A STUDY ON GRAPHS OF RINGS

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A STUDY ON GRAPHS OF RINGS

By

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requirements for the award of Bachelor of Science (Hons.)
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DECLARATION OF ORIGINALITY

I hereby declare that this project report entitled **"A study on graphs of rings"** is my own work except for citations and quotations which have been duly acknowledged. I also declare that it has not been previously and concurrently submitted for any other degree or award at UTAR or other institutions.

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APPROVAL FOR SUBMISSION

I certify that this project report entitled **"A study on graphs of rings"** was prepared by **LAU ZHOU SHENG** has met the required standard for submission in partial fulfilment of the requirements for the award of Bachelor of Science (Hons.) Applied Mathematics With Computing at Universiti Tunku Abdul Rahman.

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ABSTRACT

In this project, we are studied about the connection between ring and graph theory. This project will involve some knowledge on ring and graph theories. We said a ring R satisfied the properties which are abelian group under addition and closed under multiplication operation. On the other hand, graph theory is a study of the graph which made up of vertices, edges and its properties. The relationship between commutative ring and graph theory were firstly introduced by Beck in 1988. Later, N.Ashrafi(2010) has carried a research on unit graphs associated with rings. Let x and y be arbitrary vertices in R, such that x and y are adjacent if and only if $x+y$ is a unit in R. Besides, an element x is said to be clean if there exists an idempotent$e \in R$ such that $x-e$ is a unit in R. Clean rings is firstly defined by Nicholson in 1977. In 2013, Diesel(2013) has introduced nil clean rings and strongly nil clean rings. A ring R is called nil clean ring if for each $x \in R$ such that $x = n + e$, for some idempotent$e \in R$ and nilpotent $n \in R$. Further later, Danchev(2015) has introduced weakly nil clean ring.If $r \in R$ and there exists an idempotent $e \in R$ and nilpotent$n \in R$ such that $r = n \pm e$. In 2017, Basnet(2017) conducted a research on nil clean ring with graph. He denoted nil clean graph of ring R as $G_N(R)$. Let x and y to be distinct vertices of elements from nil clean ring R, such that x adjacent to y if and only if $x + y$ is a nil clean element in R. The $g(x)$-nil clean is firstly introduced by L.Fan(2008). An element $r \in R$ is called $g(x)$-nil clean if $r = n + s$ for some nilpotent $n \in R$ and $s \in R$ such that $g(s) = 0$, where $g(x) \in Z(R)[x]$. We then conduct a research on specifically on $g(x) = x^2 - 2x$, which is $x(x-2)$-nil clean graph of ring. Let R be$g(x)$-nil clean ring where $g(x) = x(x-2)$ and p and q to be distinct vertices of elements from R, such that p is adjacent to q if and only if $p+q \in R$ for some nilpotent$p$ and $g(q) = 0$. In this project, we generalized the properties of $g(x)$-nil clean graph such as connectedness, completeness, cycle, path and diameter and the adjacency matrix.

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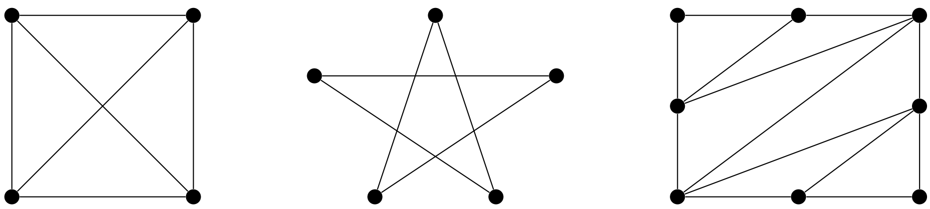
CHAPTER 1: INTRODUCTION

A Ring R is a set with 2 operations, addition and multiplication, $(R, +, \cdot)$. Besides that, R is also satisfied 2 important properties which is abelian group under addition and closed under multiplicative. We note that an abelian group closed under addition, $(R, +)$, indicates that the ring R is closed under addition and every element has an additive inverse. Furthermore, a ring is closed under multiplicative property. Moreover, in the multiplicative property it fulfils the properties of multiplication are associative and distributive. All the rings we considered are commutative with identity.

A graph is defined as to be a ordered pair of (V, E), where V is the finite set of vertices or points of the graph and E is set of unordered pair of elements of V that called edges or lines. For example, let $V = \{1, 2, 3, 4\}$, if it exists an edges or a line between vertices 1 and 3, then E will be $\{(1, 3)\}$ or $\{(3, 1)\}$, and will be written as $E = \{(1, 3)\}$ or $E = \{(3, 1)\}$. To avoid ambiguity, this type of graph can be called undirected graph. For illustration, we consider the graph below.



In this project, we focus on the study of the relationship between commutative nil clean rings and its graph properties (in Basnet(2017)). We note that girth, diameter, chromatic index and other related graph properties will be the parts of this research studies in the project that related to graph theory.

1-1 Objectives

In project I, we will be learning some basic knowledges on ring theory and graph theory. Moreover, the main task of this project is to learn and understand the proving methods in the paper by Basnet(2017) on nil clean graph of rings.

In project II, we will be applying the knowledge and methods of proving from the paper by Basnet(2017) on nil clean graph of rings into other type of ring. In this project, we will be applying the knowledge on $x(x-2)$-nil clean graph of rings with some extension of other knowledges by using the appropriate methods from Basnet(2017) on nil clean graph of rings.

1-2 Project Scopes

In this project I, we will focus on the property of the nil clean graph of rings and its relationship with the graph properties. For example, girth of graphs, chromatic index of graphs and diameter of graphs which related to the graph theory.

In project II, we will focus on the properties of other type of rings, specifically $x(x-2)$-nil clean graph of rings, and its relationship with the graph properties. In this project, we have investigate about the properties of graph which related to graph theory such as connectedness of graphs, completeness of graphs, paths and cycles of graphs and diameter of graphs with the extension of properties of adjacency matrix of the $x(x-2)$-nil clean graph of rings.

1-3 Planning

The following Table 1 and Table 2 show the action plan for project I and project II. The highlighted part represented the tasks that carried out during the particular week. The main focus in Project I is reading and collecting the research materials.

Besides, Project I provides a good opportunity in learning the proving methods and skill of writing from published research paper, in this project we will be referencing from the paper published by Basnet(2017) on nil clean graph of rings, that will be helpful in our Project II.

Furthermore, in project II, we continue our research on a different structured rings with the application of the proving methods that we have learn from the Basnet(2017) on nil clean graph of rings. With the help of the existing theorems and lemmas in the Bastnet(2017) on nil clean graph of rings, we are able to have some extensions of our own theorems and lemmas in the continuation of this project.

Task	Week													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Registration for Project	■													
Submission of biweekly report			■		■		■		■		■		■	
Reading and collecting research materials		■	■	■	■	■	■		■	■				
Study on literature review					■	■	■	■						
Mock presentation for proposal							■							
Submission of proposal							■							
Analyse research findings								■	■	■	■			
Mock presentation of interim report											■			
Submission of interim report												■		
Oral presentation of Project I													■	

Table 1: Plan for Project I

Task	Week 1	2	3	4	5	6	7	8	9	10	11	12	13	14
Continuation of Project I	■	■	■	■	■	■	■	■	■					
Collecting and reading research materials	■	■	■	■	■	■	■	■	■					
Submission of Mid-Semester Monitoring form							■							
Preparation of project poster									■	■				
Submission of project poster											■			
Submission of final project												■		
Oral presentation of Project II													■	

Table 2: Plan for Project II

Chapter 2: Literature Review

Let $G(R)$ be an undirected graph, let $V(G)$ be the set of vertices and let $E(G)$ be the set of edges. If x and y in $V(G)$, and represent elements in R and edges between the points will have an edge if an only if $xy = 0$. The relationships between a commutative ring and graph theory are first introduced by Beck (in Beck(1988)). In this paper, Beck has presented the idea of coloring of a commutative rings.

Later, N.Ashrafi(2010) has carried a research on unit graphs associated with rings. Let $G(R)$ denotes an unit graph with a set of vertices comes from the set elements of R. Let x and y be arbitrary distinct vertices from R, such that x and y are adjacent if and only if $x + y$ is a unit of R. In addition, N.Ashrafi(2010) also investigated other properties of graph such as connectedness, chromatic index, diameter, girth and planarity of $G(R)$.

Let the sets of idempotents and nilpotents of R to be denoted by $Idem(R)$ and $Nil(R)$, respectively. Nicholson(1977) has defined that an element x in a ring R is said to be clean if there exist an $e \in Idem(R)$ such that $x - e$ is a unit of R. Later in 2013, Diesel(2013) introduced a new variants, nil clean rings and strongly nil clean rings. A ring R is called nil clean ring if for each $x \in R$ such that $x = n + e$, for some $n \in Nil(R)$ and $e \in Idem(R)$.

Further later in 2015, Danchev(2015) generalized the notion of nil clean ring into weakly nil clean ring. An element $r \in R$ is called weakly nil clean if there exists an $e \in Idem(R)$ and $n \in Nil(R)$ such that $r = n + e$ or $r = n - e$.

Furthermore, in 2017, Basnet(2017) did a research on a relationship between nil clean ring and graph. In the paper, Basnet(2017) denoted a nil clean graph by $G_N(R)$, let the set of nil clean elements denote by $N(R)$. He further investigates the properties of graph of $G_N(R)$, such as girth, diameter, dominating sets and other related properties. Let x and y to be distinct vertices of the elements from nil clean ring R, such that x adjacent to y if and only if $x + y \in N(R)$. In the same year, Basnet(2017) carried out another research on the relationship between weakly nil clean ring and graph. The weakly nil clean graph denoted by $G_{WN}(R)$ and let a set of weakly nil clean elements denote by $WN(R)$. If x and y to be the distinct vertices of the elements from the weakly nil clean ring R such that x adjacent to y if and only if $x + y \in WN(R)$ or $x - y \in WN(R)$.

On the other hand, there are some notations and definition used in this project. Let G denote the graph, for any $x \in V(G)$, the degree of x denotes by $deg(x)$ which defined as the number of edges that connected to x. Besides, the neighbours set of x is denoted as $N_G(x) := \{y \in V(G) | y$ is adjacent to $x\}$ and the set $N_G[x] = N_G(x) \cup \{x\}$.

Next, a complete graph is a simple undirected graph which have no loops and every distinct vertices are connected by a unique edges. For illustration, we consider the graphs below.

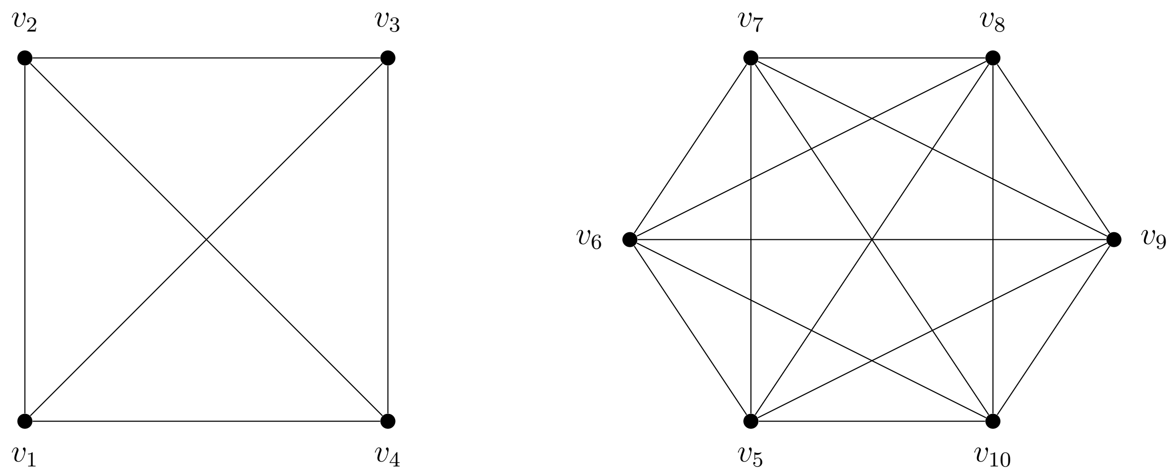


Figure 1: Graph with 4 and 6 vertices with unique edges for every vertices

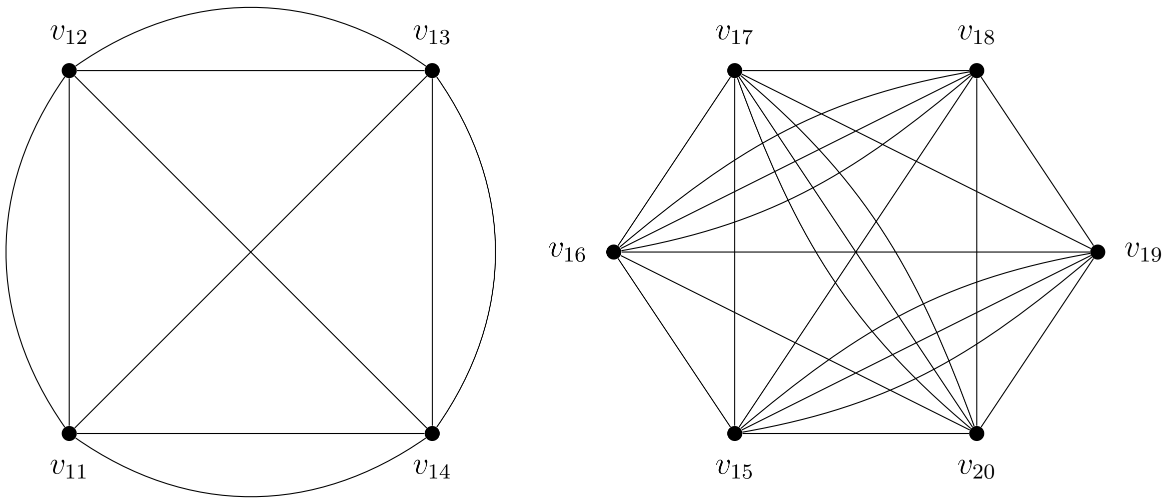


Figure 2: Graph with 4 and 6 vertices with multiple edges for every vertices

From the graphs above, we can see that every vertices in Figure 1 have an unique edges connected to it, for example, only have one edge between v_2 and v_3. So, Figure 1 is a complete graph. However, Figure 2 is not a complete graph because there exists at least one vertices that have at least one edges connected to it, for example, there are multiple edges connected between v_6 and v_7.

A nil clean graph of a ring R, denoted by $G_N(R)$, is defined by setting R as vertex set and 2 distinct vertices x and y are adjacent if $x + y$ is a nil clean element in R. Moreover, loops not considered. For illustration, we consider $GF(25)$ and $GF(27)$ which is a finite field with 25 and 27 elements.

$$\begin{aligned} GF(25) &\cong \mathbb{Z}_5[x]/\langle x^2 + x + 1\rangle \\ &= \{ax + b + \langle x^2 + x + 1\rangle : a, b \in \mathbb{Z}_5\} \end{aligned}$$

Let $\beta = x + \langle x^2 + x + 1\rangle$. Then $GF(25) = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \beta, 2\beta, 3\beta, 4\beta, 1+\beta, 2+\beta, 3+\beta, 4+\beta, 1+2\beta, 2+2\beta, 3+2\beta, 4+2\beta, 1+3\beta, 2+3\beta, 3+3\beta, 4+3\beta, 1+4\beta, 2+4\beta, 3+4\beta, 4+4\beta\}$

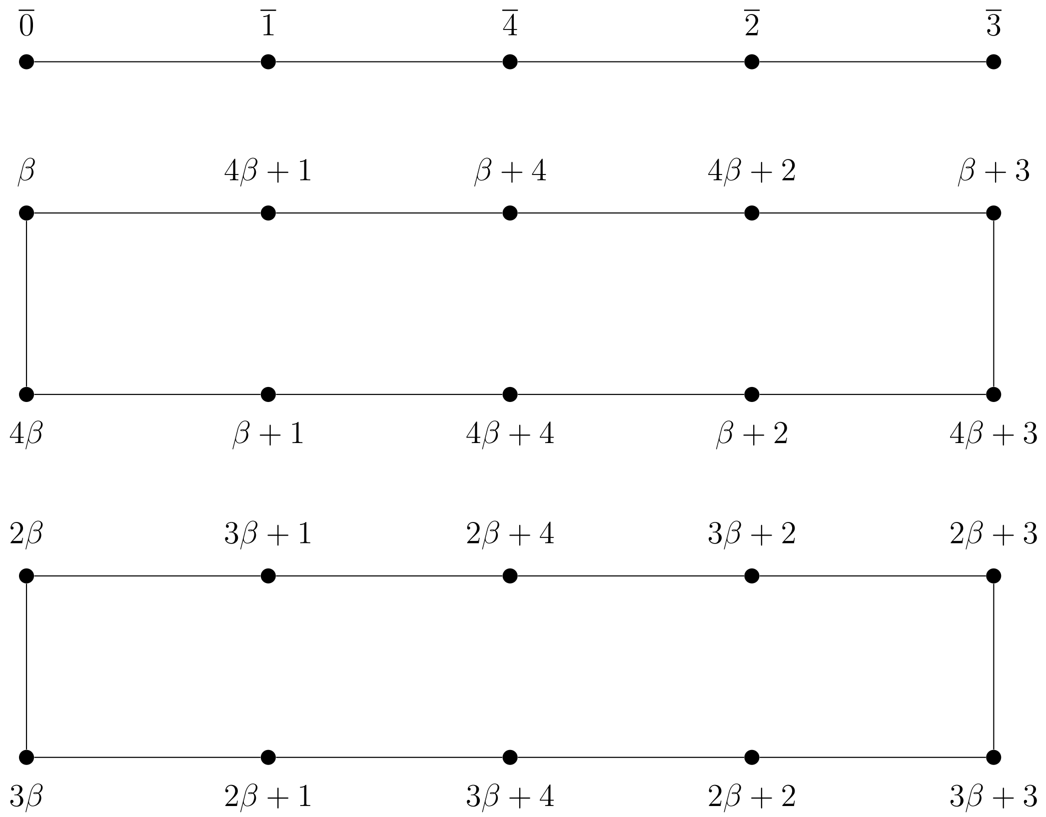


Figure 3: Nil clean graph of $GF(25)$

$$\begin{aligned} GF(27) &\cong \mathbb{Z}_3[x]/\langle x^3+2x^2+1\rangle \\ &= \{ax^2+bx+c+\langle x^3+2x^2+1\rangle : a,b,c \in \mathbb{Z}_3\} \end{aligned}$$

Let $\gamma = x^2+\langle x^3+2x^2+1\rangle$ and $\delta = x+\langle x^3+2x^2+1\rangle$. Then $GF(27) = \{\overline{0}, \overline{1}, \overline{2}, \delta, \delta+1, \delta+2, 2\delta, 2\delta+1, 2\delta+2, \gamma, \gamma+1, \gamma+2, \gamma+\delta, \gamma+\delta+1, \gamma+\delta+2, \gamma+2\delta, \gamma+2\delta+1, \gamma+2\delta+2, 2\gamma, 2\gamma+1, 2\gamma+2, 2\gamma+\delta, 2\gamma+\delta+1, 2\gamma+\delta+2, 2\gamma+2\delta, 2\gamma+2\delta+1, 2\gamma+2\delta+2\}$

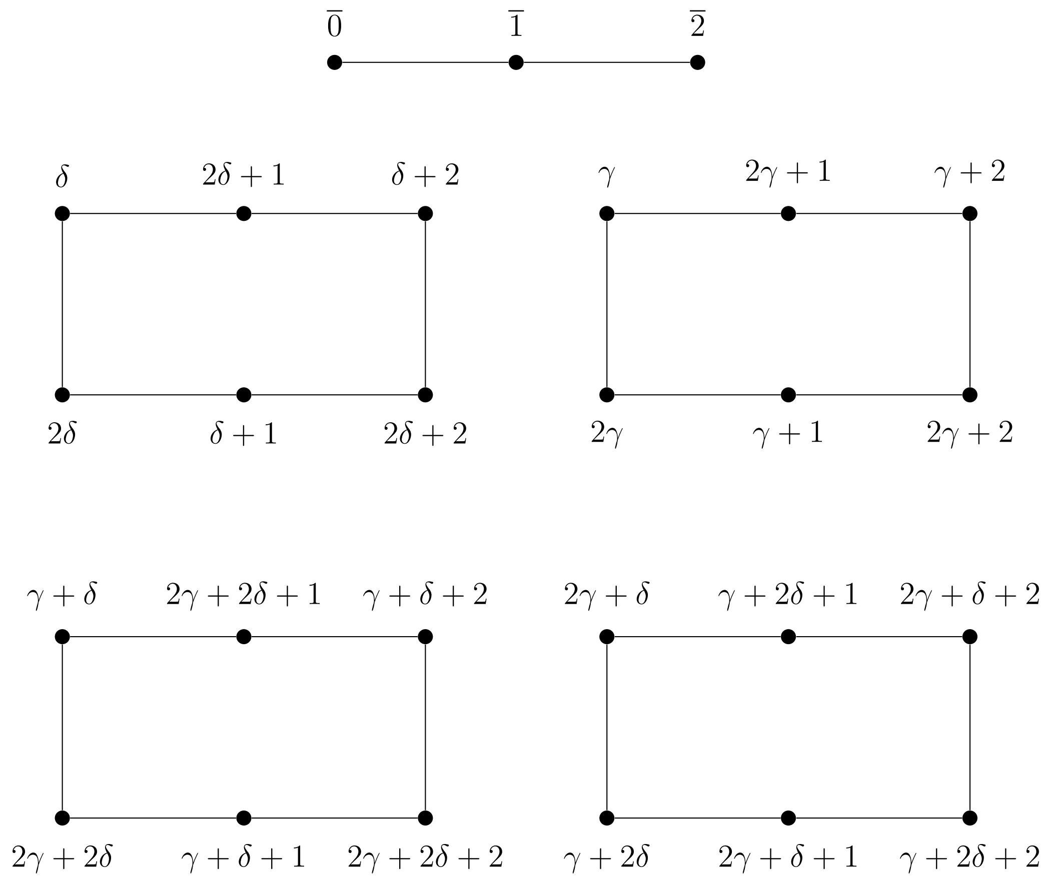


Figure 4: Nil clean graph of $GF(27)$

A girth is the shortest cycle that can be found in the graph. For illustration, we consider the Nil clean graph of $GF(25)$ from Figure 3 and $GF(27)$ from from Figure 4. From the Nil clean graph of $GF(25)$, it has the shortest cycle of 10 cycles. On the other hand, Nil clean graph of $GF(27)$ has the shortest cycle of 6 cycles. Therefore, the girth of $GF(25)$ is 10 and the girth of $GF(27)$ is 6.

Chromatic index of $G_N(R)$, $\chi'(G_N(R))$, is the minimum number of colours needed for $E(G_N(R))$ such that if $e, f \in E(G_N(R))$ and e and f are adjacent, then colour of e

will not same as the colour of f. Let Δ be the maximum vertex degree of $G_N(R)$, then Vizings theorem says that $\Delta \leq \chi'(G_N(R)) \leq \Delta + 1$; graphs that satisfied $\chi'(G) = \Delta$ are called graphs of class 1, those that satisfied $\chi'(G) = \Delta+1$ are called graph of class 2. For illustration, we consider part of the nil clean graph of $GF(25)$ with multiple edges connected between vertices.

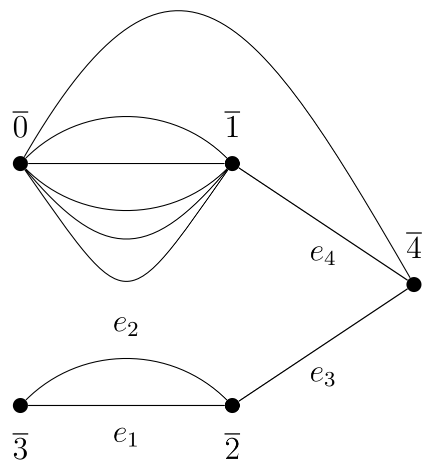


Figure 5: Part of Nil clean graph of $GF(25)$

From the figure above, we notice that vertex $\overline{2}$ have 3 edges connected to it, e_1, e_2 and e_3, this means the 3 edges are adjacent to each other. So, edges e_1, e_2 and e_3 will not have the same colour. However, edges e_1 and e_4 can have the same colour because they are not adjacent to each other. Next, according to Vizings Theorem, the maximum vertex degree will be vertices $\overline{0}$ and $\overline{1}$ which have a vertex degree of 6. So, $\Delta = 6$, and the graph satisfies $6 \leq \chi'(G) \leq 7$.

The diameter of the $G_N(R)$ is the shortest path between 2 vertices and if $x, y \in V(G_N(R))$ and the shortest path between x and y is denoted by $d(x, y)$. If there is no path between x and y is said that $d(x, y) = \infty$. The $diam(G_N(R))$ indicates the maximum of distances of each pair of distinct vertices in $G_N(R)$.

CHAPTER 3: PRELIMINARY RESULTS

3-1 Methodology

Preliminary methods will involve reading and understanding of various concepts in ring theory. This will be followed by reading of the research article Basnet (2017) and understanding of techniques used by others. The main work on research problems, which will form the contain of this project, will involve analytical thinking. Besides, in the construction of graphs of rings, we will be using MATLAB as our primary tool.

3-2 Some properties of nil-clean graphs

In the following, we study and investigate on the properties of nil-clean graphs.

Theorem 3.1 *The nil clean graph $G_N(R)$ is a complete graph if and only if R is a nil clean ring.*

Proof: ($\Rightarrow$): Let $G_N(R)$ be a complete nil clean graph of ring R. Then it implies that for all $r \in R$ are **nil clean elements**. Without lose of generality of **nil clean elements**, there exists one path from r is connected to 0 such that $r = r + 0$ which is nil clean, hence R is nil clean.

($\Leftarrow$): Conversely, let R be a nil clean ring. To form a graph from the R, let the arbitrary elements $x, y \in R$ and x and y are connected if and only if $x+y$ is a **nil clean element**. So, this implies that every pairs of distinct element $r \in R$ must have a unique edges and it form the complete nil clean graph $G_N(R)$.

Lemma 3.1 *Let $G_N(R)$ be the nil clean graph of a ring R. For $x \in R$ we have the following:*

(I) If $2x$ is nil clean, then $deg(x) = |NC(R)| - 1$.

(II) If $2x$ is not nil clean, then $deg(x) = |NC(R)|$.

Proof: Let $x \in R$, but clearly $x + R = R$. Then for every $y \in NC(R)$, there exists a unique element $x_y \in R$ such that $x + x_y = y$. Thus, we have $deg(x) \leq |NC(R)|$

For *(I)*: Now, we let $\{x_1, x_2, x_3, ..., x\} \subseteq R$ and $\{x_1 + x, x_2 + x, x_3 + x, ..., 2x\} \subseteq NC(R)$. Since we are not considering any loops, we illustrate the graph in the following:

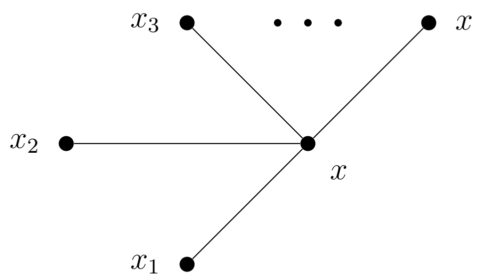


Figure 6: Nil clean graph with x elements including x

By the definition of $N_{G_N(R)}(x) = \{y \in V(G_N(R))|y$ is adjacent to $x\}$, we know that $y = \{1, 2, 3, ..., x\}$. We have $deg(x) = |N_{G_N(R)}(x)| = |N_{G_N(R)}[x]| - 1 = |NC(R)| - 1$

For *(II)*: Now, we let $\{x_1, x_2, x_3, ..., x_y, x\} \subseteq R$ and $\{x_1+x, x_2+x, x_3+x, ..., x+x_y\} \subseteq NC(R)$ but $2x \notin NC(R)$. Since we are not considering any loops, we have

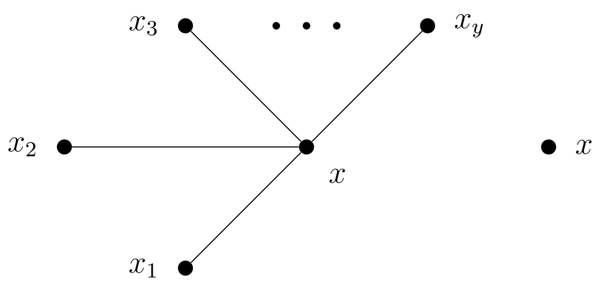


Figure 7: Nil clean graph with x elements excluding x

From the definition of $N_{G_N(R)}(x) = \{y \in V(G_N(R))|y$ is adjacent to $x\}$, we know that $y = \{x_1, x_2, x_3, ..., x_y\}$. We have $deg(x) = |N_{G_N(R)}(x)| = |NC(R)|$

Lemma 3.2 *A ring R is a finite commutative reduced ring with no non trivial idempotents if and only if R is a finite fields.*

Proof: Let R be a **finite commutative reduced ring**. This implies that R has **no non-zero nilpotent element**. If R is a finite commutative reduced ring with no non trivial idempotents implies nilpotent is $\overline{0}$ and idempotents are $\overline{0}$ and $\overline{1}$.

($\Rightarrow$): Let $\overline{0} \neq x \in R$. We have a set $A = \{x^k : k \in \mathbb{N}\}$ is a finite set. Therefore there exists $m > l$ such that $x^l = x^m$. We illustrate the example in the following.

Example 1 Let $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$, since 0 is too obvious to be calculate, so we ignore 0 from the set. So, $\mathbb{Z}_5 = \{1, 2, 3, 4\}$. Let $x = 3$, then there exists $m > l$ such that $x^l = x^m$. We let $m = 5$ and $l = 1$.

So,

$$3^1 = 3 \mod 5 = 3$$
$$3^5 = 243 \mod 5 = 3 \mod 5 = 3$$

Therefore,

$$3^1 = 3^5$$

Note that:

$$\begin{aligned}
x^l &= x^m \\
&= x^{m+l-l} \\
&= x^{m-l} \cdot x^l \\
&= x^{m-l} \cdot x^m \\
&= x^{(2m-l)-l+l} \\
&= x^{2m-l-l} \cdot x^l \\
&= x^{2(m-l)} \cdot x^m \\
&= x^{(3m-2l)-l+l} \\
&= x^{3(m-l)} \cdot x^l \\
&\vdots \\
&= x^{k(m-l)+l}, k \in \mathbb{N} \qquad (3.1)
\end{aligned}$$

Now we have:

$$\begin{aligned}
[x^{l(m-l)}]^2 &= x^{l(m-l)} \cdot x^{l(m-l)} \\
&= x^{l(m-l)+l(m-l)+l-l} \\
&= x^{l(m-l)+1} \cdot x^{l(m-l)-1} \\
&= x^l \cdot x^{l(m-l)-l} \quad \text{(From (3.1):} x^l = x^{k(m-l)+l}) \\
&= x^{l(m-l)}
\end{aligned}$$

which indicate $x^{l(m-l)} \in Idem(R)$. Thus, $x^{l(m-l)} = \overline{1}$. This gives that x is a unit in R. Since we know that $x^{l(m-l)} = \overline{1}$, we also know that $x^l \cdot x^{l(m-l)-l} = \overline{1}$, so this indicates that $x^{l(m-l)-l}$ is an inverse for x. Therefore, R is a finite field.

($\Leftarrow$): Let R be a finite field. Then that every elements in R will have an inverse. Let $x \in R$ has its inverse. Let $x \neq 0 \in R$ as a nilpotent element. Then $x^n = 0$ for some $n \in \mathbb{N}$. Since x has an inverse, then we can say that $x^n(x^{-1})^{n-1} = 0(x^{-1})^{n-1}$. This implies that $x = 0$, which is a contradiction. Therefore, R has no non-zero nilpotent element. So, R will be a finite commutative ring with no non trivial idempotents.

3-2-1 Invariants of nil clean graphs

In this section, we study the properties of nil-clean graphs related to invariants of graph theory.

3-2-1-1 Girth of nil-clean graphs

In the following, we show the theorems that studied by Basnet (2017) that related to girth.

Theorem 3.2 *The following hold true for nil clean graph $G_N(R)$ of R:*

(I) If R is not a field, then girth of $G_N(R)$ is equal to 3.

(II) If R is a field, then

(a) girth is $2p$ if $R \cong GF(p^k)$(field of order p^k), where p is a odd prime and $k > 1$;

(b) girth is infinite, in fact $G_N(R)$ is a path, otherwise.

Proof: For *(I)*: from **Lemma 3.2**, we know that if R is not a field, then R will not be a finite commutative reduced rings with no non trivial idempotents. This implies that R will have at least one non trivial nilpotent or non trivial idempotent.

Case 1: If there exists a non trivial nilpotent, says, $n \in R$, then we have

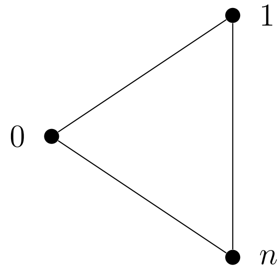


Figure 8: Girth of $G_N(R)$ with non trivial nilpotent

So, the girth of $G_N(R)$ is 3.

Case 2: If there exists a non trivial idempotent, says, $e \in R$, then we have

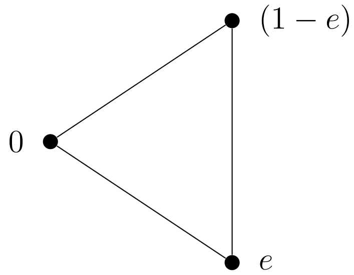


Figure 9: Girth of $G_N(R)$ with non trivial idempotent

So, the girth of $G_N(R)$ is also 3.

For *(II)*: note that the set of nil clean elements of finite field is $\{\overline{0}, \overline{1}\}$, so, the nil clean graph for $\mathbb{Z}_p$, where p is prime, is shown as the following:

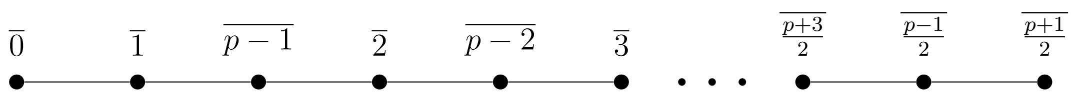


Figure 10: Nil clean graph of $\mathbb{Z}_p$

From Figure 10, we can see that the girth of $G_N(\mathbb{Z}_p)$ is infinite. Hence, (b) holds true. From the characteristic of finite field that the nil clean graph of $GF(p^k)$ for $p > 2$, we can clearly see that the graph will be disconnected. Furthermore, the disconnected graph will be consisting of a path of length $p-1$ and $(\frac{p^{k-1}-1}{2})$ number of $2p$ cycles. Let $A \subseteq GF(p^k)$ where $GF(p^k) = \mathbb{Z}_p[x]/\langle f(x)\rangle$, $f(x)$ is a irreducible polynomial of degree k over $\mathbb{Z}_p$. Now, A will consist all the linear combination of $x, x^2, ..., x^{k-1}$ with coefficient of from $\mathbb{Z}_p$ such that $g(x) + \langle f(x)\rangle \in A$ then $-g(x) + \langle f(x)\rangle \notin A$. Next, we can express A as $A = \{\overline{g_i(x)} = g_i(x) + \langle f(x)\rangle | 1 \leq i \leq (\frac{p^{k-1}-1}{2})\}$. So, we have

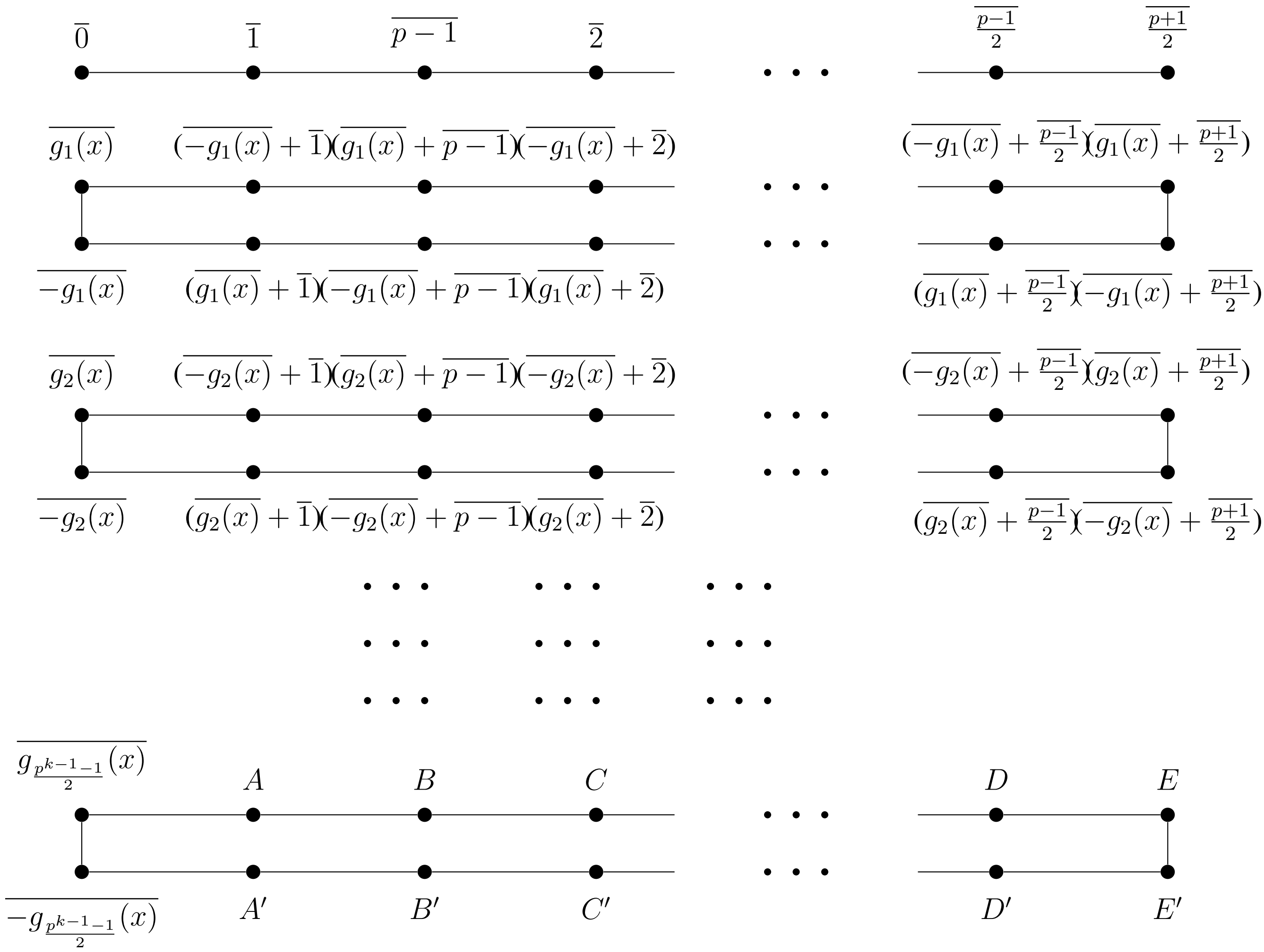


Figure 11: Nil clean graph of $GF(p^k)$

Here, $A = \overline{-g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{1}$, $B = \overline{g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{p-1}$, $C = \overline{-g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{2}$,

$D = \overline{-g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{\frac{p-1}{2}}$, $E = \overline{g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{\frac{p+1}{2}}$,

$A' = \overline{g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{1}$, $B' = \overline{-g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{p-1}$, $C' = \overline{g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{2}$,

$D' = \overline{g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{\frac{p-1}{2}}$, $E' = \overline{-g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{\frac{p+1}{2}}$,

From Figure 11, we can see that the girth will be $2p$ provided p is odd prime for $R \cong GF(p^k)$

Theorem 3.3 *$G_N(R)$ is bipartite if and only if R is a field.*

Proof: ($\Rightarrow$): Let $G_N(R)$ be bipartite, so girth of $G_N(R)$ will be a even number. Hence, by **Theorem 3.2**, R is a field.

($\Leftarrow$): By **Theorem 3.2**, if R is a field, then $R \cong GF(p^k)$ and $G_N(R)$ will have girth with even length. In fact, $G_N(R)$ can be in a form of bipartite graph.

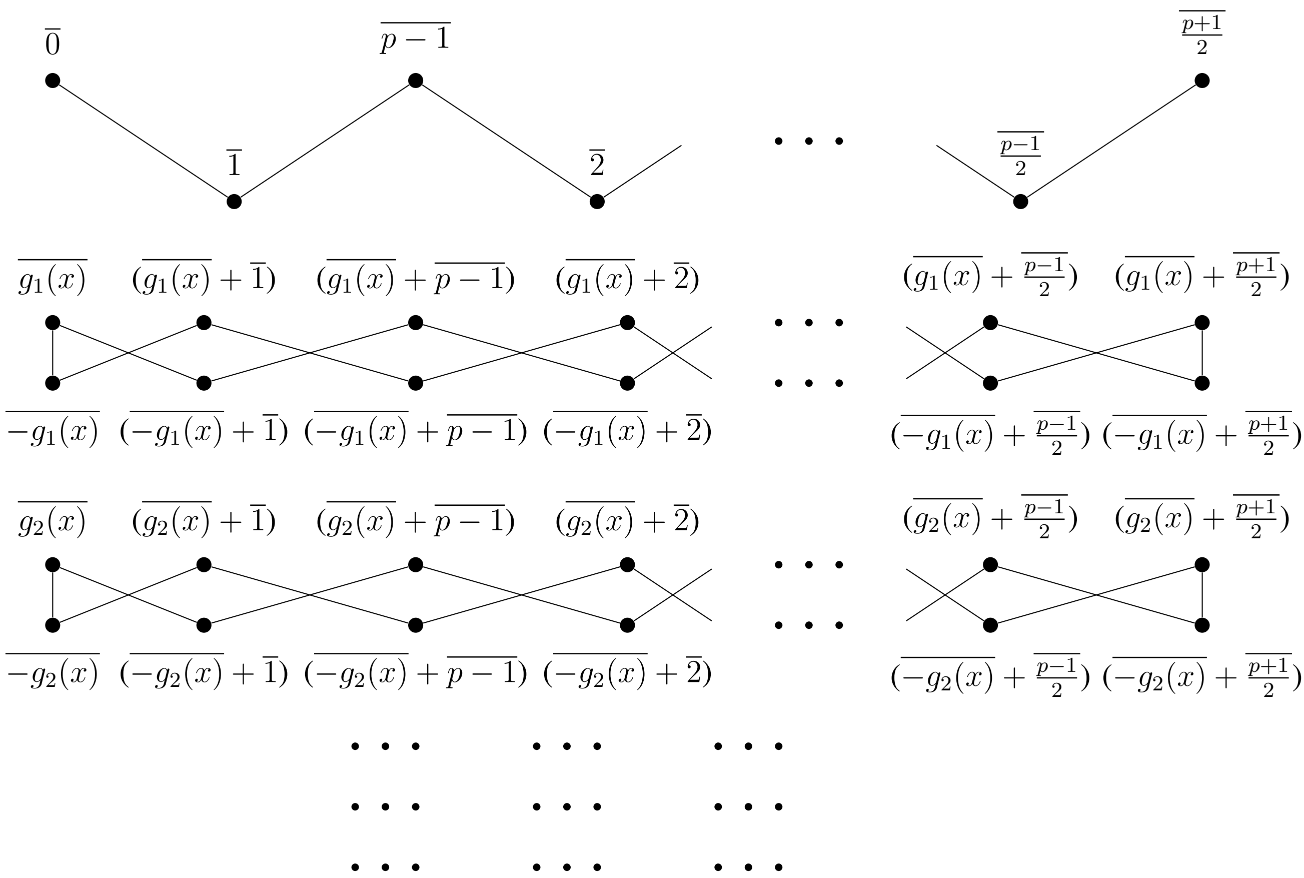


Figure 12: Bipartite nil clean graph of $GF(p^k)$

From the definition of bipartite graph, let subset $V_1 = \{\overline{0}, \overline{p-1}, \overline{\frac{p+1}{2}}, \overline{g_1(x)}, (\overline{g_1(x)}+\overline{1}), ...\}$ and subset $V_2 = \{\overline{1}, \overline{2}, \overline{\frac{p-1}{2}}, \overline{-g_1(x)}, (\overline{-g_1(x)}+\overline{1}), ...\}$ and every edge in $G_N(GF(p^k))$ has the form $e = (x, y) \in E(G_N(GF(p^k)))$, where $x \in V_1$ and $y \in V_2$. Moreover, there are no two vertices both in V_1 or both in V_2 are adjacent.

3-2-1-2 Chromatic Index of nil-clean graph

In the following, we show the theorem that studied by Basnet (2017) that related to chromatic index.

Theorem 3.4 *Let R be a finite commutative ring then nil clean graph of R is of class* 1.

Proof: We colour the edge ab with the colour $a + b$. By this colouring, every 2 distinct edges ab and ac has different colour. Let C be the set of colours and let $\{x, x_1, x_2, x_3, ..., x_n\} \subseteq R$ such that $\{x + x_1, x + x_2, x + x_3, ..., x + x_n\} \subset NC(R)$, we will have

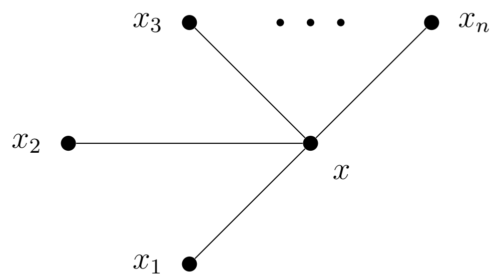


Figure 13: $G_N(R)$ with $n + 1$ elements

So, we have $C = \{x + x_1, x + x_2, x + x_3, ..., x + x_n\}$. Then, $\chi'(G_N(R)) \leq |C|$ which indicates nil clean graph have $|C|$-edges colouring. Since $C \subset NC(R)$, this indicates that $|C| \leq |NC(R)|$ and result in $\chi'(G_N(R)) \leq |C| \leq |NC(R)|$. By **Lemma 3.1**, we know that $deg(x) \leq |NC(R)|$ and $deg(x) \leq |NC(R)| - 1$, then implies $deg(x) \leq deg(x) + 1 \leq |NC(R)|$. So, it is true for $deg(x) = \Delta = |C|$ and $\Delta \leq |NC(R)|$, then $\chi'(G_N(R)) \leq \Delta$. By Vizings theorem, we have $\chi'(G_N(R)) \geq \Delta$. This will result in $\chi'(G_N(R)) = \Delta$, so $G_N(R)$ is class 1.

3-2-1-3 Diameter of nil clean graph

In the following, we show some results that studied by Basnet (2017) that related to diameter.

Lemma 3.3 *R is nil clean ring if and only if $diam(G_N(R)) = 1$.*

Proof: ($\Rightarrow$): Let R be a nil clean ring, then $\forall r \in R$ must be nil clean elements. We know that $r = n + e$ where $n \in Nil(R)$ and $e \in Idem(R)$. So, we have

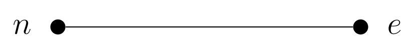


Figure 14: Nil clean graph of elements n and e

which give the result of $diam(G_N(R)) = 1$.

($\Leftarrow$): Let $diam(G_N(R)) = 1$, which indicates the maximum distances of each pair of distinct vertices in $G_N(R)$ must be 1. Let arbitrary $x, y \in R$ and since $diam(G_N(R)) = 1$, so $r = x + y$ must be nil clean element. So, $\forall r \in R$ must be nil clean elements. Therefore, R is **nil clean ring**.

Theorem 3.5 *Let R be a non nil clean, weak nil clean ring with no non trivial idempotents then $diam(G_N(R)) = 2$.*

Proof: At first, let R be a **weak nil clean ring with no non trivial idempotents**. Then, we let arbitrary $a, b \in R$ and for some $n_1, n_2 \in Nil(R)$. We have $a = n_1, n_1 + 1$ or $n_1 - 1$ and $b = n_2, n_2 + 1$ or $n_2 - 1$. We have the following table

	n_1	$n_1 + 1$	$n_1 - 1$
n_2	(n_1)——(n_2) $d(a,b) = 1$	$(n_1 + 1)$——(n_2) $d(a,b) = 1$	$(n_1 - 1)$—(1)—(n_2) $d(a,b) = 2$
$n_2 + 1$	(n_1)——$(n_2 + 1)$ $d(a,b) = 1$	$(n_1 + 1)$—(-1)—$(n_2 + 1)$ $d(a,b) = 2$	$(n_1 - 1)$——$(n_2 + 1)$ $d(a,b) = 1$
$n_2 - 1$	(n_1)—(1)—$(n_2 - 1)$ $d(a,b) = 2$	$(n_1 + 1)$——$(n_2 - 1)$ $d(a,b) = 1$	$(n_1 - 1)$—(1)—$(n_2 - 1)$ $d(a,b) = 2$

Table 3: All combination of $d(a, b)$

Thus, from Table 3 we can conclude that $diam(G_N(R)) \leq 2$.

Now, let R be a **non nil clean ring with no non trivial idempotents**. Since **non nil clean** indicates $n, n + 1 \notin R$, then we have at least one $x \in R$ such that $x = n - 1$. So, $d(0, x)$ will be equal to 2 since (0)—(1)—$(n - 1)$, therefore $diam(G_N(R)) \geq 2$.

So, if R is **weak nil clean ring with no non trivial idempotents**, $diam(G_N(R)) \leq 2$ and if R is **non nil clean ring with no non trivial idempotents**, $diam(G_N(R)) \geq 2$. Hence, a **non nil clean, weak nil clean ring with no non trivial idempotents** will have $diam(G_N(R)) = 2$.

Theorem 3.6 *Let $R = A \times B$, such that A is nil clean and B is weak nil clean with no non trivial idempotents, then $diam(G_N(R)) = 2$.*

Proof: Since A has non trivial idempotents, let $e \in Idem(A)$, then we have $Idem(R) = \{(e, 0_B), (e, 1_B) | e \in Idem(A)\}$. Now let $(a_1, b_1), (a_2, b_2) \in R$, if $(a_1, b_1) + (a_2, b_2)$ is nil clean indicates that (a_1, b_1) — (a_2, b_2), then $d((a_1, b_1), (a_2, b_2)) = 1$ in $G_N(R)$. However, if $(a_1, b_1) + (a_2, b_2)$ is not nil clean, it will be a result from $b_1 + b_2$ is not nil clean, because R is closed under addition. i.e Let $n, n+e \in A$ where $n \in Nil(A), e \in Idem(A)$. So, now we have $a_1 = n_1, n_1 + e_1$ and $a_2 = n_2, n_2 + e_2$. Clearly, $a_1 + a_2$ must be nil clean.

Since B is weak nil clean with no non trivial idempotents, let $b_1 = n_1, n_1 + 1, n_1 - 1$ and $b_2 = n_2, n_2 + 1, n_2 - 1$, where $n_1, n_2 \in Nil(B)$. So, we have the following cases:

CASE I: If $b_1 = n_1 + 1$ and $b_2 = n_2 + 1$, we have the path (a_1, b_1) — $(0, -1)$ — (a_2, b_2) in $G_N(R)$, thus $d((a_1, b_1), (a_2, b_2)) \leq 2$.

CASE II: If $b_1 = n_1 - 1$ and $b_2 = n_2 - 1$, we have the path (a_1, b_1) — $(0, 1)$ — (a_2, b_2) in $G_N(R)$, thus $d((a_1, b_1), (a_2, b_2)) \leq 2$.

CASE III: If $b_1 = n_1 - 1$ and $b_2 = n_2$, we have the path (a_1, b_1) — $(0, 1)$ — (a_2, b_2) in $G_N(R)$, thus $d((a_1, b_1), (a_2, b_2)) \leq 2$.

CASE IV: If $b_1 = n_1$ and $b_2 = n_2 - 1$, we have the path (a_1, b_1) — $(0, 1)$ — (a_2, b_2) in $G_N(R)$, thus $d((a_1, b_1), (a_2, b_2)) \leq 2$.

Therefore $diam(G_N(R)) \leq 2$, R is not nil clean implies $diam(G_N(R)) \geq 2$. Thus, $diam(G_N(R)) = 2$

CHAPTER 4: $g(x)$-NIL CLEAN GRAPH

4-1 Introduction

Let R be a ring and let $g(x)$ be a polynomial in $Z(R)[x]$, where $Z(R)$ denote as center of R. In L. Fan(2008), an element $r \in R$ is called $g(x)$-nil clean if $r = n + s$ for some $n \in Nil(R)$ and $s \in R$ such that $g(s) = 0$. The ring R is $g(x)$-nil clean if every element in R is $g(x)$-nil clean.

Clearly, if $g(x) = x^2 - x$, then $g(x)$-nil clean rings are nil clean. However, in general, $g(x)$-nil clean rings are not necessarily nil clean. We note this with the following example,

Example 2 *Let $\mathbb{Z}_{(7)} = \{m/n \mid m, n \in \mathbb{Z}, gcd(7, n) = 1\}$ and let C_3 be the cyclic group of order 3. By Wang and Chen (2004), the group ring $\mathbb{Z}_{(7)}C_3$ is $(x^6 - 1)$-clean. However, $\mathbb{Z}_{(7)}C_3$ is not clean by Hans and Nicholson (2001).*

Let $g(x)$-nil clean graph of ring R denote by $G_{N^\star}(R)$ and a set of $g(x)$-nil clean elements of ring R denote by $N^\star(R)$. Let x and y to be distinct vertices of the elements from $g(x)$-nil clean ring R, such that x adjacent to y if and only if $x + y \in N^\star(R)$.

4-2 On $x(x-2)$-nil clean graphs

In this section, we mainly focus on the $g(x) = x^2 - 2x \in Z(R)[x]$. Hence, all the $g(x)$ mentioned in the remaining of this chapter, the $g(x)$ is defined by $g(x) = x^2 - 2x \in Z(R)[x]$. Next, we will provide some examples on $x(x-2)$-nil clean graphs.

Example 3 We denoted that $GF(25)$ is a finite field with 25 elements. We show that the $GF(25)$ is a $x(x-2)$-nil clean graph.

$$\begin{aligned} GF(25) &\cong \mathbb{Z}_5[x]/\langle x^2 + x + 1\rangle \\ &= \{ax + b + \langle x^2 + x + 1\rangle : a, b \in \mathbb{Z}_5\} \end{aligned}$$

Let $\beta = x + \langle x^2 + x + 1\rangle$. *Then* $GF(25) = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \beta, 2\beta, 3\beta, 4\beta, 1 + \beta, 2 +$

$\beta, 3+\beta, 4+\beta, 1+2\beta, 2+2\beta, 3+2\beta, 4+2\beta, 1+3\beta, 2+3\beta, 3+3\beta, 4+3\beta, 1+4\beta, 2+4\beta, 3+4\beta, 4+4\beta\}$.

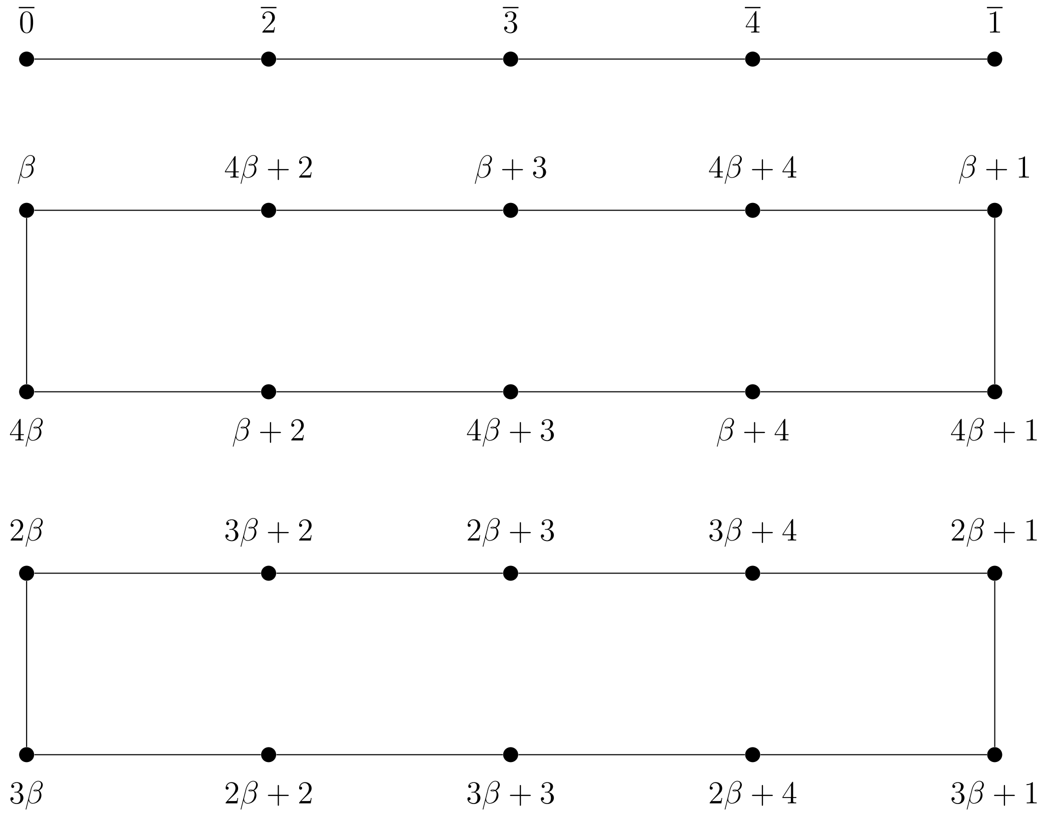


Figure 15: $g(x)$-nil clean graph of $GF(25)$

Example 4 We denoted that $GF(27)$ is a finite field with 27 elements. We show that the $GF(27)$ is a $x(x-2)$-nil clean graph.

$$\begin{aligned} GF(27) &\cong \mathbb{Z}_3[x]/\langle x^3+2x^2+1\rangle \\ &= \{ax^2+bx+c+\langle x^3+2x^2+1\rangle : a,b,c\in\mathbb{Z}_3\} \end{aligned}$$

Let $\gamma = x^2+\langle x^3+2x^2+1\rangle$ *and* $\delta = x+\langle x^3+2x^2+1\rangle$. *Then* $GF(27) = \{\overline{0}, \overline{1}, \overline{2}, \delta, \delta+1, \delta+2, 2\delta, 2\delta+1, 2\delta+2, \gamma, \gamma+1, \gamma+2, \gamma+\delta, \gamma+\delta+1, \gamma+\delta+2, \gamma+2\delta, \gamma+2\delta+1, \gamma+2\delta+2, 2\gamma, 2\gamma+1, 2\gamma+2, 2\gamma+\delta, 2\gamma+\delta+1, 2\gamma+\delta+2, 2\gamma+2\delta, 2\gamma+2\delta+1, 2\gamma+2\delta+2\}$

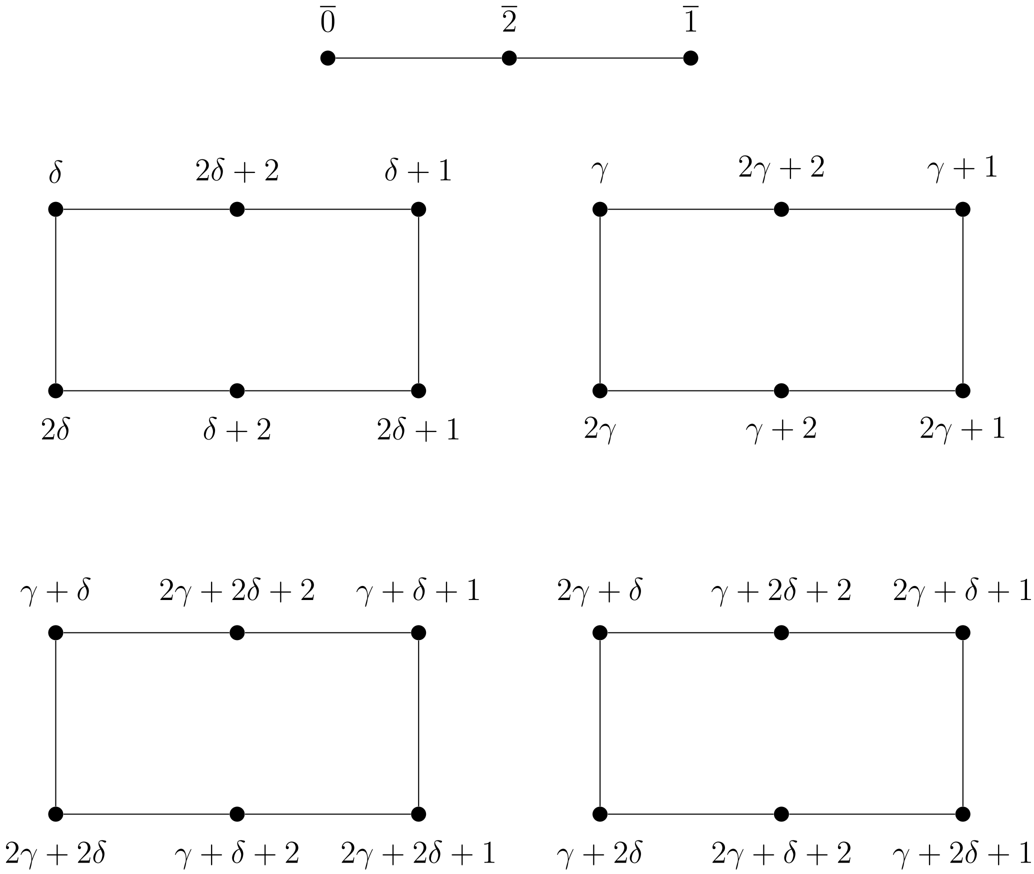


Figure 16: $g(x)$-nil clean graph of $GF(27)$

Example 5 We denoted that $GF(49)$ is a finite field with 49 elements. We show that the $GF(49)$ is a $x(x-2)$-nil clean graph.

$$\begin{aligned} GF(49) &\cong \mathbb{Z}_7[x]/\langle x^2+x+1\rangle \\ &= \{ax+b+\langle x^2+x+1\rangle : a,b\in\mathbb{Z}_7\} \end{aligned}$$

Let $\beta = x+\langle x^2+x+1\rangle$. *Then* $GF(49) = \{\overline{0},\overline{1},\overline{2},\overline{3},\overline{4},\overline{5},\overline{6},\beta,2\beta,3\beta,4\beta,5\beta,6\beta,1+\beta,2+\beta,3+\beta,4+\beta,5+\beta,6+\beta,1+2\beta,2+2\beta,3+2\beta,4+2\beta,5+2\beta,6+2\beta,1+3\beta,2+3\beta,3+3\beta,4+3\beta,5+3\beta,6+3\beta,1+4\beta,2+4\beta,3+4\beta,4+4\beta,5+4\beta,6+4\beta,1+5\beta,2+5\beta,3+5\beta,4+5\beta,5+5\beta,6+5\beta,1+6\beta,2+6\beta,3+6\beta,4+6\beta,5+6\beta,6+6\beta\}$

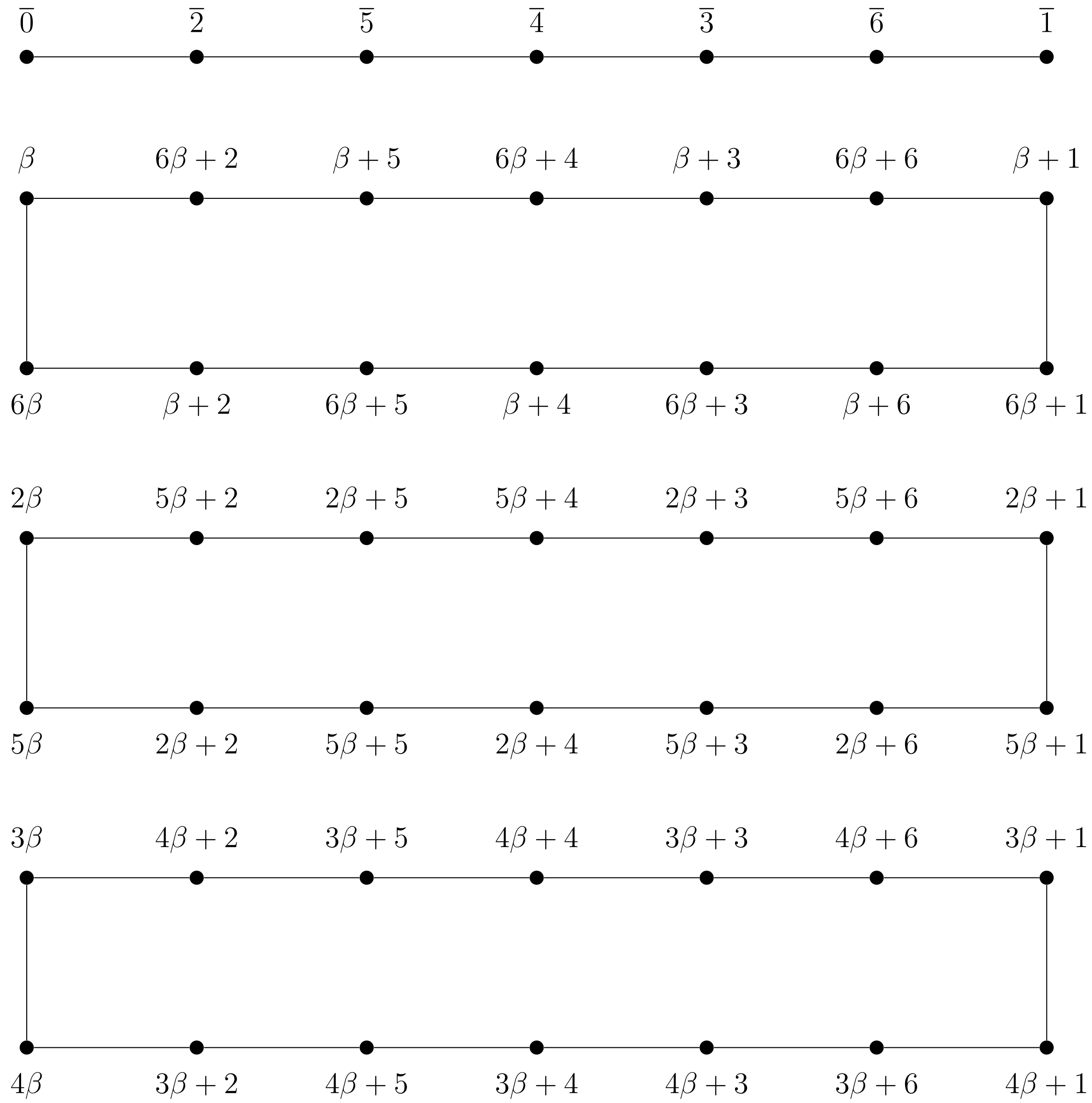
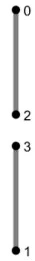


Figure 17: $g(x)$-nil clean graph of $GF(49)$

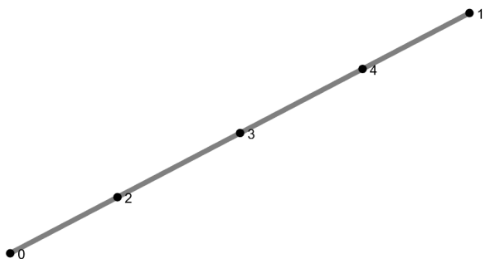
Next, we will investigate about the $x(x-2)$-nil clean graph from $\mathbb{Z}_3$ to $\mathbb{Z}_{34}$ where $\mathbb{Z}_k = \{0, 1, 2, 3, \ldots, k-1\}$ for $3 \leq k \leq 34$. The following illustration will be on the $x(x-2)$-nil clean graph form from $\mathbb{Z}_3$ to $\mathbb{Z}_{34}$.

Graph of Z_3

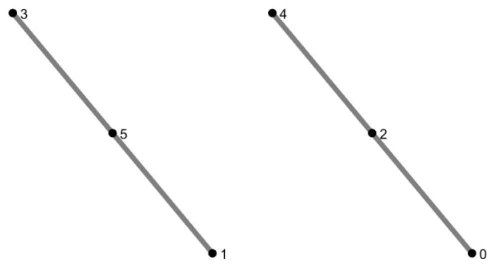
Graph of Z_4



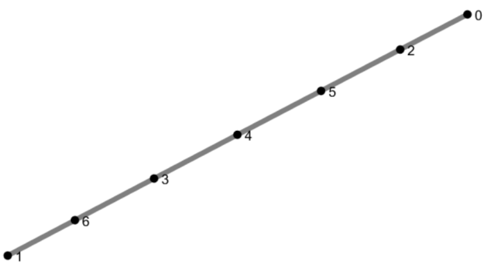
Graph of Z_5



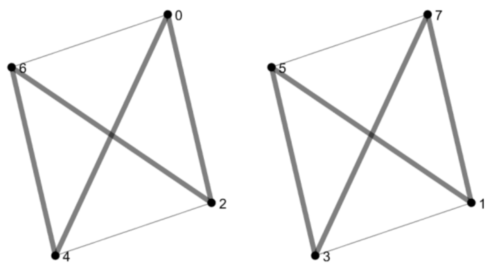
Graph of Z_6



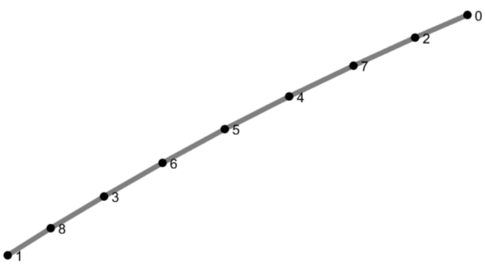
Graph of Z_7



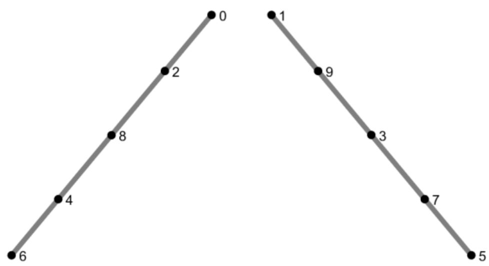
Graph of Z_8



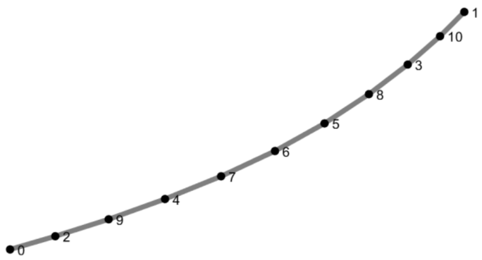
Graph of Z_9



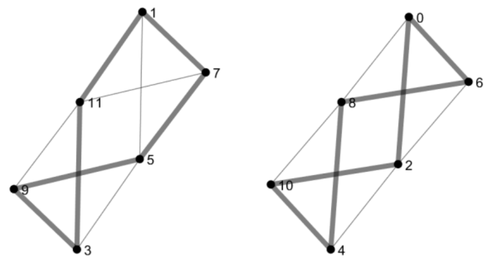
Graph of Z_{10}



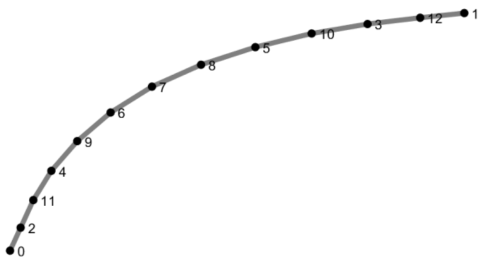
Graph of Z_{11}



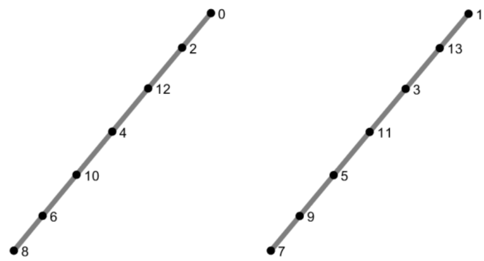
Graph of Z_{12}



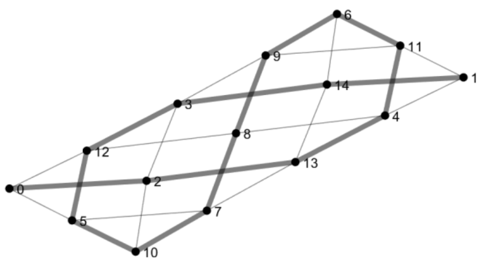
Graph of Z_{13}



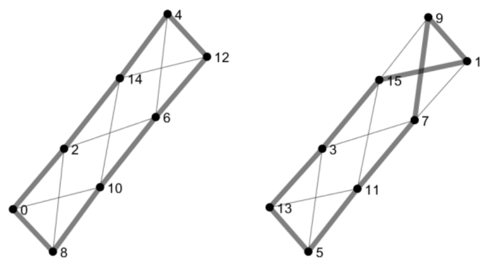
Graph of Z_{14}



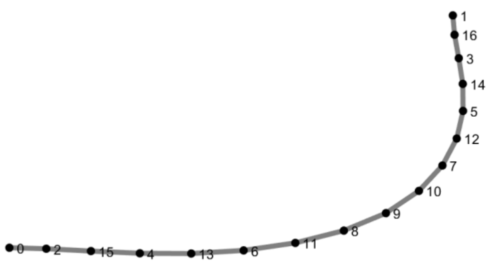
Graph of Z_{15}



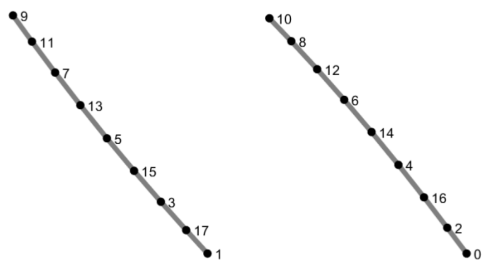
Graph of Z_{16}



Graph of Z_{17}

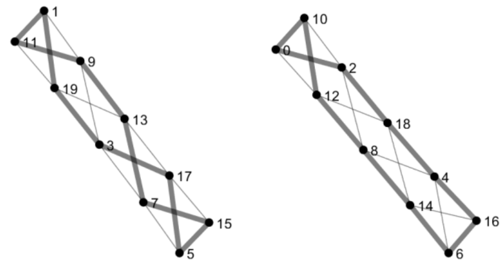


Graph of Z_{18}

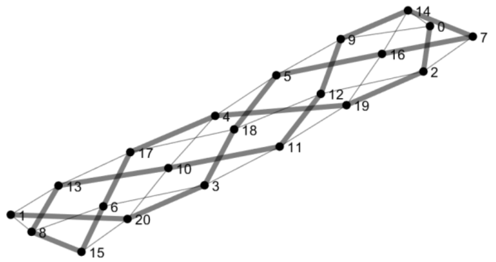


Graph of Z_{19}

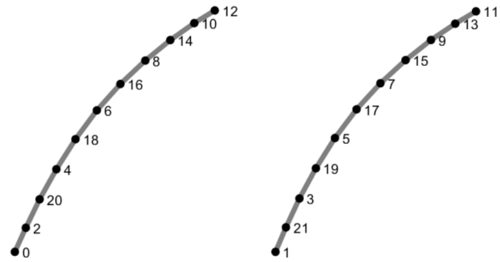
Graph of Z_{20}



Graph of Z_{21}

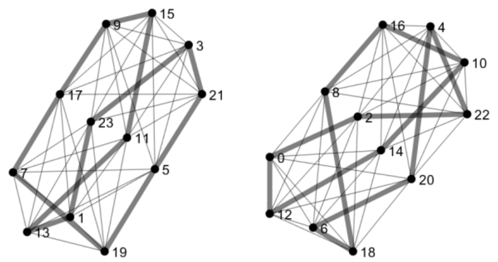


Graph of Z_{22}



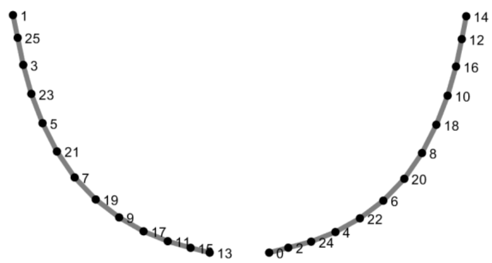
Graph of Z_{23}

Graph of Z_{24}



Graph of Z_{25}

Graph of Z_{26}



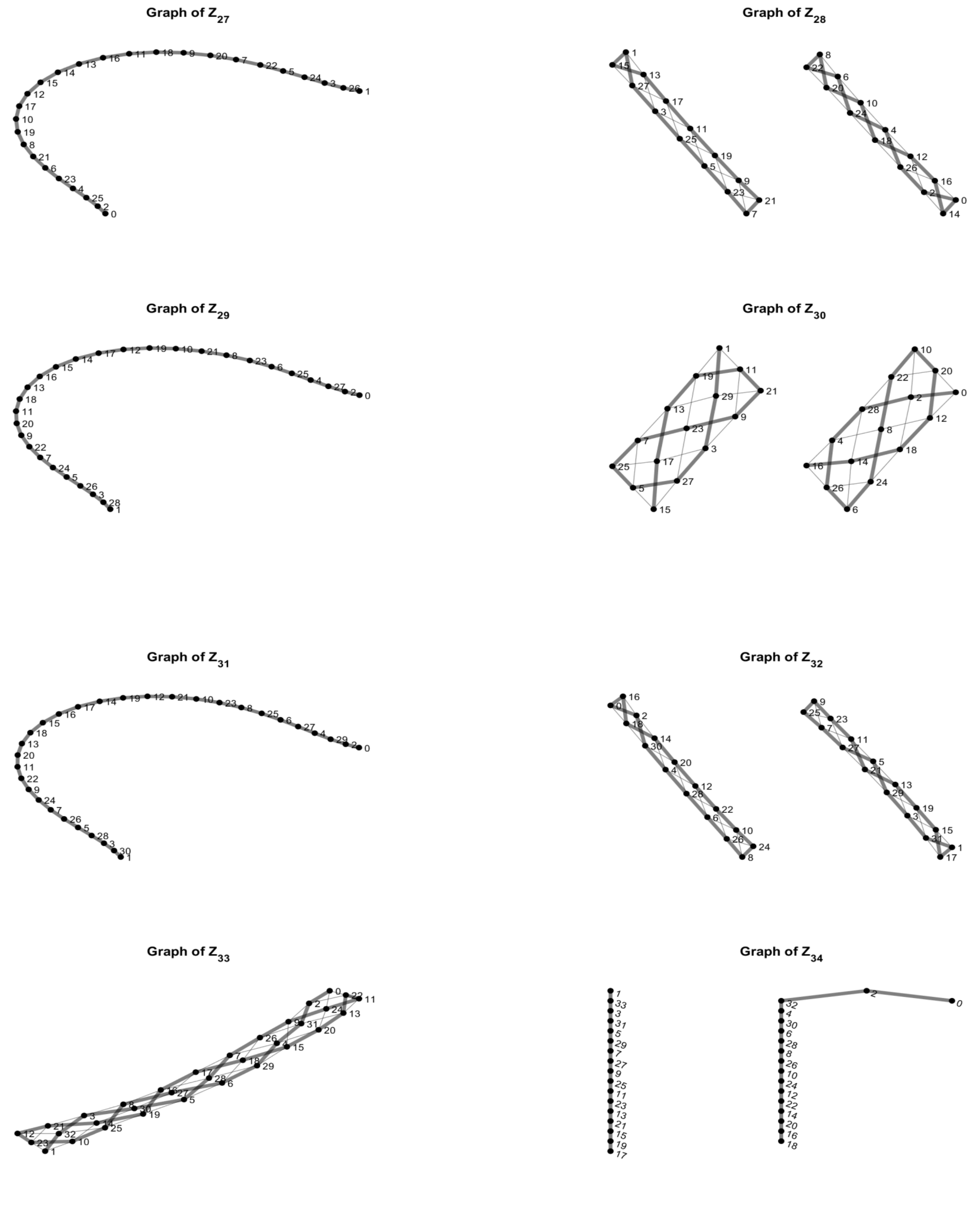


Figure 18: $g(x)$-nil clean graph of $\mathbb{Z}_3$ to $\mathbb{Z}_{34}$

4-3 Some properties of $g(x)$-nil clean graph

In the following, we mainly study and investigate on some properties of $x(x-2)$-nil clean graph. We first note the following for any $g(x) \in Z(R)$.

Theorem 4.1 *The $g(x)$-nil clean graph $G_{N^\star}(R)$ is a complete graph if and only if R is a $g(x)$-nil clean ring.*

Proof: ($\Rightarrow$): Let $G_{N^\star}(R)$ be a complete $g(x)$-nil clean graph of ring R. Then it implies that for all $r \in R$ are $g(x)$**-nil clean elements**. Without lose of generality of $g(x)$**-nil clean elements**, there exists a path from r and it is connected to 0, such that $r = r + 0$ which is $g(x)$-nil clean. Hence R is $g(x)$-nil clean.

($\Leftarrow$): Conversely, let R be a $g(x)$-nil clean ring. To form a graph in R, we let anyq arbitrary elements $x, y \in R$, x and y are connected if and only if $x + y$ is a $g(x)$**-nil clean element**. So, this implies that every pairs of distinct element $r \in R$ must have a unique edges and it form the complete $g(x)$-nil clean graph $G_{N^\star}(R)$.

Lemma 4.1 *Let $G_{N^\star}(R)$ be the $g(x)$-nil clean graph of a ring R. Then we have the following:*

(I) If $2x$ is $g(x)$-nil clean where $x \in R$, then $deg(x) = |N^\star(R)| - 1$.

(II) If $2x$ is not $g(x)$-nil clean where $x \in R$, then $deg(x) = |N^\star(R)|$.

Proof: Let $x \in R$ and clearly $x + R = R$. Then for every $y \in N^\star(R)$, there exists a unique element $x_y \in R$ such that $x + x_y = y$. Thus, we have $deg(x) \leq |N^\star(R)|$

For *(I)*: Now, we let $\{x_1, x_2, x_3, ..., x\} \subseteq R$ and $\{x_1 + x, x_2 + x, x_3 + x, ..., 2x\} \subseteq N^\star(R)$. Since we are not considering any loops, we illustrate the graph in the following:

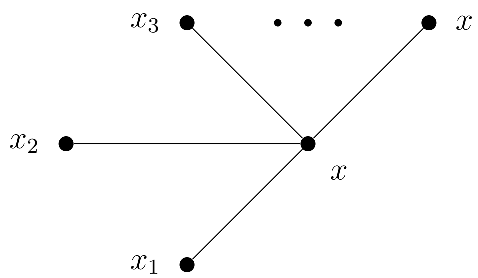


Figure 19: $g(x)$-nil clean graph with x elements including x

By the definition of $N_{G_{N^\star}(R)}(x) = \{y \in V(G_{N^\star}(R)) | y \text{ is adjacent to } x\}$, we know that $y = \{1, 2, 3, ..., x\}$. We have $deg(x) = |N_{G_{N^\star}(R)}(x)| = |N_{G_{N^\star}(R)}[x]| - 1 = |N^\star(R)| - 1$.

For *(II)*: Now, we let $\{x_1, x_2, x_3, ..., x_y, x\} \subseteq R$ and $\{x_1+x, x_2+x, x_3+x, ..., x+x_y\} \subseteq N^\star(R)$ but $2x \notin N^\star(R)$. Since we are not considering any loops, we have

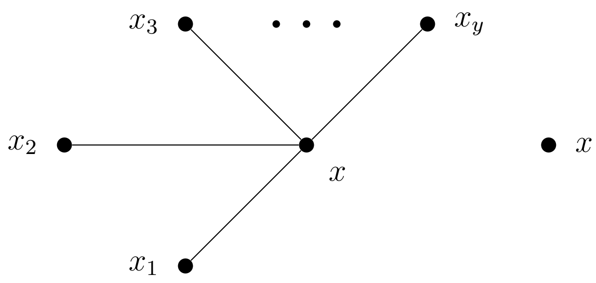


Figure 20: $g(x)$-nil clean graph with x elements excluding x

From the definition of $N_{G_{N^\star}(R)}(x) = \{y \in V(G_{N^\star}(R)) | y \text{ is adjacent to } x\}$, we know that $y = \{x_1, x_2, x_3, ..., x_y\}$. We have $deg(x) = |N_{G_{N^\star}(R)}(x)| = |N^\star(R)|$.

4-3-1 Invariants of $g(x)$-nil clean graphs

In this section, we provide some properties of $g(x)$-nil clean graphs related to invariants of graph theory.

4-3-1-1 Connected and disconnected graph of $g(x)$-nil clean graph

Proposition 4.1 *For all $n \geq 3$, then the following hold for $\mathbb{Z}_n$*

(I) If $n = 2k$, for all $k \geq 2$. Then $G_{N^\star}(\mathbb{Z}_n)$ is a disconnected graph (in particular, it can be built up into 2 parts, one part fills by even integers and the other part is filled by odd integers).

  (a) Let $n = 2k$, where $k = p^q$, and p is an odd prime and for all integer $q \geq 1$. Then $G_{N^\star}(\mathbb{Z}_n)$ is a disconnected path graph.

(II) If $n = 2k + 1$, for all $k \geq 1$, then $G_{N^\star}(\mathbb{Z}_n)$ is a connected graph.

  (a) Let $n = p^q$, where p is an odd prime and for all integer $q \geq 1$. Then $G_{N^\star}(\mathbb{Z}_n)$ is a path graph.

Proof: (I): It is obvious by taking $\mathbb{Z}_n$ (n is even) as a example in Figure 18. Furthermore, it built up into 2 parts which are the even integers part and odd integers part, it exists two path that can be illustrate in the Figure 21.

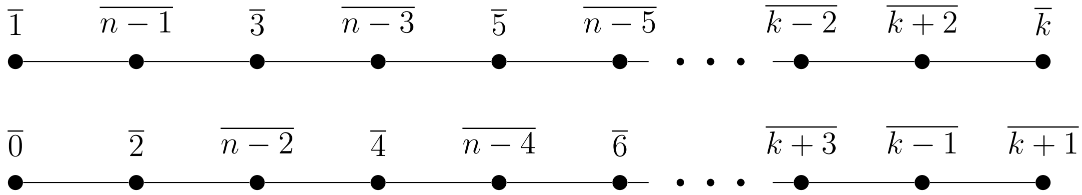


Figure 21: disconnected path graph for $g(x)$-nil clean graph of $\mathbb{Z}_{2k}$

(I)(a): Let $n = 2k$, where $k = p^q$, and p is an odd prime and for all integer $q \geq 1$. Then $G_{N^\star}(\mathbb{Z}_n)$ is a disconnected path graph and the illustration can refer to Figure 21.
(II): It is obvious by taking $\mathbb{Z}_n$ (n is odd) as a example in Figure 18. Furthermore, it exists a path that can be illustrate in the Figure 22.

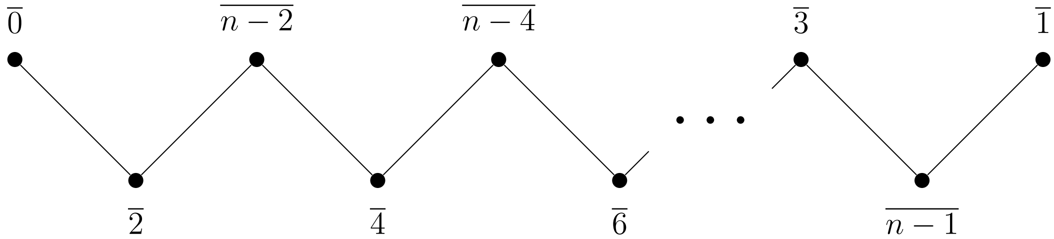


Figure 22: path graph for $g(x)$-nil clean graph of $\mathbb{Z}_{2k+1}$

(II)(a): In particular, if $n = p^q$, where p is an odd prime and for all integer $q \geq 1$. Then $G_{N^\star}(\mathbb{Z}_n)$ is a path graph and it can be illustrate in Figure 22.

4-3-1-2 Hamiltonian Path and Cycle of $g(x)$-nil clean graph

In the following, we prove the theorem that related to the hamiltonian path and hamiltonian cycle.

Theorem 4.2 *Let $n \geq 3$, then the following hold for $\mathbb{Z}_n$*

(I) If $n = 2k$, for all $k \geq 2$, then $G_{N^\star}(\mathbb{Z}_n)$ consists no hamiltonian path (In particular, $G_{N^\star}(\mathbb{Z}_n)$ is built by two distinct, symmetry connected graphs).

(a) If we consider either one of the part of $G_{N^\star}(\mathbb{Z}_n)$, then $G_{N^\star}(\mathbb{Z}_n)$ consists at least one hamiltonian path or hamiltonian cycle.

(II) If $n = 2k+1$, for all $k \geq 1$, then $G_{N^\star}(\mathbb{Z}_n)$ must consists at least one hamiltonian path.

Proof: (I): By **Proposition 4.1**, If $n = 2k$, for all $k \geq 2$, $G_{N^\star}(\mathbb{Z}_n)$ is a disconnected graph. Therefore, it consists no hamiltonian path in the graph.
(a) We now consider one of the part of $G_{N^\star}(\mathbb{Z}_n)$ where $n = 2k$ and $k = 2a$, for all $a \geq 2$ ($a \in \mathbb{Z}$). We can obtain a hamiltonian cycle in $G_{N^\star}(\mathbb{Z}_{2k})$ and illustrate the cycle in the Figure 23.

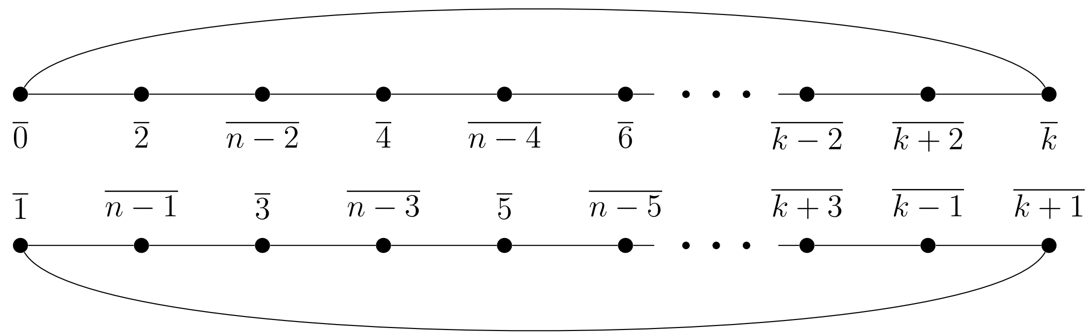


Figure 23: hamiltonian cycle for $g(x)$-nil clean graph of $\mathbb{Z}_{2k}$ where $k = 2a$

For $k = 2a + 1$, for all $a \geq 1$ ($a \in \mathbb{Z}$). We can obtain a hamiltonian path for $G_{N^\star}(\mathbb{Z}_{2k})$ and refer the illustration in Figure 21.
(II): By **Proposition 4.1**, for $n = 2k + 1$, for all $k \geq 1$, there exists at least one hamiltonian path in $G_{N^\star}(\mathbb{Z}_n)$ and illustration of hamiltonian path for $G_{N^\star}(\mathbb{Z}_n)$ is in Figure 22. Hence, $G_{N^\star}(\mathbb{Z}_n)$ is a connected graph.
This completes the proof. Furthermore, the illustration of hamiltonian paths and hamiltonian cycles of $g(x)$-nil clean graphs are presented in **Figure 18**. We note the hamiltonian path or cycle with darker line.

4-3-1-3 Diameter of $g(x)$-nil clean graph

In the following, we prove a theorem that related to the diameter.

Theorem 4.3 *Let n be a positive integer. Then the following holds for $\mathbb{Z}_n$*

(I) If $n = 2^k$, for all integer $k \geq 3$ $k \in \mathbb{Z}$, then $diam(G_{N^\star}(\mathbb{Z}_n)) = \infty$. In particular, if we consider either one of the part of $G_{N^\star}(\mathbb{Z}_n)$, then $diam(G_{N^\star}(\mathbb{Z}_n)) = 2^{k-2} - 1$.

(II) If $n = p^k$, where p is a odd prime, for all $k \geq 1$, then $diam(G_{N^\star}(\mathbb{Z}_{p^k})) = p^k - 1$.

(III) If $n = 2p^k$, where p is a odd prime and for all integer $k \geq 1$, then $diam(G_{N^\star}(\mathbb{Z}_{2p^k})) = \infty$. However, if we consider either one of the part of the disconnected graph, then $diam(G_{N^\star}(\mathbb{Z}_{2p^k})) = p^k - 1$.

Proof: (I): By hypothesis, we have $n = 2^k$, for all $k \geq 3$, then $G_{N^\star}(\mathbb{Z}_n)$ is a disconnected graph by **Proposition 4.1**. Therefore, $diam(G_{N^\star}(\mathbb{Z}_n)) = \infty$. (*I*) follows from the graph $G_{N^\star}(\mathbb{Z}_n)$ in Figure 24.

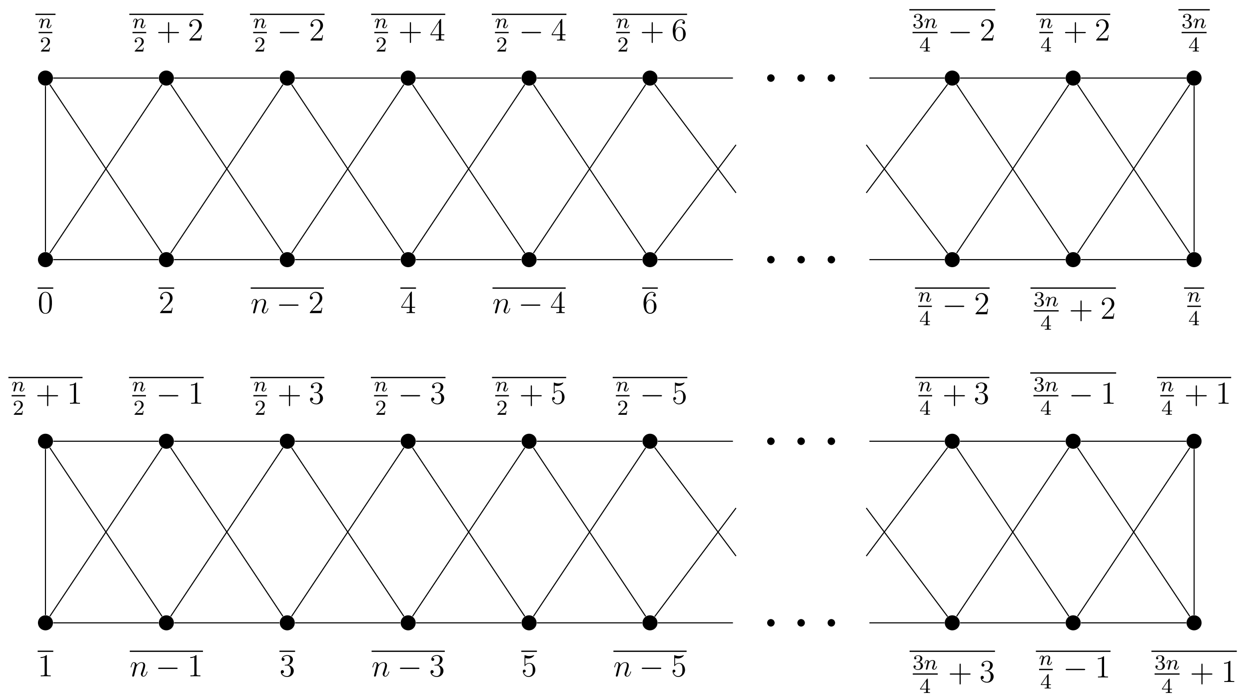


Figure 24: $g(x)$-nil clean graph of $\mathbb{Z}_n$

(II): By **Proposition 4.1**, for any integer, $n = p^k$, where p is a odd prime, for all $k \geq 1$, then $G_{N^\star}(\mathbb{Z}_{p^k})$ is a path graph. Therefore, (*II*) follows from the graph $G_{N^\star}(\mathbb{Z}_{p^k})$ in Figure 22.

(*III*) follows from the graph $G_{N^\star}(\mathbb{Z}_{2p^k})$ in Figure 25.

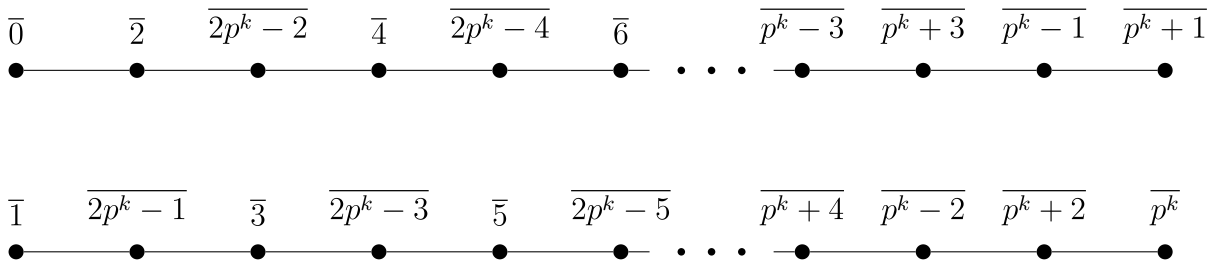


Figure 25: $g(x)$-nil clean graph of $\mathbb{Z}_{2p^k}$

A matrix is said to be a shift matrix if a matrix with ones only on the superdiagonal or subdiagonal, and zeros elsewhere. For illustration, we let matrix U_n where the superdiagonal that contains $n-1$ ones which is an upper shift matrix and matrix L_n where the subdiagonal that contains $n-1$ ones which is a lower shift matrix, as follow

$$U_n = \begin{bmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & & \\ & & 0 & \ddots & & & \\ & & & \ddots & 1 & & \\ & & & & 0 & 1 & \\ & & & & & 0 & 1 \\ & & & & & & 0 \end{bmatrix} \qquad L_n = \begin{bmatrix} 0 & & & & & & \\ 1 & 0 & & & & & \\ & 1 & 0 & & & & \\ & & 1 & \ddots & & & \\ & & & \ddots & 0 & & \\ & & & & 1 & 0 & \\ & & & & & 1 & 0 \end{bmatrix}$$

$n-1$ copies of 1 $\qquad$ $n-1$ copies of 1

In this project, we define anti-shift matrix as a matrix with ones only on the anti-superdiagonal or anti-subdiagonal, and zeros elsewhere. For illustration, we let matrix $U_n^\star$ where the anti-superdiagonal that contains $n-1$ ones which is an upper anti-shift matrix and matrix $L_n^\star$ where the anti-subdiagonal that contains $n-1$ ones which is a lower anti-shift matrix, as follow

$$U_n^\star = \begin{bmatrix} & & & & & 1 & 0 \\ & & & & 1 & 0 & \\ & & & \iddots & 0 & & \\ & & 1 & \iddots & & & \\ & 1 & 0 & & & & \\ 1 & 0 & & & & & \\ 0 & & & & & & \end{bmatrix} \qquad L_n^\star = \begin{bmatrix} & & & & & & 0 \\ & & & & & 0 & 1 \\ & & & & 0 & 1 & \\ & & & \iddots & 1 & & \\ & & 0 & \iddots & & & \\ & 0 & 1 & & & & \\ 0 & 1 & & & & & \end{bmatrix}$$

$n-1$ copies of 1 $\qquad$ $n-1$ copies of 1

4-3-1-4 Adjacency Matrix of $g(x)$-nil clean graph

Theorem 4.4 *Let n be a positive integer. Then the following holds for $\mathbb{Z}_n$*

(I) If $n = 2^k$, for all integer $k \geq 3$,$k \in \mathbb{Z}$, then $G_{N^\star}(\mathbb{Z}_{2^k})$ follow the adjacency matrix of $\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{2^k}))$.

(II) If $n = p^k$, where p is a odd prime, for all $k \geq 1$, then $G_{N^\star}(\mathbb{Z}_p)$ follow the adjacency matrix of $\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{p^k}))$ which follows form of the sum of two anti-shift matrices.

(III) If $n = 2p^k$, where p is a odd prime and for all integer $k \geq 1$, then $G_{N^\star}(\mathbb{Z}_{2p^k})$ follow the adjacency matrix of $\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{2p^k}))$.

Proof: (I): For illustration, we find the adjacency matrix of $G_{N^\star}(\mathbb{Z}_{2^k})$ where $k = 3, 4$ and 5, which are $\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_8))$, $\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{16}))$ and $\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{32}))$ as follows

$$\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_8)) = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{16})) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{32})) = \left[\begin{array}{cccccccccccccccccccccccccccccccc}
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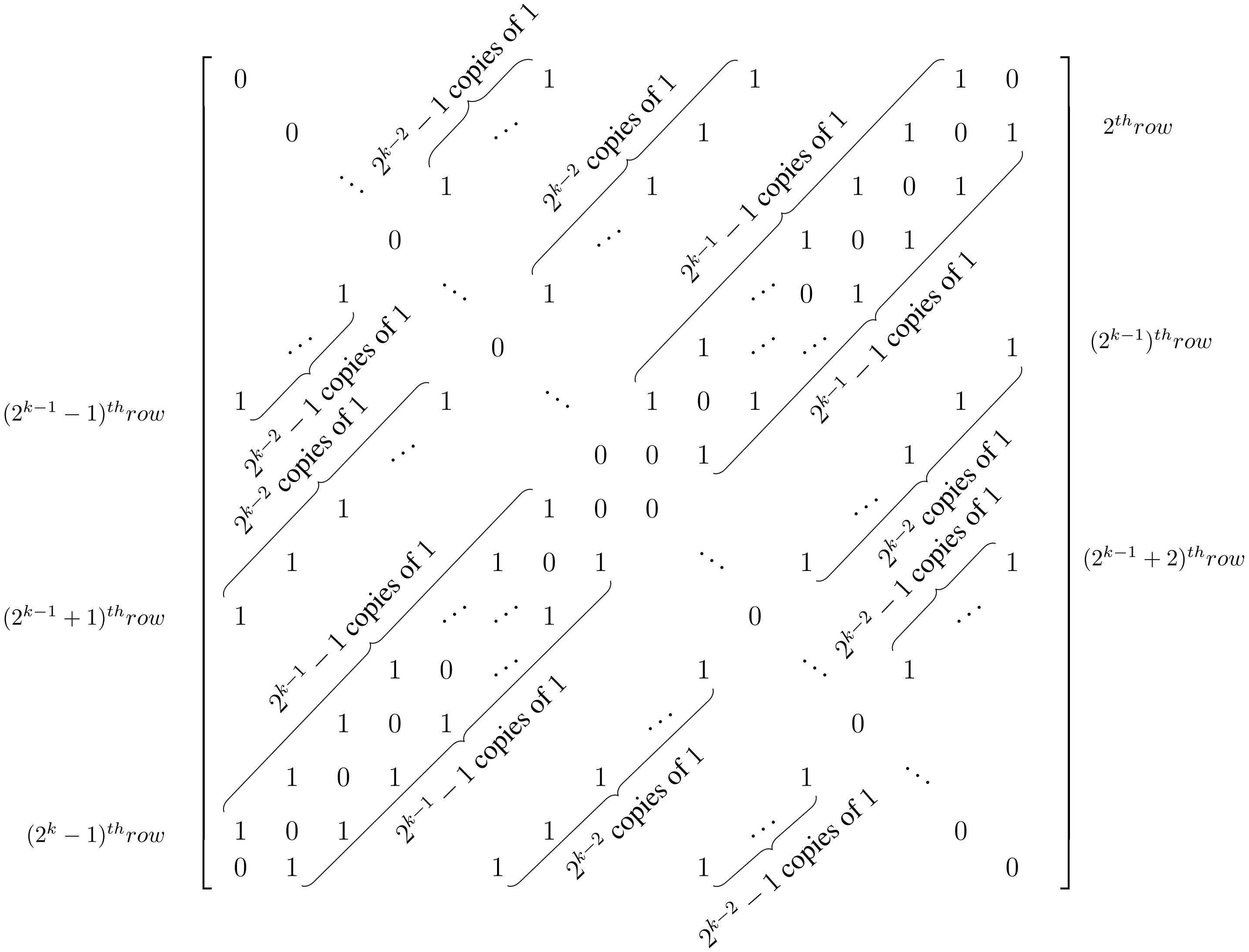
\end{array}\right]$$

Therefore, in general, $\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{2^k}))$ follows the matrix shape as follow

$$\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{2^k})) =$$



(II): For illustration, we find the adjacency matrix of $G_{N^\star}(\mathbb{Z}_{p^k})$ where $(3,1),(3,2)$ and $(7,1) \in (p,k)$, which are $\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_3))$, $\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_9))$ and $\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_7))$ as follows

$$\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_3)) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_9)) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_7)) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, in general, $\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_p))$ follows the matrix shape as follow

$$\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{p^k})) = \begin{array}{r} \\ \\ \\ \\ \\ \\ \\ \\ \\ (p^k-1)^{th}row \\ \\ \end{array} \begin{bmatrix} & & & & & & & & & 1 & 0 \\ & & & & & & & & 1 & 0 & 1 \\ & & & & & & & 1 & 0 & 1 & \\ & & & & & & 1 & 0 & 1 & & \\ & & & & & \iddots & 0 & 1 & & & \\ & & & & \iddots & \iddots & \iddots & & & & \\ & & & 1 & \iddots & \iddots & & & & & \\ & & 1 & 0 & 1 & & & & & & \\ & 1 & 0 & 1 & & & & & & & \\ 1 & 0 & 1 & & & & & & & & \\ 0 & 1 & & & & & & & & & \end{bmatrix} \begin{array}{l} \\ 2^{th}row \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}$$

$p^k - 1$ copies of 1

$p^k - 1$ copies of 1

Coincidently, it is the sum of two anti-shift matrices which can be express as $\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{p^k})) = U^\star_{p^k} + L^\star_{p^k}$ where p is a odd prime, for all $k \geq 1$.

(III): For illustration, we find the adjacency matrix of $G_{N^\star}(\mathbb{Z}_{2p^k})$ where $(3,2)$, $(5,1)$ and $(7,1) \in (p,k)$, which are $\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{18}))$, $\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{10}))$ and $\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{14}))$ as follows

$$\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{18})) = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

$$\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{10})) = \begin{bmatrix} 0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1&0&1 \\ 0&0&0&0&0&0&1&0&1&0 \\ 0&0&0&0&0&1&0&1&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0 \\ 0&0&1&0&1&0&0&0&0&0 \\ 0&1&0&1&0&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \end{bmatrix}$$

$$\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{14})) = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&0&0&0&1&0&1 \\ 0&0&0&0&0&0&0&0&0&0&1&0&1&0 \\ 0&0&0&0&0&0&0&0&0&1&0&1&0&0 \\ 0&0&0&0&0&0&0&0&1&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&1&0&0&0&0&0&0&0 \\ 0&0&0&1&0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&0&1&0&0&0&0&0&0&0&0&0 \\ 0&1&0&1&0&0&0&0&0&0&0&0&0&0 \\ 1&0&1&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

Therefore, in general, $\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{2p^k}))$ follows the matrix shape as follow

$$\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{2p^k})) = \begin{array}{r} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ (2p^k-1)^{th}row \\ \\ \end{array} \begin{bmatrix} 0 & & & & & & & & & & & & 1 & 0 \\ & 0 & & & & & & & & & & 1 & 0 & 1 \\ & & 0 & & & & & & & & 1 & 0 & 1 & \\ & & & 0 & & & & & & \iddots & \iddots & 1 & & \\ & & & & 0 & & & & \iddots & \iddots & \iddots & & & \\ & & & & & \ddots & & 1 & 0 & \iddots & & & & \\ & & & & & & 0 & 0 & 1 & & & & & \\ & & & & & 1 & 0 & 0 & & & & & & \\ & & & & \iddots & 0 & 1 & & \ddots & & & & & \\ & & & \iddots & \iddots & \iddots & & & & 0 & & & & \\ & & 1 & \iddots & \iddots & & & & & & 0 & & & \\ & 1 & 0 & 1 & & & & & & & & 0 & & \\ 1 & 0 & 1 & & & & & & & & & & 0 & \\ 0 & 1 & & & & & & & & & & & & 0 \end{bmatrix} \begin{array}{l} \\ 2^{th}row \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}$$

Theorem 4.5 *All adjacency matrix* $\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{2^k}))$, $\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_p))$ *and* $\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{2p^k}))$ *are symmetry.*

Proof: Obvious.

A matrix is said to be a block matrix if a matrix can be partitioned into several part of blocks or submatrices which are a collections of smaller matrices. For illustration, we let matrix P be a 7×7 matrix with the entries as shown below

$$P = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 5 & 6 & 7 & 8 \\ 7 & 8 & 9 & 9 & 10 & 11 & 12 \\ 1 & 2 & 3 & 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 5 & 6 & 7 & 8 \\ 7 & 8 & 9 & 9 & 10 & 11 & 12 \\ 10 & 11 & 12 & 13 & 14 & 15 & 16 \end{bmatrix}$$

P can be partitioned into submatrices P_1, P_2, P_3 and P_4 as shown below

$$P_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} P_2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix} P_3 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix} P_4 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

The partitioned P can be written as

$$P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$$

Then, a matrix is said to be a block diagonal matrix if a matrix is a $n \times n$ block matrix and having main diagonal blocks square matrices with other entries off-diagonal blocks are zero matrices. For illustration, we let Q be a 8×8 matrix with the entries as shown below

$$Q = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 3 \end{bmatrix}$$

Q can be partitioned in submatrices of one Q_1, one Q_2, one Q_3 and six Q_z which denotes zeros matrix with different size as shown below

$$Q_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} Q_2 = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix} Q_3 = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} Q_z = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The partitioned can be written as

$$\begin{bmatrix} Q_1 & Q_z & Q_z \\ Q_z & Q_2 & Q_z \\ Q_z & Q_z & Q_3 \end{bmatrix}$$

which have the general form that can be express as

$$A = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & \ddots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & A_n \end{bmatrix}, \text{ where } A_n \text{ is a square matrix of } n \times n \text{ in the main diagonal of matrix } A$$

In this project, we defined block anti-diagonal matrix as a matrix that is $n \times n$ matrix and having main anti-diagonal blocks square matrices with other entries off anti-diagonal blocks are zeros matrices which can be express in the general as

$$A' = \begin{bmatrix} 0 & 0 & 0 & A'_n \\ 0 & \iddots & \iddots & 0 \\ 0 & A'_2 & \iddots & 0 \\ A'_1 & 0 & 0 & 0 \end{bmatrix}, \text{ where } A'_n \text{ is a square matrix of } n \times n \text{ in the main anti-diagonal of matrix } A'$$

Theorem 4.6 *Let n, p, k be a positive integer, p is a odd prime and for all $k \geq 1$. For every $\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{2p^k}))$ will be a block anti-diagonal matrix.*

Proof: From theorem 4.4(III), we obtain the general form of $\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{2p^k}))$ as follow

$$\begin{array}{r c l}
 & \left[\begin{array}{cccccccccccccc}
0 & & & & & & & & & & & & 1 & 0 \\
 & 0 & & & & & & & & & & 1 & 0 & 1 \\
 & & 0 & & & & & & & & 1 & 0 & 1 & \\
 & & & 0 & & & & & & \iddots & \iddots & 1 & & \\
 & & & & 0 & & & & \iddots & \iddots & \iddots & & & \\
 & & & & & \ddots & & 1 & 0 & \iddots & & & & \\
 & & & & & & 0 & 0 & 1 & & & & & \\
 & & & & & 1 & 0 & 0 & & & & & & \\
 & & & & \iddots & 0 & 1 & & \ddots & & & & & \\
 & & & \iddots & \iddots & \iddots & & & & 0 & & & & \\
 & & 1 & \iddots & \iddots & & & & & & 0 & & & \\
 & 1 & 0 & 1 & & & & & & & & 0 & & \\
1 & 0 & 1 & & & & & & & & & & 0 & \\
0 & 1 & & & & & & & & & & & & 0
\end{array}\right] & \\
\end{array}$$

$2^{th} row$

$(2p^k - 1)^{th} row$

$p^k - 1$ copies of 1

$p^k - 1$ copies of 1

$p^k - 1$ copies of 1

$p^k - 1$ copies of 1

Coincidently, $\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{2p^k}))$ having two $p^k \times p^k$ block in the main anti-diagonal

which can form a block anti-diagonal matrix of $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ where $A = \mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{p^k}))$.
For illustration, we find the adjacency matrix of $G_{N^\star}(\mathbb{Z}_{2p^k})$ where $(3,2)$, $(5,1)$ and $(7,1) \in (p,k)$, which are $\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{18}))$, $\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{10}))$ and $\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{14}))$ as follows

$$
\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{18})) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
= \begin{bmatrix} 0 & \mathbb{M}_2(G_{N^\star}(\mathbb{Z}_9)) \\ \mathbb{M}_2(G_{N^\star}(\mathbb{Z}_9)) & 0 \end{bmatrix}
$$

$$
\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{10})) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
= \begin{bmatrix} 0 & \mathbb{M}_2(G_{N^\star}(\mathbb{Z}_5)) \\ \mathbb{M}_2(G_{N^\star}(\mathbb{Z}_5)) & 0 \end{bmatrix}
$$

$$\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{14})) = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&1\\ 0&0&0&0&0&0&0&0&0&0&1&0&1&0\\ 0&0&0&0&0&0&0&0&0&1&0&1&0&0\\ 0&0&0&0&0&0&0&0&1&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&1&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&1&0&0&0&0&0&0&0\\ 0&0&0&1&0&1&0&0&0&0&0&0&0&0\\ 0&0&1&0&1&0&0&0&0&0&0&0&0&0\\ 0&1&0&1&0&0&0&0&0&0&0&0&0&0\\ 1&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \mathbb{M}_2(G_{N^\star}(\mathbb{Z}_7)) \\ \mathbb{M}_2(G_{N^\star}(\mathbb{Z}_7)) & 0 \end{bmatrix}$$

4-3-1-5 Complete graph of $g(x)$-nil clean graph

Theorem 4.7 *Let n be a positive integer. Then the following holds for $\mathbb{Z}_n$*

(I) If $n = 2^k$, for all integer $k \geq 3$,$k \in \mathbb{Z}$, then $G_{N^\star}(\mathbb{Z}_{2^k})$ will follow the adjacency matrix of $\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{2^k}))$. In particular, $[\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{2^k}))]^{2^{k-2}-1}$ form a disconnected graph which build up of two complete graph with same size.

(II) If $n = p^k$, where p is a odd prime, for all $k \geq 1$, then $G_{N^\star}(\mathbb{Z}_{p^k})$ follow the adjacency matrix of $\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{p^k}))$. In particular, $[\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{p^k}))]^{p^k-1}$ form a disconnected graph which build up of two complete graph with different size.

(III) If $n = 2p^k$, where p is a odd prime and for all integer $k \geq 1$, then $G_{N^\star}(\mathbb{Z}_{2p^k})$ follow the adjacency matrix of $\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{2p^k}))$. In particular, $[\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{2p^k}))]^{p^k-1}$ form a disconnected graph which build up of four complete graph with different size.

Proof: (I): From theorem 4.4(I), we obtain the general form of $\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{2^k}))$ as follow

$$
\begin{array}{r}
\\ \\ \\ \\ \\ \\
(2^{k-1}-1)^{th}row \\
\\ \\ \\
(2^{k-1}+1)^{th}row \\
\\ \\ \\
(2^{k}-1)^{th}row \\
\\
\end{array}
\begin{bmatrix}
0 & & & & & & 1 & & & & 1 & & & & 1 & 0 \\
 & 0 & & & & \iddots & & & & 1 & & & & 1 & 0 & 1 \\
 & & \ddots & & 1 & & & & 1 & & & & 1 & 0 & 1 & \\
 & & & 0 & & & & \iddots & & & & 1 & 0 & 1 & & \\
 & & 1 & & \ddots & & 1 & & & & \iddots & 0 & 1 & & & \\
 & \iddots & & & & 0 & & & & 1 & \iddots & \iddots & & & & 1 \\
1 & & & & 1 & & \ddots & & 1 & 0 & 1 & & & & 1 & \\
 & & & \iddots & & & & 0 & 0 & 1 & & & & 1 & & \\
 & & 1 & & & & 1 & 0 & 0 & & & & \iddots & & & \\
 & 1 & & & & 1 & 0 & 1 & & \ddots & & 1 & & & & 1 \\
1 & & & & \iddots & \iddots & 1 & & & & 0 & & & & \iddots & \\
 & & & 1 & 0 & \iddots & & & & 1 & & \ddots & & 1 & & \\
 & & 1 & 0 & 1 & & & & \iddots & & & & 0 & & & \\
 & 1 & 0 & 1 & & & & 1 & & & & 1 & & \ddots & & \\
1 & 0 & 1 & & & & 1 & & & & \iddots & & & & 0 & \\
0 & 1 & & & & 1 & & & & 1 & & & & & & 0
\end{bmatrix}
\begin{array}{l}
\\
2^{th}row \\
\\ \\ \\
(2^{k-1})^{th}row \\
\\ \\ \\
(2^{k-1}+2)^{th}row \\
\\ \\ \\ \\ \\ \\
\end{array}
$$

As we raise the adjacency matrix to the power of $m = \{1 \le o \le 2^{k-2}-1 | k, o \in \mathbb{Z}, k \ge 3, o \text{ is odd}\}$, the anti-upper triangular of row $r_1 = \{1 \le r \le m | m, r \text{ is odd and } m, r \in \mathbb{Z}\}$, $r_2 = \{2^{k-1} - m \le r \le 2^{k-1} + m | k, m \in \mathbb{Z}, k \ge 3, \text{ and } m, r \text{ is odd }\}$ and $r_3 = \{2^k - m \le r \le 2^k - 1 | k, m \in \mathbb{Z}, k \ge 3, \text{ and } m, r \text{ is odd }\}$ and the anti-lower triangular of row $r_1' = \{2 \le r \le m+1 | m \text{ is odd}, r \text{ is even and } m, r \in \mathbb{Z}\}$, $r_2' = \{2^{k-1} - (m-1) \le r \le 2^{k-1} + (m+1) | k, m \in \mathbb{Z}, k \ge 3, m \text{ is odd }, r \text{ is even}\}$ and $r_3' = \{2^k - (m-1) \le r \le 2^k | k, m \in \mathbb{Z}, k \ge 3, m \text{ is odd }, r \text{ is even}\}$ will contains all positive integers. Since loops and multiple edges are not in our consideration, so entries $a_{i,i} = 0$ and $a_{i,j} = 1$ for all $a_{i,j} \ge 1$, for all $1 \le i \ne j \le 2^k$, otherwise 0. In short, as $m = \{1 \le o \le 2^{k-2} - 1 | k, o \in \mathbb{Z}, k \ge 3, o \text{ is odd }\}$, then

$$\text{row of anti-upper triangular} \begin{cases} 1 & , 1 \le r \le m \\ 1 & , 2^{k-1} - m \le r \le 2^{k-1} + m \\ 1 & , 2^k - m \le r \le 2^k - 1 \\ 0 & , otherwise. \end{cases}$$

$$\text{row of anti-lower triangular} \begin{cases} 1 & , 2 \le r \le m+1 \\ 1 & , 2^{k-1} - (m-1) \le r \le 2^{k-1} + (m+1) \\ 1 & , 2^k - (m-1) \le r \le 2^k \\ 0 & , otherwise. \end{cases}$$

Eventually, $[\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{2^k}))]^{2^{k-2}-1}$ will have a form as follow
After $[\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{2^k}))]^{2^{k-2}-1}$ is form, if we construct the graph from $[\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{2^k}))]^{2^{k-2}-1}$, it depicted a disconnected graph which build up of two $K_{2^{k-1}}$.

Example 6 we consider $n = 2^k$, $k = 4$, so $[\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{16}))]^3$ eventually depicts a disconnected graph which build up of two K_8.
First, we find the adjacency matrix of $G_{N^\star}(\mathbb{Z}_{16})$, $\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{16}))$, then we raise it to the power 3.

$$\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{16})) = \begin{bmatrix} 0&0&0&0&0&0&1&0&1&0&0&0&0&0&1&0 \\ 0&0&0&0&0&1&0&1&0&0&0&0&0&1&0&1 \\ 0&0&0&0&1&0&1&0&0&0&0&0&1&0&1&0 \\ 0&0&0&0&0&1&0&0&0&0&0&1&0&1&0&0 \\ 0&0&1&0&0&0&0&0&0&0&1&0&1&0&0&0 \\ 0&1&0&1&0&0&0&0&0&1&0&1&0&0&0&0 \\ 1&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&1 \\ 1&0&0&0&0&0&1&0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&1&0&1&0&0&0&0&0&1&0&1 \\ 0&0&0&0&1&0&1&0&0&0&0&0&1&0&1&0 \\ 0&0&0&1&0&1&0&0&0&0&0&0&0&1&0&0 \\ 0&0&1&0&1&0&0&0&0&0&1&0&0&0&0&0 \\ 0&1&0&1&0&0&0&0&0&1&0&1&0&0&0&0 \\ 1&0&1&0&0&0&0&0&1&0&1&0&0&0&0&0 \\ 0&1&0&0&0&0&0&1&0&1&0&0&0&0&0&0 \end{bmatrix}$$

$$[\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{16}))]^3 = \begin{bmatrix} 0&0&1&0&1&0&1&0&1&0&1&0&1&0&1&0\\ 0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&1\\ 1&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0\\ 0&1&0&0&0&1&0&1&0&1&0&1&0&1&0&1\\ 1&0&1&0&0&0&1&0&1&0&1&0&1&0&1&0\\ 0&1&0&1&0&0&0&1&0&1&0&1&0&1&0&1\\ 1&0&1&0&1&0&0&0&1&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&0&0&1&0&1&0&1&0&1\\ 1&0&1&0&1&0&1&0&0&0&1&0&1&0&1&0\\ 0&1&0&1&0&1&0&1&0&0&0&1&0&1&0&1\\ 1&0&1&0&1&0&1&0&1&0&0&0&1&0&1&0\\ 0&1&0&1&0&1&0&1&0&1&0&0&0&1&0&1\\ 1&0&1&0&1&0&1&0&1&0&1&0&0&0&1&0\\ 0&1&0&1&0&1&0&1&0&1&0&1&0&0&0&1\\ 1&0&1&0&1&0&1&0&1&0&1&0&1&0&0&0\\ 0&1&0&1&0&1&0&1&0&1&0&1&0&1&0&0 \end{bmatrix}$$

Then, we construct the graph from $[\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{16}))]^3$ as follow

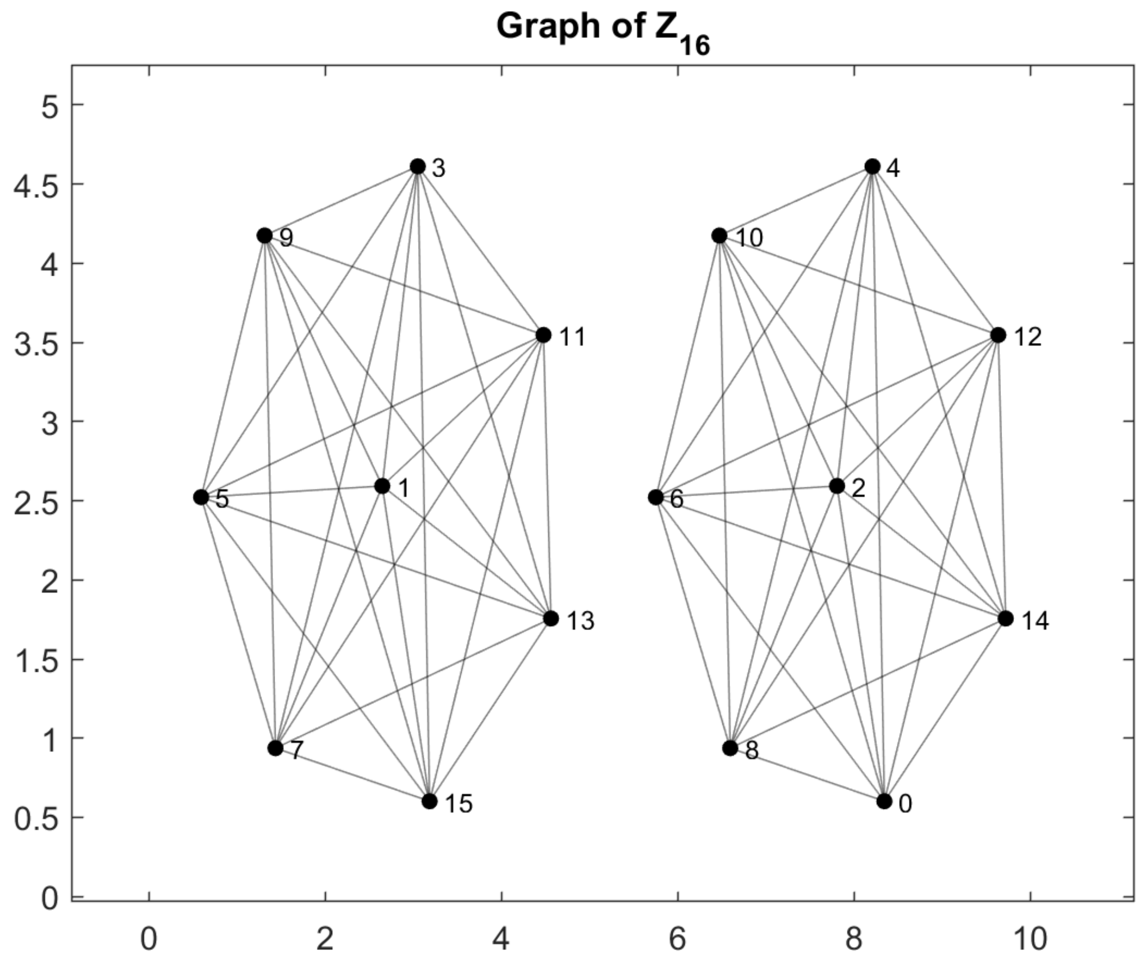


Figure 26: disconnected graph which build up of two K_8

Next, we illustrate another example on $G_{N^\star}(\mathbb{Z}_{32})$ below.

Example 7 we consider $n = 2^k$, $k = 5$, so $[\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{32}))]^7$ eventually depicts a disconnected graph which build up of two K_{16}.

First, we find the adjacency matrix of $G_{N^\star}(\mathbb{Z}_{32})$, $\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{32}))$, then we raise it to the power 7.

$$\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{32})) = \begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}$$

$$[\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{32}))]^3 = \begin{bmatrix}
0&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&1\\
1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1\\
0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0\\
0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0\\
0&0&0&0&0&1&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0\\
0&0&0&0&1&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0\\
0&0&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0\\
0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0\\
0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0\\
1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0\\
0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1\\
1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0\\
0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1\\
1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0\\
0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1\\
1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1\\
0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0\\
0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0\\
0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&0&0\\
0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&1&0&0&0&0\\
0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&1&0&0&0&0&0\\
0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&0&0&0&0&0&0&0\\
0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0\\
0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0\\
1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0\\
0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1\\
1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&1&0&0
\end{bmatrix}$$

$$[\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{32}))]^5 = \begin{bmatrix}
0&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0\\
0&0&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0&1\\
1&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0\\
0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&1\\
1&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0\\
0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1\\
0&0&0&0&1&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0\\
0&0&0&1&0&1&0&0&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&0\\
0&0&1&0&1&0&1&0&0&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&0&0\\
0&1&0&1&0&1&0&1&0&0&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&0&0&0\\
1&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0\\
0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1\\
1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0\\
0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&1\\
1&0&1&0&1&0&1&0&0&0&0&0&1&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0\\
0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&0&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1\\
1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0&0&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0\\
0&1&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&1&0&0&0&0&0&1&0&1&0&1&0&1\\
1&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0\\
0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&1\\
1&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0\\
0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1\\
0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&0&0&1&0&1&0&1&0&1&0\\
0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&0&0&1&0&1&0&1&0&0\\
0&0&1&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0&0&0&1&0&1&0&0&0\\
0&1&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&1&0&0&0&0\\
1&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0\\
0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1\\
1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0\\
0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&1\\
1&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&0&0\\
0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&1&0&1&0&0
\end{bmatrix}$$

$$[\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{32}))]^7 = \begin{bmatrix}
0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0
\end{bmatrix}$$

Then, we construct the graph from $[\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{32}))]^7$ as follow

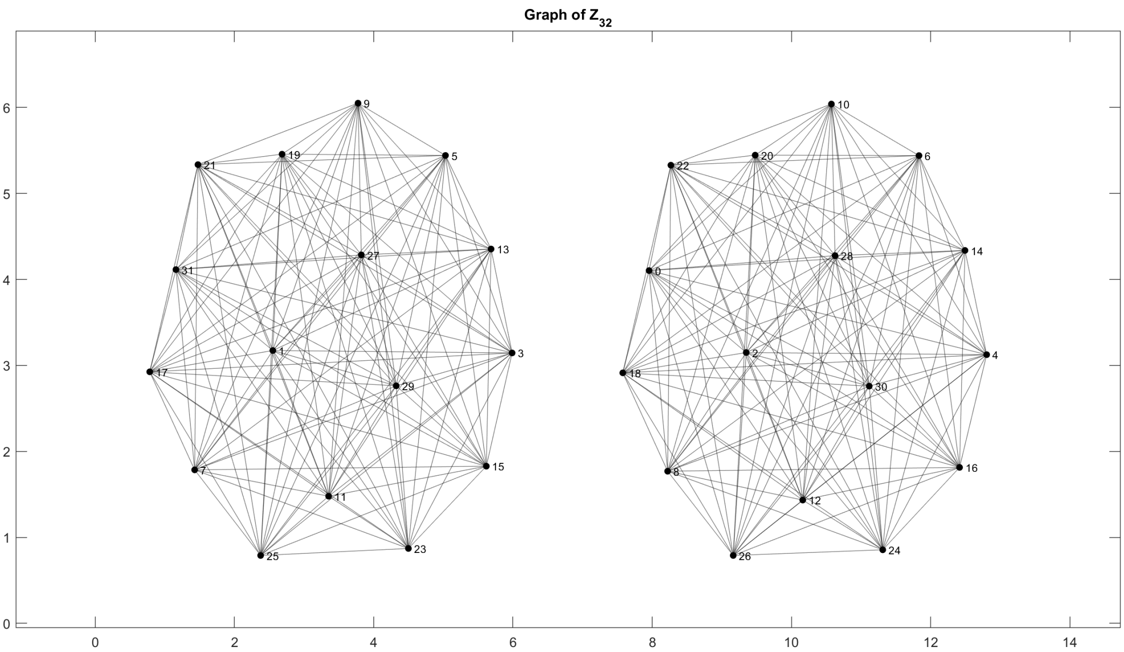


Figure 27: disconnected graph which build up of two K_{16}

(II): From theorem 4.4(II), we obtain the general form of $\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{p^k}))$ as follow

$$\begin{array}{r c l} (p^k-1)^{th}row & \begin{bmatrix} & & & & & & & & & 1 & 0 \\ & & & & & & & & 1 & 0 & 1 \\ & & & & & & & 1 & 0 & 1 & \\ & & & & & & 1 & 0 & 1 & & \\ & & & & & \iddots & 0 & 1 & & & \\ & & & & \iddots & \iddots & \iddots & & & & \\ & & & 1 & \iddots & \iddots & & & & & \\ & & 1 & 0 & 1 & & & & & & \\ & 1 & 0 & 1 & & & & & & & \\ 1 & 0 & 1 & & & & & & & & \\ 0 & 1 & & & & & & & & & \end{bmatrix} & 2^{th}row \end{array}$$

As we raise the adjacency matrix to the power of $n = \{2 \leq e \leq p^k - 1 | k \geq 1, p$ is prime, e is even and $e, p, k \in \mathbb{Z}\}$, the lower triangular of row $r_1 = \{1 \leq r \leq n + 1 | n$ is even, r is odd and $r, n \in \mathbb{Z}\}$ and upper triangular of row $r_1' = \{p^k - n \leq r \leq p^k | p$ is prime, n is even, r is odd, $k \geq 1$ and $p, k, n, r \in \mathbb{Z}\}$ will contains all positive integers. Since loops and multiple edges are not in our consideration, so entries $a_{i,i} = 0$ and $a_{i,j} = 1$ for all $a_{i,j} \geq 1$, for all $1 \leq i \neq j \leq p$, otherwise 0. Since the diagonal of the even power raised adjacency matrix will only contain zeros, then we can further conclude that the lower triangular of row $r_1 = \{3 \leq r \leq n + 1 | n$ is even, r is odd and $r, n \in \mathbb{Z}\}$ and upper triangular of row $r_1' = \{p^k - n \leq r \leq p^k - 2 | p$ is prime, n is even, r is odd and $p, k, n, r \in \mathbb{Z}\}$ will contains all ones. In short, as $n = \{2 \leq e \leq p^k - 1 | p$ is prime, e is even and $e, p, k \in \mathbb{Z}\}$, then

$$\text{row of lower triangular} \begin{cases} 1 & , 3 \leq r \leq n+1 \\ 0 & , otherwise. \end{cases}$$

$$\text{row of upper triangular} \begin{cases} 1 & , p^k - n \leq r \leq p^k - 2 \\ 0 & , otherwise. \end{cases}$$

After $[\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{p^k}))]^{p^k-1}$ is form, if we construct the graph from $[\mathbb{M}_1(G_{N^\star}(\mathbb{Z}_{p^k}))]^{p^k-1}$, it depicted a disconnected graph which build up of one $K_{\frac{p^k+1}{2}}$ and one $K_{\frac{p^k+1}{2}-1}$.

Example 8 we consider $n = p^k$, where $p = 3$ and $k = 2$, so $[\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_9))]^8$ eventually depicts a disconnected graph which build up of one K_5 and one K_4.

First, we find the adjacency matrix of $G_{N^\star}(\mathbb{Z}_9)$, $\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_9))$, then we raise it to the power 8.

$$\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_9)) = \begin{bmatrix} 0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1&0&1\\ 0&0&0&0&0&1&0&1&0\\ 0&0&0&0&1&0&1&0&0\\ 0&0&0&1&0&1&0&0&0\\ 0&0&1&0&1&0&0&0&0\\ 0&1&0&1&0&0&0&0&0\\ 1&0&1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0 \end{bmatrix}$$

$$[\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_9))]^2 = \begin{bmatrix} 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0\\ 1&0&0&0&1&0&0&0&0\\ 0&1&0&0&0&1&0&0&0\\ 0&0&1&0&0&0&1&0&0\\ 0&0&0&1&0&0&0&1&0\\ 0&0&0&0&1&0&0&0&1\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&0 \end{bmatrix}$$

$$[\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_9))]^4 = \begin{bmatrix} 0&0&1&0&1&0&0&0&0\\ 0&0&0&1&0&1&0&0&0\\ 1&0&0&0&1&0&1&0&0\\ 0&1&0&0&0&1&0&1&0\\ 1&0&1&0&0&0&1&0&1\\ 0&1&0&1&0&0&0&1&0\\ 0&0&1&0&1&0&0&0&1\\ 0&0&0&1&0&1&0&0&0\\ 0&0&0&0&1&0&1&0&0 \end{bmatrix}$$

$$[\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_9))]^6 = \begin{bmatrix} 0&0&1&0&1&0&1&0&0\\ 0&0&0&1&0&1&0&1&0\\ 1&0&0&0&1&0&1&0&1\\ 0&1&0&0&0&1&0&1&0\\ 1&0&1&0&0&0&1&0&1\\ 0&1&0&1&0&0&0&1&0\\ 1&0&1&0&1&0&0&0&1\\ 0&1&0&1&0&1&0&0&0\\ 0&0&1&0&1&0&1&0&0 \end{bmatrix}$$

$$[\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_9))]^8 = \begin{bmatrix} 0&0&1&0&1&0&1&0&1\\ 0&0&0&1&0&1&0&1&0\\ 1&0&0&0&1&0&1&0&1\\ 0&1&0&0&0&1&0&1&0\\ 1&0&1&0&0&0&1&0&1\\ 0&1&0&1&0&0&0&1&0\\ 1&0&1&0&1&0&0&0&1\\ 0&1&0&1&0&1&0&0&0\\ 1&0&1&0&1&0&1&0&0 \end{bmatrix}$$

Then, we construct the graph from $[\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_9))]^8$ as follow

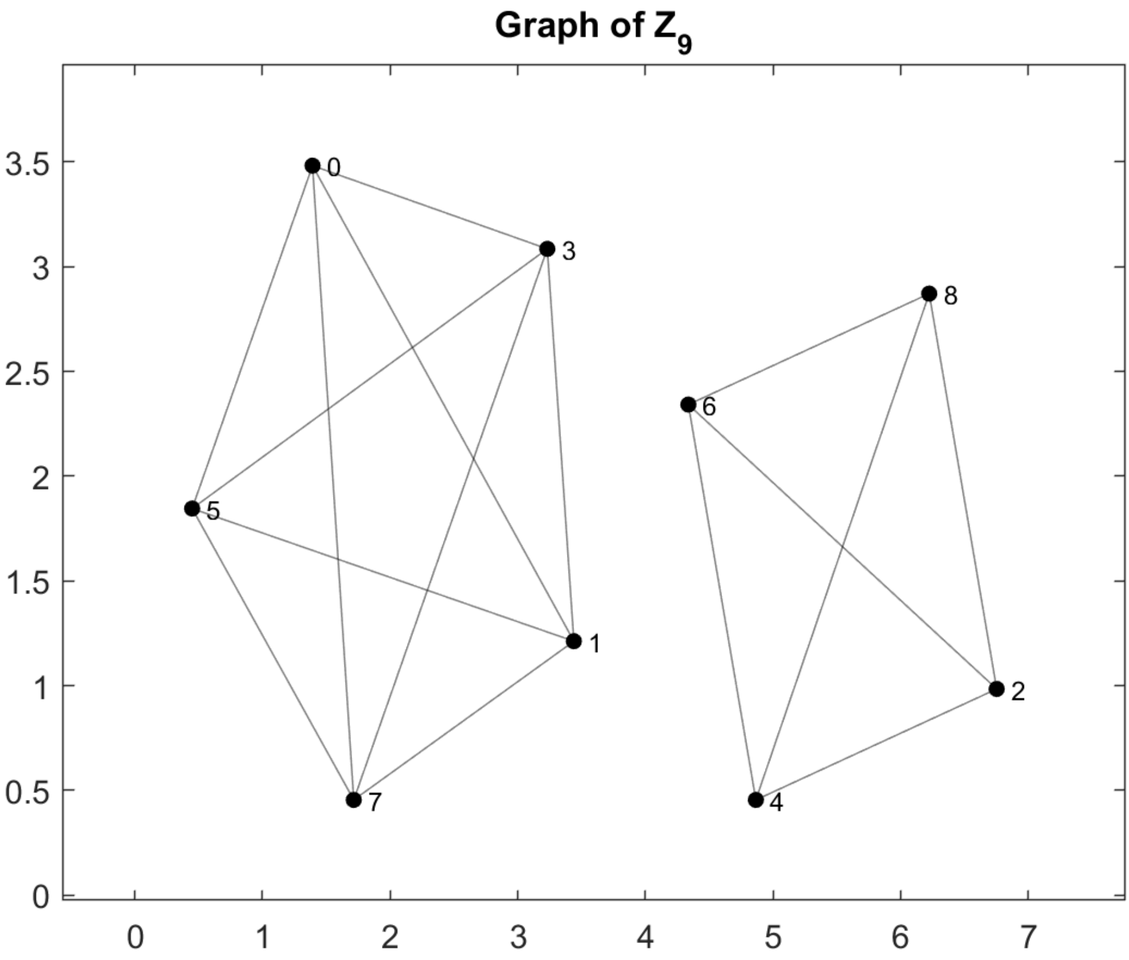


Figure 28: disconnected graph which build up of one K_5 and one K_4

Next, we illustrate another example on $G_{N^\star}(\mathbb{Z}_{11})$ below.

Example 9 we consider $n = p$, where $p = 11$ and $k = 1$, so $[\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{11}))]^{10}$ eventually depicts a disconnected graph which build up of one K_6 and one K_5.

First, we find the adjacency matrix of $G_{N^\star}(\mathbb{Z}_{11})$, $\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{11}))$, then we raise it to the power 10.

$$\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{11})) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{11}))]^2 = \begin{bmatrix} 0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0\\ 1&0&0&0&1&0&0&0&0&0&0\\ 0&1&0&0&0&1&0&0&0&0&0\\ 0&0&1&0&0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0&0&1&0&0\\ 0&0&0&0&0&1&0&0&0&1&0\\ 0&0&0&0&0&0&1&0&0&0&1\\ 0&0&0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&1&0&0 \end{bmatrix}$$

$$[\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{11}))]^4 = \begin{bmatrix} 0&0&1&0&1&0&0&0&0&0&0\\ 0&0&0&1&0&1&0&0&0&0&0\\ 1&0&0&0&1&0&1&0&0&0&0\\ 0&1&0&0&0&1&0&1&0&0&0\\ 1&0&1&0&0&0&1&0&1&0&0\\ 0&1&0&1&0&0&0&1&0&1&0\\ 0&0&1&0&1&0&0&0&1&0&1\\ 0&0&0&1&0&1&0&0&0&1&0\\ 0&0&0&0&1&0&1&0&0&0&1\\ 0&0&0&0&0&1&0&1&0&0&0\\ 0&0&0&0&0&0&1&0&1&0&0 \end{bmatrix}$$

$$[\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{11}))]^6 = \begin{bmatrix} 0&0&1&0&1&0&1&0&0&0&0\\ 0&0&0&1&0&1&0&1&0&0&0\\ 1&0&0&0&1&0&1&0&1&0&0\\ 0&1&0&0&0&1&0&1&0&1&0\\ 1&0&1&0&0&0&1&0&1&0&1\\ 0&1&0&1&0&0&0&1&0&1&0\\ 1&0&1&0&1&0&0&0&1&0&1\\ 0&1&0&1&0&1&0&0&0&1&0\\ 0&0&1&0&1&0&1&0&0&0&1\\ 0&0&0&1&0&1&0&1&0&0&0\\ 0&0&0&0&1&0&1&0&1&0&0 \end{bmatrix}$$

$$[\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{11}))]^8 = \begin{bmatrix} 0&0&1&0&1&0&1&0&1&0&0\\ 0&0&0&1&0&1&0&1&0&1&0\\ 1&0&0&0&1&0&1&0&1&0&1\\ 0&1&0&0&0&1&0&1&0&1&0\\ 1&0&1&0&0&0&1&0&1&0&1\\ 0&1&0&1&0&0&0&1&0&1&0\\ 1&0&1&0&1&0&0&0&1&0&1\\ 0&1&0&1&0&1&0&0&0&1&0\\ 1&0&1&0&1&0&1&0&0&0&1\\ 0&1&0&1&0&1&0&1&0&0&0\\ 0&0&1&0&1&0&1&0&1&0&0 \end{bmatrix}$$

$$[\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{11}))]^{10} = \begin{bmatrix} 0&0&1&0&1&0&1&0&1&0&1\\ 0&0&0&1&0&1&0&1&0&1&0\\ 1&0&0&0&1&0&1&0&1&0&1\\ 0&1&0&0&0&1&0&1&0&1&0\\ 1&0&1&0&0&0&1&0&1&0&1\\ 0&1&0&1&0&0&0&1&0&1&0\\ 1&0&1&0&1&0&0&0&1&0&1\\ 0&1&0&1&0&1&0&0&0&1&0\\ 1&0&1&0&1&0&1&0&0&0&1\\ 0&1&0&1&0&1&0&1&0&0&0\\ 1&0&1&0&1&0&1&0&1&0&0 \end{bmatrix}$$

Then, we construct the graph from $[\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{11}))]^{10}$ as follow

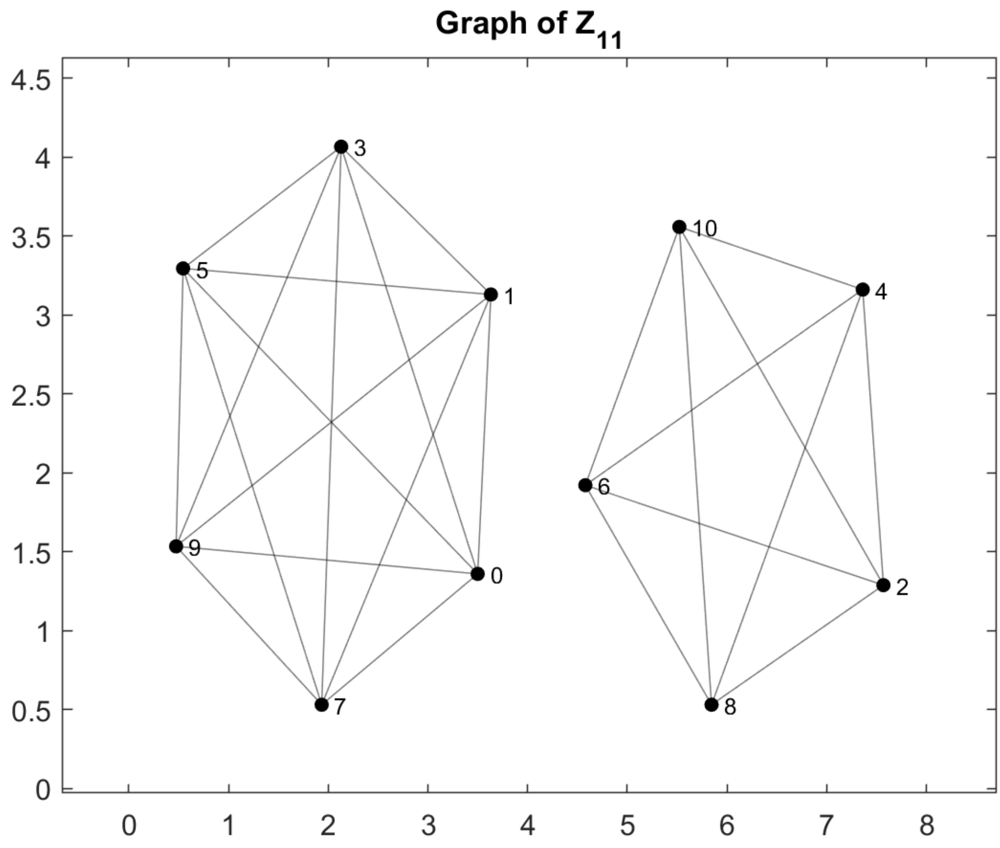


Figure 29: disconnected graph which build up of one K_6 and one K_5

(III): From theorem 4.4(III), we obtain the general form of $\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{2p^k}))$ as follow

$$
\begin{array}{r}
\\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\
(2p^k-1)^{th}row \\
\\
\end{array}
\left[\begin{array}{cccccccccccccc}
0 & & & & & & & & & & & & 1 & 0 \\
& 0 & & & & & & & & & & 1 & 0 & 1 \\
& & 0 & & & & & & & & 1 & 0 & 1 & \\
& & & 0 & & & & & & \iddots & \iddots & 1 & & \\
& & & & 0 & & & & \iddots & \iddots & \iddots & & & \\
& & & & & \ddots & & 1 & 0 & \iddots & & & & \\
& & & & & & 0 & 0 & 1 & & & & & \\
& & & & & 1 & 0 & 0 & & & & & & \\
& & & & \iddots & 0 & 1 & & \ddots & & & & & \\
& & & \iddots & \iddots & \iddots & & & & 0 & & & & \\
& & 1 & \iddots & \iddots & & & & & & 0 & & & \\
& 1 & 0 & 1 & & & & & & & & 0 & & \\
1 & 0 & 1 & & & & & & & & & & 0 & \\
0 & 1 & & & & & & & & & & & & 0
\end{array}\right]
\begin{array}{l}
\\
2^{th}row \\
\\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\
\end{array}
$$

From theorem 4.6, we also know that $\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{2p^k}))$ is a block anti-diagonal matrix which can be viewed as $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ where $A = \mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{p^k}))$. In fact, as we raise $\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{2p^k}))$ to even power, we obtain the block diagonal matrix as follow

$$\begin{aligned}
[\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{2p^k}))]^2 &= \mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{2p^k})) \times \mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{2p^k})) \\
&= \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix} \times \begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix} \\
&= \begin{bmatrix} A^2 & 0 \\ 0 & A^2 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
[\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{2p^k}))]^4 &= [\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{2p^k}))]^2 \times [\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{2p^k}))]^2 \\
&= \begin{bmatrix} A^2 & 0 \\ 0 & A^2 \end{bmatrix} \times \begin{bmatrix} A^2 & 0 \\ 0 & A^2 \end{bmatrix} \\
&= \begin{bmatrix} A^4 & 0 \\ 0 & A^4 \end{bmatrix}
\end{aligned}$$

$$\vdots$$

$$[\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{2p^k}))]^n = \begin{bmatrix} A^n & 0 \\ 0 & A^n \end{bmatrix}, \text{where } n \in \mathbb{Z},\, n \geq 2 \text{ and } n \text{ must be even}$$

Since, $p^k - 1$ is always even and we know the general calculation of $(\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{2p^k})))^{p^k-1}$ from theorem 4.7(II), so, we know that

$$\begin{aligned}
[\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{2p^k}))]^{p^k-1} &= \begin{bmatrix} A^{p^k-1} & 0 \\ 0 & A^{p^k-1} \end{bmatrix} \\
&= \begin{bmatrix} (\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{p^k})))^{p^k-1} & 0 \\ 0 & (\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{2p^k})))^{p^k-1} \end{bmatrix} \qquad (4.6a)
\end{aligned}$$

After $[\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{2p^k}))]^{p^k-1}$ is form, if we construct the graph from $[\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{2p^k}))]^{p^k-1}$, it depicted a disconnected graph which build up of two $K_{\frac{p^k+1}{2}}$ and two $K_{\frac{p^k+1}{2}-1}$.

Example 10 We consider $n = 2p^k$, where $p = 3$ and $k = 2$, so $[\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{18}))]^8$ eventually depicts a disconnected graph which build up of two K_5 and two K_4.
From example 8, we know that

$$[\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_9))]^8 = \begin{bmatrix} 0&0&1&0&1&0&1&0&1\\ 0&0&0&1&0&1&0&1&0\\ 1&0&0&0&1&0&1&0&1\\ 0&1&0&0&0&1&0&1&0\\ 1&0&1&0&0&0&1&0&1\\ 0&1&0&1&0&0&0&1&0\\ 1&0&1&0&1&0&0&0&1\\ 0&1&0&1&0&1&0&0&0\\ 1&0&1&0&1&0&1&0&0 \end{bmatrix}$$

So, we can obtain $[\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{18}))]^8$ by using $(4.6a)$

$$[\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{18}))]^8 = \begin{bmatrix} (\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_9)))^8 & 0 \\ 0 & (\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_9)))^8 \end{bmatrix}$$

$$= \begin{bmatrix} 0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0\\ 1&0&0&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0\\ 1&0&1&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0\\ 0&1&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\ 1&0&1&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0\\ 0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\ 1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&1&0&1\\ 0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&1&0\\ 0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&1&0&1\\ 0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0 \end{bmatrix}$$

Then, we construct the graph from $[\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{18}))]^8$ as follow

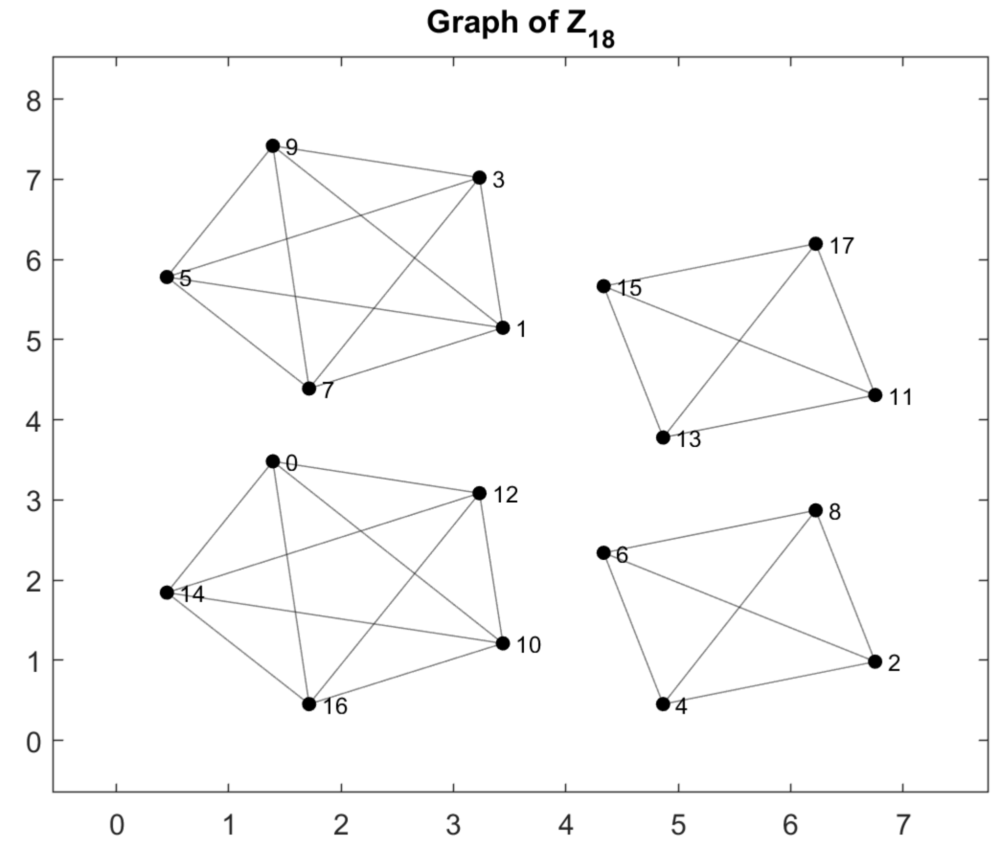


Figure 30: disconnected graph that build up of two K_5 and two K_4

Next, we illustrate another example on $G_{N^\star}(\mathbb{Z}_{22})$ below

Example 11 We consider $n = 2p^k$, where $p = 11$ and $k = 1$, so $[\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{22}))]^{10}$ eventually depicts a disconnected graph which build up of two K_6 and two K_5.
From example 9, we know that

$$[\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{11}))]^{10} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$$

So, we can obtain $[\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{22}))]^{10}$ by using $(4.6a)$

$$[\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{22}))]^{10} = \begin{bmatrix} (\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{11})))^{10} & 0 \\ 0 & (\mathbb{M}_2(G_{N^\star}(\mathbb{Z}_{11})))^{10} \end{bmatrix}$$

$$= \left[\begin{array}{cccccccccccccccccccccc}
0&0&1&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&0&0&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&1&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&0&1&0&0&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&1&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&0&1&0&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&0&1&0&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&1&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&1&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0&1&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&0&0&1&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&1&0&1&0&1&0&0
\end{array}\right]$$

Then, we construct the graph from $[\mathbb{M}_3(G_{N^\star}(\mathbb{Z}_{22}))]^{10}$ as follow

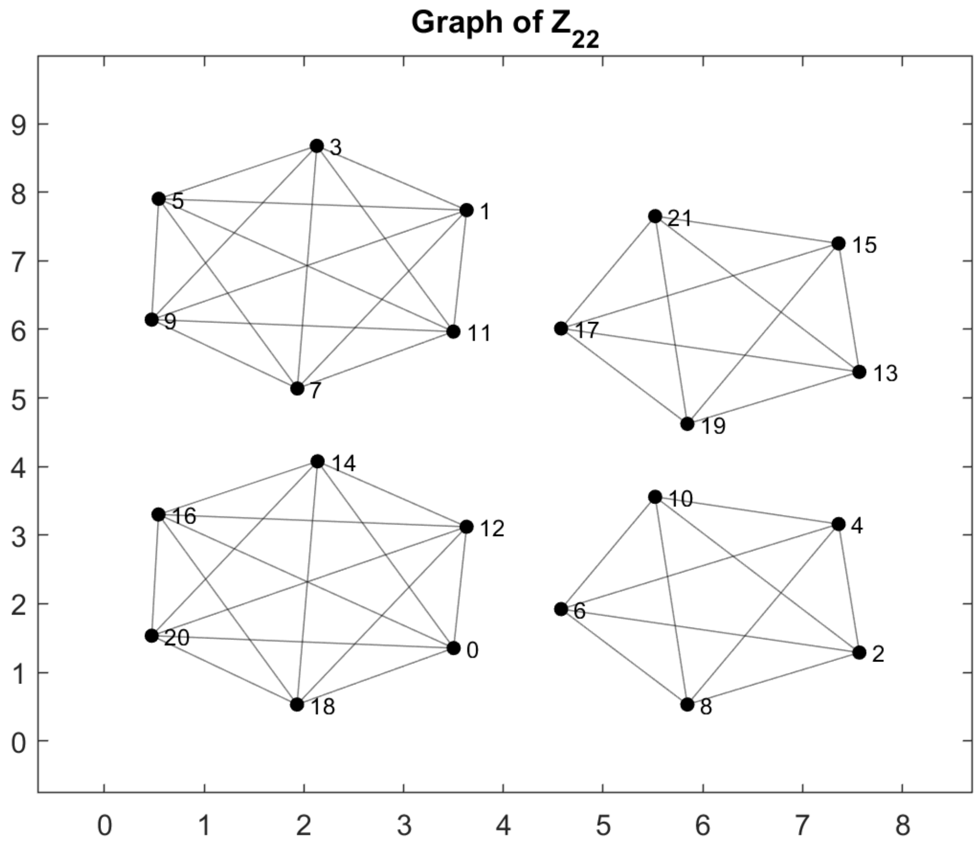


Figure 31: disconnected graph that build up of two K_6 and two K_5

CHAPTER 5: CONCLUSION

In conclusion, the highlight of this project is to study and apply the proving method to another types of rings. In the Project I, we have investigated on the published paper by Basnet(2017) on nil clean graph of rings. In the process of investigation, we have learned thoroughly about the theorems and lemmas stated with the proving methods used to explained it. In Chapter 3, we took some theorems and lemmas from Basnet(2017) on nil clean graph of rings and explained in details in order to understand the way of proving done by Basnet(2017). The theorems and lemmas that we is explained in Chapter 3 is about the girth of graphs, chromatic index of graphs and diameter of graphs which is strongly related to our ring which have a different structure from the nil clean graph of rings.

In Project II, we are able to extend from Basnet(2017) on nil clean graph of rings to $x(x-2)$-nil clean graph of rings which have a different structure than than nil clean graph of rings. With the help of those existing theorems and lemmas in the published paper by Basnet(2017), we are able to form our own theorems and lemmas with the proving methods learned from the paper that are presented in Chapter 4. By making Basnet(2017) as our main reference, we have form our own theorem on connectedness of graphs, completeness of graphs, hamiltonian cycles and paths of graphs and diameter of graphs which strongly describe our $x(x-2)$-nil clean graph of rings. Besides, we form our own theorems in generalizing the adjacency matrix on the $x(x-2)$-nil clean graph of rings which is not presented in the paper of Basnet(2017).

As the main results of this project, we can conclude that our proving methods is just one of the possible way to prove the theorem but not in general. However, there should be another way that can be used more effectively in proving those theorems and lemmas in the future.

REFERENCE

Ashraf N., Maimani H.R., Pournaki M.R., and Yassemi S., *Unit graphs associated with rings*, Communications in Algebra, 38: 2851–2871, 2010.

Basnet D.K., and Bhattacharyya J., *Nil clean graphs of rings*, Algebra Colloquium, 24: 481–492, 2017.

Basnet D.K., and Bhattacharyya J., *Weakly nil clean graphs of rings*, preprint.

Beck I., *Coloring of commutative rings*. Journal of algebra, 116, 208–226, 1988.

Danchev P.V., and McGovern W., *Commutative weakly nil clean unital rings*. Journal of Algebra, 425, 410–422, 2015.

Diesl A.J., *Nil clean rings*. Journal of Algebra, 383: 197–211, 2013.

Fan L., and Yang X., *On rings whose elements are the sum of a unit and a root of a fixed polynomial*, Communications in Algebra, 36.1: 269–278, 2008.

Han J., and Nicholson W.K., *Extensions of clean rings*, Communications in Algebra, 29: 2589–2595, 2001.

Nicholson W.K., *Lifting idempotents and exchange rings*, Transactions of the America Mathematical Society, 229: 269–278, 1977.

Wang Z., and Chen J., *A note on clean rings*, Algebra Colloquium, 14: 537–540, 2007.

APPENDIX

single_graph_generator.m

```
nodes_num = 3; % define the number of the nodes
Z = 1:nodes_num;
from =[];
to = [];
x = calc(Z,2) % return an array consists of result from X^2=2X
for x1 = Z
    for x2 = Z
        if ismember(mod(x1+x2,nodes_num),x)==1 && x1~=x2
            from = [from, x1];
            to = [to, x2];
        end
    end
end
G = simplify(graph(from,to));
h = plot(G,'k','Layout','force'); % graph generated
Adj = full(adjacency(G)); %adjacency matrix of graph generated
labelnode(h,nodes_num,{'0'});
str = strcat('Graph␣of␣','␣\bf{Z}_','{',num2str(nodes_num),'}');
title(str);

%% finding path or cycle within the graph generated
path_e = [];
path_o = [];
odd = [];
even = [];

if mod(nodes_num,2) == 0
    for j = 1:nodes_num
        if mod(j,2)==0
            even = [even,j];
        else
            odd = [odd,j];
        end
    end
```

```
    even_f = fliplr(even);
    odd_f = fliplr(odd);
    for v=1:length(even)
        drawnow;
        path_o = [path_o,[odd(v),odd_f(v)]];
        highlight(h,path_o,'EdgeColor','k','NodeColor','k','LineWidth',3)
        path_e = [path_e,[even_f(v),even(v)]];
        highlight(h,path_e,'EdgeColor','k','NodeColor','k','LineWidth',3)
    end
    drawnow;
    path_o = [path_o,
        [min(odd_f(1:((nodes_num/2)/2))),
        min(odd(1:((nodes_num/2)/2)))]];

    highlight(h,path_o,'EdgeColor','k','NodeColor','k','LineWidth',3)
    path_e = [path_e,
        [max(even(1:((nodes_num/2)/2))),
        max(even_f(1:((nodes_num/2)/2)))]];

    highlight(h,path_e,'EdgeColor','k','NodeColor','k','LineWidth',3)
else
    for j = 1:nodes_num
        if mod(j,2)==0
            even = [even,j];
        else
            odd = [odd,j];
        end
    end
    even = fliplr(even);
    for v=1:length(even)
        drawnow;
        path_o = [path_o,[odd(v),even(v)]];
        highlight(h,path_o,'EdgeColor','k','NodeColor','k','LineWidth',3)
    end
    drawnow;
    path_o = [path_o,[2,nodes_num]]
    highlight(h,path_o,'EdgeColor','k','NodeColor','k','LineWidth',3)
end
```

calc.m

```
function condition = calc(arr,number)

arr_square = mod((arr-1).^2,length(arr));
arr_2x = mod((arr-1)*number,length(arr));
same = (arr_square == arr_2x);
pos = find(same==1) ;

condition = pos-1;
```