A STUDY ON LEBESGUE MEASURE AND INTEGRATION

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DECLARATION

I sincerely declare that:

1. I am the sole writer of this report
2. All the information contained in this report is certain and correct to the knowledge of the author

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ABSTRACT

The Fundamental Theorem of Calculus reveals a significant ralationship between integration and differentiation. The functions involved are continuous. However, the following version of the Fundamental Theorem of Calculus is valid.

If F is differentiable on $[a,b]$ and if F' is Riemann integrable on $[a,b]$, then $\int_a^x F' = F(x) - F(a)$ for each $x \in [a,b]$.

Are all derivatives Riemann integrable? A brief search leads one to derivatives that are not bounded and, as a result, not Riemann integrable. However, there are even bounded derivatives existing at all points that are not Riemann integrable. Thus, the hypothesis that the derivative is Riemann integrable is essential.

The Lebesgue integral was designed to overcome the deficiencies of the Riemann integral. Are all derivatives Lebesgue integrable? The answer is no. However, all bounded derivatives are Lebesgue integrable so that the following version of the Fundamental Theorem of Calculus is valid.

If F is differentiable on $[a,b]$ and if F' is bounded on $[a,b]$, then F' is Lebesgue integrable on $[a,b]$ and $\int_a^x F' = F(x) - F(a)$ for each $x \in [a,b]$.

This discussion leads naturally to the following question. Is it possible to define an integration process for which the theorem

If F is differentiable on $[a,b]$, then the function F' is integrable on $[a,b]$ and

$\int_a^x F' = F(x) - F(a)$ for each $x \in [a,b]$.

is valid? The answer is yes.

In 20^{th} century, three integration processes have been developed for which this version of the Fundamental Theorem of Calculus is valid. These integrals, named after their principal investigators Denjoy, Perron, and Henstock, each generalize some aspect of the Lebesgue integral. Since each of these new integrals focuses on a different property of the Lebesgue integral, the definitions of the integrals are radically different. However, it turns out that all three integrals are equivalent.

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CHAPTER 1

INTRODUCTION

1.1 Problem Statement

By the end of the 19th century, some inadequacies in the Riemann theory of integration had become apparent. These failings came primarily from the fact that the collection of Riemann integrable functions became inconveniently small as mathematics developed.

For example, the set of functions for which the Newton-Leibniz formula:

$$\int_a^b F' = F(b) - F(a)$$

holds, does not include *all* differentiable functions. These inadequacies led others to invent other integration theories, the best known of which was due to Henri Lebesgue (1875-1941) and was developed in 1902.

1.2 Background Review

The idea of the Lebesgue integral is to enlarge the class of integrable functions so that $\int_a^b f(x)dx$ will be given a meaning for functions f that are not Riemann integrable. For example, let

$$f(x)=\begin{cases}1, & x \ rational,\ 0\le x\le 1\\ 0, & x \ irrational,\ 0\le x\le 1\end{cases},$$

$$\underline{\int_0^1} f(x)dx=\sup\left\{\int_0^1 \phi(x)dx \mid \phi\le f, \phi \ a \ step \ function\right\}$$
$$=0,$$

and

$$\overline{\int_0^1} f(x)dx=\inf\left\{\int_0^1 \phi(x)dx \mid f\le \phi, \phi \ a \ step \ function\right\}$$
$$=1.$$

Thus f is not Riemann integrable. However it is trivally Lebesgue integrable. f is a simple function and

$$\int_{[0,1]} f = 1\cdot\mu(\{\text{rationals}\}\cap[0,1])+0\cdot\mu(\{\text{irrationals}\}\cap[0,1])$$
$$=1\cdot 0+0\cdot 1$$
$$=0.$$

For functions that are Riemann integrable, the Lebesgue theory will assign the same numerical value to $\int_a^b f(x)dx$ as the Riemann theory.

Thus the Lebesgue integration theory can be thought of as a kind of completion of the Riemann integration theory. This can be given a precise sense in terms of the metric $d(f,g)=\int_a^b |f(x)-g(x)|dx$ on the continuous functions $C([a,b])$ so that the Lebesgue integrable functions are obtainable from the continuous functions by the same process as the real numbers are obtained from the rational numbers. However, it is best if we observe this fact after we have developed the Lebesgue theory in a more concrete way.

Indeed, the Lebesgue theory of integration has become pre-eminent in contemporary mathematical research, since it enables one to integrate a much larger collection of functions, and to take limits of integrals more freely.

The Lebesgue theory allows us to say that the sum of an absolutely convergent series is a form of integration, and this conceptual framework allows us also to give a foundation to probability theory.

1.3 Motivation/Significance of Study

Before beginning on the rather difficult path of developing the Lebesgue theory we will recall some of the weak points of the Riemann theory that can serve as motivation for seeking a better theory.

Firstly, the Riemann integral does not have satisfactory limit properties. That is, given a sequence of Riemann integrable functions $\{f_n\}$ with a limit function $f = \lim_{n\to\infty} f_n$, it does not necessarily follow that the limit function f is Riemann integrable.

Secondly, the Riemann theory of integration is the lack of a good convergence theorem. We have seen that the Riemann integral can be interchanged with a uniform limit, but in many applications this is not adequate. For example, with Fourier series we frequently do not have uniform convergence, even if the function is continuous. Of course even in the Lebesgue theory we will not be able to interchange all limits with integration. For example, if

$$f_n(x) = \begin{cases} n & \textit{if } 0 < x < 1/n, \\ 0 & \textit{otherwise}, \end{cases}$$

then $\int_0^1 f_n(x)dx = 1$ but $\lim_{n\to\infty} f_n(x) = 0$ at every point, so

$$\int_0^1 \lim_{n\to\infty} f_n(x)dx = 0 \neq 1 = \lim_{n\to\infty}\int_0^1 f_n(x)dx.$$

Nevertheless we will find two rather useful criterion for interchanging limits and integrals-the monotone convergence theorem and the dominated convergence theorem.

Thirdly, improper integrals have to be treated separately in Riemann theory. In the Lebesgue theory we will be able to treat absolutely convergent improper integrals on the same footing as proper integrals.

Fourthly, we have no reasonable criterion for deciding whether or not a function is Riemann integrable. Riemann did in fact give such a criterion, but it seems no easier to apply than to verify the definition of the Riemann integral. With the aid of the Lebesgue theory it is possible to give a criterion for the Riemann integral to exist although it must be admitted that we do not have a very good criterion for the Lebesgue integral to exist.

Finally a fifth weakness involves the theory of multiple integrals. We have postponed discussing multiple integrals until after the Lebesgue theory because the Riemann theory yields only very awkward and incomplete results.

In addition to overcoming these weaknesses, the Lebesgue theory allows a very far reaching and fruitful generalization of the concept of integration.

1.4 Objectives

The Lebesgue integral is founded on Henri Lebesgue's theory of measure in 1902. The idea of measure theory is that we want to assign a length to each subset of the real numbers. Unfortunately, this is impossible to do in a logically consistent fashion.

So measure theory tells us how to pick out which sets we can measure and how to measure them.

Lebesgue chose to partition the range rather than partitioning the domain of the function, as in the Riemann integral. Thus, for each interval in the partition, rather than asking for the value of the function between the end points of the interval in the domain, he asked how much of the domain is mapped by the function to some value between two end points in the range.

Partitioning the range of a function and counting the resultant rectangles becomes tricky since we must employ some way of determining (or measuring) how much of the domain is sent to a particular portion of a partition of the range. Measure theory addresses just this problem.

As it turns out, the Lebesgue integral solves many of the problems left by the Riemann integral.

1.5 Scope of Study

Lebesgue measure is studied in chapter three. It includes Lebesgue outer measure, Carathéodory's measurability criteria, Lebesgue measurable sets, Borel sets, the structure of Lebesgue measurable set, and an example of a Lebesgue nonmeasurable set.

In chapter four, we will look into Lebesgue measurable functions, sequences of measurable functions, approximating measurable functions, almost uniform convergence.

In chapter five, we will study Lebesgue integration, Riemann integral, Lebesgue integral for bounded functions of sets of finite measure, the Lebesgue integral for nonnegative measurable functions, the Lebesgue integral and Lebesgue integrability, and two convergence theorems.

Chapter six is the conclusion and future work.

CHAPTER 2

LITERATURE REVIEW

2.1 Lebesgue's Motivation

The span from Newton and Leibniz to Lebesgue covers only 250 years. Lebesgue published his dissertation "Intégrale, longueur, aire" ("Integral, length, area") in the *Annali di Matematica* in 1902. Lebesgue developed "measure of a set" in the first chapter and an integral based on his measure in the second.

Part of Lebesgue's motivation was two problems that had arisen with Riemann's integral. First, there were functions for which the integral of the derivative does not recover the original function and others for which the derivative of the integral is not the original. Second, the integral of the limit of a sequence of functions was not necessarily the limit of the integrals. We have seen that uniform convergence allows the interchange of limit and integral, but there are sequences that do not converge uniformly yet the limit of the integrals is equal to the integral of the limit.

Lebesgue was able to combine Darboux's work on defining the Riemann integral with Borel's research on the "content" of a set. Darboux was interested in the interplay of the definition of integral with discontinuous functions and in the convergence problems. Borel (who was Lebesgue's thesis advisor) needed to describe the size of sets of points on which a series converged; he expanded on Jordan's definition of the content of a set which itself was an expansion of Peano's definition of content measuring the size of a set. Peano's work was motivated by

Hankel's attempts to describe the size of the set of discontinuities of a Riemann integrable function and by an attempt to define integration analytically, as opposed to geometrically. Rarely, if ever, is revolutionary mathematics done in isolation.

Another problem also provided primary motivation for Lebesgue: the question of convergence and integrating series term by term. Newton had used series expansions cleverly to integrate functions when developing calculus. Fourier thought it was always valid to integrate a trigonometric series representation of a function term by term. Cauchy believed continuity of the terms sufficed; Cauchy's integral required continuity to exist. Then Abel gave an example that did not work. Weierstrass recognized that uniform convergence was the key to term-by-term integration. Dirichlet developed wildly discontinuous counterexamples. Riemann defined his integral so as not to require continuity, but uniform convergence of the series was still necessary for term-by-term integration. However, some non-uniformly convergent series could still be integrated term by term. What is the right condition? Lebesgue's theory can answer these questions.

2.2 The Contribution of Lebesgue

The Lebesgue integral is a generalization of the integral introduced by Riemann in 1854. As Riemann's theory of integration was developed during the 1870's and 1880's, a measure-theoretic viewpoint was gradually introduced. This viewpoint was made especially prominent in Camille Jordan's treatment of the integral in his *Cours d' analyse* (1893) and strongly influenced Lebesgue's outlook on these matters. The significance of placing integration theory within a measure-theoretic context was that it made it possible to see that a generalization of the notions of the integral and integrability. In 1898, Émile Borel was led through his work on complex function theory to introduce radically different notions of measure and measurability. Some mathematicians found Borel's ideas lacking in appeal and relevance, especially since they involved assigning measure zero to some dense sets. Lebesgue, however, accepted them. He completed Borel's definitions of measure and measurability so that they represented generalizations of Jordan's definitions and then used them to obtain his generalization of the Riemann integral.

After the work of Jordan and Borel, Lebesgue's generalizations were somewhat inevitable. Thus, W.H. Young and G.Vitali, independently of Lebesgue and of each other, introduced the same generalization of Jordan's theory of measure; in Young's case, it led to a generalization of the integral that was essentially the same as Lebesgue's. In Lebesgue's work, however, the generalized definition of the integral was simply the starting point of his contributions to integration theory. What made the new definition important was that Lebesgue was able to recognize in it an analytical tool capable of dealing with-and to a large extent overcoming-the numerous theoretical difficulties that had arisen in connection with Riemann's theory of integration. In fact, the problems posed by these difficulties motivated all of Lebesgue's major results.

One of the difficulties was the fundamental theorem of calculus,

$$\int_a^b f'(x)dx = f(b) - f(a).$$

The work of Dini and Volterra in the period 1878-1881 made it clear that functions exist which have bounded derivatives that are not integrable in Riemann's sense, so that the fundamental theorem becomes meaningless for these functions. Later further classes of functions were discovered; and additional problems arose in connection with Harnack's extension of the Riemann integral to unbounded functions because continuous functions with densely distributed intervals of invariability were discovered. These functions provided examples of Harnack-integrable derivatives for which the fundamental theorem is false. Lebesgue showed that for bounded derivatives these difficulties disappear entirely when integrals are taken in his sense. He also showed that the fundamental theorem is true for an unbounded, finite-valued derivative f' that is Lebesgue-integrable and this is the case if, and only if, f is of bounded variation.

Riemann's definition of the integral also raised problems in connection with the traditional theorem positing the identity of double and iterated integrals of a function of two variables. Examples were discovered for which the theorem fails to hold. As a result, the traditional formulation of the theorem had to be modified, and the modifications became drastic when Riemann's definition was extended to unbounded functions. Although Lebesgue himself did not resolve this infelicity, it was his treatment of the problem that formed the foundation for Fubini's proof (1907) that the Lebesgue integral does not make it possible to restore to the theorem the simplicity of its traditional formulation.

CHAPTER 3

LEBESGUE MEASURE

Lebesgue measure is studied in this chapter. We state the necessary eight properties of measure in the first section. Next, we define Lebesgue outer measure and list down its eight properties follow by the proofs. In the third section, Lebesgue measurable sets are defined by Carathéodory's measurability criteria. Also, the collection of sets that satisfy the criteria forms a σ-algebra and the Lebesgue outer measure is countably additive on this σ-algebra. Next, Borel sets and Borel σ-algebra are introduced follow by the properties of Lebesgue measure. In Section 3.5, we look into the structure of Lebesgue measurable sets of real numbers. Finally, we conclude this chapter with an example of Lebesgue nonmeasurable set.

Caution: In what follows a "measurable set" means a "Lebesgue measurable set of real numbers".

3.1 Properties of Measure

Lebesgue measure is an extended real-valued set function, a function from a collection of sets into $[0,\infty]$. Measure is based on the lengths of open intervals as these intervals are the basic building blocks of open sets in the reals. The best measure μ would satisfy eight properties:

1. $\mu(A)$ *is defined for every set A of real numbers (we can "measure" all sets);*

2. $0 \le \mu(A) \le \infty$ *(nonnegative extended real-valued; length is nonnegative and the* *"length" of ℝ is ∞);*

3. $\mu(A) \le \mu(B)$ *provided* $A \subset B$ *(monotonic);*
4. $\mu(\phi) = 0$*;*
5. $\mu(\{a\}) = 0$ *(points are dimensionless);*
6. $\mu(I) = l(I)$, *I an interval (the measure of an interval should be its length);*
7. $\mu(c + A) = \mu(A)$ *(translation invariance; location does not affect length, should not affect the measure);*
8. $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$ *for any mutually disjoint sequence* $\{A_k\}$ *of subsets of real numbers (countable additivity).*

3.2 Lebesgue Outer Measure

A collection of open intervals $\{I_k \mid k = 1,2,\ldots\}$ *covers* a set A if $A \subseteq \bigcup_{k=1}^{\infty} I_k$. Since the intervals are open, we call $\{I_k\}$ an *open cover* of A. Define the *length* l of the open interval $I = (a,b)$ to be $l(I) = b - a$. We combine open covers and length to measure the size of a set. Since the cover contains the set, we will call it the *outer measure*. The outer measure is extremely close to the measure of Jordan defined in 1892.

Definition 3.2.1 (Lebesgue Outer Measure) *For any set* $A \subseteq$ ℝ, *define the Lebesgue outer measure* μ^* *of* A *to be*

$$\mu^*(A) = \inf\left\{ \sum_{k=1}^{\infty} l(I_k) \mid A \subset \bigcup_{k=1}^{\infty} I_k,\, I_k \text{open intervals} \right\}$$

the infimum of the sums of the lengths of open covers of A.

μ^* has the following properties:

1. μ^* *is defined for every set of real numbers;*
2. $0 \le \mu^*(A) \le \infty$ *(nonnegative and extended real-valued);*

3. $\mu^*(A) \le \mu^*(B)$ *provided* $A \subset B$ *(monotonic);*

4. $\mu^*(\phi) = 0$*;*

5. $\mu^*(\{a\}) = 0$ *(points are dimensionless);*

6. $\mu^*(I) = l(I)$, *I an interval (the Lebesgue outer measure of an interval is its length);*

7. $\mu^*(c + A) = \mu^*(A)$ *(translation invariant);*

8. $\mu^*\left(\bigcup_{k=1}^{\infty} A_k\right) \le \sum_{k=1}^{\infty} \mu^*(A_k)$ *for any sequence of sets* $\{A_k\}$ *of real numbers (countable subadditivity).*

Proof.

3. Monotonicity, property 3, is an immediate consequence of the observation that every open cover of B will be an open cover of A.

4. and 5. Since the empty set is a subset of every set, we have

$$\phi \subset \{a\} \subset (a - \varepsilon, a + \varepsilon).$$

By monotonicity,

$$0 \le \mu^*(\phi) \le \mu^*(\{a\}) \le \mu^*((a - \varepsilon, a + \varepsilon)) \le 2\varepsilon.$$

Since this is true for arbitrary $\varepsilon > 0$,

$$\mu^*(\phi) = 0 \text{ and } \mu^*(\{a\}) = 0.$$

6. First, consider a bounded, closed interval $I = [a,b]$. For any $\varepsilon > 0$,

$$[a,b] \subset (a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}).$$

Hence, $\mu^*(I) \le b - a + \varepsilon.$ Since $\varepsilon > 0$ is arbitrary, $\mu^*([a,b]) \le b - a.$

Now, let $\{I_k\}$ be an open cover of $[a,b]$. The Heine-Borel theorem states that since $[a,b]$ is closed and bounded there is a finite subcover $\{I_k \mid k = 1,2,\ldots,N\}$ for I. Order the intervals so they overlap, starting with the first containing a and ending with the last containing b. Thus

$$\begin{aligned}\sum_{k=1}^{N} l(I_k) &= (b_1 - a_1) + (b_2 - a_2) + \cdots + (b_N - a_N) \\ &= b_N - (a_N - b_{N-1}) - (a_{N-1} - b_{N-2}) - \cdots - (a_2 - b_1) - a_1 \\ &\geq b_N - a_1 \\ &> b - a\end{aligned}$$

Thus $\mu^*(I) \geq b - a,$ which combines with the first inequality to yield $\mu^*(I) = b - a.$

Second, let l be any bounded interval and let $\varepsilon > 0.$ There is a closed interval $J \subset I$ such that $l(I) - \varepsilon < l(J).$ Then

$$l(I) - \varepsilon < l(J) = \mu^*(J) \leq \mu^*(I) \leq \mu^*(\overline{I}) = l(\overline{I}) = l(I)$$

or

$$l(I) - \varepsilon < \mu^*(I) \leq l(I).$$

Since $\varepsilon > 0$ is arbitrary, we have $\mu^*(I) = l(I).$

Last, if l is an infinite interval, for each $n \in \mathbb{N},$ there is a closed interval $J \subset I$ with $\mu^*(J) = n.$ Then $n = \mu^*(J) \leq \mu^*(I)$ implies that $\mu^*(I) = \infty.$

7. Translation invariance, property 7, is based on the fact that length, $l,$ is translation invariant: If $I = (a,b),$ then $c + I = (a + c, b + c),$ and $l(I) = l(c + I).$ If I is $(b,\infty), (-\infty,a),$ or $(-\infty,+\infty),$ then $c + I$ is $(b + c,\infty), (-\infty, a + c),$ or $(-\infty,+\infty),$ respectively, and again $l(I) = l(c + I).$ If A is an arbitrary subset of $\mathbb{R}$ with $A \subset \bigcup_{k=1}^{\infty} I_k,$ then

$$c + A \subset \bigcup_{k=1}^{\infty} (c + I_k),$$

and

$$\mu^*(c + A) \leq \sum_{k=1}^{\infty} l(c + I_k) = \sum_{k=1}^{\infty} l(I_k).$$

This tells us that $\mu^*(c + A)$ is a lower bound for the "lengths" of covers of $A,$ and because $\mu^*(A)$ is the greatest lower bound of such numbers,

$$\mu^*(c + A) \leq \mu^*(A).$$

By starting with a cover $\{J_k\}$ of $c + A,$ we have $A \subset \bigcup_{k=1}^{\infty} (J_k - c),$

and so

$$\mu^*(A) \le \sum_{k=1}^{\infty} l(J_k - c)$$
$$= \sum_{k=1}^{\infty} l(J_k).$$

This tells us that $\mu^*(A)$ is a lower bound for the "lengths" of covers of $c + A$, and because $\mu^*(c+A)$ is the greatest lower bound of such numbers,

$$\mu^*(A) \le \mu^*(c+A).$$

We conclude $\mu^*(c+A) = \mu^*(A)$.

8. We must show $\mu^*\left(\bigcup_{k=1}^{\infty} A_k\right) \le \sum_{k=1}^{\infty} \mu^*(A_k)$ for any sequence of sets of real numbers. Of course if the series $\sum_{k=1}^{\infty} \mu^*(A_k)$ diverges the argument is immediate, so assume $\sum_{k=1}^{\infty} \mu^*(A_k) < \infty$ and let $\varepsilon > 0$. For each nonempty A_k, choose an open cover $\{I_{kn}\}$ so that

$$A_k \subset \bigcup_{n=1}^{\infty} I_{kn}$$

and

$$\mu^*(A_k) \le \sum_{n=1}^{\infty} l(I_{kn}) < \mu^*(A_k) + \frac{\varepsilon}{2^k}.$$

We may do this by the definition of greatest lower bound. The collection

$$\begin{aligned} \{&I_{11}, I_{12}, \ldots, I_{1n}, \ldots; \\ &I_{21}, I_{22}, \ldots, I_{2n}, \ldots; \\ &\vdots \\ &I_{k1}, I_{k2}, \ldots, I_{kn}, \ldots\} \end{aligned}$$

is a countable collection of open intervals that cover the set

$$\bigcup_{k=1}^{\infty} A_k :$$

$$A_1 \subset \bigcup_{n=1}^{\infty} I_{1n}, A_2 \subset \bigcup_{n=1}^{\infty} I_{2n}, \ldots, A_k \subset \bigcup_{n=1}^{\infty} I_{kn}$$

and

$$\bigcup_{k=1}^{\infty} A_k \subset \bigcup_{k=1}^{\infty}\left(\bigcup_{n=1}^{\infty} I_{kn}\right).$$

$$\begin{aligned}
\bigcup_{k=1}^{\infty}\left(\bigcup_{n=1}^{\infty} I_{kn}\right) = \quad & I_{11} \cup I_{12} \cup \cdots \cup I_{1n} \cup \cdots \\
\cup \quad & \\
& I_{21} \cup I_{22} \cup \cdots \cup I_{2n} \cup \cdots \\
\cup \quad & \\
& \cdots \\
\cup \quad & \\
& I_{k1} \cup I_{k2} \cup \cdots \cup I_{kn} \cup \cdots \\
\cup \quad & \\
& \cdots \\
= \quad & I_{11} \cup I_{21} \cup I_{12} \cup I_{31} \cup I_{22} \cup I_{13} \cup \cdots \\
= \quad & \bigcup_{i=1}^{\infty} I_{\phi(i)}.
\end{aligned}$$

Since $\bigcup_{i=1}^{\infty} I_{\phi(i)}$ is an open cover of the set $\bigcup_{k=1}^{\infty} A_k$,

$$\begin{aligned}
\mu^*\left(\bigcup_{k=1}^{\infty} A_k\right) &\le \sum_{i=1}^{\infty} l(I_{\phi(i)}) \\
&= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} l(I_{\phi(i)}) \\
&< \sum_{k=1}^{\infty}\left(\mu^*(A_k) + \frac{\varepsilon}{2^k}\right) \\
&= \sum_{k=1}^{\infty} \mu^*(A_k) + \varepsilon.
\end{aligned}$$

We conclude $\mu^*\left(\bigcup_{k=1}^{\infty} A_k\right) \le \sum_{k=1}^{\infty} \mu^*(A_k).$ □

Theorem 3.2.2 *The outer measure of a countable set is zero.*

Proof. Let A be any countable set of real numbers. Since A is countable, so we can enumerate $A = \{a_1, a_2, \ldots\}$.

Let $\varepsilon > 0$. Now

$$I_1 = (a_1 - \frac{\varepsilon}{2}, a_1 + \frac{\varepsilon}{2})$$
$$I_2 = (a_2 - \frac{\varepsilon}{4}, a_2 + \frac{\varepsilon}{4})$$
$$I_3 = (a_3 - \frac{\varepsilon}{8}, a_3 + \frac{\varepsilon}{8})$$
$$\dots$$
$$I_k = (a_k - \frac{\varepsilon}{2^k}, a_k + \frac{\varepsilon}{2^k}).$$

Then

$$A \subset \bigcup_{k=1}^{\infty} I_k.$$

So,

$$\mu^*(A) \le \mu^*(\bigcup_{k=1}^{\infty} I_k) \le \varepsilon.$$

Since this is true for arbitrary $\varepsilon > 0$, we conclude that $\mu^*(A) = 0$. □

As previously indicated, we can only have a workable measure theory if we restrict attention to a class of reasonable sets. This class should be closed under countable intersection and countable union. In fact, we formalize this idea in next section.

3.3 Lebesgue Measurable Sets

In 1914, Carathéodory formulated a measurability criteria.

Definition 3.3.1 (Carathéodory's Condition) *A set E is Lebesgue measurable iff for every set $X \subseteq \mathbb{R}$, we have*

$$\mu^*(X) = \mu^*(X \cap E) + \mu^*(X \cap E^c).$$

Let M be the family of all Lebesgue measurable sets.

Informally, a set is measurable if it splits every other set into two pieces with measures that add correctly. The definition of measurable is symmetric: if E is measurable, so is E^c; i.e., if $E \in M$, then $E^c \in M$. Also, it is easily seen that ϕ and $\mathbb{R} \in M$.

In 1915, Giuseppe Vitali gave the first example of a Lebesgue nonmeasurable set of real numbers. In Section 3.6, we will have a fuller discussion.

Theorem 3.3.2 *If* $\mu^*(E)=0,$ *then* E *is measurable.*

Proof. For any set X, it is true that

$$\mu^*(X) = \mu^*((X \cap E) \cup (X \cap E^c)) \leq \mu^*(X \cap E) + \mu^*(X \cap E^c).$$

Since $X \cap E \subseteq E$, we see that $\mu^*(X \cap E) \leq \mu^*(E) = 0.$ Thus $\mu^*(X \cap E) = 0.$ Now note that $X \cap E^c \subseteq X$, so $\mu^*(X \cap E^c) \leq \mu^*(X).$

Hence,

$$\mu^*(X) \leq \mu^*(X \cap E) + \mu^*(X \cap E^c) \leq \mu^*(X).$$

Thus $E \in M$. □

But we certainly cannot base an integration theory on the collection $\{\phi, \mathbb{R}\}$. It is time to define a sigma algebra (σ-algebra) and investigate why they are so important.

Definition 3.3.3 *In a space* Ω*, a collection* $\mathcal{O}$ *of subsets of* Ω *is said to be a σ-algebra, provided:*

1. $\phi \in \mathcal{O}$;

2. If $A \in \mathcal{O}$*, then* $\Omega \cap A^c \in \mathcal{O}$;

3. If $\{A_k\}$ *is a sequence of sets in* $\mathcal{O}$*, then* $\bigcup_{k=1}^{\infty} A_k \in \mathcal{O}$.

Lemma 3.3.4 *The intersection of two Lebesgue measurable sets is measurable.*

Proof. Let E_1, E_2 be Lebesgue measurable sets. We must show $E_1 \cap E_2 \in M$.

Since for any set X,

$$X \cap \mathbb{R} = X$$

$$X \bigcap ((E_1 \cap E_2) \bigcup (E_1 \cap E_2)^c) = X$$

$$(X \bigcap (E_1 \cap E_2)) \bigcup (X \bigcap (E_1 \cap E_2)^c) = X,$$

and the Lebesgue outer measure is subadditive,

$$\mu^*(X) \le \mu^*(X \bigcap (E_1 \cap E_2)) + \mu^*(X \bigcap (E_1 \cap E_2)^c).$$

It is sufficient to show

$$\mu^*(X \bigcap (E_1 \cap E_2)) + \mu^*(X \bigcap (E_1 \cap E_2)^c) \le \mu^*(X) < \infty.$$

Measurability of E_2 implies that

$$\begin{aligned} \mu^*(X \bigcap (E_1 \cap E_2)) &= \mu^*((X \cap E_1) \bigcap E_2) \\ &= \mu^*(X \cap E_1) - \mu^*((X \cap E_1) \bigcap E_2^c). \end{aligned}$$

Since

$$\begin{aligned} X \bigcap (E_1 \cap E_2)^c &= X \bigcap (E_1^c \bigcup E_2^c) \\ &= X \bigcap ((E_1^c \bigcup E_1) \bigcap (E_1^c \bigcup E_2^c)) \\ &= X \bigcap (E_1^c \bigcup (E_1 \cap E_2^c)) \\ &= (X \cap E_1^c) \bigcup ((X \cap E_1) \bigcap E_2^c), \end{aligned}$$

and μ^* is subadditive,

$$\mu^*(X \bigcap (E_1 \cap E_2)^c) \le \mu^*(X \cap E_1^c) + \mu^*((X \cap E_1) \bigcap E_2^c).$$

Adding,

$$\begin{aligned} \mu^*(X \bigcap (E_1 \cap E_2)) + \mu^*(X \bigcap (E_1 \cap E_2)^c) &\le \mu^*(X \cap E_1) + \mu^*(X \cap E_1^c) \\ &= \mu^*(X), \end{aligned}$$

where the last equality follows from Lebesgue measurability of E_1.

Thus, $E_1, E_2 \in M$ implies $E_1 \cap E_2 \in M$. □

Lemma 3.3.5 *The union of two Lebesgue measurable sets is measurable.*

Proof. Let E_1, E_2 be Lebesgue measurable sets. We must show $E_1 \bigcup E_2 \in M$.

$$E_1, E_2 \in M \text{ implies } E_1^c, E_2^c \in M.$$

By Lemma 3.3.4,

$$E_1^c \bigcap E_2^c \in M$$

implies

$$(E_1^c \bigcap E_2^c)^c \in M,$$

which implies $E_1 \bigcup E_2 \in M$. □

Finite intersections and unions follow by induction.

Theorem 3.3.6 (Carathéodory, 1918) *The collection of sets* $E \subseteq \mathbb{R}$, *that satisfy Carathéodory's condition;*

$$\mu^*(X) = \mu^*(X \cap E) + \mu^*(X \cap E^c)$$

for every subset X *of* $\mathbb{R}$, *forms a σ-algebra,* M.

Proof. We show that the collection of Lebesgue measurable sets M is a σ-algebra. This entails three arguments:

i. The empty set is Lebesgue measurable: $\phi \in M$.

$$\mu^*(X \cap \phi) + \mu^*(X \cap \phi^c) = \mu^*(\phi) + \mu^*(X) = \mu^*(X).$$

ii. If a set is Lebesgue measurable, then its complement is Lebesgue measurable: If $E \in M$, then $\mathbb{R} \cap E^c \in M$. Carathéodory's criteria is symmetric in E and $\mathbb{R} \cap E^c$;

$$\mu^*(X \cap E) + \mu^*(X \cap E^c) = \mu^*(X \cap (\mathbb{R} \cap E^c)^c) + \mu^*(X \cap (\mathbb{R} \cap E^c)).$$

iii. Suppose $\{E_k\}$ is a mutually disjoint sequence of sets from M. We are trying to show

$$\bigcup_{k=1}^{\infty} E_k \in M,$$

that is,

$$\mu^*(X) \geq \mu^*\left(X \cap \left[\bigcup_{k=1}^{\infty} E_k \right] \right) + \mu^*\left(X \cap \left[\bigcup_{k=1}^{\infty} E_k \right]^c \right).$$

We have shown that $\bigcup_{k=1}^{n} E_k \in M$, i.e.,

$$\mu^*(X) = \mu^*\left(X \cap \left[\bigcup_{k=1}^{n} E_k \right] \right) + \mu^*\left(X \cap \left[\bigcup_{k=1}^{n} E_k \right]^c \right).$$

Claim: $\mu^*\left(X \cap \left[\bigcup_{k=1}^{n} E_k \right] \right) = \sum_{k=1}^{n} \mu^*(X \cap E_k).$

Certainly true for $n = 1$ and we assume true for $n-1$ sets E_k. We split

$$X \cap \left[\bigcup_{k=1}^{n} E_k \right]$$

in an additive manner with $E_n \in M$.

$$\mu^*\left(X \cap \left[\bigcup_{k=1}^{n} E_k \right] \right) = \mu^*\left(\left(X \cap \left[\bigcup_{k=1}^{n} E_k \right] \right) \cap E_n \right) + \mu^*\left(\left(X \cap \left[\bigcup_{k=1}^{n} E_k \right] \right) \cap E_n{}^c \right)$$

$$= \mu^*(X \cap E_n) + \mu^*\left(X \cap \left[\bigcup_{k=1}^{n-1} E_k \right] \right) \qquad (E_k \textit{ mutually disjoint})$$

$$= \mu^*(X \cap E_n) + \sum_{k=1}^{n-1} \mu^*(X \cap E_k) \qquad (\textit{induction hypothesis})$$

$$= \sum_{k=1}^{n} \mu^*(X \cap E_k).$$

The claim is valid.

Now split X in an additive manner with $\bigcup_{k=1}^{n} E_k \in M$;

$$\mu^*(X) = \mu^*\left(X \cap \left[\bigcup_{k=1}^{n} E_k \right] \right) + \mu^*\left(X \cap \left[\bigcup_{k=1}^{n} E_k \right]^c \right).$$

$$\geq \mu^*\left(X \cap \left[\bigcup_{k=1}^{n} E_k \right] \right) + \mu^*\left(X \cap \left[\bigcup_{k=1}^{\infty} E_k \right]^c \right) \qquad (\textit{monotonicity})$$

$$= \sum_{k=1}^{n} \mu^*(X \cap E_k) + \mu^*\left(X \cap \left[\bigcup_{k=1}^{\infty} E_k \right]^c \right)$$

independent of n.

Therefore,

$$\mu^*(X) \geq \sum_{k=1}^{\infty} \mu^*(X \cap E_k) + \mu^*\left(X \cap \left[\bigcup_{k=1}^{\infty} E_k \right]^c \right)$$

$$\geq \mu^*\left(X \cap \left[\bigcup_{k=1}^{\infty} E_k \right] \right) + \mu^*\left(X \cap \left[\bigcup_{k=1}^{\infty} E_k \right]^c \right) \qquad (\textit{subadditivity}).$$

The reverse inequality follows from the subadditivity. We have completed argument for showing M is a σ-algebra of subsets of ℝ. □

Theorem 3.3.7 (Carathéodory, 1918) *The Lebesgue outer measure is countably additive on M, that is,*

$$\mu^*\left(\bigcup_{k=1}^{\infty} E_k \right) = \sum_{k=1}^{\infty} \mu^*(E_k)$$

for any mutually disjoint sequence of sets $\{E_k\}$ in M.

Proof. Let $\{E_k\}$ be a sequence of mutually disjoint sets from M. We must show

$$\mu^*\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu^*(E_k).$$

But in Theorem 3.3.6, we showed

$$\mu^*\left(X \cap \left[\bigcup_{k=1}^{n} E_k\right]\right) = \sum_{k=1}^{n} \mu^*(X \cap E_k)$$

for any $X \subset \mathbb{R}$. Replacing X with $\mathbb{R}$,

$$\mu^*\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu^*(E_k).$$

Finite additivity holds.

Then,

$$\begin{aligned}
\sum_{k=1}^{n} \mu^*(E_k) &= \mu^*(\bigcup_{k=1}^{n} E_k) \\
&\leq \mu^*(\bigcup_{k=1}^{\infty} E_k) \qquad (monotonicity) \\
&\leq \sum_{k=1}^{\infty} \mu^*(E_k) \qquad (subadditivity)
\end{aligned}$$

independent of n. Thus,

$$\mu^*\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu^*(E_k)$$

and the conclusion follows. □

We have shown that the Lebesgue outer measure μ^*, written μ when restricted to the σ-algebra M of subsets of $\mathbb{R}$ satisfying Carathéodory's condition, is countably additive on M, that is, μ is countably additive on the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$.

Definition 3.3.8 (Lebesgue Measure) *The Lebesgue measure μ is the restriction of the outer measure μ^* to the measurable sets M. That is, for $E \in M$, set $\mu(E) = \mu^*(E)$.*

We will show intervals are Lebesgue measurable in Proposition 3.3.9.

Theorem 3.3.9 *Intervals are Lebesgue measurable.*

Proof. The main idea of the proof is that for an interval I that is the union of two disjoint intervals I_1 and I_2, length, $l(I)$ is additive:

If $I = I_1 \bigcup I_2, I_1 \bigcap I_2 = \phi$, then $l(I) = l(I_1) + l(I_2)$. We must show intervals like $(a,b), (a,\infty)$, etc. satisfy Carathéodory's condition. Our argument will deal with (a,∞). Thus, we must show

$$\mu^*(X) = \mu^*(X \bigcap (a,\infty)) + \mu^*(X \bigcap (a,\infty)^c)$$

for every subset X of $\mathbb{R}$. Again, because of subadditivity, we need only show $\mu^*(X) \geq \mu^*(X \bigcap (a,\infty)) + \mu^*(X \bigcap (a,\infty)^c)$ for every subset X of $\mathbb{R}$ with $\mu^*(X) < \infty$.

By the definition of Lebesgue outer measure, we have an open cover $\bigcup_{k=1}^{\infty} I_k$ of X so that

$$\mu^*(X) \leq \sum_{k=1}^{\infty} l(I_k) < \mu^*(X) + \varepsilon.$$

Consider $I_k \bigcap (a,\infty)$ and $I_k \bigcap (a,\infty)^c$.

$I_k \bigcap (a,\infty)$ is either empty or an open interval and $X \bigcap (a,\infty) \subset \bigcup_{k=1}^{\infty} (I_k \bigcap (a,\infty))$.

$I_k \bigcap (a,\infty)^c$ is either empty or an open interval and $X \bigcap (a,\infty)^c \subset \bigcup_{k=1}^{\infty} (I_k \bigcap (a,\infty)^c)$.

Thus

$$\begin{aligned}
\mu^*(X \bigcap (a,\infty)) + \mu^*(X \bigcap (a,\infty)^c) &\leq \mu^*\left(\bigcup_{k=1}^{\infty} (I_k \bigcap (a,\infty))\right) + \mu^*\left(\bigcup_{k=1}^{\infty} (I_k \bigcap (a,\infty)^c)\right) \\
&\leq \sum_{k=1}^{\infty} l(I_k \bigcap (a,\infty)) + \sum_{k=1}^{\infty} l(I_k \bigcap (a,\infty)^c) \\
&= \sum_{k=1}^{\infty} \left[l(I_k \bigcap (a,\infty)) + l(I_k \bigcap (a,\infty)^c) \right] \\
&= \sum_{k=1}^{\infty} l(I_k) \\
&< \mu^*(X) + \varepsilon.
\end{aligned}$$

Similarly, we can show

$$\mu^*(X) = \mu^*(X \cap (-\infty, b)) + \mu^*(X \cap (-\infty, b)^c),$$

which implies $(-\infty, b)$ is measurable.

Now, $(a, \infty) \in M$ implies $(-\infty, a] \in M$.

Thus

$$(-\infty, b) \cap (-\infty, a] \in M.$$

i) If $a < b$, then $(-\infty, b) \cap (-\infty, a] = [a, b) \in M$;

ii) If $a = b$, then $(-\infty, b) \cap (-\infty, a] = \phi \in M$;

iii) If $a > b$, then $(-\infty, b) \cap (-\infty, a] = (b, a] \in M$;

and the proof is complete. □

We have a smallest σ-algebra that contains the collection of open intervals of $\mathbb{R}$. This smallest σ-algebra is called the family of Borel sets, $\mathbb{B}$.

Definition 3.3.10 *The σ-algebra generated by the collection of all open intervals of $\mathbb{R}$ is called the Borel σ-algebra $\mathbb{B}$.*

We are going to show Borel sets are Lebesgue measurable in Theorem 3.3.11.

Theorem 3.3.11 *Every Borel set of real numbers is Lebesgue measurable.*

Proof. Immediately follows from Theorem 3.3.9. □

3.4 Properties of Lebesgue Measure

Some previous results are gathered along with some new results to be proved below, that are useful in determining the measure of specific sets of real numbers.

Theorem 3.4.1 *The following sets are assumed to be Lebesgue measurable sets of real numbers:*

1. $\mu(\phi) = \mu(\{a\}) = 0.$

2. $\mu(I) = l(I)$.

3. $\mu(\textit{countable set}) = 0$.

4. $\mu(\textit{subset of a set of measure zero}) = 0$.

5. $\mu\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty} \mu(E_k)$ *with equality whenever the sequence of sets* $\{E_k\}$ *is mutually disjoint.*

6. $\mu(E_1 \cup E_2) + \mu(E_1 \cap E_2) = \mu(E_1) + \mu(E_2)$.

7. $\mu(E_1) \le \mu(E_2)$ *if* $E_1 \subset E_2$. *If in addition* $\mu(E_2) < \infty$, *then* $\mu(E_2) - \mu(E_1) = \mu(E_2 - E_1)$.

8. If $E_1 \subset E_2 \subset E_3 \subset \cdots$, *then* $\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \mu(\lim_{k\to\infty} E_k) = \lim_{k\to\infty} \mu(E_k)$.

9. If $E_1 \supset E_2 \supset E_3 \supset \cdots$ *and* $\mu(E_1) < \infty$, *then* $\mu\left(\bigcap_{k=1}^{\infty} E_k\right) = \mu(\lim_{k\to\infty} E_k) = \lim_{k\to\infty} \mu(E_k)$.

10. If $\mu\left(\bigcup_{k=1}^{\infty} E_k\right) < \infty$, *then* $\limsup_{k\to\infty} \mu(E_k) \le \mu\left(\limsup_{k\to\infty} E_k\right)$.

11. $\mu\left(\liminf_{k\to\infty} E_k\right) \le \liminf_{k\to\infty} \mu(E_k)$.

12. If $\liminf_{k\to\infty} E_k = \limsup_{k\to\infty} E_k$ *and* $\mu\left(\bigcup_{k=1}^{\infty} E_k\right) < \infty$, *then* $\mu(\lim_{k\to\infty} E_k) = \lim_{k\to\infty} \mu(E_k)$.

Proof.

Parts 1 through 5 have been discussed earlier.

6. $E_1 \cup E_2 = (E_1 \cap E_2^c) \cup (E_2 \cap E_1^c) \cup (E_1 \cap E_2)$.

Thus

$$\begin{aligned}\mu(E_1 \cup E_2) + \mu(E_1 \cap E_2) &= \mu(E_1 \cap E_2^c) + \mu(E_1 \cap E_2) + \mu(E_2 \cap E_1^c) + \mu(E_2 \cap E_1) \\ &= \mu(E_1) + \mu(E_2).\end{aligned}$$

7. Follows immediately from part 5:

Since $E_1 \subset E_2, E_2 = (E_2 \cap E_1^c) \cup E_1$, and

$$\mu(E_2) = \mu((E_2 \cap E_1^c) \cup E_1)$$
$$= \mu(E_2 \cap E_1^c) + \mu(E_1).$$

If $\mu(E_2) < \infty$, we have $\mu(E_1) < \infty$ and we may subtract.

8. If $\mu(E_N) = \infty$ for some N, then $\mu(E_k) = \infty$ for all $k \geq N$ and $\lim_{k\to\infty} \mu(E_k) = \infty.$

Since

$$E_N \subset \bigcup_{k=1}^{\infty} E_k,\ \mu(E_N) \leq \mu\left(\bigcup_{k=1}^{\infty} E_k\right) \text{and thus}\ \mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \mu(\lim_{k\to\infty} E_k) =$$

$\lim_{k\to\infty} \mu(E_k) = \infty.$ So we may suppose $\mu(E_k) < \infty$ for all k.

Since

$$\bigcup_{k=1}^{\infty} E_k = E_1 \cup (E_2 \cap E_1^c) \cup (E_3 \cap E_2^c) \cup \cdots$$

and the sets $E_{k+1} \cap E_k{}^c$ are mutually disjoint, we have

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \mu(E_1) + \sum_{k=1}^{\infty} \mu(E_{k+1} \cap E_k^c) \qquad (5)$$
$$= \mu(E_1) + \sum_{k=1}^{\infty} [\mu(E_{k+1}) - \mu(E_k)] \qquad (7)$$
$$= \mu(E_1) + \lim_{k\to\infty} \sum_{k=1}^{n} [\mu(E_{k+1}) - \mu(E_k)]$$
$$= \mu(E_1) + \lim_{k\to\infty} \mu(E_{n+1}) - \mu(E_1)$$
$$= \lim_{k\to\infty} \mu(E_k).$$

9. Since

$$E_1 \cap \left(\bigcap_{k=1}^{\infty} E_k\right)^c = (E_1 \cap E_2^c) \cup (E_2 \cap E_3^c) \cup \cdots$$

and the sets $E_k \cap E_{k+1}^c$ are mutually disjoint, we have

$$\mu(E_1) - \mu\left(\bigcap_{k=1}^{\infty} E_k\right) = \mu\left(E_1 \cap \left(\bigcap_{k=1}^{\infty} E_k\right)^c\right) \qquad (7)$$

$$= \sum_{k=1}^{\infty} \mu(E_k \cap E_{k+1}^c) \qquad (5)$$

$$= \sum_{k=1}^{\infty} [\mu(E_k) - \mu(E_{k+1})] \qquad (7)$$

$$= \lim_{k\to\infty} \sum_{k=1}^{n} [\mu(E_k) - \mu(E_{k+1})]$$

$$= \mu(E_1) - \lim_{k\to\infty} \mu(E_k).$$

Since $\mu(E_1) < \infty$, we may subtract and the conclusion follows.

10. Recall

$$\limsup_{k\to\infty} E_k = \bigcap_{k\geq 1}\left(\bigcup_{m\geq k} E_m\right).$$

Then

$$\bigcup_{m\geq 1} E_m \supset \bigcup_{m\geq 2} E_m \supset \cdots \text{and}\, E_k \subset \bigcup_{m\geq k} E_m.$$

Thus

$$\mu(E_k) \leq \mu\left(\bigcup_{m\geq k} E_m\right)$$

and, hence,

$$\limsup_{k\to\infty} \mu(E_k) \leq \limsup \mu\left(\bigcup_{m\geq k} E_m\right)$$

$$= \lim_{k\to\infty} \mu\left(\bigcup_{m\geq k} E_m\right)$$

$$= \mu\left(\bigcap_{k\geq 1}\left(\bigcup_{m\geq k} E_m\right)\right) \qquad (9)$$

$$= \mu\left(\limsup_{k\to\infty} E_k\right).$$

11. Recall $\liminf\limits_{k\to\infty} E_k = \bigcup_{k\geq 1}\left(\bigcap_{m\geq k} E_m\right).$

Then

$$\bigcap_{m\geq 1} E_m \subset \bigcap_{m\geq 2} E_m \subset \cdots \text{and} \bigcap_{m\geq k} E_m \subset E_k.$$

Thus

$$\mu\left(\bigcap_{m\geq k} E_m\right) \leq \mu(E_k).$$

Hence

$$\liminf_{k\to\infty} \mu\left(\bigcap_{m\geq k} E_m\right) \leq \liminf_{k\to\infty} \mu(E_k).$$

But

$$\begin{aligned}\liminf_{k\to\infty} \mu\left(\bigcap_{m\geq k} E_m\right) &= \lim \mu\left(\bigcap_{m\geq k} E_m\right) \\ &= \mu\left(\bigcup_{k\geq 1}\left(\bigcap_{m\geq k} E_m\right)\right) \qquad (8) \\ &= \mu\left(\liminf_{k\to\infty} E_k\right).\end{aligned}$$

and the conclusion follows.

12.

$$\begin{aligned}\limsup_{k\to\infty} \mu(E_k) &\leq \mu\left(\limsup_{k\to\infty} E_k\right) \qquad (10) \\ &= \mu\left(\bigcap_{k\geq 1}\left(\bigcup_{m\geq k} E_m\right)\right) \\ &= \mu\left(\bigcup_{k\geq 1}\left(\bigcap_{m\geq k} E_m\right)\right) \\ &= \mu\left(\liminf_{k\to\infty} E_k\right) \\ &\leq \liminf_{k\to\infty} \mu(E_k).\end{aligned}$$

□

3.5 Structure of Lebesgue Measurable Sets

Are there relationships between topological properties (open, closed, etc.) and Lebesgue measurability? The next theorem shows that Lebesgue measurable subsets of ℝ are “almost open”, “almost closed”, and so forth.

Theorem 3.5.1 *For an arbitrary subset E of $\mathbb{R}$, the following statements are equivalent:*

1. *E is Lebesgue measurable in the sense of Carathéodory;*
2. *Given $\varepsilon > 0$ we can determine an open set $G \subset \mathbb{R}$ with $E \subset G$ and $\mu^*(G \cap E^c) < \varepsilon$ ("exterior" approximation by open sets);*
3. *Given $\varepsilon > 0$ we can determine a closed set $F \subset \mathbb{R}$ with $F \subset E$ and $\mu^*(E \cap F^c) < \varepsilon$ ("interior" approximation by closed sets);*
4. *There is G_δ set B_1 with $E \subset B_1$ and $\mu^*(B_1 \cap E^c) = 0$ (B_1 is a countable intersection of open sets; if we relax "open", we can obtain a very good approximation by Borel sets);*
5. *There is F_σ set B_2 with $B_2 \subset E$ and $\mu^*(E \cap B_2^c) = 0$ (B_2 is a countable union of closed sets; if we relax "closed", we have very good approximation from the "inside" by Borel sets).*

Proof.

$1. \Rightarrow 2.$ Assume E is a Lebesgue measurable subset of $\mathbb{R}$ with $\mu(E) < \infty$.

By the definition of Lebesgue outer measure, we have an open cover $G = \bigcup_{k=1}^{\infty} I_k$ so that

$$E \subset G \text{ and } \mu^*(G) < \mu(E) + \varepsilon.$$

Since

$$G = (G \cap E^c) \cup E$$

and G is Lebesgue measurable (Theorem 3.3.11),

$$\mu(G) = \mu(G \cap E^c) + \mu(E).$$

Because $\mu(E) < \infty$, we may subtract and obtain $\mu^*(G \cap E^c) < \varepsilon$.

If $\mu(E) = \infty$, let $E_k = E \cap [-k, k]$. E_k is Lebesgue measurable, $\mu(E_k) < \infty$,

and by what we just showed we have an open set G_k so that

$$E_k \subset G_k \text{ and } \mu(G_k \cap E_k^c) < \frac{\varepsilon}{2^k}.$$

Since $E = \bigcup_{k=1}^{\infty} E_k \subset \bigcup_{k=1}^{\infty} G_k = G$, it follows that

$$\begin{aligned}
\mu^*(G \cap E^c) &= \mu(G \cap E^c) \\
&\le \mu\left(\bigcup_{k=1}^{\infty} (G_k \cap E_k^c)\right) \\
&\le \sum_{k=1}^{\infty} \mu(G_k \cap E_k^c) \\
&< \varepsilon.
\end{aligned}$$

We have constructed an open set $G = \bigcup_{k=1}^{\infty} G_k$ with the desired properties.

$2. \Rightarrow 3.$ Follows by "complementation": Apply part 2 to E^c. We have

$$E^c \subset G \text{ and } \mu^*(G \cap (E^c)^c) < \varepsilon.$$

But then

$$G^c \subset E$$

and

$$\begin{aligned}
\mu^*(E \cap (G^c)^c) &= \mu^*(G \cap (E^c)^c) \\
&< \varepsilon,
\end{aligned}$$

and with $F = G^c$, the argument is complete.

$3. \Rightarrow 4.$ Let $E \subset \mathbb{R}$ and apply part 3 to E^c. We have a sequence of closed sets $\{F_k\}$ so that

$$F_k \subset E^c \text{ and } \mu^*(E^c \cap F_k^c) < \frac{1}{k}.$$

Let $B_1 = \bigcap_{k=1}^{\infty} F_k^c$. B_1 is a G_δ set, $E \subset B_1$, and

$$\begin{aligned}
\mu^*(B_1 \cap E^c) &= \mu^*(E^c \cap B_1) \\
&= \mu^*\left(E^c \cap \left(\bigcap_{k=1}^{\infty} F_k^c\right)\right) \\
&\le \mu^*(E^c \cap F_k^c) \\
&< \frac{1}{k}, \qquad k = 1,2,\ldots.
\end{aligned}$$

So $\mu^*(B_1 \cap E^c) = 0.$

$4. \Rightarrow 5.$ Follows by complementation as in $2. \Rightarrow 3.$

$5. \Rightarrow 1.$ We must show $\mu^*(X \cap E) + \mu^*(X \cap E^c) = \mu^*(X)$ for every subset X of $\mathbb{R}$. Let E be an arbitrary subset of $\mathbb{R}$ and B_2 be the F_σ set guaranteed by part 5: $B_2 \subset E, \mu^*(E \cap B_2^c) = 0$.

Since B_2 is Lebesgue measurable (Theorem 3.3.11),

$$\mu^*(X \cap E) = \mu^*((X \cap E) \cap B_2) + \mu^*((X \cap E) \cap B_2^c).$$

Because

$$(X \cap E) \cap B_2^c \subset E \cap B_2^c,$$

so

$$\mu^*((X \cap E) \cap B_2^c) = 0,$$

and we have $\mu^*(X \cap E) = \mu^*((X \cap E) \cap B_2)$.

On the other hand,

$$\begin{aligned}\mu^*(X \cap E^c) &= \mu^*((X \cap E^c) \cap B_2) + \mu^*((X \cap E^c) \cap B_2^c) \\ &= \mu^*((X \cap E^c) \cap B_2) + \mu^*(X \cap E^c).\end{aligned}$$

This implies

$$\mu^*((X \cap E^c) \cap B_2) = 0.$$

Thus

$$\begin{aligned}\mu^*(X \cap E^c) &= \mu^*((X \cap E^c) \cap B_2) + \mu^*((X \cap E^c) \cap B_2^c) \\ &= 0 + \mu^*((X \cap E^c) \cap B_2^c) \\ &\le \mu^*(X \cap B_2^c).\end{aligned}$$

Thus

$$\begin{aligned}\mu^*(X \cap E) + \mu^*(X \cap E^c) &\le \mu^*((X \cap E) \cap B_2) + \mu^*(X \cap B_2^c) \\ &\le \mu^*(X \cap B_2) + \mu^*(X \cap B_2^c) \\ &= \mu^*(X).\end{aligned}$$

because B_2 is Lebesgue measurable. This yields Carathéodory's condition on E since $\mu^*(X) \le \mu^*(X \cap E) + \mu^*(X \cap E^c)$ by subadditivity. □

The next result relates Borel sets and Lebesgue measurable sets.

Theorem 3.5.2 *Every Lebesgue measurable set of real numbers is the union of a Borel set and a set with Lebesgue measure zero (Lebesgue measure is the completion of Borel measure).*

Proof. Let E be the Lebesgue measurable set of real numbers. We then have a Borel set $(F_\sigma) B$ so that

$$B \subset E \text{ and } \mu(E \cap B^c) = 0 \text{ (Theorem 3.5.1).}$$

But

$$E = B \cup (E \cap B^c);$$

B is our desired Borel set and $E \cap B^c$ is the Lebesgue measurable set with Lebesgue measure zero. □

The last theorem of this chapter states that sets of finite Lebesgue measure are "almost" finite unions of intervals.

Theorem 3.5.3 *Suppose E is any subset of $\mathbb{R}$ with $\mu^*(E) < \infty$. Then E is a Lebesgue measurable set of real numbers iff we have a finite union of open intervals U so that*

$$\mu^*(E \cap U^c) + \mu^*(U \cap E^c) < \varepsilon,$$

for any $\varepsilon > 0$.

Proof. We first assume E is Lebesgue measurable.
Since E is Lebesgue measurable, we have an open set G so that

$$E \subset G \text{ and } \mu(G \cap E^c) < \frac{\varepsilon}{2} \text{ (Theorem 3.5.1).}$$

Since every nonempty open set of real numbers is a countable union of disjoint open intervals,

$$E \subset \bigcup_{k=1}^{\infty} I_k = G \text{ and } \mu\left(\left(\bigcup_{k=1}^{\infty} I_k\right) \cap E^c\right) < \frac{\varepsilon}{2}.$$

But $\bigcup_{k=1}^{\infty} I_k = \left(\left(\bigcup_{k=1}^{\infty} I_k\right) \cap E^c\right) \cup E$, and, consequently,

$$\mu(E) \leq \mu\left(\bigcup_{k=1}^{\infty} I_k\right) = \sum_{k=1}^{\infty} \mu(I_k) < \mu(E) + \frac{\varepsilon}{2} < \infty,$$

that is, the series $\sum_{k=1}^{\infty}\mu(I_k)$ converges. Choose N so that

$$\sum_{k=N+1}^{\infty}\mu(I_k) < \frac{\varepsilon}{2} \text{ and define } U = \bigcup_{k=1}^{N} I_k.$$

Note:

i. $U \bigcap \left(\bigcup_{k=N+1}^{\infty} I_k \right) = \phi.$

ii. $U \bigcap E^c \subset G \bigcap E^c$ and thus $\mu^*(U \bigcap E^c) < \frac{\varepsilon}{2}.$

iii. $E \bigcap U^c = E \bigcap \left(\bigcup_{k=N+1}^{\infty} I_k \right) \subset \bigcup_{k=N+1}^{\infty} I_k.$

Then

$$\mu^*(E \bigcap U^c) + \mu^*(U \bigcap E^c) < \varepsilon.$$

Conversely, assume we have a finite union of open intervals U so that

$$\mu^*(E \bigcap U^c) + \mu^*(U \bigcap E^c) < \frac{\varepsilon}{2}.$$

We will construct an open set G so that

$$E \subset G, \mu^*(G \bigcap E^c) < \varepsilon,$$

and then conclude from Theorem 3.5.1 that E must be Lebesgue measurable.

By the definition of Lebesgue outer measure we have an open set O_1 so that

$$E \bigcap U^c \subset \mathrm{O}_1 \text{ and } \mu^*(E \bigcap U^c) \le \mu^*(\mathrm{O}_1) < \mu^*(E \bigcap U^c) + \frac{\varepsilon}{2}.$$

Let $G = \mathrm{O}_1 \bigcup U$.

G is an open subset of $\mathbb{R}$,

$$E \subseteq U \bigcup (E \bigcap U^c) \subset U \bigcup \mathrm{O}_1 = G,$$

and

$$\begin{aligned} \mu^*(G \bigcap E^c) &= \mu^*((U \bigcup \mathrm{O}_1) \bigcap E^c) \\ &= \mu^*((U \bigcap E^c) \bigcup (\mathrm{O}_1 \bigcap E^c)) \\ &\le \mu^*(U \bigcap E^c) + \mu^*(\mathrm{O}_1) \\ &< \varepsilon. \end{aligned}$$

□

We have completed our development of Lebesgue measure. Knowing what Legesgue measurable sets are, we are now able to discuss measurable functions in next chapter.

3.6 A Lebesgue Nonmeasurable Set

Giuseppe Vitali discovered the first example of a Lebesgue nonmeasurable set of real numbers in1905. In the next few years, several mathematicians such as Van Vleck (1908) and F.Bernstein (1908) among others discovered such sets. All of their constructions used the Axiom of Choice: for any nonempty collection C of sets, there is a choice function f such that $f(A) \in A$ for each $A \in C$. In 1970, Solovay showed that the Axiom of Choice was required to construct a Lebesgue nonmeasurable set of real numbers. The construction involves the notions of equivalence relations and equivalence classes.

Now, we construct a Lebesgue nonmeasurable set of real numbers in $(-1,1)$.

Define $x \sim y$ if $x - y$ is rational.

For $x \in (-1,1)$, define $R_x = \{y \in I \mid y - x = r, r \text{ rational}\}$.

The following are nine properties of R_x. We will prove them.

1. Every real number $x \in (-1,1)$ *belongs to one of the sets* R_x;

2. $\bigcup_{x \in (-1,1)} R_x = (-1,1)$;

3. If $x_1 \sim x_2$ *with* $x_1 x_2 \in (-1,1)$, *then* $R_{x_1} = R_{x_2}$;

4. If $R_{x_1} \cap R_{x_2} \neq \phi$, *then* $R_{x_1} = R_{x_2}$;

5. Each set R_x *is countable;*

6. If $x \in (-1,1)$ *is rational, then* R_x *is the set of rationals of* $(-1,1)$;

7. If $x \in (-1,1)$ *is irrational, then every element of* R_x *is an irrational number in* $(-1,1)$;

8. If $x_1, x_2 \in (-1,1)$ *with* $x_1 - x_2$ *an irrational number, then* $R_{x_1} \bigcap R_{x_2} = \phi$;

9. The collection of distinct sets R_x *is uncountable.*

Proof.

1. Let $y \in (-1,1)$. Then $y \in R_y$ since $y - y = 0$ is rational.

2. Let $x \in (-1,1)$. Then

$$\begin{aligned} &\{x\} \subseteq R_x \subseteq (-1,1) \\ \Rightarrow &\bigcup_{x\in(-1,1)} \{x\} \subseteq \bigcup_{x\in(-1,1)} R_x \subseteq (-1,1) \\ \Rightarrow &(-1,1) \subseteq \bigcup_{x\in(-1,1)} R_x \subseteq (-1,1). \end{aligned}$$

This implies

$$\bigcup_{x\in(-1,1)} R_x = (-1,1).$$

3. Suppose $x_1 \sim x_2$. Then $x_1 - x_2$ is rational, this implies

$$R_{x_2} = \{x_1 \in I \mid x_1 - x_2 = r, r\ rational\} \neq \phi.$$

and

$$R_{x_1} = \{x_2 \in I \mid x_2 - x_1 = r, r\ rational\} \neq \phi$$

Claim $R_{x_1} = R_{x_2}$.

Let $y \in R_{x_1}$, then $y - x_1$ is rational, this implies $y \in R_{x_2}$, and so $R_{x_1} \subseteq R_{x_2}$ since y is arbitrary.

Similarly, let $y' \in R_{x_2}$, then $y' - x_2$ is rational, this implies $y' \in R_{x_1}$, and so $R_{x_2} \subseteq R_{x_1}$ since y' is arbitrary.

Thus

$$R_{x_1} = R_{x_2}.$$

4. Since $R_{x_1} \bigcap R_{x_2} \neq \phi$, there exists $z \in R_{x_1} \bigcap R_{x_2}$ such that $z \in R_{x_1}$ and $z \in R_{x_2}$. This implies $z - x_1$ is rational and $z - x_2$ is rational. So

$$(z-x_2)-(z-x_1)=x_1-x_2$$

is rational. By definition, $x \sim y,$ and from 3.,

$$R_{x_1}=R_{x_2}.$$

5. If $x \in (-1,1)$ is rational, then in order for $y \in R_x$ and $y-x$ to be rational, where $y \in (-1,1)$, y must be rationals in $(-1,1)$. Thus $R_x = \{$ rationals in $(-1,1)\}$ is a countable set, for rational $x \in (-1,1)$.

If $x \in (-1,1)$ is irrational, then in order for $y \in R_x$ and $y-x$ to be rational, where $y \in (-1,1)$, y must equal to $x+r_n$, for some $-2<r_n<2$, where r_n is rational. Thus R_x is a countable set, for irrational $x \in (-1,1)$.

6. Follows immediately from 5.

7. Follows immediately from 5.

8. We prove by contrapositive. Suppose $R_{x_1} \cap R_{x_2} \neq \phi$. Then by 4., $R_{x_1}=R_{x_2}$. Let $z \in R_{x_1}$, then $z \in R_{x_2}$ since $R_{x_1}=R_{x_2}$. This implies $z-x_1$ is rational and $z-x_2$ is rational. . So

$$(z-x_2)-(z-x_1)=x_1-x_2$$

is rational. By definition, $x \sim y$.

9. Suppose $\bigcup\limits_{x\in(-1,1),\, R_x distinct} R_x = (-1,1)$ is countable. But this contradicts the fact that $(-1,1)$ is uncountable. Thus the collection of distinct sets R_x is uncountable. □

In conclusion, we have decomposed $(-1,1)$ into an uncountable collection of pairwise disjoint sets, each of these sets is itself countable, one such set consisting of the rationals in $(-1,1)$, and each of the others consisting only of irrational numbers.

Pick a point from each of these disjoint subsets R_x, call this set N. N is an uncountable set, a subset of $(-1,1)$, and $N \bigcap R_x$ is a single point. We intend to show N is nonmeasurable.

Enumerate the rationals in $(-2,2): r_1, r_2, r_3, \ldots$. Define $N + r_n = \{x + r_n \mid x \in N\}$, $-2 < r_n < 2, r_n$ rational. Since $N \subset (-1,1), N + r_n \subset (-3,3)$. Claim $(N + r_n) \bigcap (N + r_m) = \phi$ if $r_n \neq r_m$. Suppose $(N + r_n) \bigcap (N + r_m) \neq \phi$. Then there exists $z = x + r_n = y + r_m$, or $x - y$ is rational, with $x, y \in N$. This implies $R_x = R_y$ with $x, y \in R_x = R_y$. But N is constructed by taking points from mutually disjoint sets, so x must equal y, or $r_n = r_m$. This contradicts that $r_n \neq r_m$. So,

$$(-1,1) \subset \bigcup_{n=1}^{\infty} (N + r_n) \subset (-3,3).$$

Suppose N is measurable, $N + r_n$ is also measurable. We have

$$\begin{aligned} 2 = \mu((-1,1)) &\leq \mu\left(\bigcup_{n=1}^{\infty} (N + r_n) \right) \\ &= \sum_{n=1}^{\infty} \mu(N + r_n) \\ &= \sum_{n=1}^{\infty} \mu(N) \\ &\leq \mu((-3,3)) = 6. \end{aligned}$$

This implies $2 \leq \sum_{n=1}^{\infty} \mu(N) \leq 6$.

The left-hand inequality, $2 \leq \sum_{n=1}^{\infty} \mu(N)$, implies $\mu(N) > 0$. The right-hand inequality $\sum_{n=1}^{\infty} \mu(N) \leq 6$, implies $\mu(N) = 0$. $\mu(N) > 0$ and $\mu(N) = 0$ cannot hold at the same time. Thus N must be nonmeasurable.

CHAPTER 4

LEBESGUE MEASURABLE FUNCTIONS

In this chapter Lebesgue measurable functions are introduced follow by sequences of Lebesgue measurable functions. Characteristic function, simple function and Approximation Theorem are mentioned in Section 4.3. We conclude this chapter with Egoroff's theorem and Lusin's theorem.

Caution: In what follows a "measurable function" means a "Lebesgue measurable function". For any function, the domain will always be a subset of $\mathbb{R}$ and the range will be a subset of $\mathbb{R}$ or $\mathbb{R}^e$ (real-valued or extended real-valued).

4.1 Measurable Functions

We begin this section by giving the definition of a Lebesgue measurable function and its equivalent forms.

Definition 4.1.1 *An extended real-valued function f, defined on a Lebesgue measurable set of real numbers E, is said to be Lebesgue measurable on E if*

$$f^{-1}((c,\infty]) = \{x \in E \mid f(x) > c\}$$

is a Lebesgue measurable subset of E for every real number c.

Theorem 4.1.2 *Suppose f is an extended real-valued function whose domain is a Lebesgue measurable set of real numbers E, and c is any real number. Then the following statements are equivalent:*

1. f is a Lebesgue measurable function on E.

2. $f^{-1}((c,\infty]) = \{x \in E \mid f(x) > c\}$ is a Lebesgue measurable subset of E.

3. $f^{-1}([c,\infty]) = \{x \in E \mid f(x) \geq c\}$ is a Lebesgue measurable subset of E.

4. $f^{-1}([-\infty,c)) = \{x \in E \mid f(x) < c\}$ is a Lebesgue measurable subset of E.

5. $f^{-1}([-\infty,c]) = \{x \in E \mid f(x) \leq c\}$ is a Lebesgue measurable subset of E.

Proof.

$1. \Leftrightarrow 2.$ Definition 4.1.1.

$2. \Rightarrow 3.$ $f^{-1}([c,\infty]) = f^{-1}\left(\bigcap_{k=1}^{\infty}\left(c-\frac{1}{k},\infty\right]\right) = \bigcap_{k=1}^{\infty} f^{-1}\left(\left(c-\frac{1}{k},\infty\right]\right).$

$3. \Rightarrow 4.$ $f^{-1}([-\infty,c)) = f^{-1}([c,\infty]^c) = \mathbb{R} \bigcap (f^{-1}([c,\infty]))^c.$

$4. \Rightarrow 5.$ $f^{-1}([-\infty,c]) = f^{-1}\left(\bigcap_{k=1}^{\infty}\left[-\infty,c+\frac{1}{k}\right)\right) = \bigcap_{k=1}^{\infty} f^{-1}\left(\left[-\infty,c+\frac{1}{k}\right)\right).$

$5. \Rightarrow 2.$ $f^{-1}((c,\infty]) = f^{-1}([-\infty,c]^c) = \mathbb{R} \bigcap (f^{-1}([-\infty,c]))^c.$ □

Theorem 4.1.3 *Continuous functions defined on measurable sets are measurable functions.*

Proof. Let f be a continuous function on the measurable set E, and c any real number. We must show $A = \{x \in E \mid f(x) > c\}$ is a measurable subset of E.

If $A = \phi$, the proof is immediate since the empty set is measurable.

Otherwise, let $x \in A$. Then $f(x) > c$, and because f is continuous at x, we have $\delta_x > 0$ so that for $z \in (x-\delta(x), x+\delta(x)) \bigcap E, f(z) > c.$

Thus

$$\begin{aligned} A &= \bigcup_{x \in A}((x-\delta(x), x+\delta(x)) \bigcap E) \\ &= \left(\bigcup_{x \in A}(x-\delta(x), x+\delta(x))\right) \bigcap E. \end{aligned}$$

A is the intersection of an open set (measurable) and the measurable set E. The set A is measurable and the argument is complete. □

We can weaken continuity to continuity except on a set of measure zero, commonly referred to as "continuous almost everywhere." Sets of measure zero do not affect measurability of a function.

Definition 4.1.4 *A property is said to hold almost everywhere on a measurable set if the set of points where it fails to hold has measure zero. In particular, two functions f and g are said to be equal almost everywhere if they have the same domain and $\mu(\{x \mid f(x) \neq g(x)\}) = 0$. We sometimes write $f = g$ a.e. on E.*

Theorem 4.1.5 *Suppose f and g are extended real-valued functions defined on a measurable set E. If f is a measurable function on E and if $f = g$ except on a set of measure zero, then g is a measurable function on E.*

Proof. Let c be any real number. We must show $\{x \in E \mid g(x) > c\}$ is a measurable subset of E. Define $A = \{x \in E \mid f \neq g\}$. By assumption, A is measurable with measure zero. Then $g = f$ on the measurable set $E \bigcap A^c = \{x \in E \mid f = g\}$, and

$$\begin{aligned}\{x \in E \mid g(x) > c\} &= \{x \in E \bigcap A^c \mid g(x) > c\} \bigcup \{x \in A \mid g(x) > c\} \\ &= \{x \in E \bigcap A^c \mid f(x) > c\} \bigcup \{x \in A \mid g(x) > c\} \\ &= (\{x \in E \mid f(x) > c\} \bigcap (E \bigcap A^c)) \bigcup \{x \in A \mid g(x) > c\}.\end{aligned}$$

The set $\{x \in A \mid g(x) > c\}$ is measurable since it is a subset of a set of measure zero. Because f is a measurable function on E, $\{x \in E \mid f(x) > c\}$ is a measurable subset of E, as is $E \bigcap A^c$. □

Proposition 4.1.6 *Let f be a Lebesgue measurable function defined on a Lebesgue measurable set E. If A is any Lebesgue measurable subset of E, then f is a Lebesgue measurable function on A.*

Proof. $\{x \in A \mid f(x) > c\} = \{x \in E \mid f(x) > c\} \bigcap A.$

By assumptions, $\{x \in E \mid f(x) > c\}$ and A are Lebesgue measurable subsets of E. Thus, $\{x \in A \mid f(x) > c\}$ is a Lebesgue measurable subset of E, and so f is a Lebesgue measurable function on A.

□

Theorem 4.1.7 *Suppose f and g are real-valued measurable functions, defined on a measurable set* E, *and k is any real number. Then the following functions are measurable functions on* E :

$$f + k, kf, |f|, f^2, \frac{1}{g}(g \neq 0\, on\, E), f + g, f \cdot g, \frac{f}{g}(g \neq 0\, on\, E).$$

Proof. The arguments are sketched:

i. $\{x \in E \mid f(x) + k > c\} = \{x \in E \mid f(x) > c - k\}$.

ii. If $k = 0$, then $kf = 0$ and $\{x \in E \mid kf(x) > c\} = \begin{cases} \phi, c \geq 0 \\ E, c < 0. \end{cases}$

If $k > 0$, then $\{x \in E \mid kf(x) > c\} = \left\{x \in E \mid f(x) > \frac{c}{k}\right\}$.

If $k < 0$, then $\{x \in E \mid kf(x) > c\} = \left\{x \in E \mid f(x) < \frac{c}{k}\right\}$.

iii. $\{x \in E \mid \; |f(x)| > c\} = \begin{cases} E, & c < 0 \\ \{x \in E \mid f(x) > c\} \bigcup \{x \in E \mid f(x) < -c\}, & c \geq 0. \end{cases}$

iv. $\{x \in E \mid f^2(x) > c\} = \begin{cases} E, & c < 0 \\ \{x \in E \mid \; |f(x)| > \sqrt{c}\}, & c \geq 0. \end{cases}$

v. $\left\{x \in E \;\middle|\; \frac{1}{g(x)} > c\right\} = \begin{cases} \{x \in E \mid g(x) > 0\}, & c = 0 \\ \{x \in E \mid g(x) > 0\} \bigcap \left\{x \in E \mid g(x) < \frac{1}{c}\right\}, & c > 0 \\ \{x \in E \mid g(x) > 0\} \bigcup \left\{x \in E \mid g(x) < \frac{1}{c}\right\}, & c < 0. \end{cases}$

vi. We use the fact that the rationals are dense in $\mathbb{R}$ and are a countable subset of $\mathbb{R}$ and

$$\{x \mid f(x) < g(x)\} = \bigcup_{r_k} (\{x \in E \mid f(x) < r_k\} \cap \{x \in E \mid r_k < g(x)\}),$$

where r_k is rational.

Then

$$\begin{aligned}\{x \in E \mid f(x) + g(x) > c\} &= \{x \in E \mid c - g(x) < f(x)\} \\ &= \bigcup_{r_k} (\{x \in E \mid c - g(x) < r_k\} \cap \{x \in E \mid r_k < f(x)\}) \\ &= \bigcup_{r_k} (\{x \in E \mid c - r_k < g(x)\} \cap \{x \in E \mid r_k < f(x)\}),\end{aligned}$$

where r_k is rational.

vii.

$$\{x \in E \mid [f(x) + g(x)]^2 > c\} = \begin{cases} E, & c < 0 \\ \quad \{x \in E \mid f(x) + g(x) > \sqrt{c}\} & \\ \bigcup \{x \in E \mid f(x) + g(x) < -\sqrt{c}\}, & c \geq 0. \end{cases}$$

$$= \begin{cases} E, & c < 0 \\ \quad \{x \in E \mid f(x) + g(x) > \sqrt{c}\} & \\ \bigcup \{x \in E \mid \sqrt{c} + f(x) < -g(x)\}, & c \geq 0. \end{cases}$$

Thus $(f + g)^2$ is a measurable function on E.

$$\{x \in E \mid -[f(x) - g(x)]^2 > c\} = \begin{cases} \phi, & c > 0 \\ \{x \in E \mid -\sqrt{-c} < f(x) - g(x) < \sqrt{-c}\}, & c \leq 0. \end{cases}$$

$$= \begin{cases} \phi, & c > 0 \\ \quad \{x \in E \mid -\sqrt{-c} < f(x) - g(x)\} & \\ \bigcap \{x \in E \mid f(x) - g(x) < \sqrt{-c}\}, & c \leq 0. \end{cases}$$

Thus $-(f - g)^2$ is a measurable function on E.

$$\{x \in E \mid f(x)\cdot g(x) > c\} = \left\{ x \in E \,\middle|\, \frac{[f(x)+g(x)]^2 - [f(x)-g(x)]^2}{4} > c \right\}$$
$$= \{x \in E \mid 4c + [f(x)-g(x)]^2 < [f(x)+g(x)]^2\}$$
$$= \bigcup_{r_k} \left(\{x \in E \mid 4c + [f(x)-g(x)]^2 < r_k\} \cap \{x \in E \mid r_k < [f(x)+g(x)]^2\} \right)$$
$$= \bigcup_{r_k} \left(\{x \in E \mid -[f(x)-g(x)]^2 > 4c - r_k\} \cap \{x \in E \mid r_k < [f(x)+g(x)]^2\} \right),$$

where r_k is rational.

Similar reasoning for $\dfrac{f}{g} = f \cdot \left(\dfrac{1}{g}\right)$. □

If we replace measurability of f and g in Theorem 4.1.7 with continuity, we still have a valid proposition. The operations performed with measurable functions in Theorem 4.1.7 in no way distinguish measurable functions from continuous functions.

Proposition 4.1.8 *Suppose f and g are measurable functions defined on a measurable set E. Then the following functions are measurable on E:*

1. $\max(f, g), \min(f, g)$.
2. $f^+ = \max(f, 0), f^- = -\min(f, 0), |f|$.

Note: $f = f^+ - f^-, |f| = f^+ + f^-$.

Proof.

1. $\{x \in E \mid \max(f(x), g(x)) > c\} = \{x \in E \mid f(x) > c\} \cup \{x \in E \mid g(x) > c\}$

and $\{x \in E \mid \min(f(x), g(x)) > c\} = \{x \in E \mid f(x) > c\} \cap \{x \in E \mid g(x) > c\}$.

2. $\{x \in E \mid |f(x)| > c\} = \begin{cases} \{x \in E \mid f(x) > c\} \cup \{x \in E \mid f(x) < -c\}, & c \geq 0 \\ E, & c < 0. \end{cases}$

□

4.2 Sequences of Measurable Functions

We are ready to discuss sequences of Lebesgue measurable functions. Pointwise limits preserve measurability with some relatively mild additional conditions imposed on the sequence). The next theorem is crucial.

Theorem 4.2.1 *Suppose* $\{f_k\}$ *is a sequence of measurable functions defined on a measurable set* E. *Then the following functions are measurable functions on* E:

1. $\overline{f_k} = \sup\{f_k, f_{k+1}, \ldots\}$ *and* $\underline{f}_k = \inf\{f_k, f_{k+1}, \ldots\}$ *for* $k = 1,2,\ldots;$

2. $\limsup_{k\to\infty} f_k = \lim_{k\to\infty} \overline{f_k}$ *and* $\liminf_{k\to\infty} f_k = \lim_{k\to\infty} \underline{f_k}$.

3. If $\lim_{k\to\infty} f_k$ *(finite or infinite) exists for every point of* E, *then the limit function* $\lim_{k\to\infty} f_k$ *is a measurable function on* E.

4. If f *is a function defined on* E *and* $f = \lim_{k\to\infty} f_k$ *almost everywhere on* E, *then* f *is a measurable function on* E.

Proof.

1. $\{x \in E \mid \overline{f_k} > c\} = \bigcup_{n=k}^{\infty} \{x \in E \mid f_n(x) > c\}$

and $\{x \in E \mid \underline{f_k} > c\} = \bigcap_{n=k}^{\infty} \{x \in E \mid f_n(x) > c\}$.

2. The sequence $\{\overline{f_k}\}$ is a nonincreasing sequence of measurable functions and since $\limsup_{k\to\infty} f_k = \lim_{k\to\infty} \overline{f_k} = \inf\{\overline{f_1}, \overline{f_2}, \ldots\}$, measurability follows from part 1.

The sequence $\{\underline{f_k}\}$ is a nondecreasing sequence of measurable functions and since $\liminf_{k\to\infty} f_k = \lim_{k\to\infty} \underline{f_k} = \sup\{\underline{f_1}, \underline{f_2}, \ldots\}$, measurability follows from part 1.

3. $\liminf_{k\to\infty} f_k = \lim_{k\to\infty} f_k = \limsup_{k\to\infty} f_k$.

4. Suppose a function f on E is the almost everywhere limit of $\{f_k\}$ and let $A = \{x \in E \mid \lim_{k\to\infty} f_k(x)$ is not defined or $\lim_{k\to\infty} f_k(x) \neq f(x)\}$. The A has measure zero. Define a new sequence of functions $\{g_k\}$ on E by

$$g_k(x) = \begin{cases} f_k(x), & x \notin A \\ 0, & x \in A, \end{cases}$$

and let g be given by

$$g(x) = \begin{cases} f(x), & x \notin A \\ 0, & x \in A. \end{cases}$$

Since each function g_k equals a measurable function, f_k, almost everywhere on E, g_k is measurable (Theorem 4.1.5).

If $x \in A$, $\lim_{k\to\infty} g_k(x) = 0 = g(x)$.

If $x \notin A$, $\lim_{k\to\infty} g_k(x) = \lim_{k\to\infty} f_k(x) = f(x) = g(x)$,

that is,

$$\lim_{k\to\infty} g_k = g \text{ for every point of } E.$$

By part 3, g is measurable on E.

By Theorem 4.1.5, since f equals to g a.e. on E, and so f is measurable on E. □

4.3 Approximating Measurable Functions

In this section, we will show that every measurable function is the limit of a sequence of simple functions.

Definition 4.3.1 *Let A be any set of real number. The characteristic function on A, denoted χ_A, is defined as follows:*

$$\chi_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A. \end{cases}$$

Definition 4.3.2 *Suppose* $E = \bigcup_{k=1}^{n} E_k$, *where the sets* E_k *are measurable, mutually disjoint, subsets of* $\mathbb{R}$, *and* $c_1, c_2, \ldots, c_n$ *are real numbers. Then a function* φ *defined on* E *by* $\varphi(x) = \sum_{k=1}^{n} c_k \chi_{E_k}(x)$, *is called a simple function.*

A simple function assumes a finite number of real values and assumes each of these on a measurable set, that is,

$$\varphi(x) = c_k \text{ on } E_k, 1 \le k \le n.$$

Theorem 4.3.3 *If* χ_E *is a characteristic function defined on a measurable set* E, *then* χ_E *is a measurable function on* E.

Proof.

$$\{x \in E \mid \chi_E(x) > c\} = \begin{cases} E, & c < 0 \\ E, & 0 \le c < 1 \\ \phi, & c \ge 1. \end{cases}$$

□

Theorem 4.3.4 *If* φ *is a simple function defined on a measurable set* E, *then* φ *is a measurable function on* E.

Proof. Let φ be a simple function defined on E by $\varphi(x) = \sum_{k=1}^{n} c_k \chi_{E_k}(x)$

and E be a measurable set such that

$$E = \bigcup_{k=1}^{n} E_k,$$

where the sets $E_k \subseteq \mathbb{R}$ are measurable and mutually disjoint and $c_1, c_2, \ldots, c_n$ be any real numbers.

Now, we have

$$\{x \in E_k \mid \chi_{E_k} > c_k^{'}\} \text{ for } k = 1,2,\ldots,n.$$

and so by Theorem 4.3.3, χ_{E_k} is a measurable function on E_k for $k = 1,2,\ldots,n.$

By Theorem 4.1.7, $c_k \chi_{E_k}$ is a measurable function on $E_k \subseteq E$ for $k = 1,2,\ldots,n.$

and so by applying Theorem 4.1.7 again, we conclude φ is a measurable function on E. □

Theorem 4.3.5 (Approximation Theorem, Lebesgue) *Let f be a measurable function defined on a measurable set E. Then there exists a sequence of simple functions $\{\varphi_k\}$ on E, so that*

$$\lim_{k\to\infty} \varphi_k = f \text{ (finite or infinite)}$$

for all $x \in E$.

If f is bounded on E, then

$$\lim_{k\to\infty} \varphi_k = f \text{ (unif)}$$

on E.

If f is nonnegative, the sequence $\{\varphi_k\}$ may be constructed so that it is a monotonically increasing sequence.

Proof. Suppose that f is nonnegative on E. We want to construct a monotonically increasing sequence $\{\varphi_k\}$ with $\lim_{k\to\infty} \varphi_k = f$. Divide the range of f and approximate by *level curves*. Since $f(E) \subseteq [0,\infty]$, we partition $[0,\infty]$:

Step 1.

$$\begin{aligned}[0,\infty] &= [0,1) \cup [1,\infty] \\ &= \left[0,\frac{1}{2}\right) \cup \left[\frac{1}{2},1\right) \cup [1,\infty].\end{aligned}$$

Define $E_{11} = f^{-1}\left(\left[0,\frac{1}{2}\right)\right), E_{12} = f^{-1}\left(\left[\frac{1}{2},1\right)\right), E_1 = f^{-1}([1,\infty]),$

and $\varphi_1 = 0 \cdot \chi_{E_{11}} + \frac{1}{2} \cdot \chi_{E_{12}} + 1 \cdot \chi_{E_1}$. Clearly $\varphi_1 \le f$ on E.

Step 2.

$$[0,\infty] = [0,1) \cup [1,2) \cup [2,\infty]$$
$$= \left[0,\frac{1}{4}\right) \cup \left[\frac{1}{4},\frac{1}{2}\right) \cup \left[\frac{1}{2},\frac{3}{4}\right) \cup \left[\frac{3}{4},1\right) \cup \left[1,\frac{5}{4}\right) \cup \left[\frac{5}{4},\frac{6}{4}\right) \cup \left[\frac{6}{4},\frac{7}{4}\right) \cup \left[\frac{7}{4},\frac{8}{4}\right) \cup [2,\infty].$$

We have decomposed $[0,\infty]$ into $\underbrace{2^2+2^2}_{2}+1$ subintervals at the 2nd step.

Form pre-images

$$E_{21} = f^{-1}\left(\left[0,\frac{1}{4}\right)\right), E_{22} = f^{-1}\left(\left[\frac{1}{4},\frac{1}{2}\right)\right), \ldots, E_{28} = f^{-1}\left(\left[\frac{7}{4},\frac{8}{4}\right)\right), E_2 = f^{-1}([2,\infty]).$$

Define $\varphi_2 = 0 \cdot \chi_{E_{21}} + \frac{1}{4} \cdot \chi_{E_{22}} + \cdots + \frac{7}{4} \cdot \chi_{E_{28}} + 2 \cdot \chi_{E_2}$,

or

$$\varphi_2 = \sum_{i=1}^{2 \cdot 2^2} \frac{i-1}{2^2} \cdot \chi_{E_{2i}} + 2\chi_{E_2}.$$

Note that

$$E_{1i} = E_{2\ \ 2i-1} \cup E_{2\ \ 2i}$$

for $i = 1,2$.

$$\vdots$$

Step k. $[0,\infty] = [0,1) \cup [1,2) \cup \cdots \cup [k-1,k) \cup [k,\infty]$ and partition into $\underbrace{2^k + 2^k + \cdots + 2^k}_{k} + 1 = k \cdot 2^k + 1$ disjoint subintervals and form inverse images.

Thus,

$$\varphi_k = \sum_{i=1}^{k \cdot 2^k} \frac{i-1}{2^k} \chi_{E_{ki}} + k\chi_{E_k}.$$

Note that $E_{ki} = E_{k+1\ \ 2i-1} \cup E_{k+1\ \ 2i}$. To construct φ_{k+1}, divide the intervals $\left[\frac{i-1}{2^k}, \frac{i}{2^k}\right)$ in half, and then φ_k to φ_{k+1} at those x's where φ_k changes.

Surely φ_k are nonnegative simple functions. We must show $\varphi_k \le \varphi_{k+1}$ and $\lim_{k\to\infty}\varphi_k = f$ on E.

Now, we are going to prove $\varphi_k \le \varphi_{k+1}$ on E. Recall that $E_{ki} = E_{k+1\ \ 2i-1} \cup E_{k+1\ \ 2i}$. If $x_0 \in E_{ki}$, for some i, then $\varphi_k(x_0) = \frac{i-1}{2^k}$ and so $\varphi_{k+1}(x_0) = \frac{2i-2}{2^{k+1}} = \frac{i-1}{2^k}$ or $\varphi_{k+1}(x_0) = \frac{2i-1}{2^{k+1}}$, and thus $\varphi_k(x_0) \le \varphi_{k+1}(x_0)$. If $x_0 \notin E_{ki}$, for $i = 1,2,\ldots,k\cdot 2^k$, then $x_0 \in E_k = f^{-1}([k,\infty]) = f^{-1}([k,k+1)) \cup f^{-1}([k+1,\infty])$.

Thus

$$x_0 \in f^{-1}([k,k+1)),$$

and so $\varphi_k(x_0) = k$ and $\varphi_{k+1}(x_0) = \frac{j}{2^{k+1}} > \frac{2k\cdot 2^k}{2^{k+1}} = k = \varphi_k(x_0)$

or

$$x_0 \in f^{-1}([k+1,\infty]),$$

and so $\varphi_{k+1}(x_0) = k+1 > k = \varphi_k(x_0)$.

We are left with proving $\lim_{k\to\infty}\varphi_k = f$ on E. If $f(x_0) = \infty$, then $\varphi_k(x_0) = k \forall k$ and $\lim_{k\to\infty}\varphi_k(x_0) = \infty$. If $f(x_0) < \infty$, then for $k > f(x_0), 0 \le f(x_0) - \varphi_k(x_0) < \frac{1}{2^k}$ and $\lim_{k\to\infty}\varphi_k(x_0) = f(x_0)$.

If f is nonnegative and bounded on E, say $, 0 \le f \le M$ on E, then for all $k > M, 0 \le f(x) - \varphi_k(x) < \frac{1}{2^k}$ for all $x \in E$, that is, $\lim_{k\to\infty}\varphi_k = f\,(unif)$ on E.

In the general case (f may be negative), recall that $f = f^+ - f^-$ where f^+, f^- are nonnegative measurable functions on E (Proposition 4.1.8). Apply the above arguments to f^+ and f^-, noting that the difference of simple functions is again a simple function. This completes the proof. □

4.4 Almost Uniform Convergence

In this section, we will prove a remarkable theorem due to Egoroff: If we have point-wise convergence of a sequence of measurable functions on a set of finite measure, then we have uniform convergence on a "large" subset of that set.

Theorem 4.4.1 (Egoroff, 1911) *Let E be a measurable set of real numbers with finite measure. If $\{f_k\}$ is a sequence of measurable functions which converge to a real-valued function f almost everywhere on E, then, given $\varepsilon > 0$, there exists a measurable subset E_ε of E such that $\mu(E \cap E_\varepsilon) < \varepsilon$ and the sequence $\{f_k\}$ converges uniformly to f on E_ε.*

Proof. By using the sequence $\{f_k\}$, we want to construct a monotonically decreasing sequence of nonnegative measurable functions, $\{g_k\}$. This new sequence will be shown to converge uniformly to zero on E_ε, from which it will immediately follow that $\lim_{k\to\infty} f_k = f(unif)$ on E_ε.

Let $A = \{x \in E \mid \lim_{k\to\infty} f_k \neq f\}$. Since A has measure zero, $\mu(E) = \mu(E \cap A^c)$, and $\lim_{k\to\infty} f_k = f$ on $E \cap A^c$. Since the limit of a sequence of measurable functions is measurable (Theorem 4.2.1), f is a measurable function on $E \cap A^c$. Define $g_k = \sup\{|f_k - f|, |f_{k+1} - f|, \ldots\}$ for $k = 1,2,\ldots$, we have from Theorem 4.1.7, Proposition 4.1.8 and Theorem 4.2.1, that g_k is measurable on $E \cap A^c$. Furthermore, $0 \le g_{k+1} \le g_k$ and $\lim_{k\to\infty} g_k = 0$ on $E \cap A^c$. This implies the sequence g_k is a monotone decreasing sequence of nonnegative measurable functions converging to zero on $E \cap A^c$.

The technical aspect of the argument begins: Let $\varepsilon > 0$ be given.

Stage 1:

Construct an increasing sequence of measurable subsets of $E \cap A^c$: $E_k^1 \equiv \{x \in E \cap A^c \mid g_k(x) < 1\}$. Clearly $E_1^1 \subset E_2^1 \subset \cdots$, $\bigcup_{k=1}^{\infty} E_k^1 = E \cap A^c, E_k^1$ is measurable. By Theorem 3.4.1, $\lim_{k\to\infty} \mu(E_k^1) = \mu(E \cap A^c)$, so for k sufficiently large, say K_1, $0 \le \mu(E \cap A^c) - \mu(E_{K_1}^1) < \frac{\varepsilon}{2}$. $0 \le g_k < 1$ for all $k \ge K_1$.

Stage 2:

Construct another increasing sequence of measurable subsets of $E \cap A^c$: $E_k^2 \equiv \left\{x \in E \cap A^c \mid g_k(x) < \frac{1}{2}\right\}$. Clearly $E_1^2 \subset E_2^2 \subset \cdots$, $\bigcup_{k=1}^{\infty} E_k^2 = E \cap A^c, E_k^2$ is measurable. By Theorem 3.4.1, $\lim_{k\to\infty} \mu(E_k^2) = \mu(E \cap A^c)$, so for k sufficiently large, say K_2, $0 \le \mu(E \cap A^c) - \mu(E_{K_2}^2) < \frac{\varepsilon}{2^2}$. $0 \le g_k < \frac{1}{2}$ for all $k \ge K_2$ on $E_{K_2}^2$.

$$\vdots$$

Stage n:

Construct another increasing sequence of measurable subsets of $E \cap A^c$: $E_k^n \equiv \left\{x \in E \cap A^c \mid g_k(x) < \frac{1}{n}\right\}$. Clearly $E_1^n \subset E_2^n \subset \cdots$, $\bigcup_{k=1}^{\infty} E_k^n = E \cap A^c, E_k^n$ is measurable. By Theorem 3.4.1, $\lim_{k\to\infty} \mu(E_k^n) = \mu(E \cap A^c)$, so for k sufficiently large, say K_n, $0 \le \mu(E \cap A^c) - \mu(E_{K_n}^n) < \frac{\varepsilon}{2^n}$ with $0 \le g_k < \frac{1}{n}$ for all $k \ge K_n$ on $E_{K_n}^n$.

Each of the sets $E_{K_1}^1, E_{K_2}^2, \ldots, E_{K_n}^n, \ldots$ is "almost" $E \cap A^c$. We will show that

$$E_\varepsilon = \bigcap_{n=1}^{\infty} E_{K_n}^n$$

is "almost" $E \cap A^c$ and that we have uniform convergence on E_ε.

$$E = (E \cap A^c) \cup A$$
$$\Rightarrow E \cap E_\varepsilon^c = [(E \cap A^c) \cap E_\varepsilon^c] \cup (A \cap E_\varepsilon^c)$$

Thus

$$\begin{aligned}
\mu(E \cap E_\varepsilon^c) &= \mu([(E \cap A^c) \cap E_\varepsilon^c] \cup (A \cap E_\varepsilon^c)) \\
&= \mu[(E \cap A^c) \cap E_\varepsilon^c] + \mu(A \cap E_\varepsilon^c) \\
&\le \mu\left((E \cap A^c) \cap \left(\bigcap_{n=1}^{\infty} E_{K_n}^n\right)^c\right) + \mu(A) \\
&= \mu\left((E \cap A^c) \cap \left(\bigcap_{n=1}^{\infty} E_{K_n}^n\right)^c\right) + 0 \\
&= \mu\left(\bigcup_{n=1}^{\infty} ((E \cap A^c) \cap E_{K_n}^{n\ c})\right) \\
&\le \sum_{n=1}^{\infty} \mu((E \cap A^c) \cap E_{K_n}^{n\ c}) \\
&= \sum_{n=1}^{\infty} (\mu(E \cap A^c) - \mu(E_{K_n}^n)) \\
&< \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} \\
&= \varepsilon.
\end{aligned}$$

It remains to show uniform convergence on $E_\varepsilon = \bigcap_{n=1}^{\infty} E_{K_n}^n$.

Let $\delta > 0$ be given. We want to show $|g_k - 0| < \delta$ on $E_\varepsilon = \bigcap_{n=1}^{\infty} E_{K_n}^n$ for k sufficiently large.

Choose N so that $\frac{1}{N} < \delta$. Recall $E_{K_N}^N = \left\{ x \in E \cap A^c \mid g_{K_N} < \frac{1}{N} \right\}$. For all $k \ge K_N$, $E_\varepsilon = \bigcap_{n=1}^{\infty} E_{K_n}^n \subset E_{K_N}^N$, and $g_k \le g_{K_N}$. This implies

$$|g_k - 0| < \delta \text{ for all } k \ge K_N,$$

and all

$$x \in \bigcap_{n=1}^{\infty} E_{K_n}^n.$$

□

In 1912, Lusin proved that measurable functions are "almost" continuous. We use Egoroff s Theorem (Theorem 4.4.1) to establish this result. Before that, we have the next lemma.

Lemma 4.4.2 *Let* $\{f_n\}$ *be a sequence of real-valued functions, each of which is continuous at a point* $c \in E$. *Suppose* $\lim_{n\to\infty} f_n = f\ (unif)$ *on* E. *Then* f *is continuous at* $c \in E$.

Proof. $|f(x)-f(c)| \le |f(x)-f_n(x)| + |f_n(x)-f_n(c)| + |f_n(c)-f(c)|$. The first and third terms on the right-hand side are small by uniform convergence on E for $n \ge N(\varepsilon)$. Once $N(\varepsilon)$ is selected, the middle term can be made small under the assumption that $f_{N(\varepsilon)}$ is continuous at $c \in E$. □

Theorem 4.4.3 (Lusin, 1912) *Let* E *be a measurable set of real numbers with finite measure. If* f *is a real-valued measurable function defined on* E, *then we may construct a closed subset* E_ε *of* E *so that* $\mu(E \cap E_\varepsilon) < \varepsilon$ *and* f *is continuous on* E_ε.

Proof.

The idea here is to approximate f with a sequence of simple functions $\{\varphi_k\}$ (Theorem 4.3.5), each being continuous except possibly at a finite set of points, and thus the set of discontinuities of all members, being a countable set, has measure zero.

Cover this set with discontinuities, with a sequence of open intervals $\{I_k\}$ so that

$$\mu\left(\bigcup_{k=1}^{\infty} I_k\right) < \frac{\varepsilon}{2}.$$

On the closed set $E \cap \left(\bigcup_{k=1}^{\infty} I_k\right)^c$, the simple functions are continuous and converge to f.

Apply Egoroff s Theorem (Theorem 4.4.1), we have E_ε so that

$$E_\varepsilon \subset E \cap \left(\bigcup_{k=1}^{\infty} I_k \right)^c, \mu\left(\left(E \cap \left(\bigcup_{k=1}^{\infty} I_k \right)^c \right) \cap E_\varepsilon \right) < \frac{\varepsilon}{2}.$$

This implies

$$\begin{aligned}
&\mu\left(\left(\bigcup_{k=1}^{\infty} I_k \right) \cup E_\varepsilon \right) \leq \mu\left(\bigcup_{k=1}^{\infty} I_k \right) + \mu(E_\varepsilon) \\
&\Rightarrow -\mu\left(\left(\bigcup_{k=1}^{\infty} I_k \right) \cup E_\varepsilon \right) \geq -\mu\left(\bigcup_{k=1}^{\infty} I_k \right) - \mu(E_\varepsilon) \\
&\Rightarrow \mu(E) - \mu\left(\left(\bigcup_{k=1}^{\infty} I_k \right) \cup E_\varepsilon \right) \geq -\mu\left(\bigcup_{k=1}^{\infty} I_k \right) + \mu(E) - \mu(E_\varepsilon) \\
&\Rightarrow \mu(E) - \mu\left(\left(\bigcup_{k=1}^{\infty} I_k \right) \cup E_\varepsilon \right) \geq -\mu\left(\bigcup_{k=1}^{\infty} I_k \right) + \mu(E \cap E_\varepsilon^c).
\end{aligned}$$

Thus

$$\begin{aligned}
\mu(E \cap E_\varepsilon^c) &\leq \mu(E) - \mu\left(\left(\bigcup_{k=1}^{\infty} I_k \right) \cup E_\varepsilon \right) + \mu\left(\bigcup_{k=1}^{\infty} I_k \right) \\
&= \mu\left(E \cap \left(\bigcup_{k=1}^{\infty} I_k \right)^c \cap E_\varepsilon^c \right) + \mu\left(\bigcup_{k=1}^{\infty} I_k \right) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon,
\end{aligned}$$

and so $\varphi_k \to f$ uniformly on E_ε.

Since the uniform limit of a sequence of continuous functions is continuous (Lemma 4.4.2), the proof is complete. □

CHAPTER 5

LEBESGUE INTEGRATION

The Riemann integral is introduced in Section 5.1. A similar approach via simple functions yields the Lebesgue integral for bounded functions on Lebesgue measurable sets of finite measure in the next section. Next, we restrict our attention to nonnegative Lebesgue measurable functions whose domain need not have finite measure, and then in the next section remove the condition that f be nonnegative. Finally, Section 5.5 is concerned with the applications of Lebesgue Dominated Convergence Theorem.

5.1 The Riemann Integral

The Riemann integral of a step function is defined in the obvious way. The extension to more general bounded functions f on $[a,b]$ is via approximation from above and below by step functions.

Definition 5.1.1 *A real-valued function ϕ with domain $[a,b]$ is called a step function if there is a partition*

$$a = x_0 < x_1 < \cdots < x_n = b$$

of the interval such that ϕ is constant on each subinterval $I_k = (x_{k-1}, x_k)$;
that is,

$$\phi(x) = c_k \, for \, x \in I_k, k = 1,2,\ldots,n,$$

with $\phi(x_k) = d_k, k = 0,1,\ldots,n.$

Definition 5.1.2 *Let ϕ be a step function on $[a,b]$:*

$$\phi(x)=\begin{cases} c_k, & x_{k-1}<x<x_k,\ k=1,2,\dots,n \\ d_k, & x=x_k, \qquad\ \ k=0,1,\dots,n. \end{cases}$$

The Riemann integral of ϕ on $[a,b]$, denoted by $\int_a^b \phi(x)dx=\sum_{k=1}^{n} c_k(x_k-x_{k-1})$.

We could write $\phi=\sum_{k=1}^{n} c_k\chi_{(x_{k-1},x_k)}+\sum_{k=0}^{n} d_k\chi_{\{x_k\}}$,

and

$$\begin{aligned}\int_a^b \phi(x)dx &= \sum_{k=1}^{n} c_k\mu((x_{k-1},x_k))+\sum_{k=0}^{n} d_k\mu(\{d_k\}) \\ &= \sum_{k=1}^{n} c_k(x_k-x_{k-1}).\end{aligned}$$

The step function's values at the endpoints of the subintervals have no bearing on the existence or value of the Riemann integral of a step function (d_k does not appear in the definition of the integral).

The value of the Riemann integral of a step function is independent of the choice of the partition of $[a,b]$ as long as the step function is constant on the open subintervals of the partition.

More formally, the Riemann integral of a step function is well defined; it is independent of the particular representation of ϕ.

Lemma 5.1.3 *If ϕ and ψ are two step functions, then there is a common partition $P=\{x_0<x_1<\cdots<x_n\}$ such that ϕ and ψ are step functions with partition P, so that*

$$\phi=\sum_{j=1}^{n} a_j\chi_{(x_{j-1},x_j)}+\sum_{i=0}^{n}\phi(x_i)\chi_{\{x_i\}}$$

and

$$\psi=\sum_{j=1}^{n} b_j\chi_{(x_{j-1},x_j)}+\sum_{i=0}^{n}\psi(x_i)\chi_{\{x_i\}}.$$

Proof. Suppose ϕ and ψ are two step functions so that $\phi(x) = c_j$, where $j \in (z_{j-1}, z_j)$ for some finite partition $P_1 = \{z_0 < z_1 < \cdots < z_{n_1}\}$ $(j = 1,2,\ldots,n_1)$ and $\psi(x) = d_k$, where $k \in (z_{k-1}, z_k)$ for some finite partition $P_2 = \{y_0 < y_1 < \cdots < y_{n_2}\}$ $(k = 1,2,\ldots,n_2)$.

$$\phi(x) = 0 \text{ for } x > z_{n_1} \text{ or } x < z_0.$$

$$\psi(x) = 0 \text{ for } x > y_{n_2} \text{ or } x < y_0.$$

Then $P_1 \cup P_2$ is finite and so can be rearranged to a finite partition

$$P = P_1 \cup P_2 = \{x_0 < x_1 < \cdots < x_n\}$$

which works for both step functions ϕ and ψ, with $a_i = c_j$ for some j or $a_i = 0$; and $b_i = d_k$ for some k or $b_i = 0$. □

Theorem 5.1.4 *If ϕ and ψ are step functions on $[a,b]$ and k is any real number, then*

1. $(k\phi)$ is a step function on $[a,b]$, and $\int_a^b (k\phi)(x)dx = k\int_a^b \phi(x)dx$ (homogeneous);

2. $(\phi + \psi)$ is a step function on $[a,b]$, and $\int_a^b (\phi + \psi)(x)dx = \int_a^b \phi(x)dx + \int_a^b \psi(x)dx$ (additive);

3. $\int_a^b \phi(x)dx \le \int_a^b \psi(x)dx$ if $\phi \le \psi$ on $[a,b]$ (monotone);

4. If $a < c < b$, the integrals $\int_a^c \phi(x)dx, \int_c^b \phi(x)dx$ exist, and $\int_a^c \phi(x)dx + \int_c^b \phi(x)dx = \int_a^b \phi(x)dx$ (additive on the domain).

Proof. Let ϕ and ψ be two arbitrary step functions on $[a,b]$:

$$\phi(x) = \begin{cases} c_k, & x_{k-1} < x < x_k, \quad k = 1,2,\ldots,n \\ d_k, & x = x_k, \quad k = 0,1,\ldots,n. \end{cases}$$

$$\psi(x) = \begin{cases} c_k', & x_{k-1}' < x < x_k', \quad k = 1,2,\ldots,n \\ d_k', & x = x_k', \quad k = 0,1,\ldots,n. \end{cases}$$

The respective Riemann integrals of ϕ and ψ on $[a,b]$, are denoted by $\int_a^b \phi(x)dx = \sum_{k=1}^{n} c_k(x_k - x_{k-1})$ and $\int_a^b \psi(x)dx = \sum_{k=1}^{n} c_k{}'(x_k{}' - x_{k-1}{}')$.

1. Clearly, $k\phi$ is a step function on $[a,b]$. $\int_a^b (k\phi)(x)dx = \sum_{k=1}^{n} kc_k(x_k - x_{k-1}) =$ $k\sum_{k=1}^{n} c_k(x_k - x_{k-1}) = k\int_a^b \phi(x)dx.$

2. By Lemma 5.1.3, there exists a common partition $P = \{x_0 < x_1 < \cdots < x_n\}$ for ϕ and ψ, so that

$$\phi = \sum_{j=1}^{n} a_j \chi_{(x_{j-1},x_j)} + \sum_{i=0}^{n} \phi(x_i)\chi_{\{x_i\}}$$

and

$$\psi = \sum_{j=1}^{n} b_j \chi_{(x_{j-1},x_j)} + \sum_{i=0}^{n} \psi(x_i)\chi_{\{x_i\}},$$

with respective Riemann integrals

$$\int_a^b \phi(x)dx = \sum_{j=1}^{n} a_j(x_j - x_{j-1})$$

and

$$\int_a^b \psi(x)dx = \sum_{j=1}^{n} b_j(x_j - x_{j-1}).$$

Thus

$$\phi + \psi = \sum_{j=1}^{n} (a_j + b_j)\chi_{(x_{j-1},x_j)} + \sum_{i=0}^{n} [\phi(x_i) + \psi(x_i)]\chi_{\{x_i\}}$$

and so $\phi + \psi$ is a step function on $[a,b]$.

Also

$$\int_a^b (\phi+\psi)(x)dx = \sum_{j=1}^{n} (a_j + b_j)(x_j - x_{j-1})$$

$$= \sum_{j=1}^{n} [a_j(x_j - x_{j-1}) + b_j(x_j - x_{j-1})]$$

$$= \sum_{j=1}^{n} a_j(x_j - x_{j-1}) + \sum_{j=1}^{n} b_j(x_j - x_{j-1})$$

$$= \int_a^b \phi(x)dx + \int_a^b \psi(x)dx.$$

3. Suppose $\phi \le \psi$, then by Lemma 5.1.3, we have

$$\phi = \sum_{j=1}^{n} a_j \chi_{(x_{j-1},x_j)} + \sum_{i=0}^{n} \phi(x_i)\chi_{\{x_i\}}$$

and

$$\psi = \sum_{j=1}^{n} b_j \chi_{(x_{j-1},x_j)} + \sum_{i=0}^{n} \psi(x_i)\chi_{\{x_i\}},$$

with $a_j \le b_j$ and respective Riemann integrals

$$\int_a^b \phi(x)dx = \sum_{j=1}^{n} a_j(x_j - x_{j-1})$$

and

$$\int_a^b \psi(x)dx = \sum_{j=1}^{n} b_j(x_j - x_{j-1}).$$

Since $a_j \le b_j$, this implies $\int_a^b \phi(x)dx \le \int_a^b \psi(x)dx.$

4. If c is one of the endpoints of subintervals between a and b, then we are done. Assume c is any point inside some subinterval (x_{n_1}, x_{n_1+1}) of $[a,b]$ for some positive integer n_1, then we have:

$$\int_a^c \phi(x)dx = \sum_{k=1}^{n_1} c_k(x_k - x_{k-1}) + c_{n_1+1}(c - x_{n_1}),$$

and

$$\int_c^b \phi(x)dx = \sum_{k=n_1+2}^{n} c_k(x_k - x_{k-1}) + c_{n_1+1}(x_{n_1+1} - c).$$

Adding them gives

$$\int_a^c \phi(x)dx + \int_c^b \phi(x)dx = \int_a^b \phi(x)dx.$$

□

We now define the Riemann integral for more general functions f on $[a,b]$. Since we will be approximating f from above and below by step functions it is imperative that f be a bounded function on $[a,b]$.

Definition 5.1.5 *Let f be a bounded function on $[a,b]$, say, $\alpha \le f(x) \le \beta$, for $x \in [a,b]$. Let ϕ, ψ denote arbitrary step functions on $[a,b]$ such that $\phi \le f \le \psi$.*

The lower Riemann integral of f on $[a,b]$, $\underline{\int_a^b} f(x)dx$, is denoted by

$$\underline{\int_a^b} f(x)dx = \sup\left\{ \int_a^b \phi(x)dx \mid \phi \le f, \phi \text{ a step function} \right\}.$$

The upper Riemann integral of f on $[a,b]$, $\overline{\int_a^b} f(x)dx$, is denoted by

$$\overline{\int_a^b} f(x)dx = \inf\left\{ \int_a^b \psi(x)dx \mid f \le \psi, \psi \text{ a step function} \right\}.$$

We would hope that the approximation from "above" and "below" approach a common value, to be called the Riemann integral of f on $[a,b]$.

Definition 5.1.6 *A bounded function f is Riemann integrable on $[a,b]$ whenever $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$. Denote the common value by $\int_a^b f(x)dx$; $\underline{\int_a^b} f(x)dx = \int_a^b f(x)dx = \overline{\int_a^b} f(x)dx$.*

We have defined what it means for a bounded function f on $[a,b]$ to be Riemann integrable; a common value for the lower and upper Riemann integrals. An equivalent condition, that is frequently easier to apply, is given by Theorem 5.1.6.

Theorem 5.1.7 *A bounded function f is Riemann integrable iff for every* $\varepsilon > 0,$ *we have step functions* ϕ *and* $\psi, \phi \le f \le \psi$ *on* $[a,b],$ *so that*

$$0 \le \int_a^b \psi(x)dx - \int_a^b \phi(x)dx = \int_a^b [\psi(x) - \phi(x)]dx < \varepsilon.$$

Proof. Assume the bounded function f is Riemann integrable on $[a,b]$ and let $\varepsilon > 0$ be given. From Definition 5.1.4, we have step functions $\hat{\phi}$ and $\hat{\psi}, \hat{\phi} \le f \le \hat{\psi}$, so that

$$\begin{aligned}\int_a^b f(x)dx - \frac{\varepsilon}{2} = \underline{\int_a^b} f(x)dx - \frac{\varepsilon}{2} < \int_a^b \hat{\phi}(x)dx \le \underline{\int_a^b} f(x)dx \\ \le \overline{\int_a^b} f(x)dx \le \int_a^b \hat{\psi}(x)dx < \overline{\int_a^b} f(x)dx + \frac{\varepsilon}{2} \\ = \int_a^b f(x)dx + \frac{\varepsilon}{2}.\end{aligned}$$

So

$$\begin{aligned}\int_a^b \hat{\phi}(x)dx \le \int_a^b \hat{\psi}(x)dx < \overline{\int_a^b} f(x)dx + \frac{\varepsilon}{2} \\ \Rightarrow 0 \le \int_a^b \hat{\psi}(x)dx - \int_a^b \hat{\phi}(x)dx < \underline{\int_a^b} f(x)dx - \int_a^b \hat{\phi}(x)dx + \frac{\varepsilon}{2} \\ < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ = \varepsilon.\end{aligned}$$

Thus

$$0 \le \int_a^b \psi(x)dx - \int_a^b \phi(x)dx = \int_a^b [\psi(x) - \phi(x)]dx < \varepsilon.$$

Conversely, let $\varepsilon > 0$ be given and assume we have step functions ϕ and $\psi, \phi \le f \le \psi$ on $[a,b],$ so that

$$0 \le \int_a^b \psi(x)dx - \int_a^b \phi(x)dx = \int_a^b [\psi(x) - \phi(x)]dx < \varepsilon.$$

But, for any bounded function f on $[a,b],$

$$\int_a^b \phi(x)dx \le \underline{\int_a^b} f(x)dx \le \overline{\int_a^b} f(x)dx \le \int_a^b \psi(x)dx.$$

So

$$\begin{aligned} 0 \le \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx &\le \int_a^b \psi(x)dx - \underline{\int_a^b} f(x)dx \\ &\le \int_a^b \psi(x)dx - \int_a^b \phi(x)dx \\ &= \int_a^b [\psi(x) - \phi(x)]dx \\ &< \varepsilon. \end{aligned}$$

We conclude that

$$0 \le \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx < \varepsilon.$$

By the arbitrary nature of ε,

$$\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx = \int_a^b f(x)dx,$$

that is, f, is Riemann integrable on $[a,b]$. □

Theorem 5.1.8 *Every continuous function f on* $[a,b]$ *is Riemann integrable.*

Proof. Since f is a continuous function on a compact interval $[a,b]$, so f is bounded, $\underline{\int_a^b} f(x)dx$ and $\overline{\int_a^b} f(x)dx$ are well-defined and f is uniformly continuous on $[a,b]$.

Thus given $\varepsilon > 0$, choose $\delta > 0$, so that

$$| f(x) - f(y) | < \frac{\varepsilon}{b-a}$$

whenever $| x - y | < \delta$, for all $x, y \in [a,b]$.

Take any partition $P = \{x_0 < x_1 < \cdots < x_n\}$ of $[a,b]$ so that $| x_k - x_{k-1} | < \delta$, $k = 1,2,\ldots,n$ and, thus,

$$-\frac{\varepsilon}{b-a} < f(x) - f(y) < \frac{\varepsilon}{b-a} \text{ for all } x, y \in (x_{k-1}, x_k),$$

that is, $f(x) < f(y) + \dfrac{\varepsilon}{b-a}$ for all $x \in (x_{k-1}, x_k)$. Therefore, $\sup\limits_{(x_{k-1},x_k)} f \le f(y) + \dfrac{\varepsilon}{b-a}$.

Similarly, $f(y) - \dfrac{\varepsilon}{b-a} < f(x)$ for all $x \in (x_{k-1}, x_k)$. We conclude that $f(y) - \dfrac{\varepsilon}{b-a} \le \inf\limits_{(x_{k-1},x_k)} f$.

It follows that $\sup\limits_{(x_{k-1},x_k)} f - \inf\limits_{(x_{k-1},x_k)} f \le \dfrac{2\varepsilon}{b-a}$.

Define $\psi = \sup f, \phi = \inf f$ on (x_{k-1}, x_k) and f, otherwise.

So $\int\limits_a^b \psi(x)dx - \int\limits_a^b \phi(x)dx \le \dfrac{2\varepsilon}{b-a}(b-a) = 2\varepsilon$, and now apply Theorem 5.1.7 to conclude f is Riemann integrable on $[a,b]$. □

Theorem 5.1.9 *Every monotone function f on $[a,b]$ is Riemann integrable.*

Proof. Without loss of generality, assume f is nondecreasing. Then $f(a) \le f(x) \le f(b)$ for all $x \in [a,b]$ and f is bounded.

Form the partition $a < a + \dfrac{b-a}{n} < a + 2\left(\dfrac{b-a}{n}\right) < \cdots < a + n\left(\dfrac{b-a}{n}\right) = b.$

Define $\phi(x) = f\left(a + (k-1)\left(\dfrac{b-a}{n}\right)\right)$ and $\psi(x) = f\left(a + k\left(\dfrac{b-a}{n}\right)\right)$ for $x_{k-1} = a + (k-1)\left(\dfrac{b-a}{n}\right) < x < a + k\left(\dfrac{b-a}{n}\right) = x_k$ and f, otherwise.

Hence

$$\int_a^b \psi(x)dx - \int_a^b \phi(x)dx = \left(f\left(a + \frac{b-a}{n}\right) \cdot \frac{b-a}{n} + f\left(a + 2 \cdot \frac{b-a}{n}\right) \cdot \frac{b-a}{n} \right.$$
$$\left. + \cdots + f\left(a + (n-1) \cdot \frac{b-a}{n}\right) \cdot \frac{b-a}{n} + f(b) \cdot \frac{b-a}{n} \right)$$
$$- \left(f(a) \cdot \frac{b-a}{n} + f\left(a + \frac{b-a}{n}\right) \cdot \frac{b-a}{n} + \cdots + f\left(a + (n-1) \cdot \frac{b-a}{n}\right) \cdot \frac{b-a}{n} \right)$$
$$\leq (f(b) - f(a)) \cdot \frac{b-a}{n}.$$

By Theorem 5.1.7, f is Riemann integrable on $[a,b]$. □

The next result characterizes Riemann integrability.

Theorem 5.1.10 (Lebesgue, 1902) *A bounded function on a closed bounded interval is Riemann integrable iff the function is continuous almost everywhere. (The set of discontinuities is limited to a set of Lebesgue measure zero.)*

Proof. We will use Theorem 5.1.7: A bounded function f on $[a,b]$ is Riemann integrable iff we can approximate f from below and above by step functions whose integrals can be made arbitrarily close to each other: If we have step functions $\phi \leq f \leq \psi$, the difference $\psi - \phi$ is a measure of how much f may vary on any subinterval of a partition.

The standard means of measuring this variation of f on an interval J is by calculating $\sup\{f(x) \mid x \in J \bigcap [a,b]\} - \inf\{f(x) \mid x \in J \bigcap [a,b]\}$, traditionally denoted by $\omega_f(J)$. This real number $\omega_f(J)$ (f is bounded on $[a,b]$) is nonnegative. It is natural then to measure the variation of f at a point $x_0 \in [a,b]$, denoted by $\omega_f(x_0)$, as $\omega_f(x_0) = \inf\{\omega_f(I) \mid I$ any open interval containing $x_0\}$.

We would expect that for a function continuous at x_0, $\omega_f(x_0) = 0$. In fact, for a bounded function f on $[a,b]$, f is continuous at x_0 iff $\omega_f(x_0) = 0$.

Here is a rough sketch of the argument to establish this result.

Suppose f is continuous at $x_0 \in [a,b]$ and $\varepsilon > 0$ is given. We have $\delta > 0$ so that for $x \in (x_0 - \delta, x_0 + \delta) \bigcap [a,b], -\varepsilon < f(x) - f(x_0) < \varepsilon,$ that is

$$f(x) < f(x_0) + \varepsilon,$$

and

$$\sup\{f(x) \mid x \in (x_0 - \delta, x_0 + \delta) \bigcap [a,b]\} \leq f(x_0) + \varepsilon.$$

On the other hand,

$$f(x_0) - \varepsilon < f(x),$$

and

$$\inf\{f(x) \mid x \in (x_0 - \delta, x_0 + \delta) \bigcap [a,b]\} \geq f(x_0) - \varepsilon.$$

Subtracting gives $\omega_f((x_0 - \delta, x_0 + \delta)) \leq 2\varepsilon.$ Then $\omega_f(x_0) \leq \omega_f((x - \delta, x + \delta)) \leq 2\varepsilon$ for every $\varepsilon > 0.$ So $\omega_f(x_0) = 0.$

Conversely, suppose $\omega_f(x_0) = 0$ for some $x_0 \in [a,b]$ and let $\varepsilon > 0.$ Then $0 = \omega_f(x_0) = \inf\{\omega_f(I) \mid I$ any open interval containing $x_0\}$ implies we have an open interval I^* containing x_0 so that $0 \leq \omega_f(I^*) < \varepsilon.$ Since I^* is open, choose $\delta > 0$ so that $(x_0 - \delta, x_0 + \delta) \subset I^*.$ Then $x \in (x_0 - \delta, x_0 + \delta) \bigcap [a,b] \Rightarrow -\varepsilon < f(x) - f(x_0) < \varepsilon,$ in other words, f is continuous at x_0 and the result is established.

At points x of discontinuity of $f, \omega_f(x) > 0.$ This implies that the set of discontinuities of f, say, $D,$ may be written as

$$D = \bigcup_{n=1}^{\infty} D_n,$$

where $D_n = \left\{x \in [a,b] \mid \omega_f(x) \geq \dfrac{1}{n}\right\}.$

We claim D_n is a closed set in $[a,b],$ or, equivalenly, $D_n^c = \left\{x \in [a,b] \mid \omega_f(x) < \dfrac{1}{n}\right\}$ is an open set in $[a,b].$ Let $x \in D_n^c.$ We will determine a $\delta > 0$ so that $(x - \delta, x + \delta) \bigcap [a,b] \subset D_n^c.$

Since $\omega_f(x) = \inf\{\omega_f(I) \mid I$ any open interval containing $x\} < \frac{1}{n}$, it follows that $\omega_f(x) \le \omega_f(I^*) < \omega_f(x) + \varepsilon$, where $\varepsilon > 0$ is arbitrary. Take $\varepsilon = \frac{1}{n} - \omega_f(x) > 0$, we have an open interval I^* containing x so that $\omega_f(I^*) < \frac{1}{n}$. Since I^* is open, there exists a $\delta > 0$ such that $(x-\delta, x+\delta) \bigcap [a,b] \subset I^* \bigcap [a,b]$. We are done if we can show $(x-\delta, x+\delta) \bigcap [a,b] \subset D_n^c$.

Let $z \in (x-\delta, x+\delta) \bigcap [a,b]$, then $\omega_f(z) \le \omega_f((x-\delta, x+\delta)) \le \omega_f(I^*) < \frac{1}{n}$ implies $z \in D_n^c$. Thus D_n is a closed set in $[a,b]$.

The technical aspect of the argument begins: First suppose f is continuous a.e. on $[a,b]$, that is, the set of discontinuities of f, D, has measure zero, and let $|f| < B$ on $[a,b]$. We want to show f is Riemann integrable on $[a,b]$. Since $D_n = \left\{ x \in [a,b] \mid \omega_f(x) \ge \frac{1}{n} \right\}$ is a subset of D, D_n has measure zero. Thus we have $D_n \subset \bigcup_{k=1}^{\infty} I_k$, I_k open intervals, $\sum_{k=1}^{\infty} l(I_k) < \frac{\varepsilon}{4B}$. But D_n is a closed and bounded subset of $[a,b]$, hence compact. It follows that we have a finite subcover of D_n by the open intervals, I_k, that is,

$$D_n \subset \bigcup_{i=1}^{m} I_{k_i}, I_{k_i} \text{ open intervals.}$$

Then the set $[a,b] \bigcap (I_{k_1} \bigcup I_{k_2} \bigcup \cdots \bigcup I_{k_m})^c$ is a finite union of closed intervals $J_1, J_2, \ldots, J_L$. That is

$$[a,b] = J_1 \bigcup J_2 \bigcup \cdots \bigcup J_L \bigcup I_{k_1} \bigcup I_{k_2} \bigcup \cdots \bigcup I_{k_m}.$$

Recall that all points of D_n are in $I_{k_1} \bigcup I_{k_2} \bigcup \cdots \bigcup I_{k_m}$. Thus $\omega_f(x) < \frac{1}{n}$ on $J_1 \bigcup J_2 \bigcup \cdots \bigcup J_L$. This means $\omega_f(x) < \frac{1}{n}$ for each $x \in J_i$, and so we have an open interval containing I_x so that $\omega_f(I_x) < \frac{1}{n}$. But then the collection $\{I_x\}$ is an open cover of the closed, bounded, and hence compact set, J_i. A finite subcover must cover J_i. We have a partition of J_i, and the "sup of f"-"inf of f" over any of the subintervals of this partition is less than $\frac{1}{n}$. Do this for each $J_i, i = 1,2,\ldots,L$. We have a finite collection of subintervals whose union is $J_1 \bigcup J_2 \bigcup \cdots \bigcup J_L$, and on any one of these subintervals, say J^*,

$$\omega_f(J^*) = \sup\{f(x) \mid x \in J^* \bigcap [a,b]\} - \inf\{f(x) \mid x \in J^* \bigcap [a,b]\} < \frac{1}{n}.$$

Now define step functions ϕ, ψ in the obvious way:

$\phi = \inf f$ on the subintervals of $J_1, J_2, \ldots, J_L$ and $I_{k_1}, I_{k_2}, \cdots, I_{k_m}$,

$\psi = \sup f$ on the subintervals of $J_1, J_2, \ldots, J_L$ and $I_{k_1}, I_{k_2}, \cdots, I_{k_m}$.

Then

$$\begin{aligned}\int_a^b [\psi(x) - \phi(x)]dx &\le \frac{1}{n}\sum_{i=1}^{L} l(\text{ subintervals of } J_i) + 2B\sum_{n=1}^{m} l(I_{k_n}) \\ &\le \frac{1}{n}(b-a) + 2B\left(\frac{\varepsilon}{4B}\right) \\ &< \varepsilon\end{aligned}$$

for n sufficiently large.

By Theorem 5.1.7, f is Riemann integrable on $[a,b]$.

Next suppose f is Riemann integrable on $[a,b]$. We want to show the bounded function f is continuous a.e. on $[a,b]$. Since the set of discontinuities of $f, D,$ is given by $D = \bigcup_{n=1}^{\infty} D_n = \bigcup_{n=1}^{\infty}\left\{x \in [a,b] \mid \omega_f(x) \ge \frac{1}{n}\right\}$, if we can show $\mu(D_n) = 0$, we would be done, since a countable union of sets of measure zero is a set of measue zero. Fix an n, say, N, and consider the set

$$D_N = \left\{x \in [a,b] \mid \omega_f(x) \ge \frac{1}{N}\right\}, \text{with } \varepsilon > 0.$$

By asumption, f is Riemann integrable on $[a,b]$, so we have step functions $\phi \le f \le \psi$ with $\int_a^b [\psi(x)-\phi(x)]dx < \frac{\varepsilon}{N}$. Let $a = x_0 < x_1 < \cdots < x_n = b$ be a partition associated with ϕ and ψ, and split the collection $\{(x_0,x_1),\ldots,(x_{n-1},x_n)\}$ into two subcollections C_1 and C_2, as follows: If $(x_{k-1},x_k) \cap D_N \ne \phi$, put it in C_1. Otherwise, put it in C_2. Then every point of D_N is in C_1 or a member of the set $\{a, x_1, x_2, \ldots, x_{n-1}, b\}$. So

$$\int_a^b [\psi(x)-\phi(x)]dx = \sum_{C_1} + \sum_{C_2} \quad < \frac{\varepsilon}{N}.$$

For $\sum_{C_1}$, some point of D_N is in each subinterval, i.e., each subinterval of C_1 contains a point x with $\omega_f(x) \ge \frac{1}{N}$. But then $\psi - \phi \ge \frac{1}{N}$ on this subinterval. As a result,

$$\frac{\varepsilon}{N} > \sum_{C_1} \ge \frac{1}{N} \sum (\text{ lengths of subintervals that contain points of } D_N),$$

so $\sum_{C_1}(x_k - x_{k-1}) < \varepsilon$. The intervals of C_1, along with the finite set $\{a, x_1, x_2, \ldots, x_{n-1}, b\}$ contain all points of D_N. So $\mu(D_N) < \varepsilon$, this completes the proof. □

We now show that the integral properties that hold for step functions (Theorem 5.1.4) remain valid for Riemann integrable functions.

Theorem 5.1.11 *If bounded functions f and g are Riemann integrable on $[a,b]$, and k is any real number, then*

1. (kf) is Riemann integrable on $[a,b]$, and

$$\int_a^b (kf)(x)dx = k\int_a^b f(x)dx \qquad \textit{(homogeneous);}$$

2. $(f+g)$ is Riemann integrable on $[a,b]$, and

$$\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx \qquad \textit{(additive);}$$

3. $\int_a^b f(x)dx \le \int_a^b g(x)dx$ *if* $f \le g$ *on* $[a,b]$ *(monotone);*

4. If $a < c < b$, f *is Riemann integrable on* $[a,c]$ *and* $[c,b]$, *and*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \qquad \textit{(additive on the domain);}$$

5. If $\alpha \le f(x) \le \beta$ *on* $[a,b]$, *then*

$$\alpha(b-a) \le \int_a^b f(x)dx \le \beta(b-a) \qquad \textit{(mean value).}$$

Proof.

1. $k = 0$ is obvious. Assume $k > 0$ and f is Riemann integrable on $[a,b]$. Let $\varepsilon > 0$ be given. By Theorem 5.1.7, we have step functions ϕ and ψ on $[a,b]$ so that $\phi \le f \le \psi$,

$$\int_a^b \phi(x)dx \le \int_a^b f(x)dx \le \int_a^b \psi(x)dx,$$

and

$$\int_a^b \psi(x)dx - \int_a^b \phi(x)dx < \frac{\varepsilon}{k}.$$

Since $\int_a^b (k\phi)(x)dx = k\int_a^b \phi(x)dx$ and $\int_a^b (k\psi)(x)dx = k\int_a^b \psi(x)dx$ by Theorem 5.1.4,

$$\begin{aligned}\int_a^b (k\phi)(x)dx &= k\int_a^b \phi(x)dx \\ &\le k\int_a^b f(x)dx \\ &\le k\int_a^b \psi(x)dx \\ &= \int_a^b (k\psi)(x)dx\end{aligned}$$

and

$$\int_a^b (k\psi)(x)dx - \int_a^b (k\phi)(x)dx < \varepsilon.$$

But,

$$k\phi \le kf \le k\psi$$

on $[a,b]$. That is, we have step functions $(k\phi),(k\psi)$ so that

$$\int_a^b (k\psi)(x)dx - \int_a^b (k\phi)(x)dx < \varepsilon.$$

By Theorem 5.1.7, (kf) is Riemann integrable on $[a,b]$.

Thus

$$\int_a^b (k\phi)(x)dx \le \int_a^b (kf)(x)dx \le \int_a^b (k\psi)(x)dx.$$

Previously we have

$$\int_a^b (k\phi)(x)dx \le k\int_a^b f(x)dx \le \int_a^b (k\psi)(x)dx.$$

So

$$-\int_a^b (k\psi)(x)dx \le -k\int_a^b f(x)dx \le -\int_a^b (k\phi)(x)dx.$$

Now

$$\int_a^b (k\phi)(x)dx - \int_a^b (k\psi)(x)dx \le \int_a^b (kf)(x)dx - k\int_a^b f(x)dx \le \int_a^b (k\psi)(x)dx - \int_a^b (k\phi)(x)dx$$

$$\begin{aligned}
&\Rightarrow -\varepsilon < -\left[\int_a^b (k\psi)(x)dx - \int_a^b (k\phi)(x)dx\right] \\
&\qquad \le \int_a^b (kf)(x)dx - k\int_a^b f(x)dx \\
&\qquad \le \int_a^b (k\psi)(x)dx - \int_a^b (k\phi)(x)dx < \varepsilon \\
&\Rightarrow -\varepsilon < \int_a^b (kf)(x)dx - k\int_a^b f(x)dx < \varepsilon \\
&\Rightarrow 0 \le \left|\int_a^b (kf)(x)dx - k\int_a^b f(x)dx\right| < \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary,

$$\int_a^b (kf)(x)dx - k\int_a^b f(x)dx = 0 \text{ and so } \int_a^b (kf)(x)dx = k\int_a^b f(x)dx.$$

Assume $k < 0$ and f is Riemann integrable on $[a,b]$. Let $\varepsilon > 0$ be given. By Theorem 5.1.7, we have step functions ϕ and ψ on $[a,b]$ so that $\phi \le f \le \psi$,

$$\int_a^b \phi(x)dx \le \int_a^b f(x)dx \le \int_a^b \psi(x)dx,$$

and

$$\int_a^b \psi(x)dx - \int_a^b \phi(x)dx < \frac{\varepsilon}{-k}.$$

Since $\int_a^b (k\phi)(x)dx = k\int_a^b \phi(x)dx$ and $\int_a^b (k\psi)(x)dx = k\int_a^b \psi(x)dx$ by Theorem 5.1.4,

$$\begin{aligned}\int_a^b (k\psi)(x)dx &= k\int_a^b \psi(x)dx \\ &\le k\int_a^b f(x)dx \\ &\le k\int_a^b \phi(x)dx \\ &= \int_a^b (k\phi)(x)dx\end{aligned}$$

and

$$\int_a^b (k\phi)(x)dx - \int_a^b (k\psi)(x)dx < \varepsilon.$$

But,

$$k\psi \le kf \le k\phi$$

on $[a,b]$. That is, we have step functions $(k\psi), (k\phi)$ so that

$$\int_a^b (k\phi)(x)dx - \int_a^b (k\psi)(x)dx < \varepsilon.$$

By Theorem 5.1.7, (kf) is Riemann integrable on $[a,b]$.

Thus

$$\int_a^b (k\psi)(x)dx \le \int_a^b (kf)(x)dx \le \int_a^b (k\phi)(x)dx.$$

Previously we have

$$\int_a^b (k\psi)(x)dx \le k\int_a^b f(x)dx \le \int_a^b (k\phi)(x)dx.$$

So

$$-\int_a^b (k\phi)(x)dx \le -k\int_a^b f(x)dx \le -\int_a^b (k\psi)(x)dx.$$

Now

$$\int_a^b (k\psi)(x)dx - \int_a^b (k\phi)(x)dx \le \int_a^b (kf)(x)dx - k\int_a^b f(x)dx \le \int_a^b (k\phi)(x)dx - \int_a^b (k\psi)(x)dx$$

$$\begin{aligned}
\Rightarrow -\varepsilon &< -\left[\int_a^b (k\phi)(x)dx - \int_a^b (k\psi)(x)dx\right] \\
&\le \int_a^b (kf)(x)dx - k\int_a^b f(x)dx \\
&\le \int_a^b (k\phi)(x)dx - \int_a^b (k\psi)(x)dx < \varepsilon \\
\Rightarrow -\varepsilon &< \int_a^b (kf)(x)dx - k\int_a^b f(x)dx < \varepsilon \\
\Rightarrow 0 &\le \left|\int_a^b (kf)(x)dx - k\int_a^b f(x)dx\right| < \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary,

$$\int_a^b (kf)(x)dx - k\int_a^b f(x)dx = 0 \text{ and so} \int_a^b (kf)(x)dx = k\int_a^b f(x)dx.$$

2. By Theorem 5.1.7, we have

$$\phi_f \le f \le \psi_f \text{ and} \int_a^b \phi_f(x)dx \le \int_a^b f(x)dx \le \int_a^b \psi_f(x)dx,$$

and

$$\phi_g \le g \le \psi_g \text{ and} \int_a^b \phi_g(x)dx \le \int_a^b g(x)dx \le \int_a^b \psi_g(x)dx.$$

Adding, $\phi_f + \phi_g \le f + g \le \psi_f + \psi_g$, by Theorem 5.1.4 yields

$$
\begin{aligned}
\int_a^b \phi_f(x)dx + \int_a^b \phi_g(x)dx &= \int_a^b (\phi_f + \phi_g)(x)dx \\
&\leq \underline{\int_a^b} (f+g)(x)dx \\
&\leq \overline{\int_a^b} (f+g)(x)dx \\
&\leq \int_a^b (\psi_f + \psi_g)(x)dx \\
&= \int_a^b \psi_f(x)dx + \int_a^b \psi_g(x)dx.
\end{aligned}
$$

Apply Theorem 5.1.7 again yields existence of $\int_a^b (f+g)(x)dx,$ and it along with $\int_a^b f(x)dx + \int_a^b g(x)dx,$ lies between $\int_a^b \phi_f(x)dx + \int_a^b \phi_g(x)dx$ and $\int_a^b \psi_f(x)dx + \int_a^b \psi_g(x)dx.$

3.

$$
\begin{aligned}
\int_a^b f(x)dx = \underline{\int_a^b} f(x)dx &= \sup\left\{ \int_a^b \phi(x)dx \mid \phi \leq f, \phi \text{ } a \text{ } step \text{ } function \right\} \\
&\leq \sup\left\{ \int_a^b \phi(x)dx \mid \phi \leq g, \phi \text{ } a \text{ } step \text{ } function \right\}, f \leq g \\
&= \underline{\int_a^b} g(x)dx \\
&= \int_a^b g(x)dx.
\end{aligned}
$$

4. Let $a < c < b.$ Since f is Riemann integrable on $[a,b],$ we have step functions $\phi, \psi,$ a common partition including $c,$ so that $\phi \leq f \leq \psi$ on $[a,b]$ and

$$\int_a^b \psi(x)dx - \int_a^b \phi(x)dx < \varepsilon$$

by Theorem 5.1.7. But

$$\int_a^c \phi(x)dx \leq \underline{\int_a^c} f(x)dx \leq \overline{\int_a^c} f(x)dx \leq \int_a^c \psi(x)dx$$

and

$$
\begin{aligned}
0 &\le \int_a^c \psi(x)dx - \int_a^c \phi(x)dx \\
&= \int_a^c (\psi - \phi)(x)dx \\
&\le \int_a^b (\psi - \phi)(x)dx \\
&< \varepsilon
\end{aligned}
$$

by Theorem 5.1.4 and so $\int_a^c f(x)dx$ is Riemann integrable on $[a,c]$ by Theorem 5.1.7.

On the other hand

$$\int_c^b \phi(x)dx \le \underline{\int_c^b} f(x)dx \le \overline{\int_c^b} f(x)dx \le \int_c^b \psi(x)dx$$

and

$$
\begin{aligned}
0 &\le \int_c^b \psi(x)dx - \int_c^b \phi(x)dx \\
&= \int_c^b (\psi - \phi)(x)dx \\
&\le \int_a^b (\psi - \phi)(x)dx \\
&< \varepsilon
\end{aligned}
$$

by Theorem 5.1.4 and so $\int_c^b f(x)dx$ is Riemann integrable on $[c,b]$ by Theorem 5.1.7.

So we have

$$\int_a^c \phi(x)dx \le \int_a^c f(x)dx \le \int_a^c \psi(x)dx$$

and

$$\int_c^b \phi(x)dx \le \int_c^b f(x)dx \le \int_c^b \psi(x)dx.$$

Adding them gives

$$\int_a^b \phi(x)dx \le \int_a^c f(x)dx + \int_c^b f(x)dx \le \int_a^b \psi(x)dx.$$

Since f is Riemann integrable on $[a,b]$, therefore

$$\int_a^b \phi(x)dx \le \int_a^b f(x)dx \le \int_a^b \psi(x)dx.$$

This implies

$$-\int_a^b \psi(x)dx \le -\int_a^b f(x)dx \le -\int_a^b \phi(x)dx$$

and so

$$\int_a^b \phi(x)dx - \int_a^b \psi(x)dx \le \int_a^c f(x)dx + \int_c^b f(x)dx - \int_a^b f(x)dx \le \int_a^b \psi(x)dx - \int_a^b \phi(x)dx$$

$$\begin{aligned}
&\Rightarrow -\varepsilon < -\left[\int_a^b \psi(x)dx - \int_a^b \phi(x)dx\right] \\
&\quad \le \int_a^c f(x)dx + \int_c^b f(x)dx - \int_a^b f(x)dx \\
&\quad \le \int_a^b \psi(x)dx - \int_a^b \phi(x)dx < \varepsilon \\
&\Rightarrow 0 \le \left|\int_a^c f(x)dx + \int_c^b f(x)dx - \int_a^b f(x)dx\right| < \varepsilon.
\end{aligned}$$

By the arbitrary nature of $\varepsilon > 0$,

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

5. The constant functions α and β are step functions satisfying $\alpha \le f \le \beta$ on $[a,b]$. From part 3,

$$\alpha(b-a) = \int_a^b \alpha dx \le \int_a^b f(x)dx \le \int_a^b \beta dx = \beta(b-a)$$

and so

$$\alpha(b-a) \le \int_a^b f(x)dx \le \beta(b-a).$$

□

The uniform limit of a sequence of Riemann integrable functions is Riemann integrable and the integral of the limit is the limit of the integral.

Theorem 5.1.12 *Suppose* $\{f_k\}$ *is a sequence of Riemann integrable functions on* $[a,b]$. *If* $\lim_{k\to\infty} f_k = f\,(unif)$ *on* $[a,b]$, *then*

1. $(\lim_{k\to\infty} f_k)$ *is Riemann integrable on* $[a,b]$;

2. $\lim_{k\to\infty}\int_a^b f_k(x)dx = \int_a^b f(x)dx = \int_a^b (\lim_{k\to\infty} f_k)(x)dx.$

Proof.

1. Let $\varepsilon > 0$. From uniform convergence of $\{f_k\}$ on $[a,b]$, we have a positive integer K such that

$$f_k(x) - \frac{\varepsilon}{4(b-a)} \le f(x) \le f_k(x) + \frac{\varepsilon}{4(b-a)}$$

for all $x \in [a,b], k \ge K$. So f is bounded on $[a,b]$, and since f_k is Riemann integrable on $[a,b]$, there exists step functions ϕ_k and $\psi_k, \phi_k \le f_k \le \psi_k$ on $[a,b]$, and $\int_a^b [\psi_k(x) - \phi_k(x)]dx < \frac{\varepsilon}{2}$, for all $k \ge K$.

But

$$\begin{aligned}\phi_k(x) - \frac{\varepsilon}{4(b-a)} &\le f_k(x) - \frac{\varepsilon}{4(b-a)}\\ &\le f(x)\\ &\le f_k(x) + \frac{\varepsilon}{4(b-a)}\\ &\le \psi_k(x) + \frac{\varepsilon}{4(b-a)}\end{aligned}$$

on $[a,b]$, for all $k \ge K$.

This shows there are step functions $\phi_k - \frac{\varepsilon}{4(b-a)}$ and $\psi_k + \frac{\varepsilon}{4(b-a)}$, bracketing f, and

$$\int_a^b \left[\left(\psi_k(x) + \frac{\varepsilon}{4(b-a)}\right) - \left(\phi_k(x) - \frac{\varepsilon}{4(b-a)}\right)\right]dx = \int_a^b [\psi_k(x) - \phi_k(x)]dx + \frac{\varepsilon}{2} < \varepsilon,$$

for all $k \ge K$. So f is Riemann integrable on $[a,b]$.

2. Integrate $f_k(x) - \frac{\varepsilon}{4(b-a)} \le f(x) \le f_k(x) + \frac{\varepsilon}{4(b-a)}$, we obtain

$$\int_a^b f_k(x)dx - \frac{\varepsilon}{4} \le \int_a^b f(x)dx \le \int_a^b f_k(x)dx + \frac{\varepsilon}{4},$$

for all $k \ge K$, that is,

$$\left| \int_a^b f(x)dx - \int_a^b f_k(x)dx \right| \le \frac{\varepsilon}{4},$$

for all $k \ge K$, and this completes the proof. □

Next two theorems are the Fundamental Theorem of Calculus.

Theorem 5.1.13 (Fundamental Theorem of Calculus for Riemann Integral Part 1) *Let f be a bounded function on $[a,b]$. If f is Riemann integrable on $[a,b]$ and $F(x) = \int_a^x f(t)dt$, then F is continuous on $[a,b]$. In addtition, if f is continuous at $x_0 \in (a,b)$, then F is differentiable at $x_0 \in (a,b)$ and $F'(x_0) = f(x_0)$.*

Proof. Firstly, we show F is continuous on $[a,b]$, or equivalently, $\lim_{x \to x_0}[F(x) - F(x_0)] = 0$ for $x_0 \in (a,b)$.

Since f is bounded on $[a,b]$, $-B \le f(t) \le B$, $a \le t \le b$. Integrate f since f is Riemann integrable on $[a,b]$, by assumption, and so

$$\int_{x_0}^x (-B)dt \le \int_{x_0}^x f(t)dt \le \int_{x_0}^x Bdt, \quad a \le x_0 \le x \le b,$$

that is,

$$-B(x - x_0) \le F(x) - F(x_0) \le B(x - x_0), \quad a \le x_0 \le x \le b.$$

This means

$$\lim_{x \to x_0^+}[F(x) - F(x_0)] = 0.$$

On the other hand,

$$\int_x^{x_0}(-B)dt \le \int_x^{x_0} f(t)dt \le \int_x^{x_0} Bdt, \qquad a \le x \le x_0 \le b,$$

that is,

$$-B(x_0 - x) \le F(x_0) - F(x) \le B(x_0 - x), \qquad a \le x \le x_0 \le b.$$

This means

$$\lim_{x \to x_0^-}[F(x) - F(x_0)] = 0.$$

So

$$\lim_{x \to x_0}[F(x) - F(x_0)] = 0 \text{ for } x_0 \in (a,b).$$

Secondly, we show F is differentiable at $x_0 \in (a,b)$ and $F^{'}(x_0) = f(x_0)$. Equivalently,

$$\lim_{x \to x_0}\left[\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right] = 0, \qquad x_0 \in (a,b).$$

Since we have proved f is continuous at $x_0 \in (a,b)$, thus given $\varepsilon > 0$, there exists $\delta > 0$ so that

$$f(x_0) - \varepsilon < f(t) < f(x_0) + \varepsilon, \qquad t \in (x_0 - \delta, x_0 + \delta) \bigcap [a,b].$$

Theorem 5.1.8 states that every continuous function is Riemann integrable on a closed and bounded interval. Thus, we can integrate

$$\int_{x_0}^{x}[f(x_0) - \varepsilon]dt \le \int_{x_0}^{x} f(t)dt \le \int_{x_0}^{x}[f(x_0) + \varepsilon]dt,$$

for $x \in [x_0, x_0 + \delta) \bigcap [a,b]$.

That is,

$$[f(x_0) - \varepsilon](x - x_0) \le F(x) - F(x_0) \le [f(x_0) + \varepsilon](x - x_0),$$

for $x \in [x_0, x_0 + \delta) \bigcap [a,b]$.

Therefore

$$-\varepsilon \le \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \le \varepsilon,$$

for $x \in (x_0, x_0 + \delta) \bigcap [a,b]$.

Hence

$$\lim_{x \to x_0^+} \left[\frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right] = 0.$$

On the other hand,

$$\int_x^{x_0} [f(x_0) - \varepsilon] dt \le \int_x^{x_0} f(t) dt \le \int_x^{x_0} [f(x_0) + \varepsilon] dt,$$

for $x \in (x_0 - \delta, x_0] \cap [a,b]$.

That is,

$$[f(x_0) - \varepsilon](x_0 - x) \le F(x_0) - F(x) \le [f(x_0) + \varepsilon](x_0 - x),$$

for $x \in (x_0 - \delta, x_0] \cap [a,b]$.

Therefore

$$-\varepsilon \le \frac{F(x_0) - F(x)}{x_0 - x} - f(x_0) \le \varepsilon,$$

for $x \in (x_0 - \delta, x_0) \cap [a,b]$.

Hence

$$\lim_{x \to x_0^-} \left[\frac{F(x_0) - F(x)}{x_0 - x} - f(x_0) \right] = 0.$$

This implies

$$\lim_{x \to x_0} \left[\frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right] = 0, \qquad x_0 \in (a,b).$$

$$\Rightarrow \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} - \lim_{x \to x_0} f(x_0) = 0$$

$$\Rightarrow \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = \lim_{x \to x_0} f(x_0)$$

$$\Rightarrow F'(x_0) = f(x_0).$$

□

Next is the Fundamental Theorem of Calculus we often use for evaluating integrals.

Theorem 5.1.14 (Fundamental Theorem of Calculus for Riemann Integral Part 2) *If f is continuous on $[a,b]$, differentiable on $[a,b]$, and if the derivative, f' is Riemann integrable on $[a,b]$, then $\int_a^b f'(x)dx = f(b) - f(a)$.*

Proof. By assumption, f' is Riemann integrable on $[a,b]$, so there are step functions $\phi \le f' \le \psi$ on $[a,b]$ with $\int_a^b \phi(x)dx \le \int_a^b f'(x)dx \le \int_a^b \psi(x)dx$, and $\int_a^b \psi(x)dx - \int_a^b \phi(x)dx < \varepsilon$ f or any $\varepsilon > 0$.

Take the common partition formed by ϕ and ψ, say $P = \{a, x_1, x_2, \ldots, x_{n-1}, b\}$. Suppose $\phi = \sum_{i=1}^{n} c_i \chi_{(x_{i-1}, x_i)} + \sum_{j=0}^{n} \phi(x_j)\chi_{\{x_j\}}$ and $\psi = \sum_{i=1}^{n} d_i \chi_{(x_{i-1}, x_i)} + \sum_{j=0}^{n} \psi(x_j)\chi_{\{x_j\}}$.

Since

$$\begin{aligned} f(b) - f(a) &= \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})] \\ &= \sum_{i=1}^{n} f'(\xi_i)(x_i - x_{i-1}), \qquad x_{i-1} < \xi_i < x_i, \end{aligned}$$

by the Mean Value Theorem, thus $c_i \le f' \le d_i$ on $(x_i - x_{i-1})$, and so

$$\begin{aligned} &\sum_{i=1}^{n} c_i(x_i - x_{i-1}) \le \sum_{i=1}^{n} f'(\xi_i)(x_i - x_{i-1}) \le \sum_{i=1}^{n} d_i(x_i - x_{i-1}) \\ \Rightarrow &\sum_{i=1}^{n} c_i(x_i - x_{i-1}) \le f(b) - f(a) \le \sum_{i=1}^{n} d_i(x_i - x_{i-1}), \end{aligned}$$

that is,

$$\int_a^b \phi(x)dx \le f(b) - f(a) \le \int_a^b \psi(x)dx.$$

So $f(b) - f(a)$ and $\int_a^b f'(x)dx$ lie between $\int_a^b \phi(x)dx$ and $\int_a^b \psi(x)dx$.

Thus

$$\left| f(b) - f(a) - \int_a^b f'(x)dx \right| < \varepsilon.$$

By the nature of arbitrary $\varepsilon > 0$,

$$\int_a^b f'(x)dx = f(b) - f(a).$$

□

This concludes our treatment of the Riemann integral.

5.2 Lebesgue Integral for Bounded Functions on Lebesgue Measurable Sets of Finite Measure

In this section, the Lebesgue integral for bounded functions f on a set E of finite Lebesgue measure is developed. The treatment parallels that of the Riemann integral, replacing step functions with simple functions.

Definition 5.2.1 *Suppose ϕ is a simple function defined on a measurable set E, that is,*

$$\phi(x) = \sum_{k=1}^{n} c_k \chi_{E_k}(x),$$

with $\bigcup_{k=1}^{n} E_k = E, E_k$ mutually disjoint, $\mu(E) < \infty, c_k$ real.

The Lebesgue integral of simple function ϕ on a measurable set E, $\int_E \phi$, is defined as

$$\int_E \phi = \sum_{k=1}^{n} c_k \mu(E_k).$$

Proposition 5.2.2 *The Lebesgue integral of a simple function defined on a Lebesgue measurable set of finite measure is independent of the representation.*

Proof. Suppose $E = \bigcup_{i=1}^{n} E_i = \bigcup_{j=1}^{m} F_j$. Note that $E = \bigcup_{i=1}^{n}\bigcup_{j=1}^{m}(E_i \cap F_j) = \bigcup_{j=1}^{m}\bigcup_{i=1}^{n}(E_i \cap F_j).$

Now suppose $\phi = \sum_{i=1}^{n} c_i \chi_{E_i} = \sum_{j=1}^{m} d_j \chi_{F_j}, \{E_i\}$ and $\{F_j\}$ mutually disjoint collection of Lebesgue measurable sets with $\mu(E) < \infty.$ We want to show

$$\sum_{i=1}^{n} c_i \mu(E_i) = \sum_{j=1}^{m} d_j \mu(F_j).$$

Since

$$E = \bigcup_{i=1}^{n} E_i = \bigcup_{j=1}^{m} F_j,$$

$$\begin{aligned}
\sum_{i=1}^{n} c_i \mu(E_i) &= \sum_{i=1}^{n} \left[c_i \sum_{j=1}^{m} \mu(E_i \cap F_j) \right] \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} c_i \mu(E_i \cap F_j) \\
&= \sum_{j=1}^{m} \sum_{i=1}^{n} d_j \mu(E_i \cap F_j) \\
&= \sum_{j=1}^{m} \left[d_j \sum_{i=1}^{n} \mu(E_i \cap F_j) \right] \\
&= \sum_{j=1}^{m} d_j \mu(F_j),
\end{aligned}$$

since if $E_i \cap F_j \neq \phi$, then there exists $x \in E_i \cap F_j$ such that

$$\phi(x) = c_i \chi_{E_i} = c_i = d_j \chi_{F_j} = d.$$

Our argument is complete.

□

Theorem 5.2.3 If ϕ, ψ are simple functions defined on a set E with finite measure, and k is any real number, then

1. $(k\phi)$ *is a simple function on* E, *and* $\int_E (k\phi) = k\int_E \phi$ *(homogeneous);*

2. $(\phi + \psi)$ *is a simple function on* E, *and* $\int_E (\phi + \psi) = \int_E \phi + \int_E \psi$ *(additive);*

3. $\int_E \phi \leq \int_E \psi$ *if* $\phi \leq \psi$ *on* E *(monotone);*

4. If E_1 *and* E_2 *are disjoint measurable subsets of* E *with* $E = E_1 \cup E_2$, *the integrals* $\int_{E_1} \psi$ *and* $\int_{E_2} \psi$ *exist, and* $\int_E \psi = \int_{E_1} \psi + \int_{E_2} \psi$ *(additive on the domain).*

Proof. Suppose $\phi = \sum_{i=1}^{n} c_i \chi_{E_i}, \psi = \sum_{j=1}^{m} d_j \chi_{F_j}$, where $E = \bigcup_{i=1}^{n} E_i = \bigcup_{j=1}^{m} F_j, \{E_i\}$ and $\{F_j\}$ are mutually disjoint collections of measurable subsets of E.

1. $\int_E (k\phi) = \sum_{i=1}^{n} (kc_i)\chi_{E_i} = k\sum_{i=1}^{n} c_i \chi_{E_i} = k\int_E \phi.$

2. Let $A_{ij} = E_i \bigcap F_j$. The nonempty sets in the collections of $A_{ij}, 1 \le i \le n, 1 \le j \le m,$ are mutually disjoint measurable sets whose union is E. Then

$$(\phi + \psi) = \sum_{i=1}^{n} \sum_{j=1}^{m} (c_i + d_j)\chi_{A_{ij}},$$

and

$$\begin{aligned}
\int_E (\phi + \psi) &= \sum_{i=1}^{n} \sum_{j=1}^{m} (c_i + d_j)\mu(A_{ij}) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} c_i \mu(E_i \bigcap F_j) + \sum_{j=1}^{m} \sum_{i=1}^{n} d_j \mu(E_i \bigcap F_j) \\
&= \sum_{i=1}^{n} c_i \sum_{j=1}^{m} \mu(E_i \bigcap F_j) + \sum_{j=1}^{m} d_j \sum_{i=1}^{n} \mu(E_i \bigcap F_j) \\
&= \sum_{i=1}^{n} c_i \mu(E_i) + \sum_{j=1}^{m} d_i \mu(F_j) \\
&= \int_E \phi + \int_E \psi.
\end{aligned}$$

3. If $\phi \le \psi$, then $\psi - \phi$ is a nonnegative simple function on E, whose integral will be nonnegative by the definition of the integral, and then from parts 1 and 2, we have

$$0 \le \int_E (\psi - \phi) = \int_E \psi + \int_E (-\phi) = \int_E \psi - \int_E \phi$$

and so

$$\int_E \phi \le \int_E \psi.$$

4. Observe that $E = E_1 \bigcup E_2, E_1 \bigcap E_2 = \phi.$

$$\begin{aligned}\int_E \psi &= \sum_{j=1}^{m} d_j \mu(F_j) \\ &= \sum_{j=1}^{m} d_j \mu((F_j \cap E_1) \cup (F_j \cap E_2)) \\ &= \sum_{j=1}^{m} d_j \mu(F_j \cap E_1) + \sum_{j=1}^{m} d_j \mu(F_j \cap E_2).\end{aligned}$$

But $\{F_j \cap E_1\}, \{F_j \cap E_2\}$ are collections of mutually disjoint measurable subsets of E_1, E_2, respectively, with

$$E_1 = \bigcup_{j=1}^{m} (F_j \cap E_1), E_2 = \bigcup_{j=1}^{m} (F_j \cap E_2), \text{and since}$$

the integral is independent of representation, we have

$$\int_{E_1} \psi = \sum_{j=1}^{m} d_j \mu(F_j \cap E_1), \int_{E_2} \psi = \sum_{j=1}^{m} d_j \mu(F_j \cap E_2),$$

and so

$$\int_E \psi = \int_{E_1} \psi + \int_{E_2} \psi.$$

□

Now, we extend the definition of the Lebesgue integral from simple functions to bounded functions.

Definition 5.2.4 *Suppose f is a bounded function defined on a measurable set E with finite measure; say $\alpha \le f \le \beta$ on $E, \mu(E) < \infty$. Let ϕ and ψ denote simple functions such that $\phi \le f \le \psi$ on E.*

The lower Lebesgue integral of f on E, $\underline{\int}_E f$, is given by

$$\underline{\int}_E f = \sup\left\{ \int_E \phi \mid \phi \le f, \phi \textit{ a simple function} \right\}$$

The upper Lebesgue integral of f on E, $\overline{\int}_E f$, is given by

$$\overline{\int}_E f = \inf\left\{ \int_E \psi \mid f \le \psi, \psi \textit{ a simple function} \right\}$$

We would hope that the approximation from "above" and "below" approach a common value, to be called the Lebesgue integral of f on a measurable set E with finite measure.

Definition 5.2.5 *A bounded function f on a measurable set E with finite measure is Lebesgue integrable on E if $\underline{\int}_E f = \overline{\int}_E f$. Denote the common value by $\int_E f$.*

The next theorem shows Riemann integrability implies Lebesgue integrability.

Theorem 5.2.6 *Let f be a bounded function on $[a,b]$. If f is Riemann integrable on $[a,b]$, then f is Lebesgue integrable on $[a,b]$, and $\int_a^b f(x)dx = \int_{[a,b]} f$.*

Proof.

$$\begin{aligned}
\underline{\int_a^b} f(x)dx &= \sup\left\{\int_a^b \phi(x)dx \mid \phi \le f, \phi \text{ a step function}\right\} \\
&= \sup\left\{\int_{[a,b]} \phi \mid \phi \le f, \phi \text{ a step function}\right\} \\
&= \sup\left\{\int_{[a,b]} \phi \mid \phi \le f, \phi \text{ a simple function}\right\} \\
&= \underline{\int}_{[a,b]} f \\
&\le \overline{\int}_{[a,b]} f \\
&= \inf\left\{\int_{[a,b]} \psi \mid f \le \psi, \psi \text{ a simple function}\right\} \\
&= \inf\left\{\int_{[a,b]} \psi \mid f \le \psi, \psi \text{ a step function}\right\} \\
&= \inf\left\{\int_a^b \psi(x)dx \mid f \le \psi, \psi \text{ a step function}\right\} \\
&= \overline{\int_a^b} f(x)dx.
\end{aligned}$$

Since f is Riemann integrable, $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$

This implies

$$\underline{\int}_{[a,b]} f = \overline{\int}_{[a,b]} f,$$

and

$$\int_a^b f(x)dx = \int_{[a,b]} f.$$

□

Next, we have Theorem 5.2.7 on the criteria for Lebesgue integrability.

Theorem 5.2.7 *Let f be a bounded function on a set E with finite measure. f is Lebesgue integrable on E iff for every $\varepsilon > 0$, there exists simple functions ϕ and ψ, $\phi \leq f \leq \psi$ on E such that*

$$0 \leq \int_E \psi - \int_E \phi = \int_E (\psi - \phi) < \varepsilon.$$

Proof. Suppose the bounded function f is Lebesgue integrable on the measurable set E, $\mu(E) < \infty$, and let $\varepsilon > 0$. By the definition of infimum and supremum, we have simple functions ϕ and ψ, $\phi \leq f \leq \psi$ on E such that

$$\int_E f - \frac{\varepsilon}{2} = \underline{\int}_E f - \frac{\varepsilon}{2} < \int_E \phi \leq \underline{\int}_E f$$
$$\leq \overline{\int}_E f \leq \int_E \psi < \overline{\int}_E f + \frac{\varepsilon}{2} = \int_E f + \frac{\varepsilon}{2}.$$

Thus

$$\int_E \phi \leq \int_E \psi < \overline{\int}_E f + \frac{\varepsilon}{2}.$$

Therefore

$$\int_E \phi \le \int_E \psi < \overline{\int}_E f + \frac{\varepsilon}{2}$$

$$\begin{aligned}\Rightarrow 0 \le \int_E \psi - \int_E \phi &< \overline{\int}_E f - \int_E \phi + \frac{\varepsilon}{2}\\ &= \underline{\int}_E f - \int_E \phi + \frac{\varepsilon}{2}\\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\\ &= \varepsilon.\end{aligned}$$

Hence

$$0 \le \int_E \psi - \int_E \phi = \int_E (\psi - \phi) < \varepsilon.$$

Conversely, let $\varepsilon > 0$ be given with simple functions ϕ and $\psi, \phi \le f \le \psi$ on E such that

$$0 \le \int_E \psi - \int_E \phi = \int_E (\psi - \phi) < \varepsilon.$$

By the definition of infimum and supremum,

$$\int_E \phi \le \underline{\int}_E f \le \overline{\int}_E f \le \int \psi.$$

Hence

$$0 \le \overline{\int}_E f - \underline{\int}_E f \le \int \psi - \underline{\int}_E f < \int \psi - \int \phi = \int_E (\psi - \phi) < \varepsilon$$

and the conclusion follows from the arbitrary nature of $\varepsilon > 0$. □

The next theorem illustrates the characterization of Lebesgue integrability in terms of Lebesgue measurable functions.

Theorem 5.2.8 *Let f be a bounded function on E with finite measure. Then f is Lebesgue integrable on E iff f is Lebesgue measurable on E.*

Proof. Let $|f| \le M$ on E and assume f is measurable on E. We need to show f is Lebesgue integrable on E by constructing simple functions ϕ and $\psi, \phi \le f \le \psi$ on E so that

$$0 \le \int_E (\psi - \phi) < \varepsilon.$$

Let $E_k = \left\{ x \in E \mid \frac{k-1}{n} M < f(x) \le \frac{k}{n} M \right\}, -n \le k \le n.$ Then $E = \bigcup_{k=-n}^{n} E_k$, E_k mutually disjoint measurable sets. Define ϕ, ψ :

$$\phi = \frac{M}{n} \sum_{k=-n}^{n} (k-1)\chi_{E_k} \text{ and } \psi = \frac{M}{n} \sum_{k=-n}^{n} k\chi_{E_k}.$$

Clearly $\phi \le f \le \psi$.

Since

$$\psi - \phi = \frac{M}{n} \sum_{k=-n}^{n} \chi_{E_k}$$

and so

$$0 \le \int_E (\psi - \phi) = \frac{M}{n} \sum_{k=-n}^{n} \mu(E_k) = \frac{M}{n} \mu(E) < \varepsilon,$$

for sufficiently large n.

Hence f is Lebesgue integrable on E by Theorem 5.2.7.

Now, assume f is Lebesgue integrable and bounded on set E with finite measure. We need to show f is a Lebesgue measurable function on E.

Since f is bounded and Lebesgue integrable on set E, there exists simple functions ϕ_n and ψ_n so that $\phi_n \le f \le \psi_n$ on E, $\int_E \phi_n \le \int_E f \le \int_E \psi_n$, and

$$\int_E (\psi_n - \phi_n) < \frac{1}{n}, \; n = 1,2,3,\ldots.$$

Define two measurable functions (Theorem 4.2.1):

$$\phi^* = \sup\{\phi_1, \phi_2, \ldots\} \text{ and } \psi^* = \inf\{\psi_1, \psi_2, \ldots\}.$$

Clearly $\phi_n \le \phi^* \le f \le \psi^* \le \psi_n$ on E for all $n \ge 1$. We need to show $\phi^* = \psi^*$ almost everywhere on E and thus conclude $f = \psi^*$ almost everywhere on E, and by Theorem 4.1.5, f will be measurable on E. Consider

$$\{x \in E \mid \psi^*(x) - \phi^*(x) > 0\} = \bigcup_{positive\ \ integer\ \ m} \left\{x \in E \mid \psi^*(x) - \phi^*(x) > \frac{1}{m}\right\}$$

$$\subset \bigcup_{positive\ \ integer\ \ m} \left\{x \in E \mid \psi_n(x) - \phi_n(x) > \frac{1}{m}\right\}$$

for all $n \geq 1$. We are done if we show $\left\{x \in E \mid \psi^*(x) - \phi^*(x) > \frac{1}{m}\right\}$ has measure zero. This set is measurable because $\psi^* - \phi^*$ is a measurable function on E.

By Theorem 5.2.3, we can split E into two disjoint measurable subsets E_1 and E_2 with

$$E_1 = \left\{x \in E \mid \psi_n(x) - \phi_n(x) > \frac{1}{m}\right\} \text{ and } E_2 = \left\{x \in E \mid \psi_n(x) - \phi_n(x) \leq \frac{1}{m}\right\}.$$

Thus

$$\begin{aligned}
\frac{1}{n} > \int_E (\psi_n - \phi_n) &= \int_{E_1} (\psi_n - \phi_n) + \int_{E_2} (\psi_n - \phi_n) \\
&\geq \int_{E_1} (\psi_n - \phi_n) \\
&= \sum_{j=1}^{p} c_{1j} \mu(E_{1j}) \\
&> \sum_{j=1}^{p} \frac{1}{m} \mu(E_{1j}) \\
&= \frac{1}{m} \mu(E_1) \\
&= \frac{1}{m} \mu\left(\left\{x \in E \mid \psi_n(x) - \phi_n(x) > \frac{1}{m}\right\}\right),
\end{aligned}$$

since on $E_1, \psi_n - \phi_n = \sum_{j=1}^{p} c_{1j} \chi_{E_{1j}}$, where E_{1j} mutually disjoint measurable sets, $E_1 = \bigcup_{j=1}^{p} E_{1j}$ and $c_{1j} > \frac{1}{m}$.

Hence

$$\mu\left(\left\{x \in E \mid \psi_n(x) - \phi_n(x) > \frac{1}{m}\right\}\right) < \frac{m}{n} \text{ for all } n \geq 1,$$

i.e., $\mu\left(\left\{x \in E \mid \psi_n(x) - \phi_n(x) > \frac{1}{m}\right\}\right) = 0$ and so $\left\{x \in E \mid \psi^*(x) - \phi^*(x) > \frac{1}{m}\right\}$ has measure zero since a countable union of sets of measure zero is a measurable set of measure zero. This implies $\phi^* = \psi^*$ almost everywhere on E and thus $f = \psi^*$ almost everywhere on E. By Theorem 4.1.5, f is Lebesgue measurable on E. □

The Lebesgue integral has the properties of linearity and monoticity.

Theorem 5.2.9 *If the bounded functions f and g are Lebesgue measurable on $E, \mu(E) < \infty,$ and k is any real number, then f and g are Lebesgue integrable on E and*

1. *(kf) is Lebesgue integrable on E, and $\int_E (kf) = k\int_E f$ (homogeneous);*
2. *$(f+g)$ is Lebesgue integrable on E, and $\int_E (f+g) = \int_E f + \int_E g$ (additive);*
3. *$\int_E f \le \int_E g$ if $f \le g$ on E (monotone);*
4. *If E_1 and E_2 are disjoint measurable subsets of E with $E = E_1 \cup E_2$, f is Lebesgue integrable on E_1 and E_2, and $\int_E f = \int_{E_1} f + \int_{E_2} f$ (additive on the domain);*
5. *If $\alpha \le f \le \beta$ on E, then $\alpha\mu(E) \le \int_E f \le \beta\mu(E)$ (mean value).*

Proof.

1. $k > 0$ is obvious. Assume $k > 0$. Since f is Lebesgue integrable on E, there exists simple functions ϕ and ψ so that $\phi \le f \le \psi, \int_E \phi \le \int_E f \le \int_E \psi$ and $\int_E (\psi - \phi) < \dfrac{\varepsilon}{k}.$

But then

$$k\phi \le kf \le k\psi, k\int_E \phi \le k\int_E f \le k\int_E \psi \text{ and } k\int_E (\psi - \phi) = \int_E (k\psi - k\phi) < \varepsilon.$$

The last inequality implies kf is Lebesgue integrable. Thus

$$k\int_E \phi = \int_E k\phi \le \int_E kf \le \int_E k\psi = k\int_E \psi.$$

Hence

$$k\int_E \phi - k\int_E \psi \le \int_E kf - k\int_E f \le k\int_E \psi - k\int_E \phi$$
$$\Rightarrow -\varepsilon < -k\left(\int_E \psi - \int_E \phi\right) = k\int_E \phi - k\int_E \psi \le \int_E kf - k\int_E f \le k\int_E \psi - k\int_E \phi = k\left(\int_E \psi - \int_E \phi\right) < \varepsilon$$
$$\Rightarrow \left|\int_E kf - k\int_E f\right| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\int_E kf - k\int_E f = 0$ and so $\int_E kf = k\int_E f$.

Assume $k < 0$. Since f is Lebesgue integrable on E, there exists simple functions ϕ and ψ so that $\phi \le f \le \psi, \int_E \phi \le \int_E f \le \int_E \psi$ and $\int_E (\psi - \phi) < \dfrac{\varepsilon}{-k}$.

But then

$$k\psi \le kf \le k\phi, k\int_E \psi \le k\int_E f \le k\int_E \phi \text{ and } k\int_E (\phi - \psi) = \int_E (k\phi - k\psi) < \varepsilon.$$

The last inequality implies kf is Lebesgue integrable. Thus

$$\int_E k\psi = k\int_E \psi \le \int_E kf \le k\int_E \phi = \int_E k\phi.$$

Hence

$$\begin{aligned}
&k\int_E \psi - k\int_E \phi \le \int_E kf - k\int_E f \le k\int_E \phi - k\int_E \psi \\
&\Rightarrow -\varepsilon < -k\left(\int_E \phi - \int_E \psi\right) = k\int_E \psi - k\int_E \phi \le \int_E kf - k\int_E f \le k\int_E \phi - k\int_E \psi = k\left(\int_E \phi - \int_E \psi\right) < \varepsilon \\
&\Rightarrow \left|\int_E kf - k\int_E f\right| < \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $\int_E kf - k\int_E f = 0$ and so $\int_E kf = k\int_E f$.

2. f is Lebesgue integrable implies $\phi_f \le f \le \psi_f, \int_E \phi_f \le \int_E f \le \int_E \psi_f$ with

$$\int_E (\psi_f - \phi_f) < \frac{\varepsilon}{2}.$$

g is Lebesgue integrable implies $\phi_g \le g \le \psi_g, \int_E \phi_g \le \int_E g \le \int_E \psi_g$ with

$$\int_E (\psi_g - \phi_g) < \frac{\varepsilon}{2}.$$

Adding, we obtain $\phi_f + \phi_g \le f + g \le \psi_f + \psi_g, \int_E (\phi_f + \phi_g) \le \int_E f + \int_E g \le \int_E (\psi_f + \psi_g)$ with $\int_E [(\psi_f + \psi_g) - (\phi_f + \phi_g)] < \varepsilon$. The last inequality implies $f + g$ is Lebesgue integrable on E and

$$\int_E (\phi_f + \phi_g) \le \int_E (f + g) \le \int_E (\psi_f + \psi_g).$$

Thus $\int_E f + \int_E g$ and $\int_E (f + g)$ lie between $\int_E (\phi_f + \phi_g)$ and $\int_E (\psi_f + \psi_g)$ and so

$$\int_E (f + g) = \int_E f + \int_E g,$$

by the arbitrary nature of $\varepsilon > 0$.

3. By parts 1 and 2, $\int_E g - \int_E f = \int_E (g-f)$. Since $g - f \geq 0$, let $\phi \equiv 0$. Then $0 = \int_E \phi \leq \int_E (g-f)$.

4. Since f is Lebesgue integrable on E, there exists simple functions $\phi \leq f \leq \psi$ on E so that

$$\int_E \phi \leq \int_E f \leq \int_E \psi$$

and $\int_E (\psi - \phi) < \varepsilon$.

Since $\int_E \phi = \int_{E_1} \phi + \int_{E_2} \phi$ and $\int_E \psi = \int_{E_1} \psi + \int_{E_2} \psi$ (Theorem 5.2.3), we conclude $\phi \leq f \leq \psi$ on E_1 and $\int_{E_1} (\psi - \phi) < \varepsilon$, and $\phi \leq f \leq \psi$ on E_2 and $\int_{E_2} (\psi - \phi) < \varepsilon$. Hence f is Lebesgue integrable on E_1 and E_2 and

$$\int_E \phi = \int_{E_1} \phi + \int_{E_2} \phi \leq \int_{E_1} f + \int_{E_2} f \leq \int_{E_1} \psi + \int_{E_2} \psi = \int_E \psi.$$

$\int_{E_1} f + \int_{E_2} f$ and $\int_E f$ lie between $\int_E \phi$ and $\int_E \psi$. Therefore $\int_E f = \int_{E_1} f + \int_{E_2} f$ by the arbitrary nature of $\varepsilon > 0$.

5. From part 3, $\alpha \mu(E) = \int_E \alpha \leq \int_E f \leq \int_E \beta = \beta \mu(E)$. □

We conclude this section with the next theorem.

Theorem 5.2.10 *If f is a bounded, Lebesgue integrable function on a set E of finite measure, and g is a bounded function on E such that $g = f$ almost everywhere on E, then g is Lebesgue integrable on E and $\int_E g = \int_E f$.*

Proof. The function f is Lebesgue measurable by Theorem 5.2.8 and application of Theorem 4.1.5 yields measurability for g, and thus integrability for g (Theorem 5.2.8).

Let $A=\{x \mid f(x)\neq g(x)\}$. The set A has measure zero, thus A is Lebesgue measurable by Theorem 3.3.2, and so $E\bigcap A^c$ is measurable by Lemma 3.3.4and

$$\begin{aligned}\int_E f &= \int_{E\cap A^c} f + \int_A f\\ &= \int_{E\cap A^c} f + 0\\ &= \int_{E\cap A^c} g\\ &= \int_{E\cap A^c} g + 0\\ &= \int_{E\cap A^c} g + \int_A g\\ &= \int_E g,\end{aligned}$$

by Theorem 5.2.9. □

5.3 Lebesgue Integral for Nonnegative Measurable Functions

We begin this section by defining nonnegative simple function on ℝ and the Lebesgue integral of a nonnegative simple function ϕ defined on a Lebesgue measurable set E.

Definition 5.3.1 *Let ϕ be a nonnegative simple function on ℝ, that is, $\phi(x)=\sum_{k=1}^{n} c_k\chi_{E_k}(x)$, where E_k are mutually disjoint Lebesgue measurable subsets of ℝ, ℝ$=\bigcup_{k=1}^{n} E_k$, and c_k are nonnegative real numbers.*

Definition 5.3.2 *The Lebesgue integral of a nonnegative simple function ϕ, on a Lebesgue measurable set E, written $\int_E \phi$, is defined by*

$$\int_E \phi = \sum_{k=1}^{n} c_k \mu(E\cap E_k),$$

where $\phi=\sum_{k=1}^{n} c_k\chi_{E_k}$, E_k mutually disjoint, ℝ$=\bigcup_{k=1}^{n} E_k$, $c_k\geq 0$.

Theorem 5.3.3 *The Lebesgue integral of a nonnegative simple function defined on a Lebesgue measurable set is independent of representation.*

Proof. Suppose

$$\phi = \sum_{k=1}^{n} c_k \chi_{E_k}, c_k \geq 0$$

and

$$\phi = \sum_{j=1}^{m} d_j \chi_{F_j}, d_j \geq 0$$

with

$$\mathbb{R} = \bigcup_{k=1}^{n} E_k = \bigcup_{j=1}^{m} F_j,$$

E_k and F_j mutually disjoint Lebesgue measurable subsets of $\mathbb{R}$. We need to show

$$\sum_{k=1}^{n} c_k \mu(E \cap E_k) = \sum_{j=1}^{m} d_j \mu(E \cap F_j).$$

Note that

$$E_k = E_k \cap \left(\bigcup_{j=1}^{m} F_j \right) = \bigcup_{j=1}^{m} (E_k \cap F_j)$$

and

$$F_j = F_j \cap \left(\bigcup_{k=1}^{n} E_k \right) = \bigcup_{k=1}^{n} (E_k \cap F_j).$$

Hence

$$\sum_{k=1}^{n} c_k \mu(E \cap E_k) = \sum_{k=1}^{n} c_k \mu\left(E \cap \left(\bigcup_{j=1}^{m} (E_k \cap F_j) \right) \right)$$
$$= \sum_{k=1}^{n} c_k \mu\left(\bigcup_{j=1}^{m} (E \cap E_k \cap F_j) \right)$$
$$= \sum_{k=1}^{n} c_k \sum_{j=1}^{m} \mu(E \cap E_k \cap F_j)$$
$$= \sum_{k=1}^{n} \sum_{j=1}^{m} d_j \mu(E \cap E_k \cap F_j)$$
$$= \sum_{j=1}^{m} \sum_{k=1}^{n} d_j \mu(E \cap E_k \cap F_j)$$
$$= \sum_{j=1}^{m} d_j \sum_{k=1}^{n} \mu(E \cap E_k \cap F_j)$$
$$= \sum_{j=1}^{m} d_j \mu\left(\bigcup_{k=1}^{n} (E \cap E_k \cap F_j) \right)$$
$$= \sum_{j=1}^{m} d_j \mu\left(E \cap \left(\bigcup_{k=1}^{n} (E_k \cap F_j) \right) \right)$$
$$= \sum_{j=1}^{m} d_j \mu(E \cap F_j)$$

since, for $E \cap E_k \cap F_j \neq \phi, c_k = c_k \chi_{E_k} = \phi = d_j \chi_{F_j} = d_j$, and if $E \cap E_k \cap F_j = \phi$, no contribution because $\mu(\phi) = 0.$ □

Theorem 5.3.4 *If ϕ, ψ are nonnegative simple functions on $\mathbb{R}$, if E is any Lebesgue measurable subset of $\mathbb{R}$, and k is any nonnegative real number, then*

1. $(k\phi)$ is a nonnegative simple function on E, and

$$\int_E (k\phi) = k \int_E \phi \qquad \textit{(homogeneous);}$$

2. $(\phi + \psi)$ is a nonnegative simple function on E, and

$$\int_E (\phi + \psi) = \int_E \phi + \int_E \psi \qquad \textit{(additive);}$$

3. $\int_E \phi \leq \int_E \psi$ if $0 \leq \phi \leq \psi$ on E (monotone);

4. If E_1 and E_2 are disjoint Lebesgue measurable subsets of E with $E = E_1 \cup E_2$, the integrals $\int_E \psi = \int_{E_1} \psi + \int_{E_2} \psi$ (additive on the domain).

Proof.

1. Suppose $\phi = \sum_{i=1}^{n} c_i \chi_{E_i}, c_i \geq 0.$ Then $k\phi = \sum_{i=1}^{n} kc_i \chi_{E_i}$ and

$$\begin{aligned}\int_E (k\phi) &= \sum_{i=1}^{n} (kc_i)\mu(E \cap E_i) \\ &= k\sum_{i=1}^{n} c_i \mu(E \cap E_i) \\ &= k\int_E \phi.\end{aligned}$$

2. Let $\phi = \sum_{k=1}^{n} c_k \chi_{E_k}$ and $\psi = \sum_{j=1}^{m} d_j \chi_{F_j}$ with $0 \leq c_k, d_j$. The idea is to form the $n \cdot m$ sets:

$$\begin{aligned}&E_1 \cap F_1, E_1 \cap F_2, \ldots, E_1 \cap F_m \\ &E_2 \cap F_1, E_2 \cap F_2, \ldots, E_2 \cap F_m \\ &\vdots \\ &E_n \cap F_1, E_n \cap F_2, \ldots, E_n \cap F_m.\end{aligned}$$

If $E_k \cap F_j \neq \phi$, define $\phi + \psi$ as $c_k + d_j$. The nonempty $E_k \cap F_j$ are mutually disjoint Lebesgue measurable subsets of $\mathbb{R}$,

$$\mathbb{R} = \bigcup_{k=1}^{n} \bigcup_{j=1}^{m} (E_k \cap F_j),$$

and

$$\phi + \psi = \sum_{k=1}^{n} \sum_{j=1}^{m} (c_k + d_j) \chi_{E_k \cap F_j}.$$

Hence

$$\begin{aligned}\int_E (\phi + \psi) &= \sum_{k=1}^{n} \sum_{j=1}^{m} (c_k + d_j) \mu(E_k \cap F_j \cap E) \\ &= \sum_{k=1}^{n} c_k \sum_{j=1}^{m} \mu(E_k \cap F_j \cap E) + \sum_{j=1}^{m} d_j \sum_{k=1}^{n} \mu(E_k \cap F_j \cap E) \\ &= \sum_{k=1}^{n} c_k \mu(E_k \cap E) + \sum_{j=1}^{m} d_j \mu(F_j \cap E) \\ &= \int_E \phi + \int_E \psi.\end{aligned}$$

3. Suppose $\phi = \sum_{k=1}^{n} c_k \chi_{E_k}, E_k$ mutually disjoint, and $\psi = \sum_{j=1}^{m} d_j \chi_{F_j}, F_j$ mutually disjoint, where

$$\mathbb{R} = \bigcup_{k=1}^{n} E_k = \bigcup_{j=1}^{m} F_j .$$

Since $0 \le \phi \le \psi, 0 \le c_k \le d_j$ on nonempty $E_k \cap F_j$ and thus

$$\begin{aligned}
\int_E \phi = \sum_{k=1}^{n} c_k \mu(E_k \cap E) &= \sum_{k=1}^{n} c_k \sum_{j=1}^{m} \mu(E_k \cap F_j \cap E) \\
&\le \sum_{j=1}^{m} d_j \sum_{k=1}^{n} \mu(E_k \cap F_j \cap E) \\
&= \sum_{j=1}^{m} d_j \mu(F_j \cap E) \\
&= \int_E \psi .
\end{aligned}$$

4.

$$\begin{aligned}
\int_E \psi &= \sum_{j=1}^{m} d_j \mu(F_j \cap E) \\
&= \sum_{j=1}^{m} d_j \mu(F_j \cap (E_1 \cup E_2)) \\
&= \sum_{j=1}^{m} d_j [\mu(F_j \cap E_1) + \mu(F_j \cap E_2)] \\
&= \sum_{j=1}^{m} d_j \mu(F_j \cap E_1) + \sum_{j=1}^{m} d_j \mu(F_j \cap E_2) \\
&= \int_{E_1} \psi + \int_{E_2} \psi .
\end{aligned}$$

□

Next, we define the Lebesgue integral of a nonnegative measurable function. We give two commonly used definitions and show their equivalence.

Definition 5.3.5 *If f is a nonnegative measurable function, defined on a Lebesgue measurable set E, the Lebesgue integral of f over E, $\int_E f$, is given by*

$$\int_E f \equiv \sup\left\{ \int_E \phi \mid \phi \le f, \phi \text{ nonnegative and simple} \right\}$$

Definition 5.3.6 *If f is a nonnegative measurable function, defined on a Lebesgue measurable set E, and ϕ_n is a nonnegative monotone sequence of simple functions, $0 \le \phi_n \le \phi_{n+1}$ on E, with*

$$\lim_{n\to\infty} \phi_n(x) = f(x) \qquad \textit{finite or infinite}$$

for all $x \in E$, the Lebesgue integral of f over E, $\int_E f$, is given by

$$\int_E f \equiv \lim_{n\to\infty} \int_E \phi_n = \int_E (\lim_{n\to\infty} \phi_n).$$

Lemma 5.3.7 *Let $a, b > 0$. If $a \ge \alpha b \, \forall 0 < \alpha < 1$, then $a \ge b$.*

Proof. Assume $a < b$. Then $0 < \alpha < 1 \Rightarrow 0 < \alpha b < b$. Since $a < b$, it follows that $0 < a < \alpha b$ or $0 < \alpha b < a < b$. When $0 < a < \alpha b$, a contradiction occurs. When $0 < \alpha b < a < b$, take $0 < \alpha = \frac{a}{b} < 1$ since α is arbitrary.Then $0 < \frac{a}{b} \cdot b = a < a < b$, a contradiction occurs. □

Proposition 5.3.8 $\int_E f$*, as given by Definition 5.3.6 is well-defined.*

Proof. Suppose we have sequences of simple functions $\{\phi_n\}, \left\{\hat{\phi}_m\right\}, 0 \le \phi_n \le \phi_{n+1}$ and $0 \le \hat{\phi}_m \le \hat{\phi}_{m+1}$ on E with $\lim_{n\to\infty} \phi_n = \lim_{m\to\infty} \hat{\phi}_m = f$.

We claim

$$\lim_{n\to\infty} \int_E \phi_n = \lim_{m\to\infty} \int_E \hat{\phi}_m.$$

Pick $0 \le \phi_n \le f$. Then $f = \lim_{m\to\infty} \hat{\phi}_m \ge \phi_n$ on E. We will show

$$\lim_{m\to\infty} \int_E \hat{\phi}_m \ge \int_E \phi_n.$$

Since ϕ_n is nonnegative and simple, we have

$$\int_E \phi_n = \sum_{k=1}^{N} c_k \mu(E \cap E_k), \; c_k \ge 0,$$

where $E=\bigcup_{k=1}^{N}(E\bigcap E_k)$ and the nonempty $E\bigcap E_k$ are mutually disjoint Lebesgue measurable subsets of $E.$ Hence we must show

$$\lim_{m\to\infty}\int_E \hat{\phi}_m \geq \sum_{k=1}^{N} c_k\mu(E\bigcap E_k).$$

But the integral is additive on the domain (Theorem 5.3.4). Thus

$$\int_E \hat{\phi}_m = \sum_{k=1}^{N}\int_{E\bigcap E_k}\hat{\phi}_m.$$

Our claim will be justified provided we can show

$$\lim_{m\to\infty}\int_{E\bigcap E_k}\hat{\phi}_m \geq c_k\mu(E\bigcap E_k),\ c_k \geq 0.$$

It is immediate if $c_k = 0$. Assume $c_k > 0$. Let $0<\alpha<1$. (This idea has been attributed to W. Rudin.) We construct a sequence of sets $\{B_m\}$ as follows:

$$B_m = \left\{x \in E\bigcap E_k \mid \hat{\phi}_m(x) \geq \alpha c_k\right\}.$$

B_m is measurable, $B_m \subseteq B_{m+1}$ since $\hat{\phi}_m \leq \hat{\phi}_{m+1}$, and $E\bigcap E_k = \bigcup_{m=1}^{\infty} B_m$. This is because $p\in E\bigcap E_k, \phi_n(p)=c_k$ since $p\in E_k$ and $\phi_n(x)=\sum_{k=1}^{l} c_k\chi_{E_k}$ and so

$$\lim_{m\to\infty}\hat{\phi}_m(p) = f(p) \geq \phi_n(p) = c_k > \alpha c_k,$$

i.e., $\hat{\phi}_m(p) > \alpha c_k$ for sufficiently large m, in other words, $p\in B_m$ for sufficiently large m.

So $\{B_m\}$ is an increasing sequence of Lebesgue measurable subsets of $E\bigcap E_k$. Then $\{\mu(B_m)\}$ is monotone increasing with

$$\lim_{m\to\infty}\mu(B_m) = \mu(E\bigcap E_k) \qquad \text{(Theorem 3.4.1).}$$

But $\left\{\int_{E\bigcap E_k}\hat{\phi}_m\right\}$ is also monotone increasing with

$$\int_{E\cap E_k} \hat{\phi}_m \geq \int_{B_m} \hat{\phi}_m \geq \alpha c_k \mu(B_m).$$

Hence

$$\lim_{m\to\infty} \int_{E\cap E_k} \hat{\phi}_m \geq \lim_{m\to\infty} [\alpha c_k \mu(B_m)] = \alpha c_k \mu(E \cap E_k).$$

But this holds for any α between 0 and 1. By Lemma 5.3.7, we have

$$\lim_{m\to\infty} \int_{E\cap E_k} \hat{\phi}_m \geq c_k \mu(E \cap E_k),$$

and the argument is complete. The reverse inequality is obtained by interchanging $\hat{\phi}_m$ and ϕ_n. □

Proposition 5.3.9 *The Definition 5.3.5 and 5.3.6 of the Lebesgue integral of a nonnegative measurable function are equivalent.*

Proof. By the Approximation Theorem 4.3.5, we have a monotone sequence of nonnegative simple functions, $0 \leq \hat{\phi}_n \leq \hat{\phi}_{n+1}$ on E, with

$$\lim_{n\to\infty} \hat{\phi}_n = f \text{ on } E, \text{and} \int_E \hat{\phi}_n \leq \int_E \hat{\phi}_{n+1}.$$

We must show

$$\lim_{n\to\infty} \int_E \hat{\phi}_n = \sup\left\{\int_E \phi \mid \phi \leq f\right\}$$

Suppose $0 \leq \phi^* \leq f$. Then $\lim_{n\to\infty} \hat{\phi}_n = f \geq \phi^*$ on E and the argument in Propostition 5.3.8 yields

$$\int_E \phi^* \leq \lim_{n\to\infty} \int_E \hat{\phi}_n \qquad \text{(finite or infinite).}$$

In other words, $\lim_{n\to\infty} \int_E \hat{\phi}_n$ is an upper bound for the set $\left\{\int_E \phi \mid \phi \leq f\right\}$

Hence,

$$\sup\left\{\int_E \phi \mid \phi \leq f\right\} \leq \lim_{n\to\infty} \int_E \hat{\phi}_n.$$

On the other hand, $\int_E \hat{\phi}_n \in \left\{\int_E \phi \mid \phi \le f\right\}$ for all n, and the sequence $\left\{\int_E \hat{\phi}_n\right\}$ is nondecreasing (Theorem 5.3.4). So $\lim_{n\to\infty}\int_E \hat{\phi}_n \le \sup\left\{\int_E \phi \mid \phi \le f\right\}$ Combining, $\lim_{n\to\infty}\int_E \hat{\phi}_n = \sup\left\{\int_E \phi \mid \phi \le f\right\}$ □

We also have properties of the integral.

Theorem 5.3.10 *If f and g are nonnegative measurable functions defined on a Lebesgue measurable set E, and k is any nonnegative real number, then*

1. *(kf) is nonnegative, measurable, and $\int_E (kf) = k\int_E f$ (homogeneous);*
2. *$(f+g)$ is nonnegative, measurable, and $\int_E (f+g) = \int_E f + \int_E g$ (additive);*
3. *$\int_E f \le \int_E g$ if $0 \le f \le g$ (monotone);*
4. *If E_1 and E_2 are disjoint measurable subsets of E with $E = E_1 \cup E_2$, the integrals $\int_{E_1} f$ and $\int_{E_2} f$ exist in $\mathbb{R}^e$, and*

$$\int_E f = \int_{E_1} f + \int_{E_2} f \quad \textit{(additive on domain).}$$

Proof. Measurability of the appropriate functions follows from Theorem 4.1.7.

1. By the Approximation Theorem 4.3.5, we have a sequence $\left\{\hat{\phi}_n\right\}$ of simple functions satisfying $0 \le \hat{\phi}_n \le \hat{\phi}_{n+1}$ with $\lim_{n\to\infty}\hat{\phi}_n = f$ on E. But then, $0 \le k\hat{\phi}_n \le k\hat{\phi}_{n+1}$ with $\lim_{n\to\infty}\left(k\hat{\phi}_n\right) = kf$. Using Definition 5.3.6,

$$k\int_E f = k\lim_{n\to\infty}\int_E \hat{\phi}_n = \lim_{n\to\infty}\left(k\int_E \hat{\phi}_n\right) = \lim_{n\to\infty}\int_E\left(k\hat{\phi}_n\right) = \int_E (kf).$$

2. $\lim_{n\to\infty}\phi_n = f, \lim_{n\to\infty}\psi_n = g$ implies $\lim_{n\to\infty}(\phi_n + \psi_n) = f + g$. Thus, $\lim_{n\to\infty}\int_E \phi_n = \int_E f$, $\lim_{n\to\infty}\int_E \psi_n = \int_E g$ implies $\lim_{n\to\infty}\int_E (\phi_n + \psi_n) = \int_E (f+g)$. Hence

$$\int_E f + \int_E g = \lim_{n\to\infty}\int_E \phi_n + \lim_{n\to\infty}\int_E \psi_n = \lim_{n\to\infty}\left(\int_E \phi_n + \int_E \psi_n\right) = \lim_{n\to\infty}\int_E (\phi_n + \psi_n) = \int_E (f+g).$$

3. If $0 \le f \le g$, then $\phi \le g$. Thus $\{\phi \mid \phi \le f\} \subset \{\phi \mid \phi \le g\}$. So

$$\sup\left\{\int_E \phi \mid \phi \le f\right\} \le \sup\left\{\int_E \phi \mid \phi \le g\right\},$$

that is, $\int_E f \le \int_E g$.

4. By Theorem 5.3.4, $\int_E \phi_n = \int_{E_1} \phi_n + \int_{E_2} \phi_n$. The sequences $\left\{\int_E \phi_n\right\}, \left\{\int_{E_1} \phi_n\right\}$, and $\left\{\int_{E_2} \phi_n\right\}$ are monotone increasing, limits are defined and nonnegative, possibly in the extended reals. Therefore,

$$\begin{aligned} &\int_E \phi_n = \int_{E_1} \phi_n + \int_{E_2} \phi_n \\ \Rightarrow\ &\lim_{n\to\infty}\int_E \phi_n = \lim_{n\to\infty}\int_{E_1} \phi_n + \lim_{n\to\infty}\int_{E_2} \phi_n \\ \Rightarrow\ &\int_E f = \int_{E_1} f + \int_{E_2} f. \end{aligned}$$

□

Next example serves to illustrate the concepts used.

Example 5.3.11 Calculate $\int_{(0,1]} \frac{1}{\sqrt{x}}$.

Solution. $\phi_1(x) = 1,\ 0 < x \le 1.$ $\int_{(0,1]} \phi_1 = 1.$

$$\phi_2(x) = \begin{cases} \frac{4}{2}, & 0 < x \le \left(\frac{2}{4}\right)^2 \\ \frac{3}{2}, & \left(\frac{2}{4}\right)^2 < x \le \left(\frac{2}{3}\right)^2 \\ \frac{2}{2}, & \left(\frac{2}{3}\right)^2 < x \le \left(\frac{2}{2}\right)^2. \end{cases}$$

$$\int_{(0,1]} \phi_2 = \frac{4}{2}\left[\left(\frac{2}{4}\right)^2 - 0^2\right] + \frac{3}{2}\left[\left(\frac{2}{3}\right)^2 - \left(\frac{2}{4}\right)^2\right] + \frac{2}{2}\left[\left(\frac{2}{2}\right)^2 - \left(\frac{2}{3}\right)^2\right]$$
$$= 1 + 2\left(\frac{1}{3^2} + \frac{1}{4^2}\right).$$

$$\phi_3(x) = \begin{cases} \frac{12}{4}, & 0 < x \le \left(\frac{4}{12}\right)^2 \\ \frac{11}{4}, & \left(\frac{4}{12}\right)^2 < x \le \left(\frac{4}{11}\right)^2 \\ \frac{10}{4}, & \left(\frac{4}{11}\right)^2 < x \le \left(\frac{4}{10}\right)^2 \\ \frac{9}{4}, & \left(\frac{4}{10}\right)^2 < x \le \left(\frac{4}{9}\right)^2 \\ \frac{8}{4}, & \left(\frac{4}{9}\right)^2 < x \le \left(\frac{4}{8}\right)^2 \\ \frac{7}{4}, & \left(\frac{4}{8}\right)^2 < x \le \left(\frac{4}{7}\right)^2 \\ \frac{6}{4}, & \left(\frac{4}{7}\right)^2 < x \le \left(\frac{4}{6}\right)^2 \\ \frac{5}{4}, & \left(\frac{4}{6}\right)^2 < x \le \left(\frac{4}{5}\right)^2 \\ \frac{4}{4}, & \left(\frac{4}{5}\right)^2 < x \le \left(\frac{4}{4}\right)^2. \end{cases}$$

$$\int_{(0,1]}\phi_3 = \frac{12}{4}\left[\left(\frac{4}{12}\right)^2 - 0^2\right] + \frac{11}{4}\left[\left(\frac{4}{11}\right)^2 - \left(\frac{4}{12}\right)^2\right] + \frac{10}{4}\left[\left(\frac{4}{10}\right)^2 - \left(\frac{4}{11}\right)^2\right]$$
$$+\frac{9}{4}\left[\left(\frac{4}{9}\right)^2 - \left(\frac{4}{10}\right)^2\right] + \frac{8}{4}\left[\left(\frac{4}{8}\right)^2 - \left(\frac{4}{9}\right)^2\right] + \frac{7}{4}\left[\left(\frac{4}{7}\right)^2 - \left(\frac{4}{8}\right)^2\right]$$
$$+\frac{6}{4}\left[\left(\frac{4}{6}\right)^2 - \left(\frac{4}{7}\right)^2\right] + \frac{5}{4}\left[\left(\frac{4}{5}\right)^2 - \left(\frac{4}{6}\right)^2\right] + \frac{4}{4}\left[\left(\frac{4}{4}\right)^2 - \left(\frac{4}{5}\right)^2\right]$$
$$= \left(\frac{4}{12}\right)^2\left(\frac{12}{4} - \frac{11}{4}\right) + \left(\frac{4}{11}\right)^2\left(\frac{11}{4} - \frac{10}{4}\right) + \left(\frac{4}{10}\right)^2\left(\frac{10}{4} - \frac{9}{4}\right)$$
$$+\left(\frac{4}{9}\right)^2\left(\frac{9}{4} - \frac{8}{4}\right) + \left(\frac{4}{8}\right)^2\left(\frac{8}{4} - \frac{7}{4}\right) + \left(\frac{4}{7}\right)^2\left(\frac{7}{4} - \frac{6}{4}\right)$$
$$+\left(\frac{4}{6}\right)^2\left(\frac{6}{4} - \frac{5}{4}\right) + \left(\frac{4}{5}\right)^2\left(\frac{5}{4} - \frac{4}{4}\right) + \left(\frac{4}{4}\right)^2\left(\frac{4}{4}\right)$$
$$= \left(\frac{4}{12}\right)^2\left(\frac{1}{4}\right) + \left(\frac{4}{11}\right)^2\left(\frac{1}{4}\right) + \left(\frac{4}{10}\right)^2\left(\frac{1}{4}\right)$$
$$+\left(\frac{4}{9}\right)^2\left(\frac{1}{4}\right) + \left(\frac{4}{8}\right)^2\left(\frac{1}{4}\right) + \left(\frac{4}{7}\right)^2\left(\frac{1}{4}\right)$$
$$+\left(\frac{4}{6}\right)^2\left(\frac{1}{4}\right) + \left(\frac{4}{5}\right)^2\left(\frac{1}{4}\right) + \left(\frac{4}{4}\right)^2\left(\frac{4}{4}\right)$$
$$= 1 + 4\left(\frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2} + \frac{1}{11^2} + \frac{1}{12^2}\right).$$

In general,

$$\phi_n(x) = \begin{cases} \dfrac{n\cdot 2^{n-1}}{2^{n-1}}, & 0 < x \le \left(\dfrac{2^{n-1}}{n\cdot 2^{n-1}}\right)^2 \\ \dfrac{n\cdot 2^{n-1} - k_n}{2^{n-1}}, & \left(\dfrac{2^{n-1}}{n\cdot 2^{n-1} - k_n + 1}\right)^2 < x \le \left(\dfrac{2^{n-1}}{n\cdot 2^{n-1} - k_n}\right)^2, 1 \le k_n \le (n-1)\cdot 2^{n-1}. \end{cases}$$

Now, we show $0 \le \phi_n \le \phi_{n+1}$, for all $n \ge 1$.

Clearly, $0 \le \phi_1(x) \le \phi_2(x), 0 \le \phi_2(x) \le \phi_3(x)$.

Consider ϕ_{n+1}.

When $0 < x \le \left(\dfrac{2^n}{(n+1)\cdot 2^n}\right)^2$, we have $0 < x \le \dfrac{1}{(n+1)^2}$. Thus $0 < x \le \dfrac{1}{(n+1)^2} < \dfrac{1}{n^2}$.

Hence $0 \le \phi_n(x) = n \le n+1 = \phi_{n+1}(x)$, for $0 < x \le \dfrac{1}{(n+1)^2} < \dfrac{1}{n^2}$.

When

$$\left(\frac{2^{n-1}}{n\cdot 2^{n-1} - k_n + 1}\right)^2 \le \left(\frac{2^n}{(n+1)\cdot 2^n - k_{n+1} + 1}\right)^2 < x \le \left(\frac{2^n}{(n+1)\cdot 2^n - k_{n+1}}\right)^2 \le \left(\frac{2^{n-1}}{n\cdot 2^{n-1} - k_n}\right)^2,$$

we have

$$\begin{aligned}
&\left(\frac{2^n}{(n+1)\cdot 2^n - k_{n+1}}\right)^2 \le \left(\frac{2^{n-1}}{n\cdot 2^{n-1} - k_n}\right)^2 \\
&\Rightarrow \frac{2^n}{(n+1)\cdot 2^n - k_{n+1}} \le \frac{2^{n-1}}{n\cdot 2^{n-1} - k_n} \\
&\Rightarrow n\cdot 2^n - 2k_n \le n\cdot 2^n + 2^n - k_{n+1} \\
&\Rightarrow -k_n \le 2^{n-1} - \frac{k_{n+1}}{2} \\
&\Rightarrow n\cdot 2^{n-1} - k_n \le n\cdot 2^{n-1} + 2^{n-1} - \frac{k_{n+1}}{2} \\
&\Rightarrow \frac{n\cdot 2^{n-1} - k_n}{2^{n-1}} \le \frac{n\cdot 2^{n-1} + 2^{n-1} - \frac{k_{n+1}}{2}}{2^{n-1}} \\
&\qquad\qquad = \frac{n\cdot 2^n + 2^n - k_{n+1}}{2^n} \\
&\Rightarrow 0 \le \phi_n(x) = \frac{n\cdot 2^{n-1} - k_n}{2^{n-1}} \\
&\qquad\qquad \le \frac{n\cdot 2^n + 2^n - k_{n+1}}{2^n} \\
&\qquad\qquad = \phi_{n+1}(x).
\end{aligned}$$

By induction, $0 \le \phi_n \le \phi_{n+1}$ on $(0,1]$, for all $n \ge 1$.

Next, we show $\lim_{n\to\infty} \phi_n(x) = \dfrac{1}{\sqrt{x}}$ for all $x \in (0,1]$.

When $0 < x \le \left(\dfrac{2^{n-1}}{n\cdot 2^{n-1}}\right)^2$, we have $0 < \sqrt{x} \le \dfrac{2^{n-1}}{n\cdot 2^{n-1}} = \dfrac{1}{n}$. So $\phi_n(x) = n \le \dfrac{1}{\sqrt{x}}$, for $0 < x \le \left(\dfrac{2^{n-1}}{n\cdot 2^{n-1}}\right)^2$.

When $\left(\dfrac{2^{n-1}}{n\cdot 2^{n-1} - k_n + 1}\right)^2 < x \le \left(\dfrac{2^{n-1}}{n\cdot 2^{n-1} - k_n}\right)^2$, where $1 \le k_n \le (n-1)\cdot 2^{n-1}$, we have $\dfrac{2^{n-1}}{n\cdot 2^{n-1} - k_n + 1} < \sqrt{x} \le \dfrac{2^{n-1}}{n\cdot 2^{n-1} - k_n}$. Thus $\phi_n(x) = \dfrac{n\cdot 2^{n-1} - k_n}{2^{n-1}} \le \dfrac{1}{\sqrt{x}}$, for $\left(\dfrac{2^{n-1}}{n\cdot 2^{n-1} - k_n + 1}\right)^2 < x \le \left(\dfrac{2^{n-1}}{n\cdot 2^{n-1} - k_n}\right)^2$, where $1 \le k_n \le (n-1)\cdot 2^{n-1}$.

By induction, $\phi_n(x) \le \dfrac{1}{\sqrt{x}}$, for all $n \ge 1$. So $\lim\limits_{n\to\infty} \phi_n(x) = \dfrac{1}{\sqrt{x}}$ for all $x \in (0,1]$.

$$
\begin{aligned}
\int_{(0,1]} \phi_n &= \frac{n\cdot 2^{n-1}}{2^{n-1}}\left[\left(\frac{2^{n-1}}{n\cdot 2^{n-1}}\right)^2 - 0^2\right] + \sum_{k=1}^{(n-1)\cdot 2^{n-1}} \frac{n\cdot 2^{n-1} - k}{2^{n-1}}\left[\left(\frac{2^{n-1}}{n\cdot 2^{n-1} - k}\right)^2\right. \\
&\qquad \left. - \left(\frac{2^{n-1}}{n\cdot 2^{n-1} - k + 1}\right)^2\right] \\
&= \frac{n\cdot 2^{n-1}}{2^{n-1}}\left[\left(\frac{2^{n-1}}{n\cdot 2^{n-1}}\right)^2 - 0^2\right] + \frac{n\cdot 2^{n-1} - 1}{2^{n-1}}\left[\left(\frac{2^{n-1}}{n\cdot 2^{n-1} - 1}\right)^2 - \left(\frac{2^{n-1}}{n\cdot 2^{n-1}}\right)^2\right] \\
&\quad + \frac{n\cdot 2^{n-1} - 2}{2^{n-1}}\left[\left(\frac{2^{n-1}}{n\cdot 2^{n-1} - 2}\right)^2 - \left(\frac{2^{n-1}}{n\cdot 2^{n-1} - 1}\right)^2\right] + \cdots \\
&\quad + \frac{n\cdot 2^{n-1} - \left[(n-1)\cdot 2^{n-1} - 1\right]}{2^{n-1}}\left[\left(\frac{2^{n-1}}{n\cdot 2^{n-1} - [(n-1)\cdot 2^{n-1} - 1]}\right)^2\right. \\
&\qquad \left. - \left(\frac{2^{n-1}}{n\cdot 2^{n-1} - [(n-1)\cdot 2^{n-1} - 1] + 1}\right)^2\right]
\end{aligned}
$$

$$+\frac{n\cdot 2^{n-1}-(n-1)\cdot 2^{n-1}}{2^{n-1}}\left[\left(\frac{2^{n-1}}{n\cdot 2^{n-1}-(n-1)\cdot 2^{n-1}}\right)^2\right.$$

$$\left.-\left(\frac{2^{n-1}}{n\cdot 2^{n-1}-[(n-1)\cdot 2^{n-1}-1]}\right)^2\right]$$

$$=\left(\frac{2^{n-1}}{n\cdot 2^{n-1}}\right)^2\left(\frac{n\cdot 2^{n-1}}{2^{n-1}}-\frac{n\cdot 2^{n-1}-1}{2^{n-1}}\right)+\left(\frac{2^{n-1}}{n\cdot 2^{n-1}-1}\right)^2\left(\frac{n\cdot 2^{n-1}-1}{2^{n-1}}-\frac{n\cdot 2^{n-1}-2}{2^{n-1}}\right)$$

$$+\cdots+\left(\frac{2^{n-1}}{n\cdot 2^{n-1}-[(n-1)\cdot 2^{n-1}-1]}\right)^2\left(\frac{n\cdot 2^{n-1}-[(n-1)\cdot 2^{n-1}-1]}{2^{n-1}}\right.$$

$$\left.-\frac{n\cdot 2^{n-1}-(n-1)\cdot 2^{n-1}}{2^{n-1}}\right)$$

$$+\frac{n\cdot 2^{n-1}-(n-1)\cdot 2^{n-1}}{2^{n-1}}\left(\frac{2^{n-1}}{n\cdot 2^{n-1}-(n-1)\cdot 2^{n-1}}\right)^2$$

$$=\left(\frac{2^{n-1}}{n\cdot 2^{n-1}}\right)^2\left(\frac{1}{2^{n-1}}\right)+\left(\frac{2^{n-1}}{n\cdot 2^{n-1}-1}\right)^2\left(\frac{1}{2^{n-1}}\right)$$

$$+\cdots+\left(\frac{2^{n-1}}{n\cdot 2^{n-1}-[(n-1)\cdot 2^{n-1}-1]}\right)^2\left(\frac{1}{2^{n-1}}\right)$$

$$+\frac{n\cdot 2^{n-1}-(n-1)\cdot 2^{n-1}}{2^{n-1}}\left(\frac{2^{n-1}}{n\cdot 2^{n-1}-(n-1)\cdot 2^{n-1}}\right)^2$$

$$=\frac{2^{n-1}}{(n\cdot 2^{n-1})^2}+\frac{2^{n-1}}{(n\cdot 2^{n-1}-1)^2}+\cdots+\frac{2^{n-1}}{(n\cdot 2^{n-1}-[(n-1)\cdot 2^{n-1}-1])^2}+1$$

$$=1+2^{n-1}\left[\frac{1}{(2^{n-1}+1)^2}+\cdots+\frac{1}{(n\cdot 2^{n-1})^2}\right].$$

Since

$$\begin{aligned}
\frac{1}{(2^{n-1}+1)^2}+\cdots+\frac{1}{(n\cdot 2^{n-1})^2} &= \frac{1}{(2^{n-1}+1)(2^{n-1}+1)}+\frac{1}{(2^{n-1}+2)(2^{n-1}+2)} \\
&\quad +\cdots+\frac{1}{(n\cdot 2^{n-1}-1)(n\cdot 2^{n-1}-1)}+\frac{1}{(n\cdot 2^{n-1})(n\cdot 2^{n-1})} \\
&\geq \frac{1}{(2^{n-1}+1)(2^{n-1}+2)}+\frac{1}{(2^{n-1}+2)(2^{n-1}+3)} \\
&\quad +\cdots+\frac{1}{(n\cdot 2^{n-1}-1)(n\cdot 2^{n-1})}+\frac{1}{(n\cdot 2^{n-1})(n\cdot 2^{n-1}+1)} \\
&= \left(\frac{1}{2^{n-1}+1}-\frac{1}{2^{n-1}+2}\right)+\left(\frac{1}{2^{n-1}+2}-\frac{1}{2^{n-1}+3}\right) \\
&\quad +\cdots+\left(\frac{1}{n\cdot 2^{n-1}-1}-\frac{1}{n\cdot 2^{n-1}}\right)+\left(\frac{1}{n\cdot 2^{n-1}}-\frac{1}{n\cdot 2^{n-1}+1}\right) \\
&= \frac{1}{2^{n-1}+1}-\frac{1}{n\cdot 2^{n-1}+1},
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{(2^{n-1}+1)^2}+\cdots+\frac{1}{(n\cdot 2^{n-1})^2} &= \frac{1}{(2^{n-1}+1)(2^{n-1}+1)}+\frac{1}{(2^{n-1}+2)(2^{n-1}+2)} \\
&\quad +\cdots+\frac{1}{(n\cdot 2^{n-1}-1)(n\cdot 2^{n-1}-1)}+\frac{1}{(n\cdot 2^{n-1})(n\cdot 2^{n-1})} \\
&\leq \frac{1}{2^{n-1}(2^{n-1}+1)}+\frac{1}{(2^{n-1}+1)(2^{n-1}+2)} \\
&\quad +\cdots+\frac{1}{(n\cdot 2^{n-1}-2)(n\cdot 2^{n-1}-1)}+\frac{1}{(n\cdot 2^{n-1}-1)(n\cdot 2^{n-1})} \\
&= \left(\frac{1}{2^{n-1}}-\frac{1}{2^{n-1}+1}\right)+\left(\frac{1}{2^{n-1}+1}-\frac{1}{2^{n-1}+2}\right) \\
&\quad +\cdots+\left(\frac{1}{n\cdot 2^{n-1}-2}-\frac{1}{n\cdot 2^{n-1}-1}\right)+\left(\frac{1}{n\cdot 2^{n-1}-1}-\frac{1}{n\cdot 2^{n-1}}\right) \\
&= \frac{1}{2^{n-1}}-\frac{1}{n\cdot 2^{n-1}}.
\end{aligned}$$

Thus

$$\frac{1}{2^{n-1}+1}-\frac{1}{n\cdot 2^{n-1}+1}\leq \frac{1}{(2^{n-1}+1)^2}+\cdots+\frac{1}{(n\cdot 2^{n-1})^2}\leq \frac{1}{2^{n-1}}-\frac{1}{n\cdot 2^{n-1}}$$

and so

$$\frac{1}{2^{n-1}+1}-\frac{1}{n\cdot 2^{n-1}+1}\le \frac{1}{(2^{n-1}+1)^2}+\cdots+\frac{1}{(n\cdot 2^{n-1})^2}\le \frac{1}{2^{n-1}}-\frac{1}{n\cdot 2^{n-1}}$$

$$\Rightarrow 2^{n-1}\left(\frac{1}{2^{n-1}+1}-\frac{1}{n\cdot 2^{n-1}+1}\right)\le 2^{n-1}\left[\frac{1}{(2^{n-1}+1)^2}+\cdots+\frac{1}{(n\cdot 2^{n-1})^2}\right]\le 2^{n-1}\left(\frac{1}{2^{n-1}}-\frac{1}{n\cdot 2^{n-1}}\right)$$

$$\Rightarrow \frac{1}{1+\frac{1}{2^{n-1}}}-\frac{1}{n+\frac{1}{2^{n-1}}}\le 2^{n-1}\left[\frac{1}{(2^{n-1}+1)^2}+\cdots+\frac{1}{(n\cdot 2^{n-1})^2}\right]\le 1-\frac{1}{n}$$

$$\Rightarrow 1+\frac{1}{1+\frac{1}{2^{n-1}}}-\frac{1}{n+\frac{1}{2^{n-1}}}\le 1+2^{n-1}\left[\frac{1}{(2^{n-1}+1)^2}+\cdots+\frac{1}{(n\cdot 2^{n-1})^2}\right]\le 2-\frac{1}{n}$$

$$\Rightarrow 1+\frac{1}{1+\frac{1}{2^{n-1}}}-\frac{1}{n+\frac{1}{2^{n-1}}}\le \int_{(0,1]}\phi_n \le 2-\frac{1}{n}$$

$$\Rightarrow \lim_{n\to\infty}\int_{(0,1]}\phi_n = 2.$$

Hence,

$$\begin{aligned}\int_{(0,1]}\frac{1}{\sqrt{x}} &= \lim_{n\to\infty}\int_{(0,1]}\phi_n\\ &= 2.\end{aligned}$$

□

In these calculations, many simplifications arise if $\lim_{n\to\infty}\int_E f_n = \int_E\left(\lim_{n\to\infty} f_n\right)$ is valid for monotone sequences of nonnegative measurable functions, not just monotone sequences of nonnegative simple functions.

Theorem 5.3.12 Lebesgue Monotone Convergence Theorem (LMCT), Beppo Levi, 1906. *Let* $\{f_k\}$ *be a monotone increasing sequence of nonnegative measurable functions on a Lebesgue measurable set* $E: 0\le f_1\le f_2\le\cdots$ *on* E. *Then*

$$\lim_{k\to\infty}\int_E f_k = \int_E\left(\lim_{k\to\infty} f_k\right).$$

Proof. We give two arguments. The first is based on Definition 5.3.5 for the integral of a nonnegative measurable function. Note that $\left(\lim_{k\to\infty} f_k\right)$ is nonnegative $(0\le f_k)$ and measurable on E (Theorem 4.2.1). So

$$\int_E \left(\lim_{k\to\infty} f_k\right) = \sup\left\{\int_E \phi \mid \phi \le \left(\lim_{k\to\infty} f_k\right), 0 \le \phi, \phi \text{ } simple\right\}$$

Since the integral preserves monotonicity for nonnegative measurable functions, and $0 \le f_k \le f_{k+1} \le \cdots \le \left(\lim_{k\to\infty} f_k\right)$ on E, we have $0 \le \int_E f_k \le \int_E f_{k+1} \le \cdots \le \int_E \left(\lim_{k\to\infty} f_k\right)$, that is, $\lim_{k\to\infty}\int_E f_k \le \int_E \left(\lim_{k\to\infty} f_k\right)$.

We will show $\int_E \left(\lim_{k\to\infty} f_k\right) \le \lim_{k\to\infty}\int_E f_k$ to complete the argument. Let ϕ be any simple function such that $0 \le \phi \le \left(\lim_{k\to\infty} f_k\right)$. If we can show $\lim_{k\to\infty}\int_E f_k \ge \int_E \phi$, then this would say $\lim_{k\to\infty}\int_E f_k$ is an upper bound for the set $\left\{\int_E \phi \mid \phi \le \left(\lim_{k\to\infty} f_k\right), 0 \le \phi, \phi \text{ } simple\right\}$. But the least upper bound of this set, $\int_E \left(\lim_{k\to\infty} f_k\right)$, would be less than or equal the upper bound, $\lim_{k\to\infty}\int_E f_k$, and the conclusion would follow.

We now show $\lim_{k\to\infty}\int_E f_k \ge \int_E \phi$, $\phi \le \left(\lim_{k\to\infty} f_k\right)$, $0 \le \phi$, ϕ simple. Since ϕ is nonnegative and simple, $\int_E \phi = \sum_{i=1}^{N} c_i \mu(E \cap E_i), c_i \ge 0$ and $\bigcup_{i=1}^{N}(E \cap E_i) = E$ where $E \cap E_i$ are mutually disjoint measurable subsets of E. Since $\int_E f_k$ is additive for nonnegative measurable functions (Theorem 5.3.10), it is sufficient to show

$$\lim_{k\to\infty}\left(\sum_{i=1}^{N}\int_{E\cap E_i} f_k\right) \ge \sum_{i=1}^{N} c_i \mu(E \cap E_i),$$

and this will be accomplished, if we show

$$\lim_{k\to\infty}\int_{E\cap E_i} f_k \ge c_i \mu(E \cap E_i).$$

If $c_i = 0$, done. Assume $c_i > 0$. The idea is to construct an increasing sequence of Lebesgue measurable sets $\{B_k\}$. The ingenious idea is due to W. Rudin. Let $0 < \alpha < 1$, and define $B_k = \{x \in E \cap E_i \mid f_k(x) \ge \alpha c_i\}$. $\{B_k\}$ are Lebesgue measurable, $B_k \subseteq B_{k+1}$, since $f_k \le f_{k+1}$, and $\bigcup_{k=1}^{\infty} B_k = E \bigcap E_i$. This is because if $x_0 \in E \bigcap E_i, \phi(x_0) = c_i$ since $x_0 \in E_i$ and $\phi(x) = \sum_{i=1}^{N} c_i \chi_{E_i}$ and so

$$\lim_{k\to\infty} f_k(x_0) \ge \phi(x_0) = c_i > \alpha c_i,$$

i.e., $f_k(x_0) > \alpha c_i$ for sufficiently large k, in other words, $x_0 \in B_k$ for sufficiently large k.

The sequence $\{\mu(B_k)\}$ is nondecreasing, and $\lim_{k\to\infty} \mu(B_k) = \mu(E \bigcap E_i)$ by Theorem 3.4.1. But $\int_{E\bigcap E_i} f_k \ge \int_{B_k} f_k \ge \alpha c_i \mu(B_k)$. Hence

$$\lim_{k\to\infty} \int_{E\bigcap E_i} f_k \ge \alpha c_i \lim_{k\to\infty} \mu(B_k) = \alpha c_i \mu(E \bigcap E_i),\ 0 < \alpha < 1.$$

Therefore $\lim_{k\to\infty} \int_{E\bigcap E_i} f_k \ge c_i \mu(E \bigcap E_i)$, by Lemma 5.3.7 and the theorem is proved using Definition 5.3.5.

The second argument is based on Definition 5.3.6, the Approximation Theorem 4.3.5, and extensive use of Proposition 5.3.10. From the Approximation Theorem, we have:

$$0 \le \phi_{11} \le \phi_{12} \le \cdots \le \phi_{1n} \le \cdots \le f_1, \lim_{n\to\infty} \phi_{1n} = f_1 \text{ on } E, \text{and} \lim_{n\to\infty} \int_E \phi_{1n} = \int_E f_1;$$

$$0 \le \phi_{21} \le \phi_{22} \le \cdots \le \phi_{2n} \le \cdots \le f_2, \lim_{n\to\infty} \phi_{2n} = f_2 \text{ on } E, \text{and} \lim_{n\to\infty} \int_E \phi_{2n} = \int_E f_2;$$

$$\vdots$$

$$0 \le \phi_{k1} \le \phi_{k2} \le \cdots \le \phi_{kn} \le \cdots \le f_k, \lim_{n\to\infty} \phi_{kn} = f_k \text{ on } E, \text{and} \lim_{n\to\infty} \int_E \phi_{kn} = \int_E f_k;$$

etc.

Construct a new sequence of simple functions, $\left\{\hat{\phi}_k\right\}$, with $\lim_{k\to\infty} \hat{\phi}_k = \left(\lim_{k\to\infty} f_k\right)$ as follows:

$$\hat{\phi}_1 = \phi_{11};$$

$$\hat{\phi}_2 = \max\{\phi_{12}, \phi_{22}\} \ge \phi_{12} \ge \phi_{11} = \hat{\phi}_1;$$

$$\vdots$$

$$\hat{\phi}_k = \max\{\phi_{1k}, \phi_{2k}, \ldots, \phi_{k-1k}, \phi_{kk}\} \geq \hat{\phi}_{k-1};$$

etc.

The $\hat{\phi}_k$ are simple and $0 \leq \hat{\phi}_1 \leq \hat{\phi}_2 \leq \cdots \leq \hat{\phi}_k \leq \cdots$, with $\lim_{k\to\infty} \hat{\phi}_k = \left(\lim_{k\to\infty} f_k\right)$. By Definition 5.3.6, $\lim_{k\to\infty} \int_E \hat{\phi}_k = \int_E \left(\lim_{k\to\infty} f_k\right)$. However, $0 \leq \hat{\phi}_k \leq f_k \leq \left(\lim_{k\to\infty} f_k\right)$ and, since the integral preserves monotonicity for nonnegative measurable functions, we have $\int_E \hat{\phi}_k \leq \int_E f_k \leq \int_E \left(\lim_{k\to\infty} f_k\right)$. Taking limits,

$$\lim_{k\to\infty} \int_E \hat{\phi}_k \leq \lim_{k\to\infty} \int_E f_k \leq \int_E \left(\lim_{k\to\infty} f_k\right).$$

Recalling $\lim_{k\to\infty} \int_E \hat{\phi}_k = \int_E \left(\lim_{k\to\infty} f_k\right)$ from above, we have

$$\int_E \left(\lim_{k\to\infty} f_k\right) \leq \lim_{k\to\infty} \int_E f_k \leq \int_E \left(\lim_{k\to\infty} f_k\right)$$

and so

$$\lim_{k\to\infty} \int_E f_k = \int_E \left(\lim_{k\to\infty} f_k\right).$$

□

Next example shows that we approximate unbounded Lebesgue measurable function on set of finite measure with bounded measurable function.

Example 5.3.13 Calculate $\int_{[0,1)} \frac{1}{\sqrt{1-t^2}}$.

Solution. Let $f_k(t) = \begin{cases} \dfrac{1}{\sqrt{1-t^2}}, & 0 \leq t \leq 1 - \dfrac{1}{k} \\ 0, & 1 - \dfrac{1}{k} < t < 1 \end{cases}$. Then

$$f_{k+1}(t) = \begin{cases} \dfrac{1}{\sqrt{1-t^2}}, & 0 \leq t \leq 1 - \dfrac{1}{k+1} \\ 0, & 1 - \dfrac{1}{k+1} < t < 1 \end{cases}.$$

When $0 \le t \le 1-\dfrac{1}{k}, f_k(t) = \dfrac{1}{\sqrt{1-t^2}} = f_{k+1}(t)$.

When $1-\dfrac{1}{k} < t \le 1-\dfrac{1}{k+1}, f_k(t) = 0 < \dfrac{1}{\sqrt{1-t^2}} = f_{k+1}(t)$.

When $1-\dfrac{1}{k+1} < t < 1, f_k(t) = 0 = f_{k+1}(t)$.

Thus

$$0 \le f_k \le f_{k+1} \text{ on } [0,1).$$

Clearly,

$$\lim_{k\to\infty} f_k(t) = \frac{1}{\sqrt{1-t^2}} \text{ for all } t \in [0,1).$$

Therefore

$$\begin{aligned}
\int_{[0,1)} \frac{1}{\sqrt{1-t^2}} &= \int_{[0,1)} \left(\lim_{k\to\infty} f_k\right) \\
&= \lim_{k\to\infty} \int_{[0,1)} f_k \\
&= \lim_{k\to\infty} \int_{[0,1-\frac{1}{k}]} f_k \\
&= \lim_{k\to\infty} \int_0^{1-\frac{1}{k}} \frac{1}{\sqrt{1-t^2}} dt \\
&= \lim_{k\to\infty} \left[\sin^{-1} t\right]_{t=0}^{t=1-\frac{1}{k}} \\
&= \lim_{k\to\infty} \left[\sin^{-1}\left(1-\frac{1}{k}\right) - \sin^{-1} 0\right] \\
&= \sin^{-1} 1 \\
&= \frac{\pi}{2}.
\end{aligned}$$

□

Next example shows that we approximate bounded Lebesgue measurable function on set of infinite measure with bounded measurable function.

Example 5.3.14 Calculate $\int_{[0,\infty)} e^{-t}$.

Solution. Let $f_k(t) = \begin{cases} e^{-t}, & 0 \le t \le k \\ 0, & k < t \end{cases}$. Then

$$f_{k+1}(t) = \begin{cases} e^{-t}, & 0 \le t \le k+1 \\ 0, & k+1 < t \end{cases}.$$

When $0 \le t \le k, f_k(t) = e^{-t} = f_{k+1}(t)$.

When $k < t \le k+1, f_k(t) = 0 < e^{-t} = f_{k+1}(t)$.

When $k+1 < t, f_k(t) = 0 = f_{k+1}(t)$.

Thus

$$0 \le f_k \le f_{k+1} \text{ on } [0,\infty).$$

Clearly,

$$\lim_{k\to\infty} f_k(t) = e^{-t} \text{ for all } t \in [0,\infty).$$

Therefore

$$\begin{aligned} \int_{[0,\infty)} e^{-t} &= \int_{[0,\infty)} \left(\lim_{k\to\infty} f_k\right) \\ &= \lim_{k\to\infty} \int_{[0,\infty)} f_k \\ &= \lim_{k\to\infty} \int_{[0,k]} f_k \\ &= \lim_{k\to\infty} \int_0^k e^{-t}\,dt \\ &= \lim_{k\to\infty} \left[-e^{-t}\right]_{t=0}^{t=k} \\ &= \lim_{k\to\infty} \left(1 - e^{-k}\right) \\ &= 1. \end{aligned}$$

□

Theorem 5.3.15 *If* $\{g_k\}$ *is a sequence of nonnegative measurable functions defined on a measurable set* E, *then*

$$\int_E \sum_{k=1}^{\infty} g_k = \sum_{k=1}^{\infty} \int_E g_k.$$

Proof: Let $f_n = g_1 + g_2 + \cdots + g_n$. Then $\{f_n\}$ is a monotone increasing sequence of nonnegative measurable functions defined on a measurable set E. We use LMCT.

$$\int_E \sum_{k=1}^{\infty} g_k = \int_E \left(\lim_{n\to\infty} f_n\right) = \lim_{n\to\infty}\int_E f_n = \lim_{n\to\infty}\int_E \sum_{k=1}^{n} g_k = \lim_{n\to\infty}\sum_{k=1}^{n}\int_E g_k = \sum_{k=1}^{\infty}\int_E g_k .$$

□

Theorem 5.3.16 (Fatou, 1906) *If* $\{f_k\}$ *is a sequence of nonnegative measurable functions defined on a Lebesgue measurable set* E, *then*

$$\int_E \left(\liminf_{k\to\infty} f_k\right) \le \liminf_{k\to\infty} f_k .$$

Proof. Let $\underline{f}_1 \equiv \inf\{f_1, f_2, \ldots\}$. $\underline{f}_1$ is measurable, $0 \le \underline{f}_1 \le f_n$ for all $n \ge 1$, and $\int_E \underline{f}_1 \le \int_E f_n$ for all $n \ge 1$.

In other words, $\int_E \underline{f}_1$ is a lower bound for the set $\left\{\int_E f_1, \int_E f_2, \ldots, \int_E f_n, \ldots\right\}$ We have

$$\int_E \underline{f}_1 \le \underline{\int_E f_1} \equiv \inf\left\{\int_E f_1, \int_E f_2, \ldots, \int_E f_n, \ldots\right\}$$

Define $\underline{f}_2 \equiv \inf\{f_2, f_3, \ldots\}$. $\underline{f}_2$ is measurable, $0 \le \underline{f}_1 \le \underline{f}_2$,

$$\int_E \underline{f}_1 \le \int_E \underline{f}_2 \text{ and } \underline{\int_E f_1} \le \underline{\int_E f_2} \equiv \inf\left\{\int_E f_2, \int_E f_3, \ldots\right\}$$

In general, if

$$\underline{f}_m \equiv \inf\{f_m, f_{m+1}, \ldots\},$$

then

$$0 \le \underline{f}_1 \le \underline{f}_2 \le \cdots \le \underline{f}_m \le \cdots, \text{and} \int_E \underline{f}_m \le \underline{\int_E f_m} = \inf\left\{\int_E f_m, \int_E f_{m+1}, \ldots\right\}$$

The sequences $\left\{\int_E \underline{f}_m\right\}, \left\{\underline{\int_E f_m}\right\}$ are nonnegative, monotone increasing sequences of perhaps extended-real numbers, and so have limits in the extended reals:

$$\lim_{m\to\infty}\int_E \underline{f}_m \le \lim_{m\to\infty}\underline{\int_E f_m}.$$

But, application of the LMCT tells us that

$$\lim_{m\to\infty}\int_E \underline{f}_m = \int_E\left(\lim_{m\to\infty}\underline{f}_m\right) = \int_E\left(\liminf_{k\to\infty} f_k\right)$$

Therefore

$$\int_E \left(\liminf_{k\to\infty} f_k\right) \le \lim_{m\to\infty} \underline{\int_E f_m} = \liminf_{k\to\infty} \int_E f_k,$$

and this is what we intended to prove. □

Example 5.3.17 Let $f_n = \frac{1}{n}\chi_{[0,n]}$. Then

$$\lim_{n\to\infty} \int_{[0,\infty)} f_n = 1 \ne 0 = \int_{[0,\infty)} \lim_{n\to\infty} f_n.$$

But

$$\int_{[0,\infty)} \liminf_{n\to\infty} f_n = \int_{[0,\infty)} \lim_{n\to\infty} f_n = 0 \le 1 = \liminf_{n\to\infty} \int_{[0,\infty)} f_n.$$

□

Monotone increasing is necessary.

Example 5.3.18 Let $f_n = -\frac{1}{n}\chi_{[0,n]}$. Then $f_n \to 0$ (unif) on $[0,\infty)$.

We have

$$\lim_{n\to\infty} \int_{[0,\infty)} f_n = -1 \ne 0 = \int_{[0,\infty)} \lim_{n\to\infty} f_n,$$

but

$$\int_{[0,\infty)} \liminf_{n\to\infty} f_n = 0 > -1 = \liminf_{n\to\infty} \int_{[0,\infty)} f_n.$$

□

Nonnegative is necessary, even with uniform convergence.

This concludes our treatment of the Lebesgue integral for nonnegative measurable functions defined on arbitrary measurable sets of real numbers.

5.4 Lebesgue Integral and Lebesgue Integrability

The requirement that f be nonnegative is eliminated. We discuss Lebesgue measurable functions defined on any Lebesgue measurable set of real numbers.

Definition 5.4.1 *Let f be a measurable function defined on a measurable set E. Recall that $f = f^+ - f^-$ (Propostition 4.1.8), where $f^+ = \max(f,0)$ and $f^- = -\min(f,0)$ are nonnegative measurable functions. $\int_E f^+$ and $\int_E f^-$ can be calculated according to Definition 5.3.5 or Definition 5.3.6.*

If both $\int_E f^+$ and $\int_E f^-$ are ∞, then the Lebesgue integral of f on E is not defined ($\infty - \infty$ is not defined in $\mathbb{R}^e$).

If either $\int_E f^+$ or $\int_E f^-$ (but not both) are finite, then the Lebesgue integral of f on E is defined by

$$\int_E f = \int_E f^+ - \int_E f^-.$$

If both $\int_E f^+$ and $\int_E f^-$ are finite, then f is Lebesgue integrable on E and

$$\int_E f = \int_E f^+ - \int_E f^-.$$

In this case, $\int_E f \in \mathbb{R}$.

Caution: In what follows "integrable" means "Lebesgue integrable".

Now, we discuss the relationships between $\int_E f$, $\int_E |f|$, and $\int_E g$ when $g = f$ almost everywhere on E.

Propostition 5.4.2 *Suppose f is a measurable function defined on a measurable set E. Then f is integrable on E iff $|f|$ is integrable on E.*
Moreover,

$$\left|\int_E f\right| \le \int_E |f|.$$

Proof. Assume f is integrable on E. We want to show $|f|$ is integrable on E and $\int_E |f|^+, \int_E |f|^- < \infty.$

Since f is measurable on E, $|f|$ is measurable on E (Propostition 4.1.8), $\int_E |f|^- = 0,$ and $\int_E |f|^+ = \int_E |f| = \int_E (f^+ + f^-) = \int_E f^+ + \int_E f^- < \infty$ because f is integrable on E. Thus $|f|$ is integrable on E.

Assume $|f|$ is integrable on E. We want to show f is integrable on E. Since f is measurable by assumption, and

$$\int_E f^+ + \int_E f^- = \int_E (f^+ + f^-) = \int_E |f| = \int_E |f|^+ < \infty,$$

we have $\int_E |f|^+, \int_E |f|^- < \infty.$ Thus, f is integrable on E.

$$\begin{aligned}\left|\int_E f\right| = \left|\int_E f^+ - \int_E f^-\right| &\le \left|\int_E f^+\right| + \left|\int_E f^-\right| \\ &= \int_E f^+ + \int_E f^- \\ &= \int_E (f^+ + f^-) \\ &= \int_E |f|.\end{aligned}$$

□

Proposition 5.4.3 *If f is integrable on E, then f is real-valued (finite) a.e. on E.*

Proof. We need to prove $\mu(\{x \in E \mid f(x) = \pm\infty\}) = 0.$

For $\{x \in E \mid f(x) = +\infty\}$,

$$f^+(x) - f^-(x) = +\infty$$

$$\Rightarrow f^+(x) = +\infty \text{ and } 0 \le f^-(x) < +\infty \quad \text{or} \quad 0 \le f^+(x) < +\infty \text{ and } f^-(x) = -\infty.$$

The last case is unacceptable, and so

$$\{x \in E \mid f(x) = +\infty\} = \{x \in E \mid f^+(x) = +\infty\}.$$

For $\{x \in E \mid f(x) = -\infty\}$,

$$f^+(x) - f^-(x) = -\infty$$

$$\Rightarrow f^+(x) = -\infty \text{ and } 0 \le f^-(x) < +\infty \quad \text{or} \quad 0 \le f^+(x) < +\infty \text{ and } f^-(x) = +\infty.$$

The first case is unacceptable, and so

$$\{x \in E \mid f(x) = -\infty\} = \{x \in E \mid f^-(x) = +\infty\}.$$

This implies

$$\begin{aligned}\{x \in E \mid f(x) = \pm\infty\} &= \{x \in E \mid f(x) = +\infty\} \cup \{x \in E \mid f(x) = -\infty\} \\ &= \{x \in E \mid f^+(x) = +\infty\} \cup \{x \in E \mid f^-(x) = +\infty\}.\end{aligned}$$

Then

$$\infty > \int_E f^+ \geq \int_{\{x \in E \mid f^+(x) = +\infty\}} f^+ \geq n\mu(\{x \in E \mid f^+(x) = +\infty\})$$

for all $n \geq 1$. We have a contradiction unless $\mu(\{x \in E \mid f^+(x) = +\infty\}) = 0$.

Similarly,

$$\mu(\{x \in E \mid f^-(x) = +\infty\}) = 0.$$

Hence

$$\mu(\{x \in E \mid f(x) = \pm\infty\}) = 0.$$

□

However, the converse of Proposition 5.4.3 is not true. Let's say we have $f(x) = \dfrac{1}{x}, 0 < x \leq 1$. Then f is real-valued everywhere, but is not integrable.

Example 5.4.4 Suppose

$$f(x) = \begin{cases} \dfrac{(-1)^n}{(n+1)^2}, & n\pi < x < (n+1)\pi, n = 0,1,2,\ldots \\ 0, & \textit{otherwise.} \end{cases}$$

Show f is integrable on $[0, \infty)$.

Solution. This is equivalent to show $|f|$ is integrable on $[0, \infty)$, by Proposition 5.4.2.

Then

$$|f|(x) = \begin{cases} \dfrac{1}{(n+1)^2}, & n\pi < x < (n+1)\pi, n = 0,1,2,\ldots \\ 0, & \textit{otherwise.} \end{cases}$$

Let

$$|f|_k(x) = \begin{cases} \dfrac{1}{(n+1)^2}, & n\pi < x < (n+1)\pi, n = 0,1,\ldots,k-1 \\ 0, & \textit{otherwise.} \end{cases}$$

Then

$$|f|_{k+1}(x)=\begin{cases}\dfrac{1}{(n+1)^2}, & n\pi<x<(n+1)\pi, n=0,1,\ldots,k-1,k\\ 0, & otherwise.\end{cases}$$

When $x<0$, $|f|_k(x)=0=|f|_{k+1}(x)$.

When i) $x=n\pi, n=0,1,\ldots,k$, $|f|_k(x)=0=|f|_{k+1}(x)$.

ii) $x=(k+1)\pi$, $|f|_k(x)=0=|f|_{k+1}(x)$.

When $n\pi<x<(n+1)\pi, n=0,1,\ldots,k-1$, $|f|_k(x)=\dfrac{1}{(n+1)^2}=|f|_{k+1}(x)$.

When $k\pi<x<(k+1)\pi$, $|f|_k(x)=0<\dfrac{1}{(n+1)^2}=|f|_{k+1}(x)$.

When $x>(k+1)\pi$, $|f|_k(x)=0=|f|_{k+1}(x)$.

$\therefore 0\le |f|_k\le |f|_{k+1}$ on $(-\infty,\infty)$.

$$\begin{aligned}\therefore \int_{[0,\infty)}|f| &= \int_{[0,\infty)}\lim_{k\to\infty}|f|_k\\ &=\lim_{k\to\infty}\int_{[0,\infty)}|f|_k\\ &=\lim_{k\to\infty}\int_{n\pi}^{(n+1)\pi}\frac{1}{(n+1)^2}dx\\ &=\lim_{k\to\infty}\frac{\pi}{(n+1)^2}\\ &<\infty.\end{aligned}$$

□

Proposition 5.4.5 *If f is a measurable function defined on a measurable set E, and g is integrable on E with $|f|\le|g|$, then $\int_E|f|\le\int_E|g|$ and f is integrable on E.*

Proof. We have $\int_E|f|\le\int_E|g|<\infty$ from Theorem 5.3.10 and Proposition 5.4.2. It remains to show f is integrable on E by showing f is measurable on E (given), and both $\int_E f^+$ and $\int_E f^-$ are finite. Given f is measurable on E, and $0\le\int_E f^+ + \int_E f^- = \int_E|f|<\infty$, and the argument is complete. □

Proposition 5.4.6 *If* $f = g$ *a.e. on a measurable set* E, *and if* g *is integrable on* E, *then* f *is integrable on* E *and*

$$\int_E f = \int_E g.$$

Proof. g is measurable on E by the assumption of being Lebesgue integrable on E. Since f is equal a.e. to a measurable function g, f is measurable on E (Theorem 4.1.5). Application of Proposition 4.1.8 yields measurability of f^+ and f^- on E.

Let $A = \{x \in E \mid f(x) \neq g(x)\}$. Then

$$f^+ = \max(f,0) = \max(g,0) = g^+ \text{ and } f^- = -\min(f,0) = -\min(g,0) = g^-$$

on $E \cap A^c$, and

$$\int_{E\cap A^c} f^+ = \int_{E\cap A^c} g^+ \text{ and} \int_{E\cap A^c} f^- = \int_{E\cap A^c} g^-,$$

that is,

$$f \text{ is measurable on } E \cap A^c, \int_{E\cap A^c} f^+, \int_{E\cap A^c} f^- < \infty : f \text{ is integrable on } E \cap A^c.$$

Since A is a measurable subset of E, f is measurable on A (Proposition 4.1.6), $\mu(A) = 0$, and hence $\int_A f^+, \int_A f^- = 0$. But $\int_E f^+ = \int_{E\cap A^c} f^+ + \int_A f^+ < \infty$, that is, f is integrable on E.

Then

$$\begin{aligned}
\int_E g &= \int_E g^+ - \int_E g^- \\
&= \left(\int_{E\cap A^c} g^+ + \int_A g^+\right) - \left(\int_{E\cap A^c} g^- + \int_A g^-\right) \\
&= \left(\int_{E\cap A^c} f^+ + 0\right) - \left(\int_{E\cap A^c} f^- + 0\right) \\
&= \left(\int_{E\cap A^c} f^+ + \int_A f^+\right) - \left(\int_{E\cap A^c} f^- + \int_A f^-\right) \\
&= \int_E f^+ - \int_E f^- \\
&= \int_E f.
\end{aligned}$$

□

Example 5.4.7 Suppose $f(x)=\begin{cases} x, & x \ rational,\ 0\le x\le 1 \\ 1-x, & x \ irrational,\ 0\le x\le 1 \end{cases}$,

and

$$g(x)=1-x, 0\le x\le 1.$$

Then

$$f=g \text{ a.e. on } [0,1]$$

and

$$\begin{aligned}\int_{[0,1]} f &= \int_{[0,1]} g \\ &= \int_0^1 (1-x)dx \\ &= \frac{1}{2}.\end{aligned}$$

□

Proposition 5.4.8 *If f, g are integrable on a measurable set E, and k is any real number, then*

1. *(kf) is integrable on E, and $\int_E (kf) = k\int_E f$ (homogeneous);*
2. *$(f+g)$ is integrable on E, and $\int_E (f+g) = \int_E f + \int_E g$ (additive);*
3. *$\int_E f \le \int_E g$ if $f \le g$ on E (monotone);*
4. *If E_1 and E_2 are disjoint measurable subsets of E with $E = E_1 \cup E_2$, f is integrable on E_1 and E_2, and*

$$\int_E f = \int_{E_1} f + \int_{E_2} f \text{ (additive on the domain).}$$

Proof.

1. If $k \ge 0$, then $\int_E (kf)^+ = \int_E kf^+ = k\int_E f^+ < \infty$ and $\int_E (kf)^- = k\int_E f^- < \infty$ because kf^+, kf^- are nonnegative measurable functions (Theorem 5.3.10).

By definition, (kf) is integrable on E. Moreover

$$\int_E (kf) = \int_E (kf)^+ - \int_E (kf)^- = k\int_E f^+ - k\int_E f^- = k\int_E f,$$

where the last equality is the definition of f being integrable on E.

If $k < 0, (kf)^+ = (-k)f^-, (kf)^- = (-k)f^+, \int_E (kf)^+ = -k\int_E f^- < \infty,$ and $\int_E (kf)^- = -k\int_E f^+ < \infty,$ that is, (kf) is integrable on $E.$ Then

$$\begin{aligned}\int_E (kf) &= \int_E (kf)^+ - \int_E (kf)^- \\ &= \int_E (-k)f^- - \int_E (-k)f^+ \\ &= k\left[\int_E f^+ - \int_E f^-\right] \\ &= k\int_E f.\end{aligned}$$

2. Since f, g are integrable on E, $|f|, |g|$ are integrable on E (Proposition 5.4.2). Since $\int_E |f| = \int_E |f|^+ < \infty, \int_E |g| = \int_E |g|^+ < \infty,$ and $|f+g| \le |f| + |g|,$ and so

$$\int_E |f+g| \le \int_E (|f| + |g|) = \int_E |f| + \int_E |g| < \infty \text{ (Theorem 5.3.10).}$$

But $|f+g|^+ = |f+g|$ and $|f+g|^- = 0, \int_E |f+g|^+ < \infty, \int_E |f+g|^- < \infty,$ that is, $|f+g|$ is integrable on $E,$ but then (Proposition 5.4.2) $f+g$ is integrable on $E.$

Now, $f + g = (f^+ + g^+) - (f^- + g^-),$ that is, the integrable function $(f+g)$ has been written as the difference of two nonnegative measurable functions, $(f^+ + g^+)$ and $(f^- + g^-),$ whose integrals are finite.

$$\begin{aligned}\int_E (f+g) &= \int_E (f^+ + g^+) - \int_E (f^- + g^-) \\ &= \int_E f^+ + \int_E g^+ - \int_E f^- - \int_E g^- \\ &= \int_E f + \int_E g.\end{aligned}$$

3.

Since $f \le g$ on $E, f^+ - f^- \le g^+ - g^-$, i.e., $f^+ + g^- \le g^+ + f^-$. Because $(f^+ + g^-)$, $(g^+ + f^-)$ are nonnegative measurable functions we may apply Proposition 5.4.2 to conclude

$$\int_E f^+ + \int_E g^- = \int_E (f^+ + g^-) \le \int_E (g^+ + f^-) = \int_E g^+ + \int_E f^-.$$

Since all terms are finite, we may subtract and obtain $\int_E f \leq \int_E g$.

4. By Proposition 5.4.2,

$$\begin{aligned}\int_E f &= \int_E f^+ - \int_E f^- \\ &= \int_{E_1} f^+ + \int_{E_2} f^+ - \int_{E_1} f^- - \int_{E_2} f^- \\ &= \int_{E_1} f + \int_{E_2} f.\end{aligned}$$

□

In the next section, we are going to prove the major convergence theorem of Lebesgue integration, the so-called Lebesgue Dominated Convergence Theorem (LDCT).

5.5 Convergence Theorems

"Monotone" and "nonnegative" are the restrictions that we seek to modify or eliminate, although, other requirements must be imposed. The new requirements do not severely restrict applications of the Lebesgue integral, and in fact result in a very powerful tool for analysis, the so-called Lebesgue Dominated Convergence Theorem (LDCT).

Theorem 5.5.1 (Lebesgue Dominated Convergence Theorem, 1910) *Let* $\{f_k\}$ *be a sequence of measurable functions defined on a measurable set* E, *such that*

$$\lim_{k\to\infty} f_k = f \text{ a.e. on } E.$$

Suppose we have an integrable function g *on* E *such that* $|f_k| \leq g$ *on* E.

Then f *is integrable on* E *and*

$$\int_E \left(\lim_{k\to\infty} f_k\right) = \int_E f = \lim_{k\to\infty} \int_E f_k.$$

Proof. Since f_k is a measurable function on E, it follows that $|f_k|$ is also a measurable function on E, by Proposition 4.1.8. By assumption, g is integrable on E, and so $|f_k|$ is integrable on E, this implies f_k is integrable on E (Proposition 5.4.2).

Moreover, f, as the limit of a sequence of measurable functions, is measurable (Theorem 4.2.1). Since $-g \leq f_k \leq g$ for all $k \geq 1$ on E, thus $-g \leq \lim_{k\to\infty} f_k \leq g$ or $-g \leq f \leq g$. g is integrable on E implies f is integrable on E.

For a sequence of functions $\{f_k\}$ we may construct two related monotone sequences of functions

$$\{\underline{f}_k\} \text{and} \{\overline{f}_k\},$$

where $\underline{f}_k = \inf\{f_k, f_{k+1}, \ldots\}$ and $\overline{f}_k = \sup\{f_k, f_{k+1}, \ldots\}$ respectively.

Hence,

$$-g \leq \underline{f}_k \leq \underline{f}_{k+1} \leq f_{k+1} \leq g \text{ and} -g \leq f \leq \overline{f}_{k+1} \leq \overline{f}_k \leq g \text{ on } E. \tag{1}$$

All functions being integrable on E, along with monotonicity, yield (Proposition 5.4.8)

$$-\infty < \int_E -g \leq \int_E \underline{f}_{k+1} \leq \int_E f_{k+1} \leq \int_E g \text{ and} \int_E -g \leq \int_E f \leq \int_E \overline{f}_{k+1} \leq \int_E g < +\infty.$$

If we can show $\lim_{k\to\infty} \int_E \underline{f}_k = \lim_{k\to\infty} \int_E \overline{f}_k$, then $\lim_{k\to\infty} \int_E f_k$ exists and

$$\lim_{k\to\infty} \int_E f_k = \int_E f = \int_E \left(\lim_{k\to\infty} f_k\right),$$

which is the conclusion we want. Returning to (1), we have

$$0 \leq g + \underline{f}_k \leq g + \underline{f}_{k+1} \leq 2g \text{ and } 0 \leq g - \overline{f}_k \leq g - \overline{f}_{k+1} \leq 2g.$$

Our first argument will be based on the Lebesgue Monotone Convergence Theorem. The sequences $\{g + \underline{f}_k\}$ and $\{g - \overline{f}_k\}$ are nonnegative, monotone increasing, and have limits $g + f$ and $g - f$ respectively. Apply LMCT yields

$$\int_E g + \lim_{k\to\infty} \int_E \underline{f}_k = \int_E \lim_{k\to\infty} (g + \underline{f}_k) = \int_E (g + f) = \int_E g + \int_E f$$

and

$$\int_E g - \lim_{k\to\infty} \int_E \overline{f}_k = \int_E \lim_{k\to\infty} (g - \overline{f}_k) = \int_E (g - f) = \int_E g - \int_E f,$$

that is,

$$\lim_{k\to\infty}\int_E \underline{f}_k = \int_E f = \lim_{k\to\infty}\int_E \overline{f}_k,$$

and this argument is complete.

The next argument is an application of Fatou's Theorem.

"Fatou" for $g + f_k$:

$$\begin{aligned}\int_E g + \int_E f = \int_E (g+f) = \int_E \liminf_{k\to\infty}(g+f_k) &\le \liminf_{k\to\infty}\int_E (g+f_k)\\ &= \liminf_{k\to\infty}\left(\int_E g + \int_E f_k\right)\\ &= \int_E g + \liminf_{k\to\infty}\int_E f_k,\end{aligned}$$

and so $\int_E f \le \liminf_{k\to\infty}\int_E f_k$.

"Fatou" for $g - f_k$:

$$\begin{aligned}\int_E g - \int_E f = \int_E (g-f) = \int_E \liminf_{k\to\infty}(g-f_k) &\le \liminf_{k\to\infty}\int_E (g-f_k)\\ &= \liminf_{k\to\infty}\left(\int_E g - \int_E f_k\right)\\ &= \int_E g - \limsup_{k\to\infty}\int_E f_k,\end{aligned}$$

and so $\limsup_{k\to\infty}\int_E f_k \le \int_E f$. Combining, $\limsup_{k\to\infty}\int_E f_k \le \int_E f \liminf_{k\to\infty}\int_E f_k$.

Thus

$$\lim_{k\to\infty}\int_E f_k = \int_E f$$

and hence

$$\int_E \left(\lim_{k\to\infty} f_k\right) = \int_E f = \lim_{k\to\infty}\int_E f_k.$$

□

The next two examples illustrate some applications of the convergence theorems that we have discussed in Chapter 5. They partially answer "Why Lebesgue?"

Example 5.5.2 Show $\int_{[0,\infty)} e^{-x^2} = \dfrac{\sqrt{\pi}}{2}$.

Solution. Let

$$f_k(t) = \begin{cases} \left(1 - \dfrac{t^2}{k}\right)^k, & 0 \le t \le \sqrt{k} \\ 0, & t > \sqrt{k} \end{cases}$$

and

$$g_k(t) = f_k(t) \cdot \sqrt{1 - \frac{t^2}{k}}.$$

Recall $1 + \dfrac{t}{k} \le e^{\frac{t}{k}}$, where $-k \le t$, and $\left(1 + \dfrac{t}{k}\right)^k \le e^t$, $\lim\limits_{k\to\infty}\left(1 + \dfrac{t}{k}\right)^k = e^t$.

Then

1. $f_k(t) \le e^{-t^2}$, $t \ge 0$ and $\lim\limits_{k\to\infty} f_k(t) = e^{-t^2}$, $t \ge 0$.

2. $g_k(t) \le e^{-t^2}$, $t \ge 0$ and $\lim\limits_{k\to\infty} g_k(t) = e^{-t^2}$, $t \ge 0$.

3. Integration by part will be used.

$$\begin{aligned} \int_{[0,\infty)} f_k &= \sqrt{k}\int_0^{\frac{\pi}{2}} \cos^{2k+1}(t)dt \\ &= \sqrt{k} \cdot \frac{(2k)(2k-2)\cdots 2}{(2k+1)(2k-1)\cdots 3} \end{aligned}$$

and

$$\begin{aligned} \int_{[0,\infty)} g_k &= \sqrt{k}\int_0^{\frac{\pi}{2}} \cos^{2k+2}(t)dt \\ &= \sqrt{k} \cdot \frac{(2k+1)(2k-1)\cdots 1}{(2k+2)(2k)\cdots 2} \cdot \frac{\pi}{2}. \end{aligned}$$

4. Since f_k, g_k are dominated by the integrable function e^{-t^2}, $t \ge 0$, we have

$$\begin{aligned} \lim_{k\to\infty}\int_{[0,\infty)} f_k &= \int_{[0,k]} \lim_{k\to\infty} f_k \\ &= \int_{[0,\infty)} e^{-t^2} \\ &= \lim_{k\to\infty}\int_{[0,\infty)} g_k. \end{aligned}$$

5.

$$\begin{aligned} \left(\int_{[0,\infty)} e^{-t^2}\right)^2 &= \left(\lim_{k\to\infty}\int_{[0,\infty)} f_k\right)\left(\lim_{k\to\infty}\int_{[0,\infty)} g_k\right) \\ &= \lim_{k\to\infty}\left(\int_{[0,\infty)} f_k \cdot \int_{[0,\infty)} g_k\right) \\ &= \frac{\pi}{4}. \end{aligned}$$

6.

$$\int_{[0,\infty)} e^{-t^2} = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}.$$

□

Example 5.5.3 Evaluate $\int_{[0,\infty)} e^{-t^2}\cos(t)$.

Solution.

$$e^{-t^2}\cos(t) = \lim_{n\to\infty}\left[e^{-t^2}\sum_{k=0}^{n}\frac{(-1)^k}{(2k)!}t^{2k}\right].$$

Here,

$$f_n(t) = \begin{cases} e^{-t^2}\sum_{k=0}^{n}\frac{(-1)^k}{(2k)!}t^{2k}, & 0 \le t \le n \\ 0, & n < t \end{cases},$$

and

$$\lim_{n\to\infty} f_n(t) = e^{-t^2}\cos(t),\ 0 \le t < \infty,\ |f_n(t)| \le e^{t-t^2} = e^{-\left(t-\frac{1}{2}\right)^2+\frac{1}{4}} < e\cdot e^{-\left(t-\frac{1}{2}\right)^2}.$$

Recall

$$\int_0^\infty x^p e^{-x^q}\,dx = \frac{1}{q}\Gamma\left(\frac{p+q-1}{q}\right) \text{ when } q > 0 \text{ and } p+q > 1,$$

and

$$\Gamma\left(k+\frac{1}{2}\right) = \frac{1\cdot 3\cdots(2k-1)}{2^k}\cdot\sqrt{\pi} \text{ for all positive integers } k.$$

Apply LDCT and the above facts:

$$
\begin{aligned}
\int_{[0,\infty)} e^{-t^2}\cos(t) &= \lim_{n\to\infty}\int_{[0,\infty)} f_n(t) \\
&= \lim_{n\to\infty}\int_{[0,n]}\left[e^{-t^2}\sum_{k=0}^{n}\frac{(-1)^k}{(2k)!}t^{2k}\right] \\
&= \lim_{n\to\infty}\int_{[0,n]}\left[e^{-t^2}\left(1-\frac{1}{2!}t^2+\frac{1}{4!}t^4-\frac{1}{6!}t^6+\cdots+\frac{(-1)^n}{(2n)!}t^{2n}\right)\right] \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\left[\int_{[0,n]}\left(e^{-t^2}\frac{(-1)^k}{(2k)!}t^{2k}\right)\right] \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\left[\frac{(-1)^k}{(2k)!}\int_{[0,n]} t^{2k}e^{-t^2}\right] \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\left[\frac{(-1)^k}{(2k)!}\int_0^n t^{2k}e^{-t^2}\right] \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\left[\frac{(-1)^k}{(2k)!}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{2k+2-1}{2}\right)\right] \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\left[\frac{(-1)^k}{(2k)!}\cdot\frac{1}{2}\cdot\Gamma\left(k+\frac{1}{2}\right)\right] \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\left[\frac{(-1)^k}{(2k)!}\cdot\frac{1}{2}\cdot\frac{1\cdot 3\cdots(2k-1)}{2^k}\cdot\sqrt{\pi}\right] \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\left[\frac{(-1)^k}{(2k)!}\cdot\frac{1}{2}\cdot\frac{1\cdot 3\cdots(2k-1)}{2^k}\cdot\sqrt{\pi}\cdot\frac{2\cdot 4\cdot 6\cdots(2k)}{2\cdot 4\cdot 6\cdots(2k)}\right] \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\left[\frac{(-1)^k}{(2k)!}\cdot\frac{1}{2}\cdot\frac{(2k)!}{2^k}\cdot\sqrt{\pi}\cdot\frac{1}{2^k k!}\right] \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\left[\frac{(-1)^k}{2^{2k}k!}\cdot\frac{\sqrt{\pi}}{2}\right] \\
&= \lim_{n\to\infty}\left(1-\frac{1}{2^2\cdot 1!}+\frac{1}{2^4\cdot 2!}-\cdots+\frac{(-1)^n}{2^{2n}n!}\right)\cdot\frac{\sqrt{\pi}}{2} \\
&= e^{-\frac{1}{4}}\cdot\frac{\sqrt{\pi}}{2}.
\end{aligned}
$$

□

CHAPTER 6

CONCLUSION AND FUTURE WORK

Our target for this report is the Fundamental Theorem of Calculus for Lebesgue integral. I can only finished up to Lebesgue Dominated Convergence Theorem in Chapter 5 because of time constraint. Here, I only list down all the lemmas and theorems without proofs that lead to the Fundamental Theorem of Calculus for Lebesgue integral.

6.1 Conclusion

The notions of bounded variation and absolute continuity on an interval play a key role in the theory of the Lebesgue integral. Two intervals I and J are non-overlapping if $I \bigcap J$ consists of at most one point.

Definition 6.1.1 *The variation of F on $[a,b]$ is defined by*

$$V(F,[a,b]) = \sup\left\{\sum_{i=1}^{n} | F(d_i) - F(c_i) |\right\}$$

where the supremum is over all finite collections $\{[c_i, d_i] \mid 1 \le i \le n\}$ of non-overlapping intervals in $[a,b]$. The function F is of bounded variation on $[a,b]$ if $V(F,[a,b])$ is finite.

Definition 6.1.2 *The function F is absolutely continuous on $[a,b]$ if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\sum_{i=1}^{n} |F(d_i) - F(c_i)| < \varepsilon$ whenever $\{[c_i, d_i] \mid 1 \le i \le n\}$ is a finite collection of non-overlapping intervals in $[a,b]$ that satisfy $\sum_{i=1}^{n}(d_i - c_i) < \delta$.*

The next theorem shows that absolute continuity is stronger than bounded variation. Before that, a lemma is needed.

Lemma 6.1.3 *Let $F : [a,b] \to \mathbb{R}$.*

(a) If F is of bounded variation on $[a,b]$, then F is of bounded variation on every subinterval of $[a,b]$ and $V(F,[a,b]) = V(F,[a,c]) + V(F,[c,b])$ for each $c \in [a,b]$.

(b) If F is of bounded variation on $[a,c]$ and $[c,b]$, then F is of bounded variation on $[a,b]$.

Theorem 6.1.4 *If $F : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$, then F is of bounded variation on $[a,b]$.*

A monotone function is of bounded variation on $[a,b]$. Hence, the difference of two monotone functions is of bounded variation. The converse is also true; a function of bounded variation can be written as the difference of two monotone functions.

Theorem 6.1.5 *If $F : [a,b] \to \mathbb{R}$ is of bounded variation on $[a,b]$, then there exist nondecreasing functions F_1 and F_2 such that $F = F_1 - F_2$.*

To prove a monotone function is differentiable a.e. on an interval, we need Vitali Covering Lemma.

Definition 6.1.6 *Let $E \subseteq \mathbb{R}$. A collection $\mathcal{J}$ of intervals is a Vitali cover of E if for each $x \in E$ and $\varepsilon > 0$, there exists an interval $I \in \mathcal{J}$ such that $x \in I$ and $\mu(I) < \varepsilon$.*

Lemma 6.1.7 (Vitali Covering Lemma) *Let* $E \subseteq \mathbb{R}$ *with* $\mu^*(E) < \infty$. *If* $\mathcal{J}$ *is a Vitali cover of* E, *then for each* $\varepsilon > 0$ *there exists a finite collection* $\{I_k \mid 1 \le k \le n\}$ *of disjoint intervals in* $\mathcal{J}$ *such that* $\mu^*\left(E \cap \left(\bigcup_{k=1}^{n} I_k\right)^c\right) < \varepsilon$. *In addition, there exists a sequence* $\{I_k\}$ *of disjoint intervals in* $\mathcal{J}$ *such that* $\mu^*\left(E \cap \left(\bigcup_{k=1}^{\infty} I_k\right)^c\right) = 0.$

The next definition establishes the notation for various limits of difference quotients. These derivates are often more useful than the ordinary derivative since they are defined at each point.

Definition 6.1.8 *Let* $F : [a,b] \to \mathbb{R}$. *The upper right and lower right derivates of* F *at* $x \in [a,b)$ *are defined by*

$$D^+F(x) = \lim_{\delta \to 0^+} \sup \left\{ \frac{F(y)-F(x)}{y-x} \middle| x < y < x+\delta \right\};$$

$$D_+F(x) = \lim_{\delta \to 0^+} \inf \left\{ \frac{F(y)-F(x)}{y-x} \middle| x < y < x+\delta \right\}.$$

Similarly, the upper left and lower left derivates of F *at* $x \in (a,b]$ *are defined by*

$$D^-F(x) = \lim_{\delta \to 0^+} \sup \left\{ \frac{F(y)-F(x)}{y-x} \middle| x-\delta < y < x \right\};$$

$$D_-F(x) = \lim_{\delta \to 0^+} \inf \left\{ \frac{F(y)-F(x)}{y-x} \middle| x-\delta < y < x \right\}.$$

A lemma is required before we show a nondecreasing function is differentiable a.e. on $[a,b]$.

Lemma 6.1.9 *If* F *is nondecreasing on* $[a,b]$, *then all four derivates of* F *are finite a.e. on* $[a,b]$.

Theorem 6.1.10 *If* F *is nondecreasing on* $[a,b]$, *then* F *is differentiable a.e. on* $[a,b]$.

The next theorem shows that the function F' is in fact Lebesgue integrable on $[a,b]$ and gives an upper bound for the value of its integral. Consequently, the derivative of a function of bounded variation is Lebesgue integrable on $[a,b]$.

Theorem 6.1.11 *If F is nondecreasing on $[a,b]$, then the function F' is Lebesgue integrable on $[a,b]$ and $\int_a^b F' \le F(b) - F(a)$.*

Now, we consider the first part of the Fundamental Theorem of Calculus for Lebesgue integral.

Lemma 6.1.12 *Let $f:[a,b] \to \mathbb{R}$ be bounded and measurable. If $F(x) = \int_a^x f$ for each $x \in [a,b]$, then F is absolutely continuous on $[a,b]$ and $F' = f$ a.e. on $[a,b]$.*

Theorem 6.1.13 (Fundamental Theorem of Calculus for Lebesgue Integral Part 1) *Let $f:[a,b] \to \mathbb{R}$ be Lebesgue integrable on $[a,b]$. If $F(x) = \int_a^x f$ for each $x \in [a,b]$, then F is absolutely continuous on $[a,b]$ and $F' = f$ a.e. on $[a,b]$.*

A function F with the property that $F' = 0$ a.e. on $[a,b]$ is called a singular function. One way to guarantee that a singular function is constant is to insist that it be absolutely continuous as well. After proving the next theorem, it is easy to prove the second part of the Fundamental Theorem of Calculus for Lebesgue integral.

Theorem 6.1.14 *Suppose that $F:[a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$. If $F' = 0$ a.e. on $[a,b]$, then F is constant on $[a,b]$.*

Theorem 6.1.15 (Fundamental Theorem of Calculus for Lebesgue Integral Part 2) *If* $F:[a,b]\to\mathbb{R}$ *is absolutely continuous on* $[a,b]$, *then* F' *is Lebesgue integrable on* $[a,b]$ *and* $\int_a^x F' = F(x)-F(a)$ *for each* $x\in[a,b]$.

Theorem 6.1.13 and 6.1.15 together yield the following theorem. This statement is usually referred to as the descriptive definition of the Lebesgue integral.

Theorem 6.1.16 *A function* $f:[a,b]\to\mathbb{R}$ *is Lebesgue integrable on* $[a,b]$ *iff there exists an absolutely continuous function* $F:[a,b]\to\mathbb{R}$ *such that* $F' = f$ *a.e. on* $[a,b]$.

Is it possible to define an integration process for which the theorem

If F ***is differentiable on*** $[a,b]$, ***then the function*** F' ***is integrable on*** $[a,b]$ ***and*** $\int_a^x F' = F(x)-F(a)$ ***for each*** $x\in[a,b]$.

is valid?

6.2 Future Work

In 20th century, three integration processes have been developed for which this version of the Fundamental Theorem of Calculus is valid. These integrals, named after their principal investigators Denjoy, Perron, and Henstock, each generalize some aspect of the Lebesgue integral. Since each of these new integrals focuses on a different property of the Lebesgue integral, the definitions of the integrals are radically different. However, it turns out that all three integrals are equivalent.

Here, we only introduce the Perron integral. In 1914, O. Perron developed another extension of the Lebesgue integral and proved that his integral also had the property that every derivative is integrable. His work was independent of Denjoy and hence has a very defferent flavor.

The first step is to introduce the notion of major and minor functions. These functions are defined using the upper and lower derivates that were first discussed in Section 6.1.

Definition 6.2.1 *Let* $f:[a,b]\to \mathbb{R}^e$.

(a) A function $U:[a,b]\to\mathbb{R}$ *is a major function of* f *on* $[a,b]$ *if* $\underline{D}U(x)>-\infty$ *and* $\underline{D}U(x)\ge f(x)$ *for all* $x\in[a,b]$.

(b) A function $V:[a,b]\to\mathbb{R}$ *is a minor function of* f *on* $[a,b]$ *if* $\overline{D}V(x)<+\infty$ *and* $\overline{D}V(x)\le f(x)$ *for all* $x\in[a,b]$.

We write U_a^b for $U(b)-U(a)$.

Theorem 6.2.2 *A measurable function* $f:[a,b]\to\mathbb{R}^e$ *is Lebesgue integrable on* $[a,b]$ *iff for each* $\varepsilon>0$, *there exist absolutely continuous major and minor functions* U *and* V *of* f *on* $[a,b]$ *such that* $U_a^b - V_a^b<\varepsilon$.

The Perron integral is defined in terms of major and minor functions. The generalization of the Lebesgue integral occurs by dropping the requirement that the major and minor functions be absolutely continuous. Recall that F is differentiable at c if and only if $\underline{D}F(c)$ and $\overline{D}F(c)$ are finite and equal.

Proposition 6.2.3 *Let* U *and* V *be functions defined on* $[a,b]$ *and let* $c\in[a,b]$. *Then*

(a) $\underline{D}U(c)\le\overline{D}U(c)$;

(b) $\underline{D}(-U)(c)=-\overline{D}U(c)$;

(c) $\overline{D}(U+V)(c)\le\overline{D}U(c)+\overline{D}V(c)$;

(d) $\underline{D}(U+V)(c)\ge\underline{D}U(c)+\underline{D}V(c)$;

(e) If $\underline{D}U(c)>-\infty$ *and* $\overline{D}V(c)<+\infty$, *then* $\underline{D}(U-V)(c)\ge\underline{D}U(c)-\overline{D}V(c)$.

Theorem 6.2.4 *Let* $F:[a,b]\to \mathbb{R}$. *If* $\underline{D}F\geq 0$ *on* $[a,b]$, *then* F *is nondecreasing on* $[a,b]$.

The quantities in the next definition are finite-valued.

Definition 6.2.5 *A function* $f:[a,b]\to \mathbb{R}^e$ *is Perron integrable on* $[a,b]$ *iff* f *has at least one major and one minor function on* $[a,b]$ *and the numbers*

$$\inf\{U_a^b \mid U \text{ is a major function of } f \text{ on } [a,b]\},$$

$$\sup\{V_a^b \mid V \text{ is a minor function of } f \text{ on } [a,b]\}$$

are equal. This common value is the Perron integral of f *on* $[a,b]$ *and denoted by* $\int_a^b f$.

The following theorem is an immediate consequence of the definition. In particular, a Lebesgue integrable function is Perron integrable and the integrals are equal.

Theorem 6.2.6 *A function* $f:[a,b]\to \mathbb{R}^e$ *is Perron integrable on* $[a,b]$ *iff for each* $\varepsilon>0$, *there exist a major function* U *and a minor function* V *of* f *on* $[a,b]$ *such that* $U_a^b - V_a^b < \varepsilon$.

F is differentiable on $[a,b]$ in the next theorem can be replaced by F is differentiable nearly everywhere on $[a,b]$ in Theorem 6.2.26.

Theorem 6.2.7 *Let* $F:[a,b]\to \mathbb{R}$ *be continuous on* $[a,b]$. *If* F *is differentiable on* $[a,b]$, *then* F' *is Perron integrable on* $[a,b]$ *and* $\int_a^x F' = F(x)-F(a)$ *for each* $x\in[a,b]$.

Next consider the relationship between Perron integrability and subintervals.

Theorem 6.2.8 *Let* $f:[a,b]\to \mathbb{R}^e$ *and let* $c\in(a,b)$.

(a) If f is Perron integrable on $[a,b]$, then f is Perron integrable on every subinterval of $[a,b]$.

(b) If f is Perron integrable on each of the intervals $[a,c]$ and $[c,b]$, then f is Perron integrable on $[a,b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$.

It is sufficient to consider finite-valued functions only. This will avoid difficulties when it comes to adding functions as $f+g$ may not be defined if both f and g assume infinite values.

Theorem 6.2.9 *If $f:[a,b] \to \mathbb{R}^e$ is Perron integrable on $[a,b]$, then f is finite-valued a.e. on $[a,b]$.*

Theorem 6.2.10 *Let $f:[a,b] \to \mathbb{R}^e$ be Perron integrable on $[a,b]$. If $g = f$ a.e. on $[a,b]$, then g is Perron integrable on $[a,b]$ and $\int_a^b g = \int_a^b f$.*

The next proposition lists the linearity properties of the Perron integral.

Proposition 6.2.11 *Suppose that f and g are Perron integrable on $[a,b]$. Then*

(a) kf is Perron integrable on $[a,b]$ and $\int_a^b kf = k\int_a^b f$ for each $k \in \mathbb{R}$;

(b) $f+g$ is Perron integrable on $[a,b]$ and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$;

(c) if $f \le g$ a.e. on $[a,b]$, then $\int_a^b f \le \int_a^b g$;

(d) if $f = g$ a.e. on $[a,b]$, then $\int_a^b f = \int_a^b g$.

We next consider the properties of the indefinite Perron integral.

Lemma 6.2.12 *Let* $f:[a,b]\to \mathbb{R}$ *be Perron integrable on* $[a,b]$ *and let* $F(x)=\int_a^x f$ *for each* $x\in[a,b]$. *If* U *is a major function and* V *a minor function of* f *on* $[a,b]$, *then the functions* $U-F$ *and* $F-V$ *are nondecreasing on* $[a,b]$.

Theorem 6.2.13 *Let* $f:[a,b]\to \mathbb{R}$ *be Perron integrable on* $[a,b]$ *and let* $F(x)=\int_a^x f$ *for each* $x\in[a,b]$. *Then*

(a) F *is continuous on* $[a,b]$;

(b) F *is differentiable a.e. on* $[a,b]$ *and* $F' = f$ *a.e. on* $[a,b]$;

(c) f *is measurable on* $[a,b]$.

Theorem 6.2.14 *Let* $f:[a,b]\to \mathbb{R}$ *be Perron integrable on* $[a,b]$.

(a) If f *is bounded on* $[a,b]$, *then* f *is Lebesgue integrable on* $[a,b]$.

(b) If f *is nonnegative on* $[a,b]$, *then* f *is Lebesgue integrable on* $[a,b]$.

(c) If f *is Perron integrable on every measurable subset of* $[a,b]$, *then* f *is Lebesgue integrable on* $[a,b]$.

Theorem 6.2.15 *Let* $f:[a,b]\to \mathbb{R}$ *be Perron integrable on* $[a,b]$. *If* E *is a perfect set in* $[a,b]$, *then there exists a perfect portion* $E\bigcap[c,d]$ *of* E *such that* f *is Lebesgue integrable on* $E\bigcap[c,d]$. *Moreover, the series* $\sum_{k=1}^{\infty}\omega\left(\int_{c_k}^{x} f,[c_k,d_k]\right)$ *converges where* $[c,d]-E=\bigcup_{k=1}^{\infty}(c_k,d_k)$.

Definition 6.2.16 *A function* $f:[a,b]\to \mathbb{R}^e$ *is* P_c *integrable on* $[a,b]$ *if* f *has at least one continuous major and one continuous minor function on* $[a,b]$ *and the numbers*

$$\inf\{U_a^b \mid U \text{ is a continuous major function of } f \text{ on } [a,b]\},$$

$$\sup\{V_a^b \mid V \text{ is a continuous minor function of } f \text{ on } [a,b]\}$$

are equal. This common value is the P_c *integral of* f *on* $[a,b]$.

The function f is P_c integrable on a measurable set $E \subseteq [a,b]$ if $f\chi_E$ is P_c integrable on $[a,b]$. The symbol P_c stands for Perron continuous, that is, Perron integrable with continuous major and minor functions.

Lemma 6.2.17 *Let W be a continuous function on $[a,b]$. Then for each $\varepsilon > 0$, there exists a continuous function U on $[a,b]$ such that $\underline{D}U \geq \underline{D}W$ on $[a,b]$, $\underline{D}U(b) = +\infty$, and $U_a^b < W_a^b + \varepsilon$.*

Theorem 6.2.18 *Suppose that $f : [a,b] \to \mathbb{R}$ is P_c integrable on each interval $[c,d] \subseteq (a,b)$. If $\int_c^d f$ converges to a finite limit as $c \to a^+$ and $d \to b^-$, then f is P_c integrable on $[a,b]$ and $\int_a^b f = \lim\limits_{\substack{c \to a^+ \\ d \to b^-}} \int_c^d f$.*

Theorem 6.2.19 *Let E be a bounded, closed set with bounds a and b and let $\{(a_k, b_k)\}$ be the sequence of intervals contiguous to E in $[a,b]$. Suppose that $f : [a,b] \to \mathbb{R}$ is P_c integrable on E and on each interval $[a_k, b_k]$. If the series $\sum_{k=1}^{\infty} \omega\left(\int_{a_k}^{x} f, [a_k, b_k]\right)$ converges, then f is P_c integrable on $[a,b]$ and*

$$\int_a^b f = \int_a^b f\chi_E + \sum_{k=1}^{\infty} \int_{a_k}^{b_k} f.$$

The next theorem is due to Marcinkiewicz. Theorem 6.2.2 states that a measurable function is Lebesgue integrable if it has one absolutely continuous major function and one absolutely continuous minor function. A similar result holds to the Perron integral with absolute continuity replaced by continuity.

Theorem 6.2.20 *Let $f : [a,b] \to \mathbb{R}$ be measurable. If f has at least one continuous major function and at least one continuous minor function on $[a,b]$, then f is Perron integrable on $[a,b]$.*

Now, we look at one other change in the definition of major and minor function. This change involves the derivate inequalities being satisfied at "most points" rather than at all points.

Definition 6.2.21 *Let* $f:[a,b]\to \mathbb{R}^e$.

(a) A continuous function $U:[a,b]\to\mathbb{R}$ *is an ex-major function of* f *on* $[a,b]$ *if* $\underline{D}U(x)>-\infty$ *nearly everywhere on* $[a,b]$ *and* $\underline{D}U(x)\geq f(x)$ *a.e. on* $[a,b]$.

(a) A continuous function $V:[a,b]\to\mathbb{R}$ *is an ex-minor function of* f *on* $[a,b]$ *if* $\overline{D}V(x)<+\infty$ *nearly everywhere on* $[a,b]$ *and* $\overline{D}V(x)\leq f(x)$ *a.e. on* $[a,b]$.

The ***ex*** represents extended. We can define P_x integral using the extended major and minor functions. Several lemmas are required to show every P_c integrable function is P_x integrable and its converse is also true.

Lemma 6.2.22 *Let* $W:[a,b]\to\mathbb{R}$ *be continuous on* $[a,b]$, *let* $c\in[a,b]$, *and let* $\varepsilon>0$. *Then there exist a nondecreasing, continuous function* $\psi:[a,b]\to\mathbb{R}$ *and a positive number* δ *such that* $\psi(a)=0, \psi(b)\leq\varepsilon$, *and*

$$\frac{W(x)-W(c)+\psi(x)-\psi(c)}{x-c}\geq 0$$

for all $x\in[a,b]$ *that satisfy* $0<|x-c|<\delta$.

Lemma 6.2.23 *Let* $W:[a,b]\to\mathbb{R}$ *be continuous, let* $\varepsilon>0$, *and suppose that* $\underline{D}W>-\infty$ *nearly everywhere on* $[a,b]$. *Then there exists a continuous function* $Y:[a,b]\to\mathbb{R}$ *such that* $\underline{D}Y\geq\underline{D}W$ *and* $\underline{D}Y>-\infty$ *on* $[a,b]$ *and* $Y_a^b\leq W_a^b+\varepsilon$.

Lemma 6.2.24 *Let* $f:[a,b]\to\mathbb{R}^e$. *If* W *is an ex-major function of* f *on* $[a,b]$ *and* $\varepsilon>0$, *there exists a continuous major function* U *of* f *on* $[a,b]$ *such that* $U_a^b<W_a^b+\varepsilon$.

Theorem 6.2.25 *A function* $f:[a,b]\to \mathbb{R}^e$ *is* P_c *integrable on* $[a,b]$ *iff* f *is* P_x *integrable on* $[a,b]$.

Theorem 6.2.26 *Let* $F:[a,b]\to \mathbb{R}$ *be continuous on* $[a,b]$. *If* F *is differentiable nearly everywhere on* $[a,b]$, *then* F' *is Perron integrable on* $[a,b]$ *and* $\int_a^x F' = F(x)-F(a)$ *for each* $x\in[a,b]$.

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