OPTIMAL DESIGNS OF THE VARIABLE SAMPLE SIZE (VSS) $\bar{X}$ CHART BASED ON MEDIAN RUN LENGTH AND EXPECTED MEDIAN RUN LENGTH

By

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A dissertation submitted to the Department of Physical and Mathematical Science,
Faculty of Science,
Universiti Tunku Abdul Rahman,
in partial fulfillment of the requirements for the degree of
Master of Science
February 2017

ABSTRACT

OPTIMAL DESIGNS OF THE VARIABLE SAMPLE SIZE (VSS) $\bar{X}$ CHART BASED ON MEDIAN RUN LENGTH AND EXPECTED MEDIAN RUN LENGTH

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The variable sample size (VSS) $\bar{X}$ chart, devoted to the detection of moderate mean shifts, has been widely investigated under the context of the average run length (ARL) criterion. Because the shape of the run-length distribution alters with the magnitude of the mean shifts, the ARL is a confusing measure and the use of percentiles of the run-length distribution is considered as more intuitive. This research develops two optimal designs of the VSS $\bar{X}$ chart, by minimizing (i) the median run length (MRL) and (ii) the expected MRL for both deterministic and unknown shift sizes, respectively. The 5th and 95th percentiles are also provided in order to measure the variation in the run-length distribution. Two VSS schemes are considered in this research, i.e. when the (i) small sample size (n_S) or (ii) large sample size (n_L) is predefined for the first subgroup (n_1). The Markov chain approach is adopted to evaluate the performance of these two VSS schemes. The comparative study reveals that improvements in the detection speed are found for these two VSS schemes without increasing the in-control average sample size. For moderate to large mean shifts, the optimal VSS $\bar{X}$ chart with $n_1 = n_L$ significantly outperforms the optimal EWMA $\bar{X}$ chart; while the former is comparable to the latter when $n_1 = n_S$.

ACKNOWLEDGEMENTS

First, I would like to express my sincere gratitude to my main supervisor, Dr. Teoh Wei Lin for her constant guidance on this research. Her encouragement, support and helpful guidance will always be remembered and appreciated. Special thanks for her precious time, constructive suggestions and comments throughout this research and dissertation writing.

My cordial appreciation goes to my co-supervisor, Dr. Yeong Wai Chung for providing me useful and important advice throughout my study. Also, I would like to thank Professor Dr. Michael Khoo Boon Chong from Universiti Sains Malaysia and Professor Dr. Philippe Castagliola from Université de Nantes & IRCCyN UMR CNRS 6597, Carquefou, France for their valuable suggestions and ideas to improve the style and contents of my published journal article of this research in Quality and Reliability Engineering International.

In addition, I would like to express my gratitude to Universiti Tunku Abdul Rahman (UTAR) for giving me this opportunity to carry out my research. I would like to acknowledge UTAR Research Fund (UTARRF), no. IPSR/RMC/UTARRF/2014-C2/T07 and UTAR Fundamental Research Grant Scheme (FRGS), no. FRGS/1/2015/SG04/ UTAR/02/3 for funding my research throughout my study. Also, I owe my deepest gratitude in acknowledging

Malaysian Ministry of Higher Education for awarding me with the MyMaster scholarship.

Finally, I am indebted to my beloved family members for their love, support and encouragement. Their unreserved contributions and patience lead to the successful completion of my study. This research will definitely prepare me with beneficial and precious knowledge in my future career.

APPROVAL SHEET

This dissertation entitled "**OPTIMAL DESIGNS OF THE VARIABLE SAMPLE SIZE (VSS) $\bar{X}$ CHART BASED ON MEDIAN RUN LENGTH AND EXPECTED MEDIAN RUN LENGTH"** was prepared by CHONG JIA KIT and submitted as partial fulfillment of the requirements for the degree of Master of Science at Universiti Tunku Abdul Rahman.

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SUBMISSION OF DISSERTATION

It is hereby certified that **Chong Jia Kit** (ID No: **15ADM01358**) has completed this dissertation entitled "OPTIMAL DESIGNS OF THE VARIABLE SAMPLE SIZE (VSS) $\bar{X}$ CHART BASED ON MEDIAN RUN LENGTH AND EXPECTED MEDIAN RUN LENGTH" under the supervision of Dr. Teoh Wei Lin (Supervisor) and Dr. Yeong Wai Chung (Co-supervisor) from the Department of Physical and Mathematical Science, Faculty of Science, Universiti Tunku Abdul Rahman.

I understand that University will upload softcopy of my dissertation in pdf format into UTAR Institutional Repository, which may be made accessible to UTAR community and public.

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(CHONG JIA KIT)

DECLARATION

I hereby declare that the dissertation is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UTAR or other institutions.

(CHONG JIA KIT)

Date 27th February 2017

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LIST OF NOTATIONS

The notations and abbreviations used in this dissertation are listed as follows:

ARL	Average run length
ASI	Adjusting sampling inspection
ASS	Average sample size
ATS	Average time to signal
CL_{SH}	Center line of the Shewhart $\bar{X}$ chart
$F_{RL}(\ell)$	cdf of the run length
$\Phi(\cdot)$	cdf of the standard normal random variable
L	Chart's coefficient controlling the width of the Shewhart $\bar{X}$ chart's limits
CV	Coefficient of variation
K	Control limit of the VSS $\bar{X}$ chart
cdf	Cumulative distribution function
CUSUM	Cumulative sum
τ	Desired in-control MRL
τ'	Desired in-control EMRL
δ_{opt}	Desired mean shift for which a quick detection is needed
DS	Double sampling
EARL	Expected ARL
EMRL	Expected MRL

EWMA	Exponentially weighted moving average
EQL	Extra quadratic loss
n_1	First subgroup of the VSS $\bar{X}$ chart
FRS	Fixed ratio sampling
$\mathbf{I}$	Identity matrix
ARL_0	In-control ARL
ASS_0	In-control ASS
EMRL_0	In-control EMRL
MRL_0	In-control MRL
SDRL_0	In-control SDRL
μ_0	In-control mean
σ_0	In-control standard deviation
$\sigma_0^{\ 2}$	In-control variance
$\mathbf{q}$	Initial probability vector
I_S	Interval $[-W,\ W]$ of the VSS $\bar{X}$ chart
I_L	Intervals $[-K,\ -W) \cup (W,\ K]$ of the VSS $\bar{X}$ chart
I_{ooc}	Intervals $(-\infty,\ -K) \cup (K,\ \infty)$ of the VSS $\bar{X}$ chart
ILF	Integer linear function
Φ^{-1}	Inverse cdf of the standard normal random variable
$X_{i,j}$	j^{th} observation in the i^{th} subgroup, where $i = 1, 2, \ldots,$ and $j = 1, 2, \ldots, n$
n_L	Large sample size of the VSS $\bar{X}$ chart

$\delta_{\min}$	Lower bound of the mean shift
LCL_{EWMA}	Lower control limit of the EWMA $\bar{X}$ chart
LCL_{SH}	Lower control limit of the Shewhart $\bar{X}$ chart
δ	Magnitude of a standardized mean shift
$\mathbf{Q}$	Matrix of transient probabilities
MRL	Median run length
MTS	Median time to signal
H	Multiplier controlling the width of the EWMA $\bar{X}$ chart's control limit
$N(\mu_0, \sigma_0^{\ 2})$	Normal distribution having in-control mean $\left(\mu_0\right)$ and in-control variance $\left(\sigma_0^{\ 2}\right)$
p	Number of transient states in matrix $\mathbf{P}$
ARL_1	Out-of-control ARL
ASS_1	Out-of-control ASS
EMRL_1	Out-of-control EMRL
MRL_1	Out-of-control MRL
SDRL_1	Out-of-control SDRL
μ_1	Out-of-control mean
PD-Plot	Percentiles of the distribution plot
Z_i	Plotting statistic for the i^{th} sample
pdf	Probability density function
pmf	Probability mass function

$f_{\delta}(\delta)$	pdf of the magnitude of a standardized mean shift
$f_{RL}(\ell)$	pmf of the run length
X	Quality characteristic of the subgroups
RL	Run length
$\bar{X}_i$	Sample mean of the i^{th} subgroup, where $i = 1, 2, ...$
n	Sample size
SS	Side-sensitive synthetic
n_S	Small sample size of the VSS $\bar{X}$ chart
λ	Smoothing constant of the EWMA $\bar{X}$ chart
SDRL	Standard deviation of the run length
$\boldsymbol{\pi}$	Stationary-probability vector of the VSS $\bar{X}$ chart
π_S	Stationary probability associated to sample size n_S
π_L	Stationary probability associated to sample size n_L
π_{OOC}	Stationary probability associated to sample size n_1
SAS	Statistical Analysis System
SPC	Statistical Process Control
α	Type-I error probability
β	Type-II error probability
P	Transition probability matrix
$Q_{i,j}$	Transition probability for entry (i, j) in matrix **Q**
P*	Transformation of the matrix **P** of the VSS $\bar{X}$ chart
R	Transpose of matrix **P*** and the diagonal elements are subtracted with one

$U(\delta_{\min},\ \delta_{\max})$	Uniform distribution over the shift interval $[\delta_{\min},\ \delta_{\max}]$
$\delta_{\max}$	Upper bound of the mean shift
$n_{\max}$	Upper bound of the sample size
UCL_{EWMA}	Upper control limit of the EWMA $\bar{X}$ chart
UCL_{SH}	Upper control limit of the Shewhart $\bar{X}$ chart
VD	Variable dimension
VSS	Variable sample size
VSI	Variable sampling interval
VSSI	Variable sample size and sampling interval
1	Vector with each of its elements equal to unity
VSIFT	VSI with fixed times
VSSCL	VSS and control limit
VSSVD	VSS variable dimension
W	Warning limit of the VSS $\bar{X}$ chart
WLC	Weighted loss function
$\ell_{0.05}$	5th percentile of the run-length distribution
$\ell_{0.95}$	95th percentile of the run-length distribution
ℓ_{γ}	$(100\gamma)^{\text{th}}$ percentile of the run-length distribution, where $0<\gamma<1$

CHAPTER 1

INTRODUCTION

1.1 Statistical Process Control

In this fast moving and highly competitive globalization and industrial era, the quality of products and services becomes ever-increasingly vital in various fields. Quality of products and services is one of the foremost customers' decision making factors. Therefore, quality improvement becomes an important approach in industrial and service sectors. Quality improvement is the reduction of variability in products and processes (Montgomery, 2013). The demand for continuous improvement and better quality tends to remain as an essential feature in the world of business (James and William, 2002).

In order to improve the quality of a process or product, Statistical Process Control (SPC) is applied to ascertain the predictability and stability of a process (James and William, 2002). In real life applications, variations in any process are usually unspecified. Hence, there are no processes which will be in a stable condition forever (Anand, 2003). SPC is the most popular statistical technique to improve quality characteristics of products (Montgomery, 2013). Indisputably, SPC provides an objective means of controlling quality in any transformation processes which include the information transferring, services supplying and artefacts manufacturing (Oakland, 2003). There are wide applications of SPC in manufacturing and service sectors. Some examples include the monitoring of the flow width measurement of wafers in a semiconductor manufacturing company, the improvement of suppliers' process in checking the quality of products produced by machine tools and the monitoring of the costs of processing loan applications in a bank (Montgomery, 2013).

In production processes, no matter how well the process is designed or carefully maintained, there surely exist some amounts of variability. This is because the existence of variations is unstable and unpredictable all the time (James and William, 2002). There are two main types of process variations, i.e. common causes of variation and assignable causes of variation. Anand (2003) stated that common causes of variation are a part of the normal operation, for which the effect is relatively small. Therefore, the process is considered to be statistically in-control. A stable process is operating with a constant distribution, mean and variance over time. On the contrary, assignable causes of variation produce a large variation in the output characteristics (Anand, 2003). This leads to unfavourable process performance and the process is said to be out-of-control. Operator errors, inappropriate controlled or adjusted machines and defective raw materials are the arising sources of this variation (Montgomery, 2013).

The problem-solving tools in SPC are known as the "Magnificent Seven". These seven tools include the Pareto chart, histogram or stem-and-leaf plot, check sheet, cause-and-effect diagram, defect concentration diagram, scatter diagram and control chart (see Besterfield, 2004; Montgomery, 2013). Among all the SPC tools, control charts are the most effective tools for controlling quality in any transformation process (Montgomery, 2013). Control chart is a graphical tool used to determine whether or not a process is in a state of statistical control. The key function of a control chart is to identify the occurrence of assignable causes, so that necessary corrective actions can be taken to eliminate process variations before a large number of defective items are manufactured (James and William, 2002). Control charts can be divided into two types, i.e. variables control charts and attributes control charts. A variables control chart is used to monitor quality characteristics that can be measured in terms of numerical measurements or continuous values (Besterfield, 2004). Examples of variables data are distance, length and weight. An attributes control chart deals with nominal scale of measurement (Montgomery, 2013). Attributes measurements of products or services are classified either as present or absent, acceptable or not acceptable, defective or non-defective, and etcetera (Anand, 2003).

If a practitioner is unable to differentiate between common and assignable causes of variation, this will lead to counterproductive corrective actions and result in a costly production process (Anand, 2003). Thus, Walter A. Shewhart proposed the first Shewhart $\bar{X}$ chart in the year 1924 to distinguish between these two kinds of variability (Montgomery, 2013). The Shewhart $\bar{X}$ chart only considers the current information in the process. This control chart is extensively used in manufacturing and service sectors because of the ease of implementation and interpretation (Montgomery, 2013).

1.2 Problem Statement

In view of its simplicity, the Shewhart $\bar{X}$ chart is widely used to detect large process changes; however, it is insensitive to small and moderate process mean shifts. Therefore, adaptive control charts are proposed to enhance the sensitivity of the basic Shewhart $\bar{X}$ chart. The variable sample size (VSS) $\bar{X}$ chart is one of the adaptive control charts.

To date, the design of the VSS $\bar{X}$ chart in the existing literature is based on the average run length (ARL) performance measure; for example, see Prabhu et al. (1993), Costa (1994), and Castagliola et al. (2012). However, it has been long recognized that sole dependence on the ARL is potentially confusing and is a somewhat peculiar criterion for a control chart (see Bischak and Trietsch, 2007; Chakraborti, 2007; Mei, 2008; Khoo et al., 2012; Teoh et al., 2014). This is due to the fact that when the process is in-control or the process shift is small, (i) the run-length distribution is highly skewed to the right and (ii) the value of the standard deviation of the run length (SDRL) is quite large (Montgomery, 2013).

In addition, Gan (1993) pointed out that interpretation of the ARL corresponding to a highly skewed distribution (for in-control or small shifts) is surely different with that of an almost symmetric distribution (for large shifts), leading to bewildering conclusions. For dependent data, Lai (1995) stated that the ARL metric is conceptually unacceptable and is hard to analyze for more complex detection schemes. Zhou et al. (2012) claimed that the ARL may not be a good performance metric in practice because of the unsatisfactory run-length distribution and excessive variations of the run length.

Therefore, there is a need for an alternative performance measure to be adopted to design a control chart. Since the associated run-length distribution has different skewness for different shift sizes and the run-length random variable is defined by a positive integer value, the percentiles of the run-length distribution are more appealing performance indexes compared to the ARL (see Palm, 1990; Khoo, 2004; Radson and Boyd, 2005; Zhou et al., 2012). The percentiles of the run-length distribution, which are sufficient to summarize the run-length behavior, provide extra information and comprehensive understanding of a control chart (Zhou et al., 2012).

Among the percentiles of the run-length distribution, Chakraborti (2007) indicated that it is more practical to design a control chart by using the 50th percentile, i.e the median run length (MRL); while the 25th and 75th or 5th and 95th percentiles are supplemented with the MRL in order to measure the spread and skewness of the run-length distribution. Here, 5th and 95th percentiles are the tails of the run-length distribution; whereas 25th and 75th percentiles are the first and third quartiles, respectively.

Maravelakis et al. (2005) also asserted that the MRL is a reliable indicator of a chart's performance as it is less influenced by the skewness of the run-length distribution. This is because in a right-skewed distribution, the MRL is a better and fair measure of central tendency as the value of MRL is smaller than that of the ARL. In view of these advantages, Shmueli and Cohen (2003), Golosnoy and Schmid (2007), Capizzi and Masarotto (2008), Khoo et al. (2012), and Teoh et al. (2014, 2015, 2016), advocated the use of MRL as an alternative measure to design and evaluate a control chart.

Owing to the beneficial properties of the MRL measure, this dissertation proposes the optimal designs of the VSS $\bar{X}$ chart based on the MRL and expected MRL (EMRL). Two optimal designs are developed for the VSS $\bar{X}$ chart, i.e. by minimizing (i) the out-of-control MRL (MRL_1) and (ii) the expected MRL_1 (EMRL_1), for known and unknown shift sizes, respectively.

In practice, it is common to have insufficient historical data; thus, practitioners usually do not know in advance the entity of the next shift size (Castagliola et al., 2011). Also, the shift size is unidentifiable and changes according to some undetermined stochastic models. To circumvent this problem, the expected ARL (EARL) is widely employed by many researchers as the performance metric to study the unknown shift size situation. To the best of the authors' knowledge, none of the existing literature explores the application of the EMRL as the performance measure to investigate the VSS $\bar{X}$ chart under the unknown shift size condition. Therefore, the optimal design for the unknown shift size, i.e. minimizing the EMRL_1, is vitally viewed and proposed in this research.

Also, the vast majority of existing researches on the VSS chart only consider one VSS scheme, i.e. when the small sample size (n_S) is predefined for the first subgroup (n_1). For the sake of completeness, this dissertation accounts for two VSS schemes, i.e. when (i) $n_1 = n_S$ and (ii) $n_1 = n_L$. Here, n_L is the large sample size and $n_S < n_L$.

1.3 Objectives of the Dissertation

The objectives of this dissertation are as follows:

i) To show that the MRL is an intuitive and credible measure compared to the ARL.

ii) To develop a new optimization algorithm for the VSS $\bar{X}$ charts when $n_1 = n_S$ and $n_1 = n_L$, by minimizing the MRL_1 for the known shift size.

iii) To develop a new optimization algorithm for the VSS $\bar{X}$ charts when $n_1 = n_S$ and $n_1 = n_L$, by minimizing the EMRL_1 for the unknown shift size.

iv) To compute the optimal charts' parameters of the MRL- and EMRL-based VSS $\bar{X}$ charts when $n_1 = n_S$ and $n_1 = n_L$.

1.4 Organization of the Dissertation

Chapter 1 introduces an overview of SPC. This chapter highlights the problem statement and objectives of this dissertation. Chapter 2 reviews the relevant literature on the development of the VSS-type control charts. In addition, some performance measures of a control chart are briefly introduced in this chapter. Also, this chapter discusses relevant literature review regarding the control charts designing based on the percentiles of the run-length distribution and MRL.

Chapter 3 outlines the operation of the VSS $\bar{X}$ chart's procedures. The run-length properties of the Shewhart $\bar{X}$, VSS $\bar{X}$ and exponentially weighted moving average (EWMA) $\bar{X}$ charts are also studied in this chapter. In addition, this chapter proposes two new optimal designs of the VSS $\bar{X}$ charts, i.e. by minimizing the MRL_1 and EMRL_1 for the known and unknown shift sizes, respectively.

Chapter 4 studies the performance of the VSS $\bar{X}$ chart in terms of the ARL, SDRL and percentiles of the run-length distribution. The optimal charts' parameters of the MRL- and EMRL-based VSS $\bar{X}$ charts when $n_1 = n_S$ and $n_1 = n_L$ are also tabulated in this chapter. Comparative studies of the performance of the proposed optimal MRL- and EMRL-based VSS $\bar{X}$ charts with that of the Shewhart $\bar{X}$ and optimal EWMA $\bar{X}$ charts are provided in this chapter. Furthermore, the implementation of the proposed optimal VSS $\bar{X}$ charts is illustrated with an example in this chapter. Last, some contributions of this dissertation and some recommendations for future research are summarized in Chapter 5.

The developed computer programs for the optimal VSS $\bar{X}$, Shewhart $\bar{X}$ and optimal EWMA $\bar{X}$ charts are provided in Appendices A to C, respectively. By using the ScicosLab software, these written optimization programs are used to compute the charts' parameters and the run-length properties (ARL, SDRL, average sample size (ASS), MRL or percentiles of the run-length distribution) of the optimal VSS $\bar{X}$, Shewhart $\bar{X}$ and optimal EWMA $\bar{X}$ charts. With the aid of the Statistical Analysis System (SAS), the written simulation programs are also provided in Appendices A to C in order to verify all the results obtained in this dissertation. Last, some additional results are tabulated in Appendix D.

1.5 Flow Chart of Research Methodology

Phase 1
Literature Review and Formulation of Theoretical Framework

Literature review on related VSS-type charts, various performance measures for a control chart and the MRL-based control charts.

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Develop the MRL and EMRL of the VSS $\bar{X}$ chart by means of the Markov Chain approach.

Phase 2
Optimal Designs and Framework Development

Develop two new optimal-designs algorithms of the VSS $\bar{X}$ charts when $n_1 = n_S$ and $n_1 = n_L$, by minimizing the (i) MRL_1 and (ii) EMRL_1 for both known and unknown shift sizes.

↓

Develop optimization programs with ScicosLab software to compute the optimal charts' parameters, together with their corresponding ($\ell_{0.05}$, MRL_1, $\ell_{0.95}$) or EMRL_1 of the optimal VSS $\bar{X}$ chart.

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Verify the accuracy of the computed results with that of the simulation programs written in SAS.

Phase 3
Control Charts' Performance Analyzes and Comparative Studies

Compare the performance of the proposed two optimal MRL- and EMRL-based VSS $\bar{X}$ charts with that of the Shewhart $\bar{X}$ and optimal EWMA $\bar{X}$ charts.

Phase 4
Illustrative Example

Implement the proposed optimal VSS $\bar{X}$ charts in a yoghurt manufacturing firm.

Phase 5
Dissemination of Findings

Publish research findings in an ISI-index journal, i.e. *Quality Reliability Engineering International*.

↓

Disseminate findings in dissertation.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

Relevant literature regarding this research is studied in this chapter. Since the adaptive control charts outperform the Shewhart $\bar{X}$ chart, there is a vast literature evolving around the development of various adaptive control charts. An in-depth literature review on the development of adaptive control charts was discussed by Tagaras (1998), Jensen et al. (2008), and Psarakis (2015).

The design of an adaptive control chart depends on the determination of three charting parameters, i.e. the sampling interval, sample size and control limit coefficient. During the implementation of the adaptive control chart, at least one of these three parameters will vary according to the value of the previous sample statistic. Since the adaptive control chart adopts more information compared to the static control chart, the flexibility feature of this adaptive chart provides a more powerful statistical and economical process-monitoring method (see Tagaras, 1998). The VSS chart is one of the adaptive control charts. A thorough literature review on the development of the VSS-type charts is discussed in Section 2.2.

Some performance measures are required to investigate the superiority of a control chart over other control charts. Therefore, in Section 2.3, I briefly discuss the performance or design criteria of a control chart, i.e. the ARL, SDRL and percentiles of the run-length distribution.

As discussed in the problem statement section of Chapter 1, lots of studies focus on the development of control charts designing based on the ARL criterion. It is well known that the ARL is traditionally used as a measure of the control chart's performance. However, the shape of the run-length distribution is highly skewed to the right when the process shift is in-control or small, to approximately symmetric when the process shift is large. Therefore, interpretation based on the ARL alone is misleading and complicated. It is high time that alternative measures of the control charts' performance need to be suggested. Some related studies regarding the development of the control charts' performance measures, i.e. the MRL and percentiles of the run-length distribution, are discussed in Section 2.4.

2.2 Development of the Variable-Sample-Size-Type Control Charts

The VSS scheme allows the sample size to change at difference levels; while the sampling interval and control-limit coefficient remain constant (Prabhu et al., 1993). A two-sample-size VSS chart consists of three regions, divided by the control and warning limits. These three regions are the central, warning and out-of-control regions. If the current sample statistic falls in the central region, a relaxed sampling is arranged for the next inspection epoch by taking a small sample size (n_S). If the current sample statistic falls in the warning region, a large sample size (n_L) should be selected in the next sampling time to tighten the control.

Prabhu et al. (1993) and Costa (1994) were the first to propose the VSS $\bar{X}$ chart. They studied the properties of this chart using the Markov chain approach. Prabhu et al. (1993) claimed that the VSS $\bar{X}$ chart has faster detection speed and lower costs associated with the in-control sampling, compared to the Shewhart $\bar{X}$ chart, when both control charts are applied in a real-industrial practice. Furthermore, the VSS $\bar{X}$ chart shows improvement in detecting a certain range of shifts compared to the Shewhart $\bar{X}$ chart with supplementary run rules, variable-sampling-interval (VSI) $\bar{X}$, cumulative-sum (CUSUM) and EWMA charts (see Costa, 1994).

From the economic point of view, Park and Reynolds (1994) developed an economic-design model of the VSS $\bar{X}$ chart by minimizing the expected cost per hour. They found that the VSS $\bar{X}$ chart provides greater cost savings compared to the Shewhart $\bar{X}$ chart. Reynolds (1996) manifested that the VSS $\bar{X}$ chart has a notable gain in statistical efficiency compared to the VSI with fixed times (VSIFT) $\bar{X}$ chart for detecting small mean shifts.

Zimmer et al. (1998) developed a three-state VSS scheme to improve the efficiency of the standard Shewhart $\bar{X}$ and the two-state VSS $\bar{X}$ charts. Since the additional states on the VSS scheme do not provide a great improvement, they claimed that the two-state VSS $\bar{X}$ chart proposed by Prabhu et al. (1993) is more practical and easy to be implemented in many applications. Zimmer et al. (2000) further proposed the three-state variable-sample-size-and-sampling-interval (VSSI) $\bar{X}$ and four-state VSS $\bar{X}$ charts. They concluded that adding more states on adaptive control scheme show significant improvement compared to the standard Shewhart control chart; however, the performance of the three- and four-state adaptive control schemes is slightly better than their corresponding two-state adaptive control scheme. Moreover, it is very complicated to manage more than two-state VSS scheme in monitoring a process. Therefore, the easiest method to improve the performance of the VSS scheme is to increase its average sample size (Zimmer et al., 2000).

The VSS scheme is usually designed under the assumption of normality observations. However, in some production processes, this assumption may not be true. Accordingly, Lin and Chou (2005) proposed the VSS and control limit (VSSCL) $\bar{X}$ chart under Burr distribution. For both normal and non-normal underlying distributions, the comparative studies reveal that the VSSCL $\bar{X}$ chart outperforms the VSS $\bar{X}$ and Shewhart $\bar{X}$ charts in detecting small and moderate mean shifts; while the EWMA chart is still the best for monitoring small process mean shifts.

Due to the attractive features of the VSS scheme, Castagliola et al. (2012) investigated the optimal VSS $\bar{X}$ chart with estimated process parameters. They provided new optimal chart's parameters for the VSS $\bar{X}$

chart with estimated process parameters, specially designed for practical numbers of Phase-I samples and sample sizes. In real-life applications, measurement errors may occur in any process. Hu et al. (2015) examined the impact of the VSS $\bar{X}$ chart with measurement errors by using a linearly-covariate-error model. They found that the performance of the VSS $\bar{X}$ chart is dramatically affected by measurement errors. Moreover, they exhibited that the run-length performance of the VSS $\bar{X}$ chart becomes worst when the measurement errors increase. They also suggested taking multiple measurements for each sample in order to compensate for the negative effects of measurement errors.

Recently, Costa and Machado (2016) developed a $\bar{X}$ -type control chart by incorporating both the side-sensitive synthetic (SS) rule and VSS scheme, i.e. the SSVSS $\bar{X}$ chart. They found that the SSVSS $\bar{X}$ chart surpasses the standard Shewhart $\bar{X}$, VSS $\bar{X}$ and synthetic VSS $\bar{X}$ charts, in terms of extra quadratic loss (EQL). On the contrary, the VSS EWMA chart is better than the SSVSS $\bar{X}$ chart in detecting small shifts. Due to the low rates of false alarms and inspected items, Costa and Machado (2016) highly recommended the use of SSVSS $\bar{X}$ chart in process monitoring.

Concerning the adaptive CUSUM-type control charts, Annadi et al. (1995) introduced the VSS CUSUM chart with and without fast initial response. They showed that there is an improvement for the VSS CUSUM chart compared to the CUSUM chart in detecting small mean shifts. Zhang and Wu (2007) presented a weighted loss function (WLC) CUSUM chart incorporating the VSS feature, for simultaneous monitoring of both the mean and variance shifts.

For the VSS EWMA dispersion-type chart, the optimal VSS S^2-EWMA chart designing based on the average time to signal (ATS) was discussed by Castagliola et al. (2008). They concluded that the optimal VSS S^2-EWMA chart outperforms the static S^2-EWMA chart for all levels of process variability and the VSI S^2-EWMA chart for certain process operating conditions. For the VSS EWMA $\bar{X}$ -type chart, a new VSS EWMA chart proposed by Amiri et al. (2014) used an integer linear function (ILF) to determine its sample size. The proposed chart is called the VSSILF EWMA chart. They claimed that the VSSILF EWMA chart surpasses the EWMA and the traditional VSS EWMA charts. Zhang and Song (2014) suggested the VSS EWMA median chart and compared their suggested chart with the EWMA $\bar{X}$ and VSS EWMA $\bar{X}$ charts. They found that their proposed chart is the best among all the three control charts. Kazemzadeh et al. (2016) applied the VSS feature to the combination of three univariate EWMA charts and a multivariate EWMA chart for monitoring the simple linear profiles. They found that employing the VSS feature in a control chart effectively improves the detection speed of a control chart.

The attribute-type control charts with VSS feature that are found in existing literature, include the *np* and *c* charts studied by Epprecht and Costa (2001). They investigated the properties of the VSS *np* and *c* charts. Also, they compared the performance of the proposed charts with that of the fixed-sample-parameter *np* and *c* charts. Kooli and Limam (2011) used the similar approach as discussed by Park and Reynolds (1994) to formulate an economic design for the VSS *np* chart. Based on the sensitivity analysis, they pointed out that the proposed VSS *np* chart is more economically preferable than the static chart. By means of genetic algorithm and Taguchi experiments, Zhou and Lian (2011) developed a new VSS *np* chart with adjusting sampling inspection (ASI), i.e. ASI *np* chart. The sample size is determined according to three different levels of sampling inspection, i.e. normal, reduced and tightened inspections. The ASI *np* chart outperforms the classical *np* chart for all levels of shifts, particularly in the processes with high quality and small shifts (Zhou and Lian, 2011).

When the production horizon is finite, Castagliola et al. (2013) discussed the VSS *t* chart for monitoring short production runs. The VSS *t* chart is statistically superior to the *t* chart for a broad domain of shift sizes. They also claimed that in a short production runs, lack of historical data is a common situation. Therefore, they optimized the VSS *t* chart for a selected range of shifts modeled with a uniform distribution. Castagliola et al. (2015) recently monitored the three-parameter logarithmic transformation of the coefficient of variation (CV) by employing the VSS chart. The results show the potential benefits of the VSS CV chart over the Shewhart CV, VSI CV and synthetic CV charts for detecting certain levels of shifts in the CV. Amdouni et al. (2015) extended the work of Castagliola et al. (2015) by implementing the VSS chart to monitor the CV in short production runs. They evaluated the chart's performance in terms of the ASS and truncated average run length

(TARL). Yeong et al. (2015) applied the VSS chart to monitor the CV without involving transformed statistics. This proposed chart is benefit to non-statistically trained practitioners as the transformed statistics are not required during implementation. This chart is evaluated by minimizing the out-of-control ARL (ARL_1) and out-of-control ASS (ASS_1). They indicated that less observation is needed for the VSS chart to detect a shift in the CV when the sample CV is monitored directly compared to that of the transformed CV.

For multivariate process monitoring, Faraz et al. (2010) and Seif et al. (2011) investigated the economic and economic statistical design of the VSS T^2 and VSSCL T^2 charts, respectively, by using the model proposed by Lorenzen and Vance (1986). Faraz et al. (2010) showed that the VSS T^2 chart is superior to the fixed-ratio-sampling (FRS) chart and comparative to the multivariate EWMA chart. Seif et al. (2011) demonstrated that the VSSCL T^2 chart leads to improvement compared to the FRS T^2 and VSS T^2 charts. In order to enhance the performance of the variable dimension (VD) T^2 chart for detecting small process shifts, Aparisi et al. (2014) developed the VSS variable dimension (VSSVD) T^2 chart. They showed that the detection speed of the VSSVD T^2 chart in detecting small process shifts is reduced by 30% to 80%; whereas the performance of the VD T^2 and VSSVD T^2 charts are both equivalently sensitive towards large process shifts.

2.3 Performance Measures of a Control Chart

In order to analyze the performance of a control chart, some performance measures are required. The chart's performance is usually measured in terms of the run length. The run length is a random variable which is defined as the number of sample statistics plotted on a control chart until the first out-of-control signal is detected. In the following sub-sections, various performance measures are briefly described. These include the ARL, SDRL and percentiles of the run-length distribution.

2.3.1 The Average Run Length (ARL)

Traditionally, ARL is widely used to evaluate a control chart's performance. The ARL is the expected value of the first run length. In other words, ARL is the average number of samples (subgroups) that must be plotted until the chart produces the first out-of-control signal (Montgomery, 2013). The two commonly used ARLs are the in-control $\text{ARL}\ (\text{ARL}_0)$ and the out-of-control $\text{ARL}\ (\text{ARL}_1)$.

Jensen et al. (2008) indicated that a balance between the ARL_0 and ARL_1 is very important. If the ARL_0 is too small, the control charts will produce smaller ARL_1 compared to those having large ARL_0. However, the control charts having too small ARL_0 will produce many false alarms and subsequently reduce practitioners' confidence in the monitoring system. On the other hand, if the ARL_0 is too large, though the control charts produce less false alarms, the detection ability of the control charts on process shifts will be reduced.

In practice, it is recommended to have a large value of ARL_0 in order to avoid too frequent false-alarm rates. When the process is out-of-control or a shift occurs, it is desirable to have a small value of ARL_1 (Ryan, 2000). Therefore, a quick signal indication can be obtained and assignable cause(s) can be removed as soon as possible. When all the competing control charts having the same ARL_0 value, a control chart having the lowest ARL_1 value is the best among all the competitors.

2.3.2 The Standard Deviation of the Run Length (SDRL)

When ARL is used as the performance measure, the SDRL is a favorable supplemental value to determine the spread, variability and dispersion of the run-length distribution (Jensen et al., 2008; Shu et al., 2013). The in-control and out-of-control SDRLs are denoted as SDRL_0 and SDRL_1, respectively. When compared with different control charts, the chart having the smallest SDRL_1 value is preferable. Small value of SDRL is able to ensure a consistent behavior of a control chart (Jensen et al., 2008).

2.3.3 The Percentiles of the Run-length Distribution

The percentiles of the run-length distribution give the probability of getting a signal at or before a certain number of samples. For example, if the 20th percentile of the run-length distribution is equal to 50, there is a 0.20 probability that the run length of a chart is less than 50. In addition, the percentiles of the run-length distribution are random variables with positive integers. Palm (1990) claimed that the percentiles of the run-length distribution provide extra information regarding the spread and expected behaviour of the run-length distribution.

It may be helpful to evaluate the lower percentiles of the run-length distribution, i.e. the 5th, 10th and 20th percentiles. These lower percentiles provide practical guidance regarding an analysis of early false alarms when the process is in-control (Teoh et al., 2014). However, the computation of the higher percentiles (i.e. 80th, 90th and 95th percentiles) of the run-length distribution also gives some crucial information to quality practitioners (Khoo et al., 2011).

Among all the percentiles of the run-length distribution, the 50th percentile of the run-length distribution, i.e. the MRL, is more practical to be used as the performance or design criterion of a control chart (Chakraborti, 2007). The MRL represents half of the time. Also, the MRL provides more information than the ARL. In a right-skewed distribution, the median is less than the mean. Therefore, the MRL is a fair representation of central tendency

(Chakraborti, 2007). The MRL_0 and MRL_1 represent the in-control and out-of-control MRLs, respectively. When MRL_0 is 250, practitioners will justify that in 50% of the time, a control chart will certainly detect a false alarm at the 250^{th} sample. This will increase practitioners' confidence when they encounter a few short runs without assignable causes. When MRL_0 is the same for all the control charts, a control chart outperforms its competitors if the chart has the smallest MRL_1 value among all its competitors.

If MRL is the choice of design measure of a control chart, the spread and dispersion of the run-length distribution can be measured through the difference between the 25^{th} and 75^{th} (or 5^{th} and 95^{th}) percentiles of the run-length distribution (Chakraborti, 2007). Note that the difference between the 25^{th} and 75^{th} percentiles of the run-length distribution is the interquartile range. The 25^{th} and 75^{th} percentiles represent the middle half of the run-length distribution; while the 5^{th} and 95^{th} percentiles are the extremes of the run-length distributions (Radson and Boyd, 2005). A control chart having the smallest difference between the 25^{th} and 75^{th} (or 5^{th} and 95^{th}) percentiles of the run-length distribution is the best among all its competitors. The smaller the difference between the 25^{th} and 75^{th} (or 5^{th} and 95^{th}) percentiles of the run-length distribution, the smaller the variation and spread of the run-length distribution.

2.4 Development of the Median Run Length (MRL) and Percentiles of the Run-length Distribution as the Control Charts' Performance Measures

Traditionally, the ARL is used as the design and performance criterion to evaluate a control chart. The design of a control chart solely based on the ARL measure was highly criticized by many researchers. For instance, Gan (1993, 1994) showed that the ARL is affected by the skewness of the run-length distribution. For highly right-skewed run-length distribution, the ARL is greater than the MRL; while the ARL is almost the same as the MRL for approximately symmetric run-length distribution. Bischak and Trietsch (2007) argued that the ARL is a peculiar measure when the process parameters of a control chart are estimated. This is because for a control chart with estimated process parameters, the unconditional expected run lengths no longer follow a geometric distribution. Thus, the first run length and any other run length will no longer have the same expectation, leading to ambiguous ARL measure. Capizzi and Masarotto (2008) claimed that the ARL is not a robust metric when the run-length distribution exhibits heavy-tailed behaviour. Furthermore, Mei (2008) discussed some limitations of the ARL measure in some cases of dependent observations.

In order to overcome the weakness of the ARL as the sole measurement of a control chart's performance, there are a rich literature focusing on the development of control charts designing based on the MRL and percentiles of the run-length distribution. For the $\bar{X}$ -type charts, Palm (1990) provided tables of percentiles of the run-length distribution for the Shewhart chart with supplementary run rules. Shmueli and Cohen (2003) evaluated the performance of the Shewhart chart with runs and scans rules in terms of the ARL, 5^{th}, 25^{th}, 50^{th} (MRL), 75^{th} and 95^{th} percentiles of the run-length distribution. Khoo (2004) introduced two performance measures for the Shewhart $\bar{X}$ chart, i.e. the percentage points of the number of individual units sampled and the percentage points of the time to signal. Charkraborti (2007) extended Khoo's (2004) research to investigate the impact of the Shewhart $\bar{X}$ chart with estimated parameters, in terms of the percentiles of the run-length distribution.

By using the Markov chain approach, Low et al. (2012) studied the Shewhart $\bar{X}$ chart incorporating the revised *m*-of-*k* runs rule based on MRL. The overall MRL performance shows that the revised runs rule significantly outperforms the Shewhart $\bar{X}$ chart for detecting small and moderate mean shifts, while maintaining similar sensitivity for large shifts. Khoo et al. (2012) proposed an optimization procedure for the synthetic $\bar{X}$ chart by minimizing the MRL_1, subject to a desired MRL_0 value. Both zero- and steady-state cases are considered in their study. Here, zero-state run length represents the run length of a control scheme initialized at the target value. Meanwhile, the steady-state run length is defined as the run length of a control chart evaluated after the control statistic has reached the steady-state condition (Lucas and Saccucci, 1990). Khoo et al. (2012) also provided some useful applications for the MRL performance measure.

Teoh et al. (2013) proposed two optimal-design models for the MRL-based double sampling (DS) $\bar{X}$ chart, i.e. by minimizing the (i) ASS_0 and (ii) $\text{ASS}_0+\text{ASS}_1$. Here, ASS_0 and ASS_1 represent the in-control and out-of-control ASS, respectively. The proposed optimization models lead to a substantial reduction of sampling and inspection costs. When the process parameters are estimated, Teoh et al. (2014) developed new theoretical and optimization methods for the DS $\bar{X}$ chart based on MRL. With the implementation of the proposed optimization method, they found that the MRL-based DS $\bar{X}$ chart with estimated process parameters provides easier chart's interpretation and lower false alarm rate compared to its ARL-based counterpart. Teoh et al. (2015) further suggested the optimization procedure of the DS $\bar{X}$ chart with estimated process parameters by minimizing the ASS_0, subject to the desired MRL_0 and MRL_1 values. Teoh et al. (2016) argued that the ARL is an ambiguous representation of the run length, especially for the case with estimated parameters. Thus, they provided a new design and charting parameters for the MRL-based Shewhart $\bar{X}$ chart with estimated process parameters. By taking practical number of Phase-I samples and sample sizes, the newly proposed MRL-based Shewhart $\bar{X}$ chart with estimated process parameters has similar sensitivity with its known-process-parameter counterpart.

This paragraph focusses on the EWMA- and CUSUM-type control charts. Gan (1992) derived the exact run-length distribution for the one-sided CUSUM charts when the observations are exponentially distributed. He concluded that a comprehensive understanding of the CUSUM charts can be achieved through the knowledge of run-length distribution. Because the MRL is a more credible measure compared to the ARL, Gan (1993, 1994) optimally designed the EWMA and CUSUM charts, respectively, based on MRL. Radson and Boyd (2005) proposed a new graphical representation, i.e. the percentiles of the distribution plot (PD-Plot) for the Shewhart $\bar{X}$ and EWMA $\bar{X}$ charts. This plot displays multiple pictures of data about the run-length probability distribution of a control chart. The advantage of the PD-Plots is to allow easy comparison of several run-length distributions. Chin and Khoo (2012) extended the idea of Zhang et al. (2009) to investigate the performance comparison of the optimal EWMA $\bar{X}$ and optimal EWMA t charts based on the MRL criterion. You et al. (2016) examined the optimal designs of the EWMA $\bar{X}$ chart with estimated process parameters based on MRL and EMRL. By using a reasonable number of Phase-I samples and sample sizes, they showed that the performance of the optimal MRL- and EMRL-based EWMA $\bar{X}$ chart with estimated process parameters is close to that of its known-process-parameter counterpart.

Regarding the multivariate-type control charts, Khoo and Quah (2002) comprehensively studied the percentiles of the run-length distribution for the multivariate CUSUM control charts. By means of the Markov chain approach, Lee and Khoo (2006a, b) proposed the optimal designs of the multivariate EWMA and multivariate CUSUM control charts, respectively, based on ARL and MRL. Khoo et al. (2011) developed an optimal design of the multivariate synthetic T^2 chart for monitoring the process mean vector. The proposed optimization algorithm involves minimizing the MRL_1, under the zero- and steady-state cases. Recently, Lee and Khoo (2017) proposed an optimal design of the multivariate synthetic generalized sample variance |S| (i.e. synthetic |S|) control chart by minimizing the MRL_1, for both zero- and steady-state cases. The comparative studies reveal that the multivariate synthetic |S| chart outperforms the standard |S| chart for detecting shifts in the covariance matrix.

CHAPTER 3

RESEARCH METHODOLOGY

3.1 Introduction

In this chapter, the operation of the VSS $\bar{X}$ chart is discussed in Section 3.2. Section 3.3 provides the run-length properties (i.e. the ARL, SDRL, MRL, ASS and percentiles of the run-length distribution) of the Shewhart $\bar{X}$, VSS $\bar{X}$ and EWMA $\bar{X}$ charts. As discussed in the problem-statement section of Chapter 1, the MRL is more readily comprehensible than the ARL. Therefore, the MRL is used as the alternative measure to optimally design the VSS $\bar{X}$ charts with $n_1 = n_S$ and $n_1 = n_L$ in this research. Two new optimal-design procedures for the MRL- and EMRL-based VSS $\bar{X}$ charts for both known and unknown shift sizes, respectively, are proposed in this research. These two optimal-design procedures are presented in Section 3.4.

3.2 The Operation of the Variable Sample Size (VSS) $\bar{X}$ Chart

Without loss of generality, let us assume that a quality characteristic X follows an independent and identically distributed normal distribution, i.e. $X \sim N(\mu_0, \sigma_0^{\ 2})$, where μ_0 and σ_0 are the in-control mean and standard deviation, respectively. Let $\bar{X}_i$ be the sample mean of the i^{th} subgroup, i.e.

$$\bar{X}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} X_{i,j}, \tag{3.1}$$

where $i = 1, 2, ..., j = 1, 2, ..., n_i$ and $n_i \in \{n_S, n_L\}$. The plotting statistic for the i^{th} sample is computed as follows:

$$Z_i = \frac{(\bar{X}_i - \mu_0)\sqrt{n_i}}{\sigma_0}, \tag{3.2}$$

where $Z_i \sim N\left(\delta\sqrt{n_i},\ 1\right)$ (Castagliola et al., 2012). Here, $\delta = \frac{|\mu_1 - \mu_0|}{\sigma_0}$ denotes the magnitude of mean shift in multiples of standard deviation units, where μ_1 is the out-of-control mean. When $\delta = 0$, i.e. the process is in-control, Z_i follows a standard normal, $N(0,\ 1)$ distribution. Figure 3.1 graphically displays the VSS $\bar{X}$ chart's operation. Here, K ($\geq W$) and W (> 0) are the control and warning limits, respectively. The VSS $\bar{X}$ chart is divided into three regions, i.e. the central region $\left(I_S \in [-W,\ W]\right)$, the warning region $\left(I_L \in \{[-K,\ W) \cup (W,\ K]\}\right)$ and the out-of-control region $\left(I_{\text{ooc}} \in \{(-\infty,\ -K) \cup (K,\ \infty)\}\right)$. The procedure of implementing the VSS $\bar{X}$ chart is described as follows:

1. Collect a sample, each having n_i observations.
2. Calculate the i^{th} sample mean and sample statistic as in Equations (3.1) and (3.2), respectively.
3. If $Z_i \in I_S$, the process is deemed as in-control; thus, take a small sample size for the next sample, i.e. $n_{i+1} = n_S$.

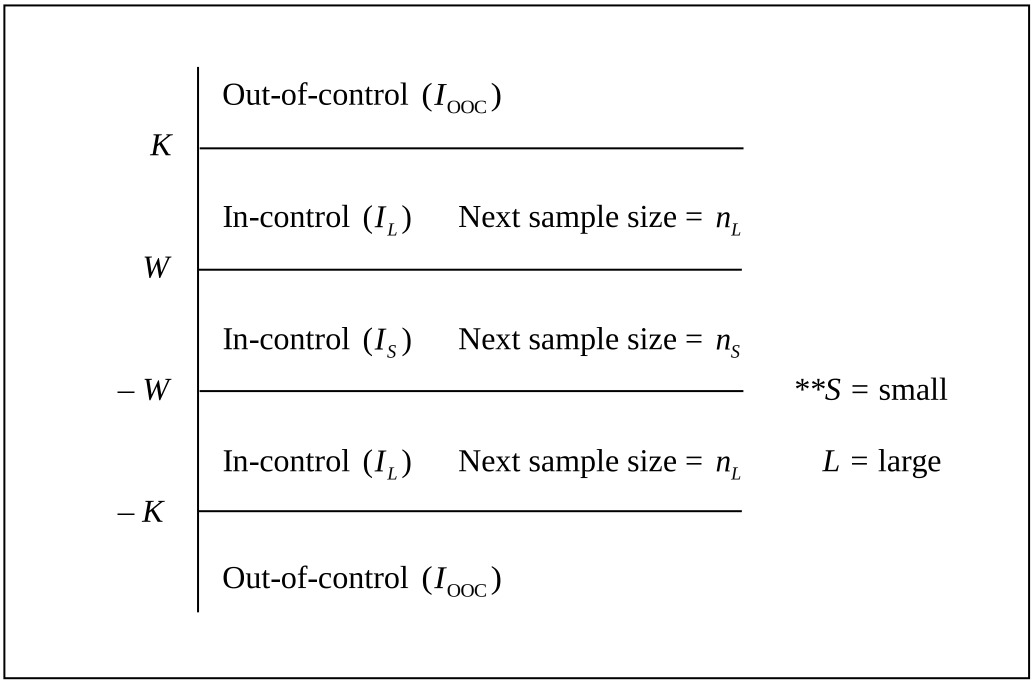

Figure 3.1: A graphical view of the VSS $\bar{X}$ chart's operation

4. If $Z_i \in I_L$, the process is still considered as in-control, but it has a higher chance to shift to an out-of-control state; thus, take a large sample size for the next sample, i.e. $n_{i+1} = n_L$, in order to tighten the control.
5. If $Z_i \in I_{\text{OOC}}$, we conclude that the process is out-of-control. Take the necessary corrective actions in order to remove the assignable cause(s).

3.3 The Run-length Properties of the Univariate Control Charts

The performances of the two optimal VSS $\bar{X}$ charts with $n_1 = n_S$ and $n_1 = n_L$ are compared to that of the Shewhart $\bar{X}$ and optimal EWMA $\bar{X}$ charts in Chapter 4. Therefore, the run-length properties, i.e. the ARL, SDRL, MRL, ASS and percentiles of the run-length distribution for the Shewhart $\bar{X}$, VSS $\bar{X}$ and EWMA $\bar{X}$ charts are delineated in this section.

3.3.1 The Shewhart $\bar{X}$ Chart

Assume that the quality characteristic $X_{i,j}$ be the j^{th} observation in the i^{th} subgroup, where $i =$ 1, 2, ... and $j =$ 1, 2, ..., n. For the Shewhart $\bar{X}$ chart, the upper (UCL_{SH}) and lower (LCL_{SH}) control limits are computed as (Montgomery, 2013)

$$UCL_{\text{SH}} / LCL_{\text{SH}} = \mu_0 \pm L \frac{\sigma_0}{\sqrt{n}}, \tag{3.3}$$

with the center line $CL_{\text{SH}} = \mu_0$ and L is the chart's coefficient controlling the width of the Shewhart $\bar{X}$ chart's limits.

Let α and β denote the probabilities of Type-I and Type-II errors, respectively. According to Montgomery (2013), the β-risk or the probability of not detecting a shift on the first subsequence sample can be expressed as

$$\beta = \Phi\left(L - \delta\sqrt{n}\right) - \Phi\left(-L - \delta\sqrt{n}\right), \tag{3.4}$$

where $\Phi(\cdot)$ is the cumulative distribution function (cdf) of the standard normal random variable. When $\delta = 0$ is substituted in Equation (3.4), the α-risk or false alarm probability can be obtained as follows:

$$\begin{aligned}
\alpha &= 1 - \beta \\
&= 1 - \left[\Phi(L) - \Phi(-L)\right] \\
&= 1 - \Phi(L) + \left[1 - \Phi(L)\right] \\
&= 2 - 2\Phi(L) \\
&= 2\left[1 - \Phi(L)\right].
\end{aligned} \tag{3.5}$$

It is well known that if the plotted statistics are independent and the chart's limits are known constants, the run lengths of the Shewhart $\bar{X}$ chart follow a geometric distribution (Montgomery, 2013). Accordingly, the probability mass function (pmf), $f_{RL}(\ell)$ and cdf, $F_{RL}(\ell)$ of the run length (*RL*) for the Shewhart $\bar{X}$ chart are defined as

$$f_{RL}(\ell) = P(RL = \ell) = \beta^{\ell-1}(1-\beta) \qquad (3.6)$$

and

$$F_{RL}(\ell) = P(RL \le \ell) = 1-\beta^{\ell}, \qquad (3.7)$$

respectively, where $\ell \in \{1, 2, 3, ...\}$.

The $(100\gamma)^{\text{th}}$ percentile of the run-length distribution can be determined as the value ℓ_γ such that (Gan, 1993)

$$P(RL \le \ell_\gamma - 1) \le \gamma \text{ and } P(RL \le \ell_\gamma) > \gamma, \qquad (3.8)$$

where γ is in the range $0 < \gamma < 1.$ The percentiles of the run-length distribution of the Shewhart $\bar{X}$ chart can be obtained by using both Equations (3.7) and (3.8). Then the MRL can be easily obtained by setting $\gamma = 0.5$ in Equation (3.8), i.e.

$$P(RL \le \text{MRL} - 1) \le 0.5 \text{ and } P(RL \le \text{MRL}) > 0.5. \qquad (3.9)$$

The computation of the ARL_0 and SDRL_0 of the Shewhart $\bar{X}$ chart are obtained as follows (Montgomery, 2013):

$$\mathrm{ARL}_0 = \frac{1}{\alpha} \tag{3.10}$$

and

$$\mathrm{SDRL}_0 = \frac{\sqrt{1-\alpha}}{\alpha}, \tag{3.11}$$

respectively. The ARL_1 and SDRL_1 of the Shewhart $\bar{X}$ chart are calculated as

$$\mathrm{ARL}_1 = \frac{1}{1-\beta} \tag{3.12}$$

and

$$\mathrm{SDRL}_1 = \frac{\sqrt{\beta}}{1-\beta}, \tag{3.13}$$

respectively.

3.3.2 The VSS $\bar{X}$ Chart

The run-length properties of the VSS $\bar{X}$ chart can be modeled by a Markov chain approach. The VSS $\bar{X}$ chart applies the 3×3 Markov transition probability matrix **P**, i.e. (Costa, 1994)

$$\mathbf{P} = \begin{pmatrix} \mathbf{Q} & \mathbf{r} \\ \mathbf{0}^T & 1 \end{pmatrix} = \left(\begin{array}{cc|c} p_S(n_S) & p_L(n_S) & 1-p_S(n_S)-p_L(n_S) \\ p_S(n_L) & p_L(n_L) & 1-p_S(n_L)-p_L(n_L) \\ \hline 0 & 0 & 1 \end{array}\right), \tag{3.14}$$

where the first two states are transient; while the third state is an absorbing state. In Equation (3.14), $\mathbf{Q}$ is the 2×2 matrix of transient probabilities, $\mathbf{0} = (0,\ 0)^T$ and $\mathbf{r}$ is the 2×1 vector fulfilling $\mathbf{r} = \mathbf{1} - \mathbf{Q1}$, where $\mathbf{1} = (1,\ 1)^T$.

The probability $p_S(n_i)$ in Equation (3.14) with $n_i \in \{n_S,\ n_L\}$ is defined as

$$\begin{aligned} p_S(n_i) &= P\left(-W \le Z_i \le W\right) \\ &= P\left(-W \le \frac{(\bar{X}_i - \mu_0)\sqrt{n_i}}{\sigma_0} \le W \right) \\ &= P\left(\mu_0 - \frac{\sigma_0}{\sqrt{n_i}} W \le \bar{X}_i \le \mu_0 + \frac{\sigma_0}{\sqrt{n_i}} W \right). \end{aligned} \tag{3.15}$$

By subtracting and multiplying each side of the inequality in Equation (3.15) with $\mu_0 + \delta\sigma_0$ and $\frac{\sqrt{n_i}}{\sigma_0}$, respectively, Equation (3.15) becomes

$$p_S(n_i) = P\left[-W - \delta\sqrt{n_i} \le \frac{\left(\bar{X}_i - \mu_0 - \delta\sigma_0\right)\sqrt{n_i}}{\sigma_0} \le W - \delta\sqrt{n_i} \right]. \tag{3.16}$$

Since $\bar{X}_i \sim N\left(\mu_0 + \delta\sigma_0,\ \frac{{\sigma_0}^2}{n_i} \right)$ and thus $\frac{\left(\bar{X}_i - \mu_0 - \delta\sigma_0\right)\sqrt{n_i}}{\sigma_0} \sim N\left(0,\ 1\right)$, Equation (3.16) is equal to

$$p_S(n_i) = \Phi\left(W - \delta\sqrt{n_i}\right) - \Phi\left(-W - \delta\sqrt{n_i}\right). \tag{3.17}$$

The probability $p_L(n_i)$ in Equation (3.14) with $n_i \in \{n_S,\ n_L\}$ is evaluated as

$$\begin{aligned} p_L(n_i) &= P\left(-K \le Z_i < -W\right) + P\left(W < Z_i \le K\right) \\ &= P\left(-K \le \frac{(\bar{X}_i - \mu_0)\sqrt{n_i}}{\sigma_0} < -W \right) + P\left(W < \frac{(\bar{X}_i - \mu_0)\sqrt{n_i}}{\sigma_0} \le K \right) \\ &= P\left(\mu_0 - \frac{\sigma_0}{\sqrt{n_i}} K \le \bar{X}_i < \mu_0 - \frac{\sigma_0}{\sqrt{n_i}} W \right) + \\ &\quad P\left(\mu_0 + \frac{\sigma_0}{\sqrt{n_i}} W < \bar{X}_i \le \mu_0 + \frac{\sigma_0}{\sqrt{n_i}} K \right). \end{aligned} \tag{3.18}$$

After subtracting and multiplying each side of the inequality in Equation (3.18) with $\mu_0 + \delta\sigma_0$ and $\frac{\sqrt{n_i}}{\sigma_0}$, respectively, we obtain

$$p_L(n_i) = P\left[-K - \delta\sqrt{n_i} \le \frac{(\bar{X}_i - \mu_0 - \delta\sigma_0)\sqrt{n_i}}{\sigma_0} < -W - \delta\sqrt{n_i} \right] + P\left[W - \delta\sqrt{n_i} < \frac{(\bar{X}_i - \mu_0 - \delta\sigma_0)\sqrt{n_i}}{\sigma_0} \le K - \delta\sqrt{n_i} \right]. \quad (3.19)$$

Since $\bar{X}_i \sim N\left(\mu_0 + \delta\sigma_0,\ \frac{\sigma_0^{\ 2}}{n_i}\right)$ and thus $\frac{(\bar{X}_i - \mu_0 - \delta\sigma_0)\sqrt{n_i}}{\sigma_0} \sim N(0,\ 1)$, Equation (3.19) is simplified to

$$p_L(n_i) = \Phi\left(-W - \delta\sqrt{n_i}\right) - \Phi\left(-K - \delta\sqrt{n_i}\right) + \Phi\left(K - \delta\sqrt{n_i}\right) - \Phi\left(W - \delta\sqrt{n_i}\right). \quad (3.20)$$

RL is the number of steps until the process reaches the absorbing state; hence, *RL* is a Discrete Phase-type random variable with parameters $(\mathbf{Q},\ \mathbf{q})$ (see Castagliola et al., 2012), where $\mathbf{q}$ is the 2×1 initial probability vector. Then the pmf and cdf of *RL*, i.e. $f_{RL}(\ell)$ and $F_{RL}(\ell)$, respectively, for the VSS $\bar{X}$ chart are equal to

$$f_{RL}(\ell) = P(RL = \ell) = \mathbf{q}^T (\mathbf{Q}^{\ell-1})\mathbf{r} \quad (3.21)$$

and

$$F_{RL}(\ell) = P(RL \le \ell) = 1 - \mathbf{q}^T \mathbf{Q}^{\ell} \mathbf{1}, \quad (3.22)$$

where $\ell \in \{1,\ 2,\ 3,\ ...\}$. If we consider the initial sample size as $n_1 = n_S$ for the first subgroup, then $\mathbf{q} = (1,\ 0)^T$; otherwise, if we assume that the initial sample size is $n_1 = n_L$ for the first subgroup, then $\mathbf{q} = (0,\ 1)^T$. Equation (3.8) together with Equation (3.22) are used to compute the percentiles of the run-length distribution of the VSS $\bar{X}$ chart; whereas Equations (3.9) and (3.22) are used to calculate the MRL values of the VSS $\bar{X}$ chart.

The computation of the ARL and SDRL for the VSS $\bar{X}$ chart is as follows:

$$\text{ARL} = \mathbf{q}^T (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{1} \tag{3.23}$$

and

$$\text{SDRL} = \sqrt{2\mathbf{q}^T (\mathbf{I} - \mathbf{Q})^{-2} \mathbf{Q}\mathbf{1} - \text{ARL}^2 + \text{ARL}}, \tag{3.24}$$

where **I** is the 2×2 identity matrix.

For a fair comparison with other control charts, it is vital to evaluate the ASS of the VSS $\bar{X}$ chart. The ASS is defined in a process functioning over an infinite horizon. Castagliola et al. (2015) showed that the ASS of the VSS schemes can be evaluated by transforming the matrix **P** in Equation (3.14) into **P*** as shown below

$$\mathbf{P^*} = \left(\begin{array}{cc|c} p_S(n_S) & p_L(n_S) & 1 - p_S(n_S) - p_L(n_S) \\ p_S(n_L) & p_L(n_L) & 1 - p_S(n_L) - p_L(n_L) \\ \hline \multicolumn{2}{c|}{\mathbf{q}^T} & 0 \end{array} \right), \tag{3.25}$$

where $\mathbf{q}^T = (1,\ 0)$ when $n_1 = n_S$ and $\mathbf{q}^T = (0,\ 1)$ when $n_1 = n_L$. Note that, there is no absorbing state in matrix **P***. When the Markov chain moves to the third state (i.e. the out-of-control state), it restarts the process in either the first or second state, depending on the value of **q**. Therefore, the stationary-probability vector, $\boldsymbol{\pi} = (\pi_S, \pi_L, \pi_{\mathrm{OOC}})^T$ is obtained from the Markov chain defined by matrix **P*** (see Castagliola et al., 2015). Here, the stationary probabilities π_S, π_L and π_{OOC} represent n_S, n_L and n_1, respectively. This stationary-probability vector ($\boldsymbol{\pi}$) can be computed as follows:

$$\boldsymbol{\pi} = \mathbf{R}^{-1}\begin{pmatrix} \mathbf{q} \\ 0 \end{pmatrix}, \tag{3.26}$$

where matrix **R** is the transpose of matrix **P***, then the diagonal elements are subtracted with one. Next, either the first or second row is replaced by ones, depending on the choice of $n_1 = n_S$ or $n_1 = n_L$. For $n_1 = n_S$, the matrix **R** can be obtained as

$$\mathbf{R} = \begin{pmatrix} 1 & 1 & 1 \\ p_L(n_S) & p_L(n_L) - 1 & 0 \\ 1 - p_S(n_S) - p_L(n_S) & 1 - p_S(n_L) - p_L(n_L) & -1 \end{pmatrix}; \tag{3.27}$$

while the matrix **R** for $n_1 = n_L$ is equal to

$$\mathbf{R} = \begin{pmatrix} p_S(n_S) - 1 & p_S(n_L) & 0 \\ 1 & 1 & 1 \\ 1 - p_S(n_S) - p_L(n_S) & 1 - p_S(n_L) - p_L(n_L) & -1 \end{pmatrix}. \tag{3.28}$$

Then the ASS of the VSS $\bar{X}$ chart can be calculated as

$$\mathrm{ASS} = (n_S,\ n_L,\ n_1)\ \boldsymbol{\pi}\ , \tag{3.29}$$

where $n_1 \in \{n_S,\ n_L\}$.

3.3.3 The Exponentially Weighted Moving Average (EWMA) $\bar{X}$ Chart

Assume that the observations $\{X_{i,1},\ X_{i,2},\ \ldots,\ X_{i,n}\}$ are taken for the i^{th} subgroup, where $i = 1,\ 2,\ \ldots$ and n is the sample size of the EWMA $\bar{X}$ chart. The plotting statistic of the EWMA $\bar{X}$ chart is (Montgomery, 2013)

$$Z_i = \lambda \bar{X}_i + (1-\lambda) Z_{i-1}, \text{ for } i = 1,\ 2,\ \ldots \tag{3.30}$$

with $Z_0 = \mu_0$, $0 < \lambda \le 1$ and $\bar{X}_i$ is the sample mean at the i^{th} subgroup. The asymptotic upper (UCL_{EWMA}) and lower (LCL_{EWMA}) control limits of the EWMA $\bar{X}$ chart are expressed as follows (Montgomery, 2013):

$$UCL_{\text{EWMA}} \,/\, LCL_{\text{EWMA}} = \mu_0 \pm H\sigma_0, \tag{3.31}$$

with the center line $CL_{\text{EWMA}} = \mu_0$ and $H = h\sqrt{\dfrac{\lambda}{n(2-\lambda)}}$, where h is the multiplier to be determined.

The Markov chain approach proposed by Brook and Evans (1972) is adopted to evaluate the ARL, SDRL, MRL and percentiles of the run-length distribution of the EWMA $\bar{X}$ chart. Let us assume that the discrete-time Markov chain of the EWMA $\bar{X}$ chart has $p+1$ states. Here, states 1, 2, …, p are transient and state $p+1$ is an absorbing state. The transition probability matrix **P** (including the absorbing state) of this discrete-time Markov chain can be expressed as (Zhang et al., 2009)

$$\mathbf{P}=\begin{pmatrix}\mathbf{Q} & \mathbf{r}\\ \mathbf{0}^T & 1\end{pmatrix}=\left(\begin{array}{cccc|c} Q_{1,1} & Q_{1,2} & \cdots & Q_{1,p} & r_1\\ Q_{2,1} & Q_{2,2} & \cdots & Q_{2,p} & r_2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ Q_{p,1} & Q_{p,2} & \cdots & Q_{p,p} & r_p\\ \hline 0 & 0 & \cdots & 0 & 1\end{array}\right), \qquad (3.32)$$

where $\mathbf{0}$ = (0, 0, …, 0) ^{T} and $\mathbf{Q}$ is a $p \times p$ transition probability matrix of the transient states. Also, in Equation (3.32), $\mathbf{r}$ is the $p \times 1$ vector fulfilling $\mathbf{r}=\mathbf{1}-\mathbf{Q}\mathbf{1}$, where $\mathbf{1}$ = (1, 1, …, 1)^T.

Zhang et al. (2009) showed that the interval between LCL_{EWMA} and UCL_{EWMA} can be divided into $p=2w+1$ subintervals of width 2∂ each (see Figure 3.2). Here, $\partial=\frac{UCL_{\text{EWMA}}-LCL_{\text{EWMA}}}{2p}$. Let H_i or H_j be the midpoint of the i^{th} or j^{th} subinterval, where i or $j=-w$, ..., -1, 0, $+1$, ..., $+w$. Then the generic element $Q_{i,j}$ of the matrix $\mathbf{Q}$ is computed as

$$\begin{aligned} Q_{i,j} = {} & \Phi\left[\left(\frac{H_j+\partial-(1-\lambda)H_i}{\lambda}-\delta\right)\sqrt{n}\right]- \\ & \Phi\left[\left(\frac{H_j-\partial-(1-\lambda)H_i}{\lambda}-\delta\right)\sqrt{n}\right]. \end{aligned} \qquad (3.33)$$

The pmf $f_{RL}(\ell)$ and cdf $F_{RL}(\ell)$ of the RL for the EWMA $\bar{X}$ chart can be obtained from Equations (3.21) and (3.22), respectively, where $\mathbf{q}$ is replaced by $p \times 1$ initial probability vector, $\mathbf{Q}$ and $\mathbf{r}$ can be found in Equation (3.32). The element q_j of the vector $\mathbf{q}$ is obtained as follows:

$$q_j=\begin{cases} 1, & \text{if } H_j-\partial<Z_0<H_j+\partial \\ 0, & \text{elsewhere}\end{cases}. \qquad (3.34)$$

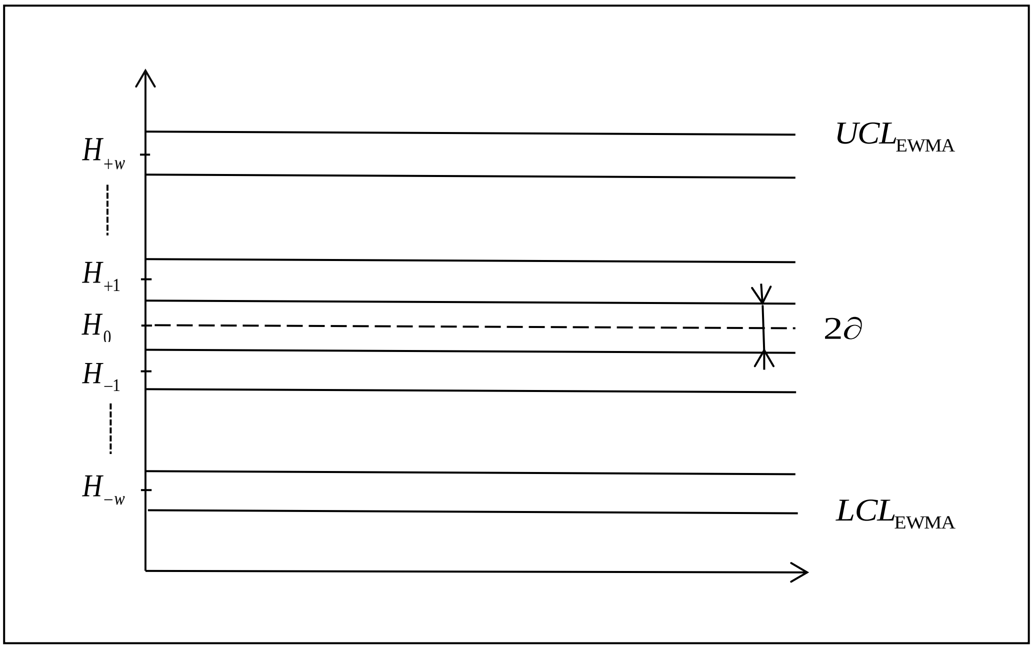

Figure 3.2: Interval between LCL_{EWMA} and UCL_{EWMA} of the EWMA $\bar{X}$ chart, divided into $p = 2w+1$ subintervals of width 2∂ each

Since $Z_0 = \mu_0$ is the initial value of the EWMA statistic, q_j equals to 1 when $j = w+1$ and zero elsewhere.

Similar to the VSS $\bar{X}$ chart, the percentiles of the run-length distribution or the MRL values of the EWMA $\bar{X}$ chart can be calculated from Equation (3.22) together with Equation (3.8) or (3.9), respectively. Whereas, the ARL and SDRL of the EWMA $\bar{X}$ chart can be computed from Equations (3.23) and (3.24), respectively. Note that, the vector **q** and matrix **Q** in these Equations (3.22), (3.23) and (3.24) need to be obtained from Subsection 3.3.3. Also, **I** in Equations (3.23) and (3.24) should be replaced with $p \times p$ identity matrix.

3.4 Optimal Designs of the VSS $\bar{X}$ Chart

In this section, two optimal designs of the VSS $\bar{X}$ chart, aiming at minimizing the (i) $\text{MRL}_1(\delta_{\text{opt}})$ and (ii) EMRL_1, for known and unknown shift sizes are developed. These two optimal-design algorithms are shown in Subsections 3.4.1 and 3.4.2, respectively. Here, $\text{MRL}_1(\delta_{\text{opt}})$ represents the out-of-control MRL for a desired magnitude of mean shift δ_{opt}, which should be detected quickly.

3.4.1 MRL Optimization of the VSS $\bar{X}$ Chart

When the size of the mean shift is known a priori, the optimal-statistical design of the VSS $\bar{X}$ chart based on MRL is outlined as follows:

$$\underset{n_S,\, n_L,\, W,\, K}{\text{Min}} \text{MRL}_1(\delta_{\text{opt}}), \tag{3.35}$$

subject to

(i) $\text{MRL}_0 = \tau,$ (3.36)

(ii) $\text{ASS}_0 = n$ and (3.37)

(iii) $1 \le n_S < n < n_L \le n_{\max},$ (3.38)

where τ and n are the desired values of the MRL_0 and ASS_0, respectively. Similar to Castagliola et al. (2013, 2015), the upper bound for the sample-size constraint in (3.38), i.e. $n_{\max}$, is restricted to 31 throughout this research. This is because small and moderate sample sizes are commonly used in industries.

By means of the above optimization model in (3.35)-(3.38), the procedure to obtain the optimal chart's parameters (n_S, n_L, W, K) of the VSS $\bar{X}$ chart with known shift sizes is as follows:

Step 1. Specify the desired values of n, $n_{\max}$, δ_{opt} and τ.

Step 2. Select a (n_S, n_L) pair based on constraint (3.38).

Step 3. Compute the initial value of K with $K = \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)$, which is derived from Equation (3.23) when $\delta = 0$. Here, $\Phi^{-1}(\cdot)$ is the inverse cdf of the standard normal random variable and α is the Type-I error probability which is determined so that $\tau \geq \frac{\ln 0.5}{\ln(1-\alpha)}$ (Chakraborti, 2007).

Step 4. Search for the initial value of W that satisfies constraint (3.37), by means of a nonlinear equation solver.

Step 5. Readjust the K and W values obtained in Steps 3 and 4 simultaneously in order to satisfy constraints (3.36) and (3.37).

Step 6. When the values of all the four chart's parameters (n_S, n_L, W, K) are preliminarily determined, calculate the objective function $\text{MRL}_1(\delta_{\text{opt}})$ with Equations (3.9) and (3.22).

Step 7. Repeat Steps (2) to (6) in order to obtain all the possible (n_S, n_L, W, K) combinations of the VSS $\bar{X}$ chart, which satisfy all the constraints in (3.36)-(3.38).

Step 8. Identify the optimal (n_S, n_L, W, K) combination(s) that minimize the $\text{MRL}_1(\delta_{\text{opt}})$ for any out-of-control conditions ($\delta \neq 0$). Since the MRL is a positive integer, there may be several optimal (n_S, n_L, W, K) combinations which have the same minimum value of MRL_1. For such a situation, select the optimal (n_S, n_L, W, K) combination, first, for having the smallest difference between the 5th and 95th percentiles of the run-length distribution and second, for having the smallest ASS_1 value.

Note that the difference between the 5th and 95th percentiles of the run-length distribution is viewed as an important criterion in this optimization algorithm. This is because once the $\mathrm{MRL}_1(\delta_{\mathrm{opt}})$ is minimized, a practitioner may opt for the optimal (n_S, n_L, W, K) combination(s) having the smallest variation in the run-length distribution. However, as the 5th and 95th percentiles of the run-length distribution are positive integers, there may again exist several optimal (n_S, n_L, W, K) combinations, for which situation, the optimal (n_S, n_L, W, K) combination having the smallest ASS_1 value will be selected.

3.4.2 Expected MRL (EMRL) Optimization of the VSS $\bar{X}$ Chart

It is really a very restrictive hypothesis to assume that the shift size is known a priori. In real life applications, the magnitude of future process changes is seldom known. When the actual mean shift differs from the desired δ_{opt} value, the optimization model in (3.35)-(3.38) may lead to a poor performance as the degree of changes or mismatch increases. For example, if $n_1 = n_S$, $n = 5$ and $\delta_{\mathrm{opt}} = 1$, Table 4.2 (see Chapter 4) gives the optimal chart's parameters (n_S, n_L, W, K) = (2, 14, 1.1420, 2.9922) and $\text{MRL}_1 = 2$. If the actual mean shift is 0.25, the MRL_1 value is 73 by using these (n_S, n_L, W, K) parameters. Then, the relative error is 100% × (73 – 51) / 51 = 43.14%. This error is absolutely undesirable. To overcome the lack of knowledge on the actual shift size, the EMRL is proposed in this research as an alternative design criterion to achieve a good detection performance for a domain of shift sizes.

The optimization model for the VSS $\bar{X}$ chart with unknown shift sizes is described as follows:

$$\underset{n_S,\, n_L,\, W,\, K}{\text{Min}} \text{EMRL}_1(\delta_{\text{opt}}), \tag{3.39}$$

subject to

(i) $\text{EMRL}_0 = \tau'$, (3.40)

(ii) $\text{ASS}_0 = n$ and (3.41)

(iii) $1 \le n_S < n < n_L \le n_{\max}$, (3.42)

where τ' is the desired in-control EMRL (EMRL_0) value. The EMRL can be computed as

$$\text{EMRL} = \int f_\delta(\delta)\,\text{MRL}\,d\delta, \tag{3.43}$$

where $f_\delta(\delta)$ is the probability density function (pdf) of δ. The proposed EMRL criterion integrates over the distribution function $f_\delta(\delta)$ for the shift range. As it is difficult to estimate the actual shape of $f_\delta(\delta)$, Castagliola et al. (2011, 2015), Celano (2010) and Ou et al. (2012) all assumed that the process mean shifts follow a uniform distribution $U\left(\delta_{\min},\ \delta_{\max}\right)$. Here, $\delta_{\min}$ and $\delta_{\max}$ are the lower and upper bounds of the mean shift, respectively.

The optimization model in (3.39)-(3.42) enables the computation of new optimal chart's parameters (n_S, n_L, *W*, *K*) of the VSS $\bar{X}$ chart, which are suitable to optimally detect a range of mean shifts without having to guess the actual value of δ. The procedure to obtain these optimal chart's parameters is illustrated as follows:

Step 1. Specify the desired values of *n*, $n_{\max}$, $\delta_{\min}$, $\delta_{\max}$ and τ'.

Steps 2 - 5. Similar to Steps 2 to 5 for the optimization model in (3.35)-(3.38) of Subsection 3.4.1, but replace constraints (3.36), (3.37) and (3.38) with constraints (3.40), (3.41) and (3.42), respectively.

Step 6. When the values of all the four chart's parameters (n_S, n_L, *W*, *K*) are preliminarily determined, calculate the objective function EMRL_1 with Equation (3.43). The Legendre-Gauss Quadrature method is employed to evaluate the integral in Equation (3.43).

Step 7. Repeat Steps (2) to (6) in order to obtain all the possible (n_S, n_L, *W*, *K*) combinations of the VSS $\bar{X}$ chart, which satisfy all the constraints (3.40)-(3.42).

Step 8. Identify the optimal (n_S, n_L, *W*, *K*) combination that minimizes the EMRL_1 over a range of mean shifts.

CHAPTER 4

RESULTS AND DISCUSSION

4.1 Introduction

In this chapter, the performances of the two VSS schemes are analyzed and discussed. Section 4.2 presents the computation of the ARL, SDRL and percentiles of the run-length distribution for the two VSS $\bar{X}$ charts with $n_1 = n_S$ and $n_1 = n_L$. Section 4.3 compares the performances of the two optimal VSS $\bar{X}$ charts with that of the Shewhart $\bar{X}$ and optimal EWMA $\bar{X}$ charts, based on the MRL and EMRL. All the results presented in this chapter and this dissertation, have been verified using the Monte Carlo simulation programs written in SAS software. Refer to Appendices A, B and C for the simulation programs of the VSS $\bar{X}$, Shewhart $\bar{X}$ and EWMA $\bar{X}$ charts, respectively. An example of application is demonstrated in Section 4.4 to illustrate the implementation of the two optimal VSS $\bar{X}$ charts.

4.2 Performance of the VSS $\bar{X}$ Charts Based on ARL, SDRL and Percentiles of the Run-length Distribution

Table 4.1 summarizes the exact values of the ARL, SDRL and percentiles of the run-length distribution for the two optimal ARL-based VSS

Table 4.1: Exact ARL, SDRL and percentiles of the run-length distribution for the VSS $\bar{X}$ chart with the optimal chart's parameters (n_S, n_L, W, K) corresponding to $\delta_{\text{opt}} \in \{0.5, 2.0\}$, $\text{ASS}_0 = 5$ and $\text{ARL}_0 = 370$

δ	ARL	SDRL	Percentiles of the run-length distribution										
			5th	10th	20th	30th	40th	50th	60th	70th	80th	90th	95th
$n_1 = n_S$, $\delta_{\text{opt}} = 0.5$, ($n_S$, n_L, W, K) = (2, 31, 1.6144, 2.9997)													
0.00	370.00	369.50	19	39	83	132	189	257	339	445	595	851	1108
0.25	80.12	78.77	5	10	19	29	42	56	74	96	128	183	237
0.50	8.85	6.87	2	3	4	5	6	7	8	10	13	18	22
0.75	4.46	2.93	2	2	2	3	3	4	4	5	6	8	10
1.00	3.24	1.84	1	2	2	2	2	3	3	4	4	6	7
1.50	2.17	0.91	1	1	2	2	2	2	2	2	3	3	4
2.00	1.64	0.63	1	1	1	1	1	2	2	2	2	2	3
$n_1 = n_S$, $\delta_{\text{opt}} = 2.0$, ($n_S$, n_L, W, K) = (4, 9, 1.2724, 2.9997)													
0.00	370.00	369.50	19	39	83	132	189	257	339	445	595	851	1108
0.25	124.71	124.00	7	14	28	45	64	87	114	150	200	286	372
0.50	23.56	22.50	2	3	6	9	13	17	22	28	37	53	68
0.75	6.31	5.05	1	2	2	3	4	5	6	7	9	13	16
1.00	2.95	1.74	1	1	2	2	2	3	3	3	4	5	6
1.50	1.56	0.61	1	1	1	1	1	1	2	2	2	2	3
2.00	1.16	0.37	1	1	1	1	1	1	1	1	1	2	2
$n_1 = n_L$, $\delta_{\text{opt}} = 0.5$, ($n_S$, n_L, W, K) = (1, 31, 1.5102, 2.9997)													
0.00	370.00	369.53	19	39	83	132	189	257	339	445	595	851	1108
0.25	71.27	74.82	1	4	13	23	34	48	65	86	117	169	221
0.50	3.75	5.02	1	1	1	1	1	2	2	3	5	9	14
0.75	1.15	0.58	1	1	1	1	1	1	1	1	1	2	2
1.00	1.00	0.08	1	1	1	1	1	1	1	1	1	1	1
1.50	1.00	0.00	1	1	1	1	1	1	1	1	1	1	1
2.00	1.00	0.00	1	1	1	1	1	1	1	1	1	1	1
$n_1 = n_L$, $\delta_{\text{opt}} = 2.0$, ($n_S$, n_L, W, K) = (4, 31, 2.1149, 2.9997)													
0.00	370.00	369.50	19	39	83	132	189	257	339	445	595	851	1107
0.25	102.29	107.17	1	6	19	33	50	69	93	124	168	242	316
0.50	6.31	9.30	1	1	1	1	1	2	3	6	10	18	26
0.75	1.21	0.91	1	1	1	1	1	1	1	1	1	2	2
1.00	1.01	0.09	1	1	1	1	1	1	1	1	1	1	1
1.50	1.00	0.00	1	1	1	1	1	1	1	1	1	1	1
2.00	1.00	0.00	1	1	1	1	1	1	1	1	1	1	1

schemes. The performances for the VSS $\bar{X}$ charts with $n_1 = n_S$ and $n_1 = n_L$ are listed in the upper and bottom parts of Table 4.1, respectively. In order to compute the optimal chart's parameters (n_S, n_L, W, K) for the VSS $\bar{X}$ chart, two ScicosLab (www.scicoslab.org) optimization programs have been written for the $n_1 = n_S$ and $n_1 = n_L$ schemes (see Appendix A.1). These programs are subjected to a desired $\text{ARL}_0 = 370$, a specific $\text{ASS}_0 = 5$ and $\delta_{\text{opt}} \in \{0.5,\ 2.0\}$. Note that δ_{opt} represents the desired mean shift for which a quick detection is needed. These optimal chart's parameters (n_S, n_L, W, K) are used to compute the ARL, SDRL and percentiles of the run-length distribution for various shift sizes, $\delta \in \{0,\ 0.25,\ 0.50,\ 0.75,\ 1.00,\ 1.50,\ 2.00\}$ in Table 4.1. The ARL and SDRL for the VSS $\bar{X}$ chart are calculated with Equations (3.23) and (3.24), respectively; while Equation (3.8) together with Equation (3.22) are employed to compute the percentiles of the run-length distribution.

From Table 4.1, I notice that the value of the ARL is much larger than the MRL when the process is in-control. The difference between the ARL and MRL values decreases when the process shift increases. From this point of view, it is important to take note that the shape and skewness of the run-length distribution change according to the magnitude of the mean shifts, ranging from highly right-skewed when the process is in-control or slightly shifted to almost symmetric when the process is out-of-control. With $\text{ARL}_0 = 370$, practitioners may wrongly interpret that a false alarm occurs at the 370$^{\text{th}}$ sample in half of the time. In fact, this value is allocated between the 60$^{\text{th}}$ and 70$^{\text{th}}$ percentiles of the run-length distribution and the false alarm actually occurs earlier, i.e. by the 257$^{\text{th}}$ sample $(\text{MRL}_0 = 257)$, in half of the time. By referring to Table 4.1, when $n_1 = n_S$, $\delta_{\text{opt}} = 2.0$ and $\delta = 1.0$, the ARL_1 is 2.95; whereas half of all the run lengths are less than 3 $(\text{MRL}_1 = 3)$, which is almost the same as the ARL value. This single example shows that in a highly right-skewed distribution, the average is always higher than the median; while in a symmetric distribution, the average is almost the same as the median. It is apparent that interpretation based on ARL for a highly right-skewed distribution is different from that of the symmetric distribution. Therefore, the MRL is a better representative of the central tendency compared to the ARL.

The percentiles of the run-length distribution provide practitioners with extra information regarding the shape, skewness, variation and behavior of a control chart. The lower percentiles such as the 5th, 10th and 20th percentiles of the run-length distribution when $\delta = 0$, provide practical guidance regarding the early false alarms in the in-control process. Let us consider the case of $n_1 = n_S$, $\delta_{\text{opt}} = 2.0$ and $\delta = 0$, there is a 10% chance that an early false alarm may occur by the 39th sample (see Table 4.1). This indicates that the lower percentiles are quite short even though the ARL_0 value (= 370) is large. Unequivocally, although the false alarm rate (= 0.0027) is low, there is a significant high percentage of false signals which occur at the beginning of process monitoring. The extra information on the early false alarms is beneficial and of great help to practitioners. With a comprehensive understanding on the behavior of the run-length distribution of a control chart, practitioners will have higher confidence in making conclusions about the actual status of a process being monitored when they encounter a few short run lengths with non-existing assignable causes. The information on the higher percentiles (i.e. 80th, 90th and 95th percentiles) of the run-length distribution also provides valuable information to practitioners. For instance, when $n_1 = n_L$,

$\delta_{\text{opt}} = 0.5$ and $\delta = 0.25$, practitioners are 95% confident to claim that an out-of-control signal will be detected by the 221$^{\text{st}}$ sample (see Table 4.1).

Practitioners should also be aware of the spread of the run-length distribution. The extremes, i.e. the 5$^{\text{th}}$ and 95$^{\text{th}}$ percentiles of the run-length distributions provide crucial information regarding the spread, variation and skewness of the run-length distribution. From Table 4.1, I observe that the difference between the 5$^{\text{th}}$ and 95$^{\text{th}}$ percentiles decreases when δ increases. For instance, when $n_1 = n_S$ and $\delta_{\text{opt}} = 0.5$, the difference between the 5$^{\text{th}}$ and 95$^{\text{th}}$ percentiles is 232 when $\delta = 0.25$; while the difference is only 3 when $\delta = 1.50$. This shows that the spread and skewness of the run-length distribution are large for in-control and slightly out-of-control cases. The variation in the run-length distribution is small when δ is large.

Since the associated run-length distribution of the VSS $\bar{X}$ chart has different skewness levels corresponding to different δ values, interpretation solely based on the ARL can be misleading and inappropriate. The ARL only provides the expected number of samples required to signal a false alarm or an out-of-control condition. It does not disclose the likelihood of obtaining a signal by a certain number of samples. There is hence a need for an alternative optimal-design criterion of the VSS $\bar{X}$ chart to be proposed in this research. The MRL is a more intuitive and credible measure as it is less influenced by the skewness of the run-length distribution. The MRL provides 50% certainty that the VSS $\bar{X}$ chart will signal by a particular number of samples for all ranges of shifts. By referring to Appendix D, Tables D.1 to D.3 exhibit additional results for the ARL, SDRL, and percentiles of the run-length distribution for the two optimal ARL-based VSS $\bar{X}$ charts when $\text{ARL}_0 \in \{250,\ 370\}$, $\delta_{\text{opt}} \in \{0.5,\ 2.0\}$ as well as $\text{ASS}_0 = 5$ and/or 10. Similar conclusions as for Table 4.1 are found for Tables D.1 to D.3 in Appendix D.

4.3 Comparative Studies

This section consists of two subsections. Subsections 4.3.1 and 4.3.2 discuss the comparative studies of the two optimal VSS $\bar{X}$ charts with the Shewhart $\bar{X}$ and optimal EWMA $\bar{X}$ charts, for known and unknown shift sizes, respectively.

4.3.1 Performance Comparisons for the Mean Shift of Known Size

Tables 4.2 and 4.3 compare the performances of the two optimal VSS schemes, Shewhart $\bar{X}$ and optimal EWMA $\bar{X}$ charts when the shift size is known a priori. Note that in this research, I consider $\text{MRL}_0 \in \{250,\ 370\}$, $n \in \{3,\ 5,\ 7,\ 10\}$ and $\delta_{\text{opt}} \in \{0.25,\ 0.50,\ 0.75,\ 1.00,\ 1.50,\ 2.00\}$. The charts' parameters L, (n_S, n_L, W, K) and (λ, H) of the Shewhart $\bar{X}$, optimal VSS $\bar{X}$ and optimal EWMA $\bar{X}$ charts are recorded in the first row of each cell in Tables 4.2 and 4.3; while the ($\ell_{0.05}$, MRL_1, $\ell_{0.95}$) values are tabulated in the second row of each cell. Here, $\ell_{0.05}$ and $\ell_{0.95}$ denote the 5$^{\text{th}}$ and 95$^{\text{th}}$ percentiles of the run-length distribution. The values of $\ell_{0.05}$ and $\ell_{0.95}$ are provided in

Table 4.2: Comparison of the ($\ell_{0.05}$, MRL_1, $\ell_{0.95}$) values for the Shewhart $\bar{X}$, optimal VSS $\bar{X}$ and optimal EWMA $\bar{X}$ charts, together with the charts' corresponding parameters when $n \in$ {3, 5, 7, 10}, $\delta_{\text{opt}} \in$ {0.25, 0.50, 0.75, 1.00, 1.50, 2.00} and $\mathrm{MRL}_0 = 250$

	Shewhart $\bar{X}$	VSS $\bar{X}$		EWMA $\bar{X}$
		$n_1 = n_S$	$n_1 = n_L$	
	L	(n_S , n_L , W, K)	(n_S , n_L , W, K)	(λ, H)
δ_{opt}	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)
		$n = 3$		
0.25	2.9923 (10, 125, 539)	(1, 31, 1.8144, 2.9922) (7, 82, 351)	(1, 31, 1.8516, 2.9922) (1, 78, 362)	(0.0798, 0.3094) (9, 27, 82)
0.50	2.9923 (4, 41, 177)	(1, 31, 1.8144, 2.9922) (2, 11, 42)	(1, 31, 1.8516, 2.9922) (1, 2, 29)	(0.1750, 0.5050) (4, 10, 27)
0.75	2.9923 (2, 15, 65)	(1, 21, 1.6303, 2.9922) (2, 5, 16)	(1, 27, 1.7844, 2.9922) (1, 1, 2)	(0.3000, 0.7074) (2, 6, 14)
1.00	2.9923 (1, 7, 28)	(2, 10, 1.5216, 2.9922) (1, 3, 8)	(2, 22, 1.9830, 2.9922) (1, 1, 1)	(0.5500, 1.0582) (1, 4, 11)
1.50	2.9923 (1, 2, 9)	(1, 6, 0.8347, 2.9922) (1, 2, 4)	(2, 10, 1.5440, 2.9922) (1, 1, 1)	(0.5500, 1.0582) (1, 2, 4)
2.00	2.9923 (1, 1, 3)	(2, 6, 1.1420, 2.9922) (1, 2, 2)	(2, 6, 1.1554, 2.9922) (1, 1, 1)	(0.5500, 1.0582) (1, 1, 2)
		$n = 5$		
0.25	2.9923 (7, 90, 389)	(1, 31, 1.4891, 2.9922) (5, 51, 215)	(1, 31, 1.5104, 2.9922) (1, 47, 217)	(0.0975, 0.2719) (7, 19, 53)
0.50	2.9923 (2, 23, 97)	(1, 31, 1.4891, 2.9922) (2, 7, 22)	(1, 31, 1.5104, 2.9922) (1, 2, 14)	(0.3000, 0.5480) (2, 7, 20)
0.75	2.9923 (1, 8, 31)	(2, 20, 1.3725, 2.9922) (2, 3, 9)	(2, 26, 1.5440, 2.9922) (1, 1, 2)	(0.5500, 0.8197) (2, 4, 12)
1.00	2.9923 (1, 3, 12)	(2, 14, 1.1420, 2.9922) (1, 2, 5)	(4, 22, 1.9354, 2.9922) (1, 1, 1)	(0.5500, 0.8197) (1, 2, 6)
1.50	2.9923 (1, 1, 3)	(4, 10, 1.3725, 2.9922) (1, 1, 2)	(4, 10, 1.3906, 2.9922) (1, 1, 1)	(0.5500, 0.8197) (1, 1, 3)
2.00	2.9923 (1, 1, 2)	(3, 6, 0.4244, 2.9922) (1, 1, 2)	(4, 6, 0.6766, 2.9922) (1, 1, 1)	(0.5500, 0.8197) (1, 1, 2)
		$n = 7$		
0.25	2.9923 (6, 69, 298)	(1, 31, 1.2721, 2.9922) (4, 37, 155)	(1, 31, 1.2879, 2.9922) (1, 34, 155)	(0.1090, 0.2464) (6, 15, 40)
0.50	2.9923 (2, 15, 62)	(2, 31, 1.3542, 2.9922) (2, 5, 15)	(2, 30, 1.3522, 2.9922) (1, 2, 10)	(0.3000, 0.4631) (2, 5, 14)
0.75	2.9923 (1, 5, 18)	(3, 20, 1.1782, 2.9922) (2, 3, 6)	(4, 26, 1.4986, 2.9922) (1, 1, 2)	(0.5500, 0.6928) (1, 3, 8)
1.00	2.9923 (1, 2, 7)	(2, 13, 0.7412, 2.992) (1, 2, 4)	(6, 22, 1.8815, 2.9922) (1, 1, 1)	(0.5500, 0.6928) (1, 2, 4)
1.50	2.9923 (1, 1, 3)	(4, 9, 0.5180, 2.9922) (1, 1, 2)	(6, 10, 1.1554, 2.9922) (1, 1, 1)	(0.5500, 0.6928) (1, 1, 2)
2.00	2.9923 (1, 1, 2)	(6, 8, 0.6679, 2.9922) (1, 1, 1)	(6, 8, 0.6766, 2.9922) (1, 1, 1)	(0.5500, 0.6928) (1, 1, 1)
		$n = 10$		
0.25	2.9923 (4, 50, 214)	(1, 31, 1.0287, 2.9922) (4, 27, 112)	(1, 31, 1.0406, 2.9922) (1, 25, 111)	(0.1450, 0.2462) (5, 12, 31)
0.50	2.9923 (1, 9, 37)	(2, 30, 1.0597, 2.9922) (2, 4, 11)	(1, 31, 1.0406, 2.9922) (1, 2, 7)	(0.5500, 0.5796) (2, 4, 13)
0.75	2.9923 (1, 3, 10)	(6, 24, 1.2118, 2.9922) (1, 2, 4)	(8, 26, 1.6043, 2.9922) (1, 1, 2)	(0.5500, 0.5796) (1, 2, 5)
1.00	2.9923 (1, 1, 4)	(9, 13, 1.1420, 2.9922) (1, 1, 3)	(9, 22, 1.7844, 2.9922) (1, 1, 1)	(0.8000, 0.7721) (1, 1, 3)
1.50	2.9923 (1, 1, 1)	(4, 11, 0.1735, 2.9922) (1, 1, 2)	(9, 11, 0.6766, 2.9922) (1, 1, 1)	(0.5500, 0.5796) (1, 1, 2)
2.00	2.9923 (1, 1, 1)	(6, 11, 0.2469, 2.9922) (1, 1, 1)	(9, 11, 0.6766, 2.9922) (1, 1, 1)	(0.5500, 0.5796) (1, 1, 1)

Table 4.3: Comparison of the ($\ell_{0.05}$, MRL_1, $\ell_{0.95}$) values for the Shewhart $\bar{X}$, optimal VSS $\bar{X}$ and optimal EWMA $\bar{X}$ charts, together with the charts' corresponding parameters when $n \in$ {3, 5, 7, 10}, $\delta_{\text{opt}} \in$ {0.25, 0.50, 0.75, 1.00, 1.50, 2.00} and $\mathrm{MRL}_0 = 370$

	Shewhart $\bar{X}$	VSS $\bar{X}$		EWMA $\bar{X}$
		$n_1 = n_S$	$n_1 = n_L$	
	L	(n_S, n_L, W, K)	(n_S, n_L, W, K)	(λ, H)
δ_{opt}	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)
		$n = 3$		
0.25	3.1099 (14, 177, 764)	(1, 31, 1.8206, 3.1098) (9, 109, 468)	(1, 31, 1.8458, 3.1098) (2, 105, 477)	(0.0813, 0.3312) (10, 31, 96)
0.50	3.1099 (5, 56, 240)	(1, 31, 1.8206, 3.1098) (3, 12, 44)	(1, 31, 1.8458, 3.1098) (1, 2, 32)	(0.1750, 0.5296) (4, 11, 30)
0.75	3.1099 (2, 20, 84)	(1, 19, 1.5840, 3.1098) (2, 5, 16)	(1, 28, 1.7970, 3.1098) (1, 1, 2)	(0.3000, 0.7385) (3, 6, 16)
1.00	3.1099 (1, 8, 35)	(2, 12, 1.6350, 3.1098) (2, 3, 8)	(2, 23, 1.9969, 3.1098) (1, 1, 1)	(0.5500, 1.1011) (2, 4, 12)
1.50	3.1099 (1, 2, 9)	(1, 7, 0.9624, 3.1098) (1, 2, 4)	(2, 11, 1.6007, 3.1098) (1, 1, 1)	(0.5500, 1.1011) (1, 2, 4)
2.00	3.1099 (1, 1, 3)	(2, 6, 1.1447, 3.1098) (1, 2, 2)	(2, 6, 1.1538, 3.1098) (1, 1, 1)	(0.8000, 1.4652) (1, 1, 3)
		$n = 5$		
0.25	3.1099 (10, 126, 544)	(1, 31, 1.4929, 3.1098) (6, 66, 280)	(1, 31, 1.5074, 3.1098) (2, 62, 281)	(0.0690, 0.2319) (9, 21, 54)
0.50	3.1099 (3, 30, 128)	(1, 31, 1.4929, 3.1098) (2, 7, 24)	(1, 31, 1.5074, 3.1098) (1, 2, 15)	(0.3000, 0.5721) (3, 8, 23)
0.75	3.1099 (1, 9, 38)	(3, 21, 1.5840, 3.1098) (2, 3, 9)	(3, 28, 1.7608, 3.1098) (1, 1, 2)	(0.5500, 0.8529) (2, 4, 13)
1.00	3.1099 (1, 4, 15)	(3, 12, 1.2146, 3.1098) (1, 2, 5)	(4, 23, 1.9527, 3.1098) (1, 1, 1)	(0.5500, 0.8529) (1, 3, 6)
1.50	3.1099 (1, 1, 4)	(4, 11, 1.4574, 3.1098) (1, 2, 2)	(4, 11, 1.4711, 3.1098) (1, 1, 1)	(0.8000, 1.1350) (1, 1, 3)
2.00	3.1099 (1, 1, 2)	(3, 6, 0.4264, 3.1098) (1, 1, 2)	(4, 6, 0.6760, 3.1098) (1, 1, 1)	(0.5500, 0.8529) (1, 1, 2)
		$n = 7$		
0.25	3.1099 (8, 96, 412)	(1, 31, 1.2752, 3.1098) (5, 48, 200)	(1, 31, 1.2858, 3.1098) (2, 44, 199)	(0.1355, 0.2966) (6, 17, 49)
0.50	3.1099 (2, 19, 80)	(2, 31, 1.3575, 3.1098) (2, 6, 16)	(1, 30, 1.2663, 3.1098) (1, 2, 11)	(0.3000, 0.4835) (2, 6, 15)
0.75	3.1099 (1, 5, 22)	(4, 21, 1.3449, 3.1098) (1, 3, 6)	(2, 27, 1.2858, 3.1098) (1, 1, 2)	(0.5500, 0.7209) (1, 3, 8)
1.00	3.1099 (1, 2, 8)	(2, 16, 0.9160, 3.1098) (2, 2, 4)	(6, 23, 1.9028, 3.1098) (1, 1, 1)	(0.5500, 0.7209) (1, 2, 4)
1.50	3.1099 (1, 1, 2)	(5, 9, 0.6701, 3.1098) (1, 1, 2)	(6, 11, 1.2858, 3.1098) (1, 1, 1)	(0.5500, 0.7209) (1, 1, 2)
2.00	3.1099 (1, 1, 1)	(6, 8, 0.6701, 3.1098) (1, 1, 1)	(6, 8, 0.6760, 3.1098) (1, 1, 1)	(0.5500, 0.7209) (1, 1, 1)
		$n = 10$		
0.25	3.1099 (5, 68, 292)	(1, 31, 1.0312, 3.1098) (4, 34, 143)	(1, 31, 1.0392, 3.1098) (2, 32, 142)	(0.1450, 0.2586) (5, 13, 34)
0.50	3.1099 (1, 11, 46)	(3, 31, 1.1447, 3.1098) (2, 4, 12)	(1, 31, 1.0392, 3.1098) (1, 2, 8)	(0.5500, 0.6031) (2, 5, 15)
0.75	3.1099 (1, 3, 12)	(6, 28, 1.3284, 3.1098) (1, 2, 4)	(6, 27, 1.3137, 3.1098) (1, 1, 2)	(0.5500, 0.6031) (1, 2, 5)
1.00	3.1099 (1, 1, 5)	(3, 18, 0.7234, 3.1098) (1, 2, 3)	(9, 23, 1.8139, 3.1098) (1, 1, 1)	(0.5500, 0.6031) (1, 2, 3)
1.50	3.1099 (1, 1, 2)	(5, 11, 0.2060, 3.1098) (1, 1, 2)	(9, 11, 0.6760, 3.1098) (1, 1, 1)	(0.5500, 0.6031) (1, 1, 2)
2.00	3.1099 (1, 1, 1)	(6, 11, 0.2490, 3.1098) (1, 1, 1)	(9, 11, 0.6760, 3.1098) (1, 1, 1)	(0.5500, 0.6031) (1, 1, 1)

these tables in order to measure the variation and spread of the run-length distribution. All the charting parameters of the Shewhart $\bar{X}$, optimal VSS $\bar{X}$ and optimal EWMA $\bar{X}$ charts attain $\text{MRL}_0 \in \{250, 370\}$ when $n \in \{3, 5, 7, 10\}$.

The optimization model in (3.35)-(3.38) (see Subsection 3.4.1 of Chapter 3) and optimization programs shown in Appendix A.2, are used to compute the optimal chart's parameters (n_S, n_L, W, K) of the two VSS schemes in Tables 4.2 and 4.3. For instance, when $n = 5$, $\text{MRL}_0 = 370$ and $\delta_{\text{opt}} = 0.75$, Table 4.3 gives ($n_S$, n_L, W, K) = (3, 21, 1.5840, 3.1098) and (n_S, n_L, W, K) = (3, 28, 1.7608, 3.1098) as the optimal chart's parameters of the VSS $\bar{X}$ chart with $n_1 = n_S$ and $n_1 = n_L$, respectively. With these optimal chart's parameters, the computed ($\ell_{0.05}$, MRL_1, $\ell_{0.95}$) values are (2, 3, 9) and (1, 1, 2) for $n_1 = n_S$ and $n_1 = n_L$, respectively. For the MRL-based Shewhart $\bar{X}$ and optimal MRL-based EWMA $\bar{X}$ charts, the written ScicosLab programs shown in Appendices B.1 and C.1, respectively, are used to compute the charts' parameters and their corresponding ($\ell_{0.05}$, MRL_1, $\ell_{0.95}$) values. Note that the formulae used to compute the ($\ell_{0.05}$, MRL_1, $\ell_{0.95}$) values of these four control charts can be obtained from Section 3.3 of Chapter 3.

From Tables 4.2 and 4.3, it is obvious that the two optimal VSS schemes and the optimal EWMA $\bar{X}$ chart generally surpass the Shewhart $\bar{X}$ chart for all levels of mean shifts. By comparing between the two optimal VSS schemes, the optimal VSS $\bar{X}$ chart with $n_1 = n_L$ generally has a shorter MRL_1 value than its counterpart with $n_1 = n_S$, for all sizes of mean shifts. Tables 4.2 and 4.3 also reveal that the optimal VSS $\bar{X}$ chart with $n_1 = n_L$ remarkably reduces the MRL_1 and $\ell_{0.95}$ values. When $\delta_{\text{opt}} \geq 0.75$ and $\delta_{\text{opt}} \geq 1.00$, the MRL_1 and $\ell_{0.95}$ values, respectively, for the VSS scheme with $n_1 = n_L$ are equal to 1. In addition, there is no variation in the run-length distribution for the VSS scheme with $n_1 = n_L$ when $\delta_{\text{opt}} \geq 1.00$. This is because the difference between $\ell_{0.05}$ and $\ell_{0.95}$ is equal to zero (see Tables 4.2 and 4.3). From this point of view, it is preferable for the VSS $\bar{X}$ chart to take a large sample size for the first subgroup in order to give additional protection against problems in the process arising during start-up.

From Tables 4.2 and 4.3, the detection speed of the optimal VSS $\bar{X}$ chart with $n_1 = n_S$ is comparable to that of the optimal EWMA $\bar{X}$ chart for $0.50 \leq \delta_{\text{opt}} \leq 2.00$. From Tables 4.2 and 4.3, the difference between $\ell_{0.05}$ and $\ell_{0.95}$ for the optimal VSS $\bar{X}$ chart with $n_1 = n_S$ is generally smaller than that of the EWMA $\bar{X}$ chart when $\delta_{\text{opt}} \geq 0.75$. This indicates that the variation in the run-length distribution of the former chart is lower than that of the latter chart. This is an advantage of the VSS $\bar{X}$ chart with $n_1 = n_S$ over the EWMA $\bar{X}$ chart toward moderate shifts.

For very small shifts, i.e. $\delta_{\text{opt}} \leq 0.25$, the optimal EWMA $\bar{X}$ chart is the best among all the competing control charts. For moderate to large shifts, i.e. $\delta_{\text{opt}} \geq 0.50$, the optimal VSS $\bar{X}$ chart with $n_1 = n_L$ significantly outperforms the Shewhart $\bar{X}$, optimal VSS $\bar{X}$ with $n_1 = n_S$ and optimal EWMA $\bar{X}$ charts. By referring to Table 4.3, when $\text{MRL}_0 = 370$, n = 3 and $\delta_{\text{opt}} = 0.50$ are considered, the MRL_1 value for the optimal VSS $\bar{X}$ chart with $n_1 = n_L$ is 2 as opposed to 56, 12 and 11 for the Shewhart $\bar{X}$, optimal VSS $\bar{X}$ with $n_1 = n_S$ and optimal EWMA $\bar{X}$ charts, respectively. From this example, it is apparent that the VSS $\bar{X}$ chart with $n_1 = n_L$ tremendously reduces the MRL_1 values of the Shewhart $\bar{X}$, optimal VSS $\bar{X}$ with $n_1 = n_S$ and optimal EWMA $\bar{X}$ charts by nearly 96.4%, 83.3% and 81.8%, respectively. Concerning the dispersion of the run-length distribution, the values of $\ell_{0.05}$ and $\ell_{0.95}$ in Tables 4.2 and 4.3 show that when $0.50 \leq \delta_{\text{opt}} \leq 2.00$, the optimal VSS $\bar{X}$ chart with $n_1 = n_L$ generally has the lowest variation in the run-length distribution compared to the other three competitive control charts; while for small shifts ($\delta_{\text{opt}} \leq 0.25$), the spread in the run-length distribution is the smallest for the optimal EWMA $\bar{X}$ chart compared to that of the other three competitive control charts.

Tables 4.2 and 4.3 indicate that the optimal VSS $\bar{X}$ chart with $n_1 = n_L$ guards well against moderate to large shifts; while the optimal EWMA $\bar{X}$ chart provides effective protection against small shifts. However, the effectiveness of the optimal VSS $\bar{X}$ chart with $n_1 = n_L$ in detecting moderate and large shifts is more meaningful than its inferiority towards small shifts. This is because moderate and large shifts are more deleterious to a process.

Also, there will be too frequent process interruptions if very small shifts are detected regularly, where it is better not to interrupt process monitoring in such a situation. Similar results which are provided in Tables D.4 to D.6 (see Appendix D.2), hold for $\text{MRL}_0 \in \{200, 300, 500\}$ and $n \in \{3, 5, 7, 10\}$.

4.3.2 Performance Comparisons for the Mean Shift of Unknown Size

Table 4.4 provides the charting parameters of the Shewhart $\bar{X}$, optimal VSS $\bar{X}$ and optimal EWMA $\bar{X}$ charts over a range of shift sizes when the intended $\text{EMRL}_0 = 370$, $n \in \{3, 5, 7, 10\}$, $\delta_{\min} = 0.1$ and $\delta_{\max} = 2.0$. For example, if $n = 3$, the optimal chart's parameters (n_S, n_L, W, K) and the corresponding EMRL_1 value for the VSS $\bar{X}$ chart with $n_1 = n_S$ are (1, 31, 1.8206, 3.1098) and 25.02, respectively. These optimal chart's parameters of the VSS $\bar{X}$ chart are obtained from the optimization model in (3.39)-(3.42) (see Subsection 3.4.2 of Chapter 3) and the optimization programs presented in Appendix A.2. For the EMRL-based Shewhart $\bar{X}$ chart, the developed ScicosLab program is shown in Appendix B.1; while that for the optimal EMRL-based EWMA $\bar{X}$ chart, is shown in Appendix C.1. Table 4.4 also presents the ($\ell_{0.05}$, MRL_1, $\ell_{0.95}$) values for the Shewhart $\bar{X}$, VSS $\bar{X}$ and EWMA $\bar{X}$ charts, for $\delta \in$ {0.25, 0.50, 0.75, 1.00, 1.50, 2.00}, corresponding to the EMRL-based charts' parameters recorded in this table. For instance, if $n = 3$, the ($\ell_{0.05}$, MRL_1, $\ell_{0.95}$) values for the VSS $\bar{X}$ chart with $n_1 = n_S$ corresponding to (n_S, n_L, W, K) = (1, 31, 1.8206, 3.1098) are (3, 12, 44) when $\delta = 0.50$.

Table 4.4: Comparison of the EMRL_1 and ($\ell_{0.05}$, MRL_1, $\ell_{0.95}$) values for the Shewhart $\bar{X}$, optimal VSS $\bar{X}$ and optimal EWMA $\bar{X}$ charts, together with the charts' corresponding parameters when $n \in$ {3, 5, 7, 10}, $\text{EMRL}_0 = 370$, $\delta_{\min} =$ 0.1 and $\delta_{\max} =$ 2.0

	Shewhart $\bar{X}$	VSS $\bar{X}$		EWMA $\bar{X}$
		$n_1 = n_S$	$n_1 = n_L$	
	L EMRL_1	(n_S, n_L, W, K) EMRL_1	(n_S, n_L, W, K) EMRL_1	(λ, H) EMRL_1
δ	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)
		$n = 3$		
	3.1099 41.18	(1, 31, 1.8206, 3.1098) 25.02	(1, 31, 1.8458, 3.1098) 21.16	(0.0726, 0.3090) 11.14
0.25	(14, 177, 764)	(9, 109, 468)	(2, 105, 477)	(10, 31, 92)
0.50	(5, 56, 240)	(3, 12, 44)	(1, 2, 32)	(6, 12, 25)
0.75	(2, 20, 84)	(2, 6, 21)	(1, 1, 2)	(4, 7, 13)
1.00	(1, 8, 35)	(2, 4, 14)	(1, 1, 1)	(3, 5, 9)
1.50	(1, 2, 9)	(1, 3, 8)	(1, 1, 1)	(3, 4, 5)
2.00	(1, 1, 3)	(1, 2, 5)	(1, 1, 1)	(2, 3, 4)
		$n = 5$		
	3.1099 28.58	(1, 31, 1.4929, 3.1098) 18.19	(1, 31, 1.5074, 3.1098) 15.58	(0.1090, 0.3073) 7.98
0.25	(10, 126, 544)	(6, 66, 280)	(2, 62, 281)	(7, 22, 64)
0.50	(3, 30, 128)	(2, 7, 24)	(1, 2, 15)	(4, 8, 16)
0.75	(1, 9, 38)	(2, 4, 13)	(1, 1, 2)	(3, 5, 9)
1.00	(1, 4, 15)	(2, 3, 9)	(1, 1, 1)	(2, 4, 6)
1.50	(1, 1, 4)	(1, 2, 6)	(1, 1, 1)	(2, 2, 3)
2.00	(1, 1, 2)	(1, 2, 4)	(1, 1, 1)	(2, 2, 3)
		$n = 7$		
	3.1099 22.01	(1, 30, 1.2559, 3.1098) 14.78	(1, 31, 1.2858, 3.1098) 12.80	(0.0956, 0.2396) 6.22
0.25	(8, 96, 412)	(5, 49, 205)	(2, 44, 199)	(7, 17, 43)
0.50	(2, 19, 80)	(2, 6, 17)	(1, 2, 11)	(4, 7, 12)
0.75	(1, 5, 22)	(2, 3, 9)	(1, 1, 2)	(3, 4, 7)
1.00	(1, 2, 8)	(2, 3, 7)	(1, 1, 1)	(2, 3, 5)
1.50	(1, 1, 2)	(1, 2, 5)	(1, 1, 1)	(2, 2, 3)
2.00	(1, 1, 1)	(1, 2, 3)	(1, 1, 1)	(1, 2, 2)
		$n = 10$		
	3.1099 16.35	(1, 31, 1.0312, 3.1098) 11.76	(1, 31, 1.0392, 3.1098) 10.20	(0.1450, 0.2586) 4.86
0.25	(5, 68, 292)	(4, 34, 143)	(2, 32, 142)	(5, 13, 34)
0.50	(1, 11, 46)	(2, 5, 13)	(1, 2, 8)	(3, 5, 9)
0.75	(1, 3, 12)	(2, 3, 7)	(1, 1, 2)	(2, 3, 5)
1.00	(1, 1, 5)	(2, 2, 6)	(1, 1, 1)	(2, 2, 3)
1.50	(1, 1, 2)	(1, 2, 4)	(1, 1, 1)	(1, 2, 2)
2.00	(1, 1, 1)	(1, 2, 3)	(1, 1, 1)	(1, 1, 2)

When $n = 5$ and the shift size is $\delta = 0.75$, Table 4.3 gives the optimal chart's parameters and MRL_1 value for the VSS $\bar{X}$ chart with $n_1 = n_S$ as (n_S, n_L, W, K) = (3, 21, 1.5840, 3.1098) and $\text{MRL}_1 = 3$, respectively; while Table

4.4 gives (n_S, n_L, W, K) = (1, 31, 1.4929, 3.1098) and $\text{MRL}_1 = 4$. I notice from this example that the global chart's parameters listed in Table 4.4 are able to provide the MRL_1 value which is close to the one obtained with a specific shift size in Table 4.3. This indicates that the global charts' parameters shown in Table 4.4 can be considered as a robust alternative to the charts' parameters presented in Table 4.3.

The results in Table 4.4 demonstrates the superiority of the optimal EWMA $\bar{X}$ chart, in comparison to the other competing control charts, for a shift domain of $[\delta_{\min} = 0.1,\ \delta_{\max} = 2.0]$. The smallest EMRL_1 value is obtained for the EWMA $\bar{X}$ chart because there is a significant improvement in the MRL_1 performance over a range of small shifts. Nevertheless, the optimal EWMA $\bar{X}$ chart is only effective in detecting small process mean shifts ($\delta \le 0.25$); while for moderate and large shifts ($\delta \ge 0.50$), the VSS $\bar{X}$ chart with $n_1 = n_L$ has the best performance, in terms of the detection speed and the variation in the run-length distribution. For example, when $n = 5$ and $\delta = 0.75$, the MRL_1 value for the VSS $\bar{X}$ chart with $n_1 = n_L$ is 1 compared to 9, 4 and 5 for the Shewhart $\bar{X}$, VSS $\bar{X}$ with $n_1 = n_S$ and EWMA $\bar{X}$ charts, respectively. For the same condition, the difference between $\ell_{0.05}$ and $\ell_{0.95}$ for the VSS $\bar{X}$ chart with $n_1 = n_L$ is 1 compared to 37, 11 and 6 for the Shewhart $\bar{X}$, VSS $\bar{X}$ with $n_1 = n_S$ and EWMA $\bar{X}$ charts, respectively. Therefore, it is found that the VSS $\bar{X}$ chart with $n_1 = n_L$ reduces the variation in the run-length distribution of the Shewhart $\bar{X}$, VSS $\bar{X}$ with $n_1 = n_S$ and EWMA $\bar{X}$ charts, by nearly

97.3%, 90.9% and 83.3%, respectively. Additional results for the Shewhart $\bar{X}$, VSS $\bar{X}$ and EWMA $\bar{X}$ charts by minimizing the EMRL_1 when $\text{EMRL}_0 \in$ {200, 250, 300, 500} and $n \in \{3, 5, 7, 10\}$, are provided in Tables D.7 to D.10 of Appendix D.3.

4.4 An Illustrative Example

This section illustrates the implementation of the optimal MRL-based VSS $\bar{X}$ charts both with $n_1 = n_S$ and $n_1 = n_L$. The example in Carot et al. (2002) is considered in this dissertation. The two optimal VSS $\bar{X}$ schemes are applied in a yoghurt manufacturing firm, in order to monitor the amount of potassium sorbate added to its products. Potassium sorbate is primarily used as a basic ingredient in food preservation. The public health institutions recommend the use of the amount of potassium sorbate in the range 0.5g to 2.0g per kg of product. The desired process parameters are assumed to be $\mu_0 = 1.5\text{g}$ and $\sigma_0 = 0.008\text{g}$ (Carot et al., 2002).

The optimal MRL-based EWMA $\bar{X}$ and Shewhart $\bar{X}$ charts are also constructed in order to enable comparisons with the two optimal VSS schemes. The data for this example are generated from a normal distribution with mean, $\mu_0 = 1.5$ and standard deviation, $\sigma_0 = 0.008$ by using the developed SAS programs. These programs for the VSS $\bar{X}$ charts with $n_1 = n_S$ and $n_1 = n_L$, Shewhart $\bar{X}$ chart and EWMA $\bar{X}$ chart can be obtained from Appendices

Table 4.5: Summary statistics of the simulated data for the amount of potassium sorbate (in grams, g) to be added in a yoghurt manufacturing process

	VSS $\bar{X}$ chart						Shewhart $\bar{X}$ chart	EWMA $\bar{X}$ chart
	$n_1 = n_S$			$n_1 = n_L$				
Subgroup number, i	n_i	$\bar{X}_i$	Z_i	n_i	$\bar{X}_i$	Z_i	$\bar{X}_i$	Z_i
1	3	1.50609	1.31921	28	1.49915	–0.56467	1.50192	1.50106
2	3	1.49547	–0.98054	3	1.49668	–0.71987	1.49426	1.49732
3	3	1.49948	–0.11233	3	1.50455	0.98525	1.49835	1.49789
4	3	1.49099	–1.95039	3	1.49698	–0.65330	1.49702	1.49741
5	21	1.50016	0.09076	3	1.49333	–1.44421	1.50278	1.50036
6	3	1.49646	–0.76710	3	1.50346	0.74957	1.49817	1.49915
7	3	1.49500	–1.08294	3	1.49751	–0.53903	1.50206	1.50075
8	3	1.49708	–0.63204	3	1.49196	–1.74140	1.49558	1.49791
9	3	1.50177	0.38230	3	1.49969	–0.06631	1.49993	1.49902
10	3	1.49494	–1.09565	3	1.50250	0.54081	1.49364	1.49606
11	3	1.50217	0.46995	3	1.51012	2.19149	1.50951	1.50346
12	3	1.50877	1.89845	28	1.50809	**5.35426**	1.50528	1.50446
13	21	1.50792	**4.53809**	28	1.50516	**3.40988**	1.50592	1.50526
14	3	1.51080	2.33824	28	1.50618	**4.08790**	1.51030	**1.50803**
15	21	1.50718	**4.11203**	28	1.50513	**3.39162**	1.50976	**1.50898**
16	3	1.51005	2.17500	28	1.50564	**3.72983**	**1.51321**	**1.51131**
17	21	1.50434	2.48422	28	1.50565	**3.73851**	1.50735	**1.50913**

Remarks: The boldfaced values denote the out-of-control cases.

A.3.3, A.3.4, B.2.2 and C.2.2, respectively. The observations for the first ten subgroups ($i = 1$ to 10) are generated under the in-control condition; whereas the observations for the 11th subgroup onwards ($i = 11$ to 17) are generated under the out-of-control condition, i.e. $\delta = 0.75$. Table 4.5 summarizes the statistics for the VSS $\bar{X}$, Shewhart $\bar{X}$ and EWMA $\bar{X}$ charts. Note that the boldfaced values represent the out-of-control cases. In this example, the dataset for all the observations used to construct the VSS $\bar{X}$, Shewhart $\bar{X}$ and EWMA $\bar{X}$ charts can be found in Tables D.11 to D.13 of Appendix D.4. Assume that the intended $\text{MRL}_0 = 370$, n = 5 and $\delta_{\text{opt}} = 0.75$. From Table 4.3, the charts' parameters for the optimal VSS $\bar{X}$ charts with $n_1 = n_S$ and $n_1 = n_L$, the Shewhart $\bar{X}$ chart and the optimal EWMA $\bar{X}$ chart are (n_S, n_L, W, K) = (3, 21, 1.5840, 3.1098), (n_S, n_L, W, K) = (3, 28, 1.7608, 3.1098), L = 3.1099

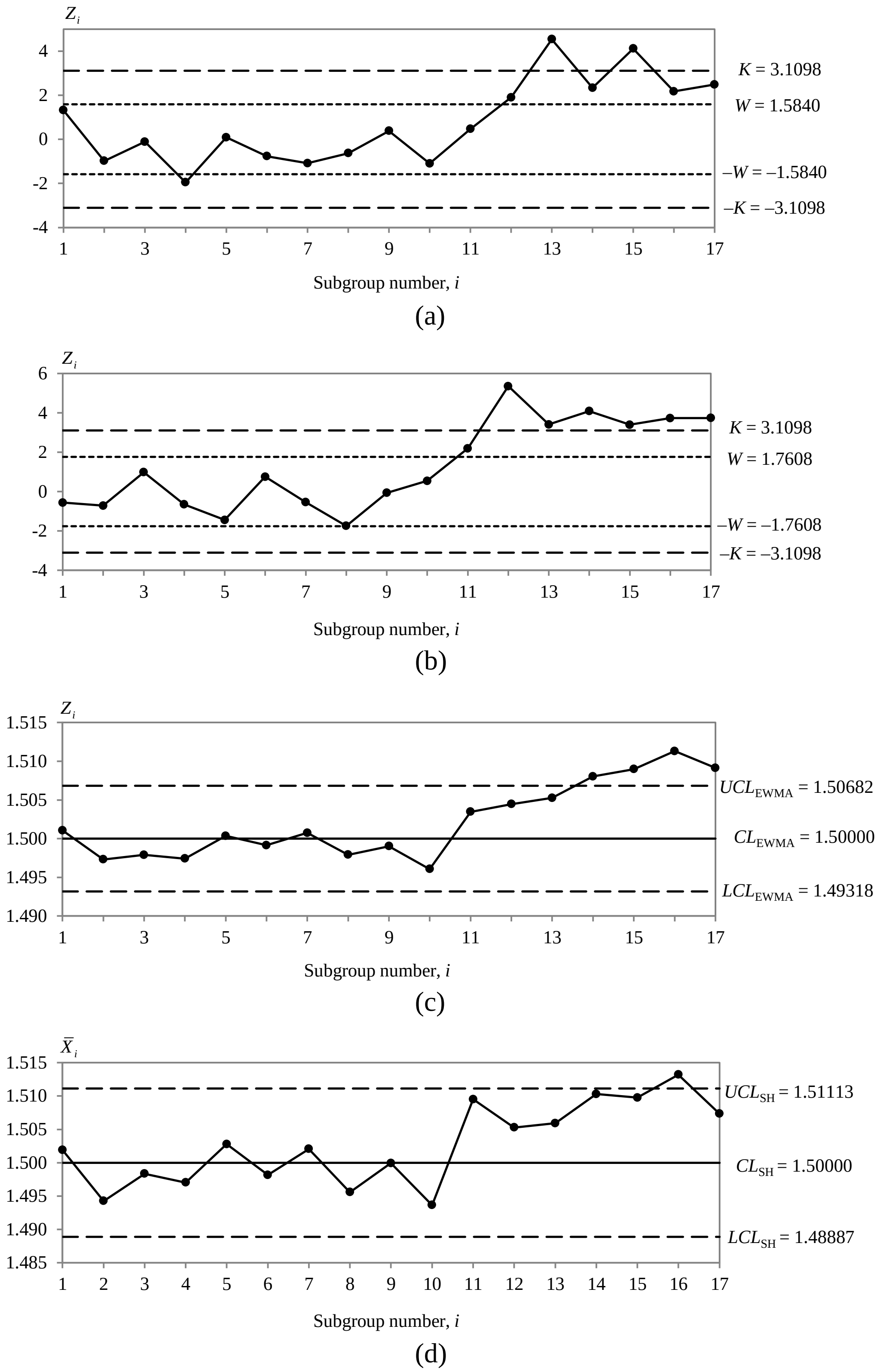

Figure 4.1: The optimal VSS $\bar{X}$ chart with (a) $n_1 = n_S$ and (b) $n_1 = n_L$, as well as (c) the optimal EWMA $\bar{X}$ chart and (d) the Shewhart $\bar{X}$ chart, for monitoring the amount of potassium sorbate to be added to a yoghurt manufacturing process

and (λ, H) = (0.5500, 0.8529), respectively. Figures 4.1(a) to (d) graphically display the optimal MRL-based VSS $\bar{X}$ charts with $n_1 = n_S$ and $n_1 = n_L$, the optimal MRL-based EWMA $\bar{X}$ chart and the Shewhart $\bar{X}$ chart, respectively. The control limits, i.e. $UCL_{\text{SH}} / LCL_{\text{SH}}$ and $UCL_{\text{EWMA}} / LCL_{\text{EWMA}}$ of the Shewhart $\bar{X}$ and optimal EWMA $\bar{X}$ charts are computed from Equations (3.3) and (3.31), respectively, in Chapter 3.

By referring to Figures 4.1(a)-(d), it is obvious that the VSS $\bar{X}$ charts with $n_1 = n_S$ and $n_1 = n_L$, Shewhart $\bar{X}$ and EWMA $\bar{X}$ charts detect the first out-of-control signal at subgroup $i = 13$ (as Z_{13} = 4.53809 > K = 3.1098), $i = 12$ (as Z_{12} = 5.35426 > K = 3.1098), $i = 16$ (as $\bar{X}_{16}$ = 1.51321 > UCL_{SH} = 1.51113) and $i = 14$ (as $Z_{14} = 1.50803$ > UCL_{EWMA} = 1.50682), respectively. Immediate investigations should be conducted to identify and eliminate the assignable cause(s). The results in this example suggest that the VSS $\bar{X}$ chart with $n_1 = n_L$ has the fastest detection speed, followed by the VSS $\bar{X}$ chart with $n_1 = n_S$, EWMA $\bar{X}$ chart and Shewhart $\bar{X}$ chart.

CHAPTER 5

CONCLUSIONS AND FUTURE RESEARCH

5.1 Introduction

An effective control chart has quick detection of process shifts. This will in turn lead to an increase in the quality and size of productions for sustainable industry. The aim of this research is to propose two new optimization algorithms for the VSS $\bar{X}$ charts with $n_1 = n_S$ and $n_1 = n_L$. These two proposed optimization algorithms include minimization of (i) the MRL_1 and (ii) EMRL_1 for known and unknown shift sizes, respectively. In this chapter, the findings and contributions of this research are summarized in Section 5.2; while some potential topics for future research are discussed in Section 5.3.

5.2 Findings and Contributions of this Dissertation

A comprehensive understanding and an in-depth knowledge of a control chart's behavior is vitally viewed by practitioners as it helps them to increase their confidence. The ARL criterion has received too much emphasis in the existing VSS charts. This will result in some important and useful information regarding the run-length properties and behavior of the VSS charts to be overlooked in their implementation by practitioners. In this research, we propose that the percentiles of the run-length distribution, particularly the 5^{th},

50^{th} (MRL) and 95^{th} percentiles are taken into account in the design of the VSS $\bar{X}$ charts as they provide several practical advantages to the shop floor personnel and quality practitioners. Consequently, we develop two optimal VSS $\bar{X}$ charts with $n_1 = n_S$ and $n_1 = n_L$ by minimizing the MRL_1 and EMRL_1, for known and unknown shift sizes, respectively. When the characteristics of the data for the out-of-control process are inadequate or unknown, the optimization design based on EMRL proposed in this research is able to cope with this random-shift size problem.

Specific optimal charts' parameters for the two VSS schemes are provided in Tables 4.2 to 4.4 of Chapter 4 and Tables D.4 to D.10 in Appendix D. These tables are useful and appealing to quality practitioners whose priority is to implement the control chart immediately. The results in Tables 4.2 to 4.4 of Chapter 4 and Tables D.4 to D.10 in Appendix D show that there is a remarkable improvement in the VSS $\bar{X}$ chart with $n_1 = n_L$, for detecting moderate to large shifts; while the performances of the VSS $\bar{X}$ chart with $n_1 = n_S$ and EWMA $\bar{X}$ chart are comparable. Undeniably, for very small shifts, one still opts for the EWMA $\bar{X}$ chart.

The optimal designs of the two VSS schemes based on MRL and EMRL suggested in this research provide an alternative to the SPC users in designing control charts. Four optimization programs for the MRL- and EMRL-based VSS schemes developed with the ScicosLab software are shown in Appendix A. These developed programs enable quick and easy computations of the optimal chart's parameters for these two VSS schemes, based on a specific or a range of shift sizes.

5.3 Recommendations for Future Research

Some potential topics for future research are discussed as follows:

(i) Optimal designs of the VSS $\bar{X}$ charts with estimated process parameters based on MRL and EMRL. In this dissertation, when the process parameters are known, two optimal designs of the VSS $\bar{X}$ charts with $n_1 = n_S$ and $n_1 = n_L$ based on MRL and EMRL are proposed. Castagliola et al. (2012) developed an optimal design of the VSS $\bar{X}$ chart with $n_1 = n_S$ based on ARL, when the process parameters are estimated. However, the ARL is a confusing measure especially for the control charts with estimated process parameters (see Bischak and Trietsch, 2007; Jensen et al., 2008; Teoh et al., 2014). Therefore, it is worthwhile to recommend new optimal designs for the VSS $\bar{X}$ charts with $n_1 = n_S$ and $n_1 = n_L$ based on MRL and EMRL, when the process parameters are estimated.

(ii) Monitoring of processes under non-normal underlying distribution. In this dissertation, the assumption of the proposed MRL- and EMRL-based VSS schemes is the normal underlying distribution. However, in some production processes, this assumption may not be true. Future research can be conducted to study the VSS $\bar{X}$ charts based on MRL and EMRL under non-normality. For example, Lin and Chou (2005) discussed the VSS and VSSCL $\bar{X}$ charts under Burr distribution in terms of the ATS and adjusted ATS. It is interesting to further study the VSS and VSSCL $\bar{X}$ charts under Burr distribution in terms of median time to signal (MTS) and adjusted MTS.

(iii) Economic design. Yeong et al. (2016) proposed the economic and economic-statistical designs of the Hotelling's T^2 chart based on EARL. Inspired by Yeong et al. (2016), future research can be performed to develop the economic and economic-statistical designs of the two VSS $\bar{X}$ charts based on MRL and EMRL. Therefore, the expected costs for the two VSS $\bar{X}$ charts can be minimized. Through these economic and economic-statistical designs, practitioners are able to select design parameters which minimize the cost by specifying an exact shift size or a range of possible shift sizes.

(iv) Monitoring of shifts in the process dispersion. In this dissertation, the two VSS $\bar{X}$ charts for monitoring the process mean shifts are investigated. Future research can be performed to develop optimal designs of the VSS dispersion-type charts based on MRL and EMRL. For example, Castagliola et al. (2008) proposed the optimal VSS S^2-EWMA chart for process dispersion based on ATS. Therefore, future research can be extended to propose the optimal VSS S^2-EWMA chart based on MRL and EMRL.

(v) Monitoring of auto-correlated processes. The performance of the VSS $\bar{X}$ charts with $n_1 = n_S$ and $n_1 = n_L$ shown in this dissertation, requires the assumption of independent observations. Montgomery (2013) stated that the assumption of independence may not be true in some manufacturing processes. For example, the consecutive measurements on product or process characteristics, such as the viscosity measurements in chemical processes, are often highly correlated (Montgomery, 2013). Accordingly, future research can be explored to design the optimal VSS $\bar{X}$ charts with $n_1 = n_S$ and $n_1 = n_L$ based on MRL and EMRL for auto-correlated data.

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APPENDIX A

PROGRAMS FOR THE VSS $\bar{X}$ CHART

A.1 Optimization Programs for the VSS $\bar{X}$ Chart Based on ARL

Two optimization programs are written using the ScicosLab software to compute the optimal charts' parameters (n_S, n_L, *W*, *K*), ARL, SDRL and ASS for the VSS $\bar{X}$ charts with $n_1 = n_S$ (see Appendix A.1.1) and $n_1 = n_L$ (see Appendix A.1.2). To obtain a particular result, we need to call the related function in the Command Window of the ScicosLab software. The function body, which contains all the ScicosLab codes, is written in the Scipad of the ScicosLab software. Note that the desired ARL_0 and δ are replaced by 'tau0' and 'delta' in these programs.

A.1.1 An Optimization Program for the ARL-based VSS $\bar{X}$ Chart with $n_1 = n_S$

For the VSS $\bar{X}$ chart with $n_1 = n_S$, the input parameters and call functions, typed at the Command Window, are listed as follows:

(i) To compute the ASS, ARL and SDRL for the given values of n_S, n_L, *W*, *K* and δ, enter

```
nS=9;nL=15;W=1.3728;K=2.9997;delta=2;
[ASS,ARL,SDRL]=rlvssxb(nS,nL,W,K,delta)
```

(ii) To compute the optimal chart's parameters (n_S, n_L, W, K) as well as the corresponding ASS_1, ARL_1 and SDRL_1 values, for the desired values of n, δ and ARL_0, enter

```
n=10;delta=2;tau0=250;
[nS,nL,W,K,ASS,ARL,SDRL]=optvssxb(n,delta,tau0)
```

The written ScicosLab program in the Scipad is shown as follows:

```
//-----------------------------------------------------------------
function [e1,e2]=momdphase(Q,q)
//-----------------------------------------------------------------
[argout,argin]=argn()
if argin~=2
  error("incorrect number of arguments")
end
q=q(:)'
W=inv(eye(Q)-Q)
z=q*W
nu1=sum(z)
e1=nu1
WQ=W*Q
z=z*WQ
nu2=2*sum(z)
e2=nu2+nu1

//-----------------------------------------------------------------
function [Q,q]=qvssxb(nS,nL,W,K,delta)
//-----------------------------------------------------------------
dsnS=delta*sqrt(nS)
dsnL=delta*sqrt(nL)
pSnS=cdfnormal(W-dsnS)-cdfnormal(-W-dsnS)
pSnL=cdfnormal(W-dsnL)-cdfnormal(-W-dsnL)
pLnS=cdfnormal(-W-dsnS)-cdfnormal(-K-dsnS)+cdfnormal(K-dsnS)-..
cdfnormal(W-dsnS)
pLnL=cdfnormal(-W-dsnL)-cdfnormal(-K-dsnL)+cdfnormal(K-dsnL)-..
cdfnormal(W-dsnL)
Q=[pSnS,pLnS;pSnL,pLnL]
q=[1;0]

//-----------------------------------------------------------------
function [ASS,ARL,SDRL]=rlvssxb(nS,nL,W,K,delta)
//-----------------------------------------------------------------
[argout,argin]=argn()
if (argin<4)|(argin>5)
  error("incorrect number of arguments")
end
if (nS<=0)|(nS~=floor(nS))
  error("argument ''nS'' must be an integer >= 1")
end
if (nL<=0)|(nL~=floor(nL))|(nL<nS)
  error("argument ''nL'' must be an integer >= nS")
end
if W<0
  error("argument ''W'' must be >= 0")
```

```
end
if K<W
  error("argument ''K'' must be >= W")
end
if argin==4
  delta=0
end
if delta<0
  error("argument ''delta'' must be >= 0")
end
[Q,q]=qvssxb(nS,nL,W,K,delta)
P=[Q,1-sum(Q,"c");1,0,0]
R=(P-eye(P))'
R(1,:)=[1,1,1]
ASS=[nS,nL,nS]*(R\[1;0;0])
[e1,e2]=momdphase(Q,q)
ARL=e1
SDRL=sqrt(e2-e1^2)

//---------------------------------------------------------------
function z=warlvssxb_(W,n,nS,nL,K)
//---------------------------------------------------------------
if (W<=0)|(W>K)
  z=%inf
else
  ASS=rlvssxb(nS,nL,W,K)
  z=ASS-n
end

//---------------------------------------------------------------
function W=warlvssxb(n,nS,nL,K)
//---------------------------------------------------------------
[argout,argin]=argn()
if argin~=4
  error("incorrect number of arguments")
end
if (nS<=0)|(nS~=floor(nS))
  error("argument ''nS'' must be an integer >= 1")
end
if (nL<=0)|(nL~=floor(nL))|(nL<nS)
  error("argument ''nL'' must be an integer >= nS")
end
if (n<nS)|(n>nL)|(n~=floor(n))
  error("argument ''n'' must be an integer in {nS,...,nL}")
end
W=simplexolve(1,warlvssxb_,list(n,nS,nL,K))

//---------------------------------------------------------------
function [nS,nL,W,K,ASS,ARL,SDRL]=optvssxb(n,delta,tau0)
//---------------------------------------------------------------
[argout,argin]=argn()
if argin~=3
  error("incorrect number of arguments")
end
if (n<=0)|(n~=floor(n))
  error("argument ''n'' must be an integer >= 1")
end
K=idfnormal(1-1/(2*tau0))
ARL=%inf
for inS=1:n-1
  for inL=n+1:31
```

```
    W1=warlvssxb(n,inS,inL,K)
    [ASS1,ARL1,SDRL1]=rlvssxb(inS,inL,W1,K,delta)
    [ASS0,ARL0]=rlvssxb(inS,inL,W1,K)

mprintf("%2d %2d %8.6f %8.6f %6.2f %6.2f %6.2f %6.2f %6.2f\n",..
[inS,inL,W1,K,ASS1,ARL1,SDRL1,ASS0,ARL0])

    if (ARL1<=ARL)
      ASS=ASS1
      ARL=ARL1
      SDRL=SDRL1
      W=W1
      nS=inS
      nL=inL
    end
  end
end
mprintf("\n\n(%2d,%2d,%6.4f,%6.4f)%6.2f(%6.2f, %6.2f)\n",[nS,..
n,W,K,ASS,ARL,SDRL])
```

A.1.2 An Optimization Program for the ARL-based VSS $\bar{X}$ Chart with $\boldsymbol{n_1 = n_L}$

For the VSS $\bar{X}$ chart with $n_1 = n_L$, the input parameters and call functions, typed at the Command Window, are described as follows:

(i) To compute the ASS, ARL and SDRL for the desired values of n_S, n_L, *W*, *K* and δ, enter

```
nS=9;nL=31;W=2.0249;K=2.9997;delta=2;
[ASS,ARL,SDRL]=rlvssxb(nS,nL,W,K,delta)
```

(ii) To compute the optimal chart's parameters (n_S, n_L, *W*, *K*) as well as the corresponding ASS_1, ARL_1 and SDRL_1 values, for the desired values of *n*, δ and ARL_0, enter

```
n=10;delta=2;tau0=250;
[nS,nL,W,K,ASS,ARL,SDRL]=optvssxb(n,delta,tau0)
```

The written ScicosLab program in the Scipad is shown as follows:

```
//-----------------------------------------------------------------
function [e1,e2]=momdphase(Q,q)
//-----------------------------------------------------------------
[argout,argin]=argn()
if argin~=2
  error("incorrect number of arguments")
end
q=q(:)'
W=inv(eye(Q)-Q)
z=q*W
nu1=sum(z)
e1=nu1
WQ=W*Q
z=z*WQ
nu2=2*sum(z)
e2=nu2+nu1

//-----------------------------------------------------------------
function [Q,q]=qvssxb(nS,nL,W,K,delta)
//-----------------------------------------------------------------
dsnS=delta*sqrt(nS)
dsnL=delta*sqrt(nL)
pSnS=cdfnormal(W-dsnS)-cdfnormal(-W-dsnS)
pSnL=cdfnormal(W-dsnL)-cdfnormal(-W-dsnL)
pLnS=cdfnormal(-W-dsnS)-cdfnormal(-K-dsnS)+cdfnormal(K-dsnS)-..
cdfnormal(W-dsnS)
pLnL=cdfnormal(-W-dsnL)-cdfnormal(-K-dsnL)+cdfnormal(K-dsnL)-..
cdfnormal(W-dsnL)
Q=[pSnS,pLnS;pSnL,pLnL]
q=[0,1]

//-----------------------------------------------------------------
function [ASS,ARL,SDRL]=rlvssxb(nS,nL,W,K,delta)
//-----------------------------------------------------------------
[argout,argin]=argn()
if (argin<4)|(argin>5)
  error("incorrect number of arguments")
end
if (nS<=0)|(nS~=floor(nS))
  error("argument ''nS'' must be an integer >= 1")
end
if (nL<=0)|(nL~=floor(nL))|(nL<nS)
  error("argument ''nL'' must be an integer >= nS")
end
if W<0
  error("argument ''W'' must be >= 0")
end
if K<W
  error("argument ''K'' must be >= W")
end
if argin==4
  delta=0
end
if delta<0
  error("argument ''delta'' must be >= 0")
end
[Q,q]=qvssxb(nS,nL,W,K,delta)
P=[Q,1-sum(Q,"c");0,1,0]
R=(P-eye(P))'
```

```
R(2,:)=[1,1,1]
ASS=[nS,nL,nL]*(R\[0;1;0])
[e1,e2]=momdphase(Q,q)
ARL=e1
SDRL=sqrt(e2-e1^2)

//-----------------------------------------------------------------
function z=warlvssxb_(W,n,nS,nL,K)
//-----------------------------------------------------------------
if (W<=0)|(W>K)
  z=%inf
else
  ASS=rlvssxb(nS,nL,W,K)
  z=ASS-n
end

//-----------------------------------------------------------------
function W=warlvssxb(n,nS,nL,K)
//-----------------------------------------------------------------
[argout,argin]=argn()
if argin~=4
  error("incorrect number of arguments")
end
if (nS<=0)|(nS~=floor(nS))
  error("argument ''nS'' must be an integer >= 1")
end
if (nL<=0)|(nL~=floor(nL))|(nL<nS)
  error("argument ''nL'' must be an integer >= nS")
end
if (n<nS)|(n>nL)|(n~=floor(n))
  error("argument ''n'' must be an integer in {nS,...,nL}")
end
W=simplexolve(1,warlvssxb_,list(n,nS,nL,K))

//-----------------------------------------------------------------
function [nS,nL,W,K,ASS,ARL,SDRL]=optvssxb(n,delta,tau0)
//-----------------------------------------------------------------
[argout,argin]=argn()
if argin~=3
  error("incorrect number of arguments")
end
if (n<=0)|(n~=floor(n))
  error("argument ''n'' must be an integer >= 1")
end
K=idfnormal(1-1/(2*tau0))
ARL=%inf
for inS=1:n-1
  for inL=n+1:31
    W1=warlvssxb(n,inS,inL,K)
    [ASS1,ARL1,SDRL1]=rlvssxb(inS,inL,W1,K,delta)
    [ASS0,ARL0]=rlvssxb(inS,inL,W1,K)

mprintf("%2d %2d %8.6f %8.6f %6.2f %6.2f %6.2f %6.2f %6.2f\n",..
[inS,inL,W1,K,ASS1,ARL1,SDRL1,ASS0,ARL0])

    if (ARL1<=ARL)
      ASS=ASS1
      ARL=ARL1
      SDRL=SDRL1
      W=W1
      nS=inS
```

```
      nL=inL
    end
  end
end
mprintf("\n\n(%2d, %2d, %6.4f, %6.4f) %6.2f(%6.2f, %6.2f)\n",..
[nS,nL,W,K,ASS,ARL,SDRL])
```

A.2 Optimization Programs for the VSS $\bar{X}$ Chart Based on MRL and EMRL

Four optimization programs are written using the ScicosLab software to compute the optimal charts' parameters (n_S, n_L, W, K), MRL, percentiles of the run-length distribution and ASS of the VSS $\bar{X}$ chart with $n_1 = n_S$ (see Appendix A.2.1) and $n_1 = n_L$ (see Appendix A.2.2). To obtain a particular result, we need to call the related function in the Command Window of the ScicosLab software. The function body, which contains all the ScicosLab codes, is written in the Scipad of the ScicosLab software. In these ScicosLab programs, 'pcrl' represents the probability γ and 'mrlini' is the initial value of the MRL. Also, 'delta', 'deltamin' and 'deltamax' in the programs denote δ, $\delta_{\min}$ and $\delta_{\max}$, respectively. The 5th and 95th percentiles of the run-length distribution are replaced by Q05 and Q95 in these programs.

A.2.1 Optimization Programs for the MRL- and EMRL-based VSS $\bar{X}$ Chart with $n_1 = n_S$

For the MRL-based VSS $\bar{X}$ chart with $n_1 = n_S$, the input parameters and call functions, typed at the Command Window, are presented as follows:

(i) To compute the MRL or the percentiles of the run-length distribution for the given values of γ, n_S, n_L, W, K and δ, enter

```
pcrl=0.05;mrlini=1;nS=9;nL=15;W=1.3728;K=2;delta=2;
X=pcrlvssxb(pcrl,mrlini,nS,nL,W,K,delta)
```

(ii) To compute the ASS for the given values of n_S, n_L, W, K and δ, enter

```
nS=2;nL=9;W=1.4540;K=3;delta=0;
ASS=assrlvssxb(nS,nL,W,K,delta)
```

(iii) To compute the optimal chart's parameters (n_S, n_L, W, K) as well as the corresponding ASS_1, MRL_1, 5^{th} and 95^{th} percentiles of the run-length distribution, for the desired values of n, MRL_0 and δ, enter

```
n=10;mrl0=200;delta=2;
[nS,nL,W,K,ASS,Q05,MRL,Q95]=optvssxb(n,mrl0,delta)
```

For the EMRL-based VSS $\bar{X}$ chart with $n_1 = n_S$, the input parameters and call functions, typed at the Command Window, are demonstrated as follows:

(i) To compute the EMRL for the given values of $\delta_{\min}$, $\delta_{\max}$, n_S, n_L, W and K, enter

```
deltamin=0.1;deltamax=2;nS=2;nL=9;W=1.4540;K=3;
EMRL=emrlvssxb(deltamin,deltamax,nS,nL,W,K)
```

(ii) To compute the optimal chart's parameters (n_S, n_L, W, K) and EMRL_1 for the desired values of n, EMRL_0, $\delta_{\min}$ and $\delta_{\max}$, enter

```
n=10;mrl0=500;deltamin=0.1;deltamax=2;
[nS,nL,W,K,EMRL]=optemrlvssxb(n,mrl0,deltamin,deltamax)
```

The written ScicosLab programs in the Scipad are shown below:

```
//------------------------------------------------------------
function F=cdfvssxb(Q,q,pct)
//------------------------------------------------------------
[argout,argin]=argn()
if argin~=3
  error("incorrect number of arguments")
end
q=q(:)'
Ft1=q*Q^pct
Ft2=sum(Ft1)
F=Ft2

//------------------------------------------------------------
function [Q,q]=qvssxb(nS,nL,W,K,delta)
//------------------------------------------------------------
dsnS=delta*sqrt(nS)
dsnL=delta*sqrt(nL)
pSnS=cdfnormal(W-dsnS)-cdfnormal(-W-dsnS)
pSnL=cdfnormal(W-dsnL)-cdfnormal(-W-dsnL)
pLnS=cdfnormal(-W-dsnS)-cdfnormal(-K-dsnS)+cdfnormal(K-dsnS)-..
cdfnormal(W-dsnS)
pLnL=cdfnormal(-W-dsnL)-cdfnormal(-K-dsnL)+cdfnormal(K-dsnL)-..
cdfnormal(W-dsnL)
Q=[pSnS,pLnS;pSnL,pLnL]
q=[1;0]

//------------------------------------------------------------
function X=pcrlvssxb(pcrl,mrlini,nS,nL,W,K,delta)
//------------------------------------------------------------
[argout,argin]=argn()
if (argin<6)|(argin>7)
  error("incorrect number of arguments")
end
if (nS<=0)|(nS~=floor(nS))
  error("argument ''nS'' must be an integer >= 1")
end
if (nL<=0)|(nL~=floor(nL))|(nL<nS)
  error("argument ''nL'' must be an integer >= nS")
end
if W<0
  error("argument ''W'' must be >= 0")
end
if K<W
  error("argument ''K'' must be >= W")
end
if argin==6
  delta=0
end
if delta<0
  error("argument ''delta'' must be >= 0")
end
[Q,q]=qvssxb(nS,nL,W,K,delta)
for X=mrlini:500000
    F=1-cdfvssxb(Q,q,X)
    if F>=pcrl
        break
     end
end

//------------------------------------------------------------
function ASS=assrlvssxb(nS,nL,W,K,delta)
```

```
//-----------------------------------------------------------------
[argout,argin]=argn()
if (argin<4)|(argin>5)
  error("incorrect number of arguments")
end
if (nS<=0)|(nS~=floor(nS))
  error("argument ''nS'' must be an integer >= 1")
end
if (nL<=0)|(nL~=floor(nL))|(nL<nS)
  error("argument ''nL'' must be an integer >= nS")
end
if W<0
  error("argument ''W'' must be >= 0")
end
if K<W
  error("argument ''K'' must be >= W")
end
if argin==4
  delta=0
end
if delta<0
  error("argument ''delta'' must be >= 0")
end
[Q,q]=qvssxb(nS,nL,W,K,delta)
P=[Q,1-sum(Q,"c");1,0,0]
R=(P-eye(P))'
R(1,:)=[1,1,1]
ASS=[nS,nL,nS]*(R\[1;0;0])

//-----------------------------------------------------------------
function [ASS,probMRL]=rlvssxb(nS,nL,W,K,mrl0,delta)
//-----------------------------------------------------------------
[argout,argin]=argn()
if (argin<5)|(argin>6)
  error("incorrect number of arguments")
end
if (nS<=0)|(nS~=floor(nS))
  error("argument ''nS'' must be an integer >= 1")
end
if (nL<=0)|(nL~=floor(nL))|(nL<nS)
  error("argument ''nL'' must be an integer >= nS")
end
if W<0
  error("argument ''W'' must be >= 0")
end
if K<W
  error("argument ''K'' must be >= W")
end
if argin==5
  delta=0
end
if delta<0
  error("argument ''delta'' must be >= 0")
end
[Q,q]=qvssxb(nS,nL,W,K,delta)
P=[Q,1-sum(Q,"c");1,0,0]
R=(P-eye(P))'
R(1,:)=[1,1,1]
ASS=[nS,nL,nS]*(R\[1;0;0])
probMRL=1-cdfvssxb(Q,q,mrl0)
```

```
//------------------------------------------------------------
function dif=searchalpha(alpha,mrl0)
//------------------------------------------------------------
if (alpha<0)|(alpha>1)
  dif=%inf
else
  dif=mrl0-log(0.5)./log(1-alpha)
end

//------------------------------------------------------------
function z=warlvssxb_(W,n,nS,nL,K)
//------------------------------------------------------------
if (W<=0)|(W>K)
  z=%inf
else
  ASS=assrlvssxb(nS,nL,W,K)
  z=ASS-n
end

//------------------------------------------------------------
function W=warlvssxb(n,nS,nL,K)
//------------------------------------------------------------
[argout,argin]=argn()
if argin~=4
  error("incorrect number of arguments")
end
if (nS<=0)|(nS~=floor(nS))
  error("argument ''nS'' must be an integer >= 1")
end
if (nL<=0)|(nL~=floor(nL))|(nL<nS)
  error("argument ''nL'' must be an integer >= nS")
end
if (n<nS)|(n>nL)|(n~=floor(n))
  error("argument ''n'' must be an integer in {nS,...,nL}")
end
W=simplexolve(1,warlvssxb_,list(n,nS,nL,K))

//------------------------------------------------------------
function z=wkarlvssxb_(WK,n,nS,nL,mrl0)
//------------------------------------------------------------
W=WK(1)
K=WK(2)
if (W<=0)|(W>K)
  z=%inf
else
  [ASS0,probMRL]=rlvssxb(nS,nL,W,K,mrl0)
  z=[(ASS0-n),(probMRL-0.5001)*10]
end

//------------------------------------------------------------
function [W,K]=wkarlvssxb(n,nS,nL,mrl0)
//------------------------------------------------------------
[argout,argin]=argn()
if argin~=4
  error("incorrect number of arguments")
end
if (nS<=0)|(nS~=floor(nS))
  error("argument ''nS'' must be an integer >= 1")
end
if (nL<=0)|(nL~=floor(nL))|(nL<nS)
  error("argument ''nL'' must be an integer >= nS")
```

```
end
if (n<nS)|(n>nL)|(n~=floor(n))
  error("argument ''n'' must be an integer in {nS,...,nL}")
end
alpha0=1/mrl0
alpha=simplexolve(alpha0,searchalpha,list(mrl0),tol=1e-6)
arl0=1/alpha
K0=idfnormal(1-1/(2*arl0))
W0=warlvssxb(n,nS,nL,K0)
WK=simplexolve([W0,K0],wkarlvssxb_,list(n,nS,nL,mrl0),tol=1e-6)
W=WK(1)
K=WK(2)

//-----------------------------------------------------------------
function [nS,nL,W,K,ASS,Q05,MRL,Q95]=optvssxb(n,mrl0,delta)
//-----------------------------------------------------------------
[argout,argin]=argn()
if argin~=3
  error("incorrect number of arguments")
end
if (n<=0)|(n~=floor(n))
  error("argument ''n'' must be an integer >= 1")
end
mprintf("delta=%3.2f\n\n",delta)
sol=[]
ii=1
MRLmin=%inf
diffPCRmin=%inf
ASSmin=%inf
for inS=1:n-1
  for inL=n+1:31
    [W1,K1]=wkarlvssxb(n,inS,inL,mrl0)
    PCR5=pcrlvssxb(0.05,1,inS,inL,W1,K1,delta)
    MRL1=pcrlvssxb(0.5,PCR5,inS,inL,W1,K1,delta)
    PCR95=pcrlvssxb(0.95,MRL1,inS,inL,W1,K1,delta)
    MRL00=pcrlvssxb(0.5,mrl0-5,inS,inL,W1,K1)
    ASS1=assrlvssxb(inS,inL,W1,K1,delta)
    ASS0=assrlvssxb(inS,inL,W1,K1)
    diffPCR=PCR95-PCR5

mprintf("%2d %2d %8.6f %8.6f %8.4f %5d %5d %5d %5.2f %5d\n",..
[inS,inL,W1,K1,ASS1,PCR5,MRL1,PCR95,ASS0,MRL00])
    sol(ii,1)=inS
    sol(ii,2)=inL
    sol(ii,3)=W1
    sol(ii,4)=K1
    sol(ii,5)=ASS1
    sol(ii,6)=PCR5
    sol(ii,7)=MRL1
    sol(ii,8)=PCR95
    sol(ii,9)=diffPCR
    ii=ii+1
    if MRL1<MRLmin
        MRLmin=MRL1
  end
  if diffPCR<diffPCRmin
        diffPCRmin=diffPCR
    end
  end
end
```

```
for jj=1:(ii-1)
  if sol(jj,7)==MRLmin
      if sol(jj,5)<ASSmin
          ASSmin=sol(jj,5)

solf=[sol(jj,1),sol(jj,2),sol(jj,3),sol(jj,4),sol(jj,5),..
sol(jj,6),sol(jj,7),sol(jj,8)]
      end
  end
end

ASSmin=%inf
for jj=1:(ii-1)
  if (sol(jj,7)==MRLmin)&(sol(jj,9)==diffPCRmin)
      if sol(jj,5)<ASSmin
          ASSmin=sol(jj,5)

solf=[sol(jj,1),sol(jj,2),sol(jj,3),sol(jj,4),sol(jj,5),..
sol(jj,6),sol(jj,7),sol(jj,8)]
      end
  end
end
nS=solf(1)
nL=solf(2)
W=solf(3)
K=solf(4)
ASS=solf(5)
Q05=solf(6)
MRL=solf(7)
Q95=solf(8)
mprintf("\n(%2d,%2d,%6.4f,%6.4f) %5.2f(%5d,%5d,%5d)\n\n\n\n",..
[nS,nL,W,K,ASS,Q05,MRL,Q95])

//---------------------------------------------------------------
function EMRL=emrlvssxb(deltamin,deltamax,nS,nL,W,K)
//---------------------------------------------------------------
[xi,wi]=quadlegendre(9,deltamin,deltamax)
EMRL=0
for il=1:9
  xil=xi(il)
  wil=wi(il)
  MRL=pcrlvssxb(0.5,1,nS,nL,W,K,xil)
  EMRL=EMRL+MRL.*wil
end
EMRL=EMRL/(deltamax-deltamin)

//---------------------------------------------------------------
function[nS,nL,W,K,EMRL]=optemrlvssxb(n,mrl0,deltamin,deltamax)
//---------------------------------------------------------------
[argout,argin]=argn()
if argin~=4
  error("incorrect number of arguments")
end
if (n<=0)|(n~=floor(n))
  error("argument ''n'' must be an integer >= 1")
end
sol=[]
EMRL=%inf
for inS=1:n-1
  for inL=n+1:31
    [W1,K1]=wkarlvssxb(n,inS,inL,mrl0)
```

```
    EMRLt=emrlvssxb(deltamin,deltamax,inS,inL,W1,K1)
    MRL00=pcrlvssxb(0.5,mrl0-5,inS,inL,W1,K1)
    ASS0=assrlvssxb(inS,inL,W1,K1)

mprintf("%2d %2d %8.6f %8.6f %8.4f %5.2f %5d\n",[inS,inL,W1,..
K1,EMRLt,ASS0,MRL00])
    if EMRLt<EMRL
        EMRL=EMRLt
        W=W1
        K=K1
        nS=inS
        nL=inL
    end
  end
end
mprintf("\n\n(%2d, %2d, %6.4f, %6.4f) %5.2f\n\n",[nS,nL,W,K,..
EMRL])

for delta={0.25,0.5,0.75,1.0,1.5,2.0}
    PCR5=pcrlvssxb(0.05,1,nS,nL,W,K,delta)
    MRL1=pcrlvssxb(0.5,PCR5,nS,nL,W,K,delta)
    PCR95=pcrlvssxb(0.95,MRL1,nS,nL,W,K,delta)
    mprintf("%3.2f (%3d, %4d, %6d)\n",[delta,PCR5,MRL1,PCR95])
end
```

A.2.2 Optimization Programs for the MRL- and EMRL-based VSS $\bar{X}$ Chart with $n_1 = n_L$

For the MRL-based VSS $\bar{X}$ chart with $n_1 = n_L$, the input parameters and call functions, typed at the Command Window, are outlined as follows:

(i) To compute the MRL or the percentiles of the run-length distribution for the given values of γ, n_S, n_L, W, K and δ, enter

```
pcrl=0.05;mrlini=1;nS=9;nL=31;W=2.0249;K=2;delta=1;
X=pcrlvssxb(pcrl,mrlini,nS,nL,W,K,delta)
```

(ii) To compute the ASS for the given values of n_S, n_L, W, K and δ, enter

```
nS=2;nL=9;W=1.4540;K=3;delta=0;
ASS=assrlvssxb(nS,nL,W,K,delta)
```

(iii) To compute the optimal chart's parameters (n_S, n_L, W, K) as well as the corresponding ASS_1, MRL_1, 5^{th} and 95^{th} percentiles of the run-length distribution, for the desired values of n, MRL_0 and δ, enter

```
n=3;mrl0=370;delta=0.25;
[nS,nL,W,K,ASS,Q05,MRL,Q95]=optvssxb(n,mrl0,delta)
```

For the EMRL-based VSS $\bar{X}$ chart with $n_1 = n_L$, the input parameters and call functions, typed at the Command Window, are presented as follows:

(i) To compute the EMRL for the given values of $\delta_{\min}$, $\delta_{\max}$, n_S, n_L, W and K, enter

```
deltamin=0.1;deltamax=2;nS=2;nL=9;W=1.4540;K=3;
EMRL=emrlvssxb(deltamin,deltamax,nS,nL,W,K)
```

(ii) To compute the optimal chart's parameters (n_S, n_L, W, K) and EMRL_1 for the desired values of n, EMRL_0, $\delta_{\min}$ and $\delta_{\max}$, enter

```
n=10;mrl0=500;deltamin=0.1;deltamax=2;
[nS,nL,W,K,EMRL]=optemrlvssxb(n,mrl0,deltamin,deltamax)
```

The written ScicosLab programs in the Scipad are presented as follows:

```
//-------------------------------------------------------------
function F=cdfvssxb(Q,q,pct)
//-------------------------------------------------------------
[argout,argin]=argn()
if argin~=3
  error("incorrect number of arguments")
end
q=q(:)'
Ft1=q*Q^pct
Ft2=sum(Ft1)
F=Ft2

//-------------------------------------------------------------
function [Q,q]=qvssxb(nS,nL,W,K,delta)
//-------------------------------------------------------------
dsnS=delta*sqrt(nS)
dsnL=delta*sqrt(nL)
pSnS=cdfnormal(W-dsnS)-cdfnormal(-W-dsnS)
pSnL=cdfnormal(W-dsnL)-cdfnormal(-W-dsnL)
pLnS=cdfnormal(-W-dsnS)-cdfnormal(-K-dsnS)+cdfnormal(K-dsnS)-..
cdfnormal(W-dsnS)
pLnL=cdfnormal(-W-dsnL)-cdfnormal(-K-dsnL)+cdfnormal(K-dsnL)-..
cdfnormal(W-dsnL)
Q=[pSnS,pLnS;pSnL,pLnL]
```

```
q=[0;1]

//------------------------------------------------------------------
function X=pcrlvssxb(pcrl,mrlini,nS,nL,W,K,delta)
//------------------------------------------------------------------
[argout,argin]=argn()
if (argin<6)|(argin>7)
  error("incorrect number of arguments")
end
if (nS<=0)|(nS~=floor(nS))
  error("argument ''nS'' must be an integer >= 1")
end
if (nL<=0)|(nL~=floor(nL))|(nL<nS)
  error("argument ''nL'' must be an integer >= nS")
end
if W<0
  error("argument ''W'' must be >= 0")
end
if K<W
  error("argument ''K'' must be >= W")
end
if argin==6
  delta=0
end
if delta<0
  error("argument ''delta'' must be >= 0")
end
[Q,q]=qvssxb(nS,nL,W,K,delta)
for X=mrlini:500000
    F=1-cdfvssxb(Q,q,X)
    if F>=pcrl
        break
     end
end

//------------------------------------------------------------------
function ASS=assrlvssxb(nS,nL,W,K,delta)
//------------------------------------------------------------------
[argout,argin]=argn()
if (argin<4)|(argin>5)
  error("incorrect number of arguments")
end
if (nS<=0)|(nS~=floor(nS))
  error("argument ''nS'' must be an integer >= 1")
end
if (nL<=0)|(nL~=floor(nL))|(nL<nS)
  error("argument ''nL'' must be an integer >= nS")
end
if W<0
  error("argument ''W'' must be >= 0")
end
if K<W
  error("argument ''K'' must be >= W")
end
if argin==4
  delta=0
end
if delta<0
  error("argument ''delta'' must be >= 0")
end
[Q,q]=qvssxb(nS,nL,W,K,delta)
```

```
P=[Q,1-sum(Q,"c");0,1,0]
R=(P-eye(P))'
R(2,:)=[1,1,1]
ASS=[nS,nL,nL]*(R\[0;1;0])

//----------------------------------------------------------------
function [ASS,probMRL]=rlvssxb(nS,nL,W,K,mrl0,delta)
//----------------------------------------------------------------
[argout,argin]=argn()
if (argin<5)|(argin>6)
  error("incorrect number of arguments")
end
if (nS<=0)|(nS~=floor(nS))
  error("argument ''nS'' must be an integer >= 1")
end
if (nL<=0)|(nL~=floor(nL))|(nL<nS)
  error("argument ''nL'' must be an integer >= nS")
end
if W<0
  error("argument ''W'' must be >= 0")
end
if K<W
  error("argument ''K'' must be >= W")
end
if argin==5
  delta=0
end
if delta<0
  error("argument ''delta'' must be >= 0")
end
[Q,q]=qvssxb(nS,nL,W,K,delta)
P=[Q,1-sum(Q,"c");0,1,0]
R=(P-eye(P))'
R(2,:)=[1,1,1]
ASS=[nS,nL,nL]*(R\[0;1;0])
probMRL=1-cdfvssxb(Q,q,mrl0)

//----------------------------------------------------------------
function dif=searchalpha(alpha,mrl0)
//----------------------------------------------------------------
if (alpha<0)|(alpha>1)
  dif=%inf
else
  dif=mrl0-log(0.5)./log(1-alpha)
end

//----------------------------------------------------------------
function z=warlvssxb_(W,n,nS,nL,K)
//----------------------------------------------------------------
if (W<=0)|(W>K)
  z=%inf
else
  ASS=assrlvssxb(nS,nL,W,K)
  z=ASS-n
end
//----------------------------------------------------------------
function W=warlvssxb(n,nS,nL,K)
//----------------------------------------------------------------
[argout,argin]=argn()
if argin~=4
  error("incorrect number of arguments")
```

```
end
if (nS<=0)|(nS~=floor(nS))
  error("argument ''nS'' must be an integer >= 1")
end
if (nL<=0)|(nL~=floor(nL))|(nL<nS)
  error("argument ''nL'' must be an integer >= nS")
end
if (n<nS)|(n>nL)|(n~=floor(n))
  error("argument ''n'' must be an integer in {nS,...,nL}")
end
W=simplexolve(1,warlvssxb_,list(n,nS,nL,K))

//-------------------------------------------------------------
function z=wkarlvssxb_(WK,n,nS,nL,mrl0)
//-------------------------------------------------------------
W=WK(1)
K=WK(2)
if (W<=0)|(W>K)
  z=%inf
else
  [ASS0,probMRL]=rlvssxb(nS,nL,W,K,mrl0)
  z=[(ASS0-n),(probMRL-0.5001)*10]
end

//-------------------------------------------------------------
function [W,K]=wkarlvssxb(n,nS,nL,mrl0)
//-------------------------------------------------------------
[argout,argin]=argn()
if argin~=4
  error("incorrect number of arguments")
end
if (nS<=0)|(nS~=floor(nS))
  error("argument ''nS'' must be an integer >= 1")
end
if (nL<=0)|(nL~=floor(nL))|(nL<nS)
  error("argument ''nL'' must be an integer >= nS")
end
if (n<nS)|(n>nL)|(n~=floor(n))
  error("argument ''n'' must be an integer in {nS,...,nL}")
end
alpha0=1/mrl0
alpha=simplexolve(alpha0,searchalpha,list(mrl0),tol=1e-6)
arl0=1/alpha
K0=idfnormal(1-1/(2*arl0))
W0=warlvssxb(n,nS,nL,K0)
WK=simplexolve([W0,K0],wkarlvssxb_,list(n,nS,nL,mrl0),tol=1e-6)
W=WK(1)
K=WK(2)

//-------------------------------------------------------------
function [nS,nL,W,K,ASS,Q05,MRL,Q95]=optvssxb(n,mrl0,delta)
//-------------------------------------------------------------
[argout,argin]=argn()
if argin~=3
  error("incorrect number of arguments")
end
if (n<=0)|(n~=floor(n))
  error("argument ''n'' must be an integer >= 1")
end
mprintf("delta=%3.2f\n\n",delta)
sol=[]
```

```
ii=1
MRLmin=%inf
diffPCRmin=%inf
ASSmin=%inf
for inS=1:n-1
  for inL=n+1:31
    [W1,K1]=wkarlvssxb(n,inS,inL,mrl0)
    PCR5=pcrlvssxb(0.05,1,inS,inL,W1,K1,delta)
    MRL1=pcrlvssxb(0.5,PCR5,inS,inL,W1,K1,delta)
    PCR95=pcrlvssxb(0.95,MRL1,inS,inL,W1,K1,delta)
    MRL00=pcrlvssxb(0.5,mrl0-5,inS,inL,W1,K1)
    ASS1=assrlvssxb(inS,inL,W1,K1,delta)
    ASS0=assrlvssxb(inS,inL,W1,K1)
    diffPCR=PCR95-PCR5

mprintf("%2d %2d %8.6f %8.6f %8.4f %5d %5d %5d %5.2f %5d\n",..
[inS,inL,W1,K1,ASS1,PCR5,MRL1,PCR95,ASS0,MRL00])
    sol(ii,1)=inS
    sol(ii,2)=inL
    sol(ii,3)=W1
    sol(ii,4)=K1
    sol(ii,5)=ASS1
    sol(ii,6)=PCR5
    sol(ii,7)=MRL1
    sol(ii,8)=PCR95
    sol(ii,9)=diffPCR
    ii=ii+1
    if MRL1<MRLmin
        MRLmin=MRL1
  end
  if diffPCR<diffPCRmin
        diffPCRmin=diffPCR
    end
  end
end

for jj=1:(ii-1)
  if sol(jj,7)==MRLmin
      if sol(jj,5)<ASSmin
          ASSmin=sol(jj,5)

solf=[sol(jj,1),sol(jj,2),sol(jj,3),sol(jj,4),sol(jj,5),..
sol(jj,6),sol(jj,7),sol(jj,8)]
      end
  end
end

ASSmin=%inf
for jj=1:(ii-1)
  if (sol(jj,7)==MRLmin)&(sol(jj,9)==diffPCRmin)
      if sol(jj,5)<ASSmin
          ASSmin=sol(jj,5)

solf=[sol(jj,1),sol(jj,2),sol(jj,3),sol(jj,4),sol(jj,5),..
sol(jj,6),sol(jj,7),sol(jj,8)]
      end
  end
end
nS=solf(1)
nL=solf(2)
W=solf(3)
```

```
K=solf(4)
ASS=solf(5)
Q05=solf(6)
MRL=solf(7)
Q95=solf(8)
mprintf("\n(%2d, %2d, %6.4f, %6.4f) %5.2f ..
(%5d, %5d, %5d)\n\n\n\n",[nS,nL,W,K,ASS,Q05,MRL,Q95])

//-------------------------------------------------------------
function EMRL=emrlvssxb(deltamin,deltamax,nS,nL,W,K)
//-------------------------------------------------------------
[xi,wi]=quadlegendre(9,deltamin,deltamax)
EMRL=0
for il=1:9
  xil=xi(il)
  wil=wi(il)
  MRL=pcrlvssxb(0.5,1,nS,nL,W,K,xil)
  EMRL=EMRL+MRL.*wil
end
EMRL=EMRL/(deltamax-deltamin)

//-------------------------------------------------------------
function[nS,nL,W,K,EMRL]=optemrlvssxb(n,mrl0,deltamin,deltamax)
//-------------------------------------------------------------
[argout,argin]=argn()
if argin~=4
  error("incorrect number of arguments")
end
if (n<=0)|(n~=floor(n))
  error("argument ''n'' must be an integer >= 1")
end
sol=[]
EMRL=%inf
for inS=1:n-1
  for inL=n+1:31
    [W1,K1]=wkarlvssxb(n,inS,inL,mrl0)
    EMRLt=emrlvssxb(deltamin,deltamax,inS,inL,W1,K1)
    MRL00=pcrlvssxb(0.5,mrl0-5,inS,inL,W1,K1)
    ASS0=assrlvssxb(inS,inL,W1,K1)

mprintf("%2d %2d %8.6f %8.6f %8.4f %5.2f %5d\n",[inS,inL,W1,..
K1,EMRLt,ASS0,MRL00])
    if EMRLt<EMRL
        EMRL=EMRLt
        W=W1
        K=K1
        nS=inS
        nL=inL
    end
  end
end
mprintf("\n\n(%2d, %2d, %6.4f, %6.4f) %5.2f\n\n",[nS,nL,W,K,..
EMRL])

for delta={0.25,0.5,0.75,1.0,1.5,2.0}
    PCR5=pcrlvssxb(0.05,1,nS,nL,W,K,delta)
    MRL1=pcrlvssxb(0.5,PCR5,nS,nL,W,K,delta)
    PCR95=pcrlvssxb(0.95,MRL1,nS,nL,W,K,delta)
    mprintf("%3.2f (%3d, %4d, %6d)\n",[delta,PCR5,MRL1,PCR95])
end
```

A.3 Monte Carlo Simulation Programs for the VSS $\bar{X}$ Chart

In this appendix, by using the SAS software, two simulation programs are written to verify the values of ARL, SDRL, MRL and percentiles of the run-length distribution computed from the theoretical method (see Subsection 3.3.2 of Chapter 3, Appendices A.1 and A.2) of the VSS $\bar{X}$ charts with $n_1 = n_S$ and $n_1 = n_L$. These two simulation programs of the VSS $\bar{X}$ charts with $n_1 = n_S$ and $n_1 = n_L$ are shown in Appendices A.3.1 and A.3.2, respectively. Also, Appendices A.3.3 and A.3.4 show two simulation programs written in SAS software for the example of application of the VSS $\bar{X}$ chart with $n_1 = n_S$ and $n_1 = n_L$, respectively.

A.3.1 A Simulation Program for the VSS $\bar{X}$ Chart with $n_1 = n_S$

In this simulation program, the ARL, SDRL, MRL and percentiles of the run-length distribution for the VSS $\bar{X}$ chart with $n_1 = n_S$, are calculated using 50001 simulation trials. The intended results can be obtained by inputting the values of the parameters 'delta', 'nS', 'nL', 'W' and 'K' in the SAS code.

```
Data VSSXbar;
mu0=0;
sigma=1;
delta=0.25;
nS=1;
nL=31;
W=1.8206;
K=3.1098;

Do m=1 to 50001;
   ok=1;i=0;
   ni=nS;
   Do while (ok=1);
        i=i+1;
        Xsum=0;
```

```
            Do j=1 to ni;
             X=delta+sigma*rannor(55555);
             Xsum=Xsum+X;
            End;

            Xbar=Xsum/ni;
            Z=(Xbar-mu0)/(sigma/sqrt(ni));

            If(Z<-K) or (Z>K) then do;
                  ARL=i;
                  output;
                  ok=0;
            end;

            If(Z<-W) or (Z>W) then ni=nL;
            else ni=nS;
          End;
End;
run;

proc univariate;
var ARL;
output pctlpts=5 50 95 pctlpre=p;

proc print;
run;
```

A.3.2 A Simulation Program for the VSS $\bar{X}$ Chart with $n_1 = n_L$

This simulation program computes the ARL, SDRL, MRL and percentiles of the run-length distribution for the VSS $\bar{X}$ chart with $n_1 = n_L$. By using 50001 simulation trials, the user needs to input the parameters ‘delta’, ‘nS’, ‘nL’, ‘W’ and ‘K’ in the SAS code in order to obtain the results.

```
Data VSSXbar;
mu0=0;
sigma=1;
delta=0.5;
nS=1;
nL=31;
W=1.8516;
K= 2.9922;

Do m=1 to 50001;
   ok=1;i=0;
   ni=nL;
   Do while (ok=1);
        i=i+1;
        Xsum=0;
        Do j=1 to ni;
         X=delta+sigma*rannor(99999);
```

```
            Xsum=Xsum+X;
           End;

           Xbar=Xsum/ni;
           Z=(Xbar-mu0)/(sigma/sqrt(ni));

           If(Z<-K) or (Z>K) then do;
                 ARL=i;
                 output;
                 ok=0;
           end;

           If(Z<-W) or (Z>W) then ni=nL;
           else ni=nS;
         End;
End;
run;

proc univariate;
var ARL;
output pctlpts=5 50 95 pctlpre=p;

proc print;
run;
```

A.3.3 A Simulation Program for the Example of Application for the VSS $\bar{X}$ Chart with $n_1 = n_S$

This simulation program computes the summary statistics for the illustrative example of the VSS $\bar{X}$ chart with $n_1 = n_S$ in Table 4.5 (see Section 4.4 of Chapter 4). The intended results can be obtained by entering the values of the parameters ‘sigma0’, ‘mu0’, ‘delta’, ‘n’, ‘nS’, ‘nL’, ‘W’ and ‘K’ in the SAS code.

```
Data VSSXb_example;
sigma0=0.008;
mu0=1.5;
delta=0.75;
mu1=mu0+delta*sigma0;
n=5;
nS=3;
nL=21;
W=1.5840;
K=3.1098;
array X{*} X1-X450;
array ni{*} ni1-ni26;
array Xbar{*} Xbar1-Xbar25;
array Z{*} Z1-Z25;
```

```
array OOC{*} OOC1-OOC25;

c=1;
nii=nS;
ni(1)=nS;

Do i=1 to 25;
   If(i<11) then mu=mu0;
                  else mu=mu1;
   Ysum=0;
   Do jj=1 to nii;
      Y=mu+sigma0*rannor(12911);
      Ysum=Ysum+Y;
        X(c)=Y;
        c=c+1;
   End;
   Xbar(i)=Ysum/nii;
   Z(i)=(Xbar(i)-mu0)/(sigma0/sqrt(nii));

   If(Z(i)<-K) or (Z(i)>K) then do;
                nii=nS;
                ni(i+1)=nS;
                OOC(i)=1;
        end;

        else if(Z(i)<-W) or (Z(i)>W) then do;
            nii=nL;
            ni(i+1)=nL;
        end;
        else do;
            nii=nS;
            ni(i+1)=nS;
        end;
End;
run;

proc print;
run;
```

A.3.4 A Simulation Program for the Example of Application for the VSS $\bar{X}$ Chart with $n_1 = n_L$

This simulation program computes the summary statistics for the illustrative example of the VSS $\bar{X}$ chart with $n_1 = n_L$ in Table 4.5 (see Section 4.4 of Chapter 4). The intended results can be obtained by entering the values of the parameters ‘sigma0’, ‘mu0’, ‘delta’, ‘n’, ‘nS’, ‘nL’, ‘W’ and ‘K’ in the SAS code.

```
Data VSSXb_example;
```

```
sigma0=0.008;
mu0=1.5;
delta=0.75;
mu1=mu0+delta*sigma0;
n=5;
nS=3;
nL=28;
W=1.7608;
K=3.1098;
array X{*} X1-X450;
array ni{*} ni1-ni26;
array Xbar{*} Xbar1-Xbar25;
array Z{*} Z1-Z25;
array OOC{*} OOC1-OOC25;

c=1;
nii=nL;
ni(1)=nL;

Do i=1 to 25;
   If(i<11) then mu=mu0;
                  else mu=mu1;
   Ysum=0;
   Do jj=1 to nii;
      Y=mu+sigma0*rannor(12911);
      Ysum=Ysum+Y;
        X(c)=Y;
        c=c+1;
   End;
   Xbar(i)=Ysum/nii;
   Z(i)=(Xbar(i)-mu0)/(sigma0/sqrt(nii));

   If(Z(i)<-K) or (Z(i)>K) then do;
                nii=nL;
                ni(i+1)=nL;
                OOC(i)=1;
        end;

        else if(Z(i)<-W) or (Z(i)>W) then do;
            nii=nL;
            ni(i+1)=nL;
        end;
        else do;
            nii=nS;
            ni(i+1)=nS;
        end;
End;
run;

proc print;
run;
```

APPENDIX B

PROGRAMS FOR THE SHEWHART $\bar{X}$ CHART

B.1 Programs for the MRL- and EMRL-based Shewhart $\bar{X}$ Chart

These programs are written using the ScicosLab software to compute the chart's parameter (L), MRL, percentiles of the run-length distribution and EMRL for the Shewhart $\bar{X}$ chart. To obtain a particular result, we need to call the related function in the Command Window of the ScicosLab software. The function body, which contains all the ScicosLab codes, is written in the Scipad of the ScicosLab software. In these ScicosLab programs, the letter 'L' represents the chart's coefficient controlling the width of the Shewhart $\bar{X}$ chart's limits. Also, in these programs, 'delta' is the magnitude of mean shifts (δ), 'pctl' denotes the probability γ and 'mrlini' is the initial value of the MRL. The 5th and 95th percentiles of the run-length distribution are represented by Q5 and Q95, respectively, in these programs. The lower ($\delta_{\min}$) and upper ($\delta_{\max}$) bounds of the mean shifts are represented by 'deltamin' and 'deltamax' in these programs, respectively.

For the MRL-based Shewhart $\bar{X}$ chart, the input parameters and call functions, typed at the Command Window, are presented as follows:

(i) To compute the MRL for the given values of n, L and δ, enter

```
n=5;L=3;delta=1;mrlini=1;
MRL=mrlXbar(n,L,delta,mrlini)
```

(ii) To compute the percentiles of the run-length distribution for the given values of *n*, *L*, δ and γ, enter

```
n=3;L=3.1098738;delta=0.25;pctl=0.05;mrlini=1;
PRL=pctlXbar(n,L,delta,pctl,mrlini)
```

(iii) To compute the chart's parameter (*L*) and MRL_1, 5th and 95th percentiles of the run-length distribution for the desired values of *n*, MRL_0 and δ, enter

```
n=3;mrl0=250;delta=0.25;
[n,L,MRL0,MRL1,Q5,Q95]=mrlXbarL(n,mrl0,delta)
```

For the EMRL-based Shewhart $\bar{X}$ chart, the intended results can be obtained by entering the input parameters and call functions in the Command Window, as described below.

(i) To compute the EMRL for the given values of $\delta_{\min}$, $\delta_{\max}$, *n* and *L*, enter

```
deltamin=0.1;deltamax=2;n=10;L=3.1098738;
EMRL=emrlxbar(deltamin,deltamax,n,L)
```

(ii) To compute the chart's parameter (*L*) and EMRL_1 for the desired values of *n*, EMRL_0, $\delta_{\min}$ and $\delta_{\max}$, enter

```
n=7;mrl0=370;deltamin=0.1;deltamax=2;
[n,L,MRL0,EMRL]=emrlXbarL(n,mrl0,deltamin,deltamax)
```

The written ScicosLab programs in the Scipad are outlined as follows:

```
//---------------------------------------------------------------
function MRL=mrlXbar(n,L,delta,mrlini)
//---------------------------------------------------------------
[argout,argin]=argn()
if argin>4
  error("incorrect number of arguments")
end
if (n<=0)|(n~=floor(n))
  error("argument ''n'' must be an integer >= 1")
end
```

```
if L<=0
  error("argument ''L'' must be an integer > 0")
end
if delta<0
  error("argument ''delta'' must be an integer >= 0")
end
if mrlini<=0
  error("argument ''mrlini'' must be an integer > 0")
end
betaxbar=cdfnormal(L-delta*sqrt(n))-cdfnormal(-L-delta*sqrt(n))
for MRL=mrlini:1000
    probmed=1-betaxbar.^MRL
    if probmed>=0.5
        break
    end
end

//---------------------------------------------------------------
function PRL=pctlXbar(n,L,delta,pctl,mrlini)
//---------------------------------------------------------------
[argout,argin]=argn()
if argin>5
  error("incorrect number of arguments")
end
if (n<=0)|(n~=floor(n))
  error("argument ''n'' must be an integer >= 1")
end
if L<=0
  error("argument ''L'' must be an integer > 0")
end
if delta<0
  error("argument ''delta'' must be an integer >= 0")
end
if (pctl<=0)|(pctl>=1)
  error("argument ''percentile'' must be an integer > 0 or < 1")
end
if mrlini<=0
  error("argument ''mrlini'' must be an integer > 0")
end
betaxbar=cdfnormal(L-delta*sqrt(n))-cdfnormal(-L-delta*sqrt(n))
for PRL=mrlini:5000
    prob_percen=1-betaxbar.^PRL
    if prob_percen>=pctl
        break
    end
end

//---------------------------------------------------------------
function dif=minLXbar(L,mrl0)
//---------------------------------------------------------------
if L<=0
  dif=%inf
else
  betat=cdfnormal(L)-cdfnormal(-L)
  dif=0.50001-(1-betat.^mrl0)
end

//---------------------------------------------------------------
function [n,L,MRL0,MRL1,Q5,Q95]=mrlXbarL(n,mrl0,delta)
//---------------------------------------------------------------
[argout,argin]=argn()
```

```
if (argin<2)|(argin>3)
  error("incorrect number of arguments")
end
if (n<=0)|(n~=floor(n))
  error("argument ''n'' must be an integer >= 1")
end
if mrl0<0
  error("argument ''mrl0'' must be an integer > 0")
end
if argin==2
  delta=0
end
if delta<0
  error("argument ''delta'' must be >= 0")
end
L=simplexolve(3,minLXbar,list(mrl0),tol=1e-6)
Q5=pctlXbar(n,L,delta,0.05,1)
MRL1=mrlXbar(n,L,delta,Q5)
Q95=pctlXbar(n,L,delta,0.95,MRL1)
MRL0=mrlXbar(n,L,0,mrl0-5)
mprintf("%8.5f (%2d, %3d, %4d) %4d\n",[L,Q5,MRL1,Q95,MRL0])

//-----------------------------------------------------------------
function EMRL=emrlxbar(deltamin,deltamax,n,L)
//-----------------------------------------------------------------
[xi,wi]=quadlegendre(9,deltamin,deltamax)
EMRL=0
for il=1:9
  xil=xi(il)
  wil=wi(il)
  MRL=mrlXbar(n,L,xil,1)
  EMRL=EMRL+MRL.*wil
end
EMRL=EMRL/(deltamax-deltamin)

//-----------------------------------------------------------------
function [n,L,MRL0,EMRL]=emrlXbarL(n,mrl0,deltamin,deltamax)
//-----------------------------------------------------------------
[argout,argin]=argn()
if (argin<3)|(argin>4)
  error("incorrect number of arguments")
end
if (n<=0)|(n~=floor(n))
  error("argument ''n'' must be an integer >= 1")
end
if mrl0<0
  error("argument ''mrl0'' must be an integer > 0")
end
L=simplexolve(3,minLXbar,list(mrl0),tol=1e-6)
EMRL=emrlxbar(deltamin,deltamax,n,L)
MRL0=mrlXbar(n,L,0,mrl0-5)
mprintf("%8.5f %6.2f %4d\n",[L,EMRL,MRL0])
```

B.2 Monte Carlo Simulation Programs for the Shewhart $\bar{X}$ Chart

In this appendix, two simulation programs are written by using the SAS software. Appendix B.2.1 is a simulation program to verify the accuracy of ($\ell_{0.05}$, MRL, $\ell_{0.95}$) computed from the theoretical method (see Subsection 3.3.1 of Chapter 3 and Appendix B.1) of the Shewhart $\bar{X}$ chart; while Appendix B.2.2 illustrates a simulation program for the example of application for the Shewhart $\bar{X}$ chart.

B.2.1 A Simulation Program for the Shewhart $\bar{X}$ Chart

In this simulation program, the 5th, 50th (MRL) and 95th percentiles of the run-length distribution, i.e. ($\ell_{0.05}$, MRL, $\ell_{0.95}$), are calculated using 50001 simulation trials. To obtain the intended results, the user needs to input the values of the parameters ‘n’, ‘L’ and ‘delta’ in the SAS code.

```
Data Xbar_pctl;
n=7;
L=3.1099;
delta=0.25;

mu0=0;
sigma0=1;
std=1/sqrt(n);
UCL=mu0+L*std;
LCL=-UCL;

Do m=1 to 50001;
      ok=1;RL=0;
      Do while(ok=1);
            RL=RL+1;
            Xsum=0;
            Do i=1 to n;
                  X=delta+sigma0*rannor(55555);
                  Xsum=Xsum+X;
            End;
            Xbar=Xsum/n;

            If (Xbar>UCL) or (Xbar<LCL) then do;
                  ARL=RL;output;
                  ok=0;
```

```
            end;
      End;
End;
run;

proc univariate;
var ARL;
output pctlpts=5 50 95 pctlpre=p;

proc print;
run;
```

B.2.2 A Simulation Program for the Example of Application for the Shewhart $\bar{X}$ Chart

This simulation program computes the summary statistics for the illustrative example of the Shewhart $\bar{X}$ chart in Table 4.5 (see Section 4.4 of Chapter 4). The intended output can be obtained by entering the values of the parameters ‘sigma0’, ‘mu0’, ‘delta’, ‘n’ and ‘L’ in the SAS code.

```
Data example_Xbar;
sigma0=0.008;
mu0=1.5;
delta=0.75;
mu1=mu0+delta*sigma0;
n=5;
L=3.10987;
std=1/sqrt(n);
UCL=mu0+L*std*sigma0;
LCL=mu0-L*std*sigma0;

array X{*} X1-X125;
array Xbar{*} Xbar1-Xbar25;

c=1;
Do i=1 to 25;
      If (i<11) then mu=mu0;
                     else mu=mu1;
      Ysum=0;
      Do j=1 to n;
            Y=mu+sigma0*rannor(12911);
            Ysum=Ysum+Y;
            X(c)=Y;
            c=c+1;
      end;
      Xbar(i)=Ysum/n;
end;
run;

proc print;
run;
```

APPENDIX C

PROGRAMS FOR THE EWMA $\bar{X}$ CHART

C.1 Optimization Programs for the MRL- and EMRL-based EWMA $\bar{X}$ Chart

These optimization programs are written using the ScicosLab software to compute the optimal (λ, H) combination, MRL, percentiles of the run-length distribution and EMRL for the EWMA $\bar{X}$ chart. To obtain a particular result, we need to call the related function in the Command Window of the ScicosLab software. The function body, which contains all the ScicosLab codes, is written in the Scipad of the ScicosLab software. In these ScicosLab programs, the symbols λ and H used in Subsection 3.3.3 of Chapter 3 are replaced by 'lam' and 'K'. Note that, the symbols 'a' and 'b' in these programs represent shifts in the process mean (δ) and standard deviation, respectively. Also, 'probperc' denotes the probability γ; while 'mrlini' and 'percini' are the initial values of the MRL and percentiles of the run-length distribution, respectively. The 5th and 95th percentiles of the run-length distribution are replaced by Q5 and Q95, respectively, in these programs. The terms 'deltamin' and 'deltamax' in the program represent $\delta_{\min}$ and $\delta_{\max}$, respectively.

For the MRL-based EWMA $\bar{X}$ chart, the input parameters and call functions, typed at the Command Window, are presented as follows:

(i) To compute the MRL for the given values of δ, *n*, λ, *H* and MRL_0, enter

```
a=0.2;b=1;n=4;lam=0.5;K=0.836;mrlini=1;mrl0=200;
MRL=mrlEWMAx(a,b,n,lam,K,mrlini,mrl0)
```

(ii) To compute the percentiles of the run-length distribution for the given values of δ, *n*, λ, *H* and γ, enter

```
a=0.5;b=1;n=3;lam=0.0734;K=0.3;percini=1;probperc=0.05;
percentile=rlEWMAx(a,b,n,lam,K,percini,probperc)
```

(iii) To compute the optimal (λ, H) combination as well as the corresponding MRL_1, 5^{th} and 95^{th} percentiles of the run-length distribution, for the desired values of δ, *n* and MRL_0, enter

```
a=0.25;b=1;n=10;mrl0=250;
[lam,K,MRL,MRL0,Q5,Q95]=mrlEWMAxoptim(a,b,n,mrl0)
```

For the EMRL-based EWMA $\bar{X}$ chart, the intended results can be obtained by entering the values of parameters and call functions at the Command Window, as described below.

(i) To compute the EMRL for the given values of $\delta_{\min}$, $\delta_{\max}$, *n*, λ and *H*, enter

```
deltamin=0.1;deltamax=2.0;b=1;n=4;lam=0.5;K=0.836;
mrlini=1;mrl0=370;
EMRL=emrlewma(deltamin,deltamax,b,n,lam,K,mrlini,mrl0)
```

(ii) To compute the optimal (λ, H) combination and EMRL_1 for the desired values of $\delta_{\min}$, $\delta_{\max}$, *n* and EMRL_0, enter

```
deltamin=0.1;deltamax=2;b=1;n=10;mrl0=370;
[lam,K,EMRL,MRL0]=emrlEWMAxoptim(deltamin,deltamax,b,..
n,mrl0)
```

The written ScicosLab programs in the Scipad are exhibited as follows:

```
//-----------------------------------------------------------------
function [Q,q]=QEWMAx(a,b,n,lam,K)
//-----------------------------------------------------------------
[argout,argin]=argn()
if argin~=5
  error("incorrect number of arguments")
end
if a<0
  error("argument ''a'' must be >= 0")
end
if b<=0
  error("argument ''b'' must be > 0")
end
if (n<=0)|(n~=floor(n))
  error("argument ''n'' must be an integer >= 1")
end
if (lam<=0)|(lam>1)
  error("argument ''lam'' must be in (0,1]")
end
if K<=0
  error("argument ''K'' must be > 0")
end
KL=-K
KU=K
m=25
p=2*m+1
d=(KU-KL)/(2*p)
h=KL+d:2*d:KU-d
Hj=ones(p,1)*h
Hi=Hj'
Q1=(Hj+d-(1-lam)*Hi)/lam
Q2=(Hj-d-(1-lam)*Hi)/lam
Q=cdfnormal((Q1-a)*sqrt(n)/b)-cdfnormal((Q2-a)*sqrt(n)/b)
q=zeros(p,1)
q(m+1)=1

//-----------------------------------------------------------------
function MRL=mrlEWMAx(a,b,n,lam,K,mrlini,mrl0)
//-----------------------------------------------------------------
[argout,argin]=argn()
if argin~=7
  error("incorrect number of arguments")
end
if a<0
  error("argument ''a'' must be >= 0")
end
if b<=0
  error("argument ''b'' must be > 0")
end
if (n<=0)|(n~=floor(n))
  error("argument ''n'' must be an integer >= 1")
end
if (lam<=0)|(lam>1)
  error("argument ''lam'' must be in (0,1]")
end
if K<=0
  error("argument ''K'' must be > 0")
end
if mrlini<=0
```

```
  error("argument ''mrlini'' must be an integer >= 1")
end
[Q,q]=QEWMAx(a,b,n,lam,K)
q=q(:)'
for MRL=mrlini:mrl0+5
    W=eye(Q)-Q^MRL
    z=q*W
    probmed=sum(z)
    if probmed>=0.5
        break
    end
end

//-----------------------------------------------------------------
function percentile=rlEWMAx(a,b,n,lam,K,percini,probperc)
//-----------------------------------------------------------------
[argout,argin]=argn()
if argin~=7
  error("incorrect number of arguments")
end
if a<0
  error("argument ''a'' must be >= 0")
end
if b<=0
  error("argument ''b'' must be > 0")
end
if (n<=0)|(n~=floor(n))
  error("argument ''n'' must be an integer >= 1")
end
if (lam<=0)|(lam>1)
  error("argument ''lam'' must be in (0,1]")
end
if K<=0
  error("argument ''K'' must be > 0")
end
if percini<=0
  error("argument ''percini'' must be an integer >= 1")
end
[Q,q]=QEWMAx(a,b,n,lam,K)
q=q(:)'
for percentile=percini:50000
    W=eye(Q)-Q^percentile
    z=q*W
    probmed=sum(z)
    if probmed>=probperc
        break
    end
end

//-----------------------------------------------------------------
function f=mrlEWMAxsolve(K,lam,n,mrl0)
//-----------------------------------------------------------------
if K<=0
  f=%inf
else
  [Q,q]=QEWMAx(0,1,n,lam,K)
  q=q(:)'
  W=eye(Q)-Q^mrl0
  z=q*W
  f=0.5001-sum(z)
end
```

```
//----------------------------------------------------------------
function mrl=mrlEWMAxoptimlam(lam,a,b,n,mrl0)
//----------------------------------------------------------------
if (lam<=0.01)|(lam>1)
  mrl=%inf
else
  K0=0.1
  K=simplexolve(K0,mrlEWMAxsolve,list(lam,n,mrl0),tol=1e-5)
  mrl=mrlEWMAx(a,b,n,lam,K,1,mrl0)
  if mrl<0
    mrl=%inf
  end
end

//----------------------------------------------------------------
function [lam,K,MRL,MRL0,Q5,Q95]=mrlEWMAxoptim(a,b,n,mrl0)
//----------------------------------------------------------------
[argout,argin]=argn()
if argin~=4
  error("incorrect number of arguments")
end
if a<0
  error("argument ''a'' must be >= 0")
end
if b<=0
  error("argument ''b'' must be > 0")
end
if (n<=0)|(n~=floor(n))
  error("argument ''n'' must be an integer >= 1")
end
if mrl0<0
  error("argument ''mrl0'' must be > 0")
end
lam0=0.05
lam=neldermead(lam0,mrlEWMAxoptimlam,list(a,b,n,mrl0),..
tol=1e-4,opt="min")
K0=0.1
K=simplexolve(K0,mrlEWMAxsolve,list(lam,n,mrl0),tol=1e-5)
Q5=rlEWMAx(a,b,n,lam,K,1,0.05)
MRL=mrlEWMAx(a,b,n,lam,K,Q5,mrl0)
Q95=rlEWMAx(a,b,n,lam,K,MRL,0.95)
MRL0=mrlEWMAx(0,1,n,lam,K,mrl0-5,mrl0)
mprintf("(%6.4f, %6.4f)(%2d, %3d, %4d) %4d\n",[lam,K,Q5,MRL,..
Q95,MRL0])

//----------------------------------------------------------------
function EMRL=emrlewma(deltamin,deltamax,b,n,lam,K,mrlini,mrl0)
//----------------------------------------------------------------
[xi,wi]=quadlegendre(9,deltamin,deltamax)
EMRL=0
for il=1:9
  xil=xi(il)
  wil=wi(il)
  MRL=mrlEWMAx(xil,b,n,lam,K,mrlini,mrl0)
  EMRL=EMRL+MRL.*wil
end
EMRL=EMRL/(deltamax-deltamin)

//----------------------------------------------------------------
function EMRL=emrlEWMAxoptimlam(lam,deltamin,deltamax,b,n,mrl0)
//----------------------------------------------------------------
```

```
if (lam<=0.01)|(lam>1)
  EMRL=%inf
else
  K0=0.1
  K=simplexolve(K0,mrlEWMAxsolve,list(lam,n,mrl0),tol=1e-5)
  if K>2
    EMRL=%inf
  else
    EMRL=emrlewma(deltamin,deltamax,b,n,lam,K,1,mrl0)
    if EMRL<0
        EMRL=%inf
      end
  end
end

//---------------------------------------------------------------
function[lam,K,EMRL,MRL0]=emrlEWMAxoptim(deltamin,deltamax,b,..
n,mrl0)
//---------------------------------------------------------------
[argout,argin]=argn()
if argin~=5
  error("incorrect number of arguments")
end
if b<=0
  error("argument ''b'' must be > 0")
end
if (n<=0)|(n~=floor(n))
  error("argument ''n'' must be an integer >= 1")
end
if mrl0<0
  error("argument ''mrl0'' must be > 0")
end
lam0=0.05
lam=neldermead(lam0,emrlEWMAxoptimlam,list(deltamin,deltamax,..
b,n,mrl0),tol=1e-4,opt="min")
K0=0.1
K=simplexolve(K0,mrlEWMAxsolve,list(lam,n,mrl0),tol=1e-5)
EMRL=emrlewma(deltamin,deltamax,b,n,lam,K,1,mrl0)
MRL0=mrlEWMAx(0,1,n,lam,K,mrl0-5,mrl0)
mprintf("(%6.4f, %6.4f) %5.2f %4d\n",[lam,K,EMRL,MRL0])
```

C.2 Monte Carlo Simulation Programs for the EWMA $\bar{X}$ Chart

In this appendix, two simulation programs related to the EWMA $\bar{X}$ chart are written using the SAS software. Appendix C.2.1 is a simulation program to verify the accuracy of the ($\ell_{0.05}$, MRL, $\ell_{0.95}$) values computed from the theoretical method (see Subsection 3.3.3 of Chapter 3 and Appendix C.1) of the EWMA $\bar{X}$ chart. Appendix C.2.2 illustrates a simulation program for the example of application of the EWMA $\bar{X}$ chart.

C.2.1 A Simulation Program for the EWMA $\bar{X}$ Chart

In this simulation program, the ($\ell_{0.05}$, MRL, $\ell_{0.95}$) values of the EWMA $\bar{X}$ chart are calculated using 50001 simulation trials. Here, the symbols λ and H used in Subsection 3.3.3 of Chapter 3 are replaced by 'lambda1' and 'L' in this program. To obtain the intended results, the user needs to enter the values of the parameters 'L', 'lambda1', 'delta' and 'n' in the SAS code.

```
Data EWMA;
L=0.4631;
lambda1=0.3;
lambda2=1-lambda1;
mu0=0;
sigma=1;
delta=0.5;
n=7;
UCL=L;
LCL=-L;

do nsim=1 to 50001;
Z=0;

do i=1 to 15000;
   Xsum=0;
   do j=1 to n;
        X=delta+sigma*RANNOR(55555);
        Xsum=Xsum+X;
   end;
   Xbar=Xsum/n;
   Z=lambda1*Xbar+lambda2*Z;

   If (Z>UCL) or (Z<LCL) then do;
                        MRL=i;
                        output;i=15001;
                        end;
    end;
end;
run;

proc univariate;
var MRL;
output pctlpts=5 50 95 pctlpre=p;

proc print;
run;
```

C.2.2 A Simulation Program for the Example of Application for the EWMA $\bar{X}$ Chart

This simulation program computes the summary statistics for the illustrative example of the EWMA $\bar{X}$ chart in Table 4.5 (see Section 4.4 of Chapter 4). The results can be obtained by entering the values of the parameters ‘K’, ‘lambda1’, ‘mu0’, ‘sigma0’, ‘delta’ and ‘n’ in the SAS code.

```
Data example_EWMA;
K=0.8529;
lambda1=0.5500;
lambda2=1-lambda1;
mu0=1.5;
sigma0=0.008;
delta=0.75;
mu1=mu0+delta*sigma0;
n=5;
UCL=mu0+K*sigma0;
LCL=mu0-K*sigma0;

array X{*} X1-X125;
array Xbar{*} Xbar1-Xbar25;
array Z{*} Z1-Z25;

c=1;
Do i=1 to 25;
      If (i<11) then mu=mu0;
                    else mu=mu1;
      Ysum=0;
      Do j=1 to n;
            Y=mu+sigma0*rannor(12911);
            Ysum=Ysum+Y;
            X(c)=Y;
            c=c+1;
      end;
      Xbar(i)=Ysum/n;
      if i=1 then Z(i)=lambda1*Xbar(i)+lambda2*mu0;
               else Z(i)=lambda1*Xbar(i)+lambda2*Z(i-1);

end;
run;

proc print;
run;
```

APPENDIX D

ADDITIONAL RESULTS

D.1 Additional Results for the Computation of ARL, SDRL and Percentiles of the Run-length Distribution of the VSS $\bar{X}$ Chart

In this appendix, additional results regarding the ARLs, SDRLs and percentiles of the run-length distribution of the two VSS schemes with $n_1 = n_S$ and $n_1 = n_L$ are provided in Tables D.1 to D.3. Tables D.1 and D.2 show the exact values of the ARL, SDRL and percentiles of the run-length distribution for the two optimal ARL-based VSS $\bar{X}$ charts when $\text{ARL}_0 = 250$ and $\text{ASS}_0 \in \{5,\ 10\}$; while those results for $\text{ARL}_0 = 370$ and $\text{ASS}_0 = 10$ are shown in Table D.3.

Table D.1: Exact ARL, SDRL and percentiles of the run-length distribution for the VSS $\bar{X}$ chart with the optimal chart's parameters (n_S, n_L, W, K) corresponding to $\delta_{\text{opt}} \in \{0.5, 2.0\}$, $\text{ASS}_0 = 5$ and $\text{ARL}_0 = 250$

			Percentiles of the run-length distribution										
δ	ARL	SDRL	5^{th}	10^{th}	20^{th}	30^{th}	40^{th}	50^{th}	60^{th}	70^{th}	80^{th}	90^{th}	95^{th}
$n_1 = n_S$, $\delta_{\text{opt}} = 0.5$, ($n_S$, n_L, W, K) = (2, 31, 1.6079, 2.8782)													
0.00	250.00	249.53	13	27	56	90	128	173	229	301	402	575	748
0.25	61.37	60.08	4	8	15	23	32	43	56	74	98	140	181
0.50	8.31	6.46	2	2	3	4	5	6	8	10	12	17	21
0.75	4.37	2.90	2	2	2	3	3	4	4	5	6	8	10
1.00	3.19	1.83	1	2	2	2	2	3	3	4	4	6	7
1.50	2.11	0.92	1	1	1	2	2	2	2	2	3	3	4
2.00	1.58	0.62	1	1	1	1	1	2	2	2	2	2	3
$n_1 = n_S$, $\delta_{\text{opt}} = 2.0$, ($n_S$, n_L, W, K) = (4, 9, 1.2680, 2.8782)													
0.00	250.00	249.53	13	27	56	90	128	173	229	301	402	575	748
0.25	89.93	89.23	5	10	21	33	46	63	82	108	144	206	268
0.50	18.65	17.62	2	3	5	7	10	13	17	22	29	49	54
0.75	5.48	4.27	1	2	2	3	3	4	5	6	8	11	14
1.00	2.72	1.56	1	1	2	2	2	2	3	3	4	5	6
1.50	1.49	0.59	1	1	1	1	1	1	2	2	2	2	2
2.00	1.13	0.34	1	1	1	1	1	1	1	1	1	2	2
$n_1 = n_L$, $\delta_{\text{opt}} = 0.5$, ($n_S$, n_L, W, K) = (1, 31, 1.5146, 2.8782)													
0.00	250.00	249.53	13	27	56	90	128	173	229	301	402	575	748
0.25	54.84	58.25	1	3	9	17	26	37	50	67	90	131	171
0.50	3.37	4.64	1	1	1	1	1	2	2	3	4	8	13
0.75	1.13	0.54	1	1	1	1	1	1	1	1	1	1	2
1.00	1.00	0.06	1	1	1	1	1	1	1	1	1	1	1
1.50	1.00	0.00	1	1	1	1	1	1	1	1	1	1	1
2.00	1.00	0.00	1	1	1	1	1	1	1	1	1	1	1
$n_1 = n_L$, $\delta_{\text{opt}} = 2.0$, ($n_S$, n_L, W, K) = (4, 31, 2.1298, 2.8782)													
0.00	250.00	249.53	13	27	56	90	128	173	229	301	402	575	748
0.25	76.38	80.88	1	4	13	24	36	51	69	93	126	182	238
0.50	5.69	8.65	1	1	1	1	1	2	2	5	9	16	24
0.75	1.18	0.89	1	1	1	1	1	1	1	1	1	1	2
1.00	1.00	0.08	1	1	1	1	1	1	1	1	1	1	1
1.50	1.00	0.00	1	1	1	1	1	1	1	1	1	1	1
2.00	1.00	0.00	1	1	1	1	1	1	1	1	1	1	1

Table D.2: Exact ARL, SDRL and percentiles of the run-length distribution for the VSS $\bar{X}$ chart with the optimal chart's parameters (n_S, n_L, W, K) corresponding to $\delta_{\text{opt}} \in \{0.5, 2.0\}$, $\text{ASS}_0 = 10$ and $\text{ARL}_0 = 250$

			Percentiles of the run-length distribution										
δ	ARL	SDRL	5th	10th	20th	30th	40th	50th	60th	70th	80th	90th	95th
		$n_1 = n_S$, $\delta_{\text{opt}} = 0.5$, ($n_S$, n_L, W, K) = (6, 31, 1.3896, 2.8782)											
0.00	250.00	249.53	13	27	56	90	128	173	229	301	402	575	748
0.25	35.31	34.06	3	5	9	13	19	25	32	42	56	80	103
0.50	4.51	2.88	2	2	2	3	3	4	4	5	6	8	10
0.75	2.35	1.02	1	1	2	2	2	2	2	3	3	4	4
1.00	1.78	0.66	1	1	1	1	2	2	2	2	2	2	3
1.50	1.22	0.42	1	1	1	1	1	1	1	1	2	2	2
2.00	1.02	0.15	1	1	1	1	1	1	1	1	1	1	1
		$n_1 = n_S$, $\delta_{\text{opt}} = 2.0$, ($n_S$, n_L, W, K) = (9, 15, 1.3679, 2.8782)											
0.00	250.00	249.53	13	27	56	90	128	173	229	301	402	575	748
0.25	49.92	49.25	3	6	12	18	26	35	46	60	80	114	148
0.50	7.74	6.80	1	2	2	3	4	6	7	9	12	17	21
0.75	2.60	1.64	1	1	1	2	2	2	3	3	4	5	6
1.00	1.56	0.71	1	1	1	1	1	1	2	2	2	2	3
1.50	1.05	0.22	1	1	1	1	1	1	1	1	1	1	2
2.00	1.00	0.03	1	1	1	1	1	1	1	1	1	1	1
		$n_1 = n_L$, $\delta_{\text{opt}} = 0.5$, ($n_S$, n_L, W, K) = (1, 31, 1.0425, 2.8782)											
0.00	250.00	249.53	13	27	56	90	128	173	229	301	402	575	748
0.25	28.72	29.66	1	2	6	9	14	20	26	35	47	67	88
0.50	2.41	2.18	1	1	1	1	1	2	2	3	3	5	7
0.75	1.11	0.36	1	1	1	1	1	1	1	1	1	1	2
1.00	1.00	0.06	1	1	1	1	1	1	1	1	1	1	1
1.50	1.00	0.00	1	1	1	1	1	1	1	1	1	1	1
2.00	1.00	0.00	1	1	1	1	1	1	1	1	1	1	1
		$n_1 = n_L$, $\delta_{\text{opt}} = 2.0$, ($n_S$, n_L, W, K) = (9, 31, 2.0371, 2.8782)											
0.00	250.00	249.53	13	27	56	90	128	173	229	301	402	575	748
0.25	43.36	45.17	1	3	8	14	21	29	39	52	71	102	134
0.50	3.36	3.87	1	1	1	1	1	2	2	3	5	8	11
0.75	1.13	0.47	1	1	1	1	1	1	1	1	1	1	2
1.00	1.00	0.06	1	1	1	1	1	1	1	1	1	1	1
1.50	1.00	0.00	1	1	1	1	1	1	1	1	1	1	1
2.00	1.00	0.00	1	1	1	1	1	1	1	1	1	1	1

Table D.3: Exact ARL, SDRL and percentiles of the run-length distribution for the VSS $\bar{X}$ chart with the optimal chart's parameters (n_S, n_L, W, K) corresponding to $\delta_{\text{opt}} \in \{0.5, 2.0\}$, $\text{ASS}_0 = 10$ and $\text{ARL}_0 = 370$

			Percentiles of the run-length distribution										
δ	ARL	SDRL	5^{th}	10^{th}	20^{th}	30^{th}	40^{th}	50^{th}	60^{th}	70^{th}	80^{th}	90^{th}	95^{th}
$n_1 = n_S$, $\delta_{\text{opt}} = 0.5$, ($n_S$, n_L, W, K) = (5, 31, 1.2944, 2.9997)													
0.00	370.00	369.53	19	39	83	132	189	257	339	445	595	851	1108
0.25	43.51	42.11	4	6	11	16	23	31	40	52	69	98	128
0.50	4.85	3.06	2	2	2	3	3	4	5	6	7	9	11
0.75	2.51	1.06	1	2	2	2	2	2	2	3	3	4	5
1.00	1.94	0.67	1	1	1	2	2	2	2	2	2	3	3
1.50	1.37	0.50	1	1	1	1	1	1	1	2	2	2	2
2.00	1.07	0.26	1	1	1	1	1	1	1	1	1	1	2
$n_1 = n_S$, $\delta_{\text{opt}} = 2.0$, ($n_S$, n_L, W, K) = (9, 15, 1.3728, 2.9997)													
0.00	370.00	369.53	19	39	83	132	189	257	339	445	595	851	1108
0.25	67.26	66.57	4	8	16	24	35	47	62	81	108	154	200
0.50	9.29	8.32	1	2	3	4	5	7	9	11	14	20	26
0.75	2.85	1.86	1	1	1	2	2	2	3	3	4	5	6
1.00	1.64	0.76	1	1	1	1	1	1	2	2	2	3	3
1.50	1.07	0.25	1	1	1	1	1	1	1	1	1	1	2
2.00	1.00	0.04	1	1	1	1	1	1	1	1	1	1	1
$n_1 = n_L$, $\delta_{\text{opt}} = 0.5$, ($n_S$, n_L, W, K) = (1, 31, 1.0405, 2.9997)													
0.00	370.00	369.53	19	39	83	132	189	257	339	445	595	851	1108
0.25	36.81	37.77	1	3	7	12	18	25	34	44	60	86	112
0.50	2.68	2.45	1	1	1	1	1	2	2	3	4	6	8
0.75	1.14	0.41	1	1	1	1	1	1	1	1	1	2	2
1.00	1.01	0.07	1	1	1	1	1	1	1	1	1	1	1
1.50	1.00	0.00	1	1	1	1	1	1	1	1	1	1	1
2.00	1.00	0.00	1	1	1	1	1	1	1	1	1	1	1
$n_1 = n_L$, $\delta_{\text{opt}} = 2.0$, ($n_S$, n_L, W, K) = (9, 31, 2.0249, 2.9997)													
0.00	370.00	369.53	19	39	83	132	189	257	339	445	595	851	1108
0.25	57.24	59.19	1	4	11	19	28	39	52	69	93	134	175
0.50	3.76	4.29	1	1	1	1	1	2	3	4	6	9	13
0.75	1.16	0.51	1	1	1	1	1	1	1	1	1	2	2
1.00	1.01	0.07	1	1	1	1	1	1	1	1	1	1	1
1.50	1.00	0.00	1	1	1	1	1	1	1	1	1	1	1
2.00	1.00	0.00	1	1	1	1	1	1	1	1	1	1	1

D.2 Additional Results for the Performance Comparisons of the Shewhart $\bar{X}$, Optimal VSS $\bar{X}$ and Optimal EWMA $\bar{X}$ Charts Based on MRL

Tables D.4 to D.6 tabulate the additional results for the comparative studies of the two optimal MRL-based VSS schemes with the MRL-based Shewhart $\bar{X}$ and optimal MRL-based EWMA $\bar{X}$ charts. In these tables, $n \in \{3, 5, 7, 10\}$, $\delta_{\text{opt}} \in \{0.25, 0.50, 0.75, 1.00, 1.50, 2.00\}$ and $\text{MRL}_0 \in \{200, 300, 500\}$ are considered.

Table D.4: Comparison of the ($\ell_{0.05}$, MRL_1, $\ell_{0.95}$) values for the Shewhart $\bar{X}$, optimal VSS $\bar{X}$ and optimal EWMA $\bar{X}$ charts, together with the charts' corresponding parameters when $n \in$ {3, 5, 7, 10}, $\delta_{\text{opt}} \in$ {0.25, 0.50, 0.75, 1.00, 1.50, 2.00} and $\mathrm{MRL}_0 = 200$

	Shewhart $\bar{X}$	VSS $\bar{X}$		EWMA $\bar{X}$
		$n_1 = n_S$	$n_1 = n_L$	
	L	(n_S, n_L, W, K)	(n_S, n_L, W, K)	(λ, H)
δ_{opt}	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)
		$n = 3$		
0.25	2.9236	(1, 31, 1.8096, 2.9235)	(1, 31, 1.8561, 2.9235)	(0.0766, 0.2909)
	(8, 103, 441)	(6, 70, 299)	(1, 66, 310)	(8, 25, 74)
0.50	2.9236	(1, 31, 1.8096, 2.9235)	(1, 31, 1.8561, 2.9235)	(0.2713, 0.6461)
	(3, 35, 150)	(2, 11, 40)	(1, 2, 28)	(3, 10, 31)
0.75	2.9236	(1, 18, 1.5484, 2.9235)	(1, 26, 1.7695, 2.9235)	(0.4375, 0.8820)
	(1, 13, 56)	(2, 5, 15)	(1, 1, 2)	(2, 6, 17)
1.00	2.9236	(1, 9, 1.1399, 2.9235)	(2, 21, 1.9655, 2.9235)	(0.5050, 0.9724)
	(1, 6, 25)	(2, 3, 8)	(1, 1, 1)	(1, 4, 9)
1.50	2.9236	(1, 6, 0.8330, 2.9235)	(2, 10, 1.5465, 2.9235)	(0.5050, 0.9724)
	(1, 2, 7)	(1, 2, 4)	(1, 1, 1)	(1, 2, 4)
2.00	2.9236	(2, 5, 0.9582, 2.9235)	(2, 6, 1.1567, 2.9235)	(0.5050, 0.9724)
	(1, 1, 3)	(1, 2, 2)	(1, 1, 1)	(1, 1, 2)
		$n = 5$		
0.25	2.9236	(1, 31, 1.4861, 2.9235)	(1, 31, 1.5128, 2.9235)	(0.1090, 0.2822)
	(6, 75, 322)	(5, 44, 186)	(1, 41, 188)	(6, 18, 51)
0.50	2.9236	(1, 31, 1.4861, 2.9235)	(1, 31, 1.5128, 2.9235)	(0.3503, 0.5899)
	(2, 20, 83)	(2, 7, 22)	(1, 2, 13)	(2, 7, 21)
0.75	2.9236	(2, 17, 1.2698, 2.9235)	(1, 25, 1.3925, 2.9235)	(0.5050, 0.7536)
	(1, 7, 27)	(2, 3, 9)	(1, 1, 2)	(2, 4, 10)
1.00	2.9236	(2, 12, 1.0268, 2.9235)	(4, 21, 1.9144, 2.9235)	(0.5050, 0.7536)
	(1, 3, 11)	(1, 2, 5)	(1, 1, 1)	(1, 2, 5)
1.50	2.9236	(4, 9, 1.2698, 2.9235)	(4, 10, 1.3925, 2.9235)	(0.5050, 0.7536)
	(1, 1, 3)	(1, 1, 2)	(1, 1, 1)	(1, 1, 3)
2.00	2.9236	(3, 6, 0.4228, 2.9235)	(4, 6, 0.6772, 2.9235)	(0.5050, 0.7536)
	(1, 1, 2)	(1, 1, 2)	(1, 1, 1)	(1, 1, 2)
		$n = 7$		
0.25	2.9236	(1, 31, 1.2698, 2.9235)	(1, 31, 1.2895, 2.9235)	(0.1090, 0.2385)
	(5, 58, 248)	(4, 32, 135)	(1, 29, 135)	(5, 14, 37)
0.50	2.9236	(1, 31, 1.2698, 2.9235)	(1, 30, 1.2698, 2.9235)	(0.3329, 0.4824)
	(1, 13, 54)	(2, 5, 15)	(1, 2, 9)	(2, 5, 14)
0.75	2.9236	(1, 17, 0.8783, 2.9235)	(4, 25, 1.4762, 2.9235)	(0.5050, 0.6369)
	(1, 4, 16)	(2, 3, 7)	(1, 1, 2)	(1, 3, 7)
1.00	2.9236	(5, 17, 1.3699, 2.9235)	(6, 21, 1.8561, 2.9235)	(0.5050, 0.6369)
	(1, 2, 7)	(1, 2, 3)	(1, 1, 1)	(1, 2, 4)
1.50	2.9236	(4, 8, 0.3107, 2.9235)	(6, 10, 1.1567, 2.9235)	(0.5050, 0.6369)
	(1, 1, 2)	(1, 1, 2)	(1, 1, 1)	(1, 1, 2)
2.00	2.9236	(6, 8, 0.6663, 2.9235)	(6, 8, 0.6772, 2.9235)	(0.5050, 0.6369)
	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)
		$n = 10$		
0.25	2.9236	(1, 31, 1.0268, 2.9235)	(1, 31, 1.0416, 2.9235)	(0.1090, 0.1995)
	(4, 42, 180)	(3, 24, 97)	(1, 21, 96)	(5, 11, 27)
0.50	2.9236	(3, 31, 1.1399, 2.9235)	(3, 30, 1.1342, 2.9235)	(0.5050, 0.5329)
	(1, 8, 32)	(2, 4, 10)	(1, 2, 7)	(2, 4, 12)
0.75	2.9236	(5, 24, 1.1088, 2.9235)	(7, 25, 1.3925, 2.9235)	(0.5050, 0.5329)
	(1, 3, 9)	(1, 2, 4)	(1, 1, 2)	(1, 2, 5)
1.00	2.9236	(9, 12, 0.9582, 2.9235)	(9, 21, 1.7498, 2.9235)	(0.9010, 0.8370)
	(1, 1, 4)	(1, 1, 3)	(1, 1, 1)	(1, 1, 3)
1.50	2.9236	(4, 11, 0.1718, 2.9235)	(9, 11, 0.6772, 2.9235)	(0.5050, 0.5329)
	(1, 1, 1)	(1, 1, 2)	(1, 1, 1)	(1, 1, 2)
2.00	2.9236	(6, 11, 0.2453, 2.9235)	(9, 11, 0.6772, 2.9235)	(0.5050, 0.5329)
	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)

Table D.5: Comparison of the ($\ell_{0.05}$, MRL_1, $\ell_{0.95}$) values for the Shewhart $\bar{X}$, optimal VSS $\bar{X}$ and optimal EWMA $\bar{X}$ charts, together with the charts' corresponding parameters when $n \in \{3, 5, 7, 10\}$, $\delta_{\text{opt}} \in \{0.25, 0.50, 0.75, 1.00, 1.50, 2.00\}$ and $\text{MRL}_0 = 300$

	Shewhart $\bar{X}$	VSS $\bar{X}$		EWMA $\bar{X}$
		$n_1 = n_S$	$n_1 = n_L$	
	L	(n_S, n_L, W, K)	(n_S, n_L, W, K)	(λ, H)
δ_{opt}	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)
		$n = 3$		
0.25	3.0475	(1, 31, 1.8176, 3.0474)	(1, 31, 1.8486, 3.0474)	(0.0800, 0.3184)
	(11, 147, 634)	(8, 94, 401)	(2, 90, 411)	(9, 29, 88)
0.50	3.0475	(1, 31, 1.8176, 3.0474)	(1, 31, 1.8486, 3.0474)	(0.2575, 0.6561)
	(4, 48, 204)	(3, 12, 43)	(1, 2, 30)	(4, 11, 35)
0.75	3.0475	(1, 20, 1.6081, 3.0474)	(2, 29, 2.1105, 3.0474)	(0.3473, 0.7926)
	(2, 17, 74)	(2, 5, 16)	(1, 1, 2)	(2, 6, 16)
1.00	3.0475	(1, 12, 1.3269, 3.0474)	(2, 23, 2.0007, 3.0474)	(0.5050, 1.0160)
	(1, 8, 31)	(2, 3, 9)	(1, 1, 1)	(2, 4, 10)
1.50	3.0475	(1, 7, 0.9613, 3.0474)	(2, 10, 1.5424, 3.0474)	(0.5050, 1.0160)
	(1, 2, 8)	(1, 2, 4)	(1, 1, 1)	(1, 2, 4)
2.00	3.0475	(2, 6, 1.1434, 3.0474)	(2, 6, 1.1546, 3.0474)	(0.9010, 1.5929)
	(1, 1, 3)	(1, 2, 2)	(1, 1, 1)	(1, 1, 3)
		$n = 5$		
0.25	3.0475	(1, 31, 1.4910, 3.0474)	(1, 31, 1.5089, 3.0474)	(0.0816, 0.2498)
	(8, 106, 455)	(6, 57, 243)	(2, 54, 245)	(8, 20, 54)
0.50	3.0475	(1, 31, 1.4910, 3.0474)	(1, 30, 1.4911, 3.0474)	(0.3518, 0.6191)
	(2, 26, 111)	(2, 7, 23)	(1, 2, 15)	(3, 8, 24)
0.75	3.0475	(3, 19, 1.5236, 3.0474)	(2, 27, 1.5634, 3.0474)	(0.5050, 0.7870)
	(1, 8, 34)	(2, 3, 9)	(1, 1, 2)	(2, 4, 11)
1.00	3.0475	(2, 15, 1.1911, 3.0474)	(4, 23, 1.9562, 3.0474)	(0.5050, 0.7870)
	(1, 3, 13)	(1, 2, 5)	(1, 1, 1)	(1, 3, 6)
1.50	3.0475	(4, 10, 1.3743, 3.0474)	(4, 10, 1.3893, 3.0474)	(0.9010, 1.2338)
	(1, 1, 4)	(1, 2, 2)	(1, 1, 1)	(1, 1, 3)
2.00	3.0475	(3, 6,0.4254, 3.0474)	(4, 6, 0.6763, 3.0474)	(0.5050, 0.7870)
	(1, 1, 2)	(1, 1, 2)	(1, 1, 1)	(1, 1, 2)
		$n = 7$		
0.25	3.0475	(1, 31, 1.2737, 3.0474)	(1, 31, 1.2868, 3.0474)	(0.1090, 0.2527)
	(6, 81, 347)	(5, 42, 175)	(2, 38, 174)	(6, 16, 42)
0.50	3.0475	(2, 31, 1.3559, 3.0474)	(2, 31, 1.3706, 3.0474)	(0.3473, 0.5189)
	(2, 17, 70)	(2, 5, 16)	(1, 2, 10)	(2, 6, 16)
0.75	3.0475	(3, 21, 1.2132, 3.0474)	(5, 27, 1.7017, 3.0474)	(0.5050, 0.6651)
	(1, 5, 20)	(2, 3, 6)	(1, 1, 2)	(1, 3, 7)
1.00	3.0475	(1, 14, 0.7308, 3.0474)	(6, 23, 1.9060, 3.0474)	(0.5050, 0.6651)
	(1, 2, 8)	(2, 2, 5)	(1, 1, 1)	(1, 2, 4)
1.50	3.0475	(5, 8, 0.4254, 3.0474)	(6, 10, 1.1546, 3.0474)	(0.5050, 0.6651)
	(1, 1, 2)	(1, 1, 2)	(1, 1, 1)	(1, 1, 2)
2.00	3.0475	(6, 8, 0.6691, 3.0474)	(6, 8, 0.6763, 3.0474)	(0.5050, 0.6651)
	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)
		$n = 10$		
0.25	3.0475	(1, 31, 1.0300, 3.0474)	(1, 31, 1.0399, 3.0474)	(0.1090, 0.2114)
	(5, 58, 247)	(4, 30, 125)	(2, 28, 124)	(5, 12, 30)
0.50	3.0475	(3, 31, 1.1434, 3.0474)	(3, 30, 1.1322, 3.0474)	(0.3000, 0.3955)
	(1, 10, 41)	(2, 4, 11)	(1, 2, 8)	(2, 4, 10)
0.75	3.0475	(3, 25, 0.9919, 3.0474)	(3, 26, 1.0306, 3.0474)	(0.5050, 0.5565)
	(1, 3, 11)	(2, 2, 5)	(1, 1, 2)	(1, 2, 5)
1.00	3.0475	(1, 12, 0.2245, 3.0474)	(9, 23, 1.8166, 3.0474)	(0.9010, 0.8725)
	(1, 1, 4)	(2, 2, 4)	(1, 1, 1)	(1, 1, 4)
1.50	3.0475	(5, 11, 0.2050, 3.0474)	(9, 11, 0.6763, 3.0474)	(0.5050, 0.5565)
	(1, 1, 1)	(1, 1, 2)	(1, 1, 1)	(1, 1, 2)
2.00	3.0475	(6, 11, 0.2480, 3.0474)	(9, 11, 0.6763, 3.0474)	(0.5050, 0.5565)
	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)

Table D.6: Comparison of the ($\ell_{0.05}$, MRL_1, $\ell_{0.95}$) values for the Shewhart $\bar{X}$, optimal VSS $\bar{X}$ and optimal EWMA $\bar{X}$ charts, together with the charts' corresponding parameters when $n \in \{3, 5, 7, 10\}$, $\delta_{\text{opt}} \in \{0.25, 0.50, 0.75, 1.00, 1.50, 2.00\}$ and $\mathrm{MRL}_0 = 500$

	Shewhart $\bar{X}$	VSS $\bar{X}$		EWMA $\bar{X}$
		$n_1 = n_S$	$n_1 = n_L$	
	L	(n_S, n_L, W, K)	(n_S, n_L, W, K)	(λ, H)
δ_{opt}	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)
		$n = 3$		
0.25	3.1977	(1, 31, 1.8240, 3.1976)	(1, 31, 1.8427, 3.1976)	(0.1045, 0.4013)
	(18, 232, 1001)	(11, 136, 582)	(3, 131, 590)	(10, 37, 123)
0.50	3.1977	(1, 31, 1.8240, 3.1976)	(1, 31, 1.8427, 3.1976)	(0.1090, 0.4117)
	(6, 70, 302)	(3, 13, 47)	(1, 2, 34)	(5, 12, 28)
0.75	3.1977	(1, 22, 1.6608, 3.1976)	(1, 30, 1.8272, 3.1976)	(0.3611, 0.8560)
	(2, 24, 103)	(2, 6, 17)	(1, 1, 2)	(3, 7, 19)
1.00	3.1977	(2, 12, 1.6375, 3.1976)	(2, 24, 2.0128, 3.1976)	(0.5050, 1.0679)
	(1, 10, 41)	(2, 3, 9)	(1, 1, 1)	(2, 4, 12)
1.50	3.1977	(1, 8, 1.0636, 3.1976)	(2, 11, 1.5987, 3.1976)	(0.5050, 1.0679)
	(1, 3, 10)	(2, 2, 4)	(1, 1, 1)	(1, 2, 5)
2.00	3.1977	(2, 7, 1.2768, 3.1976)	(2, 6, 1.1529, 3.1976)	(0.9010, 1.6715)
	(1, 1, 4)	(1, 2, 2)	(1, 1, 1)	(1, 1, 3)
		$n = 5$		
0.25	3.1977	(1, 31, 1.4950, 3.1976)	(1, 31, 1.5057, 3.1976)	(0.1087, 0.3183)
	(13, 163, 704)	(7, 81, 343)	(2, 76, 343)	(8, 24, 71)
0.50	3.1977	(1, 31, 1.4950, 3.1976)	(1, 30, 1.4881, 3.1976)	(0.2999, 0.5897)
	(3, 37, 158)	(2, 8, 25)	(1, 2, 17)	(3, 9, 25)
0.75	3.1977	(3, 26, 1.7036, 3.1976)	(3, 29, 1.7765, 3.1976)	(0.4825, 0.8019)
	(1, 11, 46)	(2, 3, 9)	(1, 1, 2)	(2, 5, 13)
1.00	3.1977	(3, 14, 1.3302, 3.1976)	(4, 24, 1.9714, 3.1976)	(0.5050, 0.8272)
	(1, 4, 17)	(1, 2, 5)	(1, 1, 1)	(1, 3, 6)
1.50	3.1977	(1, 6, 0.2501, 3.1976)	(4, 11, 1.4696, 3.1976)	(0.9010, 1.2947)
	(1, 1, 4)	(2, 2, 4)	(1, 1, 1)	(1, 1, 4)
2.00	3.1977	(3, 6, 0.4276, 3.1976)	(4, 6, 0.6756, 3.1976)	(0.5050, 0.8272)
	(1, 1, 2)	(1, 1, 2)	(1, 1, 1)	(1, 1, 2)
		$n = 7$		
0.25	3.1977	(1, 31, 1.2768, 3.1976)	(1, 31, 1.2847, 3.1976)	(0.1090, 0.2695)
	(10, 123, 528)	(6, 58, 243)	(2, 54, 242)	(7, 18, 49)
0.50	3.1977	(2, 31, 1.3593, 3.1976)	(1, 30, 1.2652, 3.1976)	(0.3606, 0.5599)
	(2, 23, 97)	(2, 6, 17)	(1, 2, 12)	(2, 7, 18)
0.75	3.1977	(4, 23, 1.4068, 3.1976)	(5, 29, 1.7388, 3.1976)	(0.5050, 0.6991)
	(1, 6, 26)	(2, 3, 6)	(1, 1, 2)	(2, 3, 8)
1.00	3.1977	(2, 12, 0.6712, 3.1976)	(6, 24, 1.9249, 3.1976)	(0.5050, 0.6991)
	(1, 3, 9)	(2, 2, 5)	(1, 1, 1)	(1, 2, 4)
1.50	3.1977	(5, 9, 0.6712, 3.1976)	(6, 11, 1.2847, 3.1976)	(0.5050, 0.6991)
	(1, 1, 2)	(1, 1, 2)	(1, 1, 1)	(1, 1, 2)
2.00	3.1977	(6, 8, 0.6712, 3.1976)	(6, 8, 0.6756, 3.1976)	(0.5050, 0.6991)
	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)	(1, 1, 2)
		$n = 10$		
0.25	3.1977	(1, 31, 1.0326, 3.1976)	(1, 31, 1.0385, 3.1976)	(0.1090, 0.2255)
	(7, 86, 370)	(5, 41, 173)	(2, 39, 172)	(6, 14, 34)
0.50	3.1977	(1, 31, 1.0326, 3.1976)	(3, 31, 1.1529, 3.1976)	(0.4696, 0.5568)
	(1, 13, 56)	(2, 5, 13)	(1, 2, 9)	(2, 5, 14)
0.75	3.1977	(4, 25, 1.0636, 3.1976)	(1, 28, 0.9693, 3.1976)	(0.5050, 0.5849)
	(1, 4, 14)	(2, 2, 5)	(1, 1, 2)	(1, 3, 6)
1.00	3.1977	(1, 14, 0.3925, 3.1976)	(9, 24, 1.8427, 3.1976)	(0.5050, 0.5849)
	(1, 2, 5)	(2, 2, 4)	(1, 1, 1)	(1, 2, 3)
1.50	3.1977	(5, 11, 0.2072, 3.1976)	(9, 11, 0.6756, 3.1976)	(0.5050, 0.5849)
	(1, 1, 2)	(1, 1, 2)	(1, 1, 1)	(1, 1, 2)
2.00	3.1977	(6, 11, 0.2501, 3.1976)	(9, 11, 0.6756, 3.1976)	(0.5050, 0.5849)
	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)	(1, 1, 1)

D.3 Additional Results for the Performance Comparisons of the Shewhart $\bar{X}$, Optimal VSS $\bar{X}$ and Optimal EWMA $\bar{X}$ Chart Based on EMRL

Tables D.7 to D.10 tabulate the additional results for the comparative studies of the two optimal EMRL-based VSS schemes with the EMRL-based Shewhart $\bar{X}$ and optimal EMRL-based EWMA $\bar{X}$ charts. In these tables, $n \in \{3, 5, 7, 10\}$, $\text{EMRL}_0 \in \{200, 250, 300, 500\}$, $\delta_{\min} = 0.1$ and $\delta_{\max} = 2.0$ are considered.

Table D.7: Comparison of the EMRL_1 and ($\ell_{0.05}$, MRL_1, $\ell_{0.95}$) values for the Shewhart $\bar{X}$, optimal VSS $\bar{X}$ and optimal EWMA $\bar{X}$ charts, together with the charts' corresponding parameters when $n \in \{3, 5, 7, 10\}$, $\text{EMRL}_0 = 200$, $\delta_{\min} = 0.1$ and $\delta_{\max} = 2.0$

	Shewhart $\bar{X}$	VSS $\bar{X}$		EWMA $\bar{X}$
		$n_1 = n_S$	$n_1 = n_L$	
	L EMRL_1	(n_S, n_L, W, K) EMRL_1	(n_S, n_L, W, K) EMRL_1	(λ, H) EMRL_1
δ	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)
		$n = 3$		
	2.9236 24.66	(1, 30, 1.7949, 2.9235) 16.72	(1, 31, 1.8561, 2.9235) 12.99	(0.0953, 0.3346) 9.01
0.25	(8, 103, 441)	(6, 71, 302)	(1, 66, 310)	(8, 26, 80)
0.50	(3, 35, 150)	(2, 11, 40)	(1, 2, 28)	(5, 10, 22)
0.75	(1, 13, 56)	(2, 6, 20)	(1, 1, 2)	(3, 6, 11)
1.00	(1, 6, 25)	(2, 4, 14)	(1, 1, 1)	(3, 5, 8)
1.50	(1, 2, 7)	(1, 3, 8)	(1, 1, 1)	(2, 3, 4)
2.00	(1, 1, 3)	(1, 2, 5)	(1, 1, 1)	(2, 3, 3)
		$n = 5$		
	2.9236 17.37	(1, 31, 1.4861, 2.9235) 12.27	(1, 31, 1.5128, 2.9235) 9.76	(0.1507, 0.3460) 6.61
0.25	(6, 75, 322)	(5, 44, 186)	(1, 41, 188)	(6, 19, 59)
0.50	(2, 20, 83)	(2, 7, 22)	(1, 2, 13)	(3, 7, 15)
0.75	(1, 7, 27)	(2, 4, 12)	(1, 1, 2)	(2, 4, 8)
1.00	(1, 3, 11)	(2, 3, 9)	(1, 1, 1)	(2, 3, 5)
1.50	(1, 1, 3)	(1, 2, 6)	(1, 1, 1)	(2, 2, 3)
2.00	(1, 1, 2)	(1, 2, 4)	(1, 1, 1)	(1, 2, 2)
		$n = 7$		
	2.9236 13.51	(1, 31, 1.2698, 2.9235) 9.92	(1, 31, 1.2895, 2.9235) 8.07	(0.2180, 0.3686) 5.55
0.25	(5, 58, 248)	(4, 32, 135)	(1, 29, 135)	(4, 15, 50)
0.50	(1, 13, 54)	(2, 5, 15)	(1, 2, 9)	(3, 5, 12)
0.75	(1, 4, 16)	(2, 3, 9)	(1, 1, 2)	(2, 3, 6)
1.00	(1, 2, 7)	(2, 3, 7)	(1, 1, 1)	(2, 2, 4)
1.50	(1, 1, 2)	(1, 2, 5)	(1, 1, 1)	(1, 2, 2)
2.00	(1, 1, 1)	(1, 2, 3)	(1, 1, 1)	(1, 1, 2)
		$n = 10$		
	2.9236 10.36	(5, 31, 1.2918, 2.9235) 7.86	(1, 31, 1.0416, 2.9235) 6.67	(0.2560, 0.3412) 4.28
0.25	(4, 42, 180)	(3, 26, 109)	(1, 21, 96)	(4, 12, 37)
0.50	(1, 8, 32)	(2, 4, 10)	(1, 2, 7)	(2, 4, 9)
0.75	(1, 3, 9)	(1, 2, 4)	(1, 1, 2)	(2, 3, 4)
1.00	(1, 1, 4)	(1, 2, 3)	(1, 1, 1)	(1, 2, 3)
1.50	(1, 1, 1)	(1, 1, 2)	(1, 1, 1)	(1, 1, 2)
2.00	(1, 1, 1)	(1, 1, 2)	(1, 1, 1)	(1, 1, 1)

Table D.8: Comparison of the EMRL_1 and ($\ell_{0.05}$, MRL_1, $\ell_{0.95}$) values for the Shewhart $\bar{X}$, optimal VSS $\bar{X}$ and optimal EWMA $\bar{X}$ charts, together with the charts' corresponding parameters when $n \in$ {3, 5, 7, 10}, $\mathrm{EMRL}_0 = 250$, $\delta_{\min} =$ 0.1 and $\delta_{\max} =$ 2.0

	Shewhart $\bar{X}$	VSS $\bar{X}$		EWMA $\bar{X}$
		$n_1 = n_S$	$n_1 = n_L$	
	L EMRL_1	(n_S, n_L, W, K) EMRL_1	(n_S, n_L, W, K) EMRL_1	(λ, H) EMRL_1
δ	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)
		$n = 3$		
	2.9923 29.75	(1, 31, 1.8144, 2.9922) 19.18	(1, 31, 1.8516, 2.9922) 15.53	(0.1079, 0.3740) 9.68
0.25	(10, 125, 539)	(7, 82, 351)	(1, 78, 362)	(8, 28, 92)
0.50	(4, 41, 177)	(2, 11, 42)	(1, 2, 29)	(5, 10, 24)
0.75	(2, 15, 65)	(2, 6, 20)	(1, 1, 2)	(3, 6, 12)
1.00	(1, 7, 28)	(2, 4, 14)	(1, 1, 1)	(3, 5, 8)
1.50	(1, 2, 8)	(1, 3, 8)	(1, 1, 1)	(2, 3, 4)
2.00	(1, 1, 3)	(1, 2, 5)	(1, 1, 1)	(2, 2, 3)
		$n = 5$		
	2.9923 20.74	(1, 31, 1.4891, 2.9922) 14.03	(1, 31, 1.5104, 2.9922) 11.52	(0.1444, 0.3473) 7.07
0.25	(7, 90, 389)	(5, 51, 215)	(1, 47, 217)	(6, 20, 63)
0.50	(2, 23, 97)	(2, 7, 22)	(1, 2, 14)	(3, 7, 16)
0.75	(1, 8, 31)	(2, 4, 12)	(1, 1, 2)	(3, 4, 8)
1.00	(1, 3, 12)	(2, 3, 9)	(1, 1, 1)	(2, 3, 5)
1.50	(1, 1, 3)	(1, 2, 6)	(1, 1, 1)	(2, 2, 3)
2.00	(1, 1, 2)	(1, 2, 4)	(1, 1, 1)	(1, 2, 2)
		$n = 7$		
	2.9923 16.22	(1, 31, 1.2721, 2.9922) 11.39	(1, 30, 1.2683, 2.9922) 9.54	(0.2141, 0.3747) 5.88
0.25	(6, 69, 298)	(4, 37, 155)	(1, 35, 158)	(5, 16, 54)
0.50	(2, 15, 62)	(2, 6, 16)	(1, 2, 10)	(3, 6, 12)
0.75	(1, 5, 18)	(2, 3, 9)	(1, 1, 2)	(2, 3, 6)
1.00	(1, 2, 7)	(2, 3, 7)	(1, 1, 1)	(2, 2, 4)
1.50	(1, 1, 2)	(1, 2, 5)	(1, 1, 1)	(1, 2, 2)
2.00	(1, 1, 1)	(1, 2, 3)	(1, 1, 1)	(1, 1, 2)
		$n = 10$		
	2.9923 12.18	(5, 31, 1.2942, 2.9922) 9.12	(1, 31, 1.0406, 2.9922) 7.75	(0.1760, 0.2776) 4.48
0.25	(4, 50, 214)	(4, 30, 126)	(1, 25, 111)	(4, 12, 33)
0.50	(1, 9, 37)	(2, 4, 11)	(1, 2, 7)	(2, 5, 9)
0.75	(1, 3, 10)	(1, 2, 5)	(1, 1, 2)	(2, 3, 5)
1.00	(1, 1, 4)	(1, 2, 3)	(1, 1, 1)	(2, 2, 3)
1.50	(1, 1, 1)	(1, 1, 2)	(1, 1, 1)	(1, 2, 2)
2.00	(1, 1, 1)	(1, 1, 2)	(1, 1, 1)	(1, 1, 2)

Table D.9: Comparison of the EMRL_1 and ($\ell_{0.05}$, MRL_1, $\ell_{0.95}$) values for the Shewhart $\bar{X}$, optimal VSS $\bar{X}$ and optimal EWMA $\bar{X}$ charts, together with the charts' corresponding parameters when $n \in$ {3, 5, 7, 10}, $\text{EMRL}_0 = 300$, $\delta_{\min} =$ 0.1 and $\delta_{\max} =$ 2.0

	Shewhart $\bar{X}$	VSS $\bar{X}$		EWMA $\bar{X}$
		$n_1 = n_S$	$n_1 = n_L$	
	L EMRL_1	(n_S, n_L, W, K) EMRL_1	(n_S, n_L, W, K) EMRL_1	(λ, H) EMRL_1
δ	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)
		n = 3		
	3.0475 34.42	(1, 31, 1.8176, 3.0474) 21.77	(1, 31, 1.8486, 3.0474) 17.90	(0.1072, 0.3820) 10.60
0.25	(11, 147, 634)	(8, 94, 401)	(2, 90, 411)	(9, 30, 100)
0.50	(4, 48, 204)	(3, 12, 43)	(1, 2, 30)	(5, 11, 25)
0.75	(2, 17, 74)	(2, 6, 21)	(1, 1, 2)	(4, 7, 12)
1.00	(1, 8, 31)	(2, 4, 14)	(1, 1, 1)	(3, 5, 8)
1.50	(1, 2, 8)	(1, 3, 8)	(1, 1, 1)	(2, 3, 5)
2.00	(1, 1, 3)	(1, 2, 5)	(1, 1, 1)	(2, 2, 3)
		n = 5		
	3.0475 24.14	(1, 31, 1.4910, 3.0474) 15.75	(1, 31, 1.5089, 3.0474) 13.20	(0.1090, 0.2990) 7.47
0.25	(8, 106, 455)	(6, 57, 243)	(2, 54, 245)	(7, 20, 59)
0.50	(2, 26, 111)	(2, 7, 23)	(1, 2, 15)	(4, 8, 16)
0.75	(1, 8, 34)	(2, 4, 13)	(1, 1, 2)	(3, 5, 8)
1.00	(1, 3, 13)	(2, 3, 9)	(1, 1, 1)	(2, 4, 6)
1.50	(1, 1, 4)	(1, 2, 6)	(1, 1, 1)	(2, 2, 3)
2.00	(1, 1, 2)	(1, 2, 4)	(1, 1, 1)	(2, 2, 2)
		n = 7		
	3.0475 18.53	(1, 31, 1.2737, 3.0474) 12.73	(1, 31, 1.2868, 3.0474) 10.87	(0.1080, 0.2513) 6.01
0.25	(6, 81, 347)	(5, 42, 175)	(2, 38, 174)	(6, 16, 42)
0.50	(2, 17, 70)	(2, 6, 16)	(1, 2, 10)	(4, 6, 12)
0.75	(1, 5, 20)	(2, 3, 9)	(1, 1, 2)	(3, 4, 7)
1.00	(1, 2, 8)	(2, 3, 7)	(1, 1, 1)	(2, 3, 4)
1.50	(1, 1, 2)	(1, 2, 5)	(1, 1, 1)	(2, 2, 3)
2.00	(1, 1, 1)	(1, 2, 3)	(1, 1, 1)	(1, 2, 2)
		n = 10		
	3.0475 13.88	(4, 31, 1.2132, 3.0474) 10.27	(1, 31, 1.0399, 3.0474) 8.83	(0.2219, 0.3276) 4.90
0.25	(5, 58, 247)	(4, 33, 136)	(2, 28, 124)	(4, 13, 40)
0.50	(1, 10, 41)	(2, 4, 11)	(1, 2, 8)	(2, 4, 9)
0.75	(1, 3, 11)	(1, 2, 5)	(1, 1, 2)	(2, 3, 5)
1.00	(1, 1, 4)	(1, 2, 3)	(1, 1, 1)	(1, 2, 3)
1.50	(1, 1, 1)	(1, 2, 2)	(1, 1, 1)	(1, 1, 2)
2.00	(1, 1, 1)	(1, 1, 2)	(1, 1, 1)	(1, 1, 1)

Table D.10: Comparison of the EMRL_1 and ($\ell_{0.05}$, MRL_1, $\ell_{0.95}$) values for the Shewhart $\bar{X}$, optimal VSS $\bar{X}$ and optimal EWMA $\bar{X}$ charts, together with the charts' corresponding parameters when $n \in \{3, 5, 7, 10\}$, $\text{EMRL}_0 = 500$, $\delta_{\min} =$ 0.1 and $\delta_{\max} =$ 2.0

	Shewhart $\bar{X}$	VSS $\bar{X}$		EWMA $\bar{X}$
		$n_1 = n_S$	$n_1 = n_L$	
	L EMRL_1	(n_S, n_L, W, K) EMRL_1	(n_S, n_L, W, K) EMRL_1	(λ, H) EMRL_1
δ	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)	($\ell_{0.05}$, MRL_1, $\ell_{0.95}$)
		n = 3		
	3.1977 53.20	(1, 31, 1.8240, 3.1976) 31.03	(1, 31, 1.8427, 3.1976) 27.07	(0.0671, 0.3068) 12.23
0.25	(18, 232, 1001)	(11, 136, 582)	(3, 131, 590)	(11, 34, 99)
0.50	(6, 70, 302)	(3, 13, 47)	(1, 2, 34)	(6, 13, 26)
0.75	(2, 24, 103)	(2, 6, 21)	(1, 1, 2)	(5, 8, 14)
1.00	(1, 10, 41)	(2, 4, 14)	(1, 1, 1)	(4, 6, 9)
1.50	(1, 3, 10)	(2, 3, 8)	(1, 1, 1)	(3, 4, 5)
2.00	(1, 1, 4)	(1, 2, 5)	(1, 1, 1)	(2, 3, 4)
		n = 5		
	3.1977 36.57	(1, 31, 1.4950, 3.1976) 22.29	(1, 31, 1.5057, 3.1976) 19.68	(0.1008, 0.3041) 8.69
0.25	(13, 163, 704)	(7, 81, 343)	(2, 76, 343)	(8, 24, 68)
0.50	(3, 37, 158)	(2, 8, 25)	(1, 2, 17)	(5, 9, 17)
0.75	(1, 11, 46)	(2, 4, 13)	(1, 1, 2)	(3, 5, 9)
1.00	(1, 4, 17)	(2, 3, 9)	(1, 1, 1)	(3, 4, 6)
1.50	(1, 1, 4)	(2, 2, 6)	(1, 1, 1)	(2, 3, 4)
2.00	(1, 1, 2)	(1, 2, 4)	(1, 1, 1)	(2, 2, 3)
		n = 7		
	3.1977 28.05	(1, 31, 1.2768, 3.1976) 18.06	(1, 31, 1.2847, 3.1976) 16.05	(0.1044, 0.2625) 6.77
0.25	(10, 123, 528)	(6, 58, 243)	(2, 54, 242)	(7, 18, 48)
0.50	(2, 23, 98)	(2, 6, 18)	(1, 2, 12)	(4, 7, 13)
0.75	(1, 6, 26)	(2, 4, 10)	(1, 1, 2)	(3, 4, 7)
1.00	(1, 3, 9)	(2, 3, 7)	(1, 1, 1)	(2, 3, 5)
1.50	(1, 1, 2)	(2, 2, 5)	(1, 1, 1)	(2, 2, 3)
2.00	(1, 1, 1)	(1, 2, 3)	(1, 1, 1)	(1, 2, 2)
		n = 10		
	3.1977 20.70	(1, 31, 1.0326, 3.1976) 14.38	(1, 31, 1.0385, 3.1976) 12.87	(0.1459, 0.2688) 5.33
0.25	(7, 86, 370)	(5, 41, 173)	(2, 39, 172)	(5, 14, 38)
0.50	(1, 13, 56)	(2, 5, 13)	(1, 2, 9)	(3, 5, 10)
0.75	(1, 4, 14)	(2, 3, 7)	(1, 1, 2)	(2, 3, 5)
1.00	(1, 2, 5)	(2, 2, 6)	(1, 1, 1)	(2, 2, 4)
1.50	(1, 1, 2)	(2, 2, 4)	(1, 1, 1)	(1, 2, 2)
2.00	(1, 1, 1)	(1, 2, 3)	(1, 1, 1)	(1, 1, 2)

D.4 Simulated Dataset for the Illustrative Example

This appendix provides the complete simulated dataset for the illustrative example discussed in Section 4.4 of Chapter 4. The simulated dataset for the two optimal MRL-based VSS $\bar{X}$ charts with $n_1 = n_S$ and $n_1 = n_L$ are shown in Tables D.11 and D.12, respectively. The simulated dataset for the optimal MRL-based EWMA $\bar{X}$ and MRL-based Shewhart $\bar{X}$ charts are provided in Table D.13. The observations for this example are generated from a normal distribution with in-control mean $\mu_0 = 1.5$ and in-control standard deviation $\sigma_0 = 0.008$. The summary statistics for this illustrative example are presented in Table 4.5 (see Section 4.4 of Chapter 4).

Table D.11: Dataset for an illustrative example for the VSS $\bar{X}$ chart with $n_1 = n_S$

Subgroup number, i	n_i	Observations										$\bar{X}_i$	Z_i
		$n_S = 3$			$n_L = 21$								
1	3	1.50778	1.51327	1.49723								1.50609	1.31921
2	3	1.49492	1.49641	1.49508								1.49547	−0.98050
3	3	1.48689	1.51443	1.49713								1.49948	−0.11230
4	3	1.47777	1.50172	1.49348								1.49099	−1.95040
5	21				1.49765	1.50375	1.49516	1.50021	1.50251	1.48404	1.49405	1.50016	0.09076
					1.50431	1.51153	1.50383	1.49848	1.49389	1.50615	1.49875		
					1.50753	1.49814	1.49405	1.49236	1.50361	1.51118	1.50214		
6	3	1.50034	1.49301	1.49602								1.49646	−0.76710
7	3	1.50191	1.49526	1.48782								1.49500	−1.08290
8	3	1.49691	1.49606	1.49827								1.49708	−0.63200
9	3	1.51605	1.49470	1.49454								1.50177	0.38230
10	3	1.50328	1.49167	1.48987								1.49494	−1.09570
11	3	1.50034	1.49503	1.51114								1.50217	0.46995
12	3	1.51091	1.50146	1.51394								1.50877	1.89845
13	21				1.51009	1.50324	1.52403	1.50309	1.49979	1.49623	1.50375	1.50792	4.53809
					1.51123	1.50087	1.51144	1.50232	1.49967	1.50842	1.51950		
					1.51381	1.51007	1.52469	1.50832	1.50845	1.49547	1.51186		
14	3	1.52123	1.50355	1.50761								1.51080	2.33824
15	21				1.51613	1.51751	1.50447	1.51511	1.50196	1.50960	1.50561	1.50718	4.11203
					1.49796	1.50630	1.51613	1.50099	1.51031	1.50805	1.50153		
					1.50832	1.50695	1.51249	1.50144	1.49825	1.51254	1.49912		
16	3	1.50260	1.50931	1.51823								1.51005	2.17500
17	21				1.50332	1.49479	1.50144	1.49849	1.50007	1.50210	1.50179	1.50434	2.48422
					1.49923	1.51115	1.50673	1.50613	1.50658	1.49355	1.51114		
					1.51007	1.50853	1.50512	1.51202	1.51154	1.50520	1.50210		

Table D.12: Dataset for an illustrative example for the VSS $\bar{X}$ chart with $n_1 = n_L$

Subgroup number, i	n_i	Observations													$\bar{X}_i$	Z_i
		$n_S = 3$			$n_L = 28$											
1	28				1.50778	1.51327	1.49723	1.49492	1.49641	1.49508	1.48689	1.51443	1.49713	1.47777	1.49915	− 0.56467
					1.50172	1.49348	1.49765	1.50375	1.49516	1.50021	1.50251	1.48404	1.49405	1.50431		
					1.51153	1.50383	1.49848	1.49389	1.50615	1.49875	1.50753	1.49814				
2	3	1.49405	1.49236	1.50361											1.49668	− 0.71987
3	3	1.51118	1.50214	1.50034											1.50455	0.98525
4	3	1.49301	1.49602	1.50191											1.49698	− 0.65330
5	3	1.49526	1.48782	1.49691											1.49333	− 1.44421
6	3	1.49606	1.49827	1.51605											1.50346	0.74957
7	3	1.49470	1.49454	1.50328											1.49751	− 0.53903
8	3	1.49167	1.48987	1.49434											1.49196	− 1.74140
9	3	1.48903	1.50514	1.50491											1.49969	− 0.06631
10	3	1.49546	1.50794	1.50409											1.50250	0.54081
11	3	1.50324	1.52403	1.50309											1.51012	2.19149
12	28				1.49979	1.49623	1.50375	1.51123	1.50087	1.51144	1.50232	1.49967	1.50842	1.51950	1.50809	5.35426
					1.51381	1.51007	1.52469	1.50832	1.50845	1.49547	1.51186	1.52123	1.50355	1.50761		
					1.51613	1.51751	1.50447	1.51511	1.50196	1.50960	1.50561	1.49796				
13	28				1.50630	1.51613	1.50099	1.51031	1.50805	1.50153	1.50832	1.50695	1.51249	1.50144	1.50516	3.40988
					1.49825	1.51254	1.49912	1.50260	1.50931	1.51823	1.50332	1.49479	1.50144	1.49849		
					1.50007	1.50210	1.50179	1.49923	1.51115	1.50673	1.50613	1.50658				
14	28				1.49355	1.51114	1.51007	1.50853	1.50512	1.51202	1.51154	1.50520	1.50210	1.49468	1.50618	4.08790
					1.51599	1.50758	1.49967	1.51497	1.50522	1.51635	1.50450	1.50775	1.49706	1.50336		
					1.51501	1.50900	1.50384	1.50740	1.50630	1.51080	1.49140	1.50290				
15	28				1.49025	1.49542	1.49945	1.50875	1.51025	1.51843	1.51881	1.48939	1.50913	1.51561	1.50513	3.39162
					1.50868	1.50893	1.51708	1.50694	1.50788	1.51783	1.51096	1.49864	1.48768	1.49887		
					1.51521	1.49706	1.51095	1.49721	1.49654	1.50231	1.51007	1.49525				
16	28				1.49595	1.51447	1.51265	1.49993	1.51026	1.52694	1.50394	1.50671	1.50447	1.50986	1.50564	3.72983
					1.50321	1.49733	1.50930	1.49951	1.49547	1.49949	1.49834	1.51021	1.50666	1.49199		
					1.49302	1.50074	1.50470	1.51654	1.51548	1.52003	1.51294	1.49775				
17	28				1.51639	1.49940	1.51209	1.48801	1.50724	1.50899	1.50013	1.49796	1.48615	1.51466	1.50565	3.73851
					1.50369	1.50079	1.51859	1.50312	1.51547	1.50943	1.48698	1.52028	1.50056	1.50232		
					1.50076	1.50013	1.51135	1.51279	1.51996	1.49982	1.50598	1.51523				

Table D.13: Dataset for an illustrative example for the EWMA $\bar{X}$ and Shewhart $\bar{X}$ charts

Subgroup number, i	Observations					$\bar{X}_i$	Z_i
	X_1	X_2	X_3	X_4	X_5		
1	1.50778	1.51327	1.49723	1.49492	1.49641	1.50192	1.50106
2	1.49508	1.48689	1.51443	1.49713	1.47777	1.49426	1.49732
3	1.50172	1.49348	1.49765	1.50375	1.49516	1.49835	1.49789
4	1.50021	1.50251	1.48404	1.49405	1.50431	1.49702	1.49741
5	1.51153	1.50383	1.49848	1.49389	1.50615	1.50278	1.50036
6	1.49875	1.50753	1.49814	1.49405	1.49236	1.49817	1.49915
7	1.50361	1.51118	1.50214	1.50034	1.49301	1.50206	1.50075
8	1.49602	1.50191	1.49526	1.48782	1.49691	1.49558	1.49791
9	1.49606	1.49827	1.51605	1.49470	1.49454	1.49993	1.49902
10	1.50328	1.49167	1.48987	1.49434	1.48903	1.49364	1.49606
11	1.51114	1.51091	1.50146	1.51394	1.51009	1.50951	1.50346
12	1.50324	1.52403	1.50309	1.49979	1.49623	1.50528	1.50446
13	1.50375	1.51123	1.50087	1.51144	1.50232	1.50592	1.50526
14	1.49967	1.50842	1.51950	1.51381	1.51007	1.51030	1.50803
15	1.52469	1.50832	1.50845	1.49547	1.51186	1.50976	1.50898
16	1.52123	1.50355	1.50761	1.51613	1.51751	1.51321	1.51131
17	1.50447	1.51511	1.50196	1.50960	1.50561	1.50735	1.50913

APPENDIX E

PUBLICATION

Teoh, W.L., Chong, J.K., Khoo, M.B.C., Castagliola, P. and Yeong, W.C., 2017. Optimal designs of the variable sample size $\bar{X}$ chart based on median run length and expected median run length. *Quality and Reliability Engineering International*, 33(1), pp. 121 – 134.