NUMERICAL INVESTIGATION OF MALAYSIAN STOCKS USING ORDER ONE UNIVERSAL PORTFOLIO STRATEGIES

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DOCTOR OF PHILOSOPHY IN SCIENCE

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NUMERICAL INVESTIGATION OF MALAYSIAN STOCKS USING ORDER ONE UNIVERSAL PORTFOLIO STRATEGIES

By

PANG SOOK THENG

A dissertation submitted to the Department of Mathematical and Actuarial Sciences, Lee Kong Chian Faculty of Engineering and Science, Universiti Tunku Abdul Rahman, in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Science
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ABSTRACT

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PANG SOOK THENG

Universal portfolio (UP) is an important investment strategy from a theoretical point of view due to Cover’s research, which proves that UP can achieve very good return for a set of stocks in the long run. The exact algorithm for Cover’s UP takes memory of order $O(n^{m-1})$ where $n$ is the number of trading days and $m$ is the number of stocks. It becomes useless for long term daily trading. To tackle the memory consumption issues, C. P. Tan in year 2013 proposed to limit UP to finite order. This research set the goal to promote active daily trading in Malaysia stock market using types of Tan’s finite order UP and its generalisation.

The goal of our research is to study Malaysia stock market using the four proposed finite order universal portfolios comprehensively and refer to constant rebalanced portfolio (CRP) as a benchmark. The research is conducted with 95 stocks data selected from KLSE (Kuala Lumpur Stock Exchange) from 1 January 2000 to 31 December 2015. The UPs are Multinomial UP, Multivariate Normal UP, Brownian-motion generated UP and UP generated by Ornstein-Uhlenbeck process. This research limits to order one UP and 3 stocks, $m = 3$. Using the basic probability theory, we have formalised the four proposed finite order universal portfolio.

To test our proposed four finite order universal portfolios, they are first benchmarked against Cover’s Dirichlet $(\alpha_1, \alpha_2, \alpha_3)$ universal portfolio strategy. Using 3 sets of stock data selected randomly from KLSE for the period from 1 January, 2003 to 30 November 2004, empirical result showed that the finite order universal portfolio can perform as good as Cover’s universal portfolio with computationally
better in speed and memory.

For the comprehensive study of the above 95 selected stocks, we first find the good parameters for each proposed universal portfolios by using the combination of the 10 selected most active stocks from the above 95 selected stocks. From numerical experiment, the good performance of the parameters is identified for the purpose of further analysis.

Next, numerical experiment for the four classes of universal portfolio strategies are studied for short term (1 year period), middle term (4 years and 8 years periods) and long term (12 years and 16 years periods) data. We employed the four portfolio strategies with their best performing parameters identified to the above 5 groups of data to learn the performance of these methods and used CRP as a benchmark. Every 3 stocks data generated by combination of the selected 95 stocks are used for study. The empirical results showed that the performances of the proposed four strategies are better than CRP in 1 year and 4 years period groups. However, the four strategies did poorly in 8 years, 12 years and 16 years period groups. When comparing to KLCI, the result shows that finite order UP performs poorly.

The research indicates that the order one universal portfolio may help promote active online trading for Malaysia stock market in short term, only if the right combination of stocks are chosen.
ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my supervisor, Dr. Chang Yun Fah and my co-supervisor, Dr. Liew How Hui, for their guidance, support, motivation, encouragement and advice. I thank my family and friends for supporting me throughout all my studies. Lastly, I would like to thank Dr. Tan Choon Peng for guiding me in the beginning of my research.

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<td>CRP</td>
<td>Constant rebalanced portfolio</td>
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<td>UP</td>
<td>Universal Portfolio</td>
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CHAPTER 1

INTRODUCTION

Portfolio is a grouping of financial assets such as stocks, bonds and cash equivalents, as well as funds counterparts, including mutual, exchange-traded and closed funds. Portfolio selection, aiming to optimise the allocation of wealth across a set of assets, is a fundamental research problem in finance (Li & Hoi 2012).

There are two major philosophies in portfolio selection. One philosophy is the mean-variance theory, which assumes that each asset follows some normal distribution and an optimal portfolio selection can be obtained based on the tolerance of risk of an investor. Another philosophy is the capital growth theory, which focuses on optimising the expected cumulative wealth for a periodic (e.g. daily) portfolio selection.

The mean-variance theory is well-established and widely used in practice. The capital growth theory, although less popular, is probably more meaningful to a developing country such as Malaysia, to promote active stock trading.

There are many portfolio selection strategies based on the capital growth theory. Li & Hoi (2012) has summarised the state-of-art portfolio selection algorithms as follows:
### Classifications

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<td><strong>Best Stocks</strong></td>
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<tr>
<td><strong>Constant Rebalanced Portfolios</strong></td>
<td><strong>Kelly (1956); Cover (1991)</strong></td>
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<td><strong>The winner</strong></td>
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<td><strong>Universal Portfolios</strong></td>
<td><strong>Cover (1991); Cover &amp; Ordentlich (1996)</strong></td>
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<td><strong>Exponential Gradient</strong></td>
<td><strong>Helmbold et al. (1998)</strong></td>
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The universal portfolio algorithm is a portfolio selection algorithm from the field of machine learning and information theory. The algorithm learns adaptively from historical data and maximizes log-optimal growth rate in the long run. The algorithm rebalances the portfolio at the beginning of each trading period. At the beginning of first trading period it starts with a naive diversification. In the following trading periods the portfolio composition depends on the historical total return of all possible Constant-Rebalanced Portfolios (CRP).

This research will investigate a variation of the universal portfolio proposed by Cover (1991) and the capital growth will be compared to Constant Rebalanced Portfolios, CRP as a benchmark.

In this chapter, we will first state the problem setting, then we will investigate the relevant research and literatures and finally, an outline of the thesis is given.
1.1 Problem Setting

Consider an \( m \)-stock market. Let \( x_n = (x_{n,i}) \) be the stock-price-relative vector on the \( n^{th} \) trading day, where \( x_{n,i} \) denotes the stock-price relative of stock \( i \) on day \( n \), which is defined to be the ratio of the closing price to its opening price on day \( n \), for \( i = 1, 2, \cdots m \). We assume that \( x_{n,i} \geq 0 \) for all \( i = 1, 2, \cdots m \) and \( n = 1, 2, \cdots \). The portfolio vector \( b_n = (b_{n,i}) \) on day \( n \) is the investment strategy used on day \( n \) when \( b_{n,i} \) is the proportion of the current wealth invested on stock \( i \) for \( i = 1, 2, \cdots m \), with \( 0 \leq b_{n,i} \leq 1 \) and \( \sum_{i=1}^{m} b_{n,i} = 1 \).

The simplest strategy is to “buy-and-hold” stocks using some portfolio \( b = (b_1, b_2, \cdots, b_m) \), where \( b_i \geq 0 \), \( \sum b_i = 1 \). In particular, when \( b = (\frac{1}{m}, \cdots, \frac{1}{m}) \), the buy-and-hold strategy is called a “uniform buy-and-hold”. Suppose that the initial wealth is \( S_0 \), then the wealth accumulate at the end of the \( n \)th day is

\[
S_n = S_0 (b_1 x_{1,1} x_{2,1} \cdots x_{n,1} + b_2 x_{1,2} x_{2,2} \cdots x_{n,2} + \cdots + b_m x_{1,m} \cdots x_{n,m}) \tag{1.1}
\]

We say that a portfolio selection algorithm “beats the market” when it outperforms uniform buy-and-hold strategy on a given market. Buy-and-hold strategy relies on the tendency of successful markets to grow. Much of modern portfolio theories focus on how to choose a good \( b \) for the buy-and-hold strategy, such as the famous Markowitz portfolio theory.

An alternative approach to the static buy-and-hold is to dynamically change the portfolio during the trading period. One of the example of this approach is
A constant-rebalanced portfolio (CRP) is a portfolio \( \mathbf{b}_j = (\mathbf{b}) \) that is constant over the trading days and the wealth at the end of \( n \)th trading day is

\[
S_n = S_0 \prod_{j=1}^{n} \mathbf{b}^j \mathbf{x}_j. \tag{1.2}
\]

Cover (1991) pointed out that the active trading strategy will perform better than “buy-and-hold” in the long run. An active trading strategy involves a sequence of universal portfolio vector \( \mathbf{b}_n = (b_{n,1}, \ldots, b_{n,m}) \), which are associated with the \( n \) trading days. Here, the value \( b_{n,i}, i = 1, 2, \ldots, m \) is the proportion of the wealth of the asset \( i \) on the \( n \)th day and \( \sum_{i=1}^{m} b_{n,i} = 1 \). The wealth at the end of \( n^{th} \) trading day \( S_n \) is given by

\[
S_n = S_0 \prod_{j=1}^{n} \mathbf{b}^j \mathbf{x}_j. \tag{1.3}
\]

The goal of our research is to study Malaysia stock market using the four proposed universal portfolios comprehensively and refer to CRP as a benchmark. The four universal portfolio strategies are investigated through a set of numer-
ical experiments concerning 95 stocks data selected from Kuala Lumpur Stock Exchange.

1.2 Literature Review

The selection of assets in the portfolio is not unique which depends on the special characteristics of the market. The relation and movement of assets in the portfolio will directly affect the expected return and variance for analysis.

Portfolio theory was first developed mathematically by Markowitz (1952). Markowitz portfolio provides a method to analyse how good in a given portfolio is based on only the means and the variance of the returns of the assets contained in the portfolio. This led to the efficient frontier where the investor could choose his preferred portfolio depending on his risk preference. The Markowitz portfolio theory shows that the selected portfolio can achieve the goal of return by ensuring that it does not exceed the tolerance risk level of the investor. This idea is later known as the expected sector for expected return-variance rule (EV-rule). Sharpe (1963) extends Markowitz work on the portfolio analysis. A simplified model of the relationships among securities for practical application of the Markowitz portfolio analysis technique is provided by Sharpe (1963).

There is a new online learning algorithm for portfolio selection on alternative measure of price relative called Cyclically Adjusted Price Relative (CAPR) proposed by Nkomo & Kabundi (2013). The CAPR is derived from a simple state-space model of stock prices and have proved that the CAPR, unlike the stan-
Standard raw price relative widely used in the machine literature, has well defined and desirable statistical properties that makes it better suited for nonparametric mean reversion strategies.

Agarwal et al. (2006) studied on-line investment algorithms first proposed by Agarwal & Hazan (2005) and extended by Hazan et al. (2006) which achieve almost the same wealth as the best constant-rebalanced portfolio determined in hindsight. These algorithms are the first to combine optimal logarithmic regret bounds with efficient deterministic computability and based on the Newton method for offline optimization.

Cover (1991) proposed the Universal Portfolio (UP) strategy and showed empirically that based on the New York Stock Exchange data over a 22 year period, this universal portfolio perform well on two-stock portfolios. Cover & Ordentlich (1996) further investigate universal portfolio and redefined the algorithm as \( \mu \)-Weighted universal portfolio and study universal portfolio with side information using conditional probability. So far, side information is not easy to determine and was not proved to be much superior, therefore, this research will not consider side information.

Kalai & Vempala (2002) pointed out that the algorithm to calculate Cover & Ordentlich (1996)’s universal portfolio strategy accurately has time (or space) complexity of \( O(n^{m-1}) \), where \( n \) is the number of trading days and \( m \) is the number of stocks.

Several studies are conducted with the objective of saving computation time.
and computer-memory space. The universal portfolio is first studied by Helm-bold et al. (1998) in employing multiplicative update rule in the portfolio derived from a frame rule in Kivinen & Warmuth (1997). The algorithm is easier to be implemented and requires less computer memory for implementation compare to Cover’s universal portfolio. Since the algorithm uses multiplicative updates on the weight distribution and does not involve any integration, it is more straightforward to implement than Cover’s algorithm.

Next, Ishijima (2001) developed several methods for computing universal port-
folios and conducted numerical experiments to show its possibility for the practi-
cal use. Ishijima applied Monte Carlo Methods for calculating universal portfo-
lios. By the virtue of Monte Carlo, they are not time-consuming even if increasing
the universe of assets. Besides, Ishijima also showed how to generate Dirichlet variates in the feasible region of constant portfolios, attention have been paid to
the generation of uniform variates which is special case of Dirichlet Variates.

The recent study with the above objective is the finite-order universal portfolio
introduced by Tan (2013). This finite order universal portfolio of order $\nu$
depends on the most recent $\nu$ days of the stock-price data, assuming that the assets are
stocks.

Tan (2002) also studied the asymptotic performance of the parametric family
of Cover-Ordentlich universal portfolios with respect to the best constant rebal-
anced portfolio. In a recent paper, Tan & Lim (2010) analysed the empirical
performance of the parametric family of Helmbold universal portfolios. Tan &
Pang (2011) compared the performances of the two parametric families of uni-
versal portfolios. Some empirical results are obtained based on some stock data
sets from the local stock exchange and the results seem to indicate that the Cover-
Ordentlich universal portfolio can outperform the Helmbold universal portfolio
by a proper choice of parameters.

The performance of the Dirichlet universal portfolio can be improved by vary-
ing the parameter vector periodically after a fixed number of days, which is known
as a trading day. After fixing a trading period, Tan & Pang (2012) showed that
using a cyclic constrained-search algorithm, a new parametric vector is chosen
that improve upon the wealth achieved in the previous period. They studied on
some selected stock data sets from the local stock exchange and results shown that
higher returns in wealth are achieved for the parameter-varying universal portfolio
over the constant-parameter universal portfolio.

In Tan & Chu (2012), the investment performance of a pair of universal port-
folios generated by the quadratic divergence with respect to a symmetric, positive
definite matrix and its inverse is studied. They have selected three stock-data
sets from Kuala Lumpur Stock Exchange and the empirical performance of the
three-stock portfolios is presented. The empirical results showed that the use of
complementary pairs of universal portfolios may increase the wealth of investors.

Tan & Lim (2013) have demonstrated another application of the mixture-
current-run universal portfolio in extracting the best daily wealth due to best pa-
rameter from the same parametric family of universal portfolios. The empirical
study showed that with the three stock data sets selected for the Kuala Lumpur
Stock Exchange, the universal portfolio that using the best current-run parameter
seem to outperform the original universal portfolio that using the best parameter
from hindsight.
In Tan et al. (2015), a study of the empirical performance of the universal portfolios generated by certain reciprocal functions of the price relatives is presented. The portfolios are obtained from the zero-gradient sets of specific logarithmic objective functions containing the estimated daily growth rate of the investment wealth. The Malaysian companies selected for this empirical study consist of five stock data sets and trading period ranging from 1 March 2006 until 2 August 2012, consisting of a total of 1500 trading days.

Yeoh et al. (2010) studied the time-varying world integration of the Malaysia stock market using a Kalman Filter approach. They estimated the time-varying world integration of the Malaysia stock market and examined if the paths of the time-varying integration match the economic events of the country, where they employed weekly time series data for the period between February 1988 and September 2009. The results showed that by using the Kalman Filter technique, the changes in the level of market integration coincided with the economic events that took place in the country and provided some evidence to the practical application and suitability of the Kalman Filter technique in studying stock market integration.

The relationship between the portfolio risk and the number of stocks in a portfolio for a given portfolio return across portfolios of the Malaysian stocks have been studied by Gupta et al. (2001). The Random Diversification Approach based on the Statman (1987) technique has been used in their study with a sample of 213 stocks traded on the Kuala Lumpur Stock Exchange (KLSE) are considered to form sets of portfolios.

Borodin et al. (2004) studied the novel algorithm for actively trading stocks.
The empirical results on historical markets provide strong evidence that this type of technical trading can beat the market and moreover can beat the best stock in the market.

From the above studies, especially the studies by Tan and his students, the dataset chosen for studying the performances of parameter for respective universal portfolios only involved small combinations of stocks and short term trading period from Malaysian stocks. Therefore, a comprehensive study of universal portfolio involve of large number of stocks, middle term and long term period will be carried out for investigation in our study.

1.3 Outline

This thesis mainly consists of seven chapters together with an introduction and conclusion. In Chapter 2, the data included in this study comprises of 95 stocks data collected from KLSE, whose was available from the Yahoo Finance historical data. The database contains of daily opening prices, daily closing prices, daily high and low, and the volume of transaction are collected.

In Chapter 3, we introduce the finite-order universal portfolios generated by two probability distributions and universal portfolios generated by stochastic process due to Tan (2013). This type of universal portfolio depends only on the positive moments of the generating probability distributions. The finite-order universal portfolio of order $\nu$ depends on the most recent $\nu$ days of the stock-price data, assuming that our assets are stocks. The constraint is there is no transaction
cost in each portfolio. In order to reduce the extensive memory and long computational time, we choose the low order universal portfolio, where the order $\nu$ is a small integer, where $\nu = 1, 2, 3$.

There are many programming languages for the implementation of the universal portfolio strategies. Python is a suitable choice because it is easy to use, powerful, and versatile. In Chapter 4, the algorithms of four proposed order one universal portfolios are written in Python code. Also, we used Dirichlet$(\alpha_1, \alpha_2, \alpha_3)$ universal portfolio strategy as a benchmark to compare with our four proposed universal portfolio strategies. The modified algorithm of Diriclet universal portfolio was obtained by Tan (2004b). We now chosen this modified algorithm for computing the three stock universal portfolio generated by Dirichlet$(\alpha_1, \alpha_2, \alpha_3)$ distribution where $\alpha_j > 0$ for $j = 1, 2, 3$. This modified algorithm is written in python using numpy module.

In Chapter 5, we present an experimental study of the four universal portfolios strategies, namely the finite order Multinomial Universal Portfolio, the finite order Multivariate Normal universal portfolio, universal portfolio generated by Brownian-motion and universal portfolio generated by Ornstein-Uhlenbeck processes. 10 most active stocks data are chosen among the 95 selected stocks from Kuala Lumpur Stock Exchange. The trading period is from 1 January 2000 to 31 December 2015. The above order one portfolio are run on 3 stock-price data sets generated by the combination of the 10 selected most active stocks. The wealths achieved after the $n$ trading days by the above portfolio strategies is compare to the CRP strategies. The well performing parameters of the above four strategies are selected for further study in Chapter 6.
In Chapter 6, the overview of the analysis are carried out. The 95 stocks with 16 years of trading period divided into 5 groups, there are 1 year trading period (short term) ranging from 1 January 2015 to 31 December 2015, 4 years period group (middle term) started from 1 January 2012 to 31 December 2015, 8 years period group (middle term) ranging from 1 January 2008 to 31 December 2015, 12 years period group (long term) started from 1 January 2004 to 31 December 2015 and 16 years period group (long term) from 1 January 2000 to 31 December 2015. These 5 groups of data are run using the proposed four universal portfolio strategies with the well performing respective parameter obtained in Chapter 5. The wealth obtained is compared to its respective benchmark, CRP. In first section of Chapter 6, the price relatives for a given stock used is the ratio price defined in section Problem Setting in this chapter. In conclusion, the empirical results showed that the performances of the proposed four strategies are better than CRP in 1 year and 4 years period groups. However, the four strategies did poorly in 8 years, 12 years and 16 years period groups.

In order to compare to benchmark, KLCI, the price relatives for a given stock is redefined as the ratio of today closing price to the day before closing price which is discussed in Chapter 2. The wealth obtained is compared to its respective benchmark, CRP and KLCI. In conclusion, only around 25% of the results showed that the performances of the proposed four strategies are better than CRP in short term, middle term and long terms period investment. This concluded that the four proposed universal portfolio strategies did poorly when compared to KLCI.
CHAPTER 2

DATA COLLECTION

In this chapter, we explain how the data were obtained and processed. This research will work solely on the analysis of the stock data from Bursa Malaysia (formerly known as KLSE), which is introduced in Section 2.1. We then explain how we pick the stocks from Bursa Malaysia in Section 2.2 and how Python was used to process the stock data for later use in Section 2.3.

2.1 Bursa Malaysia

A National Bank (Bank Negara) report in 2005 indicated that compared to other exchange markets, Malaysia was ranked twenty-third in the world. Bursa Malaysia is a prominent centre of stock markets in South East Asia. It was previously known as Kuala Lumpur Stock Exchange (KLSE) from its founding in 1930 when the Singapore Stockbrokers’ Association was set up as a formal organisation dealing in securities in Malaya (http://www.bursamalaysia.com/corporate/about-us/corporate-history/).

When Singapore broke away from Malaya federation in 1966 and became an independent country, the stock exchange remained its function until 1973, where the Stock Exchange of Malaysia and Singapore was divided into the Kuala Lumpur Stock Exchange Berhad (KLSE) and the Stock Exchange of Singapore...
(SES) with the cessation of currency interchangeability between Malaysia and Singapore.

On 14 April, 2004, the KLSE was renamed Bursa Malaysia Berhad, following a demutualisation exercise, the purpose of which was to enhance competitive position and to respond to global trends in the exchange sector by making the Exchange more customer-driven and market-oriented.

On 18 March, 2005, Bursa Malaysia was listed on the Main Board of Bursa Malaysia Securities Berhad. On 6 July 2009, Bursa Malaysia introduced an enhancement of the Kuala Lumpur Composite Index (KLCI). The KLCI adopted the FTSE’s global index standards and became known as the FTSE Bursa Malaysia KLCI. The FTSE Bursa Malaysia KLCI adopted the internationally accepted index calculation methodology to provide a more investable, tradable and transparent managed index. The constituents free float adjusted only by the investable portion included in the index calculation. With this new method the constituents of the FTSE Bursa Malaysia KLCI, also known as the FBM KLCI, shrunk from 100 to 30 companies to enhance the tradability of the index, while remaining representative of the market (Sinakalai & Suppayah 2011).

The FBM KLCI comprises thirty largest companies in Malaysia by full market capitalisation on the Bursa Malaysia’s main board. When it was launched on the 6 July 2009, it replaced the old KLCI. It started at the closing value of the old KLCI on 3rd July 2009 and inherited the full history of the old KLCI index (Wikipedia 2016b).
2.2 Data Description

The KLCI was relatively new compared to famous indices such as Dow Jones Industrial Average (https://en.wikipedia.org/wiki/Dow_Jones_Industrial_Average) and FTSE 100 Index (Wikipedia 2016a) which dated back to 1885 and 1984 respectively.

Since the FTSE 100 Index consists of 100 companies listed on the London Stock Exchange with the highest market capitalisation, we were trying to find 100 companies with the highest market capitalisation in Malaysia but from the 100 top listed companies in the main board of Bursa Malaysia given by (Capital n.d.). However, only 95 of them have data available from the Yahoo Finance (Yahoo n.d.). Five stocks listed in (Capital n.d.), i.e. JT International Bhd, IJM Land Bhd, UOA Development Bhd, Malaysia Airlines System Bhd, and Kulim Malaysia Bhd were excluded because the historical data are not available in Yahoo Finance.

We fixed our data to begin from the beginning of the year 2000. However, not all the 95 selected stocks were listing since 2000. We will tabulate the data obtained from Yahoo Finance based on the category of the stock and the earliest trading date to 31 December 2015. The average volume is calculated as the average of volume of the earliest trading day (not earlier than 1 Jan 2000) to the volume of 31 December 2015.
Table 2.1: 95 Selected Stocks Based On The Category Of The Stock

<table>
<thead>
<tr>
<th>Category</th>
<th>Code</th>
<th>Stock Name</th>
<th>Average Volume</th>
<th>Earliest Trading Date</th>
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<td></td>
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</tr>
<tr>
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<td>Guinness Anchor Bhd</td>
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<td>1 Jan 2000</td>
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<td>1 Jan 2000</td>
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<td></td>
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<td>Bristish American Tabacco (M)</td>
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<td>Category</td>
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<td>Earliest Trading Date</td>
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Table 2.1 (Continue) 95 Selected Stocks Based On The Category Of The Stock

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<th>Category</th>
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<th>Stock Name</th>
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<td>5227</td>
<td>IGB Real Estate Investment Trust</td>
<td>118431450</td>
<td>21 Sept 2012</td>
</tr>
<tr>
<td>Trading Services</td>
<td>1562</td>
<td>Berjaya Sports Toto Bhd</td>
<td>1871300</td>
<td>1 Jan 2000</td>
</tr>
<tr>
<td></td>
<td>2194</td>
<td>MMC Coparation Bhd</td>
<td>328200</td>
<td>1 Jan 2000</td>
</tr>
<tr>
<td></td>
<td>3034</td>
<td>Hap Seng Consolidated Bhd</td>
<td>850200</td>
<td>1 Jan 2000</td>
</tr>
<tr>
<td></td>
<td>3182</td>
<td>Genting Bhd</td>
<td>13869700</td>
<td>1 Jan 2000</td>
</tr>
<tr>
<td></td>
<td>3859</td>
<td>magnum Bhd</td>
<td>2109600</td>
<td>31 Dec 2007</td>
</tr>
<tr>
<td></td>
<td>4197</td>
<td>Sime Darby Bhd</td>
<td>1650850</td>
<td>1 Jan 2000</td>
</tr>
</tbody>
</table>
Table 2.1 (Continue) 95 Selected Stocks Based On The Category Of The Stock

<table>
<thead>
<tr>
<th>Category</th>
<th>Code</th>
<th>Stock Name</th>
<th>Average Volume</th>
<th>Earliest Trading Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trading Services</td>
<td>4502</td>
<td>Media Prime Bhd</td>
<td>193950</td>
<td>1 Jan 2000</td>
</tr>
<tr>
<td></td>
<td>4634</td>
<td>Pos Malaysia Bhd</td>
<td>9523550</td>
<td>1 Jan 2000</td>
</tr>
<tr>
<td></td>
<td>4677</td>
<td>YTL Coparation Bhd</td>
<td>5907750</td>
<td>1 Jan 2000</td>
</tr>
<tr>
<td></td>
<td>4863</td>
<td>Telekom Malaysia Bhd</td>
<td>4929150</td>
<td>1 Jan 2000</td>
</tr>
<tr>
<td></td>
<td>5014</td>
<td>Malaysia Airports Holdings Bhd</td>
<td>1799650</td>
<td>1 Jan 2000</td>
</tr>
<tr>
<td></td>
<td>5090</td>
<td>Media Chinese International Limited</td>
<td>134200</td>
<td>31 Dec 2007</td>
</tr>
<tr>
<td></td>
<td>5099</td>
<td>Air Asia Bhd</td>
<td>94589600</td>
<td>22 Nov 2004</td>
</tr>
<tr>
<td></td>
<td>5141</td>
<td>Dagang Enterprise Holdings Bhd</td>
<td>568050</td>
<td>3 Mar 2009</td>
</tr>
<tr>
<td></td>
<td>5186</td>
<td>Malaysia Marine and Heavy Eng Holdings Bhd</td>
<td>71266600</td>
<td>29 Oct 2010</td>
</tr>
<tr>
<td></td>
<td>5209</td>
<td>Gas Malaysia Bhd</td>
<td>43652950</td>
<td>6 Nov 2012</td>
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<tr>
<td></td>
<td>5210</td>
<td>Bumi Armada Bhd</td>
<td>281719400</td>
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</tr>
<tr>
<td></td>
<td>5218</td>
<td>Sapura kencana Petroleum Bhd</td>
<td>110011600</td>
<td>17 May 2012</td>
</tr>
<tr>
<td></td>
<td>5225</td>
<td>IHH Healthcare Bhd</td>
<td>199803750</td>
<td>25 July 2012</td>
</tr>
<tr>
<td></td>
<td>5347</td>
<td>Tenaga Nasional Bhd</td>
<td>5677850</td>
<td>31 Dec 2007</td>
</tr>
<tr>
<td></td>
<td>5657</td>
<td>Pakson Holdings Bhd</td>
<td>465350</td>
<td>1 Jan 2000</td>
</tr>
<tr>
<td></td>
<td>5681</td>
<td>Petronas Dagangan Bhd</td>
<td>301200</td>
<td>1 Jan 2000</td>
</tr>
<tr>
<td></td>
<td>6012</td>
<td>Maxis Bhd</td>
<td>154892350</td>
<td>19 Nov 2009</td>
</tr>
<tr>
<td></td>
<td>6351</td>
<td>Amway Malaysia Holdings Bhd</td>
<td>21350</td>
<td>1 Jan 2000</td>
</tr>
<tr>
<td></td>
<td>6399</td>
<td>Astro Malaysia Holdings Bhd</td>
<td>6912200</td>
<td>31 Dec 2012</td>
</tr>
<tr>
<td></td>
<td>6599</td>
<td>Aeon Co.(M) Bhd</td>
<td>650650</td>
<td>1 Jan 2000</td>
</tr>
<tr>
<td></td>
<td>7277</td>
<td>Dialog Group Bhd</td>
<td>3811900</td>
<td>1 Jan 2000</td>
</tr>
</tbody>
</table>
To select good performing parameters for the universal portfolio strategies developed in Chapter 3, we will pick 10 most active (highest average volume) and long-term (having at least 10 years being listed in the stock exchange) stocks from each categories in the above table. This is possible for all categories except the REIST category, where all stocks are listed less than 10 years. Therefore, we picked two stocks from the trading services category and did not pick any stocks from the REIST category.

Table 2.2: 10 Most Active Stocks

<table>
<thead>
<tr>
<th>Category</th>
<th>stock code</th>
<th>stock name</th>
<th>Average Volume</th>
<th>Earliest Trading Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Construction</td>
<td>5398</td>
<td>Gamuda Bhd</td>
<td>4986282.33</td>
<td>1 Jan 2000</td>
</tr>
<tr>
<td>Consumer Product</td>
<td>7084</td>
<td>QL Resources Berhad</td>
<td>2086150</td>
<td>31 Mar 2000</td>
</tr>
<tr>
<td>Finance</td>
<td>1818</td>
<td>Bursa Malaysia Bhd</td>
<td>51271350</td>
<td>18 Jan 2005</td>
</tr>
<tr>
<td>Hotel</td>
<td>5517</td>
<td>Shangri-La Hotels Malaysia Bhd</td>
<td>69100</td>
<td>1 Jan 2000</td>
</tr>
<tr>
<td>Industry Products</td>
<td>7106</td>
<td>Supermax Corporation Bhd</td>
<td>12656000</td>
<td>1 Jan 2000</td>
</tr>
<tr>
<td>IPC</td>
<td>5031</td>
<td>Time Dotcom Bhd</td>
<td>630100</td>
<td>13 Mar 2001</td>
</tr>
<tr>
<td>Plantation</td>
<td>2216</td>
<td>IJM Plantation Bhd</td>
<td>10168800</td>
<td>2 July 2003</td>
</tr>
<tr>
<td>Properties</td>
<td>5148</td>
<td>UEM Sunrise Bhd</td>
<td>9561550</td>
<td>1 Jan 2000</td>
</tr>
<tr>
<td>Trading Services</td>
<td>5099</td>
<td>Air Asia Bhd</td>
<td>94589600</td>
<td>22 Nov 2004</td>
</tr>
<tr>
<td>Trading Services</td>
<td>3182</td>
<td>Genting Bhd</td>
<td>13869700</td>
<td>1 Jan 2000</td>
</tr>
</tbody>
</table>
2.3 Generating Price Ratios

The data of all the 95 stocks were downloaded from Yahoo Finance in CSV format (https://en.wikipedia.org/wiki/Comma-separated_values). Each stock consists the daily opening prices, the daily closing prices, the daily high and low, and the volume of transaction. However, what we are interested is the ratio of the stock price. We assume that it is possible to *buy at the opening* and *sell at the closing* and use the ratio of closing price to its opening price as price relative which we defined in problem setting in Chapter 1, to study the performance of the universal portfolio strategies we investigate in Chapter 3. We also assume that the opening price of day \( n \) is same as the closing price on day \( n - 1 \) for the respective stock.

Since there are 95 stocks, the price ratio of each of the stock data will be calculated. For example, the price ratio for stock with code 5398 (GAMUDA, which belongs to the construction category) can be generated by running the following Python script.

```python
import pandas as pd

df = pd.read_csv("CONSTRUCTION/5398GAMUDA.csv")
df['Date'] = pd.to_datetime(df['Date'],format="%m/%d/%y")
df['DailyROI'] = df['Close']/df['Open']
newdf = df[['Date','DailyROI']]
newdf.to_csv("construction new/5398roi.csv")
```
However, in order to compare the performances of four proposed strategies with benchmark KLCI which we will investigate in Chapter 6, the price relative is defined in problem setting (refer Chapter 1) are redefined as the ratio of closing price on day $n$ to the closing price of day $n - 1$, where the code in line number 5 are changed accordingly. If the stock splits or bonus issue was announced by the respective company, the stock’s price has either increased or decreased sharply, the price relative will follow the ratio calculated as defined in problem setting of Chapter 1.

We then combine all the 95 files with all the price ratios for all 95 stocks (listed in the variable $stockslist$) into a large CSV file using the following Python script.

```python
stockslist = [...] list_of_csv_files ...]

import pandas as pd

maindf = pd.read_csv("IPC new/5031roi.csv")
maindf["Date"] = pd.to_datetime(maindf["Date"])
maindf = maindf.drop(maindf.columns[0], axis=1)

for thefile in stockslist:
    df = pd.read_csv(thefile)
    df["Date"] = pd.to_datetime(df["Date"])
    df = df.drop(df.columns[0], axis=1)
    df.columns = [df.columns[0], df.columns[1] + thefile[:-4].replace(" new/", "_")]
maindf = pd.merge(left=maindf, right=df, on="Date", how="outer")
maindf = maindf.sort(["Date"])
maindf.to_csv("stockslist.csv")
```
In this chapter, we first simplified the formulation of $\mu$-weighted universal portfolio Cover & Ordentlich (1996) and order $\nu$ universal portfolio Tan (2013) using probability theory and further extend to probability generated order $\nu$ universal portfolio to stochastic process based universal portfolio. We begin the section to generate the two order $\nu$ universal portfolio by probability distribution, namely Multinomial universal portfolio and Multivariate Normal universal portfolio, follow by two universal portfolio generated by stochastic process, they are universal portfolio generated by Brownian-motion and universal portfolio generated by Ornstein-Uhlenbeck process.

Consider a financial market with $m$ stocks, the wealth of a fund is invested over the $m$ shares in the market for a sequence of $n$ days trading. Let $x_n = (x_{n,1}, x_{n,2}, \ldots, x_{n,m})$ be the price relative vectors of the $n^{th}$ trading day. Here, $x_{n,1}, x_{n,2}, \ldots, x_{n,m}$ are the ratio of the closing price to the opening price for $m$ stocks respectively on $n^{th}$ trading day.

Assuming that a portfolio is self-financing and no short is allowed then the variation of the wealth of the $n^{th}$ trading day is related to the following factor

$$ b_{n,1}x_{n,1} + b_{n,2}x_{n,2} + \cdots + b_{n,m}x_{n,m} \quad (3.1) $$

where $b_{n,1} + b_{n,2} + \cdots + b_{n,m} = 1$ and $0 \leq b_{n,i} \leq 1$ for $1 \leq i \leq m$. 
Let $S_n$ be the wealth at the $n^{th}$ trading day and let $S_0$ be the initial wealth before the first trading day. By introducing the column vector notation $b_n = (b_{n1}, \ldots, b_{nm})$ from linear algebra (Goodaire 2003), (3.1) can be written as $b_n'x_n$, where $t$ denotes transpose. Therefore, the total wealth at the end of $n^{th}$ trading day becomes

$$S_n = S_0b_1'x_1b_2'x_2 \cdots b_n'x_n = S_0 \prod_{j=1}^{n} b_j'x_j \quad (3.2)$$

Universal portfolio strategy is a kind of online portfolio selection strategy (Li & Hoi 2012) and proposed in Cover (1991). Cover & Ordentlich (1996) redefined the universal portfolio strategy to $\mu$-weighted Universal Portfolio strategy where $\mu$ is a given distribution on the space of valid portfolios $\Delta_m$ (Li & Hoi 2012). The following are the wealth and update scheme used in Cover & Ordentlich (1996).

$$b_{n+1} = \frac{\int_{\Delta_m} bS_n(b)d\mu(b)}{\int_{\Delta_m} S_n(b)d\mu(b)} \quad (3.3)$$

$$S_n = \int_{\Delta_m} S_n(b)d\mu(b) \quad (3.4)$$

where $S_n(b) = \prod_{j=1}^{n} b_j'x_j$.

Let $Y = (Y_1, \ldots, Y_m)$ be an $m$-dimensional random vector such that $0 \leq Y_i \leq 1$ and $Y_1 + \cdots + Y_m = 1$. By using the notation of expected value (Wackerly et al. 2000) and replacing the random vector in (3.3) with $Y$, it can be written as

$$b_{n+1} = \frac{E[Y \prod_{j=1}^{n} Y_j'x_j]}{E[\prod_{j=1}^{n} Y_j'x_j]} = \frac{E[Y(Y_n'x_n)(Y_{n-1}'x_{n-1}) \cdots (Y_1'x_1)]}{E[(Y_n'x_n)(Y_{n-1}'x_{n-1}) \cdots (Y_1'x_1)]} \quad (3.5)$$
The $k$ component of vector $b_{n+1}$ is

$$b_{n+1,k} = \frac{E[Y_k(Y'x_n)(Y'x_{n-1}) \cdots (Y'x_1)]}{E[(Y'x_n)(Y'x_{n-1}) \cdots (Y'x_1)]}$$  \hspace{1cm} (3.6)$$

where $k = 1, 2, \ldots, m$.

### 3.1 Order $\nu$ Universal Portfolio

The idea of using a probability distribution to generate a universal portfolio was first proposed by Cover (1991). The Cover-Ordentlich universal portfolio (Cover & Ordentlich 1996) is a moving-order universal portfolio. This moving-order universal portfolio are not practical in the sense that as the number of stocks in the portfolio increases, the implementation time and the computer storage requirements grow exponentially fast. To improve on the time and memory storage performance, Tan (2013) proposed finite-order universal portfolio generated by some probability distribution. This type of universal portfolio depends only on the positive moments of the generating probability distribution. Our study only focus on Multinomial distribution generated universal portfolio and Multivariate normal distribution generated universal portfolio.

Tan (2013) introduced a variation of Cover’s universal portfolio to deal with the inefficiency of Cover’s universal portfolio. We will call it a *Probability-Based Order-$\nu$ Universal Portfolio*. In contrast to Cover, Tan (2013) introduced a random vector such that $Y_i > 0$ and define

$$b_{n+1} = \frac{E[(Y(Y'x_n)(Y'x_{n-1}) \cdots (Y'x_{n-\nu+1})]}{E[(Y_1 + Y_2 + \cdots + Y_m)(Y'x_n)(Y'x_{n-1}) \cdots (Y'x_{n-\nu+1})]}$$  \hspace{1cm} (3.7)$$
For $k$ component, (3.7) can be written as

$$b_{n+1,k} = \frac{E[(Y_k(Y^t x_n)(Y^t x_{n-1}) \cdots (Y^t x_{n-\nu+1})]}{E[(Y_1 + Y_2 + \cdots + Y_m)(Y^t x_n)(Y^t x_{n-1}) \cdots (Y^t x_{n-\nu+1})]} \quad (3.8)$$

First, the wealth formula (3.2) is written as

$$S_n = (b^t_n x_n)(b^t_{n-1} x_{n-1}) \cdots (b^t_{n-\nu+1} x_{n-\nu+1})S_{n-\nu} \quad (3.9)$$

when $\nu = 1$,

$$S_n = (b^t_n x_n) \cdot S_{n-1} \quad (3.10)$$

when $\nu = 2$,

$$S_n = (b^t_n x_n)(b^t_{n-1} x_{n-1}) \cdot S_{n-2} \quad (3.11)$$

Although this method eliminates some inefficiency in Cover’s universal portfolio, it has bootstrap issues. We pick simple constant rebalanced portfolio to bootstrap $S_0, S_1, \cdots, S_\nu$.

3.1.1 General Algorithm

The algorithm for computing (3.8) can be defined by expanding the numerator expression as follows:
\begin{equation}
E[Y_k(Y'^t x_n) \cdots (Y'^t x_{n-\nu+1})] \\
= E[Y_k \prod_{j=n-\nu+1}^n (\sum_{i=1}^m Y_i x_{j,i})] \\
= E[Y_k \sum_{i_1=1}^m \sum_{i_2=1}^m \cdots \sum_{i_\nu=1}^m Y_{i_1} \cdots Y_{i_\nu} x_{n,i_1} \cdots x_{n-\nu+1,i_\nu} E[Y_{i_1} \cdots Y_{i_\nu}]]
\end{equation}

(3.12)

The expanding of denominator of (3.8) is as follow:

\begin{equation}
E[(Y_1 + \cdots + Y_m)(Y'^t x_n) \cdots (Y'^t x_{n-\nu+1})] \\
= \sum_{j=1}^m E[Y_j(Y'^t x_n) \cdots (Y'^t x_{n-\nu+1})] \\
= \sum_{j=1}^m \sum_{i_1=1}^m \sum_{i_2=1}^m \cdots \sum_{i_\nu=1}^m x_{n,i_1} \cdots x_{n-\nu+1,i_\nu} E[Y_j Y_{i_1} \cdots Y_{i_\nu}]
\end{equation}

(3.13)

when \( \nu = 1 \), (3.8) becomes

\begin{equation}
b_{n+1,k} = \frac{\sum_{i=1}^m x_{n,i} E[Y_i Y_k]}{\sum_{i=1}^m x_{n,i} (\sum_{j=1}^m E[Y_j Y_k])}
\end{equation}

(3.14)

When \( \nu = 1 \), \( m = 3 \), let the denominator be \( \zeta \), then

\begin{equation}
\zeta = x_{n,1} E[Y_1 Y_1] + x_{n,2} E[Y_1 Y_2] + x_{n,3} E[Y_1 Y_3] \\
+ x_{n,1} E[Y_2 Y_1] + x_{n,2} E[Y_2 Y_2] + x_{n,3} E[Y_2 Y_3] \\
+ x_{n,1} E[Y_3 Y_1] + x_{n,2} E[Y_3 Y_2] + x_{n,3} E[Y_3 Y_3]
\end{equation}

(3.15)
The (3.14) becomes

\[ b_{n+1,k} = \zeta^{-1}(x_{n,1}E[Y_kY_1] + x_{n,2}E[Y_kY_2] + x_{n,3}E[Y_kY_3]) \] (3.16)

for \( k = 1, 2, 3 \), and the wealth of day \( n + 1 \) is defined as

\[ S_{n+1} = (b_{n+1}^t x_{n+1})S_n \] (3.17)

When \( \nu = 2 \), (3.8) becomes

\[ b_{n+1,k} = \frac{x_{n,1}x_{n-1,1}E[Y_kY_1] + x_{n,1}x_{n-1,2}E[Y_kY_2] + \cdots + x_{n,m}x_{n-1,m}E[Y_kY_m]}{\sum_{j=1}^m (x_{n,1}x_{n-1,1}E[Y_jY_1] + x_{n,1}x_{n-1,2}E[Y_jY_2] + \cdots + x_{n,m}x_{n-1,m}E[Y_jY_m])} \] (3.18)

and \( \nu = 2 \) and \( m = 3 \), let the denominator be \( \zeta \), then

\[ \zeta = (x_{n,1}x_{n-1,1}E[Y_1Y_1] + x_{n,1}x_{n-1,2}E[Y_1Y_1] + x_{n,3}x_{n-1,3}E[Y_1Y_3] + x_{n,2}x_{n-1,1}E[Y_1Y_2] + x_{n,2}x_{n-1,2}E[Y_1Y_2] + x_{n,2}x_{n-1,3}E[Y_1Y_3] + x_{n,3}x_{n-1,1}E[Y_3Y_1] + x_{n,3}x_{n-1,2}E[Y_3Y_2] + x_{n,3}x_{n-1,3}E[Y_3Y_3] + x_{n,1}x_{n-1,1}E[Y_2Y_1] + x_{n,1}x_{n-1,2}E[Y_2Y_1] + x_{n,3}x_{n-1,3}E[Y_2Y_3] + x_{n,2}x_{n-1,1}E[Y_2Y_2] + x_{n,2}x_{n-1,2}E[Y_2Y_2] + x_{n,2}x_{n-1,3}E[Y_2Y_3] + x_{n,3}x_{n-1,1}E[Y_3Y_1] + x_{n,3}x_{n-1,2}E[Y_3Y_2] + x_{n,3}x_{n-1,3}E[Y_3Y_3] + x_{n,1}x_{n-1,1}E[Y_3Y_1] + x_{n,1}x_{n-1,2}E[Y_3Y_1] + x_{n,3}x_{n-1,3}E[Y_3Y_3] + x_{n,2}x_{n-1,1}E[Y_3Y_2] + x_{n,2}x_{n-1,2}E[Y_3Y_2] + x_{n,2}x_{n-1,3}E[Y_3Y_3] + x_{n,3}x_{n-1,1}E[Y_3Y_1] + x_{n,3}x_{n-1,2}E[Y_3Y_2] + x_{n,3}x_{n-1,3}E[Y_3Y_3] \] (3.19)
and (3.18) becomes

\[ b_{n+1,k} = \zeta^{-1}(x_{n,1}x_{n-1,1}E[Y_kY_1Y_1] + x_{n,1}x_{n-1,2}E[Y_kY_1Y_2] + x_{n,3}x_{n-1,3}E[Y_kY_1Y_2] + x_{n,2}x_{n-1,2}E[Y_kY_2Y_1] + x_{n,2}x_{n-1,2}E[Y_kY_2Y_2] + x_{n,3}x_{n-1,2}E[Y_kY_2Y_3]) \]

(3.20)

for \( k = 1, 2, 3 \) and the wealth of day \( n + 1 \) is defined as

\[ S_{n+1} = (b_{n+1}^t x_{n+1})(b_n^t x_n)S_{n-1} \]

(3.21)

When \( \nu = 3 \), (3.8) becomes

\[ b_{n+1,k} = \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \sum_{i_3=1}^{m} x_{n,i_1}x_{n-1,i_2}x_{n-2,i_3}E[Y_kY_1Y_2Y_3] \]

(3.22)

The expansion of the algorithm for order three are similar to order one and two. When order increases, the algorithm will be changing accordingly as above and become larger due to the increasing of number of moments.

The wealth of day \( n + 1 \) is defined as

\[ S_{n+1} = (b_{n+1}^t x_{n+1})(b_n^t x_n)(b_{n-1}^t x_{n-1})S_{n-2} \]

(3.23)
3.1.2 Multinomial Universal Portfolio (MUP)

In Wackerly et al. (2000), when the random vector \((Y_1, Y_2, \ldots, Y_m)\) has a joint multinomial distribution with parameters \(N, p_1, p_2, \ldots, p_{m-1}\) where \(0 < p_i < 1\) for \(i = 1, 2, \ldots, m-1\) and \(0 < p_m = 1 - p_1 - p_2 - \cdots - p_{m-1} < 1\); \(N\) is a positive integer larger than the number of stocks in the market, \(m\). Consider joint probability function \(f(y_1, y_2, \ldots, y_m)\) of Multinomial distribution:

\[
f(y_1, y_2, \ldots, y_m) = \binom{N}{y_1, y_2, \ldots, y_m} p_1^{y_1} p_2^{y_2} \cdots p_m^{y_m} \quad (3.24)
\]

where the multinomial coefficient

\[
\binom{N}{y_1, y_2, \ldots, y_m} = \frac{N!}{y_1! y_2! \cdots y_m!} \quad (3.25)
\]

and \(y_i = 0, 1, 2, \ldots, N\) for \(i = 1, 2, \ldots, m\) subject to \(y_1 + y_2 + \cdots + y_m = N\).

The \(E[Y_1 \cdots Y_m]\) in Section 3.1.1 can be found from the joint moment generating function.

\[
M(\tau_1, \tau_2, \ldots, \tau_m) = (p_1 e^{\tau_1} + p_2 e^{\tau_2} + \cdots + p_m e^{\tau_m})^N \quad (3.26)
\]

When \(\nu = 1\), the \(E[Y_k Y_i_1 \cdots Y_i_\nu]\) in (3.12) can replace by the following moments:
\[ E(Y_k) = N p_k \]
\[ E(Y_k^2) = N(N - 1)p_k^2 + Np_i \]
\[ E(Y_kY_i) = N(N - 1)p_kp_i \quad \text{for } i \neq k \] (3.27)

When \( \nu = 2 \), the \( E[Y_kY_{i_1} \cdots Y_{i_\nu}] \) in (3.12) can be replace by the following moments:

\[ E(Y_k^2) = N(N - 1)p_k^2 + Np_k \]
\[ E(Y_kY_i) = N(N - 1)p_kp_i \quad \text{for } k \neq i \]
\[ E(Y_k^3) = N(N - 1)(N - 2)p_k^3 + 3N(N - 1)p_k^2 + Np_k \]
\[ E(Y_k^2Y_i) = N(N - 1)(N - 2)p_k^2p_i + N(N - 1)p_kp_i \quad \text{for } k \neq i \]
\[ E(Y_kY_iY_j) = N(N - 1)(N - 2)p_kp_ip_j, \quad \text{for } k \neq i, k \neq j, i \neq j \] (3.28)

When \( \nu = 3 \), the \( E[Y_kY_{i_1} \cdots Y_{i_\nu}] \) in (3.12) can replace by the following moments:
\[ E(Y_k^3) = N(N - 1)(N - 2)p_k^3 + 3N(N - 1)p_k^2 + Np_k \]
\[ E(Y_k^2Y_i) = N(N - 1)(N - 2)p_k^2p_i + N(N - 1)p_kp_i \quad \text{for } k \neq i \]
\[ E(Y_kY_iY_j) = N(N - 1)(N - 2)p_kp_i p_j \quad \text{for } k \neq i, k \neq j, i \neq j \]
\[ E(Y_k^4) = N(N - 1)(N - 2)(N - 3)p_k^4 + 6N(N - 1)(N - 2)(N - 3)p_k^3 \]
\[ + 3N(N - 1)p_k^2 + Np_k \]
\[ E(Y_k^3Y_i) = N(N - 1)(N - 2)(N - 3)p_k^3p_i + 3N(N - 1)(N - 2)p_k^2p_i \]
\[ + N(N - 1)p_kp_i \quad \text{for } k \neq i \]
\[ E(Y_k^2Y_i^2) = N(N - 1)(N - 2)(N - 3)p_k^2p_i^2 + N(N - 1)(N - 2)p_k^2p_i \]
\[ + N(N - 1)(N - 2)p_kp_i^2 + N(N - 1)p_kp_i \quad \text{for } k \neq i \]
\[ E(Y_k^3Y_iY_j) = N(N - 1)(N - 2)(N - 3)p_k^3p_i p_j \]
\[ + N(N - 1)(N - 2)p_kp_i p_j \quad \text{for } (i,j,k) \text{distinct} \]
\[ E(Y_kY_iY_jY_l) = N(N - 1)(N - 2)(N - 3)p_kp_i p_j p_l \quad \text{for } (i,j,k,l) \text{distinct}. \]

In particular, when \( m = 3 \), order =1, \( k = 1, 2, 3 \) and \( i = 1, 2, 3 \), (3.27) becomes
\[ E(Y_1) = Np_1 \]
\[ E(Y_2) = Np_2, \]
\[ E(Y_3) = Np_3 \]
\[ E(Y_1^2) = N(N - 1)p_1^2 + Np_1 \]
\[ E(Y_2^2) = N(N - 1)p_2^2 + Np_2 \]
\[ E(Y_3^2) = N(N - 1)p_3^2 + Np_3 \]
\[ E(Y_1Y_2) = N(N - 1)p_1p_2 \]
\[ E(Y_1Y_3) = N(N - 1)p_1p_3 \]
\[ E(Y_2Y_3) = N(N - 1)p_2p_3 \]

Similarly, when order increasing, all the moments can be found by using partial derivatives of the joint moment generating function of (3.26).
3.1.3 Multivariate Normal Universal Portfolio
(MVNUP)

Consider the random vector $Y = (Y_1, Y_2, \cdots, Y_m)$ with a joint multivariate normal probability density function defined as follow:

$$f(Y) = \frac{e^{-\frac{1}{2}(Y-\mu)^tK^{-1}(Y-\mu)}}{(\sqrt{2\pi})^n|K|^{1/2}}, \quad (3.31)$$

Where $K$ is the covariance matrix of $Y$, $E(Y) = \mu = (\mu_1, \mu_2, \cdots, \mu_m)$ is the mean vector. We say $Y$ has the multivariate normal distribution $N(\mu, K)$, $|\cdot|$ means determinate. In our study, we choose $m = 3$, the Multivariate Normal distribution $N(\mu, K)$ is special form of $\mu = (\mu_1, \mu_2, \mu_3)$ and $K = \text{diag}(\sigma_1^2, \sigma_2^2, \sigma_3^2)$, where $Y_1, Y_2, Y_3$ are independent. Hence, the joint moment generating function is defined as:

$$M(t_1, t_2, t_3) = e^{((t_1\mu_1 + t_2\mu_2 + t_3\mu_3) + \frac{1}{2}(\sigma_1^2t_1^2 + \sigma_2^2t_2^2 + \sigma_3^2t_3^2))} \quad (3.32)$$

When $\nu = 1$, the $E[Y_kY_i \cdots Y_{i_r}]$ in (3.12) can be obtained from the following:

$$E(Y_k) = \mu_k$$
$$E(Y_k^2) = \mu_k^2 + \sigma_k^2$$
$$E(Y_kY_i) = \mu_k\mu_i \quad \text{for} \quad i \neq k \quad (3.33)$$
$$E(Y_kY_j) = \mu_k\mu_j \quad \text{for} \quad j \neq k$$
$$E(Y_iY_j) = \mu_i\mu_j \quad \text{for} \quad i \neq j$$

for $k = 1, 2, 3$. 

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Similarly, for order two and three, all the moments of order two and three of the universal portfolio can be obtained by taking appropriate partial derivative of (3.32).

### 3.2 Stochastic Process Generated Universal Portfolios

In this section, the order-$\nu$ universal portfolio will be generalised to *stochastic process based order-$\nu$ universal portfolio*.

**Definition 3.1.** A stochastic process $\{X(t), t \in T\}$ is a collection of random variables. That is, for each $t \in T, X(t)$ is a random variable. The index $t$ is often interpreted as time and, as a result, we refer to $X(t)$ as the state of the process at time $t$.

The set $T$ is called the *index set* of the process. When $T$ is a countable set the stochastic process is said to be a *discrete-time process*. If $T$ is an interval of the real line, the stochastic process is said to be a *continuous-time process*. For instance, $\{X_n, n = 0, 1, \ldots\}$ is a discrete-time stochastic process indexed by the nonnegative integers; while $\{X(t), t \geq 0\}$ is a continuous-time stochastic process indexed by the nonnegative real numbers. The state space of a stochastic process is defined as the set of all possible values that the random variables $X(t)$ can assume. Thus, a stochastic process is a family of random variables that describes the evolution through time of some (physical) process (Ross 2007).
Let \( \{Y_{n1}\}_{n=1}^{\infty}, \{Y_{n2}\}_{n=1}^{\infty}, \cdots, \{Y_{nm}\}_{n=1}^{\infty} \) be \( m \) given independent stochastic processes. For a fixed positive \( \nu \), the \( \nu \)-order universal portfolio \( \{b_{n+1}\} \) generated by the \( m \) given stochastic processes is defined as:

\[
b_{n+1,k} = \frac{E[Y_{nk}(Y_{n1}^tX_{n})\cdots(Y_{t(n-1)}X_{n})\cdots(Y_{t(n-(\nu-1))}X_{n-(\nu-1)})]}{\sum_{j=1}^{m}E[Y_{nj}(Y_{n1}^tX_{n})\cdots(Y_{t(n-1)}X_{n})\cdots(Y_{t(n-(\nu-1))}X_{n-(\nu-1)})]} \tag{3.34}
\]

for \( k = 1, 2, \cdots, m \) and where the vector \( Y_l = (Y_{l1}, \cdots, Y_{lm}) \) for \( l = 1, 2, \cdots \).

The formula in (3.34) is the generalisation of (3.8) because it is a special case of \( Y_{nj} = Y_j, j = 1, \cdots, m \) independent of the trading day \( n \).

Expanding the numerator and denominator of (3.34), \( b_{n+1,k} \) can be written as

\[
\frac{\sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \cdots \sum_{i_{\nu}=1}^{m} (x_{n,i_1}x_{n-1,i_2} \cdots x_{n-\nu+1,i_{\nu}})E[Y_{nk}Y_{n1}Y_{n-1,i_2} \cdots Y_{n-\nu+1,i_{\nu}}]}{\sum_{j=1}^{m} \left( \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \cdots \sum_{i_{\nu}=1}^{m} (x_{n,i_1}x_{n-1,i_2} \cdots x_{n-\nu+1,i_{\nu}})E[Y_{nj}Y_{n1}Y_{n-1,i_2} \cdots Y_{n-\nu+1,i_{\nu}}] \right)} \tag{3.35}
\]

The derivation is similar to (3.12) and (3.13).

### 3.2.1 Brownian Motion, Stationary Process and Weakly Stationary Process

Let us state the definitions of various stochastic processes (Ross 2007) that we will be needed for later subsection.
Definition 3.2. A stochastic process \( \{X(t), t \geq 0\} \) is said to be a Brownian motion process with drift coefficient \( \mu \) and variance parameter \( \sigma^2 \) if

1. \( X(0) = 0 \),
2. \( \{X(t), t \geq 0\} \) has stationary and independent increments;
3. for every \( t > 0 \), \( X(t) \) is normally distributed with mean \( \mu t \) and variance \( \sigma^2 t \), i.e. \( X(t) \sim \text{Normal}(\mu t, \sigma^2 t) \).

When \( \mu = 0, \sigma = 1 \), \( \{X(t), t \geq 0\} \) is it called standard Brownian motion with \( E(X(t)) = 0 \) and \( V(X(t)) = t \).

Definition 3.3. A stochastic process \( X(t), t \geq 0 \) is called a Gaussian process, or a normal process if \( X(t_1), \cdots, X(t_n) \) has a multivariate normal distribution for all \( t_1, \cdots, t_n \).

Definition 3.4. A stochastic process \( \{X(t), t > 0\} \) is said to be a stationary process if for all \( n, s, t_1, \cdots, t_n \), the random vector \( X(t_1), \cdots, X(t_n) \) and \( X(t_1 + s), \cdots, X(t_n + s) \) have the same joint distribution.

Definition 3.5. A stochastic process \( \{Y_r\}_{r=1}^{\infty} \) is said to be weakly stationary if \( E(Y_r) = \mu \), independent of the time \( r \) and \( \text{cov}(Y_r, Y_{r+s}) \) does not depend of the \( r \) but depends on the time difference \( s \) only.

The weakly stationary process is a generalisation of the stationary process. A Brownian-motion is the integral of white noise Gaussian process whereas Ornstein-Uhlenbeck process is both a stationary process and a Gaussian.
Both Brownian-motion process and Ornstein Uhlenbeck process are stationary process. Since a stationary process is also a weakly stationary process, they are also weakly stationary.

**Definition 3.6.** If \( \{W_r\} \) is a standard Brownian-motion process, then \( \{Z_r\} \) defined by

\[
Z_r = e^{-\alpha r} W(e^{\alpha r})
\]

for \( \alpha > 0 \) is a zero-mean weakly stationary process with covariance function \( \text{cov}[Z_r, Z_{r+s}] = e^{-\alpha s} \). The process \( \{Z_r\} \) is known as the Ornstein-Uhlenbeck process.

### 3.2.2 Brownian-motion Process Generated Universal Portfolio (BMUP)

From (3.35),

\[
E[Y_{s_1 i_1} Y_{s_2 i_2} \cdots Y_{s_u i_u}] = E[Y_{s_1 i_1}] E[Y_{s_2 i_2}] \cdots E[Y_{s_u i_u}]
\]

if the \( u \) integers \( i_1, i_2, \cdots, i_u \) are distinct. Otherwise \( E[Y_{s_1 j_1} Y_{s_2 j_2} \cdots Y_{s_j j_u}] \) is determined by using the moment-generating function of \( Y_{s_1 j_1}, Y_{s_2 j_2}, \cdots, Y_{s_j j_u} \).

Specifically,

\[
E[(Y_{s_1 j_1} Y_{s_2 j_2} \cdots Y_{s_j j_u})(Y_{r_1 k} Y_{r_2 k} \cdots Y_{r_p k})] = E[Y_{s_1 j_1} Y_{s_2 j_2} \cdots Y_{s_j j_u}] \times E[Y_{r_1 k} Y_{r_2 k} \cdots Y_{r_p k}]
\]

for \( j \neq k \).

\[
E \left[ \prod_{i=1}^{q} (Y_{s_{i1 j_i}, Y_{s_{i2 j_i}} \cdots Y_{s_{iuj_i}}} ) \right] = \prod_{i=1}^{q} E(Y_{s_{i1 j_i}, \cdots Y_{s_{iuj_i}}} )
\]

(3.36)

for any set of distinct integers \( j_1, j_2, \cdots, j_q \).
When \( \{Y_{n1}\}_{n=1}^{\infty}, \{Y_{n2}\}_{n=1}^{\infty}, \ldots, \{Y_{nm}\}_{n=1}^{\infty} \) are independent Brownian motion with positive drift coefficients \( \mu_1, \mu_2, \ldots, \mu_m \) and variance parameters \( \sigma_1^2, \sigma_2^2, \ldots, \sigma_m^2 \) respectively. According to Ross (2007), the process \( \{Y_{nl}\} \) has stationary and independent increments, where \( Y_{nl} \) has a normal distribution with mean \( n\mu_l \) and variance \( n\sigma_l^2 \) for \( l = 1, 2, \ldots, m \) and \( n = 1, 2, \ldots \). The covariance of \( Y_{nl} \) and \( Y_{n'l} \) is given by

\[
\text{Cov}(Y_{nl}, Y_{n'l}) = n\sigma_l^2 \quad \text{for} \quad 0 < n_1 \leq n_2.
\]

(3.37)

Furthermore, the \( \nu \) random variables \( Y_{n-\nu+1,l}, Y_{n-\nu+2,l}, \ldots, Y_{n,l} \) have a joint multivariate normal distribution with mean vector \( \mu_l = (\mu_{l1}, \mu_{l2}, \ldots, \mu_{l\nu}) > 0 \) and \( \nu \times \nu \) covariance matrix

\[
K_l = \sigma_l^2 L = \sigma_l^2 (\lambda_{ij}) \quad \text{for} \quad l = 1, 2, \ldots, m
\]

(3.38)

where

\[
\mu_{ik} = (n - \nu + k)\mu_i \quad \text{for} \quad k = 1, 2, \ldots, \nu
\]

(3.39)

and

\[
\lambda_{ij} = \begin{cases} 
  n - \nu + i & \text{if } i \leq j, \\
  n - \nu + j & \text{if } i > j.
\end{cases}
\]

(3.40)

for \( i, j = 1, 2, \ldots, \nu \).

Note that the lambda matrix \( L = (\lambda_{ij}) \) in (3.38) does not depend on \( l \). The components \( \lambda_{ij} \) are the covariance components of \( Y_{n-\nu+1,l}, \ldots, Y_{n,l} \) that depend on time \( n \). The means \( \mu_{ik} \) are nonnegative for \( n \geq \nu = 1, k = 1, 2, \ldots, \nu \) and \( l = 1, 2, \ldots, m \). Similarly, \( \lambda_{ij} \geq 0 \) for all \( i, j = 1, 2, \ldots, \nu \) if \( n \geq \nu - 1 \).
In particular, when \( \nu = 1 \) and \( m = 3 \), (3.35) becomes

\[
bn_{n+1,k} = \frac{x_{n,1}E(Y_{nk}Y_{n1}) + x_{n,2}E(Y_{nk}Y_{n2}) + x_{n,3}E(Y_{nk}Y_{n3})}{\sum_{j=1}^{3}[x_{n,1}E(Y_{nj}Y_{n1}) + x_{n,2}E(Y_{nj}Y_{n2}) + x_{n,3}E(Y_{nj}Y_{n3})]} \quad (3.41)
\]

for \( k = 1, 2, 3 \)

Since the Brownian process are assumed to be independent, from (3.37) to (3.40), when \( \nu = 1 \), we have

\[
E(Y_{nk}^2) = n(\sigma_k^2 + n\mu_k^2)
\]

\[
E(Y_{nk}Y_{ni}) = E(Y_{nk})E(Y_{ni}) = n^2\mu_k\mu_i \quad \text{for} \ k \neq i \quad (3.42)
\]

\[
E(Y_{nk}Y_{nj}) = E(Y_{nk})E(Y_{nj}) = n^2\mu_k\mu_j \quad \text{for} \ k \neq j
\]

### 3.2.3 Asymptotic Behaviour of the Brownian-Motion-Generated Universal Portfolio

In this subsection, we will try to show that for sufficient large \( n \), the Brownian-motion generated universal portfolio will behave like the constant rebalance portfolio (define in Chapter 1).

Previously, it is stated that the \( \nu \) random variables \( Y_{n-\nu+1,l}, Y_{n-\nu+2,l}, \ldots, Y_{nl} \) are assumed to have a joint multivariate normal distribution with mean vector \( \mu_l = (\mu_{l1}, \ldots, \mu_{lv}) > 0 \) given by (3.39) and covariance matrix (3.37). Let \( \tau = (\tau_1, \ldots, \tau_v) \).
The moment-generating function \( M_l(\tau_1, \ldots, \tau_\nu) \) of the \( \nu \) random variables is given by
\[
M_l(\tau) = E(e^{\sum_{i=1}^\nu \tau_i Y_{n-\nu+i,l}}) = e^{\mu_l^t \tau + \frac{1}{2} \tau^t K_l \tau} \tag{3.43}
\]
for \( l = 1, 2, \ldots, m \).

**Lemma 3.1.** The moment \( E[Y_{n-\nu+k_1,l} Y_{n-\nu+k_2,l} \cdots Y_{n-\nu+k_\nu,l}] \) is a polynomial in \( n \) of degree \( u \) where the coefficients of \( n^u \) is \( \mu_l^u \) and the integers \( 1 \leq k_i \leq \nu \) for \( i = 1, \ldots, u, l = 1, \ldots, m \).

**Proof.** Note that
\[
\mu_l^t \tau + \frac{1}{2} \tau^t K_l \tau = \sum_{i=1}^\nu \mu_{li} \tau_i + \frac{1}{2} \sigma_l^2 \sum_{i=1}^\nu \lambda_{ii} \tau_i^2 + 2 \sigma_l^2 \sum_{i<j} \lambda_{ij} \tau_i \tau_j, \tag{3.44}
\]
the first and second order derivatives of \( M_l(\tau) \) are given by:
\[
\frac{\partial}{\partial \tau_{k_1}} M_l(\tau) = e^{\mu_l^t \tau + \frac{1}{2} \tau^t K_l \tau} [\mu_{lk_1} + \sigma_l^2 \sum_{j=1}^\nu \lambda_{k_1 j} \tau_j], \tag{3.45}
\]
\[
\frac{\partial^2}{\partial \tau_{k_1} \partial \tau_{k_2}} M_l(\tau) = e^{\mu_l^t \tau + \frac{1}{2} \tau^t K_l \tau} [\mu_{lk_1} + \sigma_l^2 \sum_{j=1}^\nu \lambda_{k_1 j} \tau_j] [\mu_{lk_2} + \sigma_l^2 \sum_{j=1}^\nu \lambda_{k_2 j} \tau_j] + e^{\mu_l^t \tau + \frac{1}{2} \tau^t K_l \tau} \sigma_l^2 \lambda_{k_1 k_2}, \tag{3.46}
\]

furthermore
\[
\left. \frac{\partial}{\partial \tau_{k_1}} M_l(\tau) \right|_{\tau=0} = \mu_{kl_1} = (n - \nu + k_1) \mu_l \tag{3.47}
\]
and
\[
\frac{\partial}{\partial \tau_{k_1}, \partial \tau_{k_2}} M_l(\tau)|_{\tau=0} = \mu_{k_1} \mu_{k_2} + \sigma^2 \lambda_{k_1 k_2},
\]

\[
= \begin{cases} 
\mu^2_l (n - \nu + k_1)(n - \nu + k_2) + \sigma^2 (n - \nu + k_1) & \text{if } k_i \leq k_2, \\
\mu^2_l (n - \nu + k_1)(n - \nu + k_2) + \sigma^2 (n - \nu + k_2) & \text{if } k_1 > k_2.
\end{cases}
\] (3.48)

It is observed that \( \frac{\partial}{\partial \tau_{k_1}} M_l(\tau) \bigg|_{\tau=0} \) and \( \frac{\partial}{\partial \tau_{k_1} \partial \tau_{k_2}} M_l(\tau) \bigg|_{\tau=0} \) are polynomials in \( n \) with degree 1 and 2 respectively. For any nonnegative integers \( s_2, s_2, \cdots, s_\nu \) summing up to \( u \), it follows that

\[
E[ Y^{s_1}_{n-\nu+1,l} Y^{s_2}_{n-\nu+2,l} \cdots Y^{s_\nu}_{n,l} ] = \frac{\partial^u M_l(\tau)}{\partial \tau_{s_1} \partial \tau_{s_2} \cdots \partial \tau_{s_\nu}} \bigg|_{\tau=0}
\] (3.49)

where the differential operator \( \frac{\partial^{s_i}}{\partial \tau_{s_i}^{s_i}} \) is omitted is \( s_1 = 0 \). For integers \( 1 \leq k_i \leq \nu \) possibly repeated, it is straightforward to deduce that \( \frac{\partial^u M_l(\tau)}{\partial \tau_{s_1} \partial \tau_{s_2} \cdots \partial \tau_{s_\nu}} \) is a sum of products where each product is of the form

\[
e^{\mu_l \tau + \frac{1}{2} \tau^2 \lambda_{k_1} \tau^{k_1}} [\mu_{rl_1} + \sigma^2 \sum_j \lambda_{r_1 j} \tau_j] \cdots [\mu_{rl_p} + \sigma^2 \sum_j \lambda_{r_p j} \tau_j] \times [\sigma^2 \lambda_{r_{p+1}}] \cdots [\sigma^2 \lambda_{r_q-1} r_q].
\] (3.50)

for some positive integers \( r_1, \cdots, r_p, r_{p+1}, \cdots, r_q \) such that each integer \( 1 \leq r_i \leq \nu \) and \( q \leq u \).
The product (3.50) evaluated at $\tau = 0$ is

$$
\mu_{lr_1} \cdots \mu_{lr_p} \sigma^2 \lambda_{rp+1} \cdots \sigma^2 \lambda_{rq-1} r_q
$$

$$
= \mu^p_l (n - \nu + r_1) \cdots (n - \nu + r_p) \sigma^2 (q-p) (n - \mu + \min(r_p; r_{p+1})) \cdots (n - \nu + \min(r_{q-1}; r_q))
$$

(3.51)

The right hand side is a polynomial in $n$ of degree $q \leq u$. The only product in $\partial^u M_l(\tau) \bigg|_{\tau = 0}$ containing the highest power of $n$ is $\mu^u_l (n - \nu + k_1) (n - \nu + k_2) \cdots (n - \nu + k_u)$ and the coefficient of $n^u$ is $\mu^u_l$, or any integers $p \leq s \leq u$, the coefficient of $n^s$ is of the form

$$
\mu^p_l \sigma^2 (s-p) c_1 c_2 \cdots c_{u-s}
$$

(3.52)

for some negative constants $c_1, c_2, \cdots c_{u-s}$ which do not depend on $n$ and

$$
|c_i| \in \{1, 2, \cdots, \nu - 1\} \quad \text{for} \quad i = 1, 2, \cdots, u - s.
$$

(3.53)

**Lemma 3.2.** The moment $E[\prod_{i=1}^q (Y_{s_1, j_1} \cdots Y_{s_{uk}, j_1})]$ for any set of distinct integers $j_1, j_2, \cdots, j_q, 1 \leq j_i \leq m, n - \nu + 1 \leq s_{ik} \leq n, u_1 + u_2 + \cdots + u_q = \nu + 1$, is a polynomial in $n$ of degree $\nu + 1$ where the coefficient of $n^{\nu+1}$ is $\mu^u_{j_1} \mu^u_{j_2} \cdots \mu^u_{j_q}$.

**Proof.** It follows from 3.1 that $E[\prod_{i=1}^q (Y_{s_1, j_i} Y_{s_{1j_i}, j_i})]$ is a polynomial in $n$ of degree $u_i$ and the coefficient of $n^{u_i}$ is $\mu^u_{j_i}$. From (3.35), $E[\prod_{i=1}^q (Y_{s_1, j_i} \cdots Y_{s_{uk}, j_i})]$ is a polynomial in $n$ of degree $\sum_{i=1}^q u_i = \nu + 1$ and the coefficient of $n^{\sum_{i=1}^q u_i}$ is $\prod_{i=1}^q \mu^u_{j_i}$. \qed

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Proposition 3.2.1. Consider the universal portfolio \( \{b_{n+1}\} \) given by (3.35) generated by \( m \) independent Brownian motions \( \{Y_{n1}\}_{n=1}^{\infty}, \{Y_{n2}\}_{n=1}^{\infty}, \ldots, \{Y_{nm}\}_{n=1}^{\infty} \), for a fixed integer \( \nu \geq 1 \). For a large \( n \), the \( k^{th} \) component of \( b_n \) behaves like

\[
b_{n,k} = \frac{\sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \cdots \sum_{i_{\nu}=1}^{m} (x_{n1} x_{n-1,i_2} \cdots x_{n-\nu+1,i_\nu})(\mu_k \mu_{i_1} \mu_{i_2} \cdots \mu_{i_\nu})}{\sum_{j=1}^{m} \left( \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \cdots \sum_{i_{\nu}=1}^{m} (x_{ni_1} x_{n-1,i_2} \cdots x_{n-\nu+1,i_\nu})(\mu_j \mu_{i_1} \mu_{i_2} \cdots \mu_{i_\nu}) \right)}
\]

(3.54)

for \( k = 1, 2, \ldots, m \) where \( \mu_1, \mu_2, \ldots, \mu_m \) are positive drift coefficients of the \( m \) Brownian motions.

Proof. It is clear from Lemma 3.2 that the component

\[
(x_{ni_1} x_{n-2,i_2} \cdots x_{n-\nu+1,i_\nu}) \times E[Y_{nk} Y_{n1} Y_{n-1,i_2} \cdots Y_{n-\nu+1,i_\nu}]
\]

given by (3.35) is a polynomial in \( n \) of degree \( \nu + 1 \) where

\[
(x_{ni_1} x_{n-2,i_2} \cdots x_{n-\nu+1,i_\nu})(\mu_k \mu_{i_1} \mu_{i_2} \cdots \mu_{i_\nu})
\]

is the coefficient of \( n^{\nu+1} \). Similarly, the denominator of \( b_{nk} \) in (3.35) is a polynomial in \( n \) of degree \( \nu + 1 \) where the coefficient of \( n^{\nu+1} \) is

\[
\sum_{j=1}^{m} \left( \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \cdots \sum_{i_{\nu}=1}^{m} (x_{ni_1} x_{n-1,i_2} \cdots x_{n-\nu+1,i_\nu})(\mu_j \mu_{i_1} \mu_{i_2} \cdots \mu_{i_\nu}) \right).
\]

In conclusion, \( b_{nk} \) is a ratio of polynomial in \( n \) of the same degree \( \nu + 1 \). Noting that for a fixed integer \( \nu \geq 1 \), the form quantities:

\[
\max_{1 \leq i \leq m} \left\{ x_{ni_1} x_{n-1,i_2} \cdots x_{n-\nu+1,i_\nu} \right\}, \quad \max_{1 \leq j \leq \nu+1} \left\{ \mu_{i_1} \mu_{i_2} \cdots \mu_{i_j} \right\},
\]

\[
\max_{1 \leq j \leq \nu} \left\{ \sigma_{i_1}^2, \sigma_{i_2}^2, \ldots, \sigma_{i_j}^2 \right\}
\]

and

\[
\max_{1 \leq j \leq \nu} \left\{ |c_1 c_2 \cdots c_j| \right\}
\]

for a fixed integer \( \nu \geq 1 \), the form quantities:
are bounded the asymptotic behaviour of $b_{n,k}$ is the ratio of the numerator coefficient of $n^{\nu+1}$ to the denominator coefficient of $n^{\nu+1}$. \hfill \Box

This proposition states that $b_n$ converges to some constant when $n \to \infty$

### 3.2.4 Ornstein Uhlenbeck Process Generated Universal Portfolio (OUUP)

A finite-order universal portfolio generated by a set of independence Brownian motions was studied in Section 3.2.2. In the portfolio, the past price relatives are weighted by the joint moments of the Brownian motions which depend on the Brownian motion parameters and the sampled times. In Section 3.2.3, asymptotically, over a long time, this portfolio converges to a constant rebalanced portfolio. We will investigate a generalisation of Brownian-motion, the Ornstein-Uhlenbeck process to see it has a better property.

According to Ross (2007), a stochastic process $\{Y_r\}_{r=1}^{\infty}$ is said to be weakly stationary if $E(Y_r) = \mu$, independent of the time $r$ and $\text{cov}(Y_r, Y_{r+s})$ does not depend of the $r$ but depends on the time difference $s$ only. For $m$ given weakly stationary processes $\{Y_{n_1}\}, \{Y_{n_2}\}, \ldots, \{Y_{nm}\}$, $E(\cdot)$ can be defined by rearranging the product of random variables $(Y_{nk}Y_{n_1}Y_{n-1,i_2} \cdots Y_{n-\nu+1,i_\nu})$ as $m$ products, where in each product, the random variables come from the same process. For weakly stationary process, $(Y_{nk}Y_{n_1}Y_{n-1,i_2} \cdots Y_{n-\nu+1,i_\nu})$ can be written as the following
for some ordered sequences of time indices \( r_1 \geq r_2 > \cdots > r_n; u_1 \geq u_2 > \cdots > u_n; v_1 \geq v_2 > \cdots > v_n; \cdots; w_1 \geq w_2 > \cdots > w_n \). Taking expected value,

\[
E(Y_{nk}Y_{ni_1}Y_{n-1,i_2}\cdots Y_{n-\nu+1,i_\nu}) = E(Y_{r_11}Y_{r_21}\cdots Y_{rn1})E(Y_{u_12}Y_{u_22}\cdots Y_{un2})
\times E(Y_{v_13}Y_{v_23}\cdots Y_{vn3}) \times \cdots \times E(Y_{w_1m}Y_{w_2m}\cdots Y_{wnm})
\] (3.56)

where the functionals \( E \) is defined as :

\[
E(Y_{q_1k}Y_{q_2k}\cdots Y_{q_{nq}k}) = \begin{cases} 
\prod_{i=1}^{n_q-1} E(Y_{q_i,k}Y_{q_{i+1}k}) & \text{if } n_q \text{ is even}, \\
E(Y_{q_{nq}k}) \prod_{i=1}^{n_q-1} E(Y_{q_i,k}Y_{q_{i+1}k}) & \text{if } n_q \text{ is odd}.
\end{cases}
\] (3.57)

for \( k = 1, 2, \cdots, m; q_1 \geq q_2 > \cdots > q_{nq} \).

The translated Ornstein-Uhlenbeck process \( \{Y_r\} \) is define by \( Y_r = Z_r + \mu \) for all \( r \) is said to have parameters \((\mu, \alpha)\) if \( E[Y_r] = \mu \) and \( E[Y_rY_{r+s}] = e^{-\alpha s} + \mu^2 \) for \( s > 0, \alpha > 0 \). It is also assumed that \( \mu > 0 \). Let \( \{Y_{n1}\}, \{Y_{n2}\}, \cdots \{Y_{nm}\} \) be \( m \) given independent (translated) Ornstein-Uhlenbeck process with parameters \((\mu_1, \alpha_1), (\mu_2, \alpha_2), \cdots, (\mu_m, \alpha_m)\) respectively., where all parameters are positive.

Consider the universal portfolio (3.35) generated by these process where \( E(\cdot) \) is defined by (3.56) and (3.57), namely
\[
E(Y_{q_1k}Y_{q_2k} \cdots , Y_{q_{nq}k}) = \begin{cases} 
\prod_{i=1}^{nq-1} \left[ e^{-\frac{\alpha_k(q_i-q_{i+1})}{2}} + \mu_k^2 \right] & \text{if } n_q \text{ is even}, \\
\mu_k \prod_{i=1}^{nq-2} \left[ e^{-\frac{\alpha_k(q_i-q_{i+1})}{2}} + \mu_k^2 \right] & \text{if } n_q \text{ is odd}. 
\end{cases}
\]

(3.58)

for \( k = 1, 2, \cdots, m \) : \( q_1 \geq q_2 > \cdots > q_{nq} \).

In particular, when \( \nu = 1, m = 3 \), (3.34) becomes

\[
b_{n+1,k} = \frac{\sum_{i=1}^{3} x_{n,i} E(Y_{nk}Y_{ni})}{\sum_{j=1}^{3} (\sum_{i=1}^{3} x_{n,i} E(Y_{nj}Y_{ni}))}
\]

(3.59)

and the (3.58) becomes

\[
E(Y_{nk}Y_{nk}) = E(Y_{nk}^2) = 1 + \mu_k^2
\]

\[
E(Y_{nk}Y_{ni}) = E(Y_{nk})E(Y_{ni}) = \mu_k \mu_i \quad \text{for } i \neq k
\]

(3.60)

\[
E(Y_{nk}Y_{nj}) = E(Y_{nk})E(Y_{nj}) = \mu_k \mu_j \quad \text{for } j \neq k
\]

for \( k = 1, 2, 3 \).
There are many programming languages for implementing the universal portfolio strategies. In the previous chapter, Python is a suitable choice because it is easy to use, powerful, versatile and being relatively popular in finance (Tuttle & Butcher 2016).

4.1 Dirichlet Universal Portfolio Strategy

We will first investigate Dirichlet universal portfolio strategy since it was proposed by Cover & Ordentlich (1996). Cover & Ordentlich (1996) presented an algorithm for generating the Dirichlet\((\frac{1}{2}, \frac{1}{2})\) two stocks universal portfolio. Chan (2002) modified this algorithm for generating any Dirichlet\((\alpha_1, \alpha_2, \cdots, \alpha_m)\) universal portfolio for \(m = 2, 3\) and 4 stocks. The modified algorithm also obtained by Tan (2004b). We choose this modified algorithm for computing the 3 stocks universal portfolio generated by Dirichlet\((\alpha_1, \alpha_2, \alpha_3)\) distribution where \(\alpha_j > 0\) for \(j = 1, 2, 3\). We present the modified algorithm of Chan (2002) for computing the three-stock universal portfolio generated by Dirichlet \((\alpha_1, \alpha_2, \alpha_3)\) distribution where \(\alpha_j > 0\) for \(j = 1, 2, 3\). Consider a sequence of price-relative vectors \(x_1, x_2, \cdots, x_n\) corresponding to \(n\) trading days in a three-stock market, where \(x_i = (x_{i,1}, x_{i,2}, x_{i,3})\) for \(i = 1, 2, \cdots, n\). We shall define three recursive functions \(X_n(l_1, l_2), C_n(l_1, l_2)\) and \(Q_n(l_1, l_2)\).
Firstly, we define

\[ X_n(l_1, l_2) = \sum_{J^n \in T_n(l_1, l_2)} \prod_{i=1}^{n} x_{ij} \] (4.1)

where \( T_n(l_1, l_2) \) is the set of all sequence \( j^n = (j_1, j_2, \cdots, j_n) \in \{1, 2, 3\}^n \) with \( l_1 \) 1’s and \( l_2 \) 2’s and \( n-l_1-l_2 \) 3’s. Secondly, we defined,

\[ C_n(l_1, l_2) = \int_{\mathcal{B}} b_1^{l_1} b_2^{l_2} b_3^{n-l_1-l_2} d\mu(b) \] (4.2)

where

\[ d\mu(b) = \frac{\Gamma(\alpha_1 + \alpha_2 + \alpha_3) b_1^{\alpha_1-1} b_2^{\alpha_2-1} b_3^{\alpha_3-1}}{\Gamma(\alpha_1) \Gamma(\alpha_2) \Gamma(\alpha_3)} db_1 db_2 \] (4.3)

Finally,

\[ Q_n(l_1, l_2) = X_n(l_1, l_2) C_n(l_1, l_2) \] (4.4)

The universal portfolio \( \{b_n\} \) can then be computed as follows:

\[ \hat{b}_n = \frac{1}{\sum_{l_1=1}^{n-1} \sum_{l_2=0}^{n-l_1-1} Q_{n-1}(l_1, l_2)} \left( \begin{array}{c} 
\sum_{l_1=0}^{n-1} \sum_{l_2=0}^{n-l_1-1} \frac{(1 + \alpha_1)}{(n + \alpha_1 + \alpha_2 + \alpha_3 - 1)} Q_{n-1}(l_1, l_2) \\
\sum_{l_1=0}^{n-1} \sum_{l_2=0}^{n-l_1-1} \frac{(l_2 + \alpha_2)}{(n + \alpha_1 + \alpha_2 + \alpha_3 - 1)} Q_{n-1}(l_1, l_2) \\
\sum_{l_1=0}^{n-1} \sum_{l_2=0}^{n-l_1-1} \frac{(n - l_1 - l_2 + \alpha_3 - 1)}{(n + \alpha_1 + \alpha_2 + \alpha_3 - 1)} Q_{n-1}(l_1, l_2) 
\end{array} \right) \] (4.5)

and the universal capital achieved by \( \hat{b}_n \) is computed as:

\[ \hat{S}_n = \sum_{l=0}^{n} Q_l(l_1, l_2) \] (4.6)
Next, we give the recursive relationships of the functions $X_n(l_1, l_2)$, $C_n(l_1, l_2)$ and $Q_n(l_1, l_2)$ and their end-point conditions.

The recursion for $X_n(l_1, l_2)$ is given by:

$$X_n(l_1, l_2) = x_{n1}X_{n-1}(l_1 - 1, l_2) + x_{n2}X_{n-1}(l_1, l_2 - 1) + x_{n3}X_{n-1}(l_1, l_2) \quad (4.7)$$

for $1 \leq l_1 \leq n - 1$ and $1 \leq l_2 \leq n - l_1 - 1$ and the endpoint condition are given by:

$$X_n(l_1, 0) = x_{n1}X_{n-1}(l_1, 0) + x_{n3}X_{n-1}(l_1, 0) \quad (4.8)$$

$$X_n(0, l_2) = x_{n2}X_{n-1}(0, l_2 - 1) + x_{n3}X_{n-1}(0, l_2) \quad (4.9)$$

The recursion for $C_n(l_1, l_2)$ given by:

$$C_n(l_1, l_2) = \frac{l_1 + \alpha_1 - 1}{n + \alpha_1 + \alpha_2 + \alpha_3 - 1}C_{n-1}(l_1 - 1, l_2) \quad (4.10)$$

for $1 \leq l_1 \leq n$ and $0 \leq l_2 \leq n - l_1$.

$$C_n(l_1, l_2) = \frac{l_2 + \alpha_2 - 1}{n + \alpha_1 + \alpha_2 + \alpha_3 - 1}C_{n-1}(l_1, l_2 - 1) \quad (4.11)$$

for $0 \leq l_1 \leq n - 1$ and $1 \leq l_2 \leq n - l_1$.

$$C_n(l_1, l_2) = \frac{n - l_1 - l_2 + \alpha_3 - 1}{n + \alpha_1 + \alpha_2 + \alpha_3 - 1}C_{n-1}(l_1, l_2) \quad (4.12)$$

for $0 \leq l_1 \leq n - 1$ and $0 \leq l_2 \leq n - l_1 - 1$.

The initial condition is

$$C_0(0, 0) = 1. \quad (4.13)$$

The recursive for $Q_n(l_1, l_2)$ given by:
\[ Q_n(l_1, l_2) = x_{n1} \frac{l_1 + \alpha_1 - 1}{n + \alpha_1 + \alpha_2 + \alpha_3 - 1} Q_{n-1}(l_1 - 1, l_2) + x_{n2} \frac{l_2 + \alpha_2 - 1}{n + \alpha_1 + \alpha_2 + \alpha_3 - 1} Q_{n-1}(l_1, l_2 - 1) + x_{n3} \frac{n - l_1 - l_2 + \alpha_3 - 1}{n + \alpha_1 + \alpha_2 + \alpha_3 - 1} Q_{n-1}(l_1, l_2) \] (4.14)

for \( 1 \leq l_1 \leq n - 1 \) and \( 1 \leq l_2 \leq n - l_1 \). The six endpoint conditions are given by

\[ Q_n(l_1, 0) = x_{n1} \frac{l_1 + \alpha_1 - 1}{n + \alpha_1 + \alpha_2 + \alpha_3 - 1} Q_{n-1}(l_1 - 1, 0) + x_{n2} \frac{l_2 + \alpha_2 - 1}{n + \alpha_1 + \alpha_2 + \alpha_3 - 1} Q_{n-1}(l_1, 0) \] (4.15)

for \( 1 \leq l_1 \leq n - 1 \).

\[ Q_n(0, l_2) = x_{n2} \frac{l_2 + \alpha_2 - 1}{n + \alpha_1 + \alpha_2 + \alpha_3 - 1} Q_{n-1}(0, l_2 - 1) + x_{n3} \frac{n - l_1 - l_2 + \alpha_3 - 1}{n + \alpha_1 + \alpha_2 + \alpha_3 - 1} Q_{n-1}(0, l_2) \] (4.16)

for \( 1 \leq l_2 \leq n - l_1 - 1 \).

\[ Q_n(l_1, l_2) = x_{n1} \frac{l_1 + \alpha_1 - 1}{n + \alpha_1 + \alpha_2 + \alpha_3 - 1} Q_{n-1}(l_1 - 1, l_2) + x_{n2} \frac{l_2 + \alpha_2 - 1}{n + \alpha_1 + \alpha_2 + \alpha_3 - 1} Q_{n-1}(l_1, l_2 - 1) \] (4.17)

for \( 1 \leq l_1, 1 \leq l_2 \) and \( l_1 + l_2 = n \).

\[ Q_n(n, 0) = x_{n1} \frac{n + \alpha_1 - 1}{n + \alpha_1 + \alpha_2 + \alpha_3 - 1} \times Q_{n-1}(n - 1, 0) \]

\[ Q_n(0, n) = x_{n2} \frac{n + \alpha_2 - 1}{n + \alpha_1 + \alpha_2 + \alpha_3 - 1} \times Q_{n-1}(0, n - 1) \] (4.18)

\[ Q_n(0, 0) = x_{n3} \frac{n + \alpha_3 - 1}{n + \alpha_1 + \alpha_2 + \alpha_3 - 1} \times Q_{n-1}(0, 0) \]

The initial condition is

\[ Q_0(0, 0) = 1. \] (4.19)
The computation of the universal portfolio $\hat{b}_n$ and the universal capital $\hat{S}_n$ through (4.8) and (4.9) respectively can be done solely by computing the quantities $Q_n(l_1, l_2)$ recursively through (4.14) to (4.18) or by computing both $X_n(l_1, l_2)$ and $C_n(l_1, l_2)$ recursively through (4.8) to (4.14). The detailed derivations of the formulae are given in Chan (2002).

This above modified algorithm is written in python using numpy module as follow.

```python
from numpy import array, zeros, arange
from time import clock

def Recursive3CCT(x,a1,a2,a3,debug=False,echo=False):
    _N = len(x[0])
    _b = zeros((3,_N+1))
    _S = zeros(_N+1)
    _S[0]=1.
    _x = x

    Qold = array([[1.]])
    Q = zeros((_N+1,_N+1))

    x1 = _x[0]
    x2 = _x[1]
    x3 = _x[2]

    begin = clock()
    for n in range(1,_N+1):
        denom = float(n+a1+a2+a3-1)
        start = clock()

        Q[0,0] = x3[n-1]*(n+a3-1)/denom*Qold[0,0]
        Q[0,n] = x2[n-1]*(n+a2-1)/denom*Qold[0,n-1]
        Q[n,0] = x1[n-1]*(n+a1-1)/denom*Qold[n-1,0]

        for l in range(1,n):
            Q[l,0] = x1[n-1] *(l+a1-1)/denom*Qold[l-1,0] \
```

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In the above code, from line number 5 to 12 is allocation of data structure to store data. Line number 20 is denominator of (4.10). While line number 24 to 41 is (4.14) to (4.18). The portfolio of (4.5) is from line number 50 to 53. The wealth is calculated by (4.6) and is line number 54.

Next, the four universal portfolios are written in python using the numpy mod-
ule. We start with order one and order two Multinomial universal portfolio. Refer order one in (3.14) and follow by order two, (3.18) in Chapter 3. We choose 3 stocks for our study where $m = 3$.

### 4.2 Multinomial Generated Universal Portfolio

#### 4.2.1 Order One Multinomial Generated Universal Portfolio

```python
from numpy import array, zeros, arange
from time import clock

def Recursive3Multinomial(x, p1, p2, NN, debug=False, echo=False):
    _N = len(x[0])
    _b = zeros((3, _N + 1))
    _S = zeros(_N + 1)
    _S[0] = 1.
    _x = x
    _p1, _p2, _p3, _NN = None, None, None, None
    order = 1
    _p1 = p1
    _p2 = p2
    _p3 = max(1. - _p1 - _p2, 0.)
    _NN = NN
    EY1 = _NN * _p1
    EY2 = _NN * _p2
    EY3 = _NN * _p3
    EY12 = _NN * _p1 * (1. - _p1) + (_NN * _p1) ** 2
    EY22 = _NN * _p2 * (1. - _p2) + (_NN * _p2) ** 2
    EY32 = _NN * _p3 * (1. - _p3) + (_NN * _p3) ** 2
    EY1Y2 = _NN * (_NN - 1.) * _p1 * _p2
    EY1Y3 = _NN * (_NN - 1.) * _p1 * _p3
    EY2Y3 = _NN * (_NN - 1.) * _p2 * _p3
```

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x1 = list(_x[0])
x2 = list(_x[1])
x3 = list(_x[2])
begin = clock()

_b[:,0] = array([1., 1., 1.])/3
_S[1] = _b[0,0]*x1[0] + _b[1,0]*x2[0] + _b[2,0]*x3[0]

for n in range(order,_N):
    if echo: start = clock()
    numer = x1[n]*x1[n-1]*EY12 + x2[n]*x2[n-1]*EY22 + x3[n]*x3[n-1]*EY32
    numer += (x1[n]*x2[n-1] + x2[n]*x1[n-1])*EY1Y2
    numer += (x1[n]*x3[n-1] + x3[n]*x1[n-1])*EY1Y3
    numer += (x2[n]*x3[n-1] + x3[n]*x2[n-1])*EY2Y3
    denom = _NN*(x1[n-1]*EY1 + x2[n-1]*EY2 + x3[n-1]*EY3)
    dotbx = numer/denom
    _S[n+1] = dotbx * _S[n]
    b1 = x1[n-1]*EY12 + x2[n-1]*EY1Y2 + x3[n-1]*EY1Y3
    b2 = x1[n-1]*EY1Y2 + x2[n-1]*EY22 + x3[n-1]*EY2Y3
    b3 = x1[n-1]*EY1Y3 + x2[n-1]*EY2Y3 + x3[n-1]*EY32
    _b[:,n] = array([b1, b2, b3])/denom
    if echo: print("Finish recursive step %3d in %.2f second.")
    if debug: print("Total time spent: %g second.")

return _S[1:]

From line number 11 to line number 16 are the parameters of multinomial
universal portfolio where \( p_1 = p_1, p_2 = p_2 \) and \( NN = N \). Line number 17
to line number 25 are the moments of the multinomial universal portfolio (3.27)
in Chapter 3. Line number 17 is when \( \nu = 1 \), refer to (3.27) in Chapter 3. Line
number 38 to number 42 represented the numerator and denominator of (3.15) and
(3.16) in Chapter 3. Line number 44 to line number 45 are the wealth obtained
after \( n + 1 \) trading days, refer to (3.17) in Chapter 3.
4.2.2 Order Two Multinomial Generated Universal Portfolio

```python
from numpy import array, zeros, ones, arange
from time import clock

def Recursive3Multinomial(x, p1, p2, NN, debug=False, echo=False):
    _order = 2
    _N = len(x[0])
    _b = zeros((3, _N+1))
    _S = ones(_N+1)
    _x = x
    _p1, _p2, _p3, _NN = None, None, None, None
    _p1 = p1
    _p2 = p2
    _p3 = max(1. - _p1 - _p2, 0.)
    if any([p1<0, p2<0, _p3<0]):
        import sys
        print("Error: Parameter is less than 0!")
        sys.exit(1)
    _p12 = p1**2
    _p22 = p2**2
    _p32 = _p3**2
    _p13 = p1**3
    _p23 = p2**3
    _p33 = _p3**3
    _NN = NN
    EY1 = _NN*_p1
    EY2 = _NN*_p2
    EY3 = _NN*_p3
    EY12 = _NN*_p1*(1.-_p1) + (_NN*_p1)**2
    EY22 = _NN*_p2*(1.-_p2) + (_NN*_p2)**2
    EY32 = _NN*_p3*(1.-_p3) + (_NN*_p3)**2
    EY13 = _NN*((_NN-1))*(_NN-2)*_p13 + 3*_NN*((_NN-1))*_p12 + _NN*_p1
    EY23 = _NN*((_NN-1))*(_NN-2)*_p23 + 3*_NN*((_NN-1))*_p22 + _NN*_p2
    EY33 = _NN*((_NN-1))*(_NN-2)*_p33 + 3*_NN*((_NN-1))*_p32 + _NN*_p3
    EY1Y2 = _NN*((_NN-1))*_p1*_p2
    EY1Y3 = _NN*((_NN-1))*_p1*_p3
    EY2Y3 = _NN*((_NN-1))*_p2*_p3
    EY12Y2 = _NN*((_NN-1))*(_NN-2)*_p12*_p2 + _NN*((_NN-1))*_p1*_p2
    EY22Y1 = _NN*((_NN-1))*(_NN-2)*_p22*_p1 + _NN*((_NN-1))*_p1*_p2
```

\[ E_{Y12Y3} = _{NN}*(_{NN}-1)*(_{NN}-2)*_{p12}*_{p3} + _{NN}*(_{NN}-1)*_{p1}*_{p3} \]
\[ E_{Y32Y1} = _{NN}*(_{NN}-1)*(_{NN}-2)*_{p32}*_{p1} + _{NN}*(_{NN}-1)*_{p1}*_{p3} \]
\[ E_{Y22Y3} = _{NN}*(_{NN}-1)*(_{NN}-2)*_{p22}*_{p3} + _{NN}*(_{NN}-1)*_{p2}*_{p3} \]
\[ E_{Y32Y2} = _{NN}*(_{NN}-1)*(_{NN}-2)*_{p32}*_{p2} + _{NN}*(_{NN}-1)*_{p2}*_{p3} \]
\[ E_{Y1Y2Y3} = _{NN}*(_{NN}-1)*(_{NN}-2)*_{p1}*_{p2}*_{p3} \]

\[ x1 = _{x}[0] \]
\[ x2 = _{x}[1] \]
\[ x3 = _{x}[2] \]

begin = clock()

\[ _{b}[i, 0] = array([1., 1., 1.])/3 \]
\[ _{S}[1] = _{b}[0,0]*x1[0] + _{b}[1,0]*x2[0] + _{b}[2,0]*x3[0] \]
\[ _{b}[i, 1] = array([1., 1., 1.])/3 \]
\[ _{S}[2] = _{S}[1]*(_{b}[0,1]*x1[1] + _{b}[1,1]*x2[1] + _{b}[2,1]*x3[1]) \]

for n in range(_order, _N):
    if echo: start = clock()

    numer = x1[n]*x1[n-1]*x1[n-2]*EY13 + x2[n]*x2[n-1]*x2[n-2]*EY23 
    += x3[n]*x3[n-1]*x3[n-2]*EY33 
    numer += x1[n]*x1[n-1]*x2[n-1]*x2[n-2]*EY12Y2 
    += x2[n]*x2[n-1]*x1[n-1]*x1[n-2]*EY12Y1 
    numer += x1[n]*x2[n-1]*x2[n-2] + x2[n]*x2[n-1]*x1[n-2] 
    += x2[n]*x1[n-1]*x2[n-2]*EY22Y1 
    numer += x1[n]*x3[n-1]*x1[n-2] + x1[n]*x1[n-1]*x3[n-2] 
    += x3[n]*x1[n-1]*x1[n-2]*EY1Y2Y3 
    numer += x1[n]*x3[n-1]*x3[n-2] + x3[n]*x3[n-1]*x1[n-2] 
    += x3[n]*x1[n-1]*x3[n-2]*EY2Y3Y1 
    numer += x2[n]*x2[n-1]*x3[n-2] + x2[n]*x3[n-1]*x2[n-2] 
    += x2[n]*x2[n-1]*x2[n-2]*EY2Y2Y3 
    numer += x3[n]*x2[n-1]*x3[n-2] + x3[n]*x3[n-1]*x2[n-2] 
    += x2[n]*x3[n-1]*x3[n-2]*EY3Y2Y1 
    numer += x1[n]*x3[n-1]*x2[n-2] + x1[n]*x2[n-1]*x3[n-2] 
    += x2[n]*x3[n-1]*x1[n-2] 

    if x != None:
        if echo: start = clock()

    dotbx = numer/denom

    _S[n+1] = dotbx * _S[n]

    if echo: time = clock() - start
    if echo: print(time)
\[ b_1 = x_1[n-1] \cdot x_1[n-2] \cdot EY_{13} + x_1[n-1] \cdot x_2[n-2] \cdot EY_{12}Y_2 + x_1[n-1] \cdot x_3[n-2] \cdot EY_{12}Y_3 \]
\[ += x_2[n-1] \cdot x_1[n-2] \cdot EY_{12}Y_2 + x_2[n-1] \cdot x_2[n-2] \cdot EY_{22}Y_1 \]
\[ b_1 += x_2[n-1] \cdot x_3[n-2] \cdot EY_{12}Y_3 + x_3[n-1] \cdot x_1[n-2] \cdot EY_{22}Y_1 \]
\[ += x_2[n-1] \cdot x_2[n-2] \cdot EY_{23} \]
\[ b_2 += x_2[n-1] \cdot x_3[n-2] \cdot EY_{22}Y_3 + x_3[n-1] \cdot x_1[n-2] \cdot EY_{12}Y_3 \]
\[ += x_3[n-1] \cdot x_2[n-2] \cdot EY_{22}Y_3 + x_3[n-1] \cdot x_3[n-2] \cdot EY_{32}Y_1 \]
\[ b_2 += x_2[n-1] \cdot x_3[n-2] \cdot EY_{22}Y_3 + x_3[n-1] \cdot x_1[n-2] \cdot EY_{32}Y_1 \]
\[ b_2 += x_2[n-1] \cdot x_3[n-2] \cdot EY_{32}Y_2 + x_3[n-1] \cdot x_3[n-2] \cdot EY_{33} \]

\[ _b[:,n] = array([b_1, b_2, b_3])/denom \]

if echo: print("Finish recursive step %3d in %.2f second." % (n, clock()-start))
print("Total time spent: %g second." % (clock()-begin))
return _S[1:]

Similary, in order two multinomial universal portfolio code, from line number 11 to line number 26 are the parameters of multinomial universal portfolio where \( p_1 = p_1, p_2 = p_2 \) and \( NN = N \). Line number 27 to line number 47 are the moments of order two multinomial generated universal portfolio, refer to (3.28) in Chapter 3. Line number 62 to number 81 represented the numerator and denominator, refer to (3.19) and (3.20) in Chapter 3. While line number 84 is the wealth obtained after \( n + 1 \) trading days, (3.21) in Chapter 3.

From the above two python codes have shown that the calculation of numerator and denominator of portfolios become complicated when order increase. These are due to the moments are increasing when order \( \nu \) increase. Therefore, only the order one of the other three proposed universal portfolios will be shown.
and discussed in the following section.

### 4.3 Order One Multivariate Normal Generated Universal Portfolio

```python
from numpy import array, zeros, arange
from time import clock

def Recursive3MultivariateNormal(x, p1, p2, p3, NN, debug=False, echo=False):
    _N = len(x[0])
    _b = zeros((3, _N+1))
    _S = zeros(_N+1)
    _S[0] = 1.
    _x = x
    _p1, _p2, _p3, _NN = None, None, None, None
    order = 1
    _p1 = p1
    _p2 = p2
    _p3 = p3
    _p12 = p1**2
    _p22 = p2**2
    _p32 = p3**2
    _NN = NN
    EY1 = _p1
    EY2 = _p2
    EY3 = _p3
    EY12 = _p12 + _NN
    EY22 = _p22 + _NN
    EY32 = _p32 + _NN
    EY1Y2 = _p1 * _p2
    EY1Y3 = _p1 * _p3
    EY2Y3 = _p2 * _p3
    x1 = list(_x[0])
    x2 = list(_x[1])
    x3 = list(_x[2])
    begin = clock()
```
From the above code, line number 10 to line number 18 are the parameters of multivariate normal universal portfolio where $\mu_1 = p_1, \mu_2 = p_2, \mu_3 = p_3$ and $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma = N N$. Line number 20 to line number 28 are the moments of the Multivariate normal generated universal portfolio (3.33) in Chapter 3. Line number 42 to number 49 represented the numerator and denominator where the numerator is (3.14) in Chapter 3. While the wealth obtained is in line number 52, refer (3.17) in Chapter 3.
4.4 Order One Brownian-motion Generated Universal Portfolio

```python
from numpy import array, zeros, arange
from time import clock

def Recursive3brownianmotion (x,p1,p2,p3,NN,MM,QQ,debug=False,echo=False):
    _N = len(x[0])
    _b = zeros((3,_N+1))
    _S = zeros(_N+1)
    _S[0] = 1.
    _x = x

    order = 1
    _p1 = p1
    _p2 = p2
    _p3 = p3
    _p12 = p1**2
    _p22 = p2**2
    _p32 = p3**2
    _NN = NN
    _MM = MM
    _QQ = QQ
    x1 = list(_x[0])
    x2 = list(_x[1])
    x3 = list(_x[2])

    begin = clock()
    _b[:,0] = array([1., 1., 1.])/3
    _S[1] = _b[0,0]*x1[0] + _b[1,0]*x2[0] + _b[2,0]*x3[0]

    for n in range(order,_N):
        if echo: start = clock()
        EY1 = n* _p1
        EY2 = n* _p2
        EY3 = n* _p3
        EY12 = n*n* _p12 + n* _NN* _NN
        EY22 = n*n* _p22 + n* _MM* _MM
        EY32 = n*n* _p32 + n* _QQ* _QQ
        EY1Y2 = n*n* _p1* _p2
        EY1Y3 = n*n* _p1* _p3
        EY2Y3 = n*n* _p2* _p3
```

From the above code, line number 12 to line number 20 are the parameters of Brownian motion generated universal portfolio where $\mu_1 = p_1$, $\mu_2 = p_2$, $\mu_3 = p_3$ and $\sigma_1^2 = NN, \sigma_2^2 = MM, \sigma_3^2 = QQ$. Line number 32 to line number 40 are the moments of the universal portfolio generated by Brownian-motion, refer (3.42) in Chapter 3. Line number 42 to number 49 represented the numerator and denominator of the portfolio (3.41) in Chapter 3. While Line number 52 is the calculation of the wealth obtained after $n + 1$ trading days, (3.17) in Chapter 3.
4.5  Order One Universal Portfolio Generated by Ornstein Uhlenbeck Process

```python
from numpy import array, zeros, arange
from time import clock

def Recursive3ornstein(x,p1,p2,p3,debug=False,echo=False):

    _N = len(x[0])
    _b = zeros((3,_N+1))
    _S = zeros(_N+1)
    _S[0] = 1.
    _x = x
    _p1, _p2, _p3, _NN, _MM, _QQ = None, None, None, None, None, None

Order = 1
    _p1 = p1
    _p2 = p2
    _p3 = p3
    _NN = NN
    _MM = MM
    _QQ = QQ
    _p12 = p1**2
    _p22 = p2**2
    _p32 = p3**2
    _p13 = p1**3
    _p23 = p2**3
    _p33 = p3**3

    EY1 = _p1
    EY2 = _p2
    EY3 = _p3
    EY12 = 1 + _p12
    EY22 = 1 + _p22
    EY32 = 1 + _p32
    EY1Y2 = _p1*_p2
    EY1Y3 = _p1*_p3
    EY2Y3 = _p2*_p3

    x1 = list(_x[0])
    x2 = list(_x[1])
    x3 = list(_x[2])
```
begin = clock()

_b[:,0] = array([1., 1., 1.])/3
_S[1] = _b[0,0]*x1[0] + _b[1,0]*x2[0] + _b[2,0]*x3[0]

for n in range(order,_N):
    if echo: start = clock()
    numer = x1[n]*x1[n-1]*EY12 + x2[n]*x2[n-1]*EY22 + x3[n]*x3[n-1]*EY32
    numer += (x1[n]*x2[n-1] + x2[n]*x1[n-1])*EY1Y2
    numer += (x1[n]*x3[n-1] + x3[n]*x1[n-1])*EY1Y3
    numer += (x2[n]*x3[n-1] + x3[n]*x2[n-1])*EY2Y3
    denom = x1[n-1]*EY12 + x2[n-1]*EY22 + x3[n-1]*EY32
    denom += (x2[n-1] + x1[n-1])*EY1Y2
    denom += (x3[n-1] + x1[n-1])*EY1Y3
    denom += (x3[n-1] + x2[n-1])*EY2Y3
    dotbx = numer/denom
    _S[n+1] = dotbx + _S[n]
    b1 = x1[n-1]*EY12 + x2[n-1]*EY1Y2 + x3[n-1]*EY1Y3
    b2 = x1[n-1]*EY1Y2 + x2[n-1]*EY22 + x3[n-1]*EY2Y3
    b3 = x1[n-1]*EY1Y3 + x2[n-1]*EY2Y3 + x3[n-1]*EY32
    _b[:,n] = array([b1, b2, b3])/denom
    if echo: print("Finish recursive step %3d in %.2f second."
    if debug: print("Total time spent: %g second."
    return _S[1:]

From the above code, from line number 13 to line number 24 are the parameters of universal portfolio generated by Ornstein Uhlenbeck process where \( \mu_1 = p_1, \mu_2 = p_2, \mu_3 = p_3 \) and \( \alpha_1 = NN, \alpha_2 = MM, \alpha_3 = QQ \). Line number 26 to line number 34 are the moments of the universal portfolio generated by Ornstein uhlenbeck process, refer (3.60) in Chapter 3. Line number 48 to number 55 represented the numerator and denominator of (3.59) in Chapter 3. While, Line number 58 is the wealth obtained after \( n + 1 \) trading days, (3.17) in Chapter 3.
In the preliminary stage of this research, a numerical experiment was done on the Dirichlet distribution based universal portfolio proposed in Cover (1991) for $m = 3$, i.e. three stocks. The Tan and Chan algorithm (Chan 2002) was found hardly worked for more than 500 trading days, i.e. $n \geq 500$. To resolve this matter, Tan et al. (2012) proposed finite order universal portfolio strategies. Much of Tan’s universal portfolio strategies are applied to a few selected stocks (refer to the literature review in Chapter 1). While we will be investigating how various strategies work on all combination of stocks.

5.1 Performance of Five Universal Portfolio Strategies for Special Sets of Stocks

In all these numerical experiments, an initial wealth of 1 is assumed. All calculations with the proposed strategies were obtained by using Python 2.7 and NumPy on a MacBook Pro with MacOS/X Version 10.8.3.
5.1.1 Performance of Dirichlet-Distribution-Based Universal Portfolio

Universal portfolio strategies can only perform well with the right parameters. In this numerical experiment, 3 sets of stocks data were selected randomly from KLSE (Kuala Lumpur Stock Exchange) and Yahoo Finance (Yahoo n.d.) for the period from 1 January, 2003 to 30 November, 2004, consisting of 500 trading days. Each set consists of 3 company stocks. Set A consists of the stocks of Malayan Banking, Genting and Amway(M) Holdings. Set B consists of the stocks of Public Bank, Sunrise and YTL Corporation. Lastly, set C consists of the stocks of Hong Leong Bank, RHB Capital and YTL Corporation. Refer to Appendix A.

Cover & Ordentlich (1996) presented an algorithm for generating the Dirichlet \((\frac{1}{2}, \frac{1}{2})\) two stock universal portfolio. Chan (2002) modified this algorithm for generating any Dirichlet \((\alpha_1, \alpha_2, \cdots, \alpha_m)\) universal portfolio for \(m = 2, 3\) and 4 stock. A general algorithm capable of handling the m-stock Dirichlet \((\alpha_1, \alpha_2, \cdots, \alpha_m)\) universal portfolio was proposed by Tan (2004b). We used the modified algorithm of Chan (2002) for computing the three-stock universal portfolio generated by Diriclet \((\alpha_1, \alpha_2, \alpha_3)\) distribution where \(\alpha_j > 0\) for \(j = 1, 2, 3\), with the starting capital \(S_0\) to be 1 unit and the initial portfolio \(b_0 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\). We can compute the investment capitals achieved by Dirichlet \((\alpha_1, \alpha_2, \alpha_3)\) Cover-Ordentlich universal portfolios for set A, set B and set C. Table 5.1 show the wealth obtained by the Dirichlet \((\alpha_1, \alpha_2, \alpha_3)\) where parameter are chosen as \(\alpha_1 = 0.01, \alpha_2 = 0.01\) and \(\alpha_3 = 8000\). From Table 5.1, the wealth obtained \(S_{500}\) is 1.8750 for data set A, 4.0761 for data set B and 4.0761 for data set C with the computation time in second.
Table 5.1: The wealth $S_{500}$ achieved by the Dirichlet (0.01,0.01,8000) UP for data set A, B and C.

<table>
<thead>
<tr>
<th></th>
<th>$S_{500}$</th>
<th>Computation Time (Seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set A</td>
<td>1.8750</td>
<td>394.77</td>
</tr>
<tr>
<td>Set B</td>
<td>4.0761</td>
<td>394.98</td>
</tr>
<tr>
<td>Set C</td>
<td>4.0761</td>
<td>394.38</td>
</tr>
</tbody>
</table>

5.1.2 Empirical Performance of Multinomial Distribution Generated Universal Portfolio

The computation of the Dirichlet universal portfolios requires a substantial amount of computer and longer computation time. To overcome this problem, the memory-saving Dirichlet universal portfolios of finite order is introduced in Tan et al. (2012). The finite-order multinomial universal portfolios having the faster computation time and reduced the computer-memory saving is studied.

A numerical experiment was carried out for the finite-order Multinomial universal portfolios on same three stock-price data sets A, B and C selected from the Kuala Lumpur Stock Exchange. The wealth $S_{500}$ achieved by selected parameters of the multinomial universal portfolio $(p_1, p_2, N)$ for algorithm given in Chapter 3 are shown in Table 5.2, Table 5.3 and Table 5.4 respectively. The computation times of the finite-order Multinomial universal portfolios are 0.014, 0.022 and 0.052 seconds for the first, second and third order respectively. From Table 5.1, the computation times of the Dirichlet universal portfolios are 394.77, 394.98 and 394.38 seconds with the maximum wealth of 1.8750, 4.0761 and 4.0761 for data set A, B and C. Thus, the wealth achieved by the finite-order multinomial universal portfolios are comparable to that of the moving-order Dirichlet universal portfolio with substantial savings in computation time. Note that, the order one Multinomial universal portfolio has better performance.
Table 5.2: The wealth $S_{500}$ achieved by the MNUP (0.8,0.01,600) for data Set A.

<table>
<thead>
<tr>
<th>Order</th>
<th>$S_{500}$</th>
<th>Computation Time (Seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.0238</td>
<td>0.014</td>
</tr>
<tr>
<td>2</td>
<td>2.0362</td>
<td>0.022</td>
</tr>
<tr>
<td>3</td>
<td>2.0315</td>
<td>0.052</td>
</tr>
</tbody>
</table>

Table 5.3: The wealth $S_{500}$ achieved by the MNUP (0.000001,0.000001,600) for data Set B.

<table>
<thead>
<tr>
<th>Order</th>
<th>$S_{500}$</th>
<th>Computation Time (Seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.0761</td>
<td>0.014</td>
</tr>
<tr>
<td>2</td>
<td>4.0354</td>
<td>0.022</td>
</tr>
<tr>
<td>3</td>
<td>4.0217</td>
<td>0.052</td>
</tr>
</tbody>
</table>

Table 5.4: The wealth $S_{500}$ achieved by the MNUP (0.000001,0.000001,600) for data Set C.

<table>
<thead>
<tr>
<th>Order</th>
<th>$S_{500}$</th>
<th>Computation Time (Seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.0761</td>
<td>0.014</td>
</tr>
<tr>
<td>2</td>
<td>4.0490</td>
<td>0.022</td>
</tr>
<tr>
<td>3</td>
<td>4.0218</td>
<td>0.052</td>
</tr>
</tbody>
</table>

5.1.3 Empirical Performance of Multivariate Normal Distribution Generated Universal Portfolio

We now investigate the finite-order multivariate normal universal portfolios. The finite-order multivariate normal universal portfolios are run on same selected three data sets with the selected values of parameter $(\mu_1, \mu_2, \mu_3, \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma)$ listed in Table 5.5, Table 5.6 and Table 5.7. Empirically, from the Table 5.5, Table 5.6 and Table 5.7, we observed the performance of the finite-order multivariate normal universal portfolios is comparable to that of the Dirichlet universal
portfolios and yet requiring substantially less computation time.

Table 5.5: The wealth $S_{500}$ achieved by the MVNUP (7.1,0.8,0.3,1.0) for data Set A.

<table>
<thead>
<tr>
<th></th>
<th>$S_{500}$</th>
<th>Computation Time (Seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>order 1</td>
<td>1.9393</td>
<td>0.014</td>
</tr>
<tr>
<td>order 2</td>
<td>1.9442</td>
<td>0.024</td>
</tr>
<tr>
<td>order 3</td>
<td>1.9405</td>
<td>0.063</td>
</tr>
</tbody>
</table>

Table 5.6: The wealth $S_{500}$ achieved by the MVNUP (0.01,0.01,1000,1.2) for data Set B.

<table>
<thead>
<tr>
<th></th>
<th>$S_{500}$</th>
<th>Computation Time (Seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>order 1</td>
<td>4.0761</td>
<td>0.015</td>
</tr>
<tr>
<td>order 2</td>
<td>4.0353</td>
<td>0.024</td>
</tr>
<tr>
<td>order 3</td>
<td>4.0218</td>
<td>0.062</td>
</tr>
</tbody>
</table>

Table 5.7: The wealth $S_{500}$ achieved by the MVNUP (0.01,0.01,1000,1.2) for data Set C.

<table>
<thead>
<tr>
<th></th>
<th>$S_{500}$</th>
<th>Computation Time (Seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>order 1</td>
<td>4.0761</td>
<td>0.015</td>
</tr>
<tr>
<td>order 2</td>
<td>4.0490</td>
<td>0.024</td>
</tr>
<tr>
<td>order 3</td>
<td>4.0219</td>
<td>0.062</td>
</tr>
</tbody>
</table>

5.1.4 Empirical Performance of Brownian-motion Generated Universal Portfolio

In this subsection, we investigate the universal portfolio generated by three independent Brownian motions with drift coefficients $\mu_1, \mu_2, \mu_3$ and variance parameters $\sigma_1^2, \sigma_2^2, \sigma_3^2$ respectively are run on the same sets A, B and C. In the fol-
ollowing tables, the notation \((\mu_1, \sigma_{11}^2; \mu_2, \sigma_{22}^2; \mu_3, \sigma_{33}^2)\) is used to denote the parameters of the 3 generating Brownian motions. For three sets of selected parameters \((1, 1.4; 1, 1.5; 1, 1.2), (1, 1.4; 10, 2.5; 100, 3.6)\) and \((25, 1.4; 0.5, 1.5; 0.5, 1.2)\) the Brownian motion universal portfolios of orders 1, 2, 3 are run on the data sets A, B and C. Assuming an initial wealth of 1 unit, the wealth \(S_{500}\) achieved by the portfolios after 500 trading days and the implementation times for data sets A, B and C are displayed in Table 5.8, Table 5.9 and Table 5.10 respectively. Comparing the wealth achieved by the multinomial universal portfolios in Section 5.1.2 and the wealth achieved in Table 5.8, Table 5.9 and Table 5.10, it is observed that 4.0335, 3.9666 and 3.9194 units are achieved by the orders 1, 2 and 3 portfolios respectively for data set B in Table 5.9. The corresponding wealth of 4.0761, 4.0354 and 4.0217 units are achieved by the \((0.000001,0.000001,600)\) multinomial universal portfolio for data set B. In Table 5.8, lower wealth of 1.6832, 1.6832 and 1.5886 units are achieved by the orders 1, 2 and 3 Brownian-motion portfolios respectively for data set A. The corresponding higher wealth of 2.0238, 2.0362 and 2.0315 units are achieved by the \((0.8,0.01,600)\) multinomial universal portfolio for data set A.
Table 5.8: The wealth $S_{500}$ achieved by the BMUP $(1,1.4;1,1.5;1,1.2)$ for data Set A.

<table>
<thead>
<tr>
<th>Order</th>
<th>$S_{500}$</th>
<th>Computation Time (Seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.6832</td>
<td>0.042</td>
</tr>
<tr>
<td>2</td>
<td>1.6832</td>
<td>0.065</td>
</tr>
<tr>
<td>3</td>
<td>1.5886</td>
<td>0.211</td>
</tr>
</tbody>
</table>

Table 5.9: The wealth $S_{500}$ achieved by the BMUP $(1,1.4;10,2.5;100,3.6)$ for data Set B.

<table>
<thead>
<tr>
<th>Order</th>
<th>$S_{500}$</th>
<th>Computation Time (Seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4.0335</td>
<td>0.025</td>
</tr>
<tr>
<td>2</td>
<td>3.9666</td>
<td>0.067</td>
</tr>
<tr>
<td>3</td>
<td>3.9194</td>
<td>0.208</td>
</tr>
</tbody>
</table>

Table 5.10: The wealth $S_{500}$ achieved by the BMUP $(25,1.4;0.5,1.5;0.5,1.2)$ for data Set C.

<table>
<thead>
<tr>
<th>Order</th>
<th>$S_{500}$</th>
<th>Computation Time (Seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.4230</td>
<td>0.027</td>
</tr>
<tr>
<td>2</td>
<td>1.4142</td>
<td>0.067</td>
</tr>
<tr>
<td>3</td>
<td>1.4451</td>
<td>0.210</td>
</tr>
</tbody>
</table>

5.1.5 Empirical Performance of Universal Portfolio Generated by Ornstein-Uhlenbeck Process

The Ornstein-Uhlenbeck process based universal portfolio is studied with the three stock-price data sets A, B and C. The parametric vector of this universal portfolio is $(\mu_1, \mu_2, \mu_3; \alpha_1, \alpha_2, \alpha_3)$. By trial and error, parametric vectors $(16,0.1,0.5; 0.2,1.2,2), (0.001,0.001,50; 0.2,1.2,2)$ and $(0.005,0.009,99; 0.2,1.2,2)$ were found to be doing well for data sets A, B and C respectively. The wealth $S_{500}$ and computation time in second after 500 trading days achieved by the orders 1, 2 and
3 portfolios for data sets A, B and C are displayed in Table 5.11, Table 5.12 and Table 5.13, respectively. The wealth achieved by the portfolios is from 2.035 to 2.037, 4.069 to 4.073 and 4.073 to 4.075 for data sets A, B and C respectively.

Table 5.11: The wealth $S_{500}$ achieved by the portfolios generated by the OU process with parametric vector $(16,0.1,0.5;0.2,1.2,2)$ for data Set A.

<table>
<thead>
<tr>
<th></th>
<th>$S_{500}$</th>
<th>Computation Time (Seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>order 1</td>
<td>2.0370</td>
<td>0.015</td>
</tr>
<tr>
<td>order 2</td>
<td>2.0368</td>
<td>0.072</td>
</tr>
<tr>
<td>order 3</td>
<td>2.0358</td>
<td>0.332</td>
</tr>
</tbody>
</table>

Table 5.12: The wealth $S_{500}$ achieved by the portfolios generated by the OU process with parametric vector $(0.001,0.001,50;0.2,1.2,2)$ for data Set B.

<table>
<thead>
<tr>
<th></th>
<th>$S_{500}$</th>
<th>Computation Time (Seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>order 1</td>
<td>4.0732</td>
<td>0.010</td>
</tr>
<tr>
<td>order 2</td>
<td>4.0711</td>
<td>0.077</td>
</tr>
<tr>
<td>order 3</td>
<td>4.0696</td>
<td>0.325</td>
</tr>
</tbody>
</table>

Table 5.13: The wealth $S_{500}$ achieved by the portfolios generated by the OU process with parametric vector $(0.005,0.009,99;0.2,1.2,2)$ for data Set C.

<table>
<thead>
<tr>
<th></th>
<th>$S_{500}$</th>
<th>Computation Time (Seconds)</th>
</tr>
</thead>
<tbody>
<tr>
<td>order 1</td>
<td>4.0746</td>
<td>0.011</td>
</tr>
<tr>
<td>order 2</td>
<td>4.0739</td>
<td>0.074</td>
</tr>
<tr>
<td>order 3</td>
<td>4.0734</td>
<td>0.334</td>
</tr>
</tbody>
</table>

5.2 Identifying “Good” Parameters

From the above results, four finite order universal portfolios strategies were found can perform better than the Cover Ordentlich Dirichlet universal portfolio for
same data sets. Noticed that the four universal portfolios with order one performed slightly better than order two and three. Therefore, this research will be limited to order one universal portfolios.

5.2.1 Representation Data

For the following sections, selection of 10 set of parameters for each universal portfolios are described. Python code are written to obtain the good performance parameters among these 10 set of parameters. We use CRP as a benchmark to evaluate and to compare with these four universal portfolio strategies. A comparison can be made with the wealth obtained by CRP and the wealth achieved by the four proposed universal portfolios. Every one year period starting from year 2000 to year 2015 of the available stock data listed in Table 2.1 in Chapter 2 are used for study. At least 20% of the wealth achieved by proposed universal portfolio performed better than wealth obtained by CRP are analysed. The good performance of the parameters is observed.

5.2.2 Parameters for Multinomial Universal Portfolio

10 set of parameters \((p_1, p_2, N)\) are used for selection of good performances, they are \(A = (0.8, 0.001, 600), B = (0.001, 0.8, 600), C = (0.5, 0.5, 600), D = (0.9, 0.1, 600), E = (0.1, 0.9, 600), F = (0.8, 0.001, 50), G = (0.001, 0.8, 50), H = (0.05, 0.95, 99), I = (0.95, 0.05, 99)\) and \(J = (0.9, 0.004, 10)\). In Section 5.1.2, \((0.8, 0.001, 600)\) performed well when ran on 3 selected stocks from KLSE. Therefore, this set of
parameter chosen for the analysis. The other nine set of parameters are formed by varying among (0.8,0.001,600).

5.2.3 Parameters for Multivariate Normal Universal Portfolio

In Section 5.1.3, good performances obtained by order one universal portfolio generated by Multivariate Normal Distribution with the set of (7.1,0.8,0.2,1.0). Therefore, the 10 sets of parameters are selected varying among (7.1,0.8,0.2,1.0), i.e. \( K = (7.1,0.8,0.2,1.0) \), \( L = (0.2,0.8,7.1,1.0) \), \( M = (0.8,7.1,0.2,1.0) \), \( N = (7.1,0.2,0.8,1.0) \), \( O = (0.8,0.2,7.1,1.0) \), \( P = (0.2,7.1,0.8,1.0) \), \( Q = (8.1,9.1,0.1,2.0) \), \( R = (0.2,8.1,9.1,2.0) \), \( S = (8.1,9.1,2.0) \) and \( T = (9.0,0.1,0.1,2.0) \).

5.2.4 Parameters for Brownian-motion Generated Universal Portfolio

In Section 5.1.4, good performances parameter of order one universal portfolio generated by Brownian-motion is (1,1.4;10,2.5;100,3.6), therefore, the 10 sets of parameters are formed among (1,1.4;10,2.5;100,3.6), i.e. \( a = (1,1.4;10,2.5;100,3.6) \), \( b = (100,1.4;1,2.5;10,3.6) \), \( c = (100,1.4;10,2.5;1,3.6) \), \( d = (10,1.4;1,2.5;100,3.6) \), \( e = (1,1.4;100,2.5;10,3.6) \), \( f = (10,1.4;100,2.5;1,3.6) \), \( g = (1,1.4;10,3.6;100,2.5) \), \( h = (1,2.5;10,1.4;100,3.6) \), \( i = (1,2.5;10,3.6;100,1.4) \) and \( j = (1,3.6;10,1.4;100,2.5) \).
5.2.5 Parameters for Universal Portfolio Generated by Ornstein Uhlenbeck Process

From Section 5.1.5, one of the parameters selected is $(16,0.1,0.5,0.2,1.2,2)$ due to good performance is obtained. The other 9 set of parameters we formed by varying among $(16,0.1,0.5,0.2,1.2,2)$. The 10 set of parameters are $k = (16,0.1,0.5,0.2,1.2,2)$, $l = (16,0.5,0.1,0.2,1.2,2)$, $m = (0.5,16,0.2,1.2,2)$, $n = (0.5,16,0.1,0.2,1.2,2)$, $o = (0.1,16,0.5,0.2,1.2,2)$, $p = (0.1,0.5,16,0.2,1.2,2)$, $q = (16,0.1,0.5,2,1.2,0.2)$, $r = (16,0.5,0.1,2,1.2,0.2)$, $s = (50,0.5,2,0.2,1.2,2)$ and $t = (50,2,0.5,0.2,1.2,2)$.

5.3 Numerical Experiment

5.3.1 Computer Program

The chosen parameters in the previous section will be used for numerical computation. The Python code is written as follow:

```python
import pandas as pd
import numpy as np

df = pd.read_csv('data00-15/klse100roi.csv')

to_find_list = ['3182', '5148', '5398', '5517', '7106',
                 '7084','1818','5031','2216','5099']

stock_data.fillna(1.0,inplace=True)
```
def CRP(x, param=[]):
    num_cols = len(x)
    _x = [list(x[i]) for i in range(num_cols)]
    num_rows = len(_x[0])
    Se = np.ones(num_rows)
    if len(param)>0 and (sum(param)-1.0)<1e-9:
        Se[0] = sum([param[i]*_x[i][0] for i in range(num_cols)])
        for n in range(num_rows-1):
            daily_roi_p = sum([param[i]*_x[i][n] for i in range(num_cols)])
            if np.isnan(daily_roi_p):
                print("\nAt n =", n, ", _x =", 
                [_x[i][n] for i in range(num_cols)])
                return None
            Se[n+1] = Se[n]*daily_roi_p
    else:
        Se[0] = (x1[0]+x2[0]+x3[0])/3
        for n in range(num_rows-1):
            Se[n+1] = Se[n]*(x1[i]+x2[i]+x3[i])/3
            if np.isnan(daily_roi_p):
                print("\nAt n =", n, ", _x =", 
                [_x[i][n] for i in range(num_cols)])
                return None
        return Se

from portfolio2015.recur3multinomial import Recursive3Multinomial
from portfolio2015.recur3multivariatNormal import Recursive3MultivariatNormal
from portfolio2015.recur3bronianmotion import Recursive3bronianmotion
from portfolio2015.recur3ornstein import Recursive3ornstein

for i, j, k in itertools.combinations(stock_roi, 3):
    x = [list(stock_data[i]), list(stock_data[j]), list(stock_data[k])]
    crp_res = CRP(x, [1./3,1./3,1./3])
    if crp_res is not None:
        print("",
    else:
        print("Data Error")

parameters = [
    [0.8, 0.001, 600], [0.001, 0.8, 600],
    [0.5, 0.5, 600], [0.9, 0.1, 600],
    [0.1, 0.9, 600], [0.8, 0.001, 50],
for p1, p2, NN in parameters:
    print("%s,%s,%s,MultiNomial,"%s")
    res = Recursive3Multinomial(x,p1,p2,NN)
    if res is not None:
        print("",%f"%res[-1])
    else:
        print("Data Error")

parameters = [
    [7.1,0.8,0.2,1.0], [0.2,0.8,7.1,1.0],
    [0.8,7.1,0.2,1.0], [7.1,0.2,0.8,1.0],
    [0.8,0.2,7.1,1.0], [0.2,7.1,0.8,1.0],
    [8.1,9.1,0.2,2.0], [0.2,8.1,9.1,2.0],
    [8.1,0.2,9.1,2.0], [9.0,0.1,0.1,2.0],
]

for p1, p2, p3, NN in parameters:
    print("%s,%s,%s,MultivariatNormal,"%s")
    res = Recursive3MultivariatNormal(x,p1,p2,p3,NN)
    if res is not None:
        print("",%f"%res[-1])
    else:
        print("Data Error")

parameters = [
    [1,10,100,1.4,2.5,3.6], [10,1,100,1.4,2.5,3.6],
    [100,1,100,1.4,2.5,3.6], [10,1,100,1.4,2.5,3.6],
    [1,100,1,100,1.4,2.5,3.6], [10,100,1,100,1.4,2.5,3.6],
    [1,10,100,1.4,3.6,2.5], [1,10,100,2.5,1.4,3.6],
    [1,10,100,2.5,3.6,1.4], [1,10,100,3.6,1.4,2.5],
]

for p1, p2, p3 in parameters:
    print("%s,%s,%s,bronianmotion,"%s")
    res = Recursive3bronianmotion(x,p1,p2,NN,MM,QQ)
    if res is not None:
        print("",%f"%res[-1])
    else:
        print("Data Error")

parameters = [
    [16,0.1,0.5,0.2,1.2,2], [16,0.5,0.1,0.2,1.2,2],
]
for p1, p2, p3 in parameters:
    print("%s,%s,%s,ornstein," % (i,j,k,[p1,p2,p3])),
    res = Recursive3ornstein(x,p1,p2,p3,NN,MM,QQ)
    if res is not None:
        print("",%f%res[-1])
    else:
        print("Data Error")

From the above python code, line number 4 is reading all the 95 stocks data from the CSV file which has been generated in Section 2.3 in Chapter 2. In line number 6 is to pick the 10 most active stocks. While in line number 9 is describing the data collected from Yahoo KLSE stocks are incomplete and sometimes wrong. In line number 41 is combination of 3 stocks which is described in Chapter 4.

The chosen parameters for each portfolios are listed from line number 52 to line number 56 for Multinomial generated universal portfolio, line number 67 to number 71 for Multivariate Normal generated universal portfolio, line number 83 to line number 87 for Brownian-motion generated universal portfolio and line number 98 to line number 102 are for Ornstein Uhlenbeck process generated universal portfolio.
5.3.2 Numerical Result and Analysis

The number of stocks and trading period will be vary based on the available data and trading period listed in Table 2.1 in Chapter 2. From period 1 January 2015 to 31 December 2015, 10 most active stocks data listed in Table 2.2 in Chapter 2 with codes 5398, 7084, 1818, 5517, 7106, 5031, 2216, 5148, 5099 and 3182 will be studied, and the 120 sets consists of 3 stocks data have been generated from the $^{10}C_3$ combination. At least 20% out of total 120 sets with the performance better the CRP portfolio strategies will be observed and studied. Only stocks data with codes 5398, 7084, 5517, 7106, 5031, 2216, 5148, 3182 are available for study from year 2004 to year 2015, refer to Table 2.2 in Chapter 2. Total of combination is 56 sets data and 20% of the good performance parameters strategy is observed. From year 2002 to year 2015 only stocks data with codes 5398, 7084, 5517, 7106, 5031, 5148 and 3182 are available with total 35 sets data, and from year 2001 to year 2015, stocks with codes 5398, 7084, 5517, 7106, 5148 and 3182 and the total 20 sets of data are studied. From year 2000 to year 2015, there are 5 stocks with codes 5398, 5517, 7106, 5148 and 3182 available for studied and total only 10 data sets consists on 3 stock data being studied. At least 20% of each universal portfolio strategy perform better than benchmark CRP are recorded, details of result can refer Appendix C. For each trading period, well performance parameters for each proposed universal portfolio are obtained and shown in the following figures.

The meaning of A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, a, b, c, d, e, f, g, h, i, k, l, m, n, o, p, q, r, s and t are given in Section 5.2.2 to Sec 5.2.5. For each parameter setting, the $W_n$ is calculated as the ratio of $S_n$ of selected universal portfolio to $S_n$ of CRP, and then a box-plot is constructed for
Figure 5.1 to Figure 5.13 compare the box-plots for $W_n$ which are selected from the good parameters of the respective four proposed universal portfolios.

**Figure 5.1: Well Performed Parameters of each Universal Portfolio in Year 2015 Period**

From Figure 5.1, good performance parameters are observed for selected parameters A, D, F, I and J for Multinomial generated universal portfolio. For Multivariate Normal generated universal portfolio, parameters K, N and T are performed good. While, for Brownian-motion generated universal portfolio, only parameter b and c are good and there are four parameters performed well for universal portfolio generated by Ornstein Uhlenbeck process, there are k, l, s and t. The performances of Ornstein Uhlenbeck process generated universal portfolio performed better than the other three universal portfolio strategies.
Boxplot of $W_n$ against Order 1 UP’s of Various Parameters Yr 2014–2015

Figure 5.2: Well Performed Parameters of each Universal Portfolio in Year 2014 to 2015 Period

From Figure 5.2, good performance parameters are observed for selected parameters D, I and J for Multinomial generated universal portfolio. For Multivariate Normal generated universal portfolio, parameters K, N and T are performed good. While, for Brownian-motion generated universal portfolio, only parameters b and c are good and there are four parameters performed well for universal portfolio generated by Ornstein Uhlenbeck process, there are k, l, s and t. Among the four strategies, we can observed that the universal portfolio generated by Ornstein Uhlenbeck process again performed better than other strategies.
Figure 5.3: Well Performed Parameters of each Universal Portfolio in Year 2013 to 2015 Period

From Figure 5.3, we can observed that the results are similar to Figure 5.2 with good performance parameters are shown for selected parameters D, I and J for Multinomial generated universal portfolio. For Multivariate Normal generated universal portfolio, parameters K, N and T are performed good. While, for Brownian-motion generated universal portfolio, only parameters b and c are good. There are four parameters performed well for universal portfolio generated by Ornstein Uhlenbeck process, there are k, l, s and t and this strategy again better better than the other three strategies.

However, none of the proposed four universal portfolio has at least 20% of the results obtained performed better than CRP for the year 2012 to year 2015, year 2011 to year 2015 and year 2010 to year 2015 periods.
From Figure 5.4, only universal portfolio generated by Ornstein Uhlenbeck process has shown the good performances with parameters k, l, s and t and have showed the similar performances among them.

From Figure 5.5, good performance parameters are observed for selected parameters A, D, F, I and J for Multinomial generated universal portfolio. For Multi-
variate normal generated universal portfolio, parameters K, L and T are performed good. While, for Brownian-motion generated universal portfolio, only parameters b and c are good and there are four parameters performed well for universal portfolio generated by Ornstein Uhlenbeck process, there are k, l, s and t with the performances better than other three strategies.

Figure 5.6: Well Performed Parameters of each Universal Portfolio in Year 2007 to 2015 Period

Figure 5.6 showed the results obtained is similar to Figure 5.5, good performance parameters are observed for selected parameters A, D, F, I and J for Multinomial generated universal portfolio. For Multivariate normal generated universal portfolio, parameters K, L and T are performed good. While, for Brownian-motion generated universal portfolio, only parameters b and c are good and there are four parameters performed well for universal portfolio generated by Ornstein Uhlenbeck process, there are k, l, s and t.
From Figure 5.7, results obtained is similar to previous period, where good performance parameters are observed for selected parameters A, D, F, I and J for Multinomial generated universal portfolio. For Multivariate Normal generated universal portfolio, parameters K, L and T are performed good. While, for Brownian-motion generated universal portfolio, only parameters a and b are good and there are four parameters performed well for universal portfolio generated by Ornstein Uhlenbeck process, there are k, l, s and t.

Figure 5.8: Well Performed Parameters of each Universal Portfolio in Year 2005 to 2015 Period
From Figure 5.8, results obtained is slightly different compare to Figure 5.7. Good performance parameters are observed for selected parameters A, E, F, I and J for Multinomial generated universal portfolio. For Multivariate Normal generated universal portfolio, parameters K, O and T are performed good. While, for Brownian-motion generated universal portfolio, only parameters b and c are good and there are four parameters performed well for universal portfolio generated by Ornstein Uhlenbeck process, there are k, l, s and t. Universal portfolio generated by Ornstein Uhlenbeck still performed better than the other three strategies.

**Figure 5.9: Well Performed Parameters of each Universal Portfolio in Year 2004 to 2015 Period**

From Figure 5.9, good performance parameters are observed for selected parameters A, D, F, I and J for Multinomial generated universal portfolio. For Multivariate Normal generated universal portfolio, parameters K, N and T are performed good. While, for Brownian-motion generated universal portfolio, only parameters b and c are good and there are four parameters performed well for universal portfolio generated by Ornstein Uhlenbeck process, there are k, l, s and t. Universal portfolio generated by Ornstein Uhlenbeck still performed better than the other three strategies. There are more outliers compared to the previous graphs.
Figure 5.10: Well Performed Parameters of each Universal Portfolio in Year 2003 to 2015 Period

From Figure 5.10, the results obtained by the four proposed strategies are different compared to the previous period. Good performance parameters are observed for selected parameters B and G for Multinomial generated universal portfolio. For Multivariate Normal generated universal portfolio, parameters L and O are performed good. While, for Brownian-motion generated universal portfolio, parameters a, d, g, h, i and j are good and there are three parameters performed well for universal portfolio generated by Ornstein Uhlenbeck process, there are m, n, and p. The universal portfolio generated by Ornstein Uhlenbeck process still performed better than the other three portfolio strategies even with the different parameters.
Figure 5.11: Well Performed Parameters of each Universal Portfolio in Year 2002 to 2015 Period

From Figure 5.11, only universal portfolio generated by Ornstein Uhlenbeck process showed the good performances with parameters m, n and p. We can concluded that the universal portfolio generated by Ornstein Uhlenbeck process consistently performed well compare to the other three strategies.

Figure 5.12: Well Performed Parameters of each Universal Portfolio in Year 2001 to 2015 Period

From Figure 5.12, good parameters are observed by B and G for universal portfolio generated by Multinomial, R of universal portfolio generated by Multivariate normal and e for universal portfolio generated by Brownian-motion. How-
ever, the universal portfolio generated by Ornstein Uhlenbeck process did not perform well in this period.

![Boxplot](image)

**Figure 5.13: Well Performed Parameters of each Universal Portfolio in Year 2000 to 2015 Period**

From Figure 5.13, good performance parameters are observed for selected parameters L, P and S for Multivariate Normal generated universal portfolio. While for Brownian-motion generated universal portfolio, parameters a, b, g, h, i and j were performed good and there are three parameters performed well for universal portfolio generated by Ornstein Uhlenbeck process, there are m, p and t. The universal portfolio generated by Brownian-motion is performed better than the other three universal portfolio strategies with 5 set of parameters shown in above figure.

In conclusion, for the further data analysis in next chapter, only the best performing parameter was chosen among all the above good performance parameters for each universal portfolio obtained in each period. Overall, from Figure 5.1 to Figure 5.13, we observed that the parameter for Multinomial generated universal portfolio is I which is (0.95, 0.05, 99) showed the good performances, for universal portfolio generated by Multivariate Normal distribution, parameter T (9.0,0.1,0.1,2.0) is observed with good performances. The other parameters per-
formed are $b$, which is $(100,1.4;1.2.5;10,3.6)$ for universal portfolio generated by Brownian motion and $k$ which is $(16,0.1,0.5,0.2,1.2,2)$ for universal portfolio generated by Ornstein Uhlenbeck process with good performances.

The detail of the results are shown in Appendix C.
CHAPTER 6

PERFORMANCE OF SELECTED UNIVERSAL PORTFOLIO STRATEGIES

In this chapter, we will study the four classes universal portfolio strategies (refer to Chapter 3) with the best performance parameters identified in Chapter 5. Numerical experiment for the four classes of portfolio strategies will be studied for short term (1 year period), middle term (4 years and 8 years periods) and long term (12 years and 16 years periods). We will employ the four portfolio strategies on the above 5 groups of data to learn the performance of these methods and use CRP as a benchmark. The experiment will be limited to order one universal portfolios and 3 stocks data are chosen for study.

We have identified the 95 most active stocks in KLSE. In Table 2.1 in Chapter 2, the first group of data starting from 1 January 2015 to 31 December 2015 consists of 95 stocks data. The second group is 4 years period starting from 1 January 2012 to 31 December 2015 consists of 90 stocks data where stock data with codes 5222, 5227, 5209, 5225 and 6399 are excluded because there were no transaction from year 2012 to 2014. The third group is 8 years trading period starting from 1 January 2008 to 31 December 2015 consists of 80 stocks data where the stock data excluded are codes 5183, 5141, 5186, 5211, 5176, 5180, 5212, 5210, 5218, 6912 and all the codes excluded in four years trading period group. The fourth group is 12 years period starting from 1 January 2004 to 31 December 2015 consists of 66 stocks data excluded the stocks with codes 3336, 5131, 7052, 1818, 5139, 5168, 6947, 2593, 5027, 5126, 5135, 5138, 1591, 3859, 5090, 5099, 5347 and all the stock excluded in 4 years and 8 years period groups.
While, the last group is 16 years of trading period starting from 1 January 2000 to 31 December 2015 consists of 63 stocks data where data are excluded in are codes 7113, 5031, 7106 and all the stock data excluded in 4 years, 8 years and 12 years periods.

6.1 Performance Analysis of Four Universal Portfolio Strategies against CRP

We will investigate the performance of the four universal portfolio strategies by analysing the basic statistics of $S_n$ against the CRP performance. The price relative for a given stock is the ratio of the closing price to its opening price on the same trading day. The summary of the analysis are calculated as log of the wealth obtained. The histogram of the log of the ratio of the wealth obtained by the four universal portfolio against the corresponding benchmark, CRP wealth are plotted.

6.1.1 Statistical Analysis of n-year Investment Based on “log $S_n$”

After running the Multinomial distribution generated universal portfolio algorithm in Chapter 4, the data stored in “resultmultinomial.csv”. The summary of statistical analysis is calculated using R commands as below:

```
> setwd("/Volumes/SAMSUNG/PhD-Thesis/code/result1year")
> df=read.csv("resultmultinomial.csv",header=F)
> x =log(df$V4)
```
> summary(x)

Min. 1st Qu. Median Mean 3rd Qu. Max.
-2.99000 -0.30040 0.07203 0.11340 0.47590 3.14600

In line number 2, the CSV file consist of the ratio of respective universal portfolio wealth to CRP wealth. While for the CRP strategy, CSV file is read with the log of the wealth obtained by this strategy. By loading other data files obtained in Chapter 4 and performing basic statistical analysis, results of summary of the statistical analysis are listed in the following tables.
Table 6.1: Short Term Period Data Analysis (1 Year Period)

<table>
<thead>
<tr>
<th>Portfolio Strategies</th>
<th>Min</th>
<th>First Quartile</th>
<th>Median</th>
<th>Mean</th>
<th>Third Quartile</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multinomial Universal Portfolio</td>
<td>-2.9900</td>
<td>-0.3004</td>
<td>0.0720</td>
<td>0.1134</td>
<td>0.4759</td>
<td>3.1460</td>
</tr>
<tr>
<td>Multivariate Normal Universal Portfolio</td>
<td>-2.9540</td>
<td>-0.2953</td>
<td>0.0672</td>
<td>0.1074</td>
<td>0.4615</td>
<td>3.0750</td>
</tr>
<tr>
<td>Brownian Motion Universal Portfolio</td>
<td>-2.8480</td>
<td>-0.2851</td>
<td>0.0578</td>
<td>0.0939</td>
<td>0.4303</td>
<td>2.8910</td>
</tr>
<tr>
<td>OU Process Universal Portfolio</td>
<td>-3.0830</td>
<td>-0.3088</td>
<td>0.0666</td>
<td>0.1075</td>
<td>0.4747</td>
<td>3.1761</td>
</tr>
<tr>
<td>$\log(S_n(CRP))$</td>
<td>-1.8430</td>
<td>-0.1040</td>
<td>0.2469</td>
<td>0.3051</td>
<td>0.6390</td>
<td>3.7700</td>
</tr>
</tbody>
</table>

Table 6.2: Middle Term Period Data Analysis (4 Years Period)

<table>
<thead>
<tr>
<th>Portfolio Strategies</th>
<th>Min</th>
<th>First Quartile</th>
<th>Median</th>
<th>Mean</th>
<th>Third Quartile</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multinomial Universal Portfolio</td>
<td>-3.6420</td>
<td>-0.3956</td>
<td>0.0653</td>
<td>0.1245</td>
<td>0.5532</td>
<td>4.3480</td>
</tr>
<tr>
<td>Multivariate Normal Universal Portfolio</td>
<td>-3.5830</td>
<td>-0.3914</td>
<td>0.0579</td>
<td>0.1161</td>
<td>0.5356</td>
<td>4.2520</td>
</tr>
<tr>
<td>Brownian Motion Universal Portfolio</td>
<td>-3.4300</td>
<td>-0.3798</td>
<td>0.0446</td>
<td>0.0980</td>
<td>0.49730</td>
<td>4.0050</td>
</tr>
<tr>
<td>OU Process Universal Portfolio</td>
<td>-3.7330</td>
<td>-0.4114</td>
<td>0.0552</td>
<td>0.1143</td>
<td>0.5492</td>
<td>4.3960</td>
</tr>
<tr>
<td>$\log(S_n(CRP))$</td>
<td>-1.8430</td>
<td>-0.1071</td>
<td>0.2430</td>
<td>0.3070</td>
<td>0.6467</td>
<td>3.7700</td>
</tr>
</tbody>
</table>

From Table 6.1 and Table 6.2, we obtained the median and mean with positive values for the four universal portfolios. This indicate that the performances of the four proposed universal portfolio strategies are better than CRP.
From Table 6.3, we observed that CRP is performed better than the four universal portfolio strategies.
Table 6.4: Long Term Period Data Analysis (12 Years Period)

<table>
<thead>
<tr>
<th>Portfolio Strategies</th>
<th>Min</th>
<th>First Quartile</th>
<th>Median</th>
<th>Mean</th>
<th>Third Quartile</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multinomial Universal Portfolio</td>
<td>-inf</td>
<td>-2</td>
<td>0</td>
<td>-inf</td>
<td>1</td>
<td>16</td>
</tr>
<tr>
<td>Multivariate Normal Universal Portfolio</td>
<td>-inf</td>
<td>-2</td>
<td>0</td>
<td>-inf</td>
<td>1</td>
<td>15</td>
</tr>
<tr>
<td>Brownian Motion Universal Portfolio</td>
<td>-inf</td>
<td>-2</td>
<td>0</td>
<td>-inf</td>
<td>1</td>
<td>16</td>
</tr>
<tr>
<td>OU Process Universal Portfolio</td>
<td>-inf</td>
<td>-2</td>
<td>0</td>
<td>-inf</td>
<td>1</td>
<td>16</td>
</tr>
<tr>
<td>$\log(S_n(CRP))$</td>
<td>-1.5850</td>
<td>-0.0472</td>
<td>0.2886</td>
<td>0.3830</td>
<td>0.7234</td>
<td>3.7470</td>
</tr>
</tbody>
</table>

Table 6.5: Long Term Period Data Analysis (16 Years Period)

<table>
<thead>
<tr>
<th>Portfolio Strategies</th>
<th>Min</th>
<th>First Quartile</th>
<th>Median</th>
<th>Mean</th>
<th>Third Quartile</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multinomial Universal Portfolio</td>
<td>-inf</td>
<td>-2</td>
<td>0</td>
<td>-inf</td>
<td>1</td>
<td>40</td>
</tr>
<tr>
<td>Multivariate Normal Universal Portfolio</td>
<td>-inf</td>
<td>-2</td>
<td>0</td>
<td>-inf</td>
<td>1</td>
<td>38</td>
</tr>
<tr>
<td>Brownian Motion Universal Portfolio</td>
<td>-inf</td>
<td>-2</td>
<td>0</td>
<td>-inf</td>
<td>1</td>
<td>39</td>
</tr>
<tr>
<td>OU Process Universal Portfolio</td>
<td>-inf</td>
<td>-2</td>
<td>0</td>
<td>-inf</td>
<td>1</td>
<td>40</td>
</tr>
<tr>
<td>$\log(S_n(CRP))$</td>
<td>-1.5850</td>
<td>-0.0548</td>
<td>0.2680</td>
<td>0.3610</td>
<td>0.6762</td>
<td>3.7470</td>
</tr>
</tbody>
</table>

From Table 6.4 and Table 6.5, we observed that the CRP is performed well compared to the four proposed strategies. There are some extreme values showed under mean, minimum value and maximum value of the four strategies. Therefore, the summary values obtained in long term period is meaningless. The maximum values obtained for four proposed strategies are much more larger than CRP. Hence, we can concluded that there are outliers and further analysis without these outlier will be carried out on next section.
In order to view the result graphically, the following histograms for short term period, middle-term period and long term period are plotted using R commands as below.

```r
> setwd("/Volumes/SAMSUNG/PhD-Thesis/code/result1year")
> df=read.csv("resultmultinomial.csv",header=F)
> setEPS()
> postscript("resultmultinomial.eps")
> x=log(df$V4)
> coeff=density(x,na.rm=TRUE)
> hist(x,freq=FALSE,ylim=c(0,0.25))
> lines(coeff)
> dev.off()
```

Refer to line number 2, CSV files of each universal portfolio’s result with their respective trading period are read and histogram are plotted in line number 7.
Figure 6.1: Short Term Period Data Analysis (1 Year Period)

Figure 6.1 allows us to make the conclusion that the four graphs are slightly skew to the left, same as the results shown in previous statistical analysis tables, we can concluded that the wealths achieved by the four universal portfolio strategies are better than the wealths achieved by CRP strategy.
Figure 6.2: Middle Term Period Data Analysis (4 Years Period)

Figure 6.2 show that the four graphs are slightly skew to the left, even the four graphs are pretty close in the shape, but the four strategies still consistently beating the performance of CRP strategy.
The third group data starting from 1 January 2008 to 31 December 2015. The financial crisis happened at the end of year 2007 and beginning of year 2008 caused a huge impact on all financial markets around the world. The crisis damaged investor confidence had an impact on global stock markets, where securities suffered large losses during the late year 2008 and early year 2009. The analysis of Figure 6.3 allow us to make the following conclusion, the four graphs are slightly skew to the right, from the summary of analysis in table 6.3 also showed that the four strategies are not perform well compared to CRP strategy.
Figure 6.4: Long Term Period Data Analysis (12 Years Period)
Figure 6.4 and Figure 6.5 show that there are outliers in the result obtained. Therefore, further analysis are carried out without the outlier to identify the performances of the four proposed universal portfolio strategies.
6.2 Performance Analysis on Data Without Outliers

In the previous section, we observe that the empirical distribution indicates that for long term (12 years and 16 years), the four order one universal portfolios are performed very badly (i.e. basically losing all money), therefore leading to some outliers where some of the ratios very small which leads to $-\infty$ and some of the ratios very big when taking logarithm. Therefore, in this section, we present the analysis for the long term period without the outliers.
Table 6.6: Long Term Period Data Analysis (12 Years Period) without Outliers

<table>
<thead>
<tr>
<th>Portfolio Strategies</th>
<th>Minimum</th>
<th>First Quartile</th>
<th>Median</th>
<th>Mean</th>
<th>Third Quartile</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multinomial</td>
<td>-4.9990</td>
<td>-1.6200</td>
<td>-0.4573</td>
<td>-0.3691</td>
<td>0.8410</td>
<td>4.9820</td>
</tr>
<tr>
<td>Universal Portfolio</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Multivariate Normal</td>
<td>-5.0000</td>
<td>-1.8530</td>
<td>-0.4408</td>
<td>-0.3500</td>
<td>0.8290</td>
<td>4.9990</td>
</tr>
<tr>
<td>Universal Portfolio</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Brownian Motion</td>
<td>-5.0000</td>
<td>-1.4940</td>
<td>-0.4199</td>
<td>-0.3222</td>
<td>0.7991</td>
<td>4.9880</td>
</tr>
<tr>
<td>Universal Portfolio</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>OU Process</td>
<td>-4.9980</td>
<td>-1.6430</td>
<td>-0.4634</td>
<td>-0.3656</td>
<td>0.8527</td>
<td>4.9900</td>
</tr>
<tr>
<td>Universal Portfolio</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>log(S_n(CRP))</td>
<td>-1.5850</td>
<td>-0.0472</td>
<td>0.2886</td>
<td>0.3830</td>
<td>0.7234</td>
<td>3.7470</td>
</tr>
</tbody>
</table>

Table 6.7: Long Term Period Data Analysis (16 Years Period) without Outliers

<table>
<thead>
<tr>
<th>Portfolio Strategies</th>
<th>Minimum</th>
<th>First Quartile</th>
<th>Median</th>
<th>Mean</th>
<th>Third Quartile</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multinomial</td>
<td>-4.9990</td>
<td>-1.9460</td>
<td>-0.5129</td>
<td>-0.3840</td>
<td>1.1190</td>
<td>4.9970</td>
</tr>
<tr>
<td>Universal Portfolio</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Multivariate Normal</td>
<td>-4.9980</td>
<td>-1.8940</td>
<td>-0.4991</td>
<td>-0.3638</td>
<td>1.1150</td>
<td>5.000</td>
</tr>
<tr>
<td>Universal Portfolio</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Brownian Motion</td>
<td>4.9990</td>
<td>-1.7900</td>
<td>-0.4645</td>
<td>-0.3383</td>
<td>-0.1075</td>
<td>4.9970</td>
</tr>
<tr>
<td>Universal Portfolio</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>OU Process</td>
<td>-5.0000</td>
<td>-1.9500</td>
<td>-0.5207</td>
<td>-0.3792</td>
<td>1.1360</td>
<td>4.9980</td>
</tr>
<tr>
<td>Universal Portfolio</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>log(S_n(CRP))</td>
<td>-1.5850</td>
<td>-0.0548</td>
<td>0.2680</td>
<td>0.3610</td>
<td>0.6762</td>
<td>3.7470</td>
</tr>
</tbody>
</table>

From the Table 6.6 and Table 6.7, the summary values obtained after removing the outliers show that the performances of CRP in 12 years and 16 years again performed better than the proposed four strategies. The following Figures show the graphs plotted without the outliers for long term period groups (12 years and 16 years ).
Figure 6.6: Long Term Period Data Analysis (12 and 16 Years Period) without Outliers for MNUP and MVNUP
From the above analysis and discussion, we can conclude that the four pro-

Figure 6.7: Long Term Period Data Analysis (12 and 16 Years Period) without Outliers for BMUP and OUUP

Form the Figure 6.6 and Figure 6.7, the spread of the graphs of four universal portfolio in 16 years period are larger than graphs in 12 years period. The data are clustered around the mean for 12 years period data. Notice that the proposed four universal portfolios performed better in 12 years period group compare to 16 years period group after removing the outliers.
posed universal portfolio strategies are perform well for the short term period and the middle term period (only 4 years period). Unfortunately, the four strategies are not perform well in 8 years period (middle term) and long term period.

6.3 Performance Analysis of Four Universal Portfolio Strategies against CRP with Redefined the Price Relative

In this section, we will try to compare the performances of four proposed strategies with benchmark KLCI. The price relatives for a given stock change to the ratio of closing price on the current trading day to the closing price on the day before. However, if the stock splits or bonus issue was announced by the respective company, the stock’s price has either increased or decreased sharply, the price relative will be calculated as the ratio of closing price to its opening price on the same trading day. The performance of the benchmark CRP is compared to the KLCI ratio where this ratio is calculated as the log of closing index to its opening index for the respective trading period.

6.3.1 Statistical Analysis of n-year Investment Based on “log \(S_n\)” with Redefined the Price Relative

Similar to previous section, the performance of the four universal portfolio strategies are analysed. The statistical analysis of the ratio of \(S_n\) against the CRP performance are performed. The summary of statistical analysis of the results obtained are calculated as log of the ratio of \(S_n\). The short term, middle term and long
term period statistical analysis are shown in the following tables. The graph of the 
log of the ratio of the wealth obtained by the four universal portfolio against the 
corresponding benchmark, CRP wealth are plotted.

### Table 6.8: Short Term Period Data Analysis (1 Year Period) with Redefined the Price Relative

<table>
<thead>
<tr>
<th>Portfolio Strategies</th>
<th>Min</th>
<th>First Quartile</th>
<th>Median</th>
<th>Mean</th>
<th>Third Quartile</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multinomial Universal Portfolio</td>
<td>-4.2260</td>
<td>-1.1140</td>
<td>-0.5467</td>
<td>-0.5935</td>
<td>-0.0218</td>
<td>2.1116</td>
</tr>
<tr>
<td>Multivariate Normal Universal Portfolio</td>
<td>-4.0850</td>
<td>-1.0720</td>
<td>-0.5272</td>
<td>-0.5671</td>
<td>-0.0178</td>
<td>2.0690</td>
</tr>
<tr>
<td>Brownian Motion Universal Portfolio</td>
<td>-3.6920</td>
<td>-0.9513</td>
<td>-0.4475</td>
<td>-0.4809</td>
<td>0.02168</td>
<td>1.9970</td>
</tr>
<tr>
<td>OU Process Universal Portfolio</td>
<td>-4.2150</td>
<td>-1.1100</td>
<td>-0.5498</td>
<td>-0.5896</td>
<td>-0.0249</td>
<td>2.1250</td>
</tr>
<tr>
<td>( \log(S_n(CRP)) )</td>
<td>0.2045</td>
<td>0.9622</td>
<td>1.5330</td>
<td>2.2890</td>
<td>2.6450</td>
<td>76.4800</td>
</tr>
<tr>
<td>( \log(KLClRatio) )</td>
<td></td>
<td></td>
<td></td>
<td>-0.0395</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

First, we analyse the performance of short term period. From the Table 6.8, 
we obtained the median and mean with negative values for the four portfolios. 
This indicate that the four proposed universal portfolio strategies are not perform 
well compare to CRP and KLCI. Noticed that CRP strategy is performed better 
than KLCI.
Table 6.9: Middle Term Period Data Analysis (4 Years Period) with Redefined the Price Relative

<table>
<thead>
<tr>
<th>Portfolio Strategies</th>
<th>Min</th>
<th>First Quartile</th>
<th>Median</th>
<th>Mean</th>
<th>Third Quartile</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multinomial Universal Portfolio</td>
<td>-4.3310</td>
<td>-1.1440</td>
<td>-0.5683</td>
<td>-0.6146</td>
<td>-0.0330</td>
<td>2.2011</td>
</tr>
<tr>
<td>Multivariate Normal Universal Portfolio</td>
<td>-4.1650</td>
<td>-1.1000</td>
<td>-0.5499</td>
<td>-0.5872</td>
<td>-0.0290</td>
<td>2.0891</td>
</tr>
<tr>
<td>Brownian Motion Universal Portfolio</td>
<td>-3.7120</td>
<td>-0.9731</td>
<td>-0.4649</td>
<td>-0.4973</td>
<td>0.0129</td>
<td>1.9970</td>
</tr>
<tr>
<td>OU Process Universal Portfolio</td>
<td>-4.3511</td>
<td>-1.1370</td>
<td>-0.5718</td>
<td>-0.6101</td>
<td>-0.0360</td>
<td>2.2176</td>
</tr>
</tbody>
</table>

\[ \log(S_n(CRP)) \]

\[ \log(KLCIRatio) \]

Next, we analyse the results obtained for middle term period. From Table 6.9 and Table 6.10, we again obtained the median and mean with negative values for the four universal portfolios strategies. This indicate that the four proposed universal portfolio strategies still performed poorly when compare to benchmark CRP and KLCI. Furthermore, CRP still performed better than KLCI.
The analysis of results obtained for long term periods are showed in Table 6.11 and Table 6.12. Both tables still showed the negative values of mean and median. Hence, we concluded that the CRP is performed well compared to the four proposed universal portfolio strategies for long term period.

In order to view the result graphically, the following histograms for short term period, middle-term period and long term period analysis are plotted.
From the Figure 6.8, the four graphs are skew to the left, these concluded that the four universal portfolio strategies are not performed good when compare to CRP strategy. There are only around 25% of the results showed the four universal portfolio strategies can beat the CRP strategy in short term investment.
Figure 6.9: Middle Term Period Data Analysis (4 Years Period) with Redefined the Price Relative
Figure 6.10: Middle Term Period Analysis (8 Years Period) with Redefined the Price Relative

For the middle term analysis, Figure 6.9 and Figure 6.10 allow us to make the following conclusion, all the histograms are skew to the left even they are close in the shape. The CRP strategy is consistently beating the four propose universal portfolio strategies. The histograms also show that around 25% of the results indicated the four universal portfolio strategies can perform better than CRP strategy in middle term investment.
Figure 6.11: Long Term Period Data Analysis (12 Years Period) with Redefined the Price Relative
For the long term investment, From Figure 6.11 to Figure 6.12, again the histograms showed the results obtained still skew to the left. This led us to conclude that the four universal portfolio strategies are still not perform better than CRP strategy in long term period.

In summary, in order to compare to KLCI, when the price relatives for a given stock changed to ratio of the closing price on the current trading day to the closing
price on the day before, the analysis of the results obtained conclude that CRP strategies are outperform the four propose universal portfolio strategies in short term, middle term and long term period investment.

6.4 Summary and Thoughts on Performance Analysis

The numerical experiment comparison of the achieved wealth by four different universal portfolio strategies against the constant rebalanced portfolio are studied. The empirical performances of the four portfolio strategies are investigated through a set of numerical experiments concerning 95 stocks data selected from Kuala Lumpur Stock Exchange.

So far, the research of universal portfolios in Malaysia was focused on determining the best performing parameters for \( m \) stocks. In the first section of this chapter, The result obtained emphasise the proposed investment strategy performed better than the constant rebalanced portfolio for short term. Also, the performance of the four proposed universal portfolio strategies performed more or less the same as the CRP. We believe that most parameters of these four order one universal portfolio strategies for 3 stocks computation will perform more or less the same. This may due to the nature of the joint distribution of the random variables \( Y_1, Y_2, Y_3 \). The four universal portfolio strategies are performed well comparing to CRP for short term. However, they are found performed terribly when trading in middle to long term.
In second section, in order to compare with benchmark KLCI, noticed that the four universal portfolio strategies did not perform well for short term, middle term and long term period. However, CRP is consistently performed good throughout the three terms.

As a conclusion, CRP is better and more stable strategy. In contrast, four universal portfolios may be able to achieve better return in wealth some of the time at the larger risks.
CHAPTER 7

CONCLUSION

The goal of our research is to study how the finite order universal portfolio strategies perform for a fund which invest on 3 out of 95 Malaysian stocks. Since universal portfolio strategies are comparable to BCRP, we will use CRP as a benchmark for comparison. The study was limited to four proposed universal portfolio strategies, the finite order Multinomial Universal Portfolio, the finite order Multivariate Normal universal portfolio, universal portfolio generated by Brownian-motion and universal portfolio generated by Ornstein-Uhlenbeck processes.

After collecting and compiling the 95 Malaysian stock data, I proceed to propose a unified framework for the research carried out by Tan (2004b), Tan et al. (2012), Tan & Lim (2011) and Tan (2013) research group. Using the basic probability theory, I have formalised the finite order universal portfolio. This type of universal portfolio depends only on the positive moments of the generating probability distribution. The moments of the proposed universal portfolio are computed for low order, i.e., order one, two and three. The price relatives used for a given stock is the ratio of closing price to the opening price on the same trading day.

Based on the formulas derived from the finite order portfolios, we have implemented them in Python programming language to test how well they perform. Cover have shown both mathematically and numerically that universal portfolio performed well. To show that the finite order universal portfolio perform well, they are compared against Cover’s Dirichlet \((\alpha_1, \alpha_2, \alpha_3)\) universal portfolio strat-
egy. I used the modified algorithm of Dirichlet \((\alpha_1, \alpha_2, \alpha_3)\) universal portfolio obtained by Tan (2004b) to perform computation on three stock data selected from Kuala Lumpur Stock Exchange. The finite order universal portfolio performed as good as Cover’s universal portfolio, but finite order universal portfolio is computationally better in speed and memory.

An experimental study of the above four universal portfolios strategies was conducted for the 10 most active stocks data among the 95 selected stocks from Kuala Lumpur Stock Exchange. The trading period is from 1 Jan 2000 to 31 Dec 2015. From the first section in Chapter 5, the proposed four finite order universal portfolios strategies are found can perform better than the Cover Ordentlich Dirichlet universal portfolio for same data sets. Noticed that the four universal portfolios with order one performed as well as order two and three. Therefore, this research will be limited to order one universal portfolio.

The above four order one universal portfolios are run on 3 stock-price data sets generated by the combination of the 10 selected most active stocks from the 95 selected stocks. Selection of good parameters for each proposed universal portfolios are carry out. We use CRP as a benchmark to evaluate and to compare with these four universal portfolio strategies. A comparison can be made with the wealth obtained by CRP and the wealths achieved by the four proposed universal portfolios. Every one year period starting from year 2000 to year 2015 of the available stock data are used for study. The best performing parameter of all four classes of universal portfolio strategies were identified from the box-plot plotted.

Next, we studied the performances of four proposed universal portfolio strategies with their best performance parameters identified in previous study. Numer-
ical experiment for the four classes of portfolio strategies are studied for short term (1 year period), middle term (4 years and 8 years periods) and long term (12 years and 16 years periods) data. We employed the four portfolio strategies on the above five groups of data to learn the performance of these methods and used CRP as a benchmark. Every 3 stocks data generated by combination of the selected of 95 stocks are chosen for study. The empirical results showed that the performances of the proposed four strategies are better than CRP in 1 year and 4 years period. However, the four strategies did poorly in 8 years, 12 years and 16 years period. Therefore, these four universal portfolio strategies are confirmed to be an extremely competitive investment strategies in short term period.

Lastly, we try to compare the performances of four proposed strategies to KLCI. We changed the price relatives for a given stock as the ratio of closing price on the current trading day to the closing price on the day before. After the investigation and analysis of the results obtained, we found that the four proposed universal portfolio strategies are not perform good compared to benchmark CRP and KLCI in short term, middle term and long term period. Furthermore, CRP is consistently performed good when compare to four proposed universal portfolio strategies and KLCI in short, middle and long term period.

In conclusion, this research has simplified the formulation of $\mu$-weighted universal portfolio Cover & Ordentlich (1996) and order $\nu$ universal portfolio Tan (2013) using probability theory and further extend to probability generated order $\nu$ universal portfolio to stochastic process based universal portfolio. The order $\nu$ universal portfolio has much better speed and memory performance as well as easy to implement in a high level programming language as illustrated in this research. The four order one universal portfolio strategies are found to perform better than CRP in the short term when used the price relatives mentioned in first
section of Chapter 6. Therefore, these four universal portfolio strategies are confirmed to be an extremely competitive investment strategies in short term period.

Stochastic processes based universal portfolio is a good generalisation order $\nu$ universal portfolio which is believed be able to perform well with the right stochastic processes. However, the derivation of the algorithm will be more complicated. Malaysian stock data are used in this research and we believe that the order $\nu$ universal portfolio may help to promote active online trading for Malaysia stock market, as well as to have more investment opportunities for Malaysian but with the right combination of stocks.

7.1 Further Research

So far, the research of universal portfolios in Malaysia was focused on determining the best performing parameters for $m \leq 10$ stocks. In analysing the performance of the four strategies, we find that they perform more or less the same as the CRP. We believe that most of the parameters for these four order one strategies with 3 stocks study will perform more or less the same. This may due to the nature of the joint distribution of the random variables $Y_1, Y_2, Y_3$.

The four universal portfolio strategies are found to perform better than CRP for short term in first section of Chapter 6. However, they are found perform poorly when trading in middle to long term. Therefore, further study for the causes of the poor performance of the proposed four strategies in middle and long term are suggest to carry out for future.
Furthermore, the four universal portfolio strategies are found to perform poorly in short term, middle term and long term when comparing to KLCI. In this situation, suggest further study to investigate the causes of the poor performance of the proposed four strategies in the future.

The result also showed that the influences of parameters of the strategies to the wealth obtained. Therefore, it is important to understand whether different values of the parameters of the universal portfolio strategies will demonstrate similar pattern. It is also important to try the order one universal portfolios on the United State and European Stocks to see if the performance with respect to CRP are similar.

The proposed four order one universal portfolio strategies did not perform well with some possible combinations of Malaysian stocks. I believe if we choose the right combination of stocks will lead to the four strategies perform as well as CRP strategy. Also, it is interesting to see if higher finite order universal portfolio strategies of mixing strategies approach will reduce the occurrence of outliers in long term.
REFERENCES


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APPENDIX A

SET A, SET B AND SET C DATASETS

Three sets of stocks data shown in the following figures were selected randomly from KLSE (Kuala Lumpur Stock Exchange) and Yahoo Finance Yahoo (n.d.) for the period from 1 January, 2003 to 30 November, 2004, consisting of 500 trading days. Each set consists of 3 company stocks. Set A consists of the stocks of Malayan Banking, Genting and Amway(M) Holdings. Set B consists of the stocks of Public Bank, Sunrise and YTL Corporation. Lastly, set C consists of the stocks of Hong Leong Bank, RHB Capital and YTL Corporation.

There is a gap shown in the graph of Public Bank and Sunrise. This may cause by the stock splits or bonus issue was announced by the respective company. When a gap in a stock occurs, it means the stock’s price has either increased or decreased sharply with no trading occurring in between. On a chart, this abrupt price movement will form an empty space or break between the prices.
APPENDIX B

TEN MOST ACTIVE STOCKS

The following figures showed the 10 most active and long-term (having at least 10 years being listed in the stock exchange) stocks from each categories among the 95 selected stocks.

There is a gap shown in the graph of Bursa Malaysia Bhd, Genting and Time Dotcom. This may cause by the stock splits or bonus issue was announced by the respective company. When a gap in a stock occurs, it means the stock’s price has either increased or decreased sharply with no trading occurring in between. On a chart, this abrupt price movement will form an empty space or break between the prices.

Stock Prices of Bursa Malaysia Bhd (Finance)

Stock Prices of IJM Plantation Bhd (Plantation)
APPENDIX C

WELL PERFORMING PARAMETERS

For selection of well performing parameters for each universal portfolios, we use CRP as a benchmark to evaluate and to compare with these four universal portfolio strategies. A comparison can be made with the wealth obtained by CRP and the wealth achieved by the four proposed universal portfolios. Every one year period starting from year 2000 to year 2015 of the available stock data listed in Table 2. At least 20% of the wealth achieved by proposed universal portfolio performed better than wealth obtained by CRP are analysed. The good performance of the parameters of each universal portfolio with average and standard deviation is observed and listed in the following table.
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APPENDIX D

PUBLICATIONS RELATED TO THIS RESEARCH


