(z_0+h)-f(z_0)}{h}$$

converges to a limit as $h \to 0$. If the limit exists, it is denoted as $f'(z_0)$, and called as derivative of f at z_0. If f is differentiable at z_0 as well as at every point of some neighborhood of z_0, it is said to be holomorphic at z_0. The function f is said to be holomorphic in Ω if f is holomorphic at every point of Ω.

The definition of holomorphic function seems no different from the real differentiable function, but it has the essence of complex differentiability compare to the real differentiability. Hence, there exists some properties that cannot be shared by real differentiable function. One of the properties is that all holomorphic functions are analytic. However, this is not true for real differentiable function, as shown in the example from previous section.

Theorem 1.2.1 (Taylor's Theorem) *Suppose that f is holomorphic in a domain Ω and that $N_R(\alpha)$ is any disk contained in Ω. Then the Taylor series for f converges to $f(z)$ for all z in $N_R(\alpha)$; that is,*

$$f(z)=\sum_{n=0}^{\infty}\frac{f^{(n)}(\alpha)}{n!}\left(z-\alpha\right)^n, \quad \textit{for all } z\in N_R(\alpha).$$

Furthemore, for any r, $0<r<R$, *the convergence is uniform on the closed subdisk* $\bar{N}_R(\alpha)=\left\{z:\left|z-\alpha\right|\le r\right\}$.

For the proof, please refer to (Mathews and Howell, 2010).

1.3 Laurent Series

If a complex function is not holomorphic at a particular point $z=z_0$, then this point is said to be a singularity or singular point of the function. There are two types of singular points, one of them is known as isolated singularity and the other known as non-isolated singularity. The point $z=z_0$ is said to be an *isolated singularity* of the function f if f analytic on a deleted neighborhood of z_0 but not analytic at z_0. For example, $z=0$ is isolated singularity of $f(z)=1/z$. On the other hand, the point $z=z_0$ is said to be a non-isolated singularity if every neighborhood of z_0 contains at least one singularity of f other than z_0.

In this section, we will discuss the power series expansion of f about an isolated singularity z_0 which is also known as the Laurent series and it will involve negative and non-negative integer powers of $z-z_0$.

Theorem 1.3.1 (Laurent's Theorem) *Let f be holomorphic in the annular domain E defined by $r < |z - z_0| < R$. Then f can be expressed as the sum of two series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$$

both series converging in the annular domain E, and converging uniformly in any closed subannulas $r < \rho_1 \le |z - z_0| \le \rho_2 < R$. The coefficients a_n are given by

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta \,, \quad n = 0, \pm 1, \pm 2, \ldots$$

where C is any positively oriented simple closed contour that lies entirely within E and has z_0 in its interior.

For the proof, please refer to (Saff and Snider, 2003).

From the power series, we see that Laurent series consists of two parts. The part of the power series with negative powers of $z - z_0$ namely,

$$\sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}$$

is known as the *principal part* of the series. Now we are going to assign different names to the isolated singularity z_0 according to the number of terms in the principal part.

An isolated singular point z_0 of the complex function f is classified depending on whether the principal part vanishes, contains a finite number or an infinite number of terms.

i) If the principal part vanishes, which means that all the coefficients a_{-n} are zero for all $n = 1, 2, 3, \dots$, then z_0 is called as a removable singularity.

ii) If the principal part contains a finite number of nonzero terms, then z_0 is called as pole. In this case, if the last nonzero coefficient is a_{-m}, and $m \geq 1$, then we say that z_0 is a pole of order m.

iii) If the principal part contains infinitely many nonzero terms, then z_0 is called an essential singularity.

Assume that $f(z)$ is holomorphic in a domain Ω and not identically zero. Then, if $f(z) = 0$, from Theorem 1.2.1, there exists a first derivative $f^{(m)}(z_0)$ which is different from zero. In this case, we say that z_0 is a zero of order m, and $f(z)$ can be expressed as $f(z) = (z - z_0)^m g(z)$, where $g(z)$ is analytic and $g(z_0) \neq 0$. There is close relationship between zeros and poles of a holomorphic function as in the following theorems.

Proposition 1.3.1 *If f is holomorphic and has a zero of order m at the point z_0, then $g(z) = 1/f(z)$ has a pole of order m at z_0.*

For the proof, please refer to (Saff and Snider, 2003).

Proposition1.3.2 *If* f *has a pole of order* m *at the point* z_0*, then* $g(z) = 1/f(z)$ *has a removable singularity at* z_0*. If we define* $g(z_0) = 0$*, then* $g(z)$ *has a zero of order* m *at* z_0*.*

For the proof, please refer to (Saff and Snider, 2003).

1.4 Harmonic Functions

The sum and product of two holomorphic functions are still holomorphic. The quotient $f(z)/g(z)$ is holomorphic in Ω provided that $g(z)$ does not vanish in Ω. For a holomorphic function $f(z)$, and if we write $f(z) = u(z) + iv(z)$, it follows that $u(z)$ and $v(z)$ are both continuous as well. From the definition, the limit $\lim_{h \to 0} \left[f(z_0 + h) - f(z_0) \right] / h$ must be the same regardless the way h approaches zero. If the real values of h are chosen, then we have a partial derivative with respect to the real part. Thus, we obtain

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

for $z = x + iy$. If approaches 0 through $h = ik$, then we have a partial derivative with respect to imaginary part. Thus, we obtain

$$f'(z) = -i \frac{\partial f}{\partial y} = -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

It follows that $f(z)$ must satisfy the partial differential equation

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$$

which resolves into the real equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The above equations are known as Cauchy-Riemann differential equations. The equations must be satisfied by real and imaginary parts of any holomorphic function. For the quantity $|f'(z)|^2$, we can see that

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

is the Jacobian of u and v with respect to x and y.

Since f is holomorphic, it follows that, u and v have continuous partial derivatives of all orders. From Clairaut's Theorem (Stewart, 2003), $u_{xy} = u_{yx}$, $v_{xy} = v_{yx}$. By using the Cauchy-Riemann equations it follows that

$$u_{xx} = v_{yx}, \qquad v_{xx} = -u_{yx}$$

$$u_{yy} = -v_{xy}, \qquad v_{yy} = u_{xy}$$

Combining Clairaut's Theorem and Cauchy-Riemann equations, we can obtain

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

If u satisfied the Laplace's equation $\Delta u = 0$, then it is said to be *harmonic*. Thus the real part and imaginary part of a holomorphic function are harmonic.

1.5 Argument Principle and Open Mapping Theorem

A function f is said to be *meromorphic* in a domain Ω provided the singularities of f are isolated poles and removable singularities. In this section, we give an important result called the Argument Principle which provides a formula on finding the difference between number of zeros and poles of a meromorphic function. Further properties of holomorphic function will be explored in this section as well.

Theorem 1.5.1 (Argument Principle) *Suppose that f is meromophic in a simple connected domain Ω. Let γ be a piecewise smooth, positively oriented, simple closed curve in Ω, which does not pass through any pole or zero of f and whose interior lies in Ω. Then*

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)}\,dz = N_0 - N_p$$

where N_0 is the total number of zeros of f inside γ and N_p is the total number of poles of f inside γ.

For the proof, please refer to (Mathews and Howell, 2010).

Theorem 1.5.1 is known as Argument Principle because it is related to the winding number of f about the origin. The winding number of a closed curve in the plane around a given point is an integer representing the total number of times that curve winds around the point counterclockwise.

Theorem 1.5.2 (Winding numbers) *Suppose that* f *is meromorphic in the simply connected domain* Ω. *If* γ *is a simple closed positively oriented contour in* Ω *such that for* $z \in \gamma$, $f(z) \neq 0$, $f(z) \neq \infty$ *and* $\alpha \notin f(\gamma)$, *then*

$$W\left(f(\gamma),\alpha\right) = \frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)-\alpha}\,dz$$

known as the winding number of $f(\gamma)$ *about* α, *counts the number of times the curve* $f(\gamma)$ *winds around the point* α. *If* $\alpha = 0$, *the integral is actually counting the number of times the curve* $f(\gamma)$ *winds around the origin.*

For the proof, please refer to (Mathews and Howell, 2010).

Any two points in the same region bounded by $f(\gamma)$ can be joined by a polygon which does not meet $f(\gamma)$. In other words, $W\left(f(\gamma),\alpha\right) = W\left(f(\gamma),\beta\right)$ if $f(\gamma)$ does not meet the line segment from α and β. If γ is a circle, it follows that $f(z)$ takes values of α and β equally many times inside γ (Ahlfors, 1979). The following theorem is the consequence of this result.

Theorem 1.5.3 *Suppose that* $f(z)$ *is analytic at* z_0, $f(z_0) = w_0$, *and that* $f(z) - w_0$ *has a zero of order* n *at* z_0. *If* $\varepsilon > 0$ *is sufficiently small, there exists a corresponding* $\delta > 0$ *such that for all* a *with* $|a - w_0| < \delta$ *the equation* $f(z) = a$ *has exactly* n *roots in the disk* $|z - z_0| < \varepsilon$.

Proof. (Ahlfors, 1979) Choose ε so that $f(z)$ is defined and analytic for $|z-z_0|\le\varepsilon$ and so that z_0 is the only zero of $f(z)-w_0$ in this disk. Let γ be the circle $|z-z_0|=\varepsilon$ and Γ its image under the mapping $w=f(z)$. Since w_0 belongs to the complement of the closed set Γ, there exists a neighborhood $|w-w_0|<\delta$ which does not intersect Γ. It follows immediately that all values a in this neighborhood are taken the same number of times inside of γ. The equation $f(z)=w_0$ has exactly n coinciding roots inside of γ, and hence every value of a is taken n times. It is understood that multiple roots are counted according to their multiplicity, but if ε is sufficiently small we can assert that all roots of the equation $f(z)=a$ are simple for $a\neq w_0$. Indeed, it is sufficient to choose ε so that $f'(z)$ does not vanish for $0<|z-z_0|<\varepsilon$. Q.E.D

Corollary 1.5.1 *A nonconstant analytic function maps open sets onto open sets.*

For the proof, please refer to (Ahlfors, 1979).

Theorem 1.5.4. (Maximum Modulus Principle) *If f is analytic and nonconstant in a region Ω, then its absolute value $|f|$ has no maximum in Ω.*

For the proof, please refer to (Ahlfors, 1979).

Corollary 1.5.2 (Minimum Modulus Principle) *If f is a nonconstant, nowhere zero, holomorphic function in domain Ω, then $|f|$ can have no local minimum in Ω.*

For the proof, please refer to (González, 1991a).

Corollary 1.5.3 *If f is a non-constant holomorphic function in a domain D, then $Re(f)$ has no local maxima and no local minima.*

For the proof, please refer to (Fisher, 1999).

Theorem 1.5.5 (Maximum Modulus Theorem) *If $f(z)$ is defined and continuous on a closed bounded set E and holomorphic on the interior of E, then the maximum of $|f(z)|$ on E is assumed on the boundary of E.*

Proof. (Ahlfors, 1979) Since E is compact, $|f(z)|$ has a maximum on E. Suppose that $f(z)$ achieved it's maximum at point z_0. The theorem is proved if z_0 is on the boundary. If z_0 is an interior point, then $|f(z_0)|$ is also the maximum of $|f(z)|$ in a disk $|z-z_0|<\delta$ contained in E. But this is not possible unless $f(z)$ is a constant in the component of the interior of E which contains z_0. It follows by continuity that $|f(z)|$ is equal to its maximum on the whole boundary of that component. This boundary is not empty and it is contained in the boundary of E. Thus the maximum is assumed at a boundary point. Q.E.D

CHAPTER 2

UNIVALENT FUNCTIONS

Some basic properties of univalent functions will be discussed in this chapter. Some examples and applications will also be given in this chapter.

Let $w = f(z)$ be a complex mapping defined in a domain Ω, and we write $z = x + iy$ and $w = u + iv$, where x, y, u and v are real numbers.

2.1 Biholomorphism

A bijective functionis a mapping that is both injective and surjective. A biholomorphism is a function that is both bijective and holomorphic. Given two open set Ω and Ω' in $\mathbb{C}$, we are interested to know how they are related. From Open Mapping Theorem, we may assume that for a biholomorphism mapping, it is also an onto mapping.

Theorem 2.1.1 *If $f : \Omega \to \Omega'$ is holomorphic and injective, then $f'(z) \neq 0$ for all $z \in \Omega$. In particular, the inverse of f defined on its range is holomorphic, and thus the inverse of a biholomorphism is also holomorphic.*

For the proof, please refer to (Stein and Shakarchi, 2003).

Geometrically, the condition $f'(z) \neq 0$ can be interpreted as conformality.

Definition 2.1.1 Let $w = f(z)$ be a complex mapping defined in a domain Ω and let $z_0 \in \Omega$. Then we say that $w = f(z)$ is *conformal* at z_0 if for every pair of smooth oriented curves γ_1 and γ_2 in Ω intersecting at z_0, the angle between γ_1 and γ_2 at z_0 is equal to the angle between the image curves γ_1' and γ_2' at $f(z_0)$ in both magnitude and orientation.

Theorem 2.1.2 *An analytic function* f *is conformal at every point* z_0 *for which* $f'(z_0) \neq 0$.

For the proof, please refer to (Saff and Snider, 2003).

From Theorem 2.1.1 and Theorem 2.1.2, we know that a biholomorphism is also a conformal mapping. If there exists a biholomorphism $f : \Omega \to \Omega'$, then we can say that Ω and Ω' are *conformal equivalent*.

2.2 Linear Fractional Transformation

One of the important examples of biholomorphism is the linear fractional transformation. The transformation is defined as follow.

Definition 2.2.1 For complex constants a, b, c, d, and $ad - bc \neq 0$, then the complex function defined as

$$f(z) = \frac{az + b}{cz + d}$$

is a *linear factional transformation*.

Linear fractional transformation is known as Möbius transformation as well. It can be showed that if $f(z)$ is a linear fractional transformation, then it's inverse $f^{-1}(z)=(dz-b)/(-cz+a)$ is again a linear fractional transformation. From the definition of the transformation, we can see that if $c=0$, then $f(z)=(az+b)/d$ is linear mapping, and thus it's a special case of linear fractional transformation. For $c\neq 0$, the transformation can be written in the form:

$$f(z)=\frac{bc-ad}{c}\cdot\frac{1}{cz+d}+\frac{a}{c}$$

From the above equation, let $A=(bc-ad)/c$ and $B=a/c$, then we can see that f is actually a composite function, $f=k\circ g\circ h$, where $k(z)=Az+B$, $g(z)=1/z$ and $h(z)=cz+d$. The domain of the linear fractional transformation is all z in the complex plane except at $z=-d/c$. Since $ad-bc\neq 0$, we can easily see that f is injective on its domain.

2.3 Univalent Functions

In this section, univalent function and the class S of univalent functions will be introduced, which is the class that we concerned the most throughout the study.

Definition 2.3.1 A holomorphic function f in a domain $\Omega\subset\mathbb{C}$ is said to be *univalent* if it is injective in Ω.

To express it more clearly, if $f(z_1) \neq f(z_2)$ for all distinct pairs of z_1 and z_2 in Ω, then we say that f is univalent. The function is said to be locally univalent at a point $z_0 \in \Omega$ if it is univalent in some neighborhood of z_0. For holomorphic function f, the condition $f'(z_0) \neq 0$ is equivalent to local univalence at z_0.

Definition 2.3.2 The class S consists of all function f such that f is univalent in the unit disk D, normalized with the condition $f(0) = 0$ and $f'(0) = 1$.

For each $f \in S$, f has a Taylor series expansion written in the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots = z + \sum_{n=2}^{\infty} a_n z^n, \qquad |z| < 1,\ a_n \in \mathbb{C}$$

Before we move to another discussion on the class S, we introduce a theorem which is related to biholomorphism between unit disk $D = \{z : |z| < 1\}$ and an open set and this theorem plays an important role in latter chapters.

Theorem 2.3.1 (Riemann Mapping Theorem) *Let Ω be a simply connected domain which is a proper subset of the complex plane. Let ζ be a given point in Ω. Then there is a unique function f which maps Ω conformally onto the unit disk and has the properties $f(\zeta) = 0$ and $f'(\zeta) > 0$.*

For the proof, please refer (Duren, 1983).

2.4 Example of Functions in the Class S

We give several examples of univalent functions in this section.

Example 2.4.1 The function $f(z) = z + az^2$ is in S for $a \in \mathbb{C}$, $|a| \le 1/2$ and not univalent in D for $|a| > 1/2$.

Proof. Obviously, $f(z)$ is a holomorphic function in D and normalized with the conditions $f(0) = 0$, $f'(0) = 1$. For $|a| \le 1/2$, observe that when $a = 0$, $f(z) = z$ is clearly a univalent function in S. For $a \ne 0$, let

$$f(z_1) = f(z_2), \qquad z_1, z_2 \in D = \{z : |z| < 1\}$$

Then we have

$$(z_1 - z_2)\left[1 + a(z_1 + z_2)\right] = 0$$

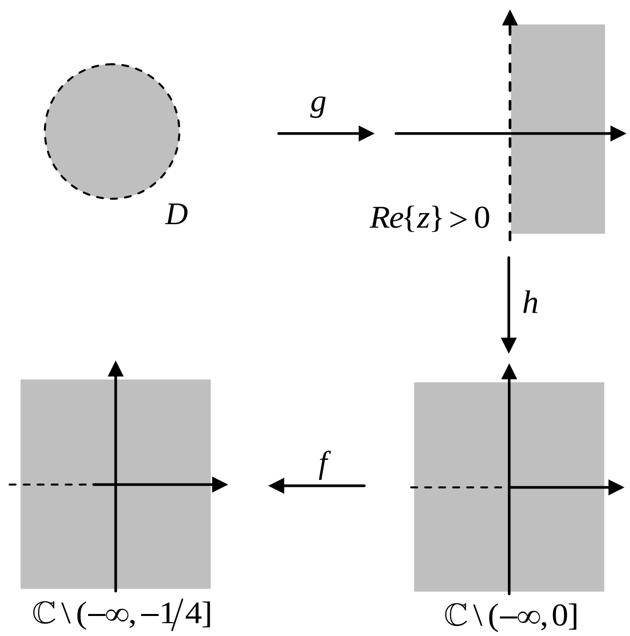
We claim that $z_1 = z_2$. If not, then $|z_1 + z_2| = 1/|a|$. From triangle inequality, $|z_1| \ge 1/|a| - |z_2| > 2 - 1 = 1$ which contradicts the fact that $|z| < 1$. Therefore $z_1 = z_2$, and hence f is univalent. Since f is in the normalized form, this implies that $f \in S$.

For $|a| > 1/2$, let $z_0 = -1/(2a)$, since $|z_0| = 1/\left(2|a|\right) < 1$ therefore $z_0 \in D$ and $f'(z_0) = 1 + 2a\left[-1/(2a)\right] = 0$ which implies that f is not local univalent. Then we conclude that f is not univalent in D for $|a| > 1/2$. Q.E.D

Example 2.4.2 The Koebe function $k(z)=\dfrac{z}{(1-z)^2}=z+2z^2+\cdots+nz^n+\cdots$ for $|z|<1$ is in S.

Proof. (Duren, 1983) Instead of using the similar method as Example 2.4.1, we prove this geometrically. Firstly, consider the following functions

$$g(z)=\frac{1+z}{1-z},\ h(z)=\left[g(z)\right]^2, \text{ and } f(z)=\frac{1}{4}\left[h(z)-1\right],$$



Observed that the function g mapped D onto the right half-plane $Re\{g\}>0$. In fact,

$$g(z)=\frac{1+z}{1-z}=\frac{1+x+iy}{1-x-iy}\cdot\frac{1-x+iy}{1-x+iy}$$

$$Re\{g(z)\}=\frac{(1-x^2-y^2)}{(1-x)^2+y^2}>\frac{0}{2-2x}>0$$

Then h mapped it onto the whole plane except for the negative real axis. By some simple calculations, we find that $f(0)=0$, $f'(0)=1$ and $f(z)=k(z)$. Therefore, $k(z)$ is univalent and normalized and hence $k(z)\in S$. Q.E.D

According to the composition of functions described as above, we know that the Koebe function mapped unit disk onto whole plane except the part of the negative real axis from $-1/4$ to negative infinity.

Other simple examples in S are listed as follow:

(i) $f(z)=z/(1-z)$, which maps D conformally onto the half-plane $Re\{w\}>-1/2$;

For $z=x+iy$,

$$f(z)=\frac{z}{1-z}=\frac{x+iy}{1-x-iy}\cdot\frac{1-x+iy}{1-x+iy}$$

$$Re\{f(z)\}=\frac{x-(x^2+y^2)}{(1-x)^2+y^2}>\frac{x-1}{2-2x}>-\frac{1}{2}$$

(ii) $f(z)=z/(1-z^2)$, which maps D conformally onto the whole plane minus the two half-lines $1/2\le Re\{w\}\le\infty$ and $-\infty\le Re\{w\}\le -1/2$.

(iii) $f(z)=\frac{1}{2}\log\left[(1+z)/(1-z)\right]$, which maps D onto the horizontal strip $-\pi/4<Im\{w\}<\pi/4$.

Let $h(z)=z/(1-z)$ and $g(z)=z/(1+iz)$, clearly h and g are in S. By some calculation, we find that $f(z)=h(z)+g(z)$ has a derivative which vanishes at $z_0 = (i+1)/2$. From the examples, we conclude that the sum of two functions in S may not be univalent, yet the class S is preserved under certain elementary transformations as listed below.

(i) *Conjugation*. If $f \in S$, and $g(z) = \overline{f(\overline{z})} = z + \overline{a}_2 z^2 + \overline{a}_3 z^3 + \cdots$, then $g \in S$.

(ii) *Rotation*. If $f \in S$ and $g(z) = e^{-i\theta} f(e^{i\theta} z)$, then $g \in S$.

(iii) *Dilation*. If $f \in S$ and $g(z) = r^{-1} f(rz)$, where $0 < r < 1$, then $g \in S$.

(iv) *Disk automorphism*. If $f \in S$, and

$$g(z) = \frac{f\left(\dfrac{z+\alpha}{1+\overline{\alpha} z}\right) - f(\alpha)}{\left(1+|\alpha|^2\right) f'(\alpha)}, \quad |\alpha| < 1,$$

then $g \in S$.

(v) *Range transformation*. If $f \in S$ and h is a univalent function on the range of f, with $h(0) = 0$ and $h'(0) = 1$, then $g = h \circ f \in S$.

(vi) *Omitted-value transformation*. If $f \in S$ and $f(z) \neq \omega$, then $g = \omega f / (\omega - f) \in S$.

(vii) *Square-root transformation*. If $f \in S$ and $g(z) = \sqrt{f(z^2)}$, then $g \in S$.

The square-root transformation needs some further explanations. Since $f(z) = 0$ only at the origin, a single-branch of the square-root may be chosen by writing

$$g(z)=\sqrt{f(z^2)}=z\left(1+a_2z^2+a_3z^4+\cdots\right)^{1/2}$$

$$=z+\frac{1}{2}a_2z^3+\left(\frac{1}{2}a_3-\frac{1}{8}a_2^2\right)z^5+\cdots$$

which implies that g is an odd analytic function. Suppose that $g(z_1)=g(z_2)$, for $z_1,z_2\in D$, then $f(z_1^2)=f(z_2^2)$. By the univalence of f, we have $z_1^2=z_2^2$ which means that $z_1=\pm z_2$. We claim that $z_1=z_2$, if not $z_1=-z_2$ would gives $g(z_1)$ $=g(-z_2)=-g(z_2)=-g(z_1)$ since $g(z)$ is an odd function. But this would contradicts the definition of g. Hence $z_1=z_2$ and $g\in S$.

In fact, there are a lot of other examples from some subclasses of S such as the class of starlike which is also our concern in the study. Moreover, one of them is the class consist only the analytic functions with negative coefficients. We will discuss about the subclasses in more detail in the latter section. In the next section, we are going to discuss a very important result that took about 70 years to prove it.

2.5 Bieberbach's Theorem

In 1916, Ludwig Bieberbach proved that for every $f(z)=z+a_2z^2+a_3z^3+\cdots$ in the class S, $|a_2|\le 2$ and equality holds if and only if f is a suitable rotation of Koebe function (Bieberbach, 1916). He conjectured that generally $|a_n|\le n$ for all $f\in S$ and it has become the famous Bieberbach's conjecture which remained unproven until 1986. Some preliminary results are needed to prove the inequality is true for the second coefficient. Firstly, consider another class of univalent functions which are defined in the exterior of the unit disk D.

Let Δ denotes the domain $\{z : |z| > 1\}$ and $\sum$ is the class of all functions of the form

$$g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n}$$

that are analytic and univalent in Δ. Let $\sum_0$ be the subclass of $\sum$ such that $g(z) \neq 0$ in for all $z \in \Delta$.

Theorem 2.5.1 *Let* $h(z) = 1/z$. *If* $f \in S$, *then* $F(z) = h \circ f \circ h \in \sum_0$. *Conversely, if* $g \in \sum_0$, *then* $G(z) = h \circ g \circ h \in S$.

Proof. (González, 1991b) We first prove that F is univalent in Δ. Let $F(\zeta_1) = F(\zeta_2)$ where $\zeta_1, \zeta_2 \in \Delta$, then $1/\zeta_1 = z_1 \in D$ and $1/\zeta_2 = z_2 \in D$. By the definition of $F(\zeta)$, we obtained $f(z_1) = 1/F(\zeta_1)$ and $f(z_2) = 1/F(\zeta_2)$. Since $F(\zeta_1) = F(\zeta_2)$, it means that $f(z_1) = f(z_2)$. By univalence of f, we have $z_1 = z_2$ and $\zeta_1 = \zeta_2$, hence F is univalent in Δ. Observed that $F(\zeta) \neq 0$ for all $\zeta \in \Delta$, since $F(\zeta_0) = 0$ for some $\zeta_0 \in \Delta$ would implies that $F(\zeta_0) f(1/\zeta_0) = 0$ which contradict the fact that $F(\zeta_0) f(1/\zeta_0) = 1$ for all $\zeta_0 \in \Delta$. Thus, $F(\zeta) \in \sum_0$. In fact, we can see that

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

$$f(1/z) = \frac{1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \cdots$$

$$F(z) = \frac{1}{f(1/z)} = z - a_2 + (a_2{}^2 - a_3) z^{-1} + \cdots$$

is in the form of power series of class Σ_0.

The converse can be shown in a similar way. We first prove that G is univalent in D. Let $G(z_1) = G(z_2)$ where $z_1, z_2 \in D$, then $1/z_1 = \zeta_1 \in \Delta$ and $1/z_2 = \zeta_2 \in \Delta$. By the definition of $G(z)$, we obtained $g(\zeta_1) = 1/G(z_1)$ and $g(\zeta_2) = 1/G(z_2)$. Since $G(z_1) = G(z_2)$, it means that $g(\zeta_1) = g(\zeta_2)$. By univalence of g, we have $z_1 = z_2$ and $\zeta_1 = \zeta_2$, hence G is univalent in D. Observed that $G(z) \neq 0$ for all $z \in D$, since $G(z_0) = 0$ for some $z_0 \in D$ would implies that $G(z_0) g(1/z_0) = 0$ which contradict the fact that $G(z_0) g(1/z_0) = 1$ for all $z_0 \in D$. Thus, $G(z)$ is in S. Q.E.D

The transformation is called an *inversion*. In fact, it establishes a one-to-one correspondence between the classes S and Σ_0. The class Σ_0 sharing the same property as S, for example, Σ_0 is preserved under the square-root transformation. We continue the preliminary result with the following theorem.

Theorem 2.5.2 (Interior Area Theorem). *Let* $f \in S$, *then the area of* $f(D)$ *is given by*

$$A = \pi \sum_{n=1}^{\infty} n \left| a_n \right|^2$$

assuming that the numbers $A_r = \pi \sum_{n=1}^{\infty} n \left| a_n \right|^2 r^{2n}$ *are bounded for* $0 < r < 1$.

Proof. (González, 1991b) We have

$$w = f(z) = z + \sum_{n=2}^{\infty} a_n z^n , \qquad |z| < 1$$

Consider the circle $C_r = \left\{ z : z = re^{i\theta}, 0 < r < 1, 0 \le \theta \le 2\pi \right\}$. Let $\Gamma_r = f(C_r)$, $D_r = Int(C_r)$, $\Phi_r = Int(\Gamma_r)$, $A_r = area\, \Phi_r$. From calculus, we have

$$A_r = \iint_{\Phi_r} du\, dv = \iint_{D_r} \left| \frac{\partial(u,v)}{\partial(x,y)} \right| dx\, dy$$

$$= \iint_{D_r} | f'(z)|^2\, dx\, dy = \int_0^{2\pi} \int_0^r \left| f'(re^{i\theta}) \right|^2 r\, dr\, d\theta$$

Since

$$f'(re^{i\theta}) = 1 + 2a_2 re^{i\theta} + \cdots + na_n r^{n-1} e^{i(n-1)\theta} + \cdots$$

We have

$$\left| f'(re^{i\theta}) \right|^2 = f'(re^{i\theta})\overline{f'(re^{i\theta})}$$

$$= \sum_{n=1}^{\infty} n^2 \left|a_n\right|^2 r^{2n-2} + \sum_{k \neq 0} c_k e^{ik\theta}$$

where the terms in the last sum involve the factors $e^{ik\theta}$ with k running through the nonzero intergers, and the coefficients c_k depending on the a_n and r. Thus we have

$$r\left| f'(re^{i\theta}) \right|^2 = \sum_{n=1}^{\infty} n^2 \left|a_n\right|^2 r^{2n-1} + \sum_{k \neq 0} rc_k e^{ik\theta}$$

By substitution in A_r and integration term by term we have

$$A_r = \pi \sum_{n=1}^{\infty} n\left|a_n\right|^2 r^{2n}$$

since $\int_0^{2\pi} e^{i\theta}\, d\theta = 0$ for $k \neq 0$. If A_r are bounded for $0 < r < 1$, and M is an upper bound, then we have

$$\pi \sum_{n=1}^{N} n\left|a_n\right|^2 r^{2n} < M$$

where N is a fixed arbitrary positive integer. The sum on the left-hand side increases monotonically with r and it is bounded. Hence, it has a limit as $r \to 1^-$, and we obtained

$$\pi \sum_{n=1}^{N} n\left|a_n\right|^2 \leq M$$

Since the partial sums $\sum_1^N n|a_n|^2$ are bounded, the series $\sum_1^\infty n|a_n|^2$ converges, and letting $N \to \infty$ we find that

$$A = \lim_{r \to 1^-} A_r = \pi \sum_{n=1}^{\infty} n|a_n|^2$$

and this concluded the proof. Q.E.D

Theorem 2.5.3 (Exterior Area Theorem) *If*

$$f(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n}$$

is in $\sum$*, then*

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1.$$

For the proof, please refer to (Conway, 1996).

Theorem 2.5.4 (Bieberbach's Theorem for the second coefficient). *If* $f \in S$*, then* $|a_2| \leq 2$ *with equality holds if and only if* f *is a rotation of the Koebe function.*

Proof. (Duren, 1983) By some calculation, a square-root transformation and inversion applied to $f \in S$ will produce a function

$$h(z) = \left[f(1/z^2) \right]^{-1/2} = z - \frac{a_2}{2} \cdot \frac{1}{z} + b_3 \cdot \frac{1}{z^3} + \cdots$$

in $\sum_0$. By Exterior Area Theorem, we have

$$\sum_{n=1}^{\infty} n|b_n| = \left|-\frac{a_2}{2}\right| + 3|b_3|^2 + \cdots \le 1$$

Thus, $|-a_2/2| \le 1$ and this implies that $|a_2| \le 2$.

Next, equality holds if and only f is a rotation of Koebe function. First, it is easy to show that the rotation of Koebe function

$$k_\theta(z) = e^{-i\theta}k(e^{i\theta}z) = \frac{z}{\left(1-e^{i\theta}z\right)^2}$$

$$= z + 2e^{i\theta}z^2 + 3e^{i\theta}z^3 + \cdots$$

has a second coefficient such that $|a_2| = 2$. Next, if $a_2 = 2e^{i\theta}$, then we have $b_n = 0$ for all $n > 2$. Therefore we have equation $g(z) = z - e^{i\theta}z$. Thus we have

$$G(z) = \frac{1}{g(1/z)} = \frac{1}{\frac{1}{z} - e^{i\theta}z} = \frac{z}{1-e^{i\theta}z^2}$$

is in S as well by Theorem 2.5.1. From square-root transformation, we know that $f\left(z^2\right) = \left[G(z)\right]^2$, hence we are able to find

$$f\left(z^2\right) = \frac{z^2}{\left(1-e^{i\theta}z^2\right)^2}$$

and this shows that

$$f(z) = \frac{z}{\left(1-e^{i\theta}z\right)^2} = e^{-i\theta}k(e^{i\theta}z)$$

which is a rotation of Koebe function. This concludes the proof. Q.E.D

The proof of the Bieberbach's conjecture is a challenging task. Lowner proved that $|a_3| \le 3$ for every f in S at 1923. The first good estimate for all the coefficients was given by Littlewood who proved that $|a_n| \le en$ in 1925. The best result dated before 1985 is provided by FitzGerald and his student Horowitz in 1978 as they proved that $|a_n| < 1.0691n$. Finally, Bieberbach's conjecture was proved by Louis de Branges in 1986. We ended the discussion of this section by stating the theorem.

Theorem 2.5.5 (Bieberbach's Theorem) *If*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

is in S, then $|a_n| \le n$. The inequality is sharp with equality occurs if and only if f is a rotation of the Koebe function.

For the proof, please refer to (De Branges, 1985).

2.6 Applications of Bieberbach's Theorem

In this section, a classical application of Bieberbach's theorem will be discussed. For holomorphic function and non-constant f on D, we know that $f(D)$ is an open set by the open mapping theorem. Since $f \in S$ with $f(0) = 0$, then its range must contain some disk centered at 0. As early as 1907, Koebe found out that the ranges of all functions in S must contain a common disk $\{w:|w|<1/4\}$. This is the famous covering theorem.

Theorem 2.6.1 (Koebe One-Quarter Theorem) *The range of every function of the class S contains the disk* $\{w:|w|<1/4\}$.

Proof. (Duren, 1983) If a function $f\in S$ omits the value $w\in\mathbb{C}$, from the omitted-value transformation,

$$g(z)=\frac{wf(z)}{w-f(z)}=z+\left(a_2+\frac{1}{w}\right)z^2+\cdots$$

is holomorphic and univalent with $g(0)=0$ and $g'(0)=1$, hence $g\in S$.

Bieberbach's Theorem gives

$$\left|a_2+\frac{1}{w}\right|\le 2$$

From triangle inequality and the inequality $|a_2|\le 2$, this shows $|1/w|\le 2+|a_2|\le 4$, thus $|w|\ge 1/4$. Hence, every omitted value must lie outside the disk $\{w:|w|<1/4\}$. Thus, the range of function f contains the disk $\{w:w<1/4\}$. Q.E.D

CHAPTER 3

AN EXPLICIT EXAMPLE OF A CLASS OF UNIVALENT FUNCTIONS

This chapter discusses about the disk automorphism. A brief introduction on disk automorphism will be given at the beginning of the chapter and later it will be used to construct an example.

3.1 Disk Automorphism

From Chapter 2, we know that functions in the class S are invariant under disk automorphism which is defined as follow

$$f(z)=\frac{z+\alpha}{1+\bar{\alpha}z}, \qquad |\alpha|<1$$

and f mapped D one to one and onto D, and mapped the origin to α.

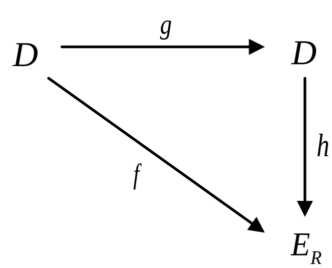
3.2 Constuction of An Explicit Example

Let $E_R = E(R,r)=\{z:|z-r|<R\}\subset\mathbb{C}$ for fixed R and r such that $0\le r<R$. From Riemann Mapping Theorem, we know that there exists a unique conformal mapping g between D and E_R such that $g(0)=0$ and $g'(0)>0$. We wish to determine when will there be a function $f\in S$ such that $f:D\to E_R$ is a biholomorphism. As it turns out, we are able to obtain a relation between R and r, an explicit expression for f, and some of its geometrical properties.

We first note that $h(z) = Rz + r$ is a conformal mapping from D to E_R. Next, consider the following mapping $g: D \to D$.

$$g(z) = \frac{z - \frac{r}{R}}{1 - \frac{r}{R} z}$$

Clearly, the composite function $f = h \circ g$ is a conformal mapping from D to E_R. But it may not lie in S. We see that



and

$$\begin{aligned} f(z) &= h \circ g(z) \\ &= R\left(\frac{z - \frac{r}{R}}{1 - \frac{r}{R} z}\right) + r \\ &= \frac{(R^2 - r^2)z}{R - rz} \end{aligned}$$

Clearly, $f(0) = 0$. Differentiating the function f, we have

$$f'(z) = \frac{R(R^2 - r^2)}{(R - rz)^2}$$

In order that f in S, we must have $f'(0)=1$, that is

$$f'(0)=\frac{R^2-r^2}{R}=1$$

which implies that $R=R^2-r^2$ or $r=\sqrt{R(R-1)}$, with $R\geq 1$

Thus we notice that in order that $f:D\to E_R$ is in S, then R and r must satisfy $r=\sqrt{R(R-1)}$ and $R\geq 1$. For $r=\sqrt{R(R-1)}$,

$$f(z)=\frac{(R^2-r^2)z}{R-rz}=\frac{Rz}{R-\left(\sqrt{R(R-1)}\right)z}$$

For $R\geq 1$, the function

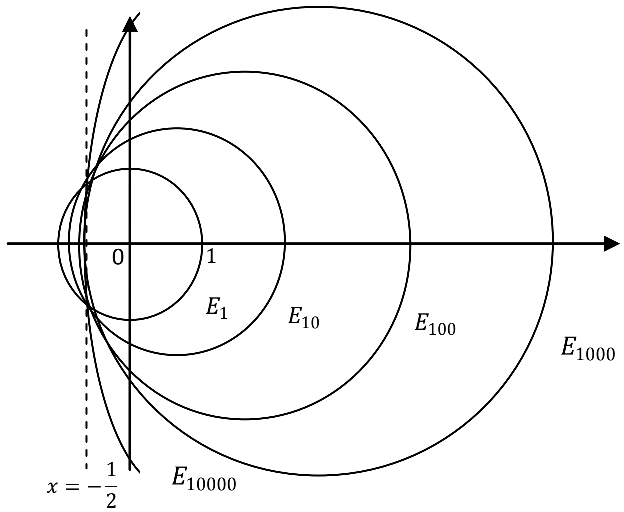
$$f_R(z)=\frac{Rz}{R-\left(\sqrt{R(R-1)}\right)z}$$

is the only conformal function from D to E_R such that $f(0)=0$ and $f'(0)=1$ and thus $f_R\in S$.

Summarizing, in order that $f:D\to E_R$ is a biholomorphism and $f\in S$, the center of the disk E_R must be located at $r=\sqrt{R(R-1)}$, and

$$f_R(z)=\frac{Rz}{R-\left(\sqrt{R(R-1)}\right)z}$$

For different radius of R, for example, when R = 1, 10, 100, 1000, or 10000, E_R can be illustrated as following:



In the diagram above, $r - R$ is the left x-intercept of E_R on the real axis. As $R \to \infty$, $r - R$ will tends to $-\frac{1}{2}$. In fact,

$$\lim_{R\to\infty} (r - R) = \lim_{R\to\infty} \frac{r^2 - R^2}{r + R}$$

$$= \lim_{R\to\infty} -\frac{R}{r + R}$$

$$= \lim_{R\to\infty} -\frac{1}{\frac{r}{R} + 1}$$

$$= -\frac{1}{2}$$

Thus, the limiting position of E_R as $R \to \infty$ will be the right open half plane bounded by $x = -1/2$. On the other hand, analytically as $R \to \infty$, $f_R(z)$ will tends to the function $f(z) = z(1 - z)^{-1}$.

In fact

$$\lim_{R\to\infty} f_R(z) = \lim_{R\to\infty} \frac{Rz}{R-rz}$$

$$= \lim_{R\to\infty} \frac{z}{1-\frac{r}{R}z}$$

$$= \frac{z}{1-z}$$

Therefore,

$$\lim_{R\to\infty} f_R(z) = f(z)$$

As in Chapter 2, the range of $f(z)$ is the half plane satisfying $Re\{f(z)\} > -\frac{1}{2}$. This shows that when $R \to \infty$, the limiting function of f_R mapped unit disk D conformally onto the right half-plane $Re\{f(z)\} > -\frac{1}{2}$. Thus, $\lim_{R\to\infty} f_R(z) = f(z)$ analytically as well as geometrically. For better understanding on the geometrical display of function $f(z) = \dfrac{z}{1-z}$, please refer to Appendix A and B.

CHAPTER 4

SUBCLASSES OF UNIVALENT FUNCTIONS

In this section, we study the starlike and convex functions which are two of the important subclasses in S. Some interesting geometrical and analytical properties of these functions will be discussed and studied in this chapter. We are able to generalize some results here.

4.1 Starlike and Convex Functions

Lemma 4.1.1. (Schwarz Lemma). *Let f be analytic in the unit disk D, with $f(0)=0$ and $|f(z)|<1$ in D. Then $|f'(0)|\le 1$ and $|f(z)|\le|z|$ in D. The equality occurs if and only if f is a rotation.*

Proof. (González, 1991a) Since $f(0)=0$, from Cauchy-Taylor expansion we have

$$f(z)=f'(0)z+\frac{f''(0)}{2!}z^2+\cdots$$

valid for $|z|<1$. Let

$$h(z)=f'(0)+\frac{f''(0)}{2!}z+\cdots$$

then $h(z)$ is analytic in $|z|<1$, and

$$h(z) = \frac{f(z)}{z} \quad \text{for } z \neq 0 \qquad \text{and} \qquad h(0) = f'(0)$$

For $|z| < 1$, choose r such that $|z| < r < 1$. $|h(\zeta)|$ attains its maximum in $|\zeta| \le r$ at some points on its boundary $|\zeta| = r$ by Maximum Modulus Theorem. Since $f(0) = 0$ and $|f(z)| < 1$, for $|z| \le r < 1$, we have

$$|h(z)| \le \max_{|\zeta| \le r} |h(\zeta)| = \max_{|\zeta| = r} \frac{|f(\zeta)|}{|\zeta|} \le \frac{1}{r}$$

The inequalities remain true when r approaches to 1. Hence we obtain

$$|h(z)| \le 1 \qquad \text{for } |z| < 1$$

Thus, it follows that

$$|f(z)| \le |z| \text{ for } 0 < |z| < 1 \qquad \text{and} \qquad |f'(0)| \le 1$$

For $z = 0$, the inequality holds since $f(0) = 0$ by assumption.

If $|f(z)| = |z|$, we have $|h(z)| = 1$ at some point z_0 in D, then $|h(z)|$ attains its maximum value at z_0. For $|f'(0)| = 1$, $|h(0)| = 1$, then $|h(z)|$ atttanis it maximum value at 0. By Maximum Modulus Principle, this is impossible unless $h(z)$ is a constant. In this case we have $h(z) = a$, where a is a constant such that $|a| = 1$. Thus, $f(z) = az$. On the other hand, if $f(z) = az$, then $|f(z)| = |z|$ for all z and $f'(z) = a$, therefore $|f'(0)| = 1$. Q.E.D

Definition 4.1.1 Let $f(z)$ and $g(z)$ be holomorphic in D. $f(z)$ is said to be *subordinate* to $g(z)$ if there exists a holomorphic function $\varphi(z)$ (not necessarily univalent) in D satisfying $\varphi(0)=0$ and $|\varphi(z)|<1$ such that

$$f(z)=g\left(\varphi(z)\right) \text{ , for } z\in D$$

If $f(z)$ is subordinated to $g(z)$, it is denoted by $f(z)\prec g(z)$.

We first go through some basic properties of subordination. Let $f(z)\prec g(z)$, since $\varphi(D)\subset D$ and $\varphi(0)=0$ it follows that $f(D)\subset g(D)$ and $f(0)=g(0)$. Moreover, $|\varphi(z)|\le|z|$ by Schwarz's Lemma and therefore

$$\left\{f(z):|z|<r\right\}\subset\left\{g(z):|z|<r\right\} \text{ , } 0<r<1$$

Notice that f and g are not assumed to be univalent in Definition 4.1.1. When the subordinating function is univalent, the above properties will lead to a principle known as *Principle of Subordination.*

Lemma 4.1.2. *Let $g(z)$ be univalent in D. Then $f(z)\prec g(z)$ if and only if $f(0)=g(0)$ and $f(D)\subset g(D)$.*

For the proof, please refer to (Jensen and Pommerenke, 1975).

Theorem 4.1.1 (Principle of Subordination) *If $g(z)$ is univalent in D, then $f(0)=g(0)$ and $f(D)\subset g(D)$ implies*

$$f(D_r)\subset g(D_r)$$

where $D_r=\left\{|z|<r, 0<r<1\right\}$.

For the proof, please refer to (Jensen and Pommerenke, 1975).

Definition 4.1.2 Let Ω be a set in $\mathbb{C}$. We say that Ω is *starlike* (with respect to origin) if the closed line segment joining the origin to each point $w \in \Omega$ lies entirely in Ω. We say that Ω is *convex* if for all $w_1, w_2 \in \Omega$, the closed line segment between w_1 and w_2 lies entirely in Ω.

Let ST denote the subclass of S which consists all the starlike functions with respect to origin and let CV denote the subclass of S which consists all the convex functions. Closely related to the classes ST and CV is the class P containing all the function g holomorphic and having positive real part in D, with $g(0)=1$. The following theorem gives an analytic description of starlike and convex function.

Theorem 4.1.2 *Let f be holomorphic in D, with $f(0)=0$ and $f'(0)=1$. Then $f \in ST$ if and only if $zf'/f \in P$.*

Proof. (Duren, 1983) First suppose that $f \in ST$. Then we claim that f maps each subdisk $|z|<r<1$ onto a starlike domain. The same assertion is that $g(z)=f(rz)$ is starlike in D. In other words, we must show that for each fixed t, where $0<t<1$ and for each $z \in D$, the point $tg(z)$ is in the range of g. But since $f \in ST$, we have $\{tf(z):|z|<1\} \subset \{f(z):|z|<1\}$, $0<t\le 1$. From Lemma 4.1.2, for arbitrary fixed t_0, we have $t_0 f \prec f$. In fact, the function $\omega(z)$ in Definition 4.1.1 can be defined as

$$\omega(z) = f^{-1}(t_0 f(z))$$

and then

$$f\left(\omega(z)\right)=t_0 f(z)$$

We have

$$\begin{aligned}\omega(0) &= f^{-1}(t_0 f(0))\\ &= f^{-1}(0)\\ &= 0\end{aligned}$$

Moreover, $\left|\omega(z)\right|=\left|f^{-1}(t_0 f(z))\right|<1$ since $t_0 f(z)\in\left\{f(z):\left|z\right|<1\right\}$. By Schwarz Lemma, $\left|\omega(z)\right|<\left|z\right|$ and $\left|\omega(rz)\right|<r$. Next, we wish to show that $t_0 g\prec g$. Observe that

$$t_0 g(z)=t_0 f(rz)=f(\omega(rz))=g(\phi(z))$$

where $\phi(z)=\omega(rz)/r$ and $\left|\phi(z)\right|<\left|z\right|$. Therefore, $t_0 g(0)\subset g(0)$ and $t_0 g(D)\subset g(D)$.

From the above arguments and Lemma 4.1.2, we have $t_0 g\prec g$. Hence, we obtain $\left\{t_0 g(z):\left|z\right|<1\right\}\subset\left\{g(z):\left|z\right|<1\right\}$ which implies that $t_0 g(z)$ is in the range of $g(z)$. This proves that f maps each circle $\left|z\right|=r<1$ onto a curve C_r that bounds a starlike domain containing the origin. It follows that $\arg f(z)$ increases as z moves around the circle $\left|z\right|=r$ in the counter-clockwise direction. In other words,

$$\frac{\partial}{\partial\theta}\left\{\arg f(re^{i\theta})\right\} \geq 0$$

Observed that for $z = re^{i\theta}$, we have

$$\frac{\partial}{\partial\theta}\left\{\arg f(re^{i\theta})\right\} = Im\left\{\frac{\partial}{\partial\theta}\log f(re^{i\theta})\right\}$$

$$= Im\left\{\frac{izf'(z)}{f(z)}\right\} = Re\left\{\frac{zf'(z)}{f(z)}\right\}$$

By applying the maximum principle for harmonic functions, we have $zf'/f \in P$.

Conversely, suppose f is a normalized holomorphic function with $zf'/f \in P$. Then f has a simple zero at the origin and no zeros elsewhere in the disk because otherwise zf'/f would have a pole. Retracing the calculation from first part, for each $r < 1$, we have

$$\frac{\partial}{\partial\theta}\left\{\arg f(re^{i\theta})\right\} > 0, \qquad 0 \leq \theta \leq 2\pi$$

Thus as z runs around the circle $|z| = r$ in the counter-clockwise direction, the point $f(z)$ traverses a closed curve C_r with increasing argument. Because f has exactly one zero inside the circle $|z| = r$, from the Argument Principle, we know that C_r surrounds the origin exactly once. But if C_r winds about the origin only once with increasing argument, self-intersection does not exists. Thus C_r is a simple closed curve which bounds a starlike domain D_r, and f assumes each value $w \in D_r$ exactly once in the disk $|z| < r$. Since this is true for every $r < 1$, it follows that f is univalent and starlike in D. This concludes the proof. Q.E.D

Theorem 4.1.3 *Let f be holomorphic in D, with $f(0)=0$ and $f'(0)=1$. Then $f \in CV$ if and only if $\left[1 + zf''/f'\right] \in P$.*

Proof. (Duren, 1983) Suppose first that $f \in CV$. We claim that f must map each sub-disk $|z| < r$ onto a convex domain. To show this, choose points z_1 and z_2 such that $|z_1| < |z_2| < r$. Let $\alpha_1 = f(z_1)$ and $\alpha_2 = f(z_2)$. Let

$$\alpha_0 = t\alpha_1 + (1-t)\alpha_2, \qquad 0 < t < 1$$

Since $f \in CV$, there is a unique point $z_0 \in D$ for which $f(z_0) = \alpha_0$. We have to show that $|z_0| < r$. Let

$$\varphi(z) = tf\,(z_1 z/z_2) + (1-t)\,f(z)$$

then $\varphi(z)$ is analytic in D, with $\varphi(0) = 0$ and $\varphi(z_2) = \alpha_0$. Since $f \in CV$, the function

$$h(z) = f^{-1}(\varphi(z))$$

is well-defined. Since $h(0) = 0$ and $|h(z)| \le 1$, we have $|h(z)| \le |z|$ from Schwarz Lemma. Therefore

$$|z_0| = |h(z_2)| \le |z_2| < r$$

And this is what we wanted to show. Hence f maps each circle $|z| = r < 1$ onto a curve C_r which bounds a convex domain. The geometry of convexity implies that the slope of the tangent to C_r is nondecreasing as the curve is traversed in the counter-clockwise direction, that is

$$\frac{\partial}{\partial\theta}\left(\arg\left\{\frac{\partial}{\partial\theta} f(re^{i\theta})\right\}\right) \geq 0$$

or

$$Im\left\{\frac{\partial}{\partial\theta}\log[ire^{i\theta} f'(re^{i\theta})]\right\} \geq 0$$

$$Im\left\{\frac{i^2re^{i\theta} f'(re^{i\theta}) + (ire^{i\theta})^2 f''(re^{i\theta})}{ire^{i\theta} f'(re^{i\theta})}\right\} \geq 0$$

$$Im\left\{i + \frac{ire^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})}\right\} \geq 0$$

which reduces to the condition

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq 0, \qquad |z| = r$$

Thus, we have $\left[1 + zf''(z)/f'(z)\right] \in P$ by the maximum principle for the harmonic functions.

Conversely, suppose f satisfied the conditions stated in the theorem and with $\left[1 + zf''(z)/f'(z)\right] \in P$. The calculation as in the first part shows that the slope of the tangent to the curve C_r increases monotonically. But as a point makes a complete circuit of C_r, the argument of the tangent vector has a net change

$$\int_0^{2\pi} \frac{\partial}{\partial\theta}\left(\arg\left\{\frac{\partial}{\partial\theta} f(re^{i\theta})\right\}\right) d\theta = \int_0^{2\pi} Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} d\theta$$

$$= Re\left\{\int_{|z|=r}\left[1+\frac{zf''(z)}{f'(z)}\right]\frac{dz}{iz}\right\} = 2\pi$$

for $z = re^{i\theta}$.

This shows that C_r is a simple closed curve bounding a convex domain. For arbitrary $r<1$, this implies that f is a univalent function with convex range.

Q.E.D

We give several examples of starlike and convex functions. The first one is the famous Koebe function. We have defined Koebe function before in Chapter 2, we wish to point out that it is starlike but not convex.

Example 4.1.1 Koebe function $k(z) = \dfrac{z}{(1-z)^2}$ is in ST but not in CV.

Proof. From previous section, we know that the image of Koebe function is the whole plane minus the part of the negative real axis from $-1/4$ to negative infinity. Thus, it is clear that Koebe function is starlike with respect to origin and not convex.

Recalled that in Example 2.4.1, we showed that $f(z)=z+az^2$ is in S for $|a|\le 1/2$ and not univalent in D for $|a|>1/2$. In fact, the condition $|a|\le 1/2$ also ensures that f belongs to class ST and it can be found that when $|a|\le 1/4$, f is a convex function. Q.E.D

Example 4.1.2 $f(z)=z+az^2$ is in ST if and only $|a|\le 1/2$.

Example 4.1.3 $f(z)=z+az^2$ is in CV if and only $|a|\le 1/4$.

For the proof of Examples 4.1.2 and 4.1.3, please refer to (Goodman, 1983).

The above two examples can be generalized. First, we have the following theorem.

Theorem 4.1.4 *The function* $f(z)=z+az^3$ *is in* S *if and only if* $|a|\le 1/3$.

Proof. It is clear that $f(z)$ is a holomorphic function in D and normalized with the conditions $f(0)=0$, $f'(0)=1$. For $|a|\le 1/3$, observe that when $a=0$, $f(z)=z$ is clearly a univalent function is S. For $a\ne 0$, let $f(z_1)=f(z_2)$, where $z_1,z_2\in D$. Then we have

$$(z_1-z_2)\left[1+a\left(z_1^{\ 2}+z_1z_2+z_2^{\ 2}\right)\right]=0$$

We claim that $z_1=z_2$. If not, then $\left|z_1^{\ 2}+z_1z_2+z_2^{\ 2}\right|=1/|a|$. From triangle inequality, $\left|z_1^{\ 2}\right|\ge 1/|a|-|z_1z_2|-\left|z_2^{\ 2}\right|>3-1-1=1$ which contradicts the fact that $|z|<1$. Therefore $z_1=z_2$, and hence f is univalent. Since f is in normalized form, this implies that $f\in S$.

For $|a|>1/3$, let $z_0=-1/(3a)$, since $|z_0|=1/\left(3|a|\right)<1$ therefore $z_0\in D$ and $f'(z_0)=1+3a\left(-1/3a\right)=0$ which implies that f is not local univalent. Then we conclude that f is not univalent in D for $|a|>1/3$. By contrapositive, the theorem is proved. Q.E.D

Theorem 4.1.5 $f(z)=z+az^3$ *is in* ST *if and only if* $|a|\le 1/3$.

Proof. If $f\in ST$, then $f\in S$. From Theorem 4.1.4, we have $|a|\le 1/3$.

Conversely, we prove that if $|a|\le 1/3$, then f is starlike. We first show that $\left|\frac{zf'}{f}-1\right|\le 1$.

$$\left|\frac{zf'}{f}-1\right|=\left|2-\frac{2}{1+az^2}\right|=\left|\frac{2az^2}{1+az^2}\right|<\frac{2|a|}{1-|a|}\le 1$$

Hence, $\frac{zf'}{f}$ lies in a circle centered at 1 with radius $r=1$ and thus $\frac{zf'}{f}\in P$ and hence f is starlike. Q.E.D

Theorem 4.1.6 $f(z)=z+az^3$ *is in* CV *if and only if* $|a|\le 1/9$.

Proof. To prove this theorem it is sufficient to prove it for the constant where it is real. Since for $a\in\mathbb{C}$, consider the function $F(z)=z+cz^3$ where $c\in\mathbb{R}$, for $a=ce^{i2\theta}$, observed that

$$f(z) = z + az^3 = z + ce^{i2\theta} z^3 = e^{-i\theta}\left(e^{i\theta} z + ce^{i3\theta} z^3\right) = e^{-i\theta} F\left(e^{i\theta} z\right)$$

We see that f is a rotation of F. Therefore, f is convex if and only if F is convex since rotation of convex function remain convex. Thus, without loss of generality, we prove the theorem in terms of real constant.

We first prove that if $F \in CV$, then $-1/9 \le c \le 1/9$. Some simple calculations show that

$$1 + \frac{zF''}{F'} = 3 - \frac{2}{1 + 3cz^2}$$

Let $z = re^{i\theta}$ and h be the real part of the holomophic function $1 + zF''/F'$, then

$$h_r(\theta) = Re\left\{1 + \frac{zF''}{F'}\right\} = 3 - \frac{2 + 6cr^2 \cos 2\theta}{1 + 9c^2 r^4 + 6cr^2 \cos 2\theta}$$

$$h_r'(\theta) = \frac{12cr^2 (9c^2 r^4 - 1) \sin 2\theta}{\left(1 + 9c^2 r^4 + 6cr^2 \cos 2\theta\right)^2}$$

According to Maximum Modulus Principle for harmonic function, it must attain its maximum and minimum values on the boundary of the unit disk, and hence $r = 1$ and $h_1'(\theta) = 0$. Therefore, $c = 0$, $\sin 2\theta = 0$ or $c^2 = 1/9$.

When $c = 0$, the function $F(z) = z$ is obviously a convex function. When $\sin 2\theta = 0$, $\cos 2\theta = \pm 1$. Then, the extremal of h_1 (minimum or maximum) occurs at

$$3 - \frac{2 + 6c}{1 + 9c^2 + 6c} = 3 - \frac{2}{1 + 3c} = \frac{1 + 9c}{1 + 3c} \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$$

for $\cos 2\theta = 1$, or

$$3 - \frac{2-6c}{1+9c^2-6c} = 3 - \frac{2}{1-3c} = \frac{1-9c}{1-3c} \cdots\cdots\cdots\cdots\cdots\cdots (2)$$

for $\cos 2\theta = -1$.

Since F is convex, $h_r(\theta) > 0$. At the boundary, $h_1 \geq 0$. By taking the value of c from three different interval, we analyze and determine the nature of the two quotients. For (1) and (2), we have

Table 1

	$c < -1/3$	$-1/3 < c < -1/9$	$c \geq -1/9$
$1+9c$	–	–	$0/+$
$1+3c$	–	$+$	$+$
$\frac{1+9c}{1+3c}$	$+$	–	$0/+$

	$c \leq 1/9$	$1/9 < c < 1/3$	$c > 1/3$
$1-9c$	$0/+$	–	–
$1-3c$	$+$	$+$	–
$\frac{1-9c}{1-3c}$	$0/+$	–	$+$

From the above table, in order to obtain $h_1 \geq 0$, we concluded that $c < -1/3$ or $c \geq -1/9$ for (1) and $c \leq 1/9$ or $c > 1/3$ for (2). Since $F \in CV$ implies $F \in S$, from Theorem 4.1.4, we have $-1/3 \leq c \leq 1/3$. By combining the inequalities, we have $-1/9 \leq c \leq 1/3$ for (1) and $-1/3 \leq c \leq 1/9$ for (2). One of them will obtain the *min* h_1 while the other one will obtain *max* h_1. To ensure that the extremal of h_1 to be existed, we have $-1/9 \le c \le 1/9$. Thus $F \in CV$ implies $-1/9 \le c \le 1/9$ .Thus $f \in CV$ since $F \in CV$ and $|a| = |ce^{i2\theta}| \le 1/9$.

Conversely, we prove that if $|a| \le 1/9$, then f is convex. We first show that $|1 + zf''/f' - 1| \le 1$.

$$\left|1+\frac{zf''}{f'}-1\right| = \left|2-\frac{2}{1+3az^2}\right| = \left|\frac{6az^2}{1+3az^2}\right| < \frac{6|a|}{1-3|a|} \le 1$$

Hence, $1+\dfrac{zf''}{f'}$ lies in a circle centered at 1 with radius $r = 1$ which means that $1+\dfrac{zf''}{f'} \in P$ and thus $f \in CV$. Q.E.D

From the Examples 4.1.2 and 4.1.3 and Theorems 4.1.5 and 4.1.6, we are able to generalize the above two theorems to higher order.

Theorem 4.1.7 *The function* $f(z) = z + az^{m+1}$ *is in* S *if and only if* $|a| \le 1/(m+1)$.

Proof. It is clear that $f(z)$ is a holomorphic function in D and normalized with the conditions $f(0) = 0$, $f'(0) = 1$. For $|a| \le 1/(m+1)$, observe that when $a = 0$, $f(z) = z$ is clearly a univalent function is S. For $a \ne 0$, let $f(z_1) = f(z_2)$, where $z_1, z_2 \in D$. Then calculations show that

$$(z_1 - z_2)\left[1 + a\left(z_1^{\,m} + z_1^{\,m-1}z_2 + \cdots + z_1 z_2^{\,m-1} + z_2^{\,m}\right)\right] = 0$$

We claim that $z_1 = z_2$. If not, then we have $|z_1^{\,m} + z_1^{\,m-1}z_2 + \cdots + z_1 z_2^{\,m-1} + z_2^{\,m}| = 1/|a|$. From triangle inequality, $|z_1^{\,m}| \ge 1/|a| - |z_1^{\,m-1}z_2| - \cdots - |z_1 z_2^{\,m-1}| - |z_2^{\,m}| > (m+1) - m = 1$

which contradicts the fact that $|z|<1$. Therefore $z_1 = z_2$, and hence f is univalent. Since f is in normalized form, this implies that $f \in S$.

For $|a|>1/(m+1)$, let $z_0 = -1/(m+1)a$, since $|z_0| = 1/(m+1)|a|<1$, therefore $z_0 \in D$ and $f'(z_0) = 1+(m+1)a\left(-1/(m+1)a\right) = 0$ which implies that f is not local univalent. Then we conclude that f is not univalent in D for $|a|>1/(m+1)$. By contrapositive, the theorem is proved. Q.E.D

Theorem 4.1.8 $f(z) = z + az^{m+1}$ *is in* ST *if and only if* $|a| \le \dfrac{1}{m+1}$.

Proof. If $f \in ST$, then $f \in S$. By Theorem 4.1.7, we have $|a| \le \dfrac{1}{m+1}$.

Conversely, we prove that if $|a| \le \dfrac{1}{m+1}$, then f is starlike. We first show that

$|zf'/f - 1| < 1$

$$\left|\frac{zf'}{f} - 1\right| = \left|m - \frac{m}{1+az^m}\right| = \left|\frac{maz^m}{1+az^m}\right| < \frac{m|a|}{1-|a|} \le 1$$

Hence, $\dfrac{zf'}{f}$ lies in a circle centered at 1 with radius $r=1$ and thus $\dfrac{zf'}{f} \in P$ and therefore f is starlike. Q.E.D

Theorem 4.1.9 $f(z)=z+az^{m+1}$ *is in* CV *if and only if* $|a|\le \dfrac{1}{(m+1)^2}$.

Proof. To prove this theorem it is sufficient to prove it for the constant where it is real. Since for $a\in\mathbb{C}$, consider the function $F(z)=z+cz^{m+1}$ where $c\in\mathbb{R}$, for $a=ce^{im\theta}$, observed that

$$f(z)=z+az^{m+1}=z+ce^{im\theta}z^{m+1}=e^{-i\theta}\left(e^{i\theta}z+ce^{i(m+1)\theta}z^{m+1}\right)=e^{-i\theta}F\left(e^{i\theta}z\right)$$

We can see that f is a rotation of F. Therefore, f is convex if and only if F is convex since rotation of convex function remain convex. Thus, without loss of generality, we prove the theorem in terms of real constant.

We first prove that if $F\in CV$, then $-1/(m+1)^2\le c\le 1/(m+1)^2$. Calculations show that

$$1+\frac{zF''}{F'}=m+1-\frac{m}{1+(m+1)cz^m}$$

Let $z=re^{i\theta}$ and h be the real part of the holomophic function $1+zF''/F'$, then

$$h_r(\theta)=Re\left\{1+\frac{zF''}{F'}\right\}=m+1-\frac{m+m(m+1)cr^m\cos m\theta}{1+(m+1)^2c^2r^{2m}+2(m+1)cr^m\cos m\theta}$$

$$h_r'(\theta)=\frac{m^2(m+1)cr^m\left((m+1)^2c^2r^{2m}-1\right)\sin m\theta}{\left(1+(m+1)^2c^2r^{2m}+2(m+1)cr^m\cos m\theta\right)^2}$$

According to Maximum Modulus Principle for harmonic function, it must attain its maximum and minimum values on the boundary of the unit disk, and hence $r=1$ and $h_1'(\theta)=0$. Therefore, $c=0$, $\sin m\theta=0$ or $c^2=1$.

When $c = 0$, the function $F(z) = z$ is obviously a starlike and convex function. When $\sin m\theta = 0$, $\cos m\theta = \pm 1$. Then, the extremal of of h_1 (minimum or maximum) occurs at

$$m+1-\frac{m+m(m+1)c}{1+(m+1)^2c^2+2(m+1)c}=\frac{1+(m+1)^2c}{1+(m+1)c}\cdots\cdots\cdots\cdots (1)$$

for $\cos m\theta = 1$

$$m+1-\frac{m-m(m+1)c}{1+(m+1)^2c^2-2(m+1)c}=\frac{1-(m+1)^2c}{1-(m+1)c}\cdots\cdots\cdots\cdots (2)$$

for $\cos m\theta = -1$

Since F is convex, $h_r(\theta) > 0$. At the boundary, $h_1 \geq 0$. By taking the value of c from three different interval, we analyze and determine the nature of the two quotients. From (1) and (2), we have

Table 2

	$c < -1/(m+1)$	$-1/(m+1) < c < -1/(m+1)^2$	$c \geq -1/(m+1)^2$
$1+(m+1)^2c$	$-$	$-$	$0/+$
$1+(m+1)c$	$-$	$+$	$+$
$\frac{1+(m+1)^2c}{1+(m+1)c}$	$+$	$-$	$0/+$

	$c \leq 1/(m+1)^2$	$1/(m+1)^2 < c < 1/(m+1)$	$c > 1/(m+1)$
$1-(m+1)^2c$	$0/+$	$-$	$-$
$1-(m+1)c$	$+$	$+$	$-$
$\frac{1-(m+1)^2c}{1-(m+1)c}$	$0/+$	$-$	$+$

From the above table, in order to obtain $h_1 \geq 0$, we concluded that $c < -\frac{1}{m+1}$ or $c \geq -\frac{1}{(m+1)^2}$ for (1) and $c \leq \frac{1}{(m+1)^2}$ or $c > \frac{1}{m+1}$ for (2). Since $F \in CV$ implies that $F \in S$, from Theorem 4.1.7, we have $-\frac{1}{m+1} \leq c \leq \frac{1}{m+1}$. By combining all the inequalities, we have $-\frac{1}{(m+1)^2} \leq c \leq \frac{1}{(m+1)}$ for (1) and $-\frac{1}{(m+1)} \leq c \leq \frac{1}{(m+1)^2}$ for (2). One of them will obtain the *min* h_1 while the other one will obtain *max* h_1. To ensure that the extremal of h_1 to be existed, we have $-\frac{1}{(m+1)^2} \leq c \leq \frac{1}{(m+1)^2}$. Therefore, $F \in CV$ implies $-\frac{1}{(m+1)^2} \leq c \leq \frac{1}{(m+1)^2}$. Therefore $f \in CV$ since $F \in CV$ and $|a| = |ce^{im\theta}| \leq \frac{1}{(m+1)^2}$.

Conversely, we prove that if $|a| \leq 1/(m+1)^2$, then f is convex. It is sufficient to show that $|1 + zf''/f' - 1| < 1$.

$$\left|1+\frac{zf''}{f'}-1\right| = \left|m - \frac{m}{1+(m+1)az^m}\right| = \left|\frac{m(m+1)az^m}{1+(m+1)az^m}\right| < \frac{m(m+1)|a|}{1-(m+1)|a|} \leq 1$$

Hence, $1+\frac{zf''}{f'}$ lies in a circle centered at 1 with radius $r = 1$ which means that

$1+\frac{zf''}{f'} \in P$ and therefore f is convex. Q.E.D

Recalled from Theorems 4.1.2 and 4.1.3, these two theorems tell us that starlike and convex mappings have a closely analytic connection and this was first discovered by Alexander in 1915.

Theorem 4.1.10 (Alexander's Theorem). *Let* f *be analytic in* D, *with* $f(0) = 0$ *and* $f'(0) = 1$. *Then* $f \in CV$ *if and only if* $zf' \in ST$.

Proof.(Duren, 1983) If $g(z) = zf'(z)$, then

$$\frac{zg'(z)}{g(z)} = 1 + \frac{zf''(z)}{f'(z)}$$

Thus, the left-hand function is analytic and has a positive real part in D if and only if the same is true for the right-hand function. Q.E.D

In fact, we are able to relate Theorems 4.1.8 and 4.1.9 using Alexander's theorem. From Theorem 4.1.9, we have $f(z) = z + az^{m+1} \in CV$ if and only if $|a| \leq \frac{1}{(m+1)^2}$. Let $h(z) = zf'(z) = z + a(m+1)z^{m+1}$. If we let $A = a(m+1)$, then $h(z)$ is starlike if and only if $|A| = |a(m+1)| \leq \frac{1}{m+1}$ which is exactly same with Theorem 4.1.8.

There are other interesting properties about ST and CV. From previous section, we know that from Bieberbach's Theorem, for all $f(z) = z + a_2 z^2 + \cdots$ in S, we have $|a_n| \leq n$ for $n = 2, 3, \ldots$. In fact, a weaker result was proved for all $f \in ST$ by Nevanlinna (Nevanlinna, 1920-1921) and for $f \in CV$ by Loewner (Loewner, 1917). The theorems are stated as following.

Theorem 4.1.11. *The coefficients of each function* $f \in ST$ *satisfy* $|a_n| \leq n$ *for* $n = 2, 3, \ldots$. *Strict inequality holds for all* n *unless* f *is a rotation of the Koebe function.*

For the proof, please refer to (Nevanlinna, 1920-1921).

Corollary 4.1.1. *If* $f \in CV$, *then* $|a_n| \leq 1$ *for* $n = 2, 3, \ldots$. *Strict inequality holds for all* n *unless* f *is a rotation of the function* h *defined by* $h(z) = z(1-z)^{-1}$.

Proof. (Duren, 1983) From Theorem 4.1.10, if $f \in CV$, then $zf' \in ST$. In fact,

$$zf'(z) = z + \sum_{n=2}^{\infty} n a_n z^n$$

From Theorem 4.1.11, we have $n|a_n| \le n$ and therefore $|a_n| \le 1$.

When the equality occurs, the function

$$h(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n$$

satisfies $zh'(z) = k(z)$ and maps D onto the half-plane $Re\{w\} > -1/2$. Q.E.D

Theorem 4.1.12 *The range of every convex function* $f \in CV$ *contains the disk* $|w| < 1/2$.

For the proof, please refer to (Duren, 1983).

4.2 A Subclass of S Consisting of only Negative Coefficient

For $0 \le \alpha \le 1$, the function $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in S$ is said to be starlike of order α if $Re\{zf'/f\} > \alpha$ and convex of order α if $Re\{1 + zf''/f'\} > \alpha$. Let $ST(\alpha)$ denote the subclass of S consisting all the functions starlike of order α and let $CV(\alpha)$ denote the subclass of S consisting all the functions convex of order α.

In 1975, Herb Silverman introduced a subclass of univalent functions consisting of functions where all coefficients are negative except the coefficient for z. The subclass is denoted as T and all functions in T can be expressed as

$$f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$$

Moreover, he introduced subclasses of T, $T^*(\alpha)$ as the class consisting all starlike functions of order α in T and $C^*(\alpha)$ as the class consisting all convex function of order α in T. He proved some coefficient inequalities that involve the above subclasses.

Theorem 4.2.1. *Let* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$. *If* $\sum_{n=2}^{\infty}(n-\alpha)|a_n|\le 1-\alpha$, *then* $f\in ST(\alpha)$.

Corollary 4.2.1. *Let* $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$. *If* $\sum_{n=2}^{\infty}n(n-\alpha)|a_n|\le 1-\alpha$, *then* $f\in CV(\alpha)$.

Theorem 4.2.2. *A function* $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$ *is in* $T^*(\alpha)$ *if and only if* $\sum_{n=2}^{\infty}(n-\alpha)|a_n|\le 1-\alpha$.

Corollary 4.2.2 *A function* $f(z)=z-\sum_{n=2}^{\infty}|a_n|z^n$ *is in* $C^*(\alpha)$ *if and only if* $\sum_{n=2}^{\infty}n(n-\alpha)|a_n|\le 1-\alpha$.

For the proof of Theorems 4.2.1 and 4.2.2; Corollaries 4.2.1 and 4.2.2, please refer to (Silverman, 1975).

4.3 Generalized Koebe Function

From previous sections, we have defined Koebe function and we know that Koebe function is a starlike function. In this section, we generalize the Koebe function as follow.

$$f_\alpha(z) = z(1-z)^{-\alpha}, \text{ where } 0 \le \alpha \le 2.$$

We wish to know whether the generalized Koebe functions remain starlike. Expanding $f_\alpha(z) = z\left(1+\alpha z + \frac{\alpha(\alpha+1)}{2!}z^2 + \cdots\right)$, then it is clear that $f_\alpha(z)$ is a well-defined single-valued function. Note that $f_2(z)$ is the Koebe function, and $f_0(z) = z$ is the identity mapping.

Theorem 4.3.1. $f_\alpha(z) = z\left(1-z\right)^{-\alpha} \in ST$ *if and only if* $0 \le \alpha \le 2$.

Proof. Suppose first that $0 \le \alpha \le 2$. It is easy to show that f_α is a normalized analytic function. We wish to prove that f_α is starlike. Using Theorem 4.1.2, it is sufficient to show that $zf_\alpha'(z)\big/f_\alpha(z) \in P$.

$$\frac{zf_\alpha(z)}{f_\alpha(z)} = \frac{1+(\alpha-1)z}{1-z}$$

$$= 1 + \frac{\alpha z}{1-z}$$

Now, we just have to prove that $Re\left\{\frac{\alpha z}{1-z}\right\} > -1$.

From Chapter 2, we have $Re\left\{\frac{z}{1-z}\right\} > -\frac{1}{2}$. Thus,

$$Re\left\{\frac{\alpha z}{1-z}\right\}=\alpha Re\left\{\frac{z}{1-z}\right\}$$

$$>-\frac{\alpha}{2}$$

Since $0\le\alpha\le2$, then $-1\le-\frac{\alpha}{2}\le0$. Thus, we have $Re\left\{\frac{\alpha z}{1-z}\right\}>-1$. Therefore, $f_\alpha\in ST$. Conversely, suppose that $f_\alpha\in ST$ which implies that f_α is univalent. Thus, $f_\alpha'(z)\ne0$, $\forall z\in D$.

$$f_\alpha'(z)=\frac{1}{(1-z)^\alpha}+\frac{\alpha z}{(1-z)^{\alpha+1}}$$

$$=\frac{1+(\alpha-1)z}{(1-z)^{\alpha+1}}$$

Since $f_\alpha'(z)\ne0$, this gives $1+(\alpha-1)z\ne0$ and thus $z\ne-(\alpha-1)^{-1}$. Therefore, the point $z_0=-(\alpha-1)^{-1}$ must lie outside of the unit disk.

$$|z_0|=\left|-\frac{1}{\alpha-1}\right|\ge1$$

$$|\alpha-1|\le1$$

Thus, $0\le\alpha\le2$. The theorem is proved. Q.E.D

Theorem 4.3.2. *If* $f_\alpha(z) = z(1-z)^{-\alpha}$ *where* $0 \le \alpha \le 2$, *then range of* f_α *contains an open disk of radius* $1/(2+\alpha)$.

Proof. By Theorem 4.3.1, we have $f_\alpha \in S$ since $f_\alpha \in ST$. Let ω be a complex number such that $\omega \ne f_\alpha(z)$ for all $z \in D$. By omitted-value transformation, we have $(\omega f_\alpha)/(\omega - f_\alpha) \in S$.

$$\frac{\omega f_\alpha(z)}{\omega - f_\alpha(z)} = \frac{\omega z}{\omega(1-z)^\alpha - z}$$

$$= z + \left(\alpha + \frac{1}{\omega}\right) z^2 + \cdots$$

By Bieberbach's Theorem, we have $|a_2| \le 2$. Therefore,

$$\left|\alpha + \frac{1}{\omega}\right| \le 2$$

$$-|\alpha| + \left|\frac{1}{\omega}\right| \le 2$$

$$|\omega| \ge \frac{1}{2+\alpha}$$

Since $\omega \ne f_\alpha(z)$, therefore every omitted value must lie outside the disk $\omega < 1/(2+\alpha)$, this proves the theorem. Q.E.D

In fact, we are able to improve the above result by using an alternative method to prove it. The theorem is stated as following.

Theorem 4.3.3. *If* $f_\alpha(z) = z(1-z)^{-\alpha}$ *where* $0 \le \alpha \le 2$, *then range of* f_α *contains an open disk of radius* $1/2^\alpha$.

Proof. For $z = re^{i\theta}$, define

$$f_{\alpha,r}(\theta) = \frac{re^{i\theta}}{(1-re^{i\theta})^\alpha}$$

$$\min_{0\le\theta\le 2\pi} \left| f_{\alpha,r} \right|^2 = \min_{0\le\theta\le 2\pi} \frac{re^{i\theta}}{(1-re^{i\theta})^\alpha} \cdot \frac{re^{-i\theta}}{(1-re^{-i\theta})^\alpha}$$

$$= r^2 \min_{0\le\theta\le 2\pi} \frac{1}{(1+r^2-2r\cos\theta)^\alpha}$$

Let

$$h(\theta) = (1+r^2-2r\cos\theta)^\alpha$$

then

$$h'(\theta) = \alpha(1+r^2-2r\cos\theta)^{\alpha-1}(2r\sin\theta)$$

Since $1+r^2-2r\cos\theta \ge 1+r^2-2r = (1-r)^2 > 0$, when $h'(\theta) = 0$, then $\sin\theta = 0$ which implies that $\theta = 0$ or $\theta = \pi$. Since $h''(0) = 2\alpha r(1-r)^{2(\alpha-1)} > 0$ and $h''(\pi)$ $= -2\alpha r(1+r)^{2(\alpha-1)} < 0$, thus $h(\theta)$ is minimum when $\theta = 0$ and $h(\theta)$ achieves its maximum when $\theta = \pi$. This gives $\min\left| f_{\alpha,r} \right|^2 = \dfrac{r^2}{(1+r)^{2\alpha}}$. As $r \to 1$, $\left| f_{\alpha,1} \right| = \dfrac{1}{2^\alpha}$.

Hence, the range of f_α contains an open disk of radius $\dfrac{1}{2^\alpha}$. Q.E.D

From Theorems 4.3.2 and 4.3.3, we can see that the the result could be different according to the method we used, but we can actually find that the method from Theorem 4.2.3 able to obtain a better scale since f_α contains a larger disk for $0 \le \alpha \le 2$.

4.4 Inequalities for Convex Function

Inequalities that involve function of the class S will be studied in this section. Furthermore, we will improve the results to convex functions. In fact, we found that the inequalities can be improved to a better scale if $f \in CV$. We first begin with the following lemma which is an application of Bieberbach's Theorem and it gives a basic estimate which leads to certain Distortion and Growth Theorem.

Lemma 4.4.1 *For each* $f \in S$ *,*

$$\left| \frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2} \right| \le \frac{4r}{1-r^2}, \qquad |z| = r < 1$$

Proof. (Duren, 1983) Given $f \in S$ and a fix $\zeta \in D$ and perform a disk automorphism to construct

$$F(z) = \frac{f\left(\frac{z+\zeta}{1+\bar{\zeta}z}\right) - f(\zeta)}{\left(1-|\zeta|^2\right) f'(\zeta)} = z + A_2(\zeta)z^2 + \cdots$$

then $F \in S$. By using Theorem 1.2.1 (Taylor's Theorem), calculation shows that

$$A_2(\zeta) = \frac{F''(0)}{2!} = \frac{1}{2}\left\{\left(1-|\zeta|^2\right)\frac{f''(\zeta)}{f'(\zeta)} - 2\bar{\zeta}\right\}$$

From Theorem 2.5.5 (Bieberbach's Theorem), $|A_2(\zeta)| \le 2$.

$$\left|\left(1-|\zeta|^2\right)\frac{f''(\zeta)}{f'(\zeta)} - 2\bar{\zeta}\right| \le 4$$

$$\left|\frac{f''(\zeta)}{f'(\zeta)} - \frac{2\bar{\zeta}}{1-|\zeta|^2}\right| \le \frac{4}{1-|\zeta|^2}$$

$$\left|\frac{\zeta f''(\zeta)}{f'(\zeta)} - \frac{2|\zeta|^2}{1-|\zeta|^2}\right| \le \frac{4|\zeta|}{1-|\zeta|^2}$$

Replacing ζ by z, the lemma is proved. Q.E.D

Lemma 4.4.2 *If f is holomorphic in D and $f'(z) \ne 0$ for all $z \in D$, then for $z = re^{i\theta} \in D$, we have*

$$Re\left\{\frac{zf''(z)}{f'(z)}\right\} = r\frac{\partial}{\partial r} Re\{\log f'(z)\}$$

Proof. (Duren, 1983) Taking the principal branch of complex logarithm function and differentiate with respect to r, we have

$$r\frac{\partial}{\partial r}\log f'(z) = r\frac{\partial}{\partial r}\left(\log|f'(z)| + iArg\ f'(z)\right)$$

Calculation shows that

$$r\frac{\partial}{\partial r}\log f'(z) = r\frac{f''(z)}{f'(z)}.\frac{\partial z}{\partial r} = r\frac{f''(z)}{f'(z)}e^{i\theta} = \frac{zf''(z)}{f'(z)}$$

Taking the real part of both side, we obtained

$$Re\left\{\frac{zf''(z)}{f'(z)}\right\} = r\frac{\partial}{\partial r}\log\left|f'(z)\right|.$$

The lemma is proved. Q.E.D

Theorem 4.4.1 (Distortion Theorem) *For each* $f \in S$,

$$\frac{1-r}{(1+r)^3} \le \left|f'(z)\right| \le \frac{1+r}{(1-r)^3}, \quad |z| = r < 1$$

For each $z \in D$, $z \ne 0$, *equality occurs if and only if* f *is a suitable rotation of the Koebe function.*

Proof. (Duren, 1983) An inequality $|\alpha| \le c$ implies that $-c \le Re\{\alpha\} \le c$. For $f \in S$, from Lemma 4.4.1, it follows that,

$$-\frac{4r}{1-r^2} \le Re\left\{\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2}\right\} \le \frac{4r}{1-r^2}$$

and thus

$$\frac{2r^2-4r}{1-r^2} \le Re\left\{\frac{zf''(z)}{f'(z)}\right\} \le \frac{2r^2+4r}{1-r^2}$$

for $z = re^{i\theta}$, $|z| < 1$.

By Lemma 4.4.2, we have

$$Re\left\{\frac{zf''(z)}{f'(z)}\right\} = r\frac{\partial}{\partial r}\log\left|f'(z)\right|$$

and hence,

$$\frac{2r-4}{1-r^2} \le \frac{\partial}{\partial r}\log\left|f'(re^{i\theta})\right| \le \frac{2r+4}{1-r^2}$$

Holding θ fixed and integrate with respect to r from 0 to R. A calculation shows the inequality

$$\log\frac{1-R}{(1+R)^3} \le \log\left|f'(Re^{i\theta})\right| \le \log\frac{1+R}{(1-R)^3}$$

for $z = Re^{i\theta}$. The Distortion Theorem follows by exponentiation.

It left only to prove the equality part of the Distortion Theorem. If $f(z) = k(z) = z/(1-z)^2$, then

$$f'(z) = \frac{1+z}{(1-z)^3}$$

Let $z = r < 1$ then we obtained the equality on the right side. On the other hand, let $z = -r > -1$ then we obtained the equality on the left side. This shows that both sides of the inequalities are sharp.

Furthermore, whenever equality occurs at upper estimate for $z = re^{i\theta}$, we have

$$\left| f'(re^{i\theta}) \right| = \frac{1+r}{(1-r)^3}$$

and thus

$$\frac{\partial}{\partial r} \log \left| f'(re^{i\theta}) \right| = \frac{2r+4}{1-r^2} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$$

From Lemma 4.4.2, we have

$$Re\left\{ \frac{e^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})} \right\} = \frac{\partial}{\partial r} \log \left| f'(re^{i\theta}) \right| \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (2)$$

For $r = 0$, choosing $\theta \in \mathbb{R}$ such that $Re\left\{ e^{i\theta} f''(0) \right\} = e^{i\theta} f''(0)$. Since $f''(0) = 2a_2$, we have $Re\left\{ e^{i\theta} f''(0) \right\} = e^{i\theta}\left(2a_2\right)$. From (1) and (2), then we obtained

$$Re\left\{ \frac{e^{i\theta} f''(0)}{f'(0)} \right\} = \frac{\partial}{\partial r} \log \left| f'(0) \right| = 4$$

From the choice of $\theta \in \mathbb{R}$, we have $2a_2 e^{i\theta} = 4$ implies that $\left| a_2 \right| = 2$. For the lower estimate, repeat the steps as above and eventually yield the same conclusion. By Bieberbach's Theorem, f is a rotation of Koebe function. This concluded the proof. Q.E.D

Next, we are going to discuss the Growth Theorem, Growth Theorem is the direct consequence of Distortion Theorem. The theorem is stated as follow.

Theorem 4.4.2 (Growth Theorem) *Suppose that* $f \in S$. *Then for* $|z| = r$,

$0 < r < 1$, *we have*

$$\frac{r}{(1+r)^2} \le |f(z)| \le \frac{r}{(1-r)^2}$$

For each $z \in D$, $z \neq 0$, *equality occurs if and only if* f *is a suitable rotation of the Koebe function.*

Proof. (Conway, 1996) Let $f \in S$ and fix $z = re^{i\theta}$ with $0 < r < 1$. Observed that

$$f(z) = \int_0^r f'(\rho e^{i\theta}) e^{i\theta} \, d\rho$$

since $f(0) = 0$. From Distortion Theorem, we obtained

$$|f(z)| \le \int_0^r |f'(\rho e^{i\theta})| \, d\rho \le \int_0^r \frac{1+\rho}{(1-\rho)^3} \, d\rho = \frac{r}{(1-r)^2}$$

The lower estimate is not as straightforward. If $|f(z)| \ge 1/4$, the proof is trivial since $r(1+r)^{-2} < 1/4$ for $0 < r < 1$. Then we obtained

$$\frac{r}{(1+r)^2} < \frac{1}{4} \le |f(z)|$$

and we are done.

For the case where $|f(z)| < 1/4$, we fix $z \in D$ and let γ be the path in D from 0 to z such that $f \circ \gamma$ is the straight line segment $[0, f(z)]$. In fact, from Theorem 2.6.1 (Koebe One-Quarter Theorem), $\gamma(t) = f^{-1}(tf(z))$, $0 \le t \le 1$. That means,

$f\left(\gamma(t)\right) = tf(z)$ for $0 \le t \le 1$. Thus, $|f(z)| = \left|\int_{\gamma} f'(w)\, dw\right| = \left|\int_0^1 f'\left(\gamma(t)\right)\gamma'(t)\, dt\right|$.

Observed that $f'\left(\gamma(t)\right)\gamma'(t) = \left[tf(z)\right]' = f(z)$ for all t. Thus we deduced that

$$
\begin{aligned}
|f(z)| &= \left|\int_0^1 f(z)\, dt\right| \\
&= \int_0^1 |f(z)|\, |dt| \\
&= \int_0^1 \left|f'\left(\gamma(t)\right)\gamma(t)\right| |dt| \\
&= \int_{\gamma} |f'(w)|\, |dw|
\end{aligned}
$$

If we take $0 \le s < t \le 1$, then $|\gamma(t) - \gamma(s)| \ge \left||\gamma(t)| - |\gamma(s)|\right|$ and so $|dw| \ge d\,|w|$. By combining all the inequalities and applying the Distortion Theorem, we obtained

$$|f(z)| = \int_{\gamma} |f'(w)|\, |dw| \ge \int_0^r \frac{1-w}{1+w}\, d\,|w| = \frac{r}{(1+r)^2}$$

and this is the lower estimate of the Growth Theorem.

Equality in either part of inequality of Growth Theorem implies equality in the corresponding part of inequality of Distortion Theorem, and Distortion Theorem implies that f is a rotational transformation of the Koebe function.

Q.E.D

The inequalities for function in the class S were discussed above. Next, we are interested to study if the inequalities can be improved for convex functions.

By similar arguments, we first consider the following lemma and it leads to improvement of inequalities.

Lemma 4.4.3 *For each* $f \in CV$,

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2}\right| \le \frac{2r}{1-r^2}, \qquad |z| = r < 1$$

Proof. Given $f \in CV$, fix $\zeta \in D$ and perform a disk automorphism to construct

$$F(z) = \frac{f\left(\frac{z+\zeta}{1+\bar{\zeta}z}\right) - f(\zeta)}{\left(1-|\zeta|^2\right) f'(\zeta)} = z + A_2(\zeta)z^2 + \cdots$$

then we can see that

$$f\left(\frac{z+\zeta}{1+\bar{\zeta}z}\right) = \left(1-|\zeta|^2\right) f'(\zeta)F(z) + f(\zeta)$$

A dilation and translation of a convex function remain convex. Thus $F \in CV$ since $f \in CV$. From Taylor's Theorem, a calculation gives

$$A_2(\zeta) = \frac{F''(0)}{2!} = \frac{1}{2}\left\{\left(1-|\zeta|^2\right)\frac{f''(\zeta)}{f'(\zeta)} - 2\bar{\zeta}\right\}$$

From Corollary 4.1.1, $|A_2(\zeta)| \le 1$.

$$\left|\left(1-|\zeta|^2\right)\frac{f''(\zeta)}{f'(\zeta)} - 2\bar{\zeta}\right| \le 2$$

$$\left|\frac{f''(\zeta)}{f'(\zeta)}-\frac{2\bar{\zeta}}{1-|\zeta|^2}\right|\le\frac{2}{1-|\zeta|^2}$$

$$\left|\frac{\zeta f''(\zeta)}{f'(\zeta)}-\frac{2|\zeta|^2}{1-|\zeta|^2}\right|\le\frac{2|\zeta|}{1-|\zeta|^2}$$

Replacing ζ by z, the lemma is proved. Q.E.D

From Theorems 4.4.1 and 4.4.2, we know that the value of $|f'(z)|$ lies between $(1-r)/(1+r)^3$ and $(1+r)/(1-r)^3$ while $|f(z)|$ lies between $r/(1+r)^2$ and $r/(1-r)^2$. By using the above lemma, we improved the inequalities as follow.

Theorem 4.4.3. *For each* $f\in CV$,

$$\frac{1}{(1+r)^2}\le|f'(z)|\le\frac{1}{(1-r)^2},\quad |z|=r<1$$

For each $z\in D$, $z\neq 0$, *equality occurs if and only if* f *is a suitable rotation of the function* h *defined by* $h(z)=z(1-z)^{-1}$.

Proof. Note that $|\alpha|\le c$ implies that $-c\le Re\{\alpha\}\le c$. For $f\in CV$, it follows from Lemma 4.4.3 that

$$-\frac{2r}{1-r^2}\le Re\left\{\frac{zf''(z)}{f'(z)}-\frac{2r^2}{1-r^2}\right\}\le\frac{2r}{1-r^2}$$

and thus

$$-\frac{2r}{1+r} \le Re\left\{\frac{zf''(z)}{f'(z)}\right\} \le \frac{2r}{1-r}$$

for $z = re^{i\theta}$, $|z| < 1$.

By Lemma 4.4.2, we have

$$Re\left\{\frac{zf''(z)}{f'(z)}\right\} = r\frac{\partial}{\partial r}\log\left|f'(z)\right|$$

and hence,

$$-\frac{2}{1+r} \le \frac{\partial}{\partial r}\log\left|f'(re^{i\theta})\right| \le \frac{2}{1-r}$$

Holding θ fixed and integrate with respect to r from 0 to R. A calculation shows the inequality

$$\log\frac{1}{(1+R)^2} \le \log\left|f'(Re^{i\theta})\right| \le \log\frac{1}{(1-R)^2}$$

for $z = re^{i\theta}$. The theorem follows by exponentiation.

It lefts only to prove the equality of the theorem. If $h(z) = z(1-z)^{-1}$, then

$$h'(z) = \frac{1}{(1-z)^2}$$

Let $z = r < 1$ then we obtained the equality on the right side. On the other hand, let $z = -r > -1$ then we obtained the equality on the left side. This shows that both sides of the inequalities are sharp.

Furthermore, whenever equality occurs at the upper estimate for $z = re^{i\theta}$, we have

$$\left| f'(re^{i\theta}) \right| = \frac{1}{(1-r)^2}$$

and thus

$$\frac{\partial}{\partial r}\log\left| f'(re^{i\theta}) \right| = \frac{\partial}{\partial r}\left(\log (1-r)^{-2}\right) = \frac{2}{1-r} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$$

From Lemma 4.4.2, we have

$$Re\left\{\frac{e^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})}\right\} = \frac{\partial}{\partial r}\log\left| f'(re^{i\theta}) \right| \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (2)$$

For $r = 0$, choosing $\theta \in \mathbb{R}$ such that $Re\left\{e^{i\theta} f''(0)\right\} = e^{i\theta} f''(0)$. Since $f''(0) = 2a_2$, we have $Re\left\{e^{i\theta} f''(0)\right\} = e^{i\theta}\left(2a_2\right)$. From (1) and (2), then we obtained

$$Re\left\{\frac{e^{i\theta} f''(0)}{f'(0)}\right\} = \frac{\partial}{\partial r}\log\left| f'(0) \right| = 2$$

From the choice of $\theta \in \mathbb{R}$, we have $2a_2 e^{i\theta} = 2$ implies that $\left|a_2\right| = 1$. By Corollary 4.1.1, f is a rotation of function h. For the lower estimate, repeat the steps as above and eventually yield the same conclusion. This concluded the proof. Q.E.D

Theorem 4.4.4 *Suppose that* $f \in CV$. *Then for* $|z| = r$, $0 \le r < 1$, *we have*

$$\frac{r}{1+r} \le \left| f(z) \right| \le \frac{r}{1-r}$$

For each $z \in D$, $z \ne 0$, *equality occurs if and only if* f *is a suitable rotation of the function* h *defined by* $h(z) = z\left(1-z\right)^{-1}$.

Proof. Let $f \in S$ and fix $z = re^{i\theta}$ with $0 < r < 1$. Observed that

$$f(z) = \int_0^r f'(\rho e^{i\theta}) e^{i\theta} \, d\rho$$

From Theorem 4.4.3, we obtained

$$|f(z)| \leq \int_0^r \left| f'(\rho e^{i\theta}) \right| d\rho \leq \int_0^r \frac{1}{(1-\rho)^2} \, d\rho = \frac{r}{1-r}$$

The lower estimate is not as straightforward. If $|f(z)| \geq 1/2$, the proof is trivial since $r(1+r)^{-1} < 1/2$ for $0 \leq r < 1$. Then we obtained

$$\frac{r}{1+r} < \frac{1}{2} \leq |f(z)|$$

and we are done.

For the case where $|f(z)| < 1/2$, we fix $z \in D$ and let γ be the path in D from 0 to z such that $f \circ \gamma$ is a straight line segment $[0, f(z)]$. In fact, from Theorem 4.1.12, $\gamma(t) = f^{-1}(tf(z))$, $0 \leq t \leq 1$. That means, $f(\gamma(t)) = tf(z)$ for $0 \leq t \leq 1$. Thus, $|f(z)| = \left| \int_\gamma f'(w) \, dw \right| = \left| \int_0^1 f'(\gamma(t)) \gamma'(t) \, dt \right|$. Observed that $f'(\gamma(t))\gamma'(t) = [tf(z)]' = f(z)$ for all *t*. Thus we deduced that

$$|f(z)| = \left|\int_0^1 f(z)\,dt\right|$$

$$= \int_0^1 |f(z)|\,|dt|$$

$$= \int_0^1 \left|f'\big(\gamma(t)\big)\gamma'(t)\right|\,|dt|$$

$$= \int_\gamma |f'(w)|\,|dw|$$

If we take $0 \le s < t \le 1$, then $|\gamma(t) - \gamma(s)| \ge \big||\gamma(t)| - |\gamma(s)|\big|$ and so $|dw| \ge d\,|w|$. By combining the inequalities and applying the Theorem 4.4.3, we obtained

$$|f(z)| = \int_\gamma |f'(w)|\,|dw| \ge \int_0^r \frac{1}{(1+w)^2}\,d\,|w| = \frac{r}{1+r}$$

and this is the lower estimate of the theorem.

Equality in either part of inequality of Theorem 4.4.4 implies equality in the corresponding part of inequality of Theorem 4.4.3, and Theorem 4.4.3 implies that f is a rotational transformation of the function h defined by $h(z) = z(1-z)^{-1}$. Q.E.D

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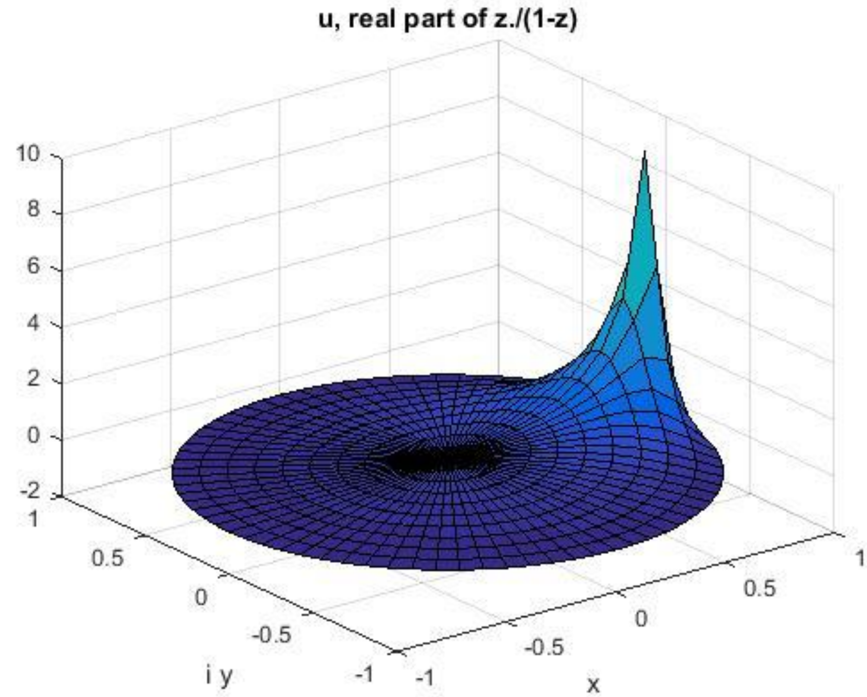
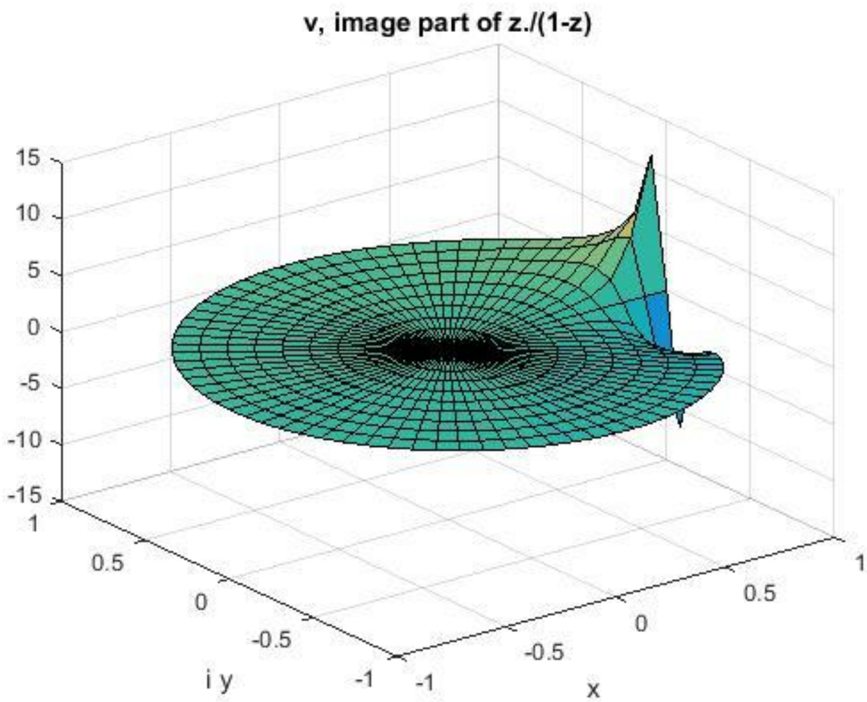
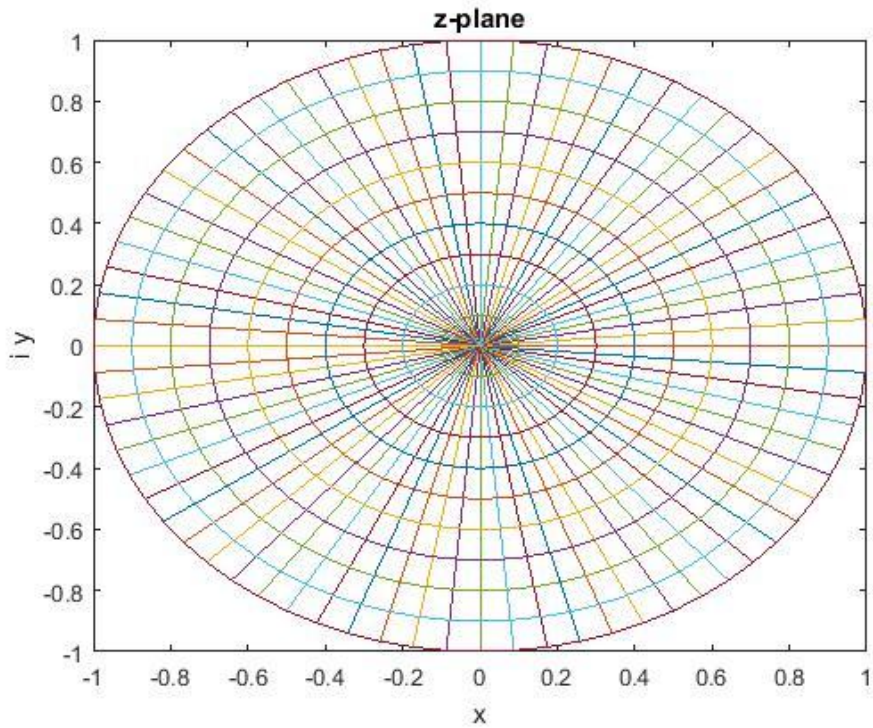
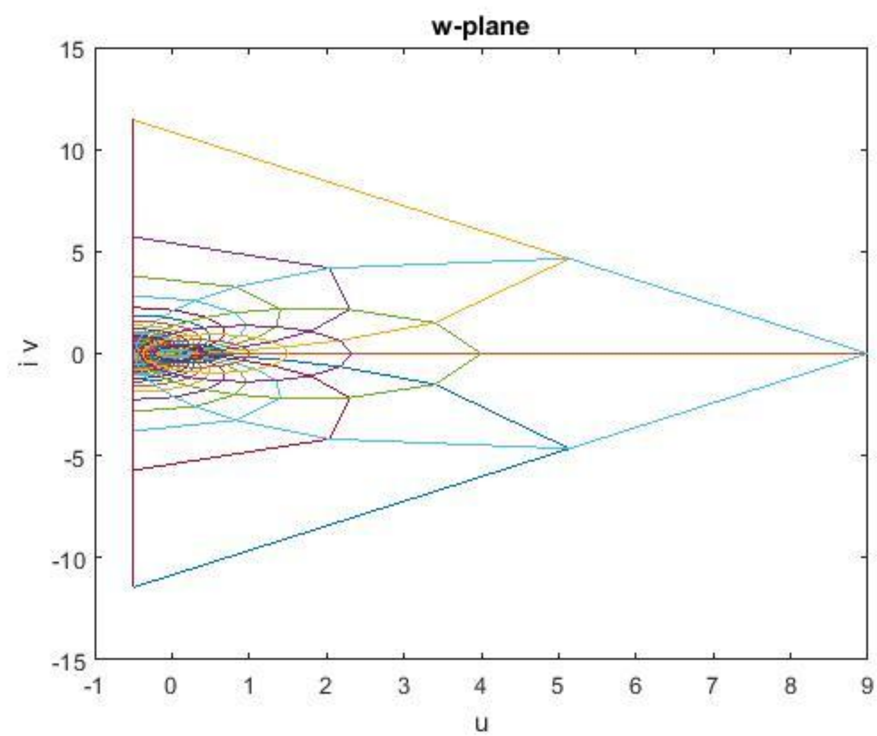
APPENDIX A


u, real part of z./(1-z)
10
8
6
4
2
0
-2
1
0.5
0
-0.5
-1
i y
-1
-0.5
0
0.5
1
x


v, image part of z./(1-z)
15
10
5
0
-5
-10
-15
1
0.5
0
-0.5
-1
i y
-1
-0.5
0
0.5
1
x

APPENDIX B


z-plane
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
i y
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
x


w-plane
15
10
5
0
-5
-10
-15
i v
-1
0
1
2
3
4
5
6
7
8
9
u

APPENDIX C

Matlab code for plot the complex function

```
Function cplxplot_polar

%cplxplot_polar – to plot the graph of complex functions w=f(z) for the given domain in the polar coordinate without and with patched grids.

f = inside('z./(1-z)'); %change this line for other function, e.g., f=sin(z)

dt=pi/36;
theta=-pi:dt:pi;
rho=0:0.1:1;
[THETA,RHO]=meshgrid(theta,rho);
[X,Y]=pol2cart(THETA,RHO);
Z=X+sqrt(-1)*Y
W=f(Z);

U=real(W);
V=imag(W);

figure(1)
surf(X,Y,U);
xlabel('x')
ylabel('i y')
title(['u, real part of ' char(f)])

figure(2)
surf(X,Y,V);
xlabel('x')
```

```
ylabel('i y')
title(['v, image part of ' char(f)])

figure(3)
plot(X,Y)
hold on
plot(X',Y')
xlabel('x')
ylabel('i y')
title('z-plane')

figure(4)
plot(U,V)
hold on
plot(U',V')
xlabel('u')
ylabel('i v')
title('w-plane')
```