A STUDY ON UNIVALENT FUNCTIONS AND FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

By

TEO YEE RUI

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ABSTRACT

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Teo Yee Rui

In this thesis, a study on univalent functions will be carried out. The class S is defined to be the set of functions holomorphic and univalent in unit disk, $\mathbb{D}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Linear Fractional Transformation in class S will be defined and its range will be studied analytically. We will introduce an equivalence relation in class S and will provide two examples to prove that starlikeness and convexity of such equivalence relation will not be preserved. It is also shown that the equivalence class of such equivalence relation is the complete solution for a particular first order ordinary differential equation. Finally, we will show that Schwarzian Derivative is invariant to the equivalence relation defined.

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APPROVAL SHEET

This dissertation entitled **"A STUDY ON UNIVALENT FUNCTIONS AND FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS"** was prepared by TEO YEE RUI and submitted as partial fulfilment for the requirements for the degree of Master of Science at Universiti Tunku Abdul Rahman.

Approved by:

____

Dr. Tan Sin Leng Date: _______________

Associate Professor / Supervisor

Department of Mathematical Science

Lee Kong Chian Faculty of Engineering and Science

Universiti Tunku Abdul Rahman

____

Dr. Leong Yoon Kwai Date: _______________

Assistant Professor / Co-Supervisor

Department of Mathematical Science

Lee Kong Chian Faculty of Engineering and Science

Universiti Tunku Abdul Rahman

LEE KONG CHIAN FACULTY OF ENGINEERING AND SCIENCE

UNIVERSITI TUNKU ABDUL RAHMAN

Date: ________________________

SUBMISSION OF DISSERTATION

It is hereby certified that **TEO YEE RUI** (ID No: **13UEM06953**) has completed this dissertation entitled **"A STUDY ON UNIVALENT FUNCTIONS AND FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS"** under the supervision of DR. TAN SIN LENG (Supervisor) from the Department of Mathematical Sciences, Lee Kong Chian Faculty of Engineering and Sciences.

I understand that the University will upload softcopy of my dissertation in PDF format into UTAR Institutional Repository, which may be made accessible to UTAR community and public.

Yours truly,

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TEO YEE RUI

DECLARATION

I hereby declare that the dissertation is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UTAR or other institutions.

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(TEO YEE RUI)

Date : ____________________

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INTRODUCTION

A study on univalent functions were carried out in this thesis. In Chapter 1 and 2, we will make a survey and discuss about ideas and theorems in complex analysis and univalent functions. Important theorems such as Open Mapping Theorem, Riemann Mapping Theorem and Bieberbach's Theorem will be discussed in detail as they are important throughout the development of the thesis.

In Chapter 3, the definition for linear fractional transformations in class S will be given. Certain subclasses of S such as S^* and C are discussed and defined geometrically as well as analytically. At the end of the chapter, convexity of linear fractional transformations in S together with its range will also be studied here.

In Chapter 4, an equivalence relation will be introduced and such equivalence class will be shown to be the complete solution for a particular first order non-linear differential equation. Then, the famous Schwarzian derivative will be shown to be invariant with respect to the equivalence relation introduced. Furthermore, example will be given to show that the starlikeness of the equivalence classes of starlike function will not be preserved. Another example will also be given to show that the convexity of the equivalence classes of convex function will not be preserved.

CHAPTER 1

HOLOMORPHIC FUNCTIONS

In this chapter, analytic functions and some of their properties will be introduced. We will mainly focus on the analyticity of complex differentiable functions and some of their results.

1.1 Real Differentiable Function

A real-valued function $f(x)$ defined on an open interval $I \subset \mathbb{R}$ is said to be *differentiable* at the point $x_0 \in I$, if

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists. The limit is denoted by $f'(x_0)$, and is called the *derivative* of $f(x)$ at x_0. If the limit does not exist, then $f(x)$ is *not differentiable* at x_0.

Continuity is a necessary condition for differentiability, that is, if a function is diffentiable at x_0, then f is continuous at $x = x_0$. However, the converse of this statement may not be true. A simple example is $f(x) = |x|$ which is continuous in $\mathbb{R}$ but not differentiable at 0.

If the derivative of f, denoted by f' is itself differentiable, then the derivative of f' is denoted by f'' and is called the second derivative of f. Continuing in this form, then we obtain functions

$$f, f', f'', f^{(3)}, \ldots, f^{(n)}, \ldots$$

where $f^{(n)}$ is called the n-th derivative of the function f. Each of the functions above is the derivative of the previous one.

We are also interested in the analyticity of a function. A real-valued function $f(x)$ on a nonempty, open interval (a,b) is said to be *analytic* if it can be represented by a Taylor series, that is,

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

that the power series will converge to f near $x_0 \in (a,b)$. It can be shown that the coefficient of the Taylor series is given by $a_n = \frac{f^{(n)}(x_0)}{n!}, n = 0,1,2,\ldots$. Furthermore, if f is analytic at x_0, then f can be proved to be infinitely differentiable on $I = (x_0 - \delta, x_0 + \delta)$ for some $\delta > 0$.

As mentioned above, analytic functions are functions that can be represented by Taylor series and infinitely differentiable. However, a real-valued function f can be infinitely differentiable and yet not analytic.

Consider the function

$$g(x) = \begin{cases} e^{(-1/x)}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

The function can be shown to be infinitely differentiable on $\mathbb{R}$ with $g^n(0) = 0$ for all non-negative integer n. If g is analytic, then the Taylor series representation of g at $x = 0$ is the zero function and it is obviously convergent for every $x \in \mathbb{R}$. But that is impossible since $g(x) = 0$ only if $x \leq 0$ and the function g is not equal to zero near $x = 0$. Therefore, g is not analytic at $x = 0$.

1.2 Complex Differentiable Function

In this section, we will look into the idea of differentiation when applied to a complex function f. Let X be an open set in the complex plane $\mathbb{C}$ and f be a complex-valued function in X. The function f is said to be *holomorphic* at the point $z_0 \in X$ if

$$\lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h}$$

exists. The limit is denoted by $f'(z_0)$, and is called the derivative of f at z_0. If f is holomorphic at every point in X, then f is said to be holomorphic in X.

The term holomorphic may be replaced by regular, analytic, or complex differentiable by other authors, but throughout this thesis, the term holomorphic will be used to represent this idea.

The definition of holomorphic function seems to have no difference from differentiable real-valued function. In fact, complex differentiability have much more interesting properties that are not seen in real differentiability. In the previous section, it is shown that differentiable real-valued function may not be analytic, whereas, complex holomorphic functions are always analytic.

One of the most important properties of complex analysis is that holomorphic functions are analytic, a property not shared by its real counterpart, as shown in the previous section. Thus, a holomorphic function can be represented by a Taylor series.

Theorem 1.2.1 (Taylor's Theorem). *Suppose that f is analytic in a domain $G \subset \mathbb{C}$ and $D(\alpha, R)$ is any open disk contained in G centered at α with radius $R > 0$. Then the Taylor series for f converges to $f(z)$ for all z in $D(\alpha, R)$, that is,*

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\alpha)}{k!}(z-\alpha)^k \quad , \quad \text{for all } z \in D(\alpha, R)$$

Furthermore, for any r, $0 < r < R$, the convergence is uniform on the closed subdisk $\overline{D}(\alpha, r) = \{z : |z-\alpha| \leq r\}$.

For the proof, please refer to (Matthews and Howell, 2012).

Since holomorphic function can be represented by a Taylor series, therefore term-by-term differentiation can be carried out within the disk of convergence $D(\alpha, R)$.

$$\frac{d}{dz}\sum_{n=0}^{\infty} a_n(z-\alpha)^n = \sum_{n=0}^{\infty} a_n \frac{d}{dz}(z-\alpha)^n.$$

Thus, unlike differentiable real-valued function, holomorphic functions are analytic and infinitely differentiable. Besides that, term-by-term integration also holds for analytic funtions. We conclude this section by stating the following theorem.

Theorem 1.2.2. *A power series $\sum_{n=0}^{\infty} a_n(z-\alpha)^n$ can be integrated term-by-term within the disk of convergence $D(\alpha, R)$, for every countour Γ lying entirely within the disk $D(\alpha, R)$, namely*

$$\int_{\Gamma}\sum_{n=0}^{\infty} a_n(z-\alpha)^n dz = \sum_{n=0}^{\infty} a_n \int_{\Gamma}(z-\alpha)^n dz$$

For the proof, please refer to (Matthews and Howell, 2000).

1.3 Singularities, Zeros and Poles

A point z_0 is said to be a *singular point*, or *singularity*, of a complex function f if $f(z)$ is not analytic at $z = z_0$. For example, the function $f(z) = \frac{1}{z+1}$ is not analytic at $z = -1$, but it is analytic at every other values of z. Thus, $z = -1$ is a singularity of $f(z)$. Generally, there are two types of singularities, namely *isolated singularity* and *non-isolated singularity*.

A point z_0 is said to be an *isolated singularity* of a complex function f if f is not analytic at z_0 but $f(z)$ is analytic everywhere in the punctured disk $D'(z_0, R)$ for a real number $R > 0$. For example, $z = -1$ is an isolated singularity for the function $f = \frac{1}{z+1}$, since f is analytic everywhere except at $z = -1$. On the other hand, the point $z = 0$ is not an isolated singularity for the function $g(z) = \ln(z)$ since $g(z)$ is not analytic at any negative real numbers. In the above case, we would called $z = 0$ an non-isolated singularity, where every neighborhood of $z = 0$ contains at least one singularity other than $z = 0$.

Next, we will be interested with the power series expansion of f about an isolated singularity z_0 and it will involve both non-negative and negative powers of $z - z_0$. It is known as Laurent series and the following theorem gives an exact definiton to the Laurent series.

Theorem 1.3.1 (Laurent's Theorem). *Suppose that* $0 \leq r < R$*, where* $r, R \in \mathbb{R}$*, and that* f *is analytic within the annulus* $A = r < |z - z_0| < R$*, then* f *has the Laurent series expansion*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n}.$$

The coefficient a_n *are given by*

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\omega)}{(\omega - z_0)^{n+1}} d\omega, n = 0, \pm 1, \pm 2, \ldots$$

where C *is a simple closed curve that lies entirely within* A *and has* z_0 *in its interior.*

For the proof, please refer to (Matthews and Howell, 2012).

We now give an example of Laurent series.

Example 1.3.2. Consider the function

$$f(z) = \frac{\cos z - 1}{z^4}$$

The Taylor series expression for $\cos z - 1$ is given by

$$f(z) = -\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots$$

Divide each term by z^4 to obtain the Laurent series

$$f(z) = \frac{-1}{2z^2} + \frac{1}{24} - \frac{z^2}{720} + \dots \quad \text{(valid for } z \neq 0\text{)}$$

From the Laurent series, we can see that it consists of two part, which are the non-negative powers of $z - z_0$ and negative powers of $z - z_0$.
Principal part refers to the part with negative powers of $z - z_0$, that is

$$\sum_{n=1}^{\infty} \frac{a_{-n}}{(z - z_0)^n}$$

We will now classify the isolated singularity $z = z_0$ according to the number of terms in the principal part. In fact, it is the principal part of the complex function that defines the function itself.

Definition 1.3.3 (Classifications of isolated singular points). Let z_0 be the isolated singularity of a complex-valued function $f(z)$ with Laurent series expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

We distinguish the following types of singularities at z_0.

1. If $a_n = 0$ for $n = -1, -2, -3, \dots$, then f has a *removable singularities* at $z = z_0$.

2. If k is a positive integer such that $a_{-k} \neq 0$ and $a_n = 0$ for $n = -k-1, -k-2, -k-3, \dots$, then f has a *pole* of order k at $z = z_0$.

3. If $a_n \neq 0$ for infinitely many negative integers n, then f has an essential *singularity* at z_0.

If a function f has a removable singularity at the point $z = z_0$, then we can define a_0 to be the value of $f(z_0)$ so that f would be holomorphic for $z = z_0$. For example, given the function $f = (\sin z)/z$, it has an isolated singularity at $z = 0$. The Laurent series of f is given by

$$f(z) = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!}\dots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!}\dots$$

We can define $f(0) = 1$ since the equation became 1 when we set $z = 0$. Therefore, the function $f(z)$ is defined and continuous for every value of z. Furthermore, f is analytic at $z = 0$ since it can be represented by a Taylor series centered at origin.

If f has a pole of order k at α, then the Laurent series for f is

$$f(z) = \sum_{n=-k}^{\infty} c_n (z-\alpha)^n$$

where $c_{-k} \neq 0$. For example

$$f(z) = \frac{\sin z}{z^3} = \frac{1}{z^2} - \frac{1}{3!} + \frac{z^2}{5!} - \frac{z^4}{7!} + \dots$$

has a pole or order 2 at $z = 0$.

If infinitely many negative powers of $(z - \alpha)$ occur in the Laurent series, then f has an essential singularity at α. For example,

$$f(z) = z^2 sin\frac{1}{z} = z - \frac{1}{3!}z^{-1} + \frac{1}{5!}z^{-3} - \frac{1}{7!}z^{-5} + \dots$$

has an essential singularity at the origin.

In the following section, we introduced the idea of *zeros* and *poles*. We began first by defining these two terms and the close relationship between them.

Definition 1.3.4. A function $f(z)$ analytic in $D(\alpha,R)$ is said to have a *zero of order k* at the point $z = z_0$ if and only if

$$f(z_0) = 0, f'(z_0) = 0, f'(z_0) = 0, \ldots, f^{(k-1)}(z_0) = 0, \text{and}, f^{(k)}(z_0) \neq 0$$

Note that a simple *zero* refers to a $zero of order one$.

Corollary 1.3.5. *If $f(z)$ is analytic and has a zero of order k at $z = \alpha$, then $g(z) = \frac{1}{f(z)}$ has a pole of order k at $z = \alpha$.*

Theorem 1.3.6. *A function $f(z)$ analytic in $D(\alpha,R)$ has a zero of order k at the point $z = \alpha$ if and only if the Taylor series representation given by*
$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n$ has

$$a_0 = 0, a_1 = 0, a_2 = 0, \ldots, a_{k-1} = 0, and, a_k \neq 0$$

Proof By Taylor's theorem, we have

$$a_n = \frac{f^{(n)}(\alpha)}{n!}.$$

The theorem follows immediately. **Q.E.D.**

Example 1.3.7. Let $f(z) = z \sin z^2$, then the Taylor series representation is given by

$$f(z) = z^3 - \frac{z^7}{3!} + \frac{z^{11}}{5!} - \frac{z^{15}}{7!} + \ldots$$

From Theorem 1.3.6, then f has an zero of order 3 at $z = 0$. Definition 1.3.4 confirms this fact because $f(0) = f'(0) = f''(0) = 0$ but $f'''(0) = 6 \neq 0$.

The next theorem provides a useful way of characterizing zeros of order k for a holomorphic function f.

Theorem 1.3.8. *A function $f(z)$ analytic in $D(\alpha,R)$ has a zero of order k at the point $z=\alpha$ if and only if f can be expressed as*

$$f(z) = (z-\alpha)^k g(z),$$

where $g(\alpha)\neq 0$ and $g(\alpha)$ is analytic in $D(\alpha,R)$.

Proof Suppose that f has a zero of order k at the point α and $f(z)=\sum_{n=0}^{\infty} c_n(z-\alpha)^n$ for $z\in D(\alpha,R)$. By Theorem 1.3.6, we have $c_n = 0$ for $0\leq n\leq k-1$ and that $c_k\neq 0$.

Then,

$$\begin{aligned} f(z) &= \sum_{n=k}^{\infty} c_n(z-\alpha)^n \\ &= \sum_{n=0}^{\infty} c_{n+k}(z-\alpha)^{n+k} \\ &= (z-\alpha)^k \sum_{n=0}^{\infty} c_{n+k}(z-\alpha)^n \end{aligned}$$

Let $g(z)=\sum_{n=0}^{\infty} c_{n+k}(z-\alpha)^n$, then

$$g(z) = \sum_{n=0}^{\infty} c_{n+k}(z-\alpha)^n = c_k + \sum_{n=1}^{\infty} c_{n+k}(z-\alpha)^n$$

for all z in $D(\alpha,R)$. Since $g(z)$ can be represented by a Taylor series centered at $z=\alpha$, then $g(z)$ is analytic in $D(\alpha,R)$, and $g(\alpha)=c_k\neq 0$.

Conversely, suppose that $f(z)=(z-\alpha)^k g(z)$. Since g is analytic at α, then it can be represented by a Taylor series

$$g(z) = \sum_{n=0}^{\infty} b_n(z-\alpha)^n$$

where $g(\alpha)=b_0\neq 0$ by assumption.

Then,

$$\begin{aligned} f(z) &= g(z)(z-\alpha)^k \\ &= \sum_{n=0}^{\infty} b_n (z-\alpha)^{n+k} \\ &= \sum_{n=k}^{\infty} b_{n-k}(z-\alpha)^n \end{aligned}$$

By Theorem 1.3.6, f has a zero of order k at the point α. This concludes the theorem. **Q.E.D.**

Next, we provide a useful way of characterizing poles of order k for a holomorphic function f.

Theorem 1.3.9. *A function $f(z)$ analytic in punctured disk $D'(\alpha,R)$ has a pole of order k at the point $z=\alpha$ if and only if f can be expressed as*

$$f(z) = \frac{g(z)}{(z-\alpha)^k},$$

where $g(\alpha) \neq 0$ and $g(\alpha)$ is analytic in $D(\alpha,R)$.

Proof Suppose that f has a pole of order k at the point α, then the Laurent series for f is defined as

$$f(z) = \frac{1}{(z-\alpha)^k} \sum_{n=0}^{\infty} c_{n-k}(z-\alpha)^n.$$

Then, let

$$h(z) = \sum_{n=0}^{\infty} c_{n-k}(z-\alpha)^n,$$

for all z in the punctured disk $D'(\alpha,R)$. Then, $h(z) = c_{-k} + \sum_{n=1}^{\infty} c_{n-k} z - \alpha^n$, and $h(\alpha) = c_{-k}$. Therefore, h is analytic in $D(\alpha,R)$ with $h(\alpha) \neq 0$.

Conversely, suppose that $f(z) = g(z)(z-\alpha)^{-k}$. Since g is analytic at the point α with $g(\alpha) \neq 0$, then it has a Taylor series representation

$$g(z) = \sum_{n=0}^{\infty} b_n (z-\alpha)^n.$$

where $b_0 \neq 0$. Then,

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} b_n (z-\alpha)^{n-k} \\ &= \sum_{n=-k}^{\infty} b_{n+k} (z-\alpha)^n \end{aligned}$$

Since $c_{-k} = b_0 \neq 0$, then f has a pole of order k at α. This concludes the theorem.
Q.E.D.

The corollary given below provides an useful relationship between the zero and pole of a function f.

Corollary 1.3.10. *If $f(z)$ has a pole of order k at $z = \alpha$, then $g(z) = \frac{1}{f(z)}$ has a removable singularity at $z = \alpha$. Furthermore, if we define $g(\alpha) = 0$, then $g(z)$ has a zero of order k.*

1.4 Open Mapping Theorem

In this section, we will further discuss about properties of holomorphic functions. We will first look into Open Mapping Theorem which will then lead to maximum modulus principle and maximum modulus theorem.

Theorem 1.4.1 (The Identity Theorem for a disk). *Suppose that f is holomorphic in the disk $D(\alpha, R)$ and that $f(\alpha) = 0$. Then either f is identically zero in the disk or its zero at α is isolated, that is, there exists a $\delta > 0$ such that $f(z) \neq 0$ for $0 < |z - \alpha| < \delta$.*

Proof Since f is holomorphic in the disk $D(\alpha, R)$, by Taylor's Theorem, we can write,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-\alpha)^n \text{ with } a_n = \frac{f^{(n)}(\alpha)}{n!} \text{ for every } z \in D(\alpha, R).$$

If the coefficients $a_n = 0$ for all $n \geq 0$, then f is identically zero in the disk $D(\alpha, R)$. In another case, there exists a smallest integer $m \geq 1$ with $a_m \neq 0$ and

$$f(z) = (z-\alpha)^m \sum_{n=m}^{\infty} a_n (z-\alpha)^{n-m} = (z-\alpha)^m g(z), \text{ for every } z \in D(\alpha, R).$$

The Taylor series for g has a radius of convergence at least R, and it follows that g is holomorphic in the disk $D(\alpha, R)$. Thus, g is continuous at α, that is, for every $\varepsilon > 0$, there exists δ such that $|z-\alpha| < \delta$ implies that $|g(z) - g(\alpha)| < \varepsilon$. Since $g(\alpha) = a_m \neq 0$, consider ε such that $\varepsilon < |g(\alpha)|$, then there exists δ such that $|g(z) - g(\alpha)| < \varepsilon < |g(\alpha)|$ for every $z \in D(\alpha, \delta)$. This implies that $g(z) \neq 0$ for every $z \in D(\alpha, \delta)$. Therefore, $0 < |z-\alpha| < \delta$ implies that $f(z) = (z-\alpha)^m g(z) \neq 0$. Hence, either f is identically zero in the disk or its zero at α is isolated. **Q.E.D.**

The following theorem which is known as Rouche's theorem plays an important role in the proof of Open Mapping Theorem.

Theorem 1.4.2 (Rouche's Theorem). *Let C be a simple closed countour lying entirely within a simply connected domain D. Suppose that f and g are holomorphic in D. If the strict inequality $|f(z) - g(z)| < |f(z)|$ holds for all z on C, then f and g have the same number of zeros (counted according to their order of multiplicities) inside C.*

For the proof, please refer to (Stein and Shakarchi, 2003).

Let U be an open set and let f be a function on U. We say that f is an *open mapping* if for every open subset U' of U, the image $f(U')$ is open. The following theorem states a very nice property where every non-constant analytic functions are open mapping.

Theorem 1.4.3 (Open Mapping Theorem). *Suppose that f is analytic and non-constant in an open set G, then $f(G)$ is open.*

Proof Choose an arbitrary $\alpha \in G$, since f is analytic and non-constant, then by Theorem 1.4.1, $f(z)-f(\alpha)$ has an isolated zero at α. Choose a radius r such that the closed disk $\overline{D}(\alpha,r)=\{z:|z-\alpha|\leq r\}$ is a subset of G and $f(z)-f(\alpha)\neq 0$ for every z on the circle $\Gamma=\{z:|z-\alpha|=r\}$. Let $m=\inf\{|f(z)-f(\alpha)|:z\in\Gamma\}$. Since Γ is a compact set and $|f(z)-f(\alpha)|$ is a continuous, real-valued function, thus, according to Bolzano-Weierstrass Theorem, there exists a $z_0\in\Gamma$ such that $m=|f(z_0)-f(\alpha)|$. Since $z_0\in\Gamma$, then $f(z_0)-f(\alpha)\neq 0$ and this implies that $m>0$. For every $\omega\in D(f(\alpha),m)$ and $z\in\Gamma$, we have

$$|f(z)-f(\alpha)|\geq m>|f(\alpha)-\omega|=|(f(\alpha)-f(z))+(f(z)-\omega)|.$$

This implies that

$$|(f(z)-f(\alpha))-(f(z)-\omega)|<|f(z)-f(\alpha)|.$$

By Rouche's Theorem, $f(z)-f(\alpha)$ and $f(z)-\omega$ has the same number of zeros, counted according to their multiplicities, inside Γ. Since $f(z)-f(\alpha)$ has a zero at α, then $f(z)-\omega$ has at least one zero inside Γ, that is, in $D(\alpha,r)$. If this zero is at $b\in D(\alpha,r)$, then we have $\omega=f(b)\in f(D(\alpha,r))$. It follows that $D(f(\alpha),m)\subseteq f(D(\alpha,r))$ since ω is an arbitrary element of $D(f(\alpha),m)$. Therefore, for every $\alpha\in G$,

$$D(f(\alpha),m)\subseteq f(D(\alpha,r))\subseteq f(G)$$

Therefore, $f(G)$ is open. **Q.E.D.**

It is noted that Open Mapping Theorem is unique for holomorphic function but it is not true for real differentiable function. Using this result, the maximum modulus principle follows easily.

Theorem 1.4.4 (Maximum Modulus Principle). *If f is non-constant, holomorphic function in a domain D, then $|f|$ can have no local maximum in D.*

Proof Suppose that there exists a $z_0 \in D$ which is a local maximum of $|f|$, that is, there exists a $r > 0$ such that $D(z_0, r) \subset D$ with $|f(z)| \leq |f(z_0)|$ for all $z \in D(z_0, r)$. By Open Mapping Theorem, $\omega_0 = f(z_0)$ is an inner point of $f(D(z_0, r))$. Thus, there exists $\rho > 0$ such that $D(\omega_0, \rho) \subset f(D(z_0, r))$. But, there are point $\omega \in D(\omega_0, \rho)$ such that $|\omega| > |\omega_0|$. Take $\omega = \omega_0 + \frac{\rho}{2} \exp^{i \text{ arg } \omega_0}$, then we have

$$|\omega| = |\omega_0 + \frac{\rho}{2} \exp^{i \text{ arg } \omega_0}| = |\omega_0| + \frac{\rho}{2} > |\omega_0|.$$

But this implies that $\omega \in D(\omega_0, \rho) \subset f(D(z_0, r))$ which means that there exists $z \in D(z_0, r)$ such that $f(z) = \omega$ and thus $|f(z)| > |f(z_0)|$, which is a contradiction. Therefore, $|f|$ can not have a local maximum in D. **Q.E.D.**

Theorem 1.4.5 (Maximum Modulus Theorem). *A non-constant function f is defined and continous on a bounded, close region K. If f is analytic in the interior of K, then the maximum value of $|f(z)|$ in K must occur on the boundary of K.*

Proof Since K is compact, then there exists $z_0 \in K$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in K$. Suppose that the maximum is attained in an interior point, that is $z_0 \in int(K)$. It would then be the global maximum of the restriction of f to $int(K)$, contradicting the maximum modulus principle. Therefore, we have $z_0 \in K \setminus int(K) = \delta K$, the boundary of K. **Q.E.D.**

A simple but important consequence of the Maximum Modulus Theorem is the Schwarz Lemma, which may be stated as follows.

Lemma 1.4.6 (Schwarz Lemma). *Let $f : D \to D$ be holomorphic with $f(0) = 0$. Then,*

1. *$|f(z)| \leq |z|$ for all $z \in D$.*

2. *If for some $z_0 \neq 0$, we have $|f(z_0)| = |z_0|$, then f is a rotation.*

3. $|f'(0)| \leq 1$ *and if equality holds, then* f *is a rotation.*

Proof We first expand f in a power series centered at the origin and convergent for all $z \in D$, that is

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots.$$

Since $f(0) = 0$, then $a_0 = 0$ and therefore the function $g(z)$ defined as

$$g(z) = \begin{cases} \frac{f(z)}{z} & , \text{ if } z \in E - \{0\} \\ f'(0) & , \text{ if } z = 0 \end{cases}$$

is holomorphic in D. If $|z| = r < 1$, then since $|f(z)| \leq 1$, we have

$$|g(z)| = \left|\frac{f(z)}{z}\right| \leq \frac{1}{r} \quad ,$$

and by Maximum Modulus Principle, we can conclude that this is true whenever $|z| \leq r$. Letting $r \to 1$ gives $|g(z)| \leq 1$ for all $z \in D$ and hence $|f(z)| \leq |z|$ for all $z \in D$. For (2), if $|f(z_0)| = |z_0|$, then $g(z)$ attains its maximum in the interior of D and again by maximum modulus principle, we have $g(z)$ is a constant function, that is $f(z) = cz$. Evaluating this expression at z_0 and taking absolute value, we find that $|c| = 1$, therefore there exists $\theta \in \mathbb{R}$ such that $c = e^{i\theta}$. This shows that $f(z) = e^{i\theta} z$, hence f is a rotation. Finally, observed that

$$g(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = f'(0),$$

hence $|f'(0)| \leq 1$. If $|f'(0)| = 1$, then $|g(0)| = 1$ and using similar technique in part (2), we obtained $f(z) = e^{i\alpha} z$, therefore f is a rotation. **Q.E.D.**

CHAPTER 2

UNIVALENT FUNCTIONS

In this chapter, the idea of univalent functions together with some of their properties will be discussed. We will begin by discussing about the Riemann Mapping Theorem. Next, some examples of univalent functions will be given. We will conclude this chapter with the introduction of Bieberbach's theorem and some applications of the Bieberbach's theroem.

2.1 Biholomorphic Mapping

A *biholomorphic mapping* is defined as a bijective holomorphic function. In other words, given two open sets E and E' in $\mathbb{C}$, biholomorphic mapping $f : E \to E'$ is the holomorphic, one-to-one and onto mapping between the two domains. We wish to determine the existence of a biholomorphic mapping between them. The existence of such function would allow us to shift our approaches to questions about holomorphic functions from one open set to another with possibly more useful properties. The unit disk $\mathbb{D} = \{z : |z| < 1\}$ would be a great candidate as it is easier to work with.

A biholomorphic mapping $f : E \to E'$ is called a *biholomorphism* and we say that E and E' are *biholomorphically equivalent* or *biholomorphic*. For biholomorphic function f, its derivative $f'(z) \neq 0$ for all z in E, and its inverse is also holomorphic.

Theorem 2.1.1. *If $f : E \to E'$ is holomorphic and injective, then $f'(z) \neq 0$ for all $z \in E$. Thus, the inverse of f defined on its range is holomorphic, and in particular the inverse of a biholomorphism is also holomorphic.*

Proof We prove this by contradiction. Suppose that $f'(z_0) = 0$ for some $z_0 \in E$. Then,

$$f(z) - f(z_0) = a(z - z_0)^k + G(z)$$

for all z near z_0 with $a \neq 0$, $k \geq 2$ and G vanishing to order $k+1$ at z_0. For sufficiently small w, we write,

$$f(z) - f(z_0) - w = F(z) + G(z)$$

where $F(z) = a(z-z_0)^k - w$. Since $|G(z)| < |F(z)|$ on a small circle centered at z_0, and F has at least two zeros inside that circle, Rouche's Theorem implies that $f(z) - f(z_0) - w$ has at least two zeros there. Since $f'(z) \neq 0$ for all $z \neq z_0$ but sufficiently close to z_0, it follows that the roots of $f(z) - f(z_0) - w$ are distinct. Hence, f is not injective, a contradiction.

Now, let $g = f^{-1}$ denote the inverse of f on its range, which will be denoted by V. Suppose that $w_0 \in V$ and w is close to w_0, and write $w = f(z)$ and $w_0 = f(z_0)$. If $w \neq w_0$, then we have

$$\frac{g(w) - g(w_0)}{w - w_0} = \frac{1}{\frac{w-w_0}{g(w)-g(w_0)}} = \frac{1}{\frac{f(z)-f(z_0)}{z-z_0}}.$$

Since $f'(z_0) \neq 0$, we may let $z \to z_0$ and conclude that g is holomorphic at w_0 with $g'(w_0) = 1/f'(g(w_0))$. **Q.E.D.**

There is a geometric consequence of holomorphic functions that satisfies this condition, known as conformality.

Definition 2.1.2. Let $\omega = f(z)$ be a complex mapping defined in a domain E and let z_0 be a point in E. Then we say that $\omega = f(z)$ is *conformal* at z_0 if for every pair of smooth oriented curves γ_1 and γ_2 in E intersecting at z_0, the angle between γ_1 and γ_2 at z_0 is equal to the angle between the image curves γ_1' and γ_2' at $f(z_0)$ in both magnitude and orientation.

Proposition 2.1.3. *If f is a holomorphic function in a domain E containing z_0, and if $f'(z_0) \neq 0$, then $\omega = f(z)$ is conformal at z_0.*

For the proof, please refer to (Matthews and Howell, 2012).

From Theorem 2.1.1 and Proposition 2.1.3, we know that biholomorphisms are *conformal mapping*. Due to this nature, if there exists a biholomorphism $f : E \to E'$, then E and E' are said to be *conformally equivalent*. It is necessary to point out that this terminology adopted here is not universal. Some authors defined conformal mapping as holomorphic mapping $f : E \to E'$ with $f'(z) \neq 0$ for all $z \in E$. This definition is less restrictive than ours. For example, consider the function $f(z) = z^2$ on the punctured plane $\mathbb{C} - \{0\}$. It is clear that $f'(z) \neq 0$, but $f(z)$ is not injective.

2.2 Riemann Mapping Theorem

The main problem is to determine conditions on an open set Ω that would ensure the existence of a biholomorphism $F : \Omega \to \mathbb{D}$.

First of all, a necessary condition is that Ω can not be the whole complex plane because Liouville's theorem states that every bounded entire function is constant. Since $\mathbb{D}$ is connected, then it is required that Ω be connected. Furthermore, since $\mathbb{D}$ is simply connected, then the same must be true for Ω. It is remarkable that these conditions on Ω are sufficient to guarantee the existence of a biholomorphism from Ω to $\mathbb{D}$.

We called a subset Ω of $\mathbb{C}$ a *proper subset* if it is non-empty and it is not the whole $\mathbb{C}$.

Theorem 2.2.1 (Riemann Mapping Theorem). *Suppose Ω is open, proper and simply connected. If $z_0 \in \mathbb{C}$, then there exists a unique biholomorphism $F : \Omega \to \mathbb{D}$ such that*

$$F(z_0) = 0 \text{ and } F'(z_0) > 0$$

For the proof, please refer to (Stein and Shakarchi, 2003).

We now give another term for injectivity.

Definition 2.2.2. A holomorphic function $f(z)$ is said to be *univalent* in a domain E if it is injective in the domain E.

By Riemann Mapping Theorem, any proper and simply connected domain, Ω can be mapped conformally onto the unit disk $\mathbb{D}$ by an univalent function $f : \Omega \to \mathbb{D}$. Therefore, any univalent function $g : \Omega \to G$ can be associated to an univalent function $h : \mathbb{D} \to G$ by the relation $f = g \circ h^{-1}$, and vice versa. For this reason, we can always direct our attention to univalent functions in $\mathbb{D}$ as it is much easier to work with. Furthermore, if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is univalent in $\mathbb{D}$, then $f(z) - a_0$ and $\frac{f(z)-a_0}{a_1}$ are univalent in $\mathbb{D}$ as well. Note that $a_1 = f'(0) \neq 0$.

Definition 2.2.3. *Class S* is the set of functions holomorphic and univalent in $\mathbb{D}$, and normalized by the condition $f(0) = 0$ and $f'(0) = 1$.

The conditions $f(0) = 0$ and $f'(0) = 1$ are known as normalized condition. Throughout this thesis, we will be studying the normalized functions in class S. For each $f \in S$, it can be represented by the Taylor series expansion of the the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n + \ldots = z + \sum_{n=2}^{\infty} a_n z^n$$

2.3 Examples of Univalent Functions in the Class S

In this section, several examples of univalent functions in the class S will be discussed.

Example 2.3.1. The Koebe function $k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + 4z^4 + \ldots$ for $z \in D$ is in S.

Solution. We begin by writing

$$k(z) = \frac{1}{4}\left(\frac{1+z}{1-z}\right)^2 - \frac{1}{4}.$$

Consider the function

$$u(z) = \frac{1+z}{1-z}$$

First, we prove the injectivity of the Koebe function. Suppose that $u(z_1) = u(z_2)$, then,

$$\frac{1+z_1}{1-z_1} = \frac{1+z_2}{1-z_2}$$

$$1+z_1-z_2-z_1z_2 = 1+z_2-z_1-z_1z_2$$

$$z_1 = z_2$$

Therefore, $u(z)$ is injective and holomorphic in $\mathbb{D}$. Next, we look at the geometry of the range of the Koebe function. Let $z = x+iy$, then we have

$$u(z) = \frac{1-x^2-y^2}{(1-x)^2+y^2} + i\frac{2y}{(1-x)^2+y^2}$$

Since we are only concerned with $z \in D$, so, we have $x^2+y^2 < 1$, then

$$\mathrm{Re}\{u(z)\} = \frac{1-(x^2+y^2)}{(1-x)^2+y^2} > 0$$

Thus, $u(z)$ maps the unit disk into the positive real plane. Then, $u^2(z)$ takes this half plane onto the entire complex plane, except for the negative real axis. Finally, $f(z) = \frac{1}{4}u^2(z) - \frac{1}{4}$ is just the normalization process. Hence, $k(z) \in S$, and its range is the whole complex plane except the negative real axis less than or equal to $-\frac{1}{4}$.

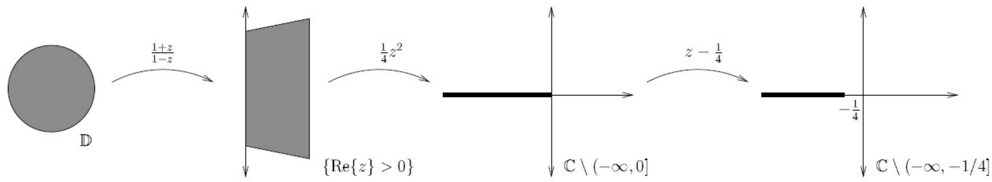

Figure 2.1 The Koebe function maps $\mathbb{D}$ conformally onto $\mathbb{C}$ $(-\inf,-1/4)$.

Remark 2.3.2. *Note that Koebe function is an important function in class S, since it provides solution to many extremal problems in class S. For example, the Bieberbach conjecture which will be discussed in the next section.*

Example 2.3.3. The function $f(z) = z(1-z)^{-1}$ for $z \in \mathbb{D}$ is in S.

Solution. First, to prove the injectivity, suppose that $f(z_1) = f(z_2)$, then,

$$\frac{z_1}{1-z_1} = \frac{z_2}{1-z_2}$$

$$z_1 - z_1z_2 = z_2 - z_1z_2$$

$$z_1 = z_2$$

Thus, f is injective. It is easy to show that $f(z)$ satisfy the normalization condition. Thus, $f(z) \in S$.

Other simple examples of functions in class S are :

1. $f(z) = z$, the identity mapping.
2. $f(z) = z(1-z^2)^{-1}$, which maps $\mathbb{D}$ conformally onto the entire plane minus the two half lines $\frac{1}{2} \le x < \infty$ and $-\infty < x \le -\frac{1}{2}$.
3. $f(z) = \frac{1}{2}\log\left[\frac{1+z}{1-z}\right]$, which maps $\mathbb{D}$ conformally onto the horizontal strip $-\frac{\pi}{4} < Im\{\omega\} < \frac{\pi}{4}$, where $\omega = f(z)$ for $z \in \mathbb{D}$.

Consider the functions $f(z) = z/(1-z)$ and $g(z) = z/(1+iz)$ which are univalent functions in S, we have

$$(f(z)+g(z))' = \frac{1}{(1-z)^2} + \frac{1}{(1+iz)^2}$$

which vanishes at $\frac{1}{2}(1+i)$. Therefore, $f+g$ is not univalent in $\mathbb{D}$ since every univalent functions satisfied $f'(z) \neq 0$. This shows that, in general, the sum of two univalent functions may not be univalent in $\mathbb{D}$.

However, functions in class S are preserved under some elementary transformation.

1. *Conjugation*. If $f \in S$ and $g(z) = \overline{f(\overline{z})} = z + \overline{a_2}z^2 + \overline{a_3}z^3 + \ldots + \overline{a_n}z^n + \ldots$, then $g(z) \in S$.
2. *Rotation*. If $f \in S$ and $g(z) = e^{-i\theta} f(e^{i\theta}z)$, then $g \in S$.
3. *Dilation*. If $f \in S$ and $g(z) = r^{-1} f(rz)$, where $0 < r < 1$, then $g \in S$.
4. *Square root transformation*. If $f \in S$ and $g = [f(z^2)]^{\frac{1}{2}}$, then $g \in S$.
5. *Range transformation* If $f \in S$ and Φ is a function holomorphic and univalent on the range of f, with $\Phi(0) = 0$ and $\Phi'(0) = 1$, then $g = \Phi \circ f \in S$.
6. *Omitted-value transformation*. If $f \in S$, and $\omega \neq f(z)$, then $g = \frac{\omega f}{\omega - f} \in S$.

Since we are using property 6 stated above quite often in the thesis, we shall prove it in the following example.

Example 2.3.4. If $f \in S$, and $\omega \neq f(z)$, then $g = \frac{\omega f}{\omega - f} \in S$.

Solution. Suppose that $f \in S$, then $\omega \neq 0$ since $f(0) = 0$. Suppose that $g(z_1) =$

$g(z_2)$, then,

$$\frac{\omega f(z_1)}{\omega - f(z_1)} = \frac{\omega f(z_2)}{\omega - f(z_2)}$$
$$\omega^2 f(z_1) - \omega f(z_1) f(z_2) = \omega^2 f(z_2) - \omega f(z_1) f(z_2)$$
$$f(z_1) = f(z_2)$$
$$z_1 = z_2$$

Therefore, $g(z)$ is injective. It is easy to show that $g(z)$ satisfied the normalization condition. Thus, $g(z) \in S$.

2.4 Bierberbach's Theorem

Perhaps one of the keystone to the study of univalent functions is the Bieberbach's Conjecture. In 1916, Ludwig Bieberbach proved that $|a_2| \leq 2$ for every function f in the class S, and equality happens only for Koebe function and its rotational transformation. In the footnote of his 1916 paper, Bieberbach stated that the condition $|a_n| \leq n$ for all f in class S is possibly true. Such footnote became the famous Bieberbach's Conjecture which remained unproven until 1985. First of all, we shall look at some preliminary result needed.

Let D^E denotes the domain $\{\zeta : 1 < |\zeta| < \infty\}$ (the exterior of $\mathbb{D}$) and Σ be the class of all functions in the form

$$\phi(\zeta) = \zeta + c_0 + \frac{c_1}{\zeta} + \frac{c_2}{\zeta} + \ldots = \zeta + \sum_{n=0}^{\infty} \frac{c_n}{\zeta^n}$$

that are holomorphic and univalent in D^E. The subclass of those functions such that $\phi(\zeta) \neq 0$ in D^E is denoted by Σ_0.

Proposition 2.4.1. *If $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ is in S, then*

$$\Phi(\zeta) = \frac{1}{f(\frac{1}{\zeta})} = \zeta - a_2 + \frac{a_2^2 - a_3}{\zeta} + \ldots$$

is in Σ_0.

Proof First, we prove that Φ is univalent in D^E. Suppose that $\Phi(\zeta_1) = \Phi(\zeta_2)$ where $\zeta_1, \zeta_2 \in D^E$. Then, we have $1/\zeta_1 = z_1$ and $1/\zeta_2 = z_2$ where $z_1, z_2 \in \mathbb{D}$. By the definition of $\Phi(\zeta)$, we have $f(z_1) = 1/\Phi(\zeta_1) = 1/\Phi(\zeta_2) = f(z_2)$. Since f is univalent, so $f(z_1) = f(z_2)$ implies that $z_1 = z_2$ and so $\zeta_1 = \zeta_2$. Therefore, Φ is univalent in D^E.

Next, we show that $\Phi(\zeta) \neq 0$ for all $\zeta \in D^E$. Suppose that there exists $\zeta_1 \in D^E$ such that $\Phi(\zeta_1) = 0$, then $\Phi(\zeta_1)f(1/\zeta_1) = 0$. But $\Phi(\zeta)f(1/\zeta) = 1$ for all $\zeta \in D^E$, thus, $\Phi(\zeta) \neq 0$ for all $\zeta \in D^E$. Therefore, $\Phi(\zeta)$ is in Σ_0. **Q.E.D.**

Theorem 2.4.2. *Let $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ be holomorphic and univalent on the circle $C_r = \{z : |z| = r\}$, and suppose that Ψ, the image of C_r under $f(z)$, is described in a positive direction as θ runs from 0 to 2π. Then the area of the domain enclosed by Ψ is given by*

$$A = \pi \sum_{n=-\infty}^{\infty} n|a_n|^2 r^{2n}.$$

Proof For $z \in C_r$, we let θ as the parameter and write $z = re^{i\theta}$ and $w = f(re^{i\theta}) = u(\theta) + iv(\theta)$, where

$$u(\theta) = \frac{1}{2} \sum_{n=-\infty}^{\infty} [a_n e^{in\theta} + \bar{a}_n e^{-in\theta}] r^n,$$
$$v(\theta) = \frac{1}{2i} \sum_{n=-\infty}^{\infty} [a_n e^{in\theta} - \bar{a}_n e^{-in\theta}] r^n.$$

By Green's theorem, the area of the domain enclosed by Ψ is given by

$$
\begin{aligned}
A &= \int_0^{2\pi} u\frac{dv}{d\theta}d\theta \\
&= \frac{1}{4}\int_0^{2\pi}[\sum_{m=-\infty}^{\infty}(a_m e^{in\theta}+\bar{a}_m e^{-in\theta})r^m] \\
&\quad \times[\sum_{n=-\infty}^{\infty}(a_n e^{in\theta}+\bar{a}_n e^{-in\theta})nr^n]d\theta \\
&= \frac{\pi}{2}\sum_{n=-\infty}^{\infty}[a_n(-na_{-n}+nr^{2n}\bar{a}_n)+\bar{a}_n(nr^{2n}a_n-n\bar{a}_{-n})] \\
&= \pi\sum_{n=-\infty}^{\infty}n|a_n|^2r^{2n},
\end{aligned}
$$

since $\sum_{n=-\infty}^{\infty}na_na_{-n}=\sum_{n=-\infty}^{\infty}n\bar{a}_n\bar{a}_{-n}=0$ and $\int_0^{2\pi}e^{ik\theta}d\theta=0$ for $k\neq 0$. **Q.E.D.**

Corollary 2.4.3. *Let*

$$f(z)=cz+c_0+\sum_{n=1}^{\infty}\frac{c_n}{z^n}\ ,c\neq 0$$

be holomorphic and univalent in $\overline{D_r^E}=\{z: r\leq|z|\leq\infty\}$. *Then the area of the complement of* $f(\overline{D_r^E})$ *is given by*

$$A_r=\pi\left(|c|^2r^2-\sum_{n=1}^{\infty}\frac{n|c_n|^2}{r^{2n}}\right).$$

In 1914, T.H. Gronwall discovered a theorem called Exterior Area Theorem which is the fundamental to the theory of univalent functions and it is important to Bieberbach's proof two years later.

Theorem 2.4.4 (Exterior Area Theorem). *If*

$$\Phi(z)=z+\sum_{n=0}^{\infty}\frac{c_n}{z^n}$$

is in Σ, i.e. Φ is holomorphic and univalent in D^E, then,

$$\sum_{n=1}^{\infty} n|c_n|^2 \leq 1.$$

Proof Note that the condition in Corollary 2.4.3 are satisfied for each $r > 1$, hence for $r > 1$, we have

$$A_r = \pi\left(r^2 - \sum_{n=1}^{\infty} \frac{n|c_n|^2}{r^{2n}}\right)$$

Since the area is always non-negative, then

$$r^2 - \sum_{n=1}^{\infty} \frac{n|c_n|^2}{r^{2n}} \geq 0$$
$$\sum_{n=1}^{\infty} \frac{n|c_n|^2}{r^{2n+2}} \leq 0$$

For a fixed but arbitrary positive integer N, we have

$$\sum_{n=1}^{N} \frac{n|c_n|^2}{r^{2n+2}} \leq 1$$

Observed that the sum of the left hand side of the inequality above increases monotonically when $r \to 1^+$ and it is bounded. Hence it has a limit as $r \to 1^+$, that is $\sum_{n=1}^{N} n|c_n|^2 \leq 1$. Since the partial sum $\sum_{n=1}^{N} n|c_n|^2$ increases monotonically and bounded above, then the series converges and we have

$$\sum_{n=1}^{\infty} n|c_n|^2 \leq 1$$

This proved the theorem. **Q.E.D.**

Lemma 2.4.5. *If*

$$f(z) = z + \sum_{n=0}^{\infty} a_n z^n$$

is in S, then

$$F(z) = [f(z^2)]^{\frac{1}{2}} = z + \frac{1}{2}a_2 z^3 + \left(\frac{1}{2}a_3 - \frac{1}{8}a_2^2\right)z^5 + \dots$$

is in S as well.

Proof This is stated in property 4 above, we shall prove it since we will use it later. Since $f(z) = 0$ only at the origin, a single-valued branch of the square root may be chosen by writing

$$\begin{aligned} F(z) &= [f(z^2)]^{\frac{1}{2}} \\ &= z(1 + a_2 z^2 + a_3 z^4 + \dots)^{\frac{1}{2}} \\ &= z + \frac{1}{2}a_2 z^3 + \left(\frac{1}{2}a_3 - \frac{1}{8}a_2^2\right)z^5 + \dots. \end{aligned}$$

Now, suppose that $F(z_1) = F(z_2)$, for $z_1, z_2 \in \mathbb{D}$, then $f(z_1^2) = f(z_2^2)$, and by the univalence of f, we have $z_1^2 = z_2^2$, and hence $z_1 = \pm z_2$. Since $F(z)$ is an odd function, so that $z_1 = -z_2$ gives $F(z_1) = -F(z_2)$, therefore we must have $z_1 = z_2$. This shows that F is univalent in $\mathbb{D}$ and since F is in normalized form, we conclude that $F \in S$. **Q.E.D.**

An inequality is said to be *sharp* if it is impossible to improve the inequality under given conditons. In other words, we can not increase its lower bound or decrease its upper bound as there exists a function such that the equality holds. A function for which equality occurs is called an *extremal function*. We now state the theorem introduced by Bieberbach in 1916 that leads to the famous Bieberbach's Conjecture in univalent function theory.

Theorem 2.4.6 (Bieberbach's Theorem for the second coefficient). *If*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

is in S, then $|a_2| \leq 2$. The inequality is sharp with equality occurs iff f is a *rotation of the Koebe function.*

Proof Suppose that $f \in S$, then by Lemma 2.4.5, $F(z) = [f(z^2)]^{\frac{1}{2}}$ is also in S. Then, we have

$$\begin{aligned} F(z) &= z + \frac{1}{2}a_2 z^3 + \big(\frac{1}{2}a_3 - \frac{1}{8}a_2^2\big)z^5 + \dots \\ &= z + 0z^2 + \frac{1}{2}a_2 z^3 + 0z^4 + \dots \end{aligned}$$

By Proposition 2.4.1, then we have

$$\Phi(\zeta) = \frac{1}{F(1/\zeta)} = \zeta - \frac{1}{2}a_2\frac{1}{\zeta} + c_3\frac{1}{\zeta^3} + c_5\frac{1}{\zeta^5} + \dots$$

is in Σ_0. By the Exterior Area Theorem, then we have

$$\sum_{n=1}^{\infty} n|c_n|^2 = |-\frac{a_2}{2}|^2 + 3|c_3|^2 + 5|c_5|^2 + \dots \leq 1.$$

Hence, $|-\frac{a_2}{2}|^2 \leq 1$, and this gives $|a_2| \leq 2$.

Next, we show that this inequality is sharp by showing that equality occurs if f is a rotational transformation of the Koebe function. If $a_2 = 2e^{i\theta}$, then we have $c_n = 0$ for all $n > 2$, by Exterior Area Theorem. Therefore, $\Phi(\zeta) = \zeta - e^{i\theta}/\zeta$. Thus,

$$F(z) = \frac{1}{\Phi(1/z)} = \frac{z}{1 - e^{i\theta}z^2}$$

Since $f(z^2) = [F(z)]^2$, then we have

$$f(z^2) = \frac{z^2}{(1 - e^{i\theta}z^2)^2}$$

and hence

$$f(z) = \frac{z}{(1 - e^{i\theta}z)^2} = e^{-i\theta}k(e^{i\theta}z) = k_\theta(z),$$

which is a rotational transformation of the Koebe function. Finally, observed that

$$\begin{aligned} k_\theta(z) &= e^{-i\theta}k(e^{i\theta}z) \\ &= \frac{z}{(1-e^{i\theta}z)^2} \\ &= z + 2e^{i\theta}z^2 + 3e^{2i\theta}z^3 + \dots \end{aligned}$$

Therefore, it is clear that $|a_2| = 2$. This completed the proof. **Q.E.D.**

The proof of the Bieberbach's conjecture is a difficult task. It was until 1923 when Lowner proved that $|a_3| \leq 3$ for every f in S, using the partial differential equation that bears his name today. Over the years between 1955 and 1972, special cases of the Bieberbach's conjecture were proved including when $n = 4$ (Garabedian and Schiffer, 1975), $n = 6$ (Pederson and Ozawa,1969) and $n = 5$ (Pederson and Schiffer, 1972). The first good estimate for all the coefficients came in 1925 by Littlewood who proved that $|a_n| \leq en$. The best result dated before 1985 is provided by FitzGerald and his student Horowitz in 1978 as they proved that $|a_n| < 1.0691n$. Finally, Bieberbach's conjecture was proved by Louis de Branges of Purdue University in 1986.

Clearly, Bieberbach's conjecture had been a vital part in the study of univalent function. We state this theorem to conclude this section.

Theorem 2.4.7 (**Bieberbach's Theorem**). *If*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

is in S, then $|a_n| \leq n$. The inequality is sharp with equality occurs iff f is a rotation of the Koebe function.

For the proof, please refer to (de Branges, 1985).

2.5 Some Applications of Bieberbach's Theorem

In this section, we will see some classical applications due to the Bieberbach's inequality $|a_2| \leq 2$. If f is holomorphic and non-constant in $\mathbb{D}$, then by open mapping theorem, we know that $f(\mathbb{D})$ is an open set. Since $f \in S$ are holomorphic and non-constant with the normalized condition $f(0) = 0$, so its range contains some open disk centered at origin. In 1907, Koebe discovered that the range of all functions in S contain an open disk $|\omega| < r$, where r is an absolute constant. This is known as the Koebe one-quarter theorem.

Theorem 2.5.1 (Koebe One-Quarter Theorem). *The range of every function of class S contains the disk* $\left\{\omega : |\omega| < \frac{1}{4}\right\}$.

Proof If a function f omits a value $\omega \in \mathbb{C}$, then by the omitted-value transformation, we have,

$$g(z) = \frac{f(z)}{\omega - f(z)} = z + \left(a_2 + \frac{1}{\omega}\right)z^2 + \ldots$$

Since $g(z) \in S$, then by Bieberbach's theorem, we have

$$\left|a_2 + \frac{1}{\omega}\right| \leq 2$$

Since $f \in S$, then $|a_2| \leq 2$, so

$$\left|\frac{1}{\omega}\right| \leq |a_2| + 2 \leq 4$$

Hence, $|\omega| \geq \frac{1}{4}$. This implies that every omitted value of $f \in S$ must lie outside the disk $\left\{\omega : |\omega| < \frac{1}{4}\right\}$. **Q.E.D.**

Another direct consequences of Theorem 2.4.6 are Growth Theorem and Distortion Theorem. The Koebe Distortion Theorem and Growth Theorem provides sharp lower and upper bound for $|f'(z)|$ and $|f(z)|$ respectively for every $f \in S$. First, we need the following lemmas.

Lemma 2.5.2. *For each $f \in S$, and $|z| = r < 1$ we have*

$$\left|\frac{zf''(z)}{f'(z)} - \frac{2r^2}{1-r^2}\right| \leq \frac{4r}{1-r^2}.$$

For the proof, please refer to (Duren, 1983).

Lemma 2.5.3. *If $f(z)$ is holomorphic in $\mathbb{D}$ and $f'(z) \neq 0$ for all $z \in \mathbb{D}$, then for $\zeta = \rho e^{i\theta} \in \mathbb{D}$, we have*

$$\rho \frac{\partial}{\partial \rho}\, ln|f'(\zeta)| = Re\left\{\frac{\zeta f''(\zeta)}{f'(\zeta)}\right\}.$$

Proof Taking the principal branch of the complex logarithmic function and differentiate with respect to ρ, we have

$$\rho \frac{\partial}{\partial \rho} \ln f'(\zeta) = \rho \frac{\partial}{\partial \rho}(\ln|f'(\zeta)| + i \arg f'(\zeta)) \tag{1}$$

By calculation, it can be show that

$$\rho \frac{\partial}{\partial \rho} \ln f'(\zeta) = \rho \frac{1}{f'(\zeta)} f''(\zeta) \frac{\partial \zeta}{\partial \rho} = \frac{f''(\zeta)}{f'(\zeta)} \rho e^{i\theta} = \zeta \frac{f''(\zeta)}{f'(\zeta)}$$

Taking the real part of (1), then we have

$$\text{Re}\left\{\frac{\zeta f''(\zeta)}{f'(\zeta)}\right\} = \rho \frac{\partial}{\partial \rho} \ln|f'(\zeta)|$$

This concluded the proof. **Q.E.D.**

Theorem 2.5.4 (Distortion Theorem). *For each $f(z) \in S$, then for $z = re^{i\theta} \in \mathbb{D}$,*

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3} \tag{2}$$

Equality occurs if and only if f is a suitable rotation of the Koebe function.

Proof [4] Since an inequality $|\alpha| \leq c$ implies $-c \leq Re\{\alpha\} \leq c$, it follows from Lemma 2.5.2 that

$$\frac{-4\rho}{1-\rho^2} \leq Re\left\{\frac{\zeta f''(\zeta)}{f'(\zeta)} - \frac{2\rho^2}{1-\rho^2}\right\} \leq \frac{4\rho}{1-\rho^2}$$

and therefore,

$$\frac{2\rho^2-4\rho}{1-\rho^2} \leq Re\left\{\frac{\zeta f''(\zeta)}{f'(\zeta)}\right\} \leq \frac{2\rho^2+4\rho}{1-\rho^2}$$

for $\zeta = \rho e^{i\theta}, |\zeta| < 1$. Because $f'(z) \neq 0$ for $z \in \mathbb{D}$ and $f'(0) = 1$, we can choose a single-valued branch of the complex logarithmic function $\ln f'(z)$ which vanishes at the origin, that is $\ln f'(z) = \ln|f'(z)| + i \operatorname{Arg} f'(z)$ with $\ln f'(0) = 0$. By Lemma 2.5.3, we have

$$\rho \frac{\partial}{\partial \rho} \ln|f'(\zeta)| = \operatorname{Re}\left\{\frac{\zeta f''(\zeta)}{f'(\zeta)}\right\}.$$

and hence,

$$\frac{2\rho-4}{1-\rho^2} \leq \frac{\partial}{\partial \rho} \ln|f'(\zeta)| \leq \frac{2\rho+4}{1-\rho^2}$$

Next, we hold θ fixed and intergrate the above equation with respect to ρ from 0 to r. It will then produces the inequality

$$\ln\left(\frac{1-r}{(1+r)^3}\right) \leq \ln|f'(re^{i\theta})| = \ln|f'(z)| \leq \ln\left(\frac{1+r}{(1-r)^3}\right)$$

for $z = re^{i\theta}$. The Distortion Theorem follows by exponentiation.

Observed that if $f(z) = k(z) = z(1-z)^{-2}$, then

$$f'(z) = k'(z) = \frac{1+z}{(1-z)^3}$$

If we set $z = r < 1$, then we obtained the equality on the right side of (1). On the other hand, setting $z = -r > -1$, we will obtained the equality on the left side of the distortion theorem. This shows that the inequalities are sharp.

Conversely, if $f \in S$ and satisfied the equality for the upper and lower estimate of inequality (1), we wish to show that f is a rotation of the Koebe function. If $f \in S$ such that $z = Re^{i\alpha}$ in $\mathbb{D}$, we have $|f'(z)| = \frac{1-R}{(1+R)^3}$, then,

$$\frac{\partial}{\partial r} \ln|f'(z)| = \frac{2r-4}{1-r^2} \tag{3}$$

holds for all $0 \leq r \leq R$. Note that

$$\frac{\partial}{\partial r} \ln|f'(re^{i\theta}| = \mathrm{Re}\left\{ e^{i\theta} \frac{f''(re^{i\theta})}{f'(re^{i\theta})} \right\}. \tag{4}$$

We can choose $\alpha \in \mathbb{R}$, such that $\mathrm{Re}\left\{e^{i\alpha} f''(0)\right\} = e^{i\alpha} f''(0) = e^{i\alpha}(2a_2)$, since $f''(0) = 2a_2$. Then,

$$\mathrm{Re}\left\{ e^{i\theta} \frac{f''(re^{i\theta})}{f'(re^{i\theta})} \right\} = \frac{2r-4}{1-r^2}$$

follows from equation (2) and (3). Taking $r = 0$, then it gives

$$\mathrm{Re}\left\{ e^{i\alpha} \frac{f''(0)}{f'(0)} \right\} = -4$$

and from our choice of $\alpha \in \mathbb{R}$, we have $2e^{i\alpha} a_2 = -4$ which implies that $|a_2| = 2$. From Bieberbach's theorem, we concluded that f is a rotational transformation of the Koebe function.

If equality holds for upper estimate of inequality (1), similar argument will gives the same conclusion, that is f is a rotational transformation of the Koebe function. This concluded the proof. **Q.E.D.**

Theorem 2.5.5 (Growth Theorem). *For each $f(z) \in S$, then for $z = re^{i\theta} \in D$ for $0 < r < 1$,*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2} \tag{5}$$

Equality occurs if and only if f is a suitable rotation of the Koebe function.

Proof [4] Let $f \in S$ and fix $z = re^{i\theta}$ with $0 < r < 1$. Observed that

$$f(z) = \int_0^r f'(\rho e^{i\theta}) e^{i\theta} d\rho.$$

Since $f(0) = 0$, by the distoriton theorem, we obtained

$$|f(z)| \leq \int_0^r |f'(\rho e^{i\theta})| d\rho \leq \int_0^r \frac{1+\rho}{(1-\rho)^3} d\rho = \frac{r}{(1-r)^2},$$

and this is the right-hand inequality. For the left-hand inequality, observed that

$$\frac{r}{(1+r)^2} < \frac{1}{4}$$

for $r > 0$. Hence, if $|f(z)| \geq \frac{1}{4}$, then $\frac{r}{(1+r)^2} < \frac{1}{4} \leq |f(z)|$.
For the case where $|f(z)| < \frac{1}{4}$, we let $f(re^{i\theta}) = Re^{i\alpha}$, where $R < \frac{1}{4}$. The Koebe one-quarter theorem implies that the straight line segment Γ from 0 to $Re^{i\alpha}$ lies entirely in the image of $\mathbb{D}$ by $f(z)$. Hence Γ corresponds to a path γ in $\mathbb{D}$, which joins $z = 0$ to $re^{i\theta}$. Thus if $t = |z|$, we deduced from Theorem 2.5.4 that

$$|f(z)| = \int_\Gamma |dw| = \int_\gamma |\frac{dw}{dz}||dz| \geq \int_\gamma \frac{1-t}{(1+t)^3} dt = \frac{r}{1+r^2}$$

which is the left hand inequality of the (4).
Equality in either part of inequality (4) implies equality in the corresponding parts of inequality (1), and hence distortion theorem implies that f is a rotational transformation of the Koebe function. **Q.E.D.**

One further inequality, which is a combined growth and distortion theorem is sometimes useful. We conclude this section by stating the theorem.

Theorem 2.5.6. *For each $f(z) \in S$, then for $z = re^{i\theta} \in D$, and $z \neq 0$,*

$$\frac{1-r}{1+r} \leq \left|\frac{zf'(z)}{f(z)}\right| \leq \frac{1+r}{1-r}$$

Equality occurs if and only if f is a suitable rotation of the Koebe function.

For the proof, please refer to (Duren, 1983).

CHAPTER 3

SPECIAL CLASSES OF UNIVALENT FUNCTIONS

In this chapter, we will first give the definition for linear fractional transformations in class S. Certain subclasses of S such as S^* and C are discussed and defined geometrically as well as analytically. Finally, convexity of linear fractional transformations in S together with its range will also be studied here.

3.1 Linear Fractional Transformation

Linear fractional transformation was first studied by Augustus Ferdinand Mobius (1790-1868). These mappings are conveniently expressed as the quotient of two linear expressions.

Definition 3.1.1. If a,b,c and d are complex constants with $ad-bc \neq 0$, then the complex function defined by

$$f(z)=\frac{az+b}{cz+d}$$

is called a linear fractional transformation.

Linear fractional transformation are also known as Mobius transformation or bilinear transformation. If $c=0$, then it is a linear transformation, which is a special case of linear fractional transformations. If $c \neq 0$, then we can write

$$f(z)=\frac{az+b}{cz+d}=\frac{bc-ad}{c}.\frac{1}{cz+d}+\frac{a}{c} \tag{1}$$

Setting $k(z)=cz+d$, $h(z)=1/z$ and $g(z)=(a/c)+[(bc-ad)/c]z$, then $f(z)=g\circ h\circ k(z)$. Thus, equation (1) is a composition of linear transformation, inversion and translation. The domain of a linear fractional transformation is the set of all complex value z such that $z \neq -d/c$. The condition $ad-bc \neq 0$ ensures that $f(z)$ would not be reduced to a constant. We now look at two examples of linear fractional transformations.

Example 3.1.2. Let $\mathbb{H}$ be the upper half plane, that is,

$$\mathbb{H} = \{\omega \in \mathbb{C} : Im(\omega) > 0\}$$

and $\mathbb{D}$ be the unit disk centered at origin, $\mathbb{D} = \{z : |z| < 1\}$. Consider

$$F(z) = \frac{i-z}{i+z} \quad \text{and} \quad G(\omega) = i\frac{1-\omega}{1+\omega}$$

then the mapping $F : \mathbb{H} \to \mathbb{D}$ is biholomorphism with inverse $G : \mathbb{D} \to \mathbb{H}$.

Solution. First, we observe that both mappings are holomorphic and injective in their respective domains. Then we note that any point in the upper half-plane is closer to i than to $-i$, so $|F(z)| < 1$ and that F maps $\mathbb{H}$ into $\mathbb{D}$. Next, we show that F maps $\mathbb{H}$ onto $\mathbb{D}$. Let $\omega \in \mathbb{D}$, then,

$$\begin{aligned} Im\left(i\frac{1-\omega}{1+\omega}\right) &= Re\left(\frac{1-u-iv}{1+u+iv}\right) \\ &= \frac{1-u^2-v^2}{(1+u)^2+v^2} > 0 \end{aligned}$$

since $|\omega| < 1$. Therefore, $i\frac{1-\omega}{1+\omega} \in \mathbb{H}$ and

$$\begin{aligned} F\left(i\frac{1-\omega}{1+\omega}\right) &= \frac{i-i\frac{1-\omega}{1+\omega}}{i+i\frac{1-\omega}{1+\omega}} \\ &= \frac{1+\omega-1+\omega}{1+\omega+1-\omega} = \omega. \end{aligned}$$

This shows that F maps $\mathbb{H}$ onto $\mathbb{D}$, and hence $F(z)$ is a biholomorphism with inverse $G(\omega)$.

Example 3.1.3. Let $\mathbb{P}$ be the positive real plane, that is,

$$\mathbb{P} = \{\omega = u + iv : u > 0\}$$

and $\mathbb{D}$ be the unit disk centered at origin, $\mathbb{D} = \{z : |z| < 1\}$. Consider

$$F(z) = \frac{1+z}{1-z} \quad \text{and} \quad G(\omega) = \frac{\omega - 1}{\omega + 1}$$

then $\mathbb{D}$ and $\mathbb{P}$ are conformally equivalent that is $F : \mathbb{D} \to \mathbb{P}$ conformally with inverse $G : \mathbb{P} \to \mathbb{D}$.

Solution. First, observe that f is injective and holomorphic in its domain. Let $z = x + iy$, then we have

$$F(z) = \frac{1-(x^2+y^2)}{(1-x)^2+y^2} + i\frac{2y}{(1-x)^2+y^2},$$

so F maps the unit disk $\mathbb{D}$ into $\mathbb{P}$ since $|z| < 1$. To show that F is onto $\mathbb{P}$, take note that for $\omega \in \mathbb{P}$, we have $|\omega - 1| < |\omega + 1|$, therefore $\frac{\omega-1}{\omega+1}$ is in $\mathbb{D}$ and $F\left(\frac{\omega-1}{\omega+1}\right) = \omega$. This shows that F is onto $\mathbb{P}$. Therefore f takes the unit disk $\mathbb{D}$ conformally to the positive real-plane $\mathbb{P}$, with inverse $G(\omega)$.

We conclude this section by stating an implicit formula to determine linear fractional transformation.

Theorem 3.1.4. *There exists a unique linear fractional transformation that maps three distinct points, z_1, z_2 and z_3, onto three distinct points, w_1, w_2 and w_3, respectively. An implicit formular for the mapping is given by*

$$\frac{z-z_1}{z-z_3}.\frac{z_2-z_3}{z_2-z_1} = \frac{w-w_1}{w-w_3}.\frac{w_2-w_3}{w_2-w_1}$$

For the proof, please refer to (Matthews and Howell, 2012).

3.2 Starlike and Convex Functions

In this section, we will discuss about starlike and convex functions which are two of the important subclasses of S.

Definition 3.2.1. A set $\Omega \in \mathbb{C}$ is said to be *starlike* (with respect to the origin) if the linear segment joining each point $\omega \in \Omega$ to origin lies entirely in Ω. Also, we say that Ω is *convex* if the linear segment joining any two points in Ω lies entirely in Ω .

The function $f : \mathbb{D} \to f(\mathbb{D})$ is called a *starlike function* if the image $F = f(\mathbb{D})$ is starlike, that is, if $\omega \in F$, then $t\omega \in F$, for $0 \le t \le 1$. Let S^* denote the class of all starlike functions in S, hence, we have $S^* \subset S$. The following theorem is a well known characterization of functions in S^*.

Theorem 3.2.2. *Let f be analytic in $\mathbb{D}$, with $f(0) = 0$ and $f'(0) = 1$. Then $f \in S^*$ if and only if*

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0.$$

Proof First, suppose that $f \in S^*$. We claim that f maps each subdisk $|z| < \rho < 1$ onto a starlike domain, that is $g(z) = f(\rho z)$ is starlike in $\mathbb{D}$. In other words, we must show that for each fixed t $(0 < t < 1)$ and for each $z \in \mathbb{D}$, then the point $tg(z)$ is in the range of g. Since $f \in S^*$, so $tf(z)$ is in the range of f, therefore $tf(z) = f(\omega(z))$ for some functions ω analytic in D and by Schwarz Lemma, satisfying $|\omega(z)| \le |z|$. Thus,

$$tg(z) = tf(\rho z) = f(\omega(\rho z)) = g(\omega_1(z))$$

where $\omega_1(z) = \omega(\rho z)/\rho$ and $|\omega_1(z)| \le |z|$.

This proves that f maps each circle $|z| = \rho < 1$ onto a curve C_ρ that bounds a starlike domain. It follows that arg $f(z)$ increases as z moves around the circle $|z| = \rho$ in the positive direction, that is

$$\frac{\delta}{\delta\theta}\left\{\arg f(\rho e^{i\theta})\right\} \geq 0.$$

Since $\log f(z) = \log|f(z)| + i \arg f(z)$, setting $z = \rho e^{i\theta}$, then we have $\log f(\rho e^{i\theta}) = \log|f(\rho e^{i\theta})| + i \arg f(\rho e^{i\theta})$. So,

$$\begin{aligned}
\frac{\delta}{\delta\theta}\left\{\arg f(\rho e^{i\theta})\right\} &= \mathrm{Im}\left\{\frac{\delta}{\delta\theta}\log f(\rho e^{i\theta})\right\} \\
&= \mathrm{Im}\left\{\frac{i\rho e^{i\theta} f'(\rho e^{i\theta})}{f(\rho e^{i\theta})}\right\} \\
&= \mathrm{Im}\left\{\frac{izf'(z)}{f(z)}\right\} \\
&= \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}
\end{aligned}$$

Thus, we have $Re\{zf'(z)/f(z)\} > 0$, by the maximum principle for harmonic functions.

Conversely, suppose that f is a normalized analytic function such that $Re\{zf'(z)/f(z)\} > 0$. Since $f(0) = 0$, then f has a simple zero at the origin and no zeros elsewhere in the disk. Retracing the calculation above, we see that for each $\rho < 1$,

$$\frac{\delta}{\delta\theta}\left\{\arg f(\rho e^{i\theta})\right\} \geq 0, \quad 0 \leq \theta \leq 2\pi$$

Thus as z runs around the circle $|z| = \rho$ in the counter-clockwise direction, the point $f(z)$ traverses a closed curve C_ρ with an increasing argument. Since f has exactly one zero inside the circle $|z| < \rho$, the argument principle states that C_ρ surrounds the origin exactly once. But if C_ρ winds about the origin only once with increasing argument, it can have no self-intersections. Thus, C_ρ is a simple closed curve which bounds a starlike domain D_ρ, and f assumes each value $w \in D_\rho$ exactly once in the disk $|z| < \rho$. Since this is true for every $\rho < 1$, it follows that f is univalent and starlike in D. This concludes the proof. **Q.E.D.**

The function $f : \mathbb{D} \to f(\mathbb{D})$ is called convex if the image $F = f(\mathbb{D})$ is convex, that is, if $f(z_1)$ and $f(z_2)$ are any two points in F, then $tf(z_2) + (1-t)f(z_1)$ is also in F for $0 \le t \le 1$. Let C denote the class of all convex functions in S, then we have $C \subset S$. Furthermore, from the geometrical point of view, it is easy to see that every convex function is starlike, and thus $C \subset S^*$. The following theorem is a well known characterization of functions in C.

Theorem 3.2.3. *Let f be analytic in $\mathbb{D}$, with $f(0) = 0$ and $f'(0) = 1$. Then $f \in C$ if and only if*

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0.$$

Proof Suppose that $f \in C$, we claim that f must map each subdisk $|z| < r$ onto a convex domain. First, choose points z_1 and z_2 with $|z_1| < |z_2| < r$. Let $w_1 = f(z_1)$ and $w_2 = f(z_2)$, and let

$$w_0 = tw_1 + (1-t)w_2 \quad , \quad 0 < t < 1$$

Since f is a convex mapping, then there is a unique point $z_0 \in \mathbb{D}$ such that $f(z_0) = w_0$. We have to show that $|z_0| < r$, but the function

$$g(z) = tf\left(\frac{z_1 z}{z_2}\right) + (1-t)f(z)$$

is analytic in $\mathbb{D}$, with $g(0) = 0$ and $g(z_2) = w_0$. Since $f \in C$, then the function $h(z) = f^{-1}(g(z))$ is well-defined. Since $h(0) = 0$ and $|h(z)| \le 1$, then by Schwarz lemma, we have $|h(z)| \le |z|$. Thus,

$$|z_0| = |f^{-1}(\omega_0)| = |h(z_2)| \le |z_2| < r.$$

Hence, f maps each circle $|z| = r < 1$ onto a curve C_r which bounds a convex domain. The convexity implies that the slope of the tangent to C_r is non-decreasing as the curve is traversed in the positive direction.

Analytically, this condition is

$$\frac{\delta}{\delta\theta}\left(\arg\left\{\frac{\delta}{\delta\theta}f(re^{i\theta})\right\}\right) \geq 0$$

Since $\log[\frac{\delta}{\delta\theta}f(re^{i\theta})] = \log|\frac{\delta}{\delta\theta}f(re^{i\theta})| + i\arg\left\{\frac{\delta}{\delta\theta}f(re^{i\theta})\right\}$, then we have

$$\begin{aligned}
&\frac{\delta}{\delta\theta}\,\mathrm{Im}\left\{\log[\frac{\delta}{\delta\theta}f(re^{i\theta})]\right\} \geq 0,\\
&\mathrm{Im}\left\{\frac{\delta}{\delta\theta}\log\left[ire^{i\theta}f'(re^{i\theta})\right]\right\} \geq 0,\\
&\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geq 0 \quad , \quad |z| = r.
\end{aligned}$$

Thus, we have $\mathrm{Re}\{1+(zf''(z))/f(z)\} > 0$, by the maximum principle for harmonic function.

Conversely, suppose f is a normalized analytic function such that

$$\mathrm{Re}\left\{1+(zf''(z))/f(z)\right\} > 0.$$

The calculation above shows that the slope of the tangent to the curve C_r increases monotonically. As a point makes a complete circuit of C_r, the argument of the tangent vector has a net change

$$\begin{aligned}
\int_0^{2\pi}\frac{\delta}{\delta\theta}\left(\arg\left\{\frac{\delta}{\delta\theta}f(re^{i\theta})\right\}\right)d\theta &= \int_0^{2\pi}\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}d\theta\\
&= \mathrm{Re}\left\{\int_{|z|=r}\left[1+\frac{zf''(z)}{f'(z)}\right]\frac{dz}{iz}\right\}\\
&= 2\pi \quad , \quad z = re^{i\theta}
\end{aligned}$$

This shows that C_r is a simple closed curve bounding a convex domain. This for arbitrary $r < 1$ implies that f is a univalent function with convex range. **Q.E.D.**

The following theorem, which is known as Alexander Theorem established a relationship between the class S^* and C. It was first observed by J.W. Alexander in 1915.

Theorem 3.2.4 (Alexander's Theorem). *Let f be analytic in D, with $f(0) = 0$ and $f'(0) = 1$. Then $f \in C$ if and only if $zf'(z) \in S^*$.*

Proof Let $g(z) = zf'(z)$ then

$$\frac{zg'(z)}{g(z)} = 1 + \frac{zf''(z)}{f'(z)}$$

Then,

$$\mathrm{Re}\left\{\frac{zg'}{g(z)}\right\} = \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\}.$$

Therefore, by Theorem 3.2.2 and 3.2.3, we have $f \in C$ if and only if $zf'(c) \in S^*$.

Q.E.D.

Near the origin, each function $f \in S$ is close to the identity mapping. It is expected that each $f \in S$ will maps small open disks $|z| = \rho$ onto curves that bound a convex domain. The following theorem gives a quantitative description towards this statement.

Theorem 3.2.5. *For every positive number $\rho \leq 2 - \sqrt{3}$, each function $f \in S$ maps the disk $|z| < \rho$ onto a convex domain. This is false for $\rho > 2 - \sqrt{3}$.*

Proof From Theorem 2.5.2, we have

$$\left|\frac{zf''}{f'} - \frac{2r^2}{1-r^2}\right| \leq \frac{4r}{1-r^2}$$

for each $f \in S$ and $|z| = r < 1$. Therefore,

$$\mathrm{Re}\left\{1 + \frac{zf''}{f'}\right\} \geq \frac{1-4r+r^2}{1-r^2}.$$

Since $1-4r+r^2>0$ for $r<2-\sqrt{3}$, thus by Theorem 3.2.3, f must maps a disk $|z|<r$ onto a convex domain. **Q.E.D.**

The number $2-\sqrt{3}=0.267...$ is known as the radius of convexity for the class S. The radius of starlikeness is known to be $\tanh\frac{\pi}{4}=0.655...$, but this result lies deeper and will not be covered in this thesis. The following theorem provides a slight improvement upon the Koebe one-quarter theorem, though it is restricted to convex functions.

Theorem 3.2.6. *The range of every function $f\in C$ contains the disk $|\omega|<1/2$.*

Proof If $f\in C$ and $f(z)\neq\omega$, then $g(z)=[f(z)-\omega]^2$ is univalent. If $g(z_1)=g(z_2)$, then either $f(z_1)=f(z_2)$ or $\frac{1}{2}[f(z_1)+f(z_2)]=\omega$. The latter is impossible for a convex function f which omits the value ω. Thus,

$$h(z)=\frac{\omega^2-g(z)}{2\omega}\in S$$

But $h(z)\neq\omega/2$ because $g(z)\neq 0$, so it follows from the Koebe one quarter theorem that $|\omega/2|\geq 1/4$, or $|\omega|\geq 1/2$. This proves the theorem. **Q.E.D.**

We conclude this section by giving some examples of starlike and convex functions.

Example 3.2.7. The Koebe function

$$k(z)=\frac{z}{(1-z)^2}\quad,\quad z\in\mathbb{D}$$

is in S^* but not in C.

Solution. From Example 2.3.1, it follows that the Koebe function is starlike but not convex from the geometry of the range of the Koebe function. However, we will give an analytic proof here. A simple calculation shows that

$$\operatorname{Re}\left\{\frac{zk'(z)}{k(z)}\right\} = \operatorname{Re}\left\{\frac{1+z}{1-z}\right\}$$

Since $\operatorname{Re}\{\frac{1+z}{1-z}\} > 0$, then, by Theorem 3.2.2, $k(z)$ is starlike.Since $\operatorname{Im}\{k(\mathbb{D})\} = \mathbb{C} \setminus (-\infty, -1/4)$, thus the closed line segment joining $-1+i$ and $-1-i$ is not in the range of $k(\mathbb{D})$. Therefore, $k(z)$ is not convex.

Example 3.2.8. Let $f(z) = z + a_2z^2 + \ldots, z \in \mathbb{D}$ satisfy

$$\sum_{n=2}^{\infty} n|a_n| \leq 1$$

then $f \in S^*$.

Solution. For $|z| < 1$, it follows that

$$\begin{aligned}
|zf'(z) - f(z)| &= \left| z + \sum_{n=2}^{\infty} na_nz^n - z - \sum_{n=2}^{\infty} (n-1)a_nz^n \right| \\
&\leq \sum_{n=2}^{\infty} (n-1)|a_n||z|^n \\
&\leq |z| - \sum_{n=2}^{\infty} |a_n||z|^n \\
&\leq |f(z)|
\end{aligned}$$

Hence, we have $|zf'(z)/f(z) - 1| \leq 1$ and this gives $\operatorname{Re}\{zf'(z)/f(z)\} > 0$. Therefore, by Theorem 3.2.2, $f(z)$ is starlike.

Example 3.2.9. Consider the function

$$f(z) = z + \frac{1}{4}z^2 \quad , \quad z \in \mathbb{D}$$

then $f \in C$.

Solution. We will prove this analytically. Let $z = x + iy$, then

$$\text{Re}\left\{\frac{zf''}{f'}+1\right\} = \text{Re}\left\{\frac{2z+2}{z+2}\right\} = \frac{2x^2+2y^2+6x+4}{(x+2)^2+y^2} > \frac{6+6x}{(x+2)^2+y^2} > 0$$

Hence, by Theorem 3.2.3, $f(z)$ is convex.

3.3 Close-to-Convex functions

Another interesting subclass of S is the class of close-to-convex function. It was introduced by Kaplan in 1952.

Definition 3.3.1. A function f analytic in the unit disk $\mathbb{D}$ is said to be close-to-convex if there exists a convex function g such that

$$\text{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > 0$$

for all $z \in \mathbb{D}$.

We shall denote by K the class of close-to-convex functions with normalized condition $f(0) = 0$ and $f'(0) = 1$. It is noted that f is not required to be univalent and the associated function g need not to be normalized.

Every close-to-convex function is univalent. We will first need the following simple but important criterion for univalence. This criterion is due to Noshiro and Warchawski.

Theorem 3.3.2 (Noshiro-Warchawski Theorem). *Suppose that f is analytic in a convex domain E and Re $\{f'(z)\} > 0$ there, then f is univalent in E.*

Proof Let z_1 and z_2 be two distinct points in E. Since E is convex, then the linear segment, L joining z_1 and z_2 lies inside of E. The line segment is given by

$\gamma = z_1 + t(z_2 - z_1)$ where $t \in [0,1]$.

$$\begin{aligned} f(z_2) - f(z_1) &= \int_\gamma f'(z)\ dz \\ &= (z_2 - z_1) \int_0^1 f'(z_1 + t(z_2 - z_1))\ dt \end{aligned}$$

Therefore,

$$\begin{aligned} |f(z_2) - f(z_1)| &= |z_2 - z_1| \left| \int_0^1 f'(z_1 + t(z_2 - z_1))\, dt \right| \\ &\geq |z_2 - z_1| \left| Re \left\{ \int_0^1 f'(z_1 + t(z_2 - z_1)) \right\} dt \right| \\ &> 0 \end{aligned}$$

since Re $\{f'(z)\} > 0$. Hence, for $z_1 \neq z_2$, then $f(z_1) \neq f(z_2)$ and we conclude that f is univalent in E. **Q.E.D.**

We are now ready to prove that close-to-convex function is univalent.

Theorem 3.3.3. *Every close-to-convex function is univalent.*

Proof Suppose that f is close-to-convex, then Re $\{f'(z)/g'(z)\} > 0$ for some convex function g. Let E be the range of g and consider the function

$$h(\omega) = f(g^{-1}(\omega)) \quad , \quad \text{for } \omega \in E.$$

Note that the inverse of g exists because $g \in C \subset K$. Since every function in C is univalent, thus g is univalent and so the inverse exists. Then,

$$h'(\omega) = \frac{f'(g^{-1}(\omega))}{g'(g^{-1}(\omega))} = \frac{f'(z)}{g'(z)}$$

for $z = g^{-1}(\omega)$ in $\mathbb{D}$. Thus, Re$\{h'(\omega)\} > 0$ in E. Thus, $h(\omega)$ is univalent, by

Noshiro-Warchawski Theorem. Let z_1 and z_2 in $\mathbb{D}$, then there exist ω_1, ω_2 in E such that $g(z_1) = \omega_1$ and $g(z_2) = \omega_2$. If $f(z_1) = f(z_2)$, then,

$$h(\omega_1) = f(z_1) = f(z_2) = h(\omega_2).$$

By the univalence of h, then $\omega_1 = \omega_2$, that is $g(z_1) = g(z_2)$. Since $g \in C$, therefore we have $z_1 = z_2$ since g is univalent. Thus, f is also univalent in $\mathbb{D}$. **Q.E.D.**

From Definition 3.3.1, it is obvious that every convex function is close-to-convex. More generally, every starlike function is close-to-convex. Let $f \in S^*$, and write

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots = z[1 + a_2 z + a_3 z^2 + \ldots] = zg'(z)$$

From the Alexander Theorem, we know that if $zg'(z) \in S^*$, then $g(z) \in C$. Consider the analytical condition for close-to-convex function, then we have

$$\mathrm{Re}\left\{\frac{f'(z)}{g'(z)}\right\} = \mathrm{Re}\left\{\frac{f'(z)}{\frac{f(z)}{z}}\right\} = \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$$

from Theorem 3.2.2 since $f \in S^*$.

The following chain of proper inclusions summarized the relationship between the special subclasses of S.

$$C \subset S^* \subset K \subset S.$$

3.4 Linear fractional transformation in Class S

From definition, class S is the set of functions holomorphic and univalent in $\mathbb{D}$, normalized by the condition $f(0) = 0$ and $f'(0) = 1$.

Theorem 3.4.1. *If f is a linear fractional transformation and $f \in S$, then*

$$f(z) = \frac{z}{\alpha z + 1}$$

where $\alpha = c/a$ and $|\alpha| \leq 1$.

Proof A linear fractional transformation is clearly 1-1, and thus, for a linear fractional transformation $f(z) = (az+b)/(cz+d)$ to be in class S, it must satisfy both the normalized condition. If $f(0) = 0$, then it follows that $b = 0$. If $f'(0) = 1$, then $ad/d^2 = 1$ and this gives $a = d$. Thus,

$$f(z) = \frac{z}{\alpha z + 1} \quad , \quad \alpha = \frac{c}{a}$$

Since $f \in S$, then f is analytic in $\mathbb{D}$. For $\alpha \neq 0$, f is not analytic at $z = -1/\alpha$. Therefore, $-1/\alpha$ does not belongs to $\mathbb{D}$. Thus, $|\alpha| \leq 1$. **Q.E.D.**

Theorem 3.4.2. *Every linear fractional transformation in class S is convex.*

Proof Consider the function

$$g(z) = \frac{z}{z+1} \quad , \quad z \in D$$

Then,

$$\operatorname{Re}\left\{1 + \frac{zg''(z)}{g'(z)}\right\} = \operatorname{Re}\left\{\frac{1-z}{1+z}\right\} > 0.$$

Thus, by Theorem 3.2.3, $g(z)$ is a convex function. From Theorem 3.4.1, we shall show that $f(z) = \frac{z}{\alpha z+1}$ is convex for all α such that $|\alpha| \leq 1$.
First, consider $|\alpha| = 1$. Let g_1 be the rotation of g, then

$$g_1(z) = e^{-i\theta} g(e^{i\theta} z) = \frac{z}{1 + e^{i\theta} z}$$

Since $g_1(z)$ is the rotational transformation of the convex function $g(z)$, thus,

$g_1(z)$ is convex.

Next, consider $|\alpha| < 1$. Let g_1 be the rotation of g and g_2 be the dilation of g_1, for $0 < r < 1$, then

$$g_2(z) = r^{-1} g_1(rz) = \frac{z}{1 + re^{i\theta} z} = \frac{z}{1 + \beta z}$$

where $\beta = re^{i\theta}$ with $|\beta| < 1$. We can see that $g_2(z)$ is a transformation of the convex function $g(z)$, and so $g_2(z)$ is convex. Therefore, every linear fractional transformation in S is convex. **Q.E.D.**

In the following section, we will provide a detail explanation on the image of the function

$$f(z) = \frac{z}{\alpha z + 1} \quad , \quad |\alpha| \leq 1.$$

Let $g(z)$ be the rotation of $f(z)$, then

$$g(z) = e^{-i\theta} f(e^{i\theta} z) = \frac{z}{\alpha e^{i\theta} z + 1} = f_\theta(z).$$

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{e^{i\theta}} & \mathbb{D} \\ g \downarrow & & \downarrow f \\ \mathbb{C} & \xrightarrow{e^{i\theta}} & \mathbb{C} \end{array}$$

The diagram above illustrate the relationship between f and g. It shows that the image of the rotation of $f(z)$ (the image of $g(z)$) is the rotation of the image of $f(z)$. If we let $\alpha = re^{i\theta}$, then it is sufficient to consider the image of $f(z)$ when $\theta = 0$.

First, consider the function $f(z)$ with $|\alpha| = 1$. Define

$$f_0(z) = \frac{z}{z+1} \quad , \quad z \in \mathbb{D}$$

Let $z = x + iy$, where $x, y \in \mathbb{R}$, then,

$$\operatorname{Re}(f_0(z)) = \frac{x^2+y^2+x}{(1+x)^2+y^2} < \frac{x^2+y^2+x}{2(x^2+y^2)+2x} = \frac{1}{2}$$

Thus, the image of f_0 is the complex half plane satisfying $\operatorname{Re}\{f_0\} < 1/2$. Let f_θ be the rotations of f_0, then the images of f_θ are rotations of the image of f_0. Thus, the boundary of f_θ is a straight line rotating clockwise along the boundary of a circle with radius $1/2$ as θ changes from 0 to 2π.

The following diagram shows the rotation of the boundary of f_θ as θ changes from 0 to 2π.

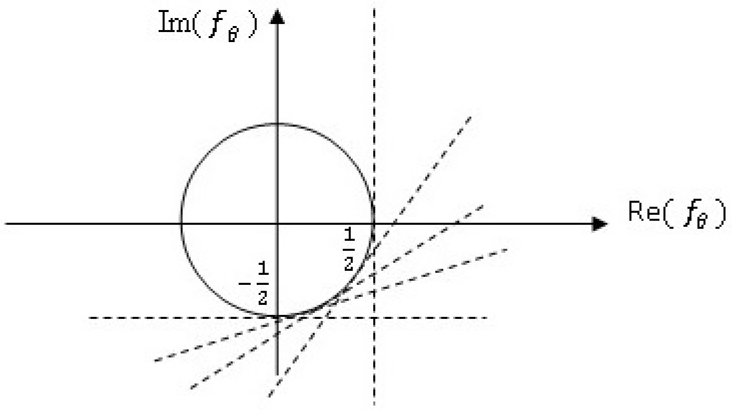

Figure 3.1 The boundary of f_θ as θ variates from 0 to 2π.

Next, consider the function $f(z)$ with $|\alpha| < 1$. Let $\alpha = re^{i\theta}$ where $|r| < 1$, then,

$$f(z) = \frac{z}{re^{i\theta}z+1} \quad , \quad z \in \mathbb{D}.$$

Define

$$f_{r,0}(z) = \frac{z}{rz+1} \quad , \quad r \in \mathbb{R} \text{ and } |r| < 1$$

Since $f_{r,0}$ is a linear fractional transformation, then $f_{r,0}$ maps circle to either straight line or circle, and it maps boundary points of $\mathbb{D}$ to boundary points of

$f_{r,0}$ as well. Consider the boundary points of $f_{r,0}$,

$$f_{r,0}(1)=\frac{1}{r+1} \quad , \quad f_{r,0}(-1)=\frac{1}{r-1} \quad , \quad f_{r,0}(i)=\frac{r}{r^2+1}+\frac{i}{r^2+1}$$

The coordinates of $f_{r,0}$ corresponding to these boundary points are

$$\left(\frac{1}{r+1},0\right) \quad , \quad \left(\frac{1}{r-1},0\right) \quad , \quad \left(\frac{r}{r^2+1},\frac{1}{r^2+1}\right)$$

respectively.

For $r \neq 0$, then $f_{r,0}$ maps $\mathbb{D}$ onto an open disk with centre (x,y) and radius k, then,

$$\left(\frac{1}{r+1}-x\right)^2+y^2=k^2$$

$$\left(\frac{1}{r-1}-x\right)^2+y^2=k^2$$

$$\left(\frac{r}{r^2+1}-x\right)^2+\left(\frac{1}{r^2+1}-y\right)^2=k^2$$

Solving these equations give $x=r/(r^2-1)$, $y=0$ and $k=1/(1-r^2)$. Therefore, the image of $f_{r,0}$ is an open disk centered at $(r/(r^2-1),0)$ with radius $k=1/(1-r^2)$.

Let $f_{r,\theta}$ be the rotations of $f_{r,0}$, then the images of $f_{r,\theta}$ are rotations of $f_{r,0}$. Thus, the images of $f_{r,\theta}$ are open disks with radius $1/(1-r^2)$ with the centre of these open disks rotating clockwise along the boundary of a circle of radius $r/(r^2-1)$ as θ changes from 0 to 2π.

The following diagram shows part of the image of $f_{r,\theta}$.

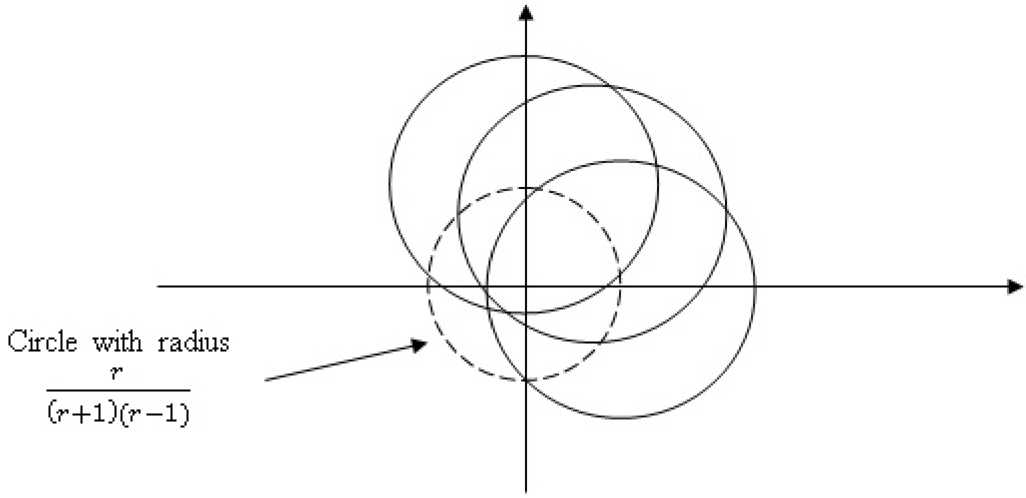

Figure 3.2 The boundary of $f_{r,\theta}$ for $|r| < 1$ as θ variates from 0 to 2π.

Finally, consider

$$f(z) = \frac{z}{rz+1} \quad , \quad |r| \leq 1 \text{ and } z \in \mathbb{D}$$

When $r = 0$, then $f(z) = z$, which is the identity mapping that maps $\mathbb{D}$ onto $\mathbb{D}$. As the value of r gradually increases from 0 to 1, the image of $f(z)$ which is an open disk will have its centre shifted from the origin to $(r/(r^2-1), 0)$ while the radius increases from 1 to $1/(1-r^2)$. This gives an explanation as r gradually increases from 0 to 1, the image of $f(z)$ will slowly "open up". As $r \to 1$, then we have

$$\lim_{r\to 1} f(z) = \frac{z}{z+1}$$

Furthermore, concerning the centre and radius of the image of $f(z)$, we have

$$\lim_{r\to 1} \frac{r}{r^2-1} = -\infty \quad \text{and} \quad \lim_{r\to 1} \frac{1}{1-r^2} = \infty$$

This shows that the boundary of the image of $f(z)$ is a complex half plane. For example, when $r = 1$, such straight line is $\mathrm{Re}(f(z)) = 1/2$.

Example 3.4.3. Consider the function

$$f(z)=\frac{z}{(z/5)+1}$$

for $z\in\mathbb{D}$. From $f(z)$, we have $r=1/5<1$. Thus, the image of $f(z)$ is an open disk. The radius and the centre of such open disk is given by

$$\text{Radius }=\frac{1}{1-(1/5)^2}=\frac{25}{24}$$

$$\text{Centre }=\left(\frac{(1/5)}{(1/5)^2-1},0\right)=\left(\frac{-5}{24},0\right)$$

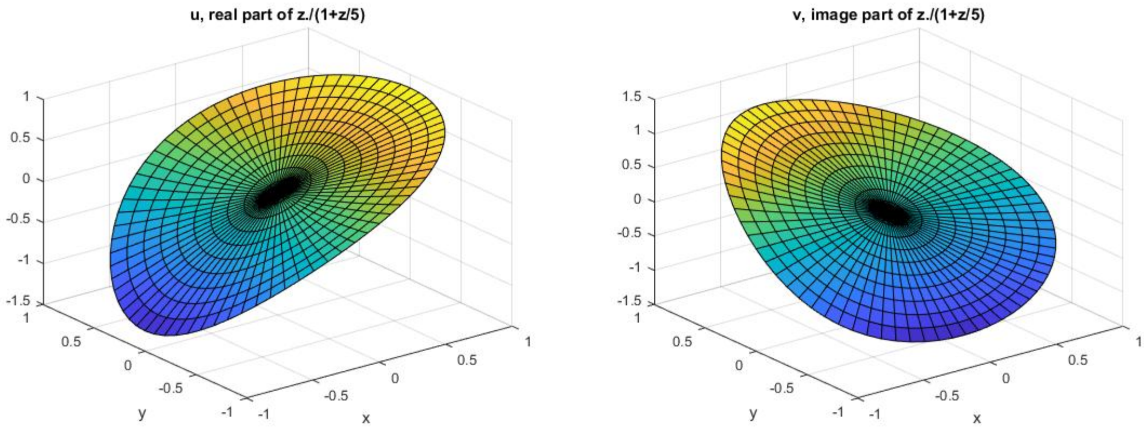

Figure 3.3 The real part, u and imaginary part, v for $f(z)=\frac{z}{(z/5)+1}$, for $z\in\mathbb{D}$.

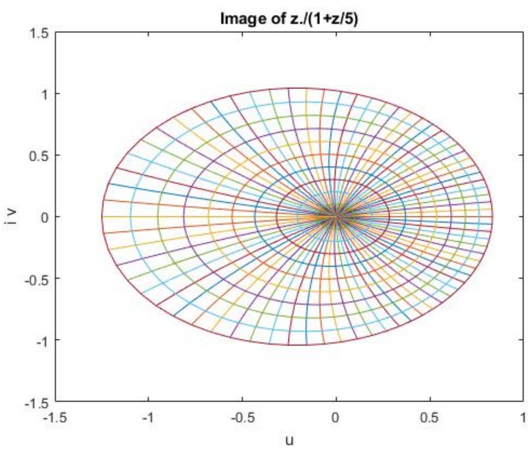

Figure 3.4 The image of $f(z)=\frac{z}{(z/5)+1}$, for $z\in\mathbb{D}$.

CHAPTER 4

SOLUTIONS FOR A PARTICULAR FIRST ORDER DIFFERENTIAL EQUATIONS IN UNIVALENT FUNCTIONS

In this chapter, we first introduce an equivalence relation and show that such equivalence class is the complete solution for a particular first order non-linear differential equation. Furthermore, the Schwarzian derivative is invariant with respect to the equivalence relation introduced. Furthermore, we give examples to show that the properties such as starlikeness and convexity of the equivalence class are not preserved.

4.1 Starlikeness and convexity of equivalence class

Definition 4.1.1. Given a set A, an equivalence relation for the set A is a binary relation $\sim$ satisfying three properties.

1. For every element $a \in A$, then $a \sim a$. (Reflexivity)

2. For every element $a, b \in A$, if $a \sim b$, then $b \sim a$. (Symmetry)

3. For every element $a, b, c \in A$, if $a \sim b$ and $b \sim c$, then $a \sim c$. (Transitivity)

Definition 4.1.2. For $f \in S$, g is related to f, or $g \sim f$ if

$$g = f_\omega = \frac{\omega f}{(\omega - f)} \quad \text{for some} \quad \omega \notin f(U)$$

where U is an open set.

The next theorem show that the definition stated above is an equivalence relation.

Theorem 4.1.3. *The relation "$\sim$" is an equivalence relation in S.*

Proof We need to prove that $\sim$ is reflexive, symmetric and transitive. First, for the reflexivity, consider

$$f = \frac{\omega f}{\omega - f} = \frac{f}{1 - \frac{1}{\omega} f} \quad , \quad \omega \notin f(\mathbb{D})$$

Since $f(\mathbb{D}) \neq \mathbb{C}$. therefore $f \sim f$ when ω tends to ∞. Thus, it is reflexive.

For the symmetry, if $f \sim g$, then

$$f = \frac{\omega g}{\omega - g} \quad , \quad \omega \notin g(\mathbb{D})$$
$$g = \frac{-\omega f}{-\omega - f}$$

Since g is well-defined, then $-\omega \notin f(\mathbb{D})$. Thus, it is symmetric.

For the transitivity, if $f \sim g$ and $g \sim h$, then

$$f = \frac{\omega_1 g}{\omega_1 - g} \quad \text{and} \quad g = \frac{\omega_2 h}{\omega_2 - h} \quad , \quad \omega_1 \notin g(D), \omega_2 \notin h(\mathbb{D}).$$

Then,

$$f = \frac{\omega_1 \left(\frac{\omega_2 h}{\omega_2 - h}\right)}{\omega_1 - \left(\frac{\omega_2 h}{\omega_2 - h}\right)}$$
$$= \frac{\gamma h}{\gamma - h} \quad , \quad \text{where } \gamma = \frac{\omega_1 \omega_2}{\omega_1 + \omega_2}$$

Since h is well defined, thus, $\gamma \notin h(\mathbb{D})$. Thus, it is transitive.

Therefore "$\sim$" is an equivalence relation in S. **Q.E.D.**

Let $\overline{f} = \{g \in S : g \sim f\}$ be the set of equivalence class containing f in S. We give a simple example on the equivalence relation.

Example 4.1.4. The Koebe function $k(z) = z(1-z)^{-2}$ is not equivalent to the identity mapping $f(z) = z$ on the unit disk $\mathbb{D}$.

Solution. Suppose that $k(z) \sim f(z)$, then,

$$\frac{z}{(1-z)^2} = \frac{\omega z}{\omega - z}$$

for some ω such that $|\omega| \geq 1$. Then,

$$\begin{aligned} \omega z - z^2 &= \omega z - 2\omega z^2 + \omega z^3 \\ z^2[\omega z - (2\omega - 1)] &= 0 \end{aligned}$$

For $z \neq 0$, then we have

$$\begin{aligned} \omega z - (2\omega - 1) &= 0 \\ z &= \frac{2\omega - 1}{\omega} \end{aligned}$$

Since

$$\begin{aligned} |2\omega - 1| - |\omega| &\geq 2|\omega| - 1 - |\omega| \\ &= |\omega| - 1 \geq 0 \end{aligned}$$

and this gives $|2\omega - 1| \geq |\omega|$. Thus,

$$|z| = \frac{|2\omega - 1|}{|\omega|} \geq 1.$$

This is a contradiction since $z \in \mathbb{D}$ implies that $|z| < 1$. Therefore, $k(z)$ is not equivalent to $f(z)$ and $k(z) \notin \overline{f}$.

Next, we give a example to prove that the starlikeness in the equivalence class of a starlike function may not be preserved. Consider again the well-known Koebe function

$$k(z) = \frac{z}{(1-z)^2} \quad , \quad z \in \mathbb{D}.$$

From Example 3.2.7, we know that $k(z) \in S^*$. The equivalence class of k is defined by

$$\overline{k} = \left\{ g \in S : g = \frac{\omega k}{\omega - k}, \omega \in \mathbb{R}, \omega \leq -\frac{1}{4} \right\}$$

Consider an element in $\overline{k}$,

$$g(z) = \frac{\omega z}{\omega(1-z)^2 - z} \quad \text{for} \quad \omega \leq -\frac{1}{4}$$

Therefore,

$$\begin{aligned} \frac{zg'(z)}{g(z)} &= \frac{z\omega^2(1-z^2)}{\omega(1-z)^2 - z^2} \cdot \frac{\omega(1-z)^2 - z}{\omega z} \\ &= \frac{\omega(1-z^2)}{\omega(1-z)^2 - z} \end{aligned}$$

Let $(zg')/g = \omega h$, then $h(z) = \frac{1-z^2}{\omega(1-z)^2 - z}$. Since $\omega \leq -1/4$, then $g \in S^*$ if and only if $\mathrm{Re}\{h(z)\} < 0$. Let $z = x + iy$, then,

$$\begin{aligned} h(z) &= \frac{1-(x+iy)^2}{\omega(1-x-iy)^2 - x - iy} \\ &= \frac{(1-x^2+y^2) - i(2xy)}{\omega[(1-x)^2 - y^2] - x - i[2\omega y(1-x) + y]} \end{aligned}$$

and,

$$\begin{aligned} \mathrm{Re}\{h(z)\} &= \mathrm{Re}\left\{ \frac{(1-x^2+y^2) - i(2xy)}{\omega[(1-x)^2 - y^2] - x - i[2\omega y(1-x) + y]} \right\} \\ &= \frac{(1-x^2+y^2)\left\{\omega[(1-x)^2 - y^2] - x\right\} + (2xy)[2\omega y(1-x) + y]}{\left\{\omega[(1-x)^2 - y^2] - x\right\}^2 + [2\omega y(1-x) + y]^2} \end{aligned}$$

Since the denominator must be greater than 0, then the problem relies on the positivity of the numerator. Let $\omega = -1, x = -0.9$ and $y = -0.9$, then the value of the numerator of $\mathrm{Re}\{h(z)\}$ is 2.1824 which is greater than 0. This shows that $g \notin S^*$ since $\mathrm{Re}\{zg'(z)/g(z)\} < 0$.

Therefore, the functions in equivalence class of Koebe function are not all starlike. Thus, starlikeness is not preserved under this equivalence relation.

Next, we consider the convexity of the equivalence class of a convex function. But first of all, we will look at an example where the convexity of the equivalence class of a convex function is preserved.

Example 4.1.5. If f is a linear fractional transformation in S, then the equivalence classes of f is convex.

Solution. From Theorem 3.4.1, if f is a linear fractional transformation in S, then it is of the form

$$f(z) = \frac{z}{\alpha z + 1} \quad , \quad |\alpha| \leq 1$$

From Theorem 3.4.2, $f(z)$ is a convex function. For $g \in \overline{f}$, then,

$$g(z) = \frac{\omega z}{(\omega\alpha - 1)z + \omega}$$

Note that $\omega \neq 0$ since $\omega \notin f(\mathbb{D})$ and $f(0) = 0$. Thus, $\omega(\omega) - 0(\omega\alpha - 1) = \omega^2 \neq 0$. Therefore, $g(z)$ is a linear fractional transformation, and thus, $g(z)$ is convex. Therefore, convexity of equivalence classes of linear fractional transformation in class S are preserved.

To conclude this section, we give an example to prove that the convexity of the equivalence class of convex function may not be preserved.

Assume that f is the designated convex function, then the equivalence class of f is

$$\overline{f} = \left\{ g \in S : g = \frac{\omega f}{\omega - f}, \omega \notin f(\mathbb{D}) \right\}$$

From Theorem 3.2.3, $g \in \overline{f}$ is a convex function if

$$\mathrm{Re}\left\{\frac{zg''(z)}{g'(z)}+1\right\}>0.$$

Then, we have

$$\begin{aligned}\mathrm{Re}\left\{\frac{zg''(z)}{g'(z)}\right\} &= \mathrm{Re}\left\{z.\left[\frac{\omega^2 f''}{(\omega-f)^2}+\frac{2\omega^2(f')^2}{(\omega-f)^3}\right].\frac{(\omega-f)^2}{\omega^2 f'}\right\}\\ &= \mathrm{Re}\left\{\frac{zf''}{f'}+\frac{2zf'}{\omega-f}\right\}\end{aligned}$$

Therefore, $g \in C$ if

$$\mathrm{Re}\left\{\frac{zg''(z)}{g'(z)}+1\right\}=\mathrm{Re}\left\{\frac{zf''}{f'}+\frac{2zf'}{\omega-f}+1\right\}>0$$

Next, consider the function $p(z)=z+z^2/4$ for $z \in \mathbb{D}$. From Example 3.2.9, then $p(z) \in C$. The equivalence class of $p(z)$ is defined as

$$\overline{p}=\left\{q \in S: q=\frac{\omega p}{\omega-p}, \omega \notin p(\mathbb{D})\right\}.$$

We wish to show that not all functions in $\overline{p}$ are convex.

Letting $z=x+iy$, then

$$\begin{aligned}\mathrm{Re}\left\{\frac{zq''}{q'}+1\right\} &= \mathrm{Re}\left\{\frac{zp''}{p'}+1\right\}+\mathrm{Re}\left\{\frac{2zp'}{\omega-p}\right\}\\ &= \mathrm{Re}\left\{\frac{z+1}{\frac{1}{2}z+1}\right\}+\mathrm{Re}\left\{\frac{2z+z^2}{\omega-z-\frac{1}{4}z^2}\right\}\\ &= \frac{(1+x)(1+\frac{1}{2}x)+\frac{1}{2}y^2}{(1+\frac{1}{2}x)^2+(\frac{1}{2}y)^2}+\\ &\qquad \frac{(2x+x^2-y^2)(\omega-x-\frac{1}{4}x^2+\frac{1}{4}y^2)-(2y+2xy)(y+\frac{1}{2}xy)}{(\omega-x-\frac{1}{4}x^2+\frac{1}{4}y^2)^2+(y+\frac{1}{2}xy)^2}\end{aligned}$$

Let $\omega=-3/4$, we show that $\omega \notin p(\mathbb{D})$. If $\omega \in p(\mathbb{D})$, then there exists $z_0 \in \mathbb{D}$ such that

$$-\frac{3}{4}=z_0+\frac{z_0^2}{4}$$

This gives $z_0 = -1$ or $z_0 = -3$, but since both values of z_0 is not in $\mathbb{D}$, thus $\omega = -3/4 \notin p(\mathbb{D})$.

Taking $\omega = -3/4, x = 0.9$ and $y = 0$, then $\text{Re}\{zq''/q'+1\} = -0.0986$. Therefore, $q(z) \notin C$ because $\text{Re}\{zq''/q'+1\} < 0$. Therefore, the equivalence class of the convex function $p(z)$ is not convex. Thus, convexity is not preserved under this equivalence relation.

4.2 Complete solution for a particular first order ordinary differential equation

In this section, we will only be interested in a particular first order ordinary differential equation.

Recall that $g \sim f$ if and only if $g = \omega f/(\omega - f)$ where $\omega \notin f(\mathbb{D})$ and the equivalence class of f in S is defined by $\overline{f} = \{g \in S : g \sim f\}$.

Theorem 4.2.1. *For $f \in S$, consider the first-order ordinary differential equation*

$$\frac{g'}{g^2} = \frac{f'}{f^2}$$

on the unit disk $\mathbb{D}$, then $\overline{f}$ provides the complete solution in S.

Proof First, we show that if $g \in \overline{f}$, then g is a solution to the ordinary differential equation. Since $g \in \overline{f}$, then

$$g = f_\omega = \frac{\omega f}{\omega - f} \quad , \quad \omega \notin f(\mathbb{D})$$

$$(\omega - f)g = \omega f$$

Differentiating both sides, then

$$(\omega - f)g' = (\omega + g)f'$$

Therefore,

$$
\begin{aligned}
(\omega - f)g' &= \left(\frac{\omega^2}{\omega - f}\right) f' \\
g' &= \left[\frac{\omega^2}{(\omega - f)^2}\right] f' . \frac{f^2}{f^2} \\
\frac{g'}{g^2} &= \frac{f'}{f^2}
\end{aligned}
$$

Thus, $g \in \overline{f}$ is a solution to the ordinary differential equation.

Next, we show that if g is a solution to the ordinary differential equation in S, then $g \in \overline{f}$. We begin by considering

$$
\begin{aligned}
h &= \frac{fg}{f - g} \\
h' &= \frac{f^2 g' - f' g^2}{(f - g)^2} \\
h' &= 0
\end{aligned}
$$

since $f^2 g' = f' g^2$. Therefore, $h(z) = \omega$, where ω is a constant. So,

$$
\begin{aligned}
\frac{fg}{g - f} &= \omega \\
fg &= \omega g - \omega f \\
g &= \frac{\omega f}{\omega - f}
\end{aligned}
$$

Since $g \in S$, so g is well-defined. Therefore, $\omega - f(z) \neq 0$ for all $z \in \mathbb{D}$, thus, $\omega \notin f(\mathbb{D})$. This concludes the theorem. **Q.E.D.**

Next, some examples are given to demonstrate the application of the Theorem 4.2.1.

Example 4.2.2. Consider the identity function $f(z) = z$ in S.

Then, consider the differential equation arises from f,

$$\frac{g'}{g^2} = \frac{1}{z^2}$$
$$z^2 g' - g^2 = 0$$

From Theorem 4.2.1, the complete solution set in S for the above differential equation is

$$\overline{f} = \left\{ \frac{\omega z}{\omega - z} : |\omega| \geq 1 \right\}$$

Example 4.2.3. Consider the Koebe function $f(z) = z/(1-z)^2 \in S$. We have

$$\frac{f'(z)}{f(z)^2} = \frac{1-z^2}{z^2}$$

Thus, the differential equation becomes

$$z^2 g' + g^2(z^2 - 1) = 0$$

From Theorem 4.2.1,

$$\overline{f} = \left\{ \frac{\omega z}{\omega(1-z)^2 - z} : \mathrm{Re}(\omega) \leq -\frac{1}{4} \right\}$$

will provide the complete solution set for the differential equation above.

Example 4.2.4. Consider the function $f(z) = \sin z$ in S. Then, consider the differential equation arises from f below

$$(\sin^2 z)g' - (\cos z)g^2 = 0$$

From Theorem 4.2.1, the complete solution for the differential equation above is given by

$$\overline{f} = \left\{ \frac{\omega \sin z}{\omega - \sin^2 z} : \omega \notin \sin z, z \in \mathbb{D} \right\}.$$

4.3 The Schwarzian Derivative

Some of the analytic criterion for univalence have limited application as they involve special properties such as starlikess and convexity. These conditions are sufficient for univalence but far from necessary. However, Z. Nehari developed a general criterion for univalence which involves the Schwarzian Derivative. This is a very useful sufficient condition which is almost necessary in a certain sense.

Definition 4.3.1. The *Schwarzian Derivative* of a locally univalent analytic complex function f is defined by

$$\begin{aligned} S(f) &= \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 \\ &= \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \end{aligned}$$

Schwarzian derivative is important primarily because it is invariance under linear fractional transformation. The following theorem due to Z. Nehari provides a interesting criterion for univalence.

Theorem 4.3.2. *Let f be analytic in $\mathbb{D}$ and suppose its Schwarzian derivative satisfies*

$$|S(f)| \leq \frac{2}{(1-|z|^2)^2}$$

Then f is univalent in $\mathbb{D}$.

For the proof, please refer to (Nehari, 1949).

We prove that Schwarzian Derivative are invariant with respect to the equivalence relation defined in Section 4.1 to conclude this thesis.

Theorem 4.3.3. *The Schwarzian Derivative is invariant under the equivalence relation "$\sim$".*

Proof For a given equivalence class arises from $f \in S$, and $g \in \overline{f}$, to show that

$$\frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

Since $g \in \overline{f}$, then

$$g = f_\omega = \frac{\omega f}{\omega - f} \quad , \quad \omega \notin f(D)$$

Then, by simple calculation,

$$\begin{aligned}
g' &= \frac{\omega^2 f'}{(\omega - f)^2} \quad \text{, and} \\
g'' &= \frac{\omega^2 f''}{(\omega - f)^2} + \frac{2(\omega f')^2}{(\omega - f)^3} \quad \text{, and} \\
g''' &= \frac{\omega^2 f'''}{(\omega - f)^2} + \frac{6\omega^2 f' f''}{(\omega - f)^3} + \frac{6\omega^2 (f')^3}{(\omega - f)^4}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{g'''}{g'} - \frac{3}{2}\left(\frac{g''}{g'}\right)^2 &= \left[\frac{\omega^2 f'''}{(\omega - f)^2} + \frac{6\omega^2 f' f''}{(\omega - f)^3} + \frac{6\omega^2 (f')^3}{(\omega - f)^4}\right] . \frac{(\omega - f)^2}{\omega^2 f'} \\
&\quad - \frac{3}{2}\left\{\left[\frac{\omega^2 f''}{(\omega - f)^2} + \frac{2(\omega f')^2}{(\omega - f)^3}\right] . \frac{(\omega - f)^2}{\omega^2 f'}\right\}^2 \\
&= \frac{f'''}{f'} + \frac{6f''}{\omega - f} + \frac{6(f')^2}{(\omega - f)^2} - \frac{3}{2}\left(\frac{f''}{f'} + \frac{2f'}{\omega - f}\right)^2 \\
&= \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2
\end{aligned}$$

The theorem is proved. **Q.E.D.**

CONCLUSION

This thesis has developed an equivalence relation that is proven to be the complete solution for a particular first order ordinary linear differential equation. Furthermore, Schwarzian Derivative is invariant with respect to such equivalence relation. However, many opportunities for extending the scope of this thesis remain. Further research along this area may be as follows.

1. Extending first order ordinary differential equation to n-th order ordinary differential equation, where $n = 2,3,4\ldots.$

2. Obtaining geometrical properties of each equivalence class in S.

3. As in Chapter 3, more explicit characterization on univalent functions should be investigated.

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