MAGIC SQUARES WITH ADDITIONAL PROPERTIES

By

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requirements for the award of Bachelor of Science (Hons.)
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DECLARATION OF ORIGINALITY

I hereby declare that this project report entitled "**MAGIC SQUARES WITH ADDITIONAL PROPERTIES**" is my own work except for citations and quotations which have been duly acknowledged. I also declare that it has not been previously and concurrently submitted for any other degree or award at UTAR or other institutions.

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Approval for Submission

I certify that this project report entitled "**MAGIC SQUARES WITH ADDITIONAL PROPERTIES**" was prepared by **AUDREY NG CHER DI** has met the required standard for submission in partial fulfilment of the requirements for the award of Bachelor of Science (Hons.) Applied Mathematics With Computing at Universiti Tunku Abdul Rahman.

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Supervisor : ______________________________

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ABSTRACT

My final year project entitled 'Magic Squares with Additional Properties' aims to research on one of the branches of mathematics which is magic squares. Since magic squares have existed for a long time, there are many interesting properties on magic squares that are found and yet to be found. We shall learn about the history and origin of magic squares. In addition, a magic square also has its uses that we can make use of. By referring to some published materials, we can learn more about it and therefore understand its properties better. For a magic square, the sum of the row entries, column entries and entries of the main diagonals has to be the same and therefore this constant sum is known as the magic sum. With this property itself, it makes this magic square seems unique and distinguished. And also in order to understand some other properties, the method of construction of magic squares must be known first. With different methods to construct these magic squares of different order, these properties can be identified throughout the way. These methods are known and its construction are detailed and clear for someone with least mathematical background to understand them. Not only that, there are fairly many types of magic squares which are interesting enough to catch anyone's attention. Such special characteristics comprising of symmetric properties to having broken diagonals in a square definitely made magic squares a distinctive part of mathematics. Lastly, we will try to modify a method of construction to produce magic squares as well.

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CHAPTER 1: INTRODUCTION

Magic square is an interesting part of mathematics where it has been gaining interest of many mathematicians throughout all these times till now. Its well-known unique properties have made it so special that many research have been done to discover more new properties. The special relationship between the sums of the numbers in the square is actually the main factor for why this such square is called magic. As a definition, a magic square is an arrangement of integers in a square of an order in which the sum of the integers for every column, row and main diagonal are the same. Each one of these integers will only occur once in the square. This constant sum is called the *magic sum*. If the entries are the consecutive integers from 1 to n^2, the magic square is said to be of order n.

1-1 Background of Magic Squares

The origin of magic squares can be known from various parts of the world. However, one of the earliest discovery of magic squares was in China during the Xia dynasty. At about 2,200 B.C. under the ruling of Emperor Yu, it is said that there was once a flood where it affected the people badly. Until one day which Emperor Yu was trying to figure out solutions to solve this matter, it is said that he witnessed a turtle appeared from the flood water. This particular turtle had a unique pattern that can be seen on its shell.The pattern looked like a 3 by 3 grid square with circular dots arranged accordingly. The legend says that Emperor Yu managed to think of a way to prevent such disaster from happening again by applying methods derived from the pattern on the turtle's shell. This pattern is hereby named as the Lo-Shu square shown in Figure 1.1. By counting the number of dots for each of the small subsequent pattern, we can create a magic square of order 3 illustrated in Figure 1.2.

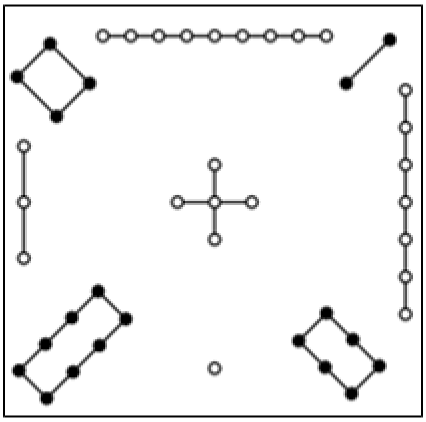

Figure 1.1: Lo-Shu square

4	9	2
3	5	7
8	1	6

Figure 1.2: Magic square of order 3

Besides than the discovery in China, magic squares are also found in Arabia. Islamic mathematicians knew about magic squares during the seventh century and they have first created the first magic squares of order 5 and 6. These magic squares can be found in the Encyclopaedia of the Brethren of Purity.

Not only that, there were sources saying that magic squares were also originated from India. The Hindus seem to create a magic square of order 4 and it was created in such a way that the broken diagonals of the square sums up to its magic sum along with its original properties. This type of magic square is called as a *pandiagonal magic square* and this will be elaborated in the chapters ahead.

1-2 Application of Magic Squares

There is some practical usage of magic squares when it comes to application. In India, magic squares are used in music composition. The numbers in the squares are replaced with musical notes and this can be applied to time cycles and additive rhythm. This links to a great rhythmic cadence, which naturally gravitates towards the very next beat after the end of third repetition (Dimond, 2013).

Not only that, the Sudoku game which is very famous that is created in Japan was derived from Latin squares. These Latin squares are also a form of magic squares. For a square with order n, there are n entries on each row and column where there is no repetition of the same integer in the same row or column.

In addition, up to this present day, the Lo-Shu square shown in Figure 1.1 is still used as a medium in the field of *feng-shui*. Feng-shui is a system of applying ways to harmonize everyone with its surrounding environment. In the Lo-Shu square, we can see that the odd-valued and even-valued entries are placed alternately. This is similar to one of the concepts from feng-shui which is yin and yang. For example, the odd-valued entries represent the Yang quality and the even-valued entries carry the Yin energy (Tchi, 2017).

CHAPTER 2: OBJECTIVES AND PLANNING

The goal of this project is to study the various methods of construction of magic squares along with its additional properties obtained after the construction is done. Different types of order of magic squares have different ways to construct them. By completing the construction, the next objective that is to be accomplished is to investigate the discovered properties of the magic squares. With that, we will try to make slight changes to one of the construction methods to create magic squares as well.

2-1 Project Scopes

In this project, the emphasis is on the construction of magic squares of odd and even order. The squares of even order is further branched out to squares of singly-even order and doubly-even order. All these three types of orders of squares are being constructed using different methods that will result in magic squares. For construction of magic squares of odd order, De la Loubere method is applied while on the other hand, Strachy's method is used to create magic squares of singly-even order. Lastly, the Generalized Doubly-Even Method is used to construct magic squares of doubly-even order.

For the discussion section, the main highlight is on analysing the special outcomes after applying the methods of construction of each order of squares. These are the special characteristics which distinguish the magic squares that uses different methods of construction. On top of that, I have made some modification to the Generalized Doubly -Even method which also results in magic squares.

2-2 Planning

2-2-1 Action plan for Project I

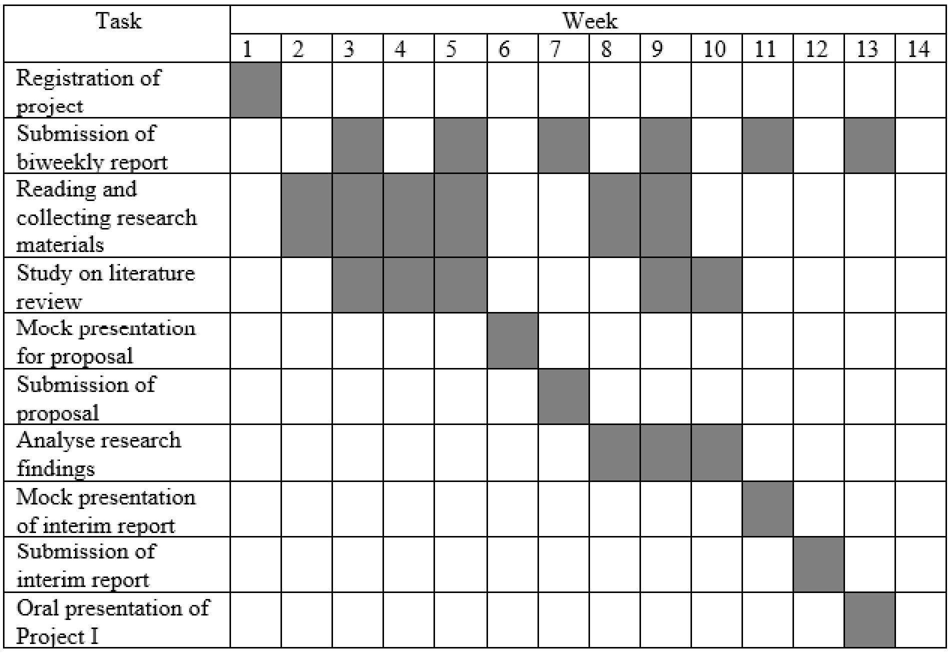

Figure 2.1: Action plan for Project I

2-2-2 Action plan for Project II

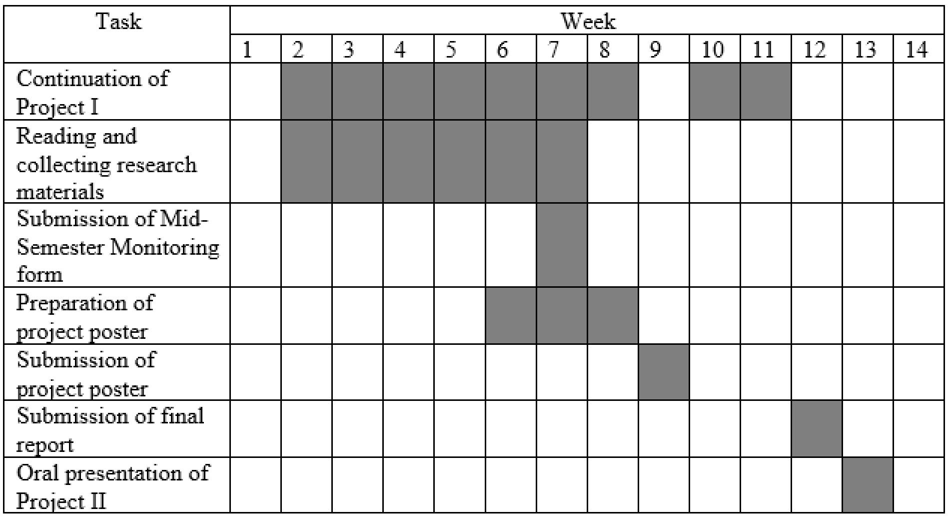

Figure 2.2: Action plan for Project II

CHAPTER 3: LITERATURE REVIEW

Until now there have been many mathematicians who performed research on magic squares and also to discover additional properties of these magic squares. Their results never failed to impress the experts and express the uniqueness behind magic squares.

As mentioned earlier, one of the earliest discovery on magic squares is the Lo-Shu square in China. From Figure 1.1 and Figure 1.2, the sum in each row, column, diagonal is 15. This sum represents the number of days in each of the 24 cycles of the Chinese solar year (Sorici, 2010). This could be just a mere coincidence or a work of nature but in either ways, this definitely brought out the uniqueness of magic squares.

Another example of a magic square called as the Franklin Square is created by a very well-known American scientist, Benjamin Franklin. Based on the information obtained in the book from Schumer (2004), Franklin's magic square of order 8 has many interesting properties to be followed up. By referring to Figure 3.1, the sum of all rows and columns is 260 and if this square is partitioned in to sub-squares of order 4, the sub-squares are pseudomagical which means the sum of each row and each column in the sub-squares are equal that is 130. However, the sum of the main diagonals is either 252 and 268 which are not equalled to 260. But this pattern is unique in the sense that the broken diagonals will somehow sum up to 252 or 268 in an alternating pattern. This shows that each magic square will have its own special properties which is not considered wrong or right but rather unique in its own way.

17	47	30	36	21	43	26	40
32	34	19	45	28	38	23	41
33	31	46	20	37	27	42	24
48	18	35	29	44	22	39	25
49	15	62	4	53	11	58	8
64	2	51	13	60	6	55	9
1	63	14	52	5	59	10	56
16	50	3	61	12	54	7	57

Figure 3.1: Franklin Square of order 8

On the construction of the magic squares, Andrews (1917) generalized some steps to take note of in the construction. There are two variables which is the initial starting number and the increment of each number in the square. With these two variables known, the summation can be easily determined. For example, the sum of each row, column and main diagonal in Figure 3.2 is 15. But if '2' is regarded as the initial number as with regular increments of 1, then the new magic square will have a magic sum of 18 as shown in Figure 3.3. Then if the increment value is changed from 1 to 2, the new magic square will have a magic sum of 30 in Figure 3.4. From here, the author is putting emphasis on the initial starting number and the increment value for the construction of magic squares because it holds a big importance in this matter.

8	1	6
3	5	7
4	9	2

Figure 3.2: Magic square with initial entry '1'

9	2	7
4	6	8
5	10	3

Figure 3.3: Magic square with initial entry ‘2’

16	2	12
6	10	14
8	18	4

Figure 3.4: Magic square with increment of 2

An article by Benjamin and Yasuda (1999) has elaborated more on square-palindromic matrices which have the magic square properties as well. They stated in a theorem that every symmetrical magic square and all 3x3 magic squares are square-palindromic.

Not only that, Laposky (1978) added that magic squares are a possible source of design. An example of an exotic geometric arrangement with magic number property is the representation of a 2-dimensional projection of a theoretical 4-dimensional figure which is a magic hypercube or tesseract.

CHAPTER 4: METHODOLOGY

There is no limit to how big a magic square can be. Hence, there are multiple ways to construct magic squares of different orders. First of all, the first step to constructing a magic square is to identify the order of the square. There are magic squares of odd order, singly-even order and doubly-even order. Once the order of the square is known, then the correct method of construction can be used accordingly. The following methods provided are so detailed and understandable for the readers to understand them. Note that the magic sum has to be constant for every row, column and main diagonal. The magic sum is calculated as :

$$S_n = \frac{1}{2}n(n^2+1)$$

4-1 Computation on Magic Sum

As mentioned before, a magic square consists of entries 1 to n^2. This means the total sum of all the entries of the square is shown as :

$$1+2+3+\cdots+(n^2-1)+n^2$$

The summation formula for this type of series is :

$$\sum_{r=1}^{k} r = \frac{1}{2}k(k+1)$$

Clearly, we can substitute the following unknowns correctly and the summation of the series is :

$$\sum_{r=1}^{n^2} r = \frac{1}{2}n^2(n^2+1)$$

This gives the total sum of the entries in the square of order n. Next, magic sum is where the sum of each row, each column and both the main diagonals are the same. Given this reason, we shall divide it with the number of rows or *n* and we can say that the magic sum is :

$$S_n = \frac{1}{2}n(n^2+1)$$

4-2 Construction of Magic Squares of Odd Order

According to Chee (1981), magic squares of odd order is when the order, n is odd. This square of this order is constructed using the method called De la Loubere. De la Loubere method makes creation of magic squares straight forward. This method was brought to France in 1688 by a French mathematician, De la Loubere as he was returning from his 1687 embassy to the kingdom of Siam. Therefore, this method is also known as the Siamese method. An example is illustrated below to explain the steps of construction accurately.

Let us take $n = 7$,

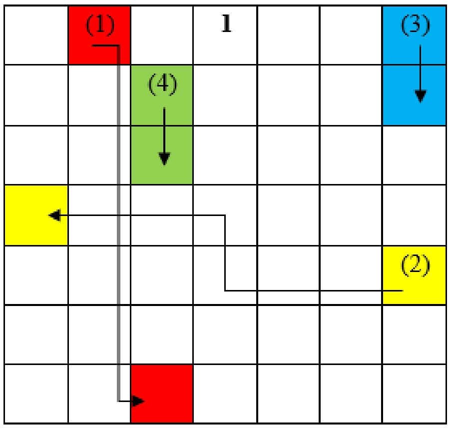

Figure 4.1: Conditions for De la Loubere method

Conditions during the filling in of the numbers shown in Figure 4.1 :

1. Once the first row is reached, the next number is filled in at the right cell of the previous entry at the last row.
2. When the far right-hand column is reached, the next number is filled in at the cell at the far left-hand column above the row of the previous entry.
3. When the top right-hand cell is filled up already, the next number is filled in at the cell right below it.
4. When a cell is already filled in, the next number is filled in at the cell right below it as well.

Steps :

1. Place '1' in the middle cell of the top row.
2. Follow condition number 1 and fill in the next number.
3. Fill in the consecutive numbers 45° diagonally towards the right.
4. Once the far right-hand column is reached, follow condition number 2.
5. Then, repeat step 3 and fill in accordingly to the conditions given.

The result of a magic square of order , n = 7 is

30	39	48	1	10	19	28
38	47	7	9	18	27	29
46	6	8	17	26	35	37
5	14	16	25	34	36	45
13	15	24	33	42	44	4
21	23	32	41	43	3	12
22	31	40	49	2	11	20

Figure 4.2: Magic square of order 7

The magic sum is also constant using this method where the sum of each row, column and main diagonal is 175.

$$S_7 = \frac{1}{2}(7)(7^2 + 1) = 175$$

This method works for constructing any magic squares of odd order. Another example of a magic square of an odd order is as the following :

47	58	69	80	1	12	23	34	45
57	68	79	9	11	22	33	44	46
67	78	8	10	21	32	43	54	56
77	7	18	20	31	42	53	55	66
6	17	19	30	41	52	63	65	76
16	27	29	40	51	62	64	75	5
26	28	39	50	61	72	74	4	15
36	38	49	60	71	73	3	14	25
37	48	59	70	81	2	13	24	35

Figure 4.3: Magic square of order 9

Notice that the last entry is always at the last row in the same column as the cell with number ‘1’. The magic sum for this order is 369.

4-3 Construction of Magic Squares of Singly-Even Order

Magic squares of this order is constructed using Strachy's method according to Kurdle and Menard (2007). Before that, singly-even order is represented by :

$$n = 2(2k + 1) = 4k + 2$$

where k is an integer and a singly-even order consists of one even component while the other is an odd component. Therefore, it is called as the singly-even order.

Let us take k = 2 , then n = 10.

Steps :

A	C
D	B

Figure 4.4: Sub-squares of order $2k + 1$

1. Divide the square of order n into sub-squares A, B, C and D of order 2k + 1 according to Figure 4.4.

2. Fill in the numbers using the De la Loubere method for each sub-square where the entries are as below :

   - A : numbers from 1 to $(2k+1)^2$ where $2k+1 = \frac{n}{2}$
   - B : numbers from $(\frac{n}{2})^2 + 1$ to $2(\frac{n}{2})^2$
   - C : numbers from $2(\frac{n}{2})^2 + 1$ to $3(\frac{n}{2})^2$
   - D : numbers from $3(\frac{n}{2})^2 + 1$ to $4(\frac{n}{2})^2$ where $4(\frac{n}{2})^2 = n^2$

   The resulting square is magic in columns as shown below.

17	24	1	8	15	67	74	51	58	65
23	5	7	14	16	73	55	57	64	66
4	6	13	20	22	54	56	63	70	72
10	12	19	21	3	60	62	69	71	53
11	18	25	2	9	61	68	75	52	59
92	99	76	83	90	42	49	26	33	40
98	80	82	89	91	48	30	32	39	41
79	81	88	95	97	29	31	38	45	47
85	87	94	96	78	35	37	44	46	28
86	93	100	77	84	36	43	50	27	34

Figure 4.5: Sub-squares using De la Loubere method

3. From Figure 4.5, take k-1 columns from the far right-hand side of sub-square B and exchange that particular column with the corresponding column of sub-square C (shown in yellow).

4. Exchange the middle cell of sub-square A with the corresponding cell in sub-square D (shown in red).

5. Let the middle cell of the first column in sub-square A and sub-square D remain at their original cells as it is.

6. Take k columns from the far left-hand side of sub-square A and exchange the columns with the corresponding columns of sub-square D.

The resulting square is a magic square of order 10.

92	99	1	8	15	67	74	51	58	40
98	80	7	14	16	73	55	57	64	41
4	81	88	20	22	54	56	63	70	47
85	87	19	21	3	60	62	69	71	28
86	93	25	2	9	61	68	75	52	34
17	24	76	83	90	42	49	26	33	65
23	5	82	89	91	48	30	32	39	66
79	6	13	95	97	29	31	38	45	72
10	12	94	96	78	35	37	44	46	53
11	18	100	77	84	36	43	50	27	59

Figure 4.6: Magic square of order 10

The magic sum for this magic square of order 10 is

$$S_{10} = \frac{1}{2}(10)(10^2 + 1) = 505$$

Another example of magic square that uses Strachy's method is of order 6 where k = 1. For this order, step number 3 is ignored.

35	1	6	26	19	24
3	32	7	21	23	25
31	9	2	22	27	20
8	28	33	17	10	15
30	5	34	12	14	16
4	36	29	13	18	11

Figure 4.7: Magic square of order 6

4-4 Construction of Magic Squares of Doubly-Even Order

Magic squares of this order is constructed using Generalized Doubly-Even Method (Kurdle and Menard, 2007). Before that, doubly-even order is represented by :

$$n = 2(2k) = 4k$$

where k is an integer and doubly-even order is made up of two even components. Thus, the reason is in the name itself.
Take for example, n = 8 with k = 2.
Steps :

1. Fill in the numbers from 1 to n^2 in an ordinary sequence as shown in Figure 4.8.
2. Partition the square into k^2 sub-squares of order 4.
3. Draw their respective main diagonals for each sub-squares.
4. For the cells that are cut by the diagonals, interchange the numbers in a reverse order with respect to the centre of the square (marked with the yellow circle).
5. For example, cell numbered '4' is one step to the left and four steps above from the centre of the square. Hence the position is renamed as [-1,4]. In reverse by changing the sign of the position, the position will be [1,-4] from the centre of the square and that will be cell numbered '61'. Thus, we interchange the numbers in both of these cells.

The resulting magic square is shown in Figure 4.9.

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

Figure 4.8: Sub-squares of order 4

64	2	3	61	60	6	7	57
9	55	54	12	13	51	50	16
17	47	46	20	21	43	42	24
40	26	27	37	36	30	31	33
32	34	35	29	28	38	39	25
41	23	22	44	45	19	18	48
49	15	14	52	53	11	10	56
8	58	59	5	4	62	63	1

Figure 4.9: Magic square of order 8

Here the magic sum of this magic square of order 8 will be

$$S_8 = \frac{1}{2}(8)(8^2 + 1) = 260$$

144	2	3	141	140	6	7	137	136	10	11	133
13	131	130	16	17	127	126	20	21	123	122	24
25	119	118	28	29	115	114	32	33	111	110	36
108	38	39	105	104	42	43	101	100	46	47	97
96	50	51	93	92	54	55	89	88	58	59	85
61	83	82	64	65	79	78	68	69	75	74	72
73	71	70	76	77	67	66	80	81	63	62	84
60	86	87	57	56	90	91	53	52	94	95	49
48	98	99	45	44	102	103	41	40	106	107	37
109	35	34	112	113	31	30	116	117	27	26	120
121	23	22	124	125	19	18	128	129	15	14	132
12	134	135	9	8	138	139	5	4	142	143	1

Figure 4.10: Magic square of order 12

Another example of a construction of a magic square using the Generalized Doubly-Even method is when n = 12 shown in Figure 4.10. For this order, the magic sum will be

$$S_{12} = \frac{1}{2}(12)(12^2 + 1) = 870$$

CHAPTER 5: RESULTS AND DISCUSSION

The method of construction of magic squares of all orders are very neat and understandable. These are the basic ways to create magic squares. De la Loubere method for constructing odd-order magic squares is fascinating where the number '1' is always placed in the middle cell of the first row while the last number entry will be at the middle cell of the last row. For squares of singly-even order, Strachy's method is applied and not all squares of singly-even order need to follow all the steps provided. Particularly for singly-even square of order 6 where k = 1, no changes need to be made on the right-hand side columns of the square. Other than this order, all squares of this order have to follow each of the steps to be magic. On the other hand, squares of doubly-even order use Generalized Doubly-Even method to be magic squares. Every interchange that is done is with respect to the centre of the whole square of order n and not to the centre of each sub-square. Thus, these methods have some points to be alert of and they are the foundation to creating magic squares.

5-1 Symmetrical Magic Squares

There are fairly many types of magic squares that possess interesting properties. In this section, we will elaborate on one of the types of magic squares known as *symmetrical* magic square. A magic square is said to be symmetrical when the entries of the symmetrical pair of cells in the square sum up to $n^2 + 1$. It is simple to identify the symmetrical pair of cells. For instance ,to find the symmetrical cell to cell (i, j), we just need find the cell $(n+1-i, n+1-j)$ and this pair of cells are said to be symmetrical with each other. Cell (i, j) denotes the position of the cell in the whole square in which i and j represent the row and column of the cell located. The following equation shows the relationship more clearly.

$$a_{i,j} + a_{n+1-i,n+1-j} = n^2 + 1$$

For example in Figure 5.1, cell (4,2) is numbered '12' and cell (2,4) is numbered '14' where the sum is 26 and this satisfies the requirements. The coloured pairs of cells show that the pairs are example of symmetrical cells.

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

Figure 5.1: Symmetrical magic square of order 5

In addition, symmetrical magic squares also portray a special property in which symmetrical patterns can be drawn on the square by joining the entries. It is flexible where we can join the entries in a natural way or by joining the odd-valued entries or by joining the even-valued entries. Some examples are given below to show this property by magic squares of different orders. For an example of an odd-ordered magic square, we will see whether there will be a unique pattern that shows symmetric property. Let us take $n = 5$.

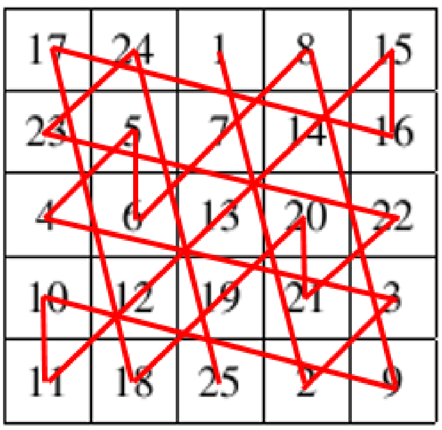

Figure 5.2: All entries joined together for order 5

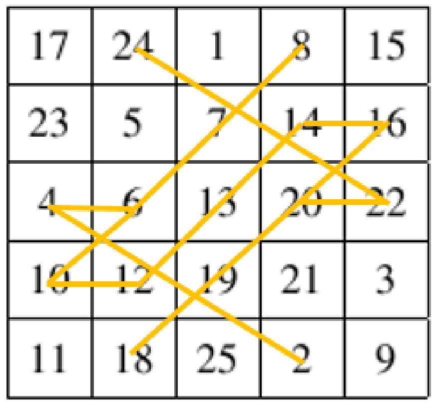

Figure 5.3: All even-valued entries joined together for order 5

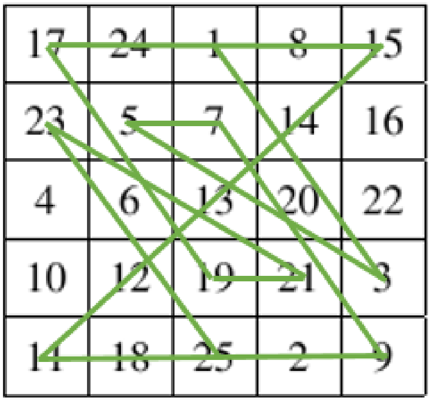

Figure 5.4: All odd-valued entries joined together for order 5

From Figures 5.2, 5.3 and 5.4, the pattern depicted is obtained through a 180° rotation about the centre of the square. Therefore, we can say the patterns on these magic squares are symmetrical.

Now, we shall check for symmetric property in a singly-even order magic square. Let us take $n = 6$.

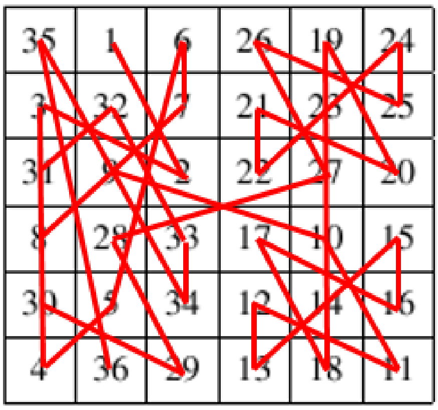

Figure 5.5: All entries joined together for order 6

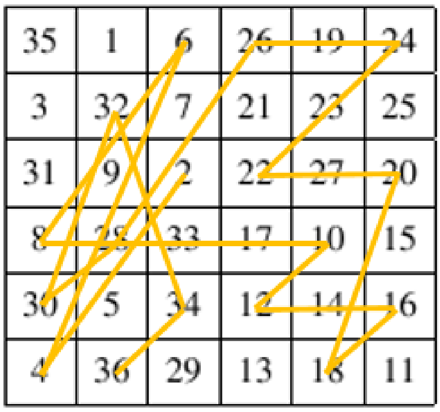

Figure 5.6: All even-valued entries joined together for order 6

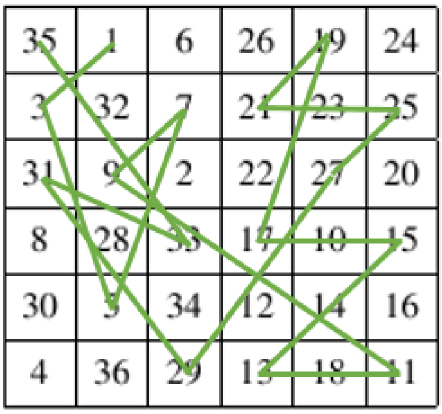

Figure 5.7: All odd-valued entries joined together for order 6

From Figure 5.5, 5.6 and 5.7, we can see that the magic square of this order is not symmetrical and there were no distinguished patterns. The patterns are also not symmetrical and so we may say that magic squares of singly-even order are not symmetrical.

Lastly , we shall check for magic squares of doubly-even order. Let us take $n = 4$.

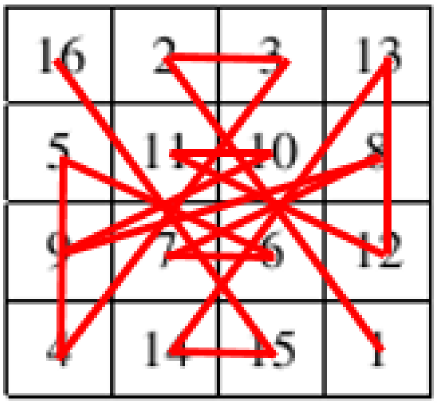

Figure 5.8: All entries joined together for order 4

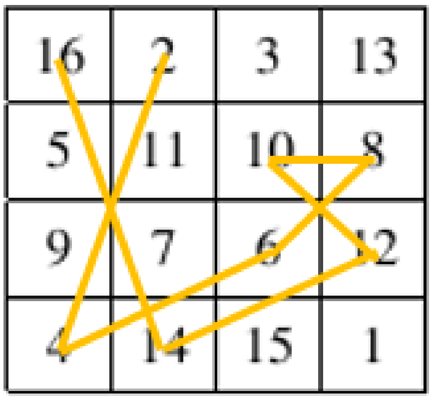

Figure 5.9: All even-valued entries joined together for order 4

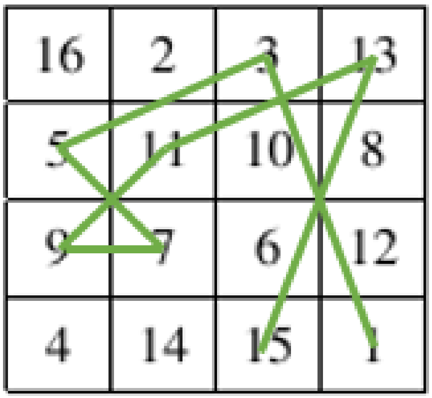

Figure 5.10: All odd-valued entries joined together for order 4

From the Figure 5.8, we can see a symmetrical pattern on the square which the pattern undergoes a rotation of 180° transformation about the centre of the square. And also the pattern shown when the even-valued entries in Figure 5.9 and odd-valued entries from Figure 5.10 joined are not symmetrical. However, both patterns can be seen that they undergo a rotation of 180° transformation if both of the figures are being compared with each other instead of individually. Hence, we may say that magic squares of doubly-even are symmetrical and the patterns shown by joining different values of entries are unique as well.

5-2 Self-Complementary Magic Squares

A magic square is called a *self-complementary* magic square when its complement can be transformed into getting its original magic square. This such relationship means that if a complement of a magic square, $\bar{A}$, is equivalent to its original magic square ,A, hence A is a self-complementary magic square. Let A be a magic square of order n.

$$\mathcal{A} \xrightarrow{n^2+1-entry\ value} \bar{\mathcal{A}} \xrightarrow{Transformation} \mathcal{A}$$

By referring to the relationship above to find the complement of the magic square, we just need to compute $n^2 + 1 - entry\ value$ for each cell in the square . Once the complement is obtained, we will try to let the complement of the magic square of order n to undergo a 180° rotation about the centre of the square or a reflection along the middle axis of the square.

For a magic square which undergoes a 180° rotation on the square, it is said to be $ro - symmetrical$. And for a magic square which undergoes a central vertical or horizontal reflection, it is said to be $ref - symmetrical$.

If n is odd, the magic square can only be ro-symmetrical if it is self-complementary. On the other hand, if n is even, the magic square can be ro-symmetrical or else ref-symmetrical if it is self-complementary.Now let us try to show some examples of self-complementary magic squares that undergo these two types of transformation.

For n = 5, given the magic square of order 5 in Figure 5.11, the complement is shown in Figure 5.12:

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

Figure 5.11: Magic square of order 5

9	2	25	18	11
3	21	19	12	10
22	20	13	6	4
16	14	7	5	23
15	8	1	24	17

Figure 5.12: Complement of order 5

From here, we would see whether its complement can return to its original form as in Figure 5.11. On top of that, a unique and symmetrical pattern can be drawn on its complement to show its symmetric property.

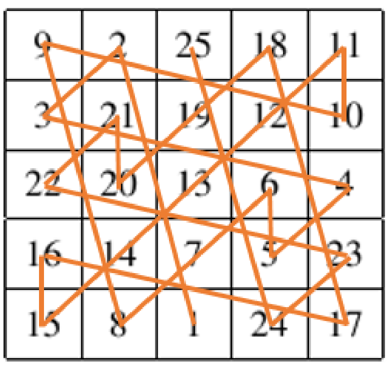

Figure 5.13: Pattern on complement of order 5

By looking at Figure 5.13, it is obvious that the magic square undergoes a 180° transformation and with this, it can return to its original form of magic square of order 5. Therefore, we can say this magic square is ro-symmetrical and it is self-complementary.

Another example is when $n = 4$.

13	3	2	16
7	12	9	6
10	5	8	11
4	14	15	1

Figure 5.14: Magic square of order 4

4	14	15	1
10	5	8	11
7	12	9	6
13	3	2	16

Figure 5.15: Complement of order 4

Once the complement of order 4 is obtained as in Figure 5.15, we can see that the complement can return to its original form of magic square by undergoing a central horizontal reflection. Hence, this magic square is ref-symmetrical and it is self-complementary.

5-3 Other Types of Magic Squares

The previous section has already discussed on symmetrical and self-complementary magic squares. Now, let us discuss on some other types of magic squares which also portray special properties within them.

First of all is a *semi-magic square*. A square is magic when these main properties are fulfilled.

1. Sum of each row is the magic sum.

2. Sum of each column is the magic sum.

3. Sum of each main diagonal is the magic sum.

By referring to the example given in Figure 5.16, a semi-magic square is a square where it only satisfies properties 1 and 2 in the above definition for magic square (Weisstein, 2017).

1	5	9
6	7	2
8	3	4

1	5	9
8	3	4
6	7	2

1	6	8
9	2	4
5	7	3

1	8	6
9	4	2
5	3	7

2	4	9
6	8	1
7	3	5

2	6	7
9	1	5
4	8	3

3	4	8
5	9	1
7	2	6

3	7	5
8	6	1
4	2	9

Figure 5.16: Semi-magic squares of order 3

Furthermore, there are also magic squares which are known as pan-diagonal magic squares that is mentioned earlier in chapter 1. This means that all of the broken diagonals of the square sums up to the magic sum of the order of the square. From Figure 5.17 and 5.18, the broken diagonals are highlighted and they summed up to the magic sum of the square.

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

Figure 5.17: Pandiagonal magic square of order 4

20	22	4	6	13
9	11	18	25	2
23	5	7	14	16
12	19	21	3	10
1	8	15	17	24

Figure 5.18: Pandiagonal magic square of order 5

5-4 Modification in Generalized Doubly-Even Method

In this section,the main emphasis is on making some modifications to the method which is used to construct magic squares of doubly-even order. This method, Generalized Doubly-Even method that is referred by Kurdle and Menard (2007) is effective in constructing magic squares of this order. However, we would like to see whether magic squares can still be produced by applying some changes to this method.

There is something to take note of regarding this method. The order of the square has to be a multiple of 8, $n = 8k$ where k = 1,2,... . This is because from the original method, the square has to be a multiple of 4. From here, we will proceed to divide the square into sub-squares of order 4. Therefore, it follows that the order of the square in the beginning has to be an order of 8. This explaination is shown clearly in Figure 5.19.

The following shows the steps to be taken to construct a magic square of a doubly-even order.

1. For a square of order n, we shall first divide it into four main partitions , A_i for *i* = 1,2,3,4. Each partition is a square of order $\frac{n}{2}$.

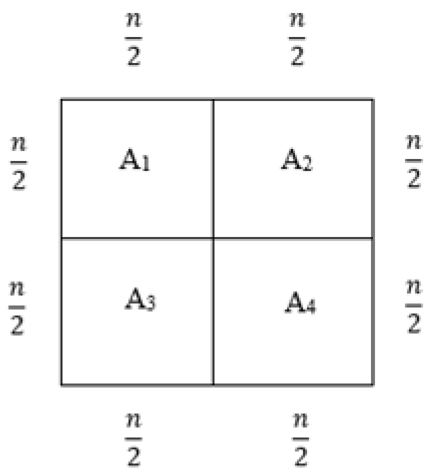

Figure 5.19: Partitions of sub-squares

2. Next, fill in each partition with the entries values as specified below.

   - Partition A_1 : Entries from 1 to $(\frac{n}{2})^2$
   - Partition A_2 : Entries from $(\frac{n}{2})^2 + 1$ to $2(\frac{n}{2})^2$

- Partition A_3 : Entries from $2(\frac{n}{2})^2 + 1$ to $3(\frac{n}{2})^2$
- Partition A_4 : Entries from $3(\frac{n}{2})^2 + 1$ to $4(\frac{n}{2})^2$

3. Next, partition each A_i into sub-squares of order 4.

4. For each of the sub-square of order 4, draw the main diagonals on them.

5. Once the diagonals are drawn, exchange the entries which are cut by the diagonals with its symmetrical cell.

With these steps, the resulting square turned out to be a magic square.Now let us try with different values n when n = 8 and 16.

For n = 8,

1	2	3	4	17	18	19	20
5	6	7	8	21	22	23	24
9	10	11	12	25	26	27	28
13	14	15	16	29	30	31	32
33	34	35	36	49	50	51	52
37	38	39	40	53	54	55	56
41	42	43	44	57	58	59	60
45	46	47	48	61	62	63	64

Figure 5.20: Partitions with the main diagonals

After applying those steps and exchanging the cells cut by the diagonals symmetrically, we obtain the following square and it is magic.

64	2	3	61	48	18	19	45
5	59	58	8	21	43	42	24
9	55	54	12	25	39	38	28
52	14	15	49	36	30	31	33
32	34	35	29	16	50	51	13
37	27	26	40	53	11	10	56
41	23	22	44	57	7	6	60
20	46	47	17	4	62	63	1

Figure 5.21: Magic square of order 8

For n = 16, the magic square after applying this modified method is shown in Figure 5.22.

256	2	3	253	252	6	7	249	192	66	67	189	188	70	71	185
9	247	246	12	13	243	242	16	73	183	182	76	77	179	178	80
17	239	238	20	21	235	234	24	81	175	174	84	85	171	170	88
232	26	27	229	228	30	31	225	168	90	91	165	164	94	95	161
224	34	35	221	220	38	39	217	160	98	99	157	156	102	103	153
41	215	214	44	45	211	210	48	105	151	150	108	109	147	146	112
49	207	206	52	53	203	202	56	113	143	142	116	117	139	138	120
200	58	59	197	196	62	63	193	136	122	123	133	132	126	127	129
128	130	131	125	124	134	135	121	64	194	195	61	60	198	199	51
137	119	118	140	141	115	114	144	201	55	54	204	205	51	50	208
145	111	110	148	149	107	106	152	209	47	46	212	213	43	42	216
104	154	155	101	100	158	159	97	40	218	219	37	36	222	223	33
96	162	163	93	92	166	167	89	32	226	227	29	28	230	231	25
169	87	86	172	173	83	82	176	233	23	22	236	237	19	18	240
177	79	78	180	181	75	74	184	241	15	14	244	245	11	10	248
72	186	187	69	68	190	191	65	8	250	251	5	4	254	255	1

Figure 5.22: Magic square of order 16

CHAPTER 6: CONCLUSION

As an overall for this final year project, the highlight of this topic is the method of construction of magic squares. Magic squares of different types of order are being constructred differently. Magic squares of odd order uses De la Loubere method whereas magic squares of singly-even order uses Strachy's method and last but not least, magic squares of doubly-even method uses the Generalized Doubly-Even method. All of these methods are so precise that they made the construction of magic squares to be fairly understandable. And also by applying some modification to the Generalized Doubly Even method to try to construct magic squares of doubly even order seemed to produce magic squares as well for the examples shown. As a main outcome for this project, we can say that this modified way is possible to be used to construct magic squares of similar order as well.

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