NEW APPROACH FOR FINDING PERFORMANCE MEASURES OF CONTINUOUS-TIME SERVER QUEUE WITH NEGATIVE CUSTOMERS

By

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ABSTRACT

NEW APPROACH FOR FINDING PERFORMANCE MEASURES OF CONTINUOUS-TIME SERVER QUEUE WITH NEGATIVE CUSTOMERS

Chin Ching Herny

A single-server continuous-time queue that adopt first come first serve (FCFS) queueing discipline with negative customers is studied. The arrival of a negative customer in the queue will remove one positive customer at the head if the system is not empty (RCH) and only positive customers will receive service. In this research, a fairly general queueing model with negative customers that can represent more wide applications in real world is solved. An alternative approach will be applied to derive a set of equations which is using to find the stationary queue length distributions of this model. In the alternative numerical approach, interarrival time and/or service time distributions of the positive customers are assumed to have Constant Asymptotic Rate (*CAR*) when time t goes to infinity. Whereas negative customer arrives to the system according to a Poisson process. Expressions will also be derived analytically to find the stationary queue length distribution for the *M*/*M*/1, *M*/*CAR*/1, *GI*/*M*/1 and *CAR*/*CAR*/1 queues with negative customers. The stationary probabilities found from the alternative and analytical approaches are used to find the waiting time distribution. Results computed by both the proposed numerical and analytical methods are compared and discussed. All the results will be verified by those obtained from the simulation procedure.

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APPROVAL SHEET

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SUBMISSION OF DISSERTATION

It is hereby certified that **CHIN CHING HERNY** (ID No: **16UEM01639**) has completed this dissertation entitled "NEW APPROACH FOR FINDING PERFORMANCE MEASURES OF CONTINUOUS-TIME SERVER QUEUE WITH NEGATIVE CUSTOMERS" under the supervision of Dr. Koh Siew Khew (Supervisor) and Dr. Goh Yong Kheng (Co-supervisor) from the Department of Mathematical and Actuarial Sciences, Lee Kong Chian Faculty of Engineering and Science, Universiti Tunku Abdul Rahman and Dr. Tan Yi Fei (External Co-Supervisor) from Faculty of Engineering, Multimedia University.

I understand that the University will upload a softcopy of my dissertation in pdf format into UTAR Institutional Repository, which may be accessible to UTAR community and public.

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DECLARATION

I, ______Chin Ching Herny______ hereby declare that the dissertation is based on my original work except for quotations and citation which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UTAR or other institutions.

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Date:____________________

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LIST OF ABBREVIATIONS

A/B/m	Kendall's notation: A denotes interarrival time distribution; B denote service time distribution; m denotes server number
A/B/m/Y/Z	Notation to describe queueing models: A denotes interarrival time distribution; B denotes service time distribution; m denotes server number; Y denotes system capacity; Z denotes queueing discipline
BICGSTAB	Biconjugate gradient stabilized method
B_n	Minimum of S_n and R
CAR	Constant asymptotic rate distribution for interarrival and/or service times
CDF	Cumulative distribution function
Δt	A very small duration of time
DST	All positive customers in the system are killed
$E(N)$	Mean queue length
$E(W_s)$	Mean sojourn time
$E(W)$	Mean waiting time
FCFS	First come first serve
$F^*(z)$	Laplace-Stieltjes transform of distribution function $F(t)$
$f(t)$	PDF of interarrival time distribution
$F(t)$	CDF of interarrival time distribution
$\bar{F}(t)$	SF of interarrival time distribution
G	General distribution for interarrival or service times
γ	Arrival rate of negative customer
γ_k	Arrival rate of negative customer when the negative arrival process is in state k
Γ_k	The state space of the negative arrival process (arrival process of negative customers) at the end of the interval τ_k
Geo	Geometrical distribution for arrival process and/or service process
GI	General and independent distribution for interarrival or service times

GLE	Geometrical linear extrapolation
G-networks	Generalized queueing networks or open networks of G-queue
$g(t)$	PDF of service time distribution
$G(t)$	CDF of service time distribution
$\bar{G}(t)$	SF of service time distribution
$h(t)$	PDF of service time distribution
$H(t)$	CDF of service time distribution
$\mathbf{I}$	Identity matrix with dimension equal to the number of stationary probabilities, p_{nirj}
I	Maximum state for positive arrival process
IID	Independent and identically distributed
$\mathbf{I}_n$	Identity matrix of the same size as the number of columns for $\mathbf{P}_n^*$
J	Maximum state for service process
K	Number of layer used in geometrical linear extrapolation
κ	Shape parameter of gamma distribution
Λ_k	The state space of the positive arrival process (arrival process of positive customers) at the end of the interval τ_k
λ	Arrival rate of positive customer
λ_k	Arrival rate of positive customer when the arrival process is in state k
LE	Linear extrapolation
LHS	Left-hand side
LST	Laplace-Stieltjes transform
m	Number of customer used in determine r_m
M	Exponentially distributed interarrival or service times
M_k	The state space of the service process at the end of the interval τ_k
μ	Service rate of positive customer
μ_k	Service rate of positive customer when the service process is in state k

N	A large number that is chosen such that $p_{N+1} \approx 0$
n_k	The number of positive customers in the system at the end of the interval τ_k
p_{ij}	Transition probabilities from i customers to j customers
$\mathbf{P}_n^*$	Column vector storing all the stationary probabilities of size n
p_n	The stationary probability of n positive customers in the system
p_{nirj}	The stationary probability when the queue size equals n, positive arrival process is in state i, negative arrival process in at state r and service process in at state j
$p_{nirj}^{(k)}$	The probability at the end of the interval τ_k when the queue size equals n, positive arrival process is in state i, negative arrival process is in state r and service process is in state j
$p_{n,\Delta t_x}$	The stationary probability with n customers in the system obtained using duration $\Delta t = \Delta t_x$
$\mathbf{P}$	Transition probability matrix for p_{ij}
$P(z)$	Probability generating function
PDF	Probability density function
$\boldsymbol{q}$	A vector that stores all q_n
$\mathbf{Q}_{n,m}$	A coefficient matrix that relates $\mathbf{P}_m^*$ to $\mathbf{P}_n^*$
$\mathbf{Q}_{n,0}^f$	A final coefficient matrix that relates $\mathbf{P}_0^*$ to $\mathbf{P}_n^*$
$\mathbf{Q}_{n,m}^s$	A coefficient matrix that relates $\mathbf{P}_m^*$ to $\mathbf{P}_n^*$ after some matrix operations
q_n	Arrival point probability
R	Minimum interarrival time of negative arrivals occur during a service process
RCE	A positive customer is eleminated from the end of the queue
RCH	A positive customer is eliminated from the head of the queue
ρ	Traffic intensity/utilization factor
$r(k)$	Hazard rate function for a discrete-time distribution
$r_{k,x}$	kth layer geometrical ratios for $k+1 \leq x \leq K+1$

r_m	Probability that exactly m customers leave the system between the arrivals of two successive positive customers given that there are at least m customers in the system at the time the arrival of the first customer
R_n	Total number of departures occur between nth and $(n+1)$th positive arrivals
$r(t)$	Hazard rate function for a continuous-time distribution
SF	Survival function
σ_k	Newly modified service rate
S_n	Service time of nth positive customer
t	Observed continuous-time
T_a	Interarrival time random variable of positive arrivals
t_a	The time point when the hazard rate of the interarrival time distribution for positive customers starts to converge to a constant value.
τ_k	The kth segmented interval along the time axis with duration of Δt
T_b	Service time random variable
T_c	Interarrival time random variable of negative arrivals
θ	Scale parameter of gamma distribution
T_n	The time between nth and $(n+1)$th positive arrivals
T_q	Random variable of total time or duration spent by the customer in the server
$T_q^{(n)}$	n-fold convolution of the distribution for T_q
t_s	The time point when the hazard rate of the service time distribution starts to converge to a constant value.
U	The maximum of I and J
$W_q(t)$	Waiting time distribution
$W_{q,n}$	Time spent in the system by the nth positive customer
X_n	The system size just right before the nth arrival
Z_k	The state vector that stores the values of n_k, Λ_k, Γ_k and M_k at the end of the interval τ_k

CHAPTER 1

INTRODUCTION

A queue is a social phenomenon that happens in our daily life which could be found in diverse areas such as telecommunication, transportation, manufacturing, computing, network etc. Most people will encounter the displeasure of waiting in the queue almost every day of their life. For example, waiting in a traffic jam, customers waiting in a holding line and lining up in a bank or post office waiting to be served. The main concern for a person who is waiting in the queue is: “ How long do I have to wait?” or “ How many person to go before my turn?”. This enigma can be answered by queueing theory through comprehensive mathematical modeling and analysis. Erlang (1909) is the pioneer who applied the probability theory to solve telephone traffic problems. The term ”server” is used to represent operators and the required connecting time is the service demanded by a user. The call duration between the operator and user is the service time.

In earlier works done by Erlang and other researchers, the main motivation for conducting research in the area of queueing theory was to solve the problem of congestion. Thereafter, more researchers devoted themselves to this field and developed general models that are applicable to other queueing problems such as server breakdown, repairable queue, queue with vacation, batch arrival and so on. In this research, a queueing system with negative arrivals is considered. Negative arrivals were first proposed and applied by Gelenbe (1989) in the neural network for traffic-rerouting. Gelenbe (1993a) then introduced the generalized queueing networks (G-networks) which is the open networks of G-queue.

Queues with negative arrivals are also called as G-queue. In G-queue, Positive customers that arrive to the system will receive service while negative customers will not receive service upon their arrivals but will remove one or more positive customers according to some killing disciplines which will be discussed in more detailed in the next chapter. Typically, there are three types of killing discipline: removal of a positive customer at the head (RCH), removal of a positive customer at the end (RCE) and disaster model (DST) in which all positive customers in the queue will be removed upon the arrival of a negative customer. The importance of the studies on negative arrivals is to reduce the congested condition compared to the one without the presence of it.

In this study, a continuous-time FCFS infinite single-server queueing system with negative arrivals and RCH killing discipline is considered. A negative customer is assumed to arrive to the system according to a Poison process with rate γ. Both the interarrival time and service time distributions of a positive customer are assumed to have a constant asymptotic rate as time t tends to infinity. The alternative approach was introduced by Koh (2013) and the constant asymptotic rate is abbreviated as " *CAR* ". The resulting queue in this study can be denoted as *CAR*/*CAR*/1 queue with negative arrivals or *CAR*/*CAR*/1 G-queue. Many distributions such as exponential, Erlang, gamma, hyperexponential etc. fulfill the characteristics of having an approximately constant hazard rate as time t tends to infinity. The method will be first applied to solve the simplest *M*/*M*/1 G-queue and further extended to a queueing model in which both the interarrival time and service time distributions follow *CAR* distribution.

1.1 Problem Statements

Queueing systems with negative customers were first applied in the continuous-time model and studied extensively due to its wide applications in areas such as neural networks, telecommunication systems, and manufacturing systems. It was later discovered that discrete-time models are more relevant to describe phenomena like digital computer network and communication systems. Hence, in recent years, research G-queue is focused on the discrete-time models. For continuous-time models, early work on G-queue assumed exponential distribution for both the interarrival time and service time. Harrison and Pitel (1996) then relaxed the exponential assumption of service time distribution and derived analytically the expressions to find the stationary queue length and sojourn time distributions. While for the discrete-time models, performance analysis of $GI/G/1$ G-queue had been studied by Zhou (2005). Queueing systems with general distribution for interarrival time and service time are versatile and can be applied to a larger class of models that reflect more real-life applications. However, based on the literature review for this study, no paper has been published on the continuous-time $GI/G/1$ queue with negative customers due to its complexity. Hence, one of the targets of this research is to study a queueing system with negative arrivals that could be applied to a queueing system with more general interarrival time and service time distributions. An alternative approach will be used to derive a set of equations to find the stationary probabilities which will be solved to obtain the stationary queue length and waiting time distributions.

All in all, the main targeted problem of this research is to obtain a queueing model that is fairly general with the inclusion of negative customers. With the aid of negative customers and fairly general distribution on both interarrival time and service time, this model can mirror the actual behavior of arrivals and their service in a queueing system. For example, customers may encounter service incompletion and leave the system. And these reasons can be server breakdown, customers choose to give up the service due to the long service time etc. Hence, negative customers are important to represent those scenario in a queueing system.

A simple figure and chart of the system model can be illustrated to show the operation of the queueing system with negative customers. Figure 1.1 below shows that the arrival of negative customers impact on the system only if there is a service in process. The positive customer that is in the service may complete its service or be removed by a negative arrival, whichever come first. In order to show the impact of negative customer to the system, the average arrival rate of the negative customers should be at least greater than the average service rate so that the congested condition of the queue can be minimized. In other word, if the average service rate is greater than the average arrival rate of the negative customers, then the negative customers may not effectively reduce the congested condition of a queue.

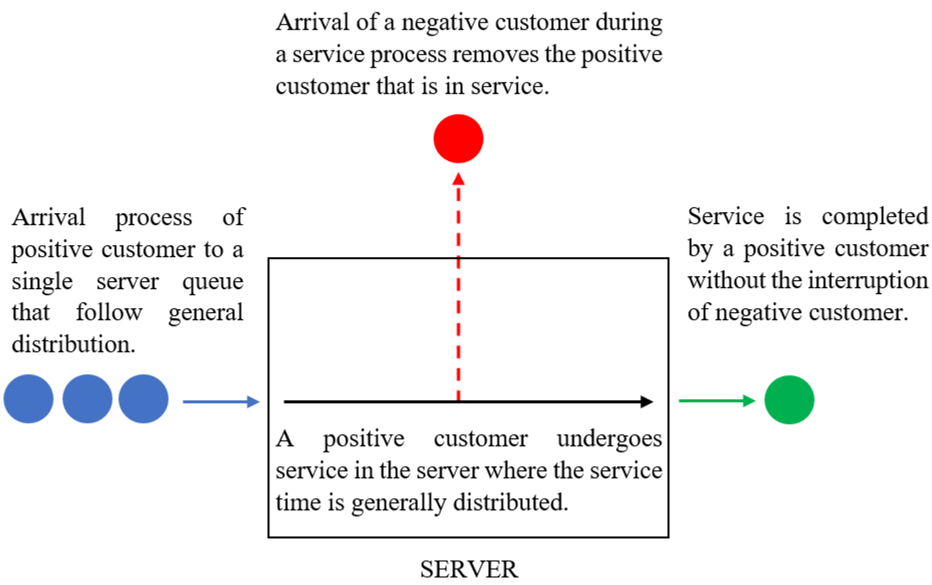

Figure 1.1: Operation of Queueing System with Negative Customers.

For example, when files are transferring from an environment to another environment (service process), some of the file may not be transferred successfully due to corruption or is removed by the anti-virus program as infection is detected. While other files then continue the transferring process until all the file are moved. For queue in clinic, phone call to customer service, etc., the ordinary customer are positive customers, the service process is given by the doctor or the service provider. While the negative customer may represent customer who leave the queue half way due to reasons such as too long waiting time, customer are in rush to run out other errands, etc.

1.2 Objectives

There are 4 objectives in this research:

1. To analyze a single-server continuous-time queueing system with negative customers in which the exponential assumption of both the interarrival time and service time distributions are relaxed.

2. To apply the alternative method in finding the stationary queue length distribution of G-queue.

3. To develop a new approach in finding the waiting time distribution of G-queue.

4. To implement RCH killing discipline in different queueing model.

1.3 Contributions

The main contribution of this research is to find the basic performance measures for a single-server continuous-time $CAR/CAR/1$ G-queue which is fairly close to the $GI/G/1$ queue that can be applied in more queueing problems. Before the exponential assumption of both the interarrival time and service time distributions which are relaxed, the alternative numerical approach will first be applied to an $M/M/1$ G-queue. It is proven mathematically that the stationary probabilities equations derived for the $M/M/1$ G-queue using the alternative numerical approach can yield the same results as the one in Harrison and Pitel (1993). The same approach is then extended to G-queue with positive customers interarrival time or service time has non-exponential distribution. Besides introducing the alternative numerical approach, an equation will also be derived analytically to find the stationary queue length distribution for the *GI*/*M*/1 G-queue. The stationary probabilities that are found in all the models can be used to find the waiting time distribution which is another new approach introduced in this dissertation. Simulation algorithms are also explored to find the stationary queue length and waiting time distributions which will be used to verify all the results obtained by both the numerical and analytical methods.

1.4 Dissertation Organization

The organization of this dissertation is as follows: Literature review is discussed in Chapter 2. An alternative approach in finding the stationary queue length distribution of an *M*/*M*/1 G-queue is discussed in Chapter 3. In Chapter 4, the exponential assumption of the service time distribution is relaxed and the stationary queue length distribution is found. Chapter 5 presents both proposed numerical method and analytical method in solving *CAR*/*M*/1 G-queue and *GI*/*M*/1 G-queue respectively. The exponential assumption on the distributions for both the interarrival time and service time are relaxed in Chapter 6 and the stationary queue length distribution is found. Chapter 7 presents the waiting time distribution of *CAR*/*CAR*/1 G-queue. Numerical examples and analysis are shown at the end of each chapters. Lastly, Chapter 8 summarizes the works that are studied in this research and presents the future works that can be carried out.

CHAPTER 2

LITERATURE REVIEW

Literature of queueing theory is first presented in Section 2.1. Then, queueing theory with negative customers is discussed in Section 2.2. The definition and examples of *CAR* distribution which is used in this research is shown in Section 2.3.

2.1 Queueing Theory

Since the introduction of Erlang's work in solving the congestion problem in telephone traffic, many researchers continue to extend his work and contribute to this field. For example, Molina (1927) applied the probability theory to telephone trunking problems by including "delayed basis". Another example is Fry (1928) who used probability theory in solving congestion problems and fluctuation phenomena in physics. Thereafter, these publications have caught the attention of other researchers to develop and refine the theory into a more complicated model that could be applied in various sophisticated situations. For example, Pollaczek (1934) started to investigate system behavior during a bounded time interval due to the lack of adequacy for the equilibrium theory in plenty of queueing phenomena. Then, he performed appreciable work to study the analytical behavior of queueing systems (Pollaczek, 1965). The complete bibliography of queueing theory was primarily outlined by the fundamental research of the 1950s and 1960s (Syski, 1960; Saaty, 1961; Saaty, 1966; Bhat, 1969). In this dissertation, only some of the publications that created milestones in the queueing theory research area are mentioned.

Bailey (1954) provided a time-dependent solution for an $M/M/1$ queue, where generating functions was used. On the other hand, Ledermann and Reuter (1954) used spectral theory to find the differential equations for a simple birth and death process. Then, the Laplace transform method was conjointly applied to solve the queueing problem. For example, Chaudhry and Templeton (1983) analyzed first bulk queue using Laplace transforms. It is noted that the use of this technique in conjunction with generating functions has become one amongst the known standard method in finding the solution to queueing systems.

Kendall (1953) introduced a notation which is known as Kendall's notation A/B/m to describe some of the basic elements of the queue. In this notation, A and B indicate the interarrival time and service time distributions, respectively. While m is the server number in the system. He initiated a probabilistic approach to analyze queueing systems and provided a demonstration of obtaining imbedded Markov chains in the queue length process of $M/G/1$ and $GI/M/s$ queues. Meanwhile, an integral equation for waiting time distribution in the $GI/G/1$ queue was derived by Lindley (1952) where imbedded Markov chain is involved. Kendall's notation was then extended by Lee (1966) to A/B/m/Y/Z where the new elements Y and Z are the system capacity and queueing discipline respectively. However, Y and Z are usually omitted when the system has infinite capacity and the queueing discipline is FCFS.

In the 1960s, due to the need for application of queueing theory in the industry, Bhat et al. (1979) studied the importance of approximations to analyze queueing systems. During that time, most of the queueing models that were deemed as appropriate in reflecting the real-life phenomena had been studied and many publications focus on the analysis of different queueing models. Not many new methodologies were introduced to solve the queueing problem. However, with the advanced development in computer technology has aided in incorporating the application of queueing theory into the industry.

Since the introduction of queueing theory, queueing theory has been applied widely in industry since its introduction. In 1977, queueing theory is applied in health care industry to solve the problems of neuronal circuitry. We may refer to Ward and Rubin (1977) where they applied a models for neurons and neuron-pair networks to spike-train data from the hippocampus of a rabbit. On the other hand, Moss (1987) used the concept of queueing theory to find the relationship among staff numbers, the process of prescription dispensing and the waiting time of outpatient. Lately, there is a study conducted by Haghighnejad et al. (2016) in Iran that applied queueing theory to investigate the waiting time in the emergency department. More literature review about application of queueing theory in health care can be found in Lakshmi and Iyer (2013).

The use of queueing theory in inventory system started by Fabens (1961) where his study is based on semi-Markov process and can be widely used to variety of models in both continuous and discrete time. The purposes of applying this theory into this field are to analyze the value of centralized inventory information, modeling for perishable inventory, inventory management, multi-stage production and so on which can be found in (Zheng and Zipkin, 1990; Schwarz, Sauer, Daduna, Kulik and Szekli, 2006; Graves, 1982; Schwarz and Daduna, 2006; Goyal and Gunasekaran, 1990)

In fact, queueing theory can be applied in diverse areas especially in the rapidly emerging and growing computer and communication field (Coffman and Hofri, 1986; Tamir and Frazier, 1992; Alfa, 2010). Other than that, it is also widely used in the manufacturing system where Palm (1947) and Benson and Cox (1951) laid the foundation to analyze manufacturing systems problem using queueing theory. The reader can refer to Papadopoulos and Heavey (1996) and

Altiok (2012) for some queueing theory applications in production and manufacturing system. For computing network, see (Buzen, 1973; Karol et al., 1987; Bolch et al., 2006). To learn more about the application of queueing theory, the reader can refer to Newell (2013).

2.2 Queue with Negative Customers

As a pioneer in introducing the queueing system with negative customers (G-queue), Gelenbe (1991) showed that an $M/M/1$ open queueing network has a product form solution. Gelenbe et al. (1991) then investigated a different type of killing strategy where a positive customer is removed from the tail of a queue. They provided necessary conditions for achieving stability of single-server G-queues. Another killing strategy considered by Gelenbe (1993b) is the batch removal by a negative customer. After Gelenbe's works were presented (Gelenbe, 1993a; Gelenbe, 1993c; Gelenbe, 1994), numerous studies related to negative customers were published. Most of these papers can be found in the bibliography of Do (2011).

Harrison and Pitel (1993) applied Laplace transform method to obtain the sojourn time distribution of an $M/M/1$ queue in which negative arrivals form a Poisson process. They also analyzed and compared basic performance measures for queueing systems with different queueing disciplines and killing strategies. In 1995, Harrison and Pitel used Laplace transform of to obtain the response time distribution in a tandem pair of two normal $M/M/1$ queues. They (Harrison and Pitel, 1996) then provided the stability conditions and derived the stationary queue length distribution for the $M/G/1$ G-queue. In the same year, there were a few publications related to an $M/G/1$ G-queue. For instance, Jain and Sigma (1996) derived a Pollaczek-Khintchine expression for an $M/G/1$ queues with disasters (DST). While Bayer and Boxma (1996) conducted Wiener-Hopf analysis for an $M/G/1$ queue in which positive customers are allowed to be killed only at the end of the service. Boucherie and Boxma (1996) also studied a work removal in an $M/G/1$ queue with negative customers and concluded that the workload distribution of this model is the same as the waiting time distribution of a $GI/G/1$ queue without negative customers.

Similar to Harrison and Pitel (1993), Shin (1998) studied multi-server G-queue where the Laplace transform of the sojourn time distributions for queues with various combination of killing strategies and queueing disciplines are obtained. However, they remained the exponential assumption of the service time distribution. Dudin and Nishimura (1999) analyzed the queue length and waiting time distributions for a more complicated queueing model in which Markovian arrival input of disasters is considered. The model was further extended by Dudin and Karolik (2001) where they investigated non-instantaneous recovery in which variants of inflow and outflow of customers during the recovery are considered. Then, Gelenbe's original concept was generalized by Zhu and Zhang (2004) where positive and negative customers may cancel out each other. In the same year, Li and Zhao (2004) extended Gelenbe original model to a queueing system with Markovian arrival input for both the positive and negative arrivals.

Some researchers considered general interarrival time instead of general service time distribution. For example, Yang and Chae (2001) investigated a $GI/M/1$ queue with negative arrivals that kill a customer at the end (RCE) and DST killing strategies. While the study of the stability condition of the embedded Markov chain in a $GI/M/1$ G-queue is done by Abbas and Assani (2010). Recently, an analysis on a $GI/M/1$ queue associated with a multi-phase service environment and disasters is studied by Jiang and Liu (2017). They obtained the cycle analysis, the sojourn time distribution and the stationary queue length distribution for this queueing model.

On the other hand, there were researchers studied the behavior of negative customers. For instance, Artalejo (2000) considered a G-queue where a random amount of work could be removed from the queue by negative customers. A clearing model for *M*/*G*/1 queues is also studied by Boxma et al. (2001) who considered DST that instantly remove the entire residual workload from the system. Zhu (2003) discussed a queueing model where negative arrivals may receive service. Liu and Wu (2009) considered a G-queue with vacation where the arrivals process of negative customer is Markovian arrival process (MAP). The author applied the matrix-analytic approach that is different from the proposed method in this dissertation. In this paper, the service behavior of the arriving customer is preemptive resume in which some customers are prioritized to receive the service from the server. On the other hand, Wang et al. (2013) studied a discrete-time queue where the positive customers and negative customers are correlated and applied this behavior together with RCH and DST killing principles.

Starting from the early of 21st century, discrete-time queues with negative customers has caught the attention of the researchers since it is more appropriate to be applied in the digital computer and communication systems. For instance, Atencia and Moreno (2004) studied a discrete-time G-queue with geometrically distributed interarrival time and service time where negative arrivals were used to represent virus in the computer system. A more complicated and complex discrete-time queueing system with negative customers is studied by Zhou (2005) in which both the service time and interarrival time have general discrete-time distribution. Other discrete-time queues that are related to negative arrivals can be seen in Tian and Zhang, 2002; Park et al., 2009; Atencia and Moreno, 2006; Li and Tian, 2007; Lee and Kim, 2014.

Meanwhile, continuous-time G-queues are further extended with different features. Kumar et al. (2006) generalized $M/G/1$ feedback G-queue. In this model, the server does not give service to any customer until the queue length hits a certain number and service is provided exhaustively thereafter. Their model was extended by Liu et al. (2009) to retrial G-queue and other features such as server breakdowns and repairs were included into the model. Wu and Lian (2013a) then considered G-queue with priority and unreliable server under Bernoulli vacation where negative arrivals indicate the breakdown at the busy server. In the same year, Wu and Lian (2013b) considered two classes of positive arrivals together with negative customers that caused server breakdown and killed a customer by interrupting the service process. Do et al. (2014) studied an $M/M/1$ retrial G-queue in which the server may undergo a working vacation that has exponential distribution while negative arrivals will eliminate a group of positive arrivals randomly from the orbit. While Zhang and Liu (2015) take into consideration vacation interruption for the same model in which lower service rate is given to positive customers during vacation. There is also a retrial $M/G/1$ G-queue with non-persistent customers studied by Gao and Wang (2014) where non-persistent customers are defined as impatient customers who wait in the queue. This is an important feature that can represent a lot of real-life application like impatient behaviors of patients in hospital emergency room and impatient call-in customer for service.

From all the studies above, it can be seen that negative customers are widely used in various queueing systems. It can act as an inhibitory signal in the neural network, a virus that deletes the certain database or to represent server breakdown in the manufacturing industry. A negative customer is also an important feature to be used in emergency navigation algorithm during evacuation (Bi, 2016). It gives re-routing decisions by considering various classes of positive customers (evacuee) who might be panic during an emergency. While in the inventory system, negative customers represent demand that opts for another store (Soujanya, 2015).

Recently, an $M/G/1$ retrial queue with negative customers is studied by Li and Zhang (2017) who considered general retrial times and working breakdowns. They obtained the necessary condition for the system stability. Negative arrivals are also researched by Fourneau and Gelenbe (2017) in queueing system together with a new type of customer called "Adders" that acts as a load regulator by changing the queue length at the service center that it visits. Instead of just study the basic performance measures, researchers like Sun and Wang (2018) studied the optimal joining strategies using a common-reward-cost structure. Although the authors considered on an $M/M/1$ G-queue, there are still a lot of work done that have been published and yet need to be included into current dissertation and project. The authors research direction are more on joining strategies instead of alternative method in solving a queueing system. All the above studies show that queueing system with negative customers is worth to be investigated.

2.3 Constant Asymptotic Rate Distribution (*CAR*)

Let $r(t)$ be the hazard rate functions of a distribution. If $r(t)$ tends to a constant when t goes to infinity, the distribution is known to have a constant asymptotic rate *CAR*. Denote $t = t_a$ as a particular point in the time axis when the hazard rate starts to converge to an approximately constant value. Figures 2.1 and 2.2 show the examples of which the continuous-time distributions have *CAR*. Whereas Figures 2.3 and 2.4 are the examples of discrete-time distributions with *CAR* characteristics.

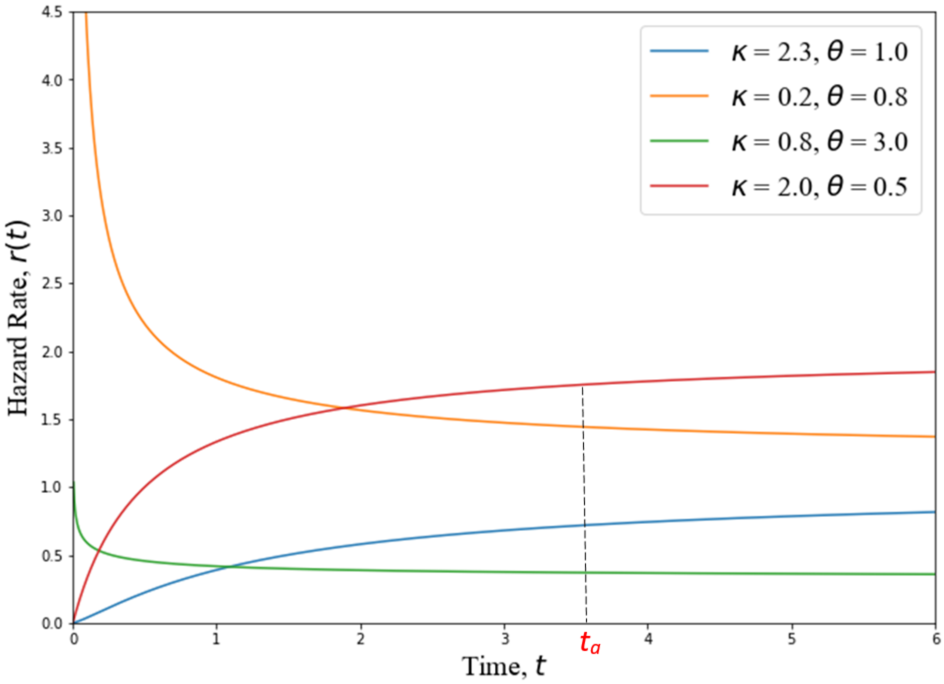

Figure 2.1: The Hazard Rate Functions of Gamma Distribution with Different Shape (κ) and Scale (θ) Parameters .

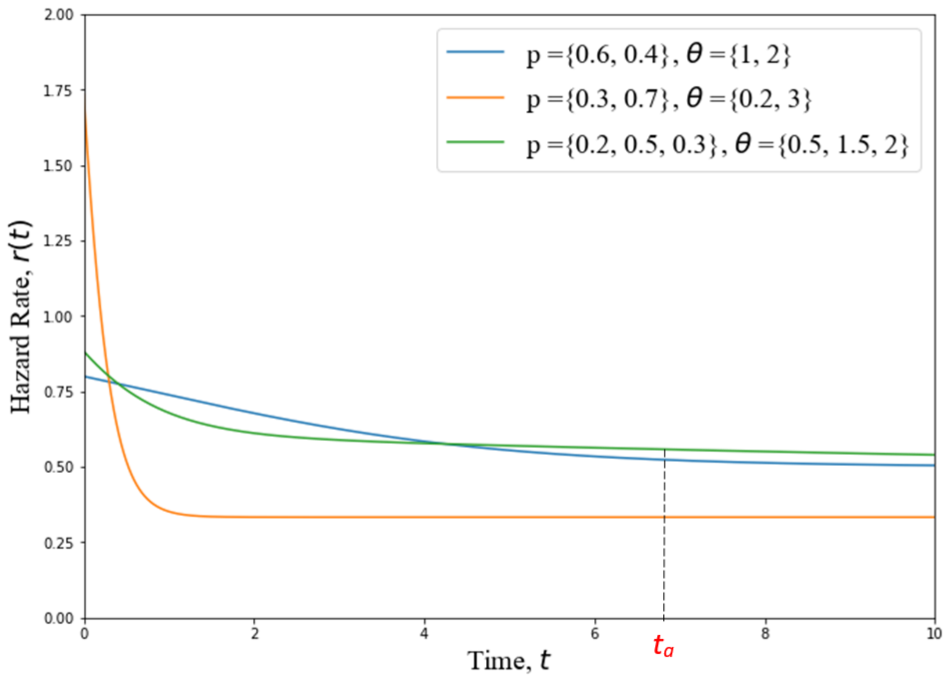

Figure 2.2: The Hazard Rate Functions of Hyperexponential Distribution with Different Combinations of Several Exponential Distributions.

From Figures 2.1 and 2.2, it can be observed that the hazard rate is converging to an approximately constant value at the time of point $t = 3$ or $t = 4$.

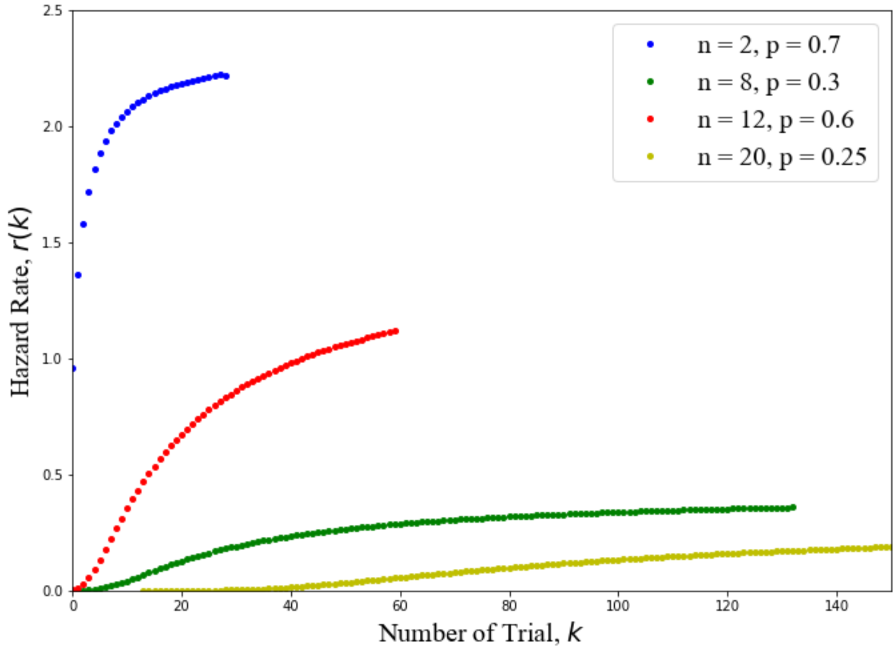

Figure 2.3: The Hazard Rate Functions of Negative Binomial Distribution with Different Success and Probability Parameters.

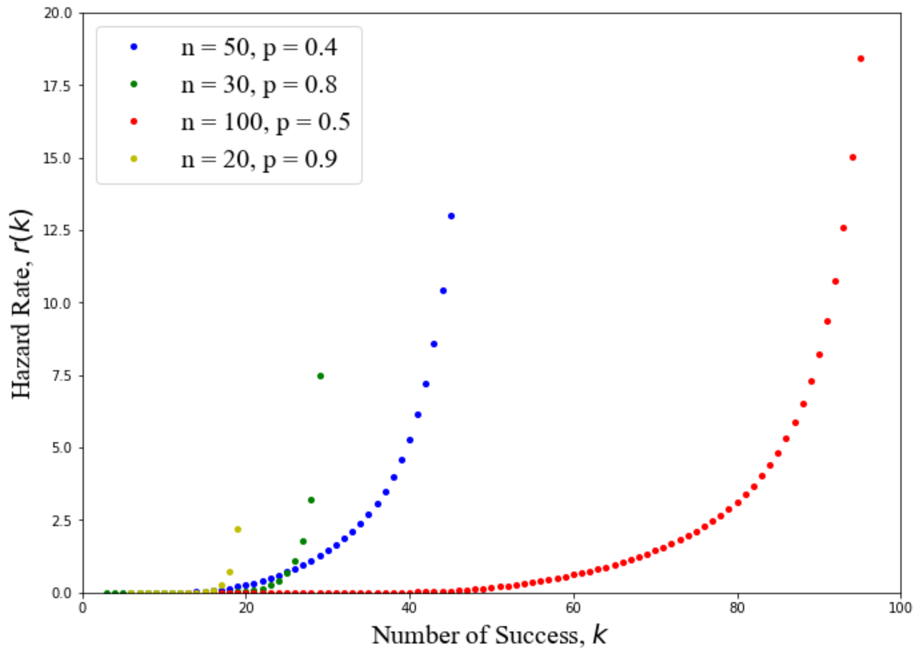

Figure 2.4: The Hazard Rate Functions of Binomial Distribution with Different Trial and Probability Parameters.

For discrete-time distribution, the point at which the hazard rate function $r(k)$ tend to a constant value depends on the number of trials or successes, k. As it can be observed from Figure 2.3, the hazard rate tends to a constant value when the number of trials increases. However, for the binomial distributions in Figure 2.4, the convergence occurs for $r(k) \to 0$ when $k \to 0$. Nonetheless, the requirement of the *CAR* distribution are fulfilled for both the discrete-time distributions.

Mathematically, the hazard rate function, $r(t)$, can be shown to have a constant value when $t \to \infty$. Consider the an Erlang-k distribution with rate λ:

$$\begin{aligned}
\lim_{t\to\infty} r(t) &= \lim_{t\to\infty} \frac{\dfrac{\lambda^k t^{k-1} e^{-\lambda t}}{(k-1)!}}{\displaystyle\sum_{n=0}^{k-1} \frac{(\lambda t)^n}{n!}} \\
&= \frac{\lambda^k}{(k-1)!} \lim_{t\to\infty} \frac{t^{k-1}}{1 + \lambda t + \dfrac{(\lambda t)^2}{2!} + \cdots + \dfrac{(\lambda t)^{k-1}}{(k-1)!}} \\
&= \frac{\lambda^k}{(k-1)!} \lim_{t\to\infty} \frac{1}{\dfrac{1}{t^{k-1}} + \dfrac{\lambda}{t^{k-2}} + \dfrac{\lambda^2}{2!t^{k-3}} + \cdots + \dfrac{\lambda^{k-1}}{(k-1)!}} \\
&= \frac{\lambda^k}{(k-1)!} \left(\frac{(k-1)!}{\lambda^{k-1}} \right) \\
&= \lambda
\end{aligned} \tag{2.1}$$

Derivation in Equation (2.1) shows that the hazard rate function of Erlang-k distribution is equal to a constant rate λ when t goes to infinity. Since Erlang-k distribution is a special case of gamma distribution, the result obtained in Equation (2.1) can be agreed with the red line shows in Figure 2.1 where the κ is 2 and λ is 1/θ. Next, consider a geometric distribution with parameter, p, then the hazard rate function of this distribution for all k is found to be:

$$\begin{aligned} r(k) &= \frac{(1-p)^k p}{1 - \sum_{n=0}^{k} (1-p)^n p} \\ &= \frac{(1-p)^k p}{1 - p - (1-p)p - (1-p)^2 p - \cdots - (1-p)^k p} \\ &= \frac{(1-p)^k p}{(1-p)^{k+1}} \\ &= \frac{p}{1-p} \end{aligned} \tag{2.2}$$

From Figures 2.1 to 2.4 as well as the Equations (2.1) and (2.2), it has been easily shown how a distribution fulfills the requirement of *CAR*. It will be presented in the next few chapters on how to find the basic performance measures for a queueing system with the interarrival time and/or service time that has a *CAR* distribution.

CHAPTER 3

STATIONARY QUEUE LENGTH DISTRIBUTION OF *M/M/*1 G-QUEUE USING RCH REMOVING DISCIPLINE

In this chapter, the alternative approach is applied in obtaining the stationary queue length distribution of a single-server infinite capacity $M/M/1$ queue with the arrival of negative customers that kill or eliminate the first customer in the queue if any (RCH). A set of equations for the stationary probabilities is derived in Section 3.1 and it is subsequently proven and solved in Section 3.2 using generating function method.

3.1 Derivation of Equations for Stationary Probabilities

In this chapter, the overview of the model can be represented in the following operation diagram, Fig 3.1.

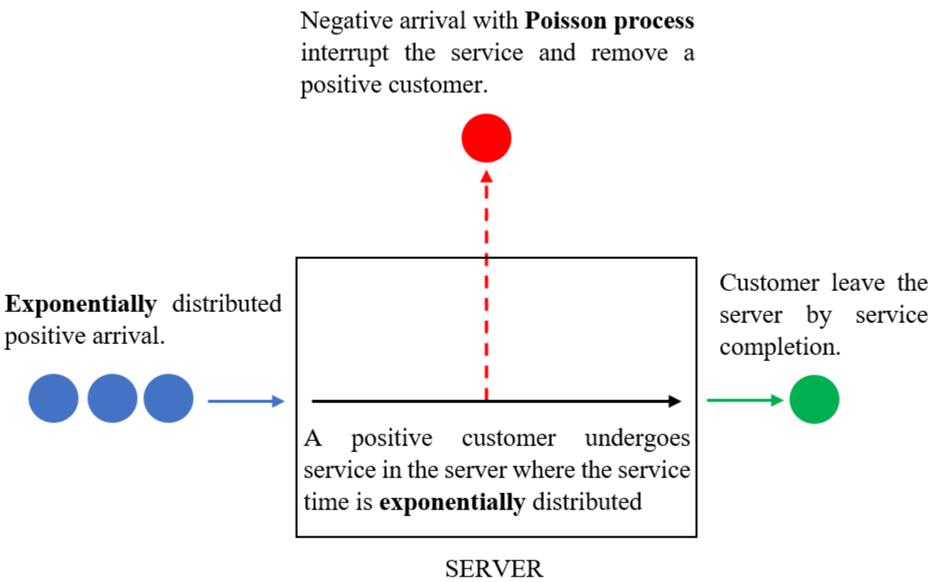

Figure 3.1: Operation of $M/M/1$ Queue with Negative Customers.

3.1.1 Transition Probabilities and Events

The time-axis is first segmented into equal size of intervals with the length of each interval is Δt. A small value of Δt will be chosen such that $\Delta t \approx 0$ and the kth interval is denoted as τ_k:

$$\tau_k = ((k-1)\Delta t, k\Delta t), \quad \text{where } k = 1, 2, 3, ... \tag{3.1}$$

The interarrival and service time of positive customers are both assumed to be exponentially distributed with rate λ and μ respectively. The negative customers arrive to the system according to a Poisson process with the rate γ. Supposed that the probability density function (PDF), survival function (SF) and hazard rate function of the interarrival time of positive customers are $f(t) = \lambda e^{-\lambda t}$, $\bar{F}(t) = e^{-\lambda t}$ and λ_k respectively. By definition of the hazard rate function, the following is obtained:

$$\lambda_k = \frac{f(k\Delta t)}{\bar{F}(k\Delta t)} = \frac{\lambda e^{-\lambda k \Delta t_1}}{e^{-\lambda k \Delta t_1}} = \lambda, \text{where } k = 1, 2, 3, ... \tag{3.2}$$

As we can see from Equation (3.2), due to the memoryless property, the hazard rate function for any random variable that follows exponential distribution is a constant which is the reciprocal of its mean regardless of the point of the time. Similarly, let the hazard rate functions for the service time of the positive customers and the interarrival time of the negative customers in the interval τ_k be μ_k and γ_k respectively. Both rates can be expressed individually as

$$\mu_k = \mu, \gamma_k = \gamma, \text{where } k = 1, 2, 3, \ldots \tag{3.3}$$

Since all the hazard rate functions are constant, the following definitions are given,

$\lambda\Delta t$	:	the probability that a positive customer arrives in an interval.
$\gamma\Delta t$	:	the probability that a negative customer arrives in an interval.
$\mu\Delta t$	:	the probability that a service is completed in an interval.

These expressions are also the transition probabilities from states to states and applied in the derivation of the stationary probabilities equations.

Since Δt is set to be a small value, only one of the following event could take place in each of the interval τ_k, for $k = 1, 2, 3, \ldots$.

1. A positive customer arrives to the system.
2. A negative customer arrives to the system and eliminates a positive customer from the head of the queue if any.
3. There is a service completion occurs and a positive customer leave the system if the system is not empty.
4. None of the above events occurs.

3.1.2 State Spaces

Next, let τ_0 be the interval before τ_1 and assume that there is a positive arrival in τ_0. The state spaces of the positive arrival process, negative arrival process and service process at the end of the interval τ_k, where $k = 1, 2, 3, \ldots$ are denoted as Λ_k, Γ_k and M_k, respectively. These state spaces are defined as follows:

$$\Lambda_k = \begin{cases} 0, & \text{if } \tau_k = \tau_0\text{; or} \\ & \text{a positive customer arrives in } \tau_k, k \geq 1. \\ 1, & \text{if no positive customer arrives in } \tau_k, k \geq 1. \end{cases} \tag{3.4}$$

$$\Gamma_k = \begin{cases} 0, & \text{if } \tau_k = \tau_0\text{; or} \\ & \text{no negative customer arrives in } \tau_k, k \geq 1. \\ 1, & \text{if a negative customer arrives in } \tau_k, k \geq 1. \end{cases} \tag{3.5}$$

$$M_k = \begin{cases} 0, & \text{if there is no customer in the system; or} \\ & \text{a service completeion occurs in } \tau_k, k \geq 1\text{; or} \\ & \text{a negative customer arrives in } \tau_k, k \geq 1; \\ 1, & \text{if no service completion occurs in } \tau_k, k \geq 1. \end{cases} \tag{3.6}$$

Let n_k be the number of positive customers in the system at the end of the interval τ_k and the state vector $Z_k = \{n_k, \Lambda_k, \Gamma_k, M_k\}$ denotes the queue length, the states of the positive arrival process, the state of the negative arrival process and the state of the service process at the end of the interval τ_k.

Suppose that at the end of the interval τ_{k-1}, the queue is not empty where $n_k \geq 1$, the positive arrival process is in state 1 where no positive customer arrives in interval τ_{k-1}, the negative arrival process is in state 0 where no negative customer arrives in interval τ_{k-1} as well and the state of the service process is 1 where no service completion occurs in the interval τ_{k-1}. Hence, the state vector at the end of the interval τ_{k-1} is $Z_{k-1} = \{n_k, 1, 0, 1\}$. Then, the possible state vectors Z_k at the end of the interval τ_k are given as follows by considering the events that could possibly take place in the interval.

1. A positive customer arrives to the system and $Z_k = \{n_k + 1, 0, 0, 1\}$.

2. A negative customer arrives to the system and $Z_k = \{n_k - 1, 1, 1, 0\}$.

3. A service completion occurs in the system and $Z_k = \{n_k - 1, 1, 0, 0\}$.

4. None of the above events occurs and $Z_k = \{n_k, 1, 0, 1\}$.

However, if the queue is empty where service is not available and $n_k = 0$, given the state vector is $Z_{k-1} = \{0, 1, 1, 0\}$ at the end of the interval τ_{k-1}, then the possible state vector Z_k formed at the end of the interval τ_k are listed as follows:

1. A positive customer arrives to the system and $Z_k = \{1, 0, 0, 1\}$.

2. A negative customer arrives to the system and $Z_k = \{0, 1, 1, 0\}$.

3. None of the above events occurs and $Z_k = \{0, 1, 0, 0\}$.

3.1.3 Stationary Probabilities

In long run, stationary probabilities can be obtained when the system is in an equilibrium state and if the system utilization is less than 1. Let $p_{nirj}^{(k)}$ be the probability that obtained at the end of the interval τ_k, where n is the number of positive customers in the system when the positive arrival process, negative arrival process and service process are in state i, r and j respectively. Then the stationary probability with states n, i, r and j is denoted as p_{nirj} and found by

$$\lim_{k \to \infty} p_{nirj}^{(k)} = p_{nirj}. \tag{3.7}$$

Suppose the state vector at the end of the interval τ_0 is $Z_0 = \{1, 1, 0, 0\}$, then the probability obtained at the end of the interval τ_0 is written as $p^{(0)}_{1100}$. A sample of equations are derived as follow:

1. A positive customer arrives to the system, $p^{(1)}_{2001} = p^{(0)}_{1100}(\lambda\Delta t)$.
2. A negative customer arrives to the system, $p^{(1)}_{0110} = p^{(0)}_{1100}(\gamma\Delta t)$.
3. A service completion occurs in the system, $p^{(1)}_{0100} = p^{(0)}_{1100}(\mu\Delta t)$.
4. None of the above events occurs, $p^{(1)}_{1101} = p^{(0)}_{1100}(1 - \lambda\Delta t - \gamma\Delta t - \mu\Delta t)$

However, assuming the state vector at the end of the interval τ_0 is $Z_0 = \{0, 1, 0, 0\}$ where the queue is empty, a sample of equations are written as follow:

1. A positive customer arrives to the system, $p^{(1)}_{1001} = p^{(0)}_{0100}(\lambda\Delta t)$.
2. A negative customer arrives to the system, $p^{(1)}_{0110} = p^{(0)}_{0100}(\gamma\Delta t)$.
3. None of the above events occurs, $p^{(1)}_{0100} = p^{(0)}_{0100}(1 - \lambda\Delta t - \gamma\Delta t)$

By using the similar transition and iteration as shown in above samples, the complete relationship of state $nirj$ can be represented in a transition diagram as shown in Fig 3.2 where the color of the arrow indicates their transition probabilities. In general, there are total of four outgoing transition probabilities in each state $nirj$.

Positive arrival, Red $\rightarrow$: $\lambda\Delta t$

Negative arrival, Blue $\rightarrow$: $\gamma\Delta t$

Service Completion, Green $\rightarrow$: $\mu\Delta t$

Nothing happen when $n = 0$, Orange $\rightarrow$: $1 - \lambda\Delta t - \gamma\Delta t$

Nothing happen when $n \geq 1$, Black $\rightarrow$: $1 - \lambda\Delta t - \gamma\Delta t - \mu\Delta t$

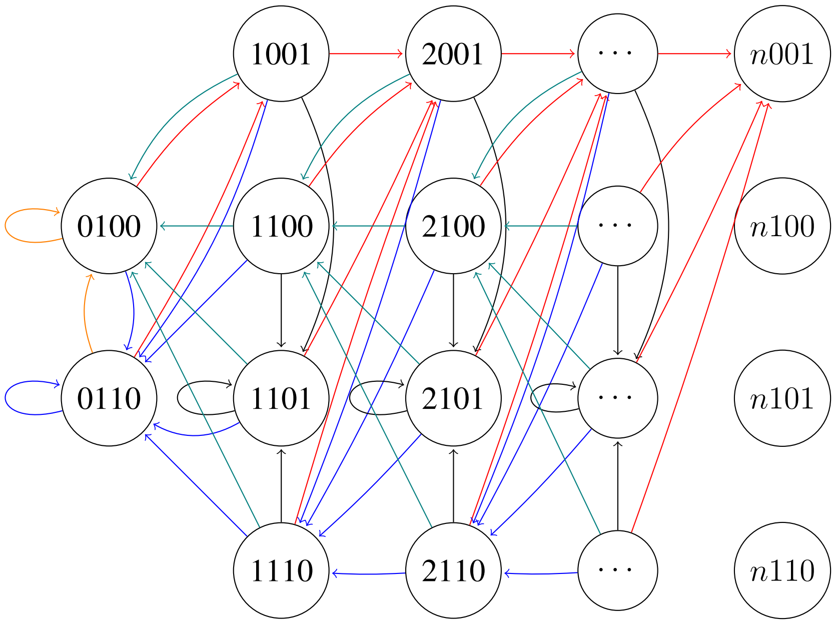

Figure 3.2: A Markov Chain of *M*/*M*/1 Queue with Negative Customers.

Then, by observing the relationship in the Markov chain above, a set of stationary probabilities can be derived and generalized as follows when $k \to \infty$. To further simplify the equations, let $p_{n***} = p_{n001} + p_{n100} + p_{n101} + p_{n110}$.

$$\begin{aligned} n = 0, \quad p_{n100} &\cong p_{(n+1)***}(\mu\Delta t) + (p_{n100} + p_{n110})(1 - \lambda\Delta t \\ &\quad -\gamma\Delta t) \quad (3.8) \\ n = 0, \quad p_{n110} &\cong (p_{(n+1)***} + p_{n100} + p_{n110})(\gamma\Delta t) \quad (3.9) \\ n = 1, \quad p_{n001} &\cong (p_{(n-1)100} + p_{(n-1)110})(\lambda\Delta t) \quad (3.10) \\ n \geq 1, \quad p_{n100} &\cong p_{(n+1)***}(\mu\Delta t) \quad (3.11) \\ n \geq 1, \quad p_{n101} &\cong p_{n***}(1 - \lambda\Delta t - \gamma\Delta t - \mu\Delta t) \quad (3.12) \\ n \geq 1, \quad p_{n110} &\cong p_{(n+1)***}(\gamma\Delta t) \quad (3.13) \\ n \geq 2, \quad p_{n001} &\cong p_{(n-1)***}(\lambda\Delta t) \quad (3.14) \end{aligned}$$

3.2 Stationary Queue Length Distribution

Equation (3.8) to (3.14) are solved to obtain the stationary queue length distribution. A probability generating function method $P(z) = \sum_{n=0}^{\infty} p_n z^n$ where z is complex with $|z| \leq 1$ is used to solve these equations. In the following, the proof that the set of equations derived has a closed form expression same as the analytic method in Harrison and Pitel (1993) is shown.

Let $p_0 = p_{0100} + p_{0110}$ and $p_n = p_{n001} + p_{n100} + p_{n101} + p_{n110}$ for $n \geq 1$ where $p_{n001}, p_{n100}, p_{n101}$ and p_{n110} are the left-hand side (LHS) components of Equations (3.10) to (3.14). Then, summing up Equations (3.8) and (3.9) obtains

$$p_0 = p_1(\mu\Delta t) + p_0(1 - \lambda\Delta t - \gamma\Delta t) + (p_1 + p_0)(\gamma\Delta t)$$

and rearranging the equation yields

$$p_1 = p_0 \frac{\lambda\Delta t}{\mu\Delta t + \gamma\Delta t} = p_0 \frac{\lambda}{\mu + \gamma} \quad (3.15)$$

By comparing Equation (3.15) to the results obtained in Harrison and Pitel (1993), it is now shown that Equations (3.8) and (3.9) can yield the following conclusions.

$$\rho = \frac{\lambda}{\mu + \gamma} \quad (3.16)$$

$$p_1 = p_0 \rho \quad (3.17)$$

where ρ is the traffic intensity or utilization factor of this queueing model and it has to be less than 1 to fulfill the stability condition.

Next, for $n \geq 1$, Equations (3.10) to (3.14) can be rewritten as follows:

$$\begin{aligned} p_{n100} &= p_{n+1}(\mu\Delta t) \\ p_{n101} &= p_n(1 - \lambda\Delta t - \gamma\Delta t - \mu\Delta t) \\ p_{n110} &= p_{n+1}(\gamma\Delta t) \\ p_{n001} &= p_{n-1}(\lambda\Delta t) \end{aligned}$$

Summing up the above equations will get

$$p_{n+1} = p_n(1+\rho) - p_{n-1}\rho, \quad n \geq 1. \tag{3.18}$$

Then, multiplying z^n into both side of Equation (3.18) obtains

$$\begin{aligned} z^n p_{n+1} &= z^n p_n(1+\rho) - z^n p_{n-1}\rho \\ z^{n+1} p_{n+1} z^{-1} &= z^n p_n(1+\rho) - z^{n-1} p_{n-1}\rho z \end{aligned}$$

By taking the sum from $n = 1$ to ∞ on the above equation yields

$$z^{-1}\sum_{n=1}^{\infty} z^{n+1}p_{n+1} = (1+\rho)\sum_{n=1}^{\infty} z^n p_n - \rho z \sum_{n=1}^{\infty} z^{n-1}p_{n-1}$$

where the summation terms can be rewritten in term of the probability generating function $P(z) = \sum_{n=0}^{\infty} p_n z^n$.

$$\begin{aligned} \sum_{n=1}^{\infty} z^{n+1}p_{n+1} &= P(z) - p_0 - p_1 z \\ \sum_{n=1}^{\infty} z^n p_n &= P(z) - p_0 \\ \sum_{n=1}^{\infty} z^{n-1}p_{n-1} &= P(z) \end{aligned}$$

Then,

$$z^{-1}[P(z) - p_0 - p_1 z] = (1+\rho)[P(z) - p_0] - \rho z P(z)$$

where from Equation (3.17), $p_1 = p_0\rho$. Hence,

$$\begin{aligned} z^{-1}[P(z) - p_0 - p_0\rho z] &= (1+\rho)[P(z) - p_0] - \rho z P(z) \\ z^{-1}P(z) - (1+\rho)P(z) + \rho z P(z) &= z^{-1}(p_0 + p_0\rho z) - (1+\rho)p_0 \\ (\rho z + z^{-1} - 1 - \rho)P(z) &= (z^{-1} + \rho - 1 - \rho)p_0 \\ P(z) &= \frac{1-z}{z^2\rho + 1 - z - z\rho}p_0 \\ P(z) &= \frac{p_0}{1 - z\rho} \qquad (3.19) \end{aligned}$$

By letting $z = 1$, $P(1) = 1$ where the sum of p_n for $n \geq 0$ must be equal to 1. Then,

$$\begin{aligned} 1 &= \frac{p_0}{1-\rho} \\ p_0 &= 1 - \rho \qquad (3.20) \end{aligned}$$

In general, the following expression can be obtained by substituting Equation (3.20) into Equation (3.19).

$$P(z) = \sum_{n=0}^{\infty}(1-\rho)\rho^n z^n \qquad (3.21)$$

where the coefficient in front of z^n is p_n, so

$$p_n = (1-\rho)\rho^n, \quad \rho < 1 \qquad (3.22)$$

Hence, the set of stationary probabilities equations that had been derived in Section 3.1 using alternative approach is proven to have the same expression (Equation (3.22)) as the one in Harrison and Pitel (1993) regardless of the value of Δt. This also concludes that the measure of effectiveness such as mean sojourn time, $E(W_s)$, of this model can be obtained by simply replacing the service rate in an ordinary *M/M/*1 queue without negative customers with the sum of service rate and arrival rate of negative customers, $\mu + \gamma$. For instance,

$$E(W_s) = \frac{\rho}{\lambda(1-\rho)} = \frac{1}{\mu+\gamma-\lambda}, \quad \rho < 1 \tag{3.23}$$

which have been verified in Harrison and Pitel (1993) as well.

As a result from Equation (3.22), the mean queue length $E(N)$ and mean waiting time $E(W)$ can be obtained as shown in Equations (3.24) and (3.25):

$$\begin{aligned} E(N) &= \sum_{n=0}^{\infty} np_n \\ &= \frac{\rho}{1-\rho}, \quad \rho < 1 \\ &= \frac{\lambda}{\mu+\gamma-\lambda} \end{aligned} \tag{3.24}$$

By Little's formulas, $E(N) = \lambda E(W)$, then

$$\begin{aligned} E(W) &= \frac{E(N)}{\lambda} \\ &= \frac{1}{\mu+\gamma-\lambda} \end{aligned} \tag{3.25}$$

Once again, Equation (3.25) is proved to have the same expression as the mean waiting time equation obtained in Harrison and Pitel (1993).

In the next chapter, this alternative approach will be extended to a queueing model with non-exponential service time.

CHAPTER 4

STATIONARY QUEUE LENGTH DISTRIBUTION OF *M/CAR/*1 G-QUEUE USING RCH REMOVING DISCIPLINE

The alternative approach discussed in Chapter 3 to obtain the stationary queue length distribution is extended to an *M/CAR/*1 queue. The same removal discipline RCH is considered. Section 4.1 shows the derivation of a set of equations for stationary probabilities and the equations is solved in Section 4.2 using matrix iterative method and the extrapolation method introduced in Section 4.3. Lastly, a numerical example is presented in Section 4.4 and compared to the results obtained by the existing method and simulation procedure in Appendix A.

4.1 Derivation of Equations for Stationary Probabilities

4.1.1 Transition Probabilities and Events

In this model, all the assumptions are the same as those made in Chapter 3 except that the service time of positive customers is assumed to have Constant Asymptotic Rate (*CAR*) Distribution that is mentioned in Chapter 2.

Initially, the time-axis is segmented as Equation (3.1) into equal length of interval with duration Δt. Next, let $g(t)$ be the probability density function (PDF) and $\bar{G}(t)$ be the survival function (SF) of the service time. Then, the hazard rate function μ_k in each small interval, τ_k is expressed as

$$\mu_k = \frac{g(k\Delta t)}{\bar{G}(k\Delta t)}, \quad 1 \leq k \leq J \tag{4.1}$$

where J is an integer that is large enough such that

$$\mu_J \simeq \lim_{k\to\infty} \mu_k. \tag{4.2}$$

J can be calculated as

$$J = \frac{t_s}{\Delta t} \tag{4.3}$$

where t_s is interpreted as the time point where the hazard rate starts to converge as shown in the figure below.

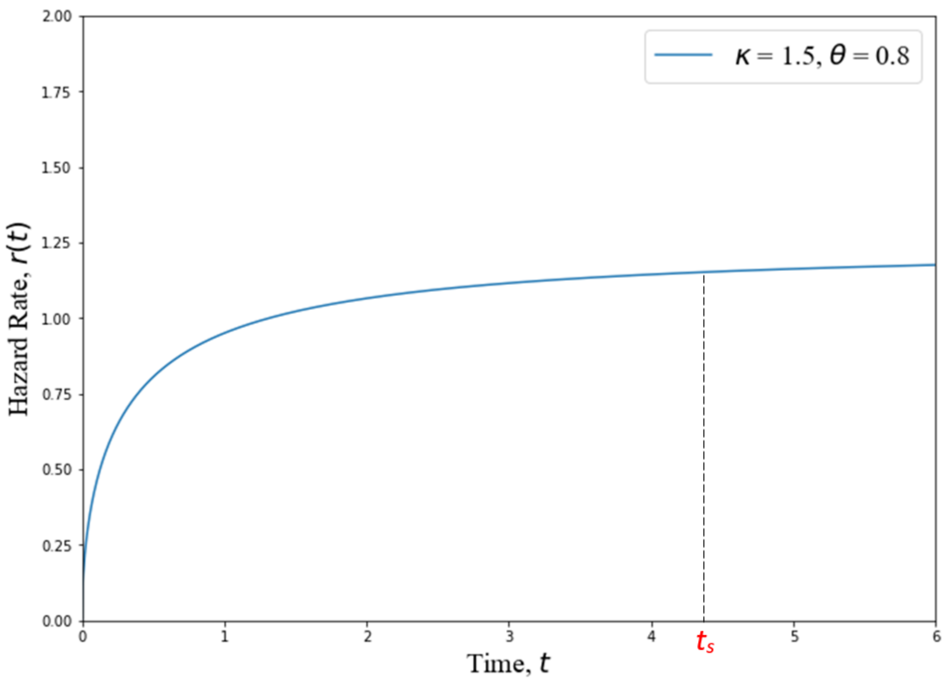

Figure 4.1: A Hazard Rate Function Graph for Gamma Distribution.

Since the interarrival time of both positive customers and negative customers are exponentially distributed, then the hazard rate functions are constants as shown in Section 3.1.1. Let λ and γ be the rate of these interarrival times respectively, the transition probabilities can be interpreted as follows:

$\lambda \Delta t$: the probability that a positive customer arrives in an interval.

$\gamma \Delta t$: the probability that a negative customer arrives in an interval.

$\mu_k \Delta t$: the conditional probability that a service is completed in the interval τ_k given that there is no service completion in $\tau_1, \tau_2, \cdots, \tau_{k-1}$.

4.1.2 State Spaces

In this model, the notations are defined for the state spaces of the positive arrival process and the negative arrival process are the same as Equations (3.4) and (3.5). Suppose that a service completion or a negative arrival occur in τ_0, then the state space of the service process, M_k is defined as follows:

$$M_k = \begin{cases} 0, & \text{if there is no customer in the system; or} \\ & \text{a service completion occurs in } \tau_k, k \geq 1\text{; or} \\ & \text{a negative customer arrives in } \tau_k, k \geq 1\text{;} \\ \min(k, J), & \text{if the next service completion does not occurs} \\ & \text{in } \tau_k, k \geq 1. \end{cases} \tag{4.4}$$

Using the same notation for the state vector, $Z_k = \{n_k, \Lambda_k, \Gamma_k, M_k\}$ which has been defined in Chapter 3, assuming that at the end of the interval τ_{k-1}, the queue length is $n_k \geq 1$, the state of the positive arrival process is 1, the negative arrival process is in state 0 and the state of the service process is $j - 1$. The state vector at the end of the interval τ_{k-1} is $Z_{k-1} = \{n_{k-1}, 1, 0, j - 1\}$. Let $j^* = \min(j, J)$, possible state vector, Z_k, that could be obtained at the end of the interval τ_k are:

1. A positive customer arrives to the system and $Z_k = \{n_k + 1, 0, 0, j^*\}$.
2. A negative customer arrives to the system and $Z_k = \{n_k - 1, 1, 1, 0\}$.
3. A service completion occurs in the system and $Z_k = \{n_k - 1, 1, 0, 0\}$.
4. None of the above events occurs and $Z_k = \{n_k, 1, 0, j^*\}$.

When the queue is empty at the end of of the interval $\tau_{k-1}, n_{k-1} = 0$, the state vector is $Z_{k-1} = \{0, 1, 1, 0\}$. The following are the possible state vector Z_k at the end of the next interval τ_k.

1. A positive customer arrives to the system and $Z_k = \{1, 0, 0, j^*\}$.
2. A negative customer arrives to the system and $Z_k = \{0, 1, 1, 0\}$.
3. None of the above events occurs and $Z_k = \{0, 1, 0, 0\}$.

Once a positive customer arrive to the system, it starts the service immediately if the queue is empty upon its arrival. Hence, the state of the service process at the end of that particular interval is $j^* = \min(1, J)$.

4.1.3 Stationary Probabilities

The probabilities $p^{(k)}_{nirj}$ can be obtained using the method similar to Section 3.1.3. For example, if the state vector at the end of the interval τ_3 is $Z_3 = \{1, 1, 0, j-1\}$ with the probability $p^{(3)}_{110(j-1)}$, then the following equations could be obtained at the end of the interval τ_4.

1. A positive customer arrives to the system, $p^{(4)}_{200j^*} = p^{(3)}_{110(j-1)}(\lambda \Delta t)$.
2. A negative customer arrives to the system, $p^{(4)}_{0110} = p^{(3)}_{110(j-1)}(\gamma \Delta t)$.
3. A completion of service occurs in the system, $p^{(4)}_{0100} = p^{(3)}_{110(j-1)}(\mu_j \Delta t)$.
4. None of the above events occurs, $p^{(4)}_{110j^*} = p^{(3)}_{110(j-1)}(1 - \lambda \Delta t - \gamma \Delta t - \mu_j \Delta t)$

When $k \to \infty$, the set of equations to find the stationary probabilities, p_{nirj} is obtained as follows:

$$\begin{aligned}
n = 0, \quad p_{n100} &\cong (p_{n100} + p_{n110})(1 - \lambda\Delta t - \gamma\Delta t) + p_{(n+1)110}\mu_1\Delta t \\
&+ p_{(n+1)001}\mu_2\Delta t + \sum_{j=0}^{J} p_{(n+1)10j}\mu_{\min(j+1,J)}\Delta t \qquad (4.5)\\
n = 0, \quad p_{n110} &\cong (p_{n100} + p_{n110} + p_{(n+1)110} + p_{(n+1)001})\gamma\Delta t \\
&+ \sum_{j=0}^{J} p_{(n+1)10j}\gamma\Delta t \qquad (4.6)\\
n = 1, \quad p_{n102} &\cong (p_{n001} + p_{n101})(1 - \lambda\Delta t - \gamma\Delta t - \mu_2\Delta t) \qquad (4.7)\\
n = 1, \quad p_{n10j} &\cong p_{n10(j-1)}(1 - \lambda\Delta t - \gamma\Delta t - \mu_j\Delta t), \\
&\text{for } 3 \le j \le J-1 \qquad (4.8)\\
n = 1, \quad p_{n10J} &\cong (p_{n10(J-1)} + p_{n10J})(1 - \lambda\Delta t - \gamma\Delta t - \mu_J\Delta t) \qquad (4.9)\\
n \ge 1, \quad p_{n001} &\cong (p_{(n-1)100} + p_{(n-1)110})\lambda\Delta t \qquad (4.10)\\
n \ge 1, \quad p_{n100} &\cong p_{(n+1)110}(\mu_1\Delta t) + \sum_{j=1}^{J} p_{(n+1)00j}\mu_{\min(j+1,J)}\Delta t \\
&+ \sum_{j=0}^{J} p_{(n+1)10j}\mu_{\min(j+1,J)}\Delta t \qquad (4.11)\\
n \ge 1, \quad p_{n101} &\cong (p_{n100} + p_{n110})(1 - \lambda\Delta t - \gamma\Delta t - \mu_1\Delta t) \qquad (4.12)\\
n \ge 1, \quad p_{n110} &\cong p_{(n+1)110}\gamma\Delta t + \sum_{j=1}^{J} p_{(n+1)00j}\gamma\Delta t \\
&+ \sum_{j=0}^{J} p_{(n+1)10j}\gamma\Delta t \qquad (4.13)\\
n = 2, \quad p_{n002} &\cong (p_{(n-1)001} + p_{(n-1)101})\lambda\Delta t \qquad (4.14)\\
n = 2, \quad p_{n00j} &\cong p_{(n-1)10(j-1)}\lambda\Delta t, \quad \text{for } 3 \le j \le J-1 \qquad (4.15)\\
n = 2, \quad p_{n00J} &\cong (p_{(n-1)10(J-1)} + p_{(n-1)10J})\lambda\Delta t \qquad (4.16)\\
n \ge 2, \quad p_{n10j} &\cong (p_{n00(j-1)} + p_{n10(j-1)})(1 - \lambda\Delta t - \gamma\Delta t - \mu_j\Delta t), \\
&\text{for } 2 \le j \le J-1 \qquad (4.17)\\
n \ge 2, \quad p_{n10J} &\cong (p_{n00(J-1)} + p_{n00J} + p_{n10(J-1)} + p_{n10J})(1 - \lambda\Delta t - \gamma\Delta t \\
&-\mu_J\Delta t) \qquad (4.18)
\end{aligned}$$

$$n \geq 3, \quad p_{n00j} \cong (p_{(n-1)00(j-1)} + p_{(n-1)10(j-1)})\lambda\Delta t, \quad \text{for } 2 \leq j \leq J-1 \tag{4.19}$$

$$n \geq 3, \quad p_{n00J} \cong (p_{(n-1)00(J-1)} + p_{(n-1)00J} + p_{(n-1)10(J-1)} + p_{(n-1)10J})\lambda\Delta t \tag{4.20}$$

4.2 Stationary Queue Length Distribution

In previous chapter, the *M*/*M*/1 queueing model was solved using generating function method. However, this method is not applicable to solve the set of equations that is derived in Section 4.1.3 for an *M*/*CAR*/1 queueing model. Hence, a matrix recursive method is used to solve this system of equations.

To solve the set of Equations (4.5) to (4.18), the following notations are first introduced:

$\mathbf{P}_n^*$: Column vector that contains all the stationary probabilities of size n.

$\mathbf{Q}_{n,m}$: Coefficient matrix that relates $\mathbf{P}_m^*$ to $\mathbf{P}_n^*$.

$\mathbf{I}_n$: Ientity matrix of the same size as the number of columns for $\mathbf{P}_n^*$

When $n = 1$, Equation (4.7) to (4.13) can be represented in the matrix form as follows:

$$\mathbf{P}_1^* = \mathbf{Q}_{1,0}\mathbf{P}_0^* + \mathbf{Q}_{1,1}\mathbf{P}_1^* + \mathbf{Q}_{1,2}\mathbf{P}_2^* \tag{4.21}$$

By simple matrix operations, $\mathbf{P}_1^*$ in (4.21) can be obtained in term of $\mathbf{P}_0^*$ and $\mathbf{P}_2^*$ and simplified to the following:

$$
\begin{aligned}
(\mathbf{I}_1 - \mathbf{Q}_{1,1})\mathbf{P}_1^* &= \mathbf{Q}_{1,0}\mathbf{P}_0^* + \mathbf{Q}_{1,2}\mathbf{P}_2^* \\
\mathbf{P}_1^* &= \mathbf{Q}_{1,0}^s\mathbf{P}_0^* + \mathbf{Q}_{1,2}^s\mathbf{P}_2^*
\end{aligned} \tag{4.22}
$$

where $\mathbf{Q}_{n,m}^s$ is the coefficient matrix that relates $\mathbf{P}_m^*$ to $\mathbf{P}_n^*$ which is obtained after performing some matrix operations that involves matrix inversion and $\mathbf{I}_n$.

When $n = 2$, Equations (4.10) to (4.18) can be expressed as follows:

$$
\mathbf{P}_2^* = \mathbf{Q}_{2,1}\mathbf{P}_1^* + \mathbf{Q}_{2,2}\mathbf{P}_2^* + \mathbf{Q}_{2,3}\mathbf{P}_3^* \tag{4.23}
$$

By substituting $\mathbf{P}_1^*$ in Equation (4.22) into Equation (4.23) and solve for $\mathbf{P}_2^*$ in term of $\mathbf{P}_0^*$ and $\mathbf{P}_3^*$, the following system of equations in the matrix form can be obtained.

$$
\begin{aligned}
\mathbf{P}_2^* &= \mathbf{Q}_{2,1}(\mathbf{Q}_{1,0}^s\mathbf{P}_0^* + \mathbf{Q}_{1,2}^s\mathbf{P}_2^*) + \mathbf{Q}_{2,2}\mathbf{P}_2^* + \mathbf{Q}_{2,3}\mathbf{P}_3^* \\
(\mathbf{I}_2 - \mathbf{Q}_{2,1}\mathbf{Q}_{1,2}^s - \mathbf{Q}_{2,2})\mathbf{P}_2^* &= \mathbf{Q}_{2,1}\mathbf{Q}_{1,0}^s\mathbf{P}_0^* + \mathbf{Q}_{2,3}\mathbf{P}_3^* \\
\mathbf{P}_2^* &= \mathbf{Q}_{2,0}^s\mathbf{P}_0^* + \mathbf{Q}_{2,3}^s\mathbf{P}_3^*
\end{aligned} \tag{4.24}
$$

For $n \geq 3$, a general form of expression can be obtained from Equations (4.10) to (4.13) and (4.17) to (4.20) and presented as:

$$
\mathbf{P}_n^* = \mathbf{Q}_{n,n-1}\mathbf{P}_{n-1}^* + \mathbf{Q}_{n,n}\mathbf{P}_n^* + \mathbf{Q}_{n,n+1}\mathbf{P}_{n+1}^* \tag{4.25}
$$

The same computation in getting Equation (4.24) is performed iteratively to obtain the following matrix equation for $n \geq 3$:

$$
\mathbf{P}_n^* = \mathbf{Q}_{n,0}^s\mathbf{P}_0^* + \mathbf{Q}_{n,n+1}^s\mathbf{P}_{n+1}^* \tag{4.26}
$$

Then, let $N{+}1$ be an integer that is large enough such that all the stationary probabilities with size $N+1$ are approximate to zero, $p_{(N+1)irj} \cong 0$ or $\mathbf{P}^*_{N+1} = \mathbf{0}$. With this assumption, Equation (4.26) can be simplified to:

$$\mathbf{P}^*_N = \mathbf{Q}^s_{N,0}\mathbf{P}^*_0 \tag{4.27}$$

Next, a backward iterative procedure is performed by substituting Equation (4.27) into (4.26) for $n = N-1$ and yields the expressions:

$$\begin{aligned}\mathbf{P}^*_{N-1} &= \mathbf{Q}^s_{N-1,0}\mathbf{P}^*_0 + \mathbf{Q}^s_{N-1,N}\mathbf{P}^*_N \\ &= \mathbf{Q}^s_{N-1,0}\mathbf{P}^*_0 + \mathbf{Q}^s_{N-1,N}\mathbf{Q}^s_{N,0}\mathbf{P}^*_0 \\ &= \mathbf{Q}^f_{N,0}\mathbf{P}^*_0 \end{aligned} \tag{4.28}$$

where $\mathbf{Q}^f_{n,0}$ is another coefficient matrix that relates $\mathbf{P}^*_0$ to $\mathbf{P}^*_n$. Then, for $n = N-2, N-3, \cdots, 1$, the following general expression can be obtained.

$$\mathbf{P}^*_n = \mathbf{Q}^f_{n,0}\mathbf{P}^*_0 \tag{4.29}$$

When $n = 0$, Equation (4.5) and (4.6) can be written as

$$\mathbf{P}^*_0 = \mathbf{Q}^s_{0,0}\mathbf{P}^*_0 + \mathbf{Q}^s_{0,1}\mathbf{P}^*_1 \tag{4.30}$$

Letting $n = 1$ in Equation (4.29) and substituting into Equation (4.30) yields

$$\mathbf{P}^*_0 = \mathbf{Q}^f_{n,0}\mathbf{P}^*_0 \tag{4.31}$$

Since the sum total of the all the probabilities is always 1, the following is obtained.

$$\sum_{n=0}^{N}\sum_i\sum_r\sum_j p_{nirj} \cong 1 \tag{4.32}$$

An inspection on the system of equations in (4.31) reveals that there is one linearly dependent equation. Hence, by substituting one of the equations in (4.31) with Equation (4.32), the value of the stationary probabilities p_{01r0} for $r = 0, 1$ can be found. By substituting the value of p_{n1r0} into Equation (4.29), the numerical values for all the p_{nirj} can be obtained. The stationary probabilities that there are n positive customers in the queue are then found by

$$p_n = \sum_i \sum_r \sum_j p_{nirj} \tag{4.33}$$

Once, stationary queue length distribution is obtained from Equation (4.33), the mean queue length $E(N)$ can be obtained using

$$\begin{aligned} E(N) &= \sum_{n=0}^{N} np_n \\ &= \sum_{n=0}^{N} \sum_i \sum_r \sum_j np_{nirj} \end{aligned} \tag{4.34}$$

4.3 K-layers Geometrical Linear Extrapolation (K-GLE)

In Equation (4.3), t_s is the point at which the hazard rate starts to converge to a constant value. As shown in Equation (4.3), J can be calculated by choosing a small value of Δt. In solving the set of Equations from (4.5) to (4.20), the smaller the value of Δt, the more precise the solution is obtained. However, a small value of Δt results in a large value of J which subsequently lead to large dimension of the matrix equation. For example, if $t_s = 1$ and $\Delta t = 0.001$, then $J = 1000$, which also indicates the dimension of a matrix is at least 1000 $\times$ 1000. To cope with this, an modified extrapolation method is be introduced. This method can avoid from solving the infinitely large dimension of matrix while obtaining the values for stationary probabilities when $\Delta t \approx 0$.

Let $p_{n,\Delta t_x} = a_x$ be the stationary probability at queue length equal to n that is obtained using $\Delta t = \Delta t_x$, for $x = 1, 2, 3, \cdots, K+2$ where $\Delta t_1 > \Delta t_2 > \cdots > c > 0$, $\Delta t_x - \Delta t_{x+1} = c$ and K is the number of layers that are needed to be performed. The selection of the value c can be vary. However, for accuracy purpose, the author has performed some verification on the selections where c should always be less than or equal to 0.01 and this also indirectly help in the selection of Δt_x.

For example, consider a 2-layers geometrical linear Extrapolation (2-GLE) and $c = 0.01$, then $\Delta t_x = \{\Delta t_1, \Delta t_2, \Delta t_3, \Delta t_4\} = \{0.04, 0.03, 0.02, 0.01\}$ based on the value of $K = 2$. By solving the value of $p_{n,\Delta t_x}$ for all n using the method introduced in the previous section, the values of $x, \Delta t_x$ and $p_{n,\Delta t_x}$ can be tabulated in Table 4.1.

Table 4.1: An Example of The Stationary Probability $p_{n,\Delta t_x}$, when $\Delta t_x = \{0.04, 0.03, 0.02, 0.01\}$.

x	1	2	3	4	5
Δt_x	0.04	0.03	0.02	0.01	0.00
$p_{n,\Delta t_x}$	a_1	a_2	a_3	a_4	$\widehat{a_5}$

The value of $\widehat{a_5}$ in Table 4.1 is an estimated value for p_0 when $\Delta t_x = 0$. This value can be extrapolated by using the existing set of values $\{a_1, a_2, a_3, a_4\}$. However, this set of values are not necessary form a linear relationship. Hence, a transformation is performed on this set of values to obtain a linear relationship. It can be noticed easily that when two values are closed, the ratio of them will be approximately equal to 1 and this strengthens the linear relationship of a set of non-linear values. In this method, the first-layer geometrical ratio, $r_{1,x}$ and

k-layer geometrical ratio, $r_{k,x}$ are given by:

$$r_{1,x} = \frac{a_x}{a_{x-1}}, \quad \text{for } x = 2, \cdots, K+2; \text{ and} \tag{4.35}$$

$$r_{k,x} = \frac{r_{k-1,x}}{r_{k-1,x-1}}, \quad \text{for } x = k+1, \cdots, K+2 \text{ and } 2 \leq k \leq K \tag{4.36}$$

Since $r_{k,x} \leq r_{k-1,x}$, the higher the layers of the geometrical ratio, the more likely for the set of values of the ratios to form a linear relationship. Table 4.2 shows the 2-layer geometrical ratios calculated from the values of a_1 to a_4 in Table 4.1.

Table 4.2: The 2 Layers Geometrical Ratios of The Stationary Probability p_n, when $\Delta t_x = \{0.04, 0.03, 0.02, 0.01\}$.

x	1	2	3	4	5
Δt_x	0.04	0.03	0.02	0.01	0.00
$p_{n,\Delta t_x}$	a_1	a_2	a_3	a_4	$\widehat{a}_5$
$r_{1,x}$		$r_{1,2} = \frac{a_2}{a_1}$	$r_{1,3} = \frac{a_3}{a_2}$	$r_{1,4} = \frac{a_4}{a_3}$	$\widehat{r}_{1,5}$
$r_{2,x}$			$r_{2,3} = \frac{r_{1,3}}{r_{1,2}}$	$r_{2,4} = \frac{r_{1,4}}{r_{1,3}}$	$\widehat{r}_{2,5}$

From Table 4.2, the extrapolated value of $\widehat{r}_{2,5}$ can be calculated using linear equations:

$$\widehat{r}_{2,5} = r_{2,4} + \frac{r_{2,4} - r_{2,3}}{0.01 - 0.02}(0 - 0.01)$$

Then, $\widehat{r}_{1,5}$ can be obtained as follows:

$$\widehat{r}_{1,5} = r_{1,4}\widehat{r}_{2,5}$$

Hence, the estimated value of the stationary probability p_n at $\Delta t = 0$ can be found by

$$\widehat{a}_5 = a_4\widehat{r}_{1,5}$$

In general, the estimation process can be written into the following two equations where capital K is the value in K-GLE method and small letter $k = K-1, K-2, \cdots, 1$:

$$\widehat{r}_{k,K+3} = r_{k,K+2}\widehat{r}_{k+1,K+3} \quad (4.37)$$

$$\widehat{a}_{K+3} = a_{K+2}\widehat{r}_{1,K+3} \quad (4.38)$$

By repeating the extrapolation and estimation process using different n for $n = 0, 1, 2, \cdots$, the stationary queue length distribution can be obtained easily without the involvement of large dimension matrix that happens when Δt goes smaller than 0.01.

In general, the procedure of obtaining stationary queue length using proposed method with K-GLE extrapolation can be written as follows:

1. Define the layer of K-GLE that is needed to be performed.
2. Define the value of c and list out Δt_x accordingly.
3. Applying the proposed method in Sections 4.2 to calculate stationary probability $p_{n,\Delta t_x}$ using all Δt_x for $n = 0, 1, 2, \cdots$.
4. Tabulate the result in Step 3 into the format as shown in Table 4.1.
5. Calculate all the required layers of geometrical ratio just like Table 4.2.
6. Perform a linear extrapolation on the Kth layer geometrical ratio to obtain $\widehat{r}_{K,K+3}$.
7. Estimate the geometrical ratio at $\Delta t = 0$ in the $(K-1)$th layer using Equation (4.37).
8. Repeat Step 7 until the $\widehat{r}_{1,K+3}$ is obtained and apply Equation (4.38) to estimate the stationary probabilities when $\Delta t = 0$.

9. Obtain the stationary queue length distribution by repeating Step 4 to Step 8 using different value of n for $n = 0, 1, 2, \cdots$.

The higher the layer of the geometrical ratios, the more accurate the numerical answers can be computed. Examples presented in Sections 4.4, 5.3 and 6.3 have shown that the estimated values with accuracy of at least 5 decimal places can be obtained for $K = 2$ or 3.

4.4 Numerical Example

Consider a queueing system in which the service time of the positive customers (T_b) has a gamma distribution with parameter $(\kappa, \theta) = (2, 1.1)$ where κ is the shape parameter and θ is the scale parameters of gamma distribution. While both the interarrival time of positive customers T_a and negative customers follow exponential distribution with rate $\lambda = 2$ and $\gamma = 2.5$ respectively. The utilization factor of the ordinary queue without negative customers is $\rho = E(T_b)/E(T_a) > 1$. If $\rho > 1$, the ordinary queue is an unstable queue. With the arrival of negative customers, the utilization factor of the queue can be reduced to $\rho < 1$.

By using the matrix recursive method and the extrapolation method introduced in Sections 4.2 and 4.3, the stationary queue length distribution of *M/CAR/*1 queieng model with negative customers is obtained. The stationary queue length distribution may also be computed using the simulation procedure and the existing generating function method proposed by Harrison and Pitel (1996). A comparison of results is tabulated in Table 4.3 where the simulation procedure in Appendix A is carried out with a running step of 10^8.

From Table 4.3, it is obvious that the stationary queue length distribution obtained using proposed method with the combination of 2-GLE is closed to the analytic method that had been published in Harrison and Pitel (1996). In Harrison and Pitel (1996), the author has expressed the stationary queue length distribution in a complex form where Laplace-Stieltjes transform (LST) is required in solving the basic performance measure. In this proposed method, only matrix operation and linear extrapolation is needed to solve the stationary queue length distribution and mean queue length of the model. However, the results obtained by another linear extrapolation (LE) and simulation procedure are slightly difference from those found by the analytic approach.

Table 4.3: Stationary Queue Length Distribution Computed from the Proposed Numerical Method, Generating Function Method and Simulation Procedure.
$[(\kappa, \theta) = (2, 1.1), \lambda = 2, \gamma = 2.5]$

Queue	Stationary Queue Length Distribution, p_n			
Length,	Numerical Method		Analytical	Simulation
n	2-GLE	LE	Method	Procedure
0	0.256889	0.256906	0.256889	0.256680
1	0.195293	0.195295	0.195293	0.195312
2	0.145393	0.145390	0.145393	0.145261
3	0.107270	0.107266	0.107270	0.107213
4	0.078828	0.078825	0.078828	0.078874
5	0.057825	0.057822	0.057825	0.057841
6	0.042385	0.042382	0.042385	0.042383
7	0.031056	0.031054	0.031056	0.031089
8	0.022752	0.022750	0.022752	0.022824
9	0.016667	0.016665	0.016667	0.016788
10	0.012209	0.012208	0.012209	0.012277
⋮	⋮	⋮	⋮	⋮
25	0.000115	0.000114	0.000115	0.000114
Mean Queue Length	2.797456	2.795095	2.797459	2.808651

In term of mean queue length, compare to the simulation procedure and linear extrapolation with matrix operation, the result obtained from matrix operation and 2-GLE method are closer to the one obtained using analytical method. While calculating the mean queue length, method in Harrison and Pitel (1996) requires the knowledge of differentiation and Laplace-Stieltjes transform (LST).

However, it is easier and simple to compute mean queue length in the alternative method as shown in Equation (4.34).

These group of result is then verified using statistical testing. Wilcoxon signed-rank test is used to determine whether the two independent results obtained using different methods having the same distribution.

For Wilcoxon signed-rank test, the general procedure is as follows:

1. Hypothesis:

   H_0 : The results obtained from the two methods have the same distribution.

   H_1 : The results obtained from the two methods do not have the same distribution.

2. Test statistics:

   (a) Obtain the difference of each pair of n.
   (b) Separate the results into signed (+ and -) and the absolute difference.
   (c) Rank the absolute difference from 1 to N (smallest to largest).
   (d) Sum the rank for negative sign (W^-) and positive sign respectively (W^+).
   (e) Obtain the test statistics as W = min(W^-,W^+).

3. Decision rules:

   - Reject null hypothesis if the critical value obtained from Wilcoxon signed-rank test table is larger than the Test statistics W or if the p-value obtained from the test is smaller than the significance value α.

- Do not reject null hypothesis if the critical value obtained from Wilcoxon signed-rank test table is smaller than the test statistics W or if the p-value obtained from the test is larger than the significance value α.

Table 4.4 below shows the test statistics and p-value obtained from the Wilcoxon signed -rank test that applied to the stationary queue length distribution obtained in Table 4.3.

Table 4.4: Wilcoxon Signed-Rank Test on the Stationary Queue Length Distributions Obtained Using *M/CAR/1*, *M/G/1* and Simulation Procedure.

Wilcoxon Signed-Rank Test	2-GLE vs Analytical Method	2-GLE vs Simulation
Test Statistics	104	165
P-value	0.464745	0.789716
Conclusion	Do not reject null hypothesis	Do not reject null hypothesis

With $\alpha = 0.05$, the critical value found in the Wilcoxon signed-rank test table is 89. From Table 4.4, it can be seen that the test statistics and p-value shows that there is not significance evidence to say the stationary queue length distribution obtained using various methods are from different distributions. This result applies to both the case where the stationary queue length distribution computed from 2-GLE is compared to the one obtained using either analytical method or simulation procedure.

In the next chapter, the interarrival time of positive customers is assumed to follow *CAR* while the service time is remained to have an exponential distribution.

CHAPTER 5

STATIONARY QUEUE LENGTH DISTRIBUTION OF *CAR/M/*1 AND *GI/M/*1 G-QUEUE USING RCH REMOVING DISCIPLINE

This chapter presents both numerical and analytical methods in solving the stationary queue length distribution of *CAR/M/*1 and G/M/1 queues with negative customers respectively. The proposed numerical approach is first discussed in Section 5.1 while the analytical method is presented in Section 5.2. The numerical results for the two methods are shown in Section 5.3 and compared to the results obtained by the simulation procedure given in Appendix A.

5.1 *CAR/M/*1 Queue with Negative Customers

5.1.1 Derivation of Equations for Stationary Probabilities

Transition Probabilities and Events

The assumptions made in this chapter are as follows:

- interarrival time of positive customers follow *CAR*.
- Service time follows exponential distribution with rate μ.
- Negative customers arrive to the queue according to Poisson Process with the rate γ.
- The single-server queue has infinite capacity with FCFS queueing discipline.

Let $f(t)$ be the probability density function (PDF) and $\bar{F}(t)$ be the survival function (SF) of the interarrival time of positive customers. Then, the hazard rate function λ_k in each small interval, τ_k is expressed as follows:

$$\lambda_k = \frac{f(k\Delta t)}{\bar{F}(k\Delta t)}, \quad 1 \leq k \leq I \tag{5.1}$$

where integer I is large enough such that

$$\lambda_I \simeq \lim_{k\to\infty} \mu_k. \tag{5.2}$$

I can be calculated as

$$I = \frac{t_a}{\Delta t} \tag{5.3}$$

where t_a is the time point defined where the hazard rate of the interarrival time distribution starts to converge just like t_s in Figure 4.1. Then, the transition probabilities can be obtained and interpreted as follows:

$\lambda_k \Delta t$: the conditional probability that a positive customer arrives in the interval τ_k given there is no positive arrival in $\tau_1, \tau_2, \cdots, \tau_{k-1}$.

$\gamma \Delta t$: the probability that a negative customer arrives in an interval.

$\mu \Delta t$: the probability that a service completion occurs in an interval.

State Spaces

Since the distributions for service time of positive customers and interarrival time of negative customers are exponential, notations for the state spaces of the service process and the negative arrival process are the same as Equations (3.5) and (3.6) respectively. Assuming that the first interval is τ_0 and there is a positive arrival in this interval. Then, the state space of positive arrival process, Λ_k is defined as follows:

$$\Lambda_k = \begin{cases} 0, & \text{if } \tau_k = \tau_0\text{; or} \\ & \text{a positive customer arrives in } \tau_k, k \geq 1. \\ \min(k, I), & \text{if no positive customer arrives in} \tau_k, k \geq 1. \end{cases} \quad (5.4)$$

Let the state vector be $Z_k = \{n_k, \Lambda_k, \Gamma_k, M_k\}$, assuming that at the end of the interval τ_{k-1}, the queue length is $n_k \geq 1$, the state of the positive arrival process is $i-1$, the negative arrival process is in state 0 and the state of the service process is 1. The state vector at the end of the interval τ_{k-1} is $Z_{k-1} = \{n_k, i-1, 0, 1\}$. Let $i^* = \min(i, I)$, the state vector at the end of the next interval τ_k can be listed as follows:

1. A positive customer arrives to the system and $Z_k = \{n_k + 1, 0, 0, 1\}$.

2. A negative customer arrives to the system and $Z_k = \{n_k - 1, i^*, 1, 0\}$.

3. A service completion occurs in the system and $Z_k = \{n_k - 1, i^*, 0, 0\}$.

4. None of the above events occurs and $Z_k = \{n_k, i^*, 0, 1\}$.

When the queue is empty at the end of of the interval $\tau_{k-1}, n_k = 0$. Assuming that the state vector is $Z_{k-1} = \{0, i-1, 1, 0\}$ at the end of the interval τ_{k-1}, the possible state vector Z_k at the end of the next interval τ_k are as follows:

1. A positive customer arrives to the system and $Z_k = \{1, 0, 0, 0\}$.

2. A negative customer arrives to the system and $Z_k = \{0, i^*, 1, 0\}$.

3. None of the above events occurs and $Z_k = \{0, i^*, 0, 0\}$.

Stationary Probabilities

The following is an example of finding the probabilities $p_{nirk}^{(k)}$ at the end of the interval τ_k for $k = 6$:

Let $Z_5 = \{2, i-1, 0, 1\}$ be the state vector at the end of the interval τ_5, then the following p_{nirj}^{6} could be obtained by taking into consideration the events that could possibly occur in the interval τ_6:

1. A positive customer arrives to the system, $p_{3001}^{(6)} = p_{2(i-1)01}^{(5)}(\lambda_i \Delta t)$.
2. A negative customer arrives to the system, $p_{1i^*10}^{(6)} = p_{2(i-1)01}^{(5)}(\gamma \Delta t)$.
3. A service completion occurs in the system, $p_{1i^*00}^{(6)} = p_{2(i-1)01}^{(5)}(\mu \Delta t)$.
4. None of the above events occurs, $p_{2i^*01}^{(6)} = p_{2(i-1)01}^{(5)}(1-\lambda_i\Delta t-\gamma\Delta t-\mu\Delta t)$

When $k \to \infty$, for all the value of n, the stationary probabilities $p_{nirj} = \lim\limits_{k\to\infty} p_{nirk}^{(k)}$ are obtained as follows for $2 \leq i \leq I-1$ and I is defined in Equation (5.3):

$$\begin{aligned} n=0, \quad p_{ni00} &\cong (p_{(n+1)(i-1)00} + p_{(n+1)(i-1)01} + p_{(n+1)(i-1)10})\mu\Delta t \\ &+ (p_{n(i-1)00} + p_{n(i-1)10})(1 - \lambda_i\Delta t - \gamma\Delta t) \qquad (5.5) \\ n=0, \quad p_{nI00} &\cong (p_{(n+1)(I-1)00} + p_{(n+1)(I-1)01} + p_{(n+1)(I-1)10})\mu\Delta t \\ &+ (p_{(n+1)I00} + p_{(n+1)I01} + p_{(n+1)I10})\mu\Delta t \\ &+ (p_{n(I-1)00} + p_{n(I-1)10} + p_{nI00} \\ &\quad + p_{nI10})(1 - \lambda_I\Delta t - \gamma\Delta t) \qquad (5.6) \\ n=0, \quad p_{ni10} &\cong (p_{(n+1)(i-1)00} + p_{(n+1)(i-1)01} + p_{(n+1)(i-1)10})\gamma\Delta t \\ &+ (p_{n(i-1)00} + p_{n(i-1)10})\gamma\Delta t \qquad (5.7) \end{aligned}$$

$$
\begin{aligned}
n=0, \quad p_{nI10} &\cong (p_{(n+1)(I-1)00} + p_{(n+1)(I-1)01} + p_{(n+1)(I-1)10})\gamma\Delta t \\
&+ (p_{(n+1)I00} + p_{(n+1)I01} + p_{(n+1)I10})\gamma\Delta t \\
&+ (p_{n(I-1)00} + p_{n(I-1)10} + p_{nI00} + p_{nI10})\gamma\Delta t \qquad (5.8)
\end{aligned}
$$

$$
n \geq 0, \quad p_{n100} \cong p_{(n+1)001}\mu\Delta t \qquad (5.9)
$$

$$
n \geq 0, \quad p_{n110} \cong p_{(n+1)001}\gamma\Delta t \qquad (5.10)
$$

$$
n=1, \quad p_{n001} \cong \sum_{i=1}^{I}(p_{(n-1)i00} + p_{(n-1)i10})\lambda_{\min(i+1,I)}\Delta t \qquad (5.11)
$$

$$
n \geq 1, \quad p_{ni00} \cong (p_{(n+1)(i-1)00} + p_{(n+1)(i-1)01} + p_{(n+1)(i-1)10})\mu\Delta t \qquad (5.12)
$$

$$
\begin{aligned}
n \geq 1, \quad p_{nI00} &\cong (p_{(n+1)(I-1)00} + p_{(n+1)(I-1)01} + p_{(n+1)(I-1)10})\mu\Delta t \\
&+ (p_{(n+1)I00} + p_{(n+1)I01} + p_{(n+1)I10})\mu\Delta t \qquad (5.13)
\end{aligned}
$$

$$
n \geq 1, \quad p_{ni10} \cong (p_{(n+1)(i-1)00} + p_{(n+1)(i-1)01} + p_{(n+1)(i-1)10})\gamma\Delta t \qquad (5.14)
$$

$$
\begin{aligned}
n \geq 1, \quad p_{nI10} &\cong (p_{(n+1)(I-1)00} + p_{(n+1)(I-1)01} + p_{(n+1)(I-1)10})\gamma\Delta t \\
&+ (p_{(n+1)I00} + p_{(n+1)I01} + p_{(n+1)I10})\gamma\Delta t \qquad (5.15)
\end{aligned}
$$

$$
n \geq 1, \quad p_{n101} \cong p_{n001}(1 - \lambda_1\Delta t - \gamma\Delta t - \mu\Delta t) \qquad (5.16)
$$

$$
\begin{aligned}
n \geq 1, \quad p_{ni01} &\cong (p_{n(i-1)00} + p_{n(i-1)01} + p_{n(i-1)10})(1 - \lambda_i\Delta t - \gamma\Delta t \\
&\quad -\mu\Delta t) \qquad (5.17)
\end{aligned}
$$

$$
\begin{aligned}
n \geq 1, \quad p_{nI01} &\cong (p_{n(I-1)00} + p_{n(I-1)01} + p_{n(I-1)10} + p_{nI00} + p_{nI01} \\
&\quad + p_{nI10})(1 - \lambda_I\Delta t - \gamma\Delta t - \mu\Delta t) \qquad (5.18)
\end{aligned}
$$

$$
\begin{aligned}
n \geq 2, \quad p_{n001} &\cong \sum_{i=1}^{I}(p_{(n-1)i00} + p_{(n-1)i01} + p_{(n-1)i10})\lambda_{\min(i+1,I)}\Delta t \\
&+ p_{(n-1)001}\lambda_1\Delta t \qquad (5.19)
\end{aligned}
$$

5.1.2 Stationary Queue Length Distribution

A backward recursive substitution method is be applied to solve the set of equations derived in Section 5.1.1 to obtain the stationary queue length distribution of *CAR*/*M*/1 queue with negative customers.

By assuming $N+1$ is an integer that is large enough such that $p_{(N+1)irj} = 0$, Equations (5.9) to (5.10) and (5.12) to (5.15) are equal to 0 when $n = N$ as they depends on $p_{(N+1)irj}$. Let $2 \leq i \leq I-1$, then

$$\begin{aligned} p_{N100} &= 0 && (5.20) \\ p_{Ni00} &= 0 && (5.21) \\ p_{NI00} &= 0 && (5.22) \\ p_{N110} &= 0 && (5.23) \\ p_{Ni10} &= 0 && (5.24) \\ p_{NI10} &= 0 && (5.25) \end{aligned}$$

By substituting Equations (5.20) to (5.25) into Equations (5.17) and (5.18), the following equations are obtained:

$$\begin{aligned} p_{Ni01} &\cong p_{N(i-1)01}(1 - \lambda_i \Delta t - \gamma \Delta t - \mu \Delta t) && (5.26) \\ p_{NI01} &\cong (p_{N(I-1)01} + p_{NI01})(1 - \lambda_i \Delta t - \gamma \Delta t - \mu \Delta t) && (5.27) \end{aligned}$$

Then, replacing the value of i with $2 \leq i \leq I-1$ in Equation (5.26) and together with Equations (5.27) and (5.16), a set of expressions in Equation (5.28) below can be yielded.

$$\begin{aligned} p_{N101} &\cong p_{N001}(1 - \lambda_1 \Delta t - \gamma \Delta t - \mu \Delta t) \\ p_{N201} &\cong p_{N101}(1 - \lambda_2 \Delta t - \gamma \Delta t - \mu \Delta t) \\ p_{N301} &\cong p_{N201}(1 - \lambda_3 \Delta t - \gamma \Delta t - \mu \Delta t) \\ &\vdots && (5.28) \\ p_{N(I-2)01} &\cong p_{N(I-3)01}(1 - \lambda_{I-2} \Delta t - \gamma \Delta t - \mu \Delta t) \\ p_{N(I-1)01} &\cong p_{N(I-2)01}(1 - \lambda_{I-1} \Delta t - \gamma \Delta t - \mu \Delta t) \\ p_{NI01} &\cong (p_{N(I-1)01} + p_{NI01})(1 - \lambda_I \Delta t - \gamma \Delta t - \mu \Delta t) \end{aligned}$$

It is obvious that each p_{Nirj} above are dependent on $p_{N(i-1)rj}$. Substituting the first expression into the second expression in Equation (5.28) will obtain:

$$p_{N201} \cong a_{N2} p_{N001} \tag{5.29}$$

where a_{N2} is the coefficient of p_{N001} that corresponds to p_{N201} which is also equal to $(1 - \lambda_1 \Delta t - \gamma \Delta t - \mu \Delta t)(1 - \lambda_2 \Delta t - \gamma \Delta t - \mu \Delta t)$. Next, substituting Equation (5.29) into the third expression of Equation (5.28) obtains:

$$p_{N301} \cong a_{N3} p_{N001} \tag{5.30}$$

where a_{N3} is the coefficient of p_{N001} that corresponds to p_{N301} which is also equal to $a_{N2}(1 - \lambda_3 \Delta t - \gamma \Delta t - \mu \Delta t)$. In general, all the expressions in Equation (5.28) can be recursively substituted and expressed in term of p_{N001} as follows:

$$p_{Ni01} \cong a_{Ni} p_{N001}, \quad 1 \leq i \leq I \tag{5.31}$$

where a_{ni} is the coefficient of p_{N001} that correspond to p_{ni01} after some evaluations.

Since all p_{Nirj} are expressed in term of p_{N001}, any $p_{(N-1)irj}$ that are dependent only on p_{Nirj} can be expressed in term of p_{N001}. When $n = N - 1$, Equations (5.9) to (5.10) and (5.12) to (5.15) can be expressed in term of p_{N001} as follows:

$$p_{(N-1)i00} \cong b_{(N-1)i} p_{N001}, \quad 1 \leq i \leq I \tag{5.32}$$

$$p_{(N-1)i10} \cong c_{(N-1)i} p_{N001}, \quad 1 \leq i \leq I \tag{5.33}$$

where b_{ni} is the coefficient of p_{N001} that corresponds to p_{ni00} and c_{ni} is the coefficient of p_{N001} that corresponds to p_{ni10} for $n < N$. Substituting Equations

(5.32) and (5.33) into Equations (5.17) and (5.18) obtains:

$$
\begin{aligned}
p_{(N-1)i01} &\cong (b_{(N-1)(i-1)}p_{N001} + p_{(N-1)(i-1)01} + c_{(N-1)(i-1)}p_{N001})(1 \\
&\quad -\lambda_i\Delta t - \gamma\Delta t - \mu\Delta t), \quad \text{for } 2 \le i \le I-1 && (5.34) \\
p_{(N-1)I01} &\cong (b_{(N-1)(I-1)}p_{N001} + p_{(N-1)(I-1)01} + c_{(N-1)(I-1)}p_{N001} \\
&\quad +b_{(N-1)I}p_{N001} + p_{(N-1)I01} + c_{(N-1)I}p_{N001})(1 \\
&\quad -\lambda_I\Delta t - \gamma\Delta t - \mu\Delta t) && (5.35)
\end{aligned}
$$

Putting Equations (5.16), (5.34) and (5.35) together form the expressions that are the same as Equations (5.28) in which each $p_{(N-1)irj}$ are dependent on $p_{(N-1)(i-1)rj}$. Then, substituting Equation (5.16) into Equation (5.34) when $i = 2$ yields

$$
p_{(N-1)201} \cong d_{(N-1)2}p_{(N-1)001} + a_{(N-1)2}p_{N001} \qquad (5.36)
$$

where $d_{(N-1)2} = (1 - \lambda_1\Delta t - \gamma\Delta t - \mu\Delta t)(1 - \lambda_2\Delta t - \gamma\Delta t - \mu\Delta t)$ is the coefficient of $p_{(N-1)001}$ that corresponds to $p_{(N-1)201}$ while $a_{(N-1)2} = (b_{(N-1)1} + c_{(N-1)1})(1 - \lambda_2\Delta t - \gamma\Delta t - \mu\Delta t)$ is the coefficient of p_{N001} that corresponds to $p_{(N-1)201}$. Then, the recursive substitution processes can be performed using the similar steps from Equations (5.29) to (5.31) to obtain general expressions as follows:

$$
\begin{aligned}
p_{(N-1)101} &\cong d_{(N-1)1}p_{(N-1)001} && (5.37) \\
p_{(N-1)i01} &\cong d_{(N-1)i}p_{(N-1)001} + a_{(N-1)i}p_{N001}, \quad 2 \le i \le I && (5.38)
\end{aligned}
$$

where d_{ni} is the coefficient of p_{n001} that corresponds to p_{ni01} and a_{ni} is the coefficient of p_{N001} that correspond to p_{ni01} after some mathematical evaluations.

Then, using Equations (5.19) for $n \ge 2$ and Equation (5.11) when $n = 1$ can yield a relationship between p_{n001} and $p_{(n-1)001}$ for $1 \le n \le N$. For instance, when $n = N$, Equation (5.19) can be expressed in term of only $p_{(N-1)001}$ and p_{N001} by substituting Equations (5.32), (5.33), (5.37) and (5.38) into it. Hence, $p_{(N-1)001}$ can be written in term of p_{N001} as well.

$$p_{(N-1)001} \cong K_{(N-1)} p_{N001} \tag{5.39}$$

where K_n is a coefficient of p_{N001} that relates to p_{n001} for $1 \leq n < N$. The substitution process from Equations (5.32) to (5.39) are repeated for $n = N - 2, N - 3, \cdots, 1$. In general, Equations (5.31) to (5.33) and (5.37) to (5.39) can be written as follows:

$$p_{ni00} \cong b_{ni} p_{N001}, \quad 1 \leq i \leq I, \quad 0 \leq n < N \tag{5.40}$$

$$p_{ni10} \cong c_{ni} p_{N001}, \quad 1 \leq i \leq I, \quad 0 \leq n < N \tag{5.41}$$

$$p_{n101} \cong d_{n1} p_{n001}, \quad 1 \leq n < N \tag{5.42}$$

$$p_{ni01} \cong d_{ni} p_{n001} + a_{ni} p_{N001}, \quad 2 \leq i \leq I, \quad 1 \leq n < N \tag{5.43}$$

$$p_{Ni01} \cong a_{Ni} p_{N001}, \quad 1 \leq i \leq I \tag{5.44}$$

$$p_{n001} \cong K_n p_{N001}, \quad 1 \leq n \leq N \tag{5.45}$$

When $n = 0$, either one of the equations from (5.6) and (5.8) will be removed as they are dependent. Then by adding an equation that the sum of all stationary probabilities is always 1, Equations (5.40) to (5.45) can be solved and the value of p_{N001} is obtained. Substituting p_{N001} back into Equations (5.40) to (5.45), all the stationary probabilities can be evaluated. Lastly, the stationary queue length distribution can be obtained by using Equation (4.33) where the mean queue length can be obtained using Equation (4.34). A similar extrapolation process can be done as mentioned in Section 4.3 in order to obtain a result when Δt approach to 0.

5.2 *GI/M/*1 Queue with Negative Customers

In this section, an imbededed Markov chain approach is used to solve an *GI/M/*1 G-queue with RCH killing discipline.

The same assumptions listed in Section 5.1.1 are used. Prior to the arrival of the nth customer, let X_n denotes the number of customers in the system. Then

$$X_{n+1} = X_n + 1 - R_n, \quad R_n \leq X_n + 1, \quad X_n \geq 0, \tag{5.46}$$

where R_n represent the total number of customers that have been served or removed during the interarrival time T_n. T_n is the time or duration between the nth and $(n+1)$th positive arrivals. With the assumption of IID on the customers interarrival times, the random variable T_n can be denoted as T, and its CDF is denoted by $F(t)$. Since the service times can be interrupted by negative arrivals, the actual times that a customer stays in the server can be expressed as $T_q = \min(T_b, T_c)$, where T_b is the service times random variable and T_c is the interarrival times random variable for negative arrivals. Using the distributions assumptions made, the CDF of the random variable T_q can then be obtained using the order statistics as the following:

$$\begin{aligned} Pr(T_q \leq t) &= 1 - Pr(T_b \geq t)Pr(T_b \geq t) \\ &= 1 - e^{-\mu t}e^{-\gamma t} \\ &= 1 - e^{-(\mu+\gamma)t} \end{aligned} \tag{5.47}$$

In Equation (5.47), it can be observed that the time spent for customers in the server, T_q, is an exponential distribution with rate $(\mu + \gamma)$. This also indicates that the random variable R_n is a Poisson process with parameter $(\mu + \gamma)$ which does not depend on the past event happened in the system.

To prove that $X_1, X_2, \dots$ is a Markov chain, it suffices to show that the future state X_{n+1} is independent of the previous state $X_{n-1}, X_{n-2}, \dots$ given the current state. From Equation (5.46), the future state X_{n+1} depends on X_n and R_n. But R_n, the total number of customers leaving the system by either completed their service or removed by negative arrivals during the interarrival time between the nth and $(n+1)$th positive customers depends only on the length of T_n but not on the past events which are the queue size at earlier arrival points, $X_{n-1}, X_{n-2}, \dots$. Hence, the collection of $\{X_0, X_1, X_2, \dots\}$ is a discrete-time Markov chain.

Next, the transition probability for this Markov chain is derived as:

$$p_{ij} = Pr\{X_{n+1} = j | X_n = i\}.$$

p_{ij} depends only on the distribution of the total number of positive arrivals that leave during an interarrival time. Define

$$\begin{aligned} r_m &= Pr\{m \text{ leaving during an interarrival time} | X_n \geq m\} \\ &= Pr\{R_n = m | X_n \geq m\}, \end{aligned}$$

where r_m can be calculated by conditioning on the duration F of the interarrival time:

$$r_m = Pr\{R_n = m | X_n \geq m\} = \int_0^\infty Pr\{R_n = m | F = t\} f(t) dt.$$

Using the conclusion drawn from Equation (5.47), $\{R_n | F = t\}$ is a Poisson process random variable with mean $(\mu + \gamma)t$. Thus

$$Pr\{R_n = m | F = t\} = \frac{e^{-(\mu+\gamma)t}((\mu + \gamma)t)^m}{m!}.$$

This gives

$$r_m = \int_0^\infty \frac{e^{-(\mu+\gamma)t}((\mu+\gamma)t)^m}{m!} dF(t). \tag{5.48}$$

Then, from Equation (5.46)

$$Pr\{X_{n+1} = j | X_n = i\} = Pr\{R_n = i + 1 - j | X_n \geq i + 1 - j\}.$$

The term r_m can be interpreted as the probability of exactly m positive customers leaving the system between the arrivals of two successive positive customers given the condition of at least m positive customers in the system upon the arrival of the first one.

Using Equation (5.48), $\{X_0, X_1, X_2, \dots\}$ can be formulated as an imbedded Markov chain with the following transition probability matrix **P**:

$$\mathbf{P} = \{p_{ij}\} = \begin{pmatrix} 1 - r_0 & r_0 & 0 & 0 & 0 & \dots \\ 1 - \sum_{m=0}^{1} r_m & r_1 & r_0 & 0 & 0 & \dots \\ 1 - \sum_{m=0}^{2} r_m & r_2 & r_1 & r_0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{5.49}$$

Let $\boldsymbol{q} = q_n, n = 0, 1, 2, \dots$ where q_n denotes the stationary probability of n positive customers in the system when a positive arrival is about to occur. Then the stationary equations are obtained as

$$\boldsymbol{q}\mathbf{P} = \boldsymbol{q} \quad \text{and} \quad \sum_{n=0}^{\infty} q_n = 1, \tag{5.50}$$

which bring forth

$$q_0 = \sum_{j=0}^{\infty} q_j \left(1 - \sum_{m=0}^{j} r_m\right) \quad \text{and} \quad q_i = \sum_{m=0}^{\infty} q_{i+m-1} r_m, \quad i \geq 1. \tag{5.51}$$

The probability generating function of r_m can then be obtained using the method of operators on Equation (5.43). Let z be the linear operator and $zq_i = q_{i+1}$. Then Equation (5.43) can be rewritten as

$$q_i - (q_{i-1}r_0 + q_i r_1 + q_{i+1}r_2 + q_{i+2}r_3 + \dots) = 0,$$

and revised as follows:

$$zq_{i-1} - (q_{i-1}r_0 + zq_{i-1}r_1 + z^2 q_{i-1}r_2 + z^3 q_{i-1}r_3 + \dots) = 0,$$

which, on dividing by q_{i-1} yields

$$z - r_0 - zr_1 - z^2 r_2 - z^3 r_3 - \dots = 0,$$

or

$$z = \sum_{m=0}^{\infty} r_m z^m. \tag{5.52}$$

Let $P(z)$ denotes the probability generating function of $\{r_m\}$. Then Equation (5.44) can be written as

$$P(z) = z = \sum_{m=0}^{\infty} r_m z^m. \tag{5.53}$$

By substituting Equation (5.40) into Equation (5.45), the following is derived.

$$
\begin{aligned}
P(z) &= \sum_{m=0}^{\infty} \int_0^{\infty} \frac{e^{-(\mu+\gamma)t}((\mu+\gamma)t)^m}{m!} dF(t) z^m \\
&= \int_0^{\infty} e^{-(\mu+\gamma)t} \sum_{m=0}^{\infty} \frac{((\mu+\gamma)tz)^m}{m!} dF(t) \\
&= \int_0^{\infty} e^{-(\mu+\gamma)t} e^{(\mu+\gamma)tz} dF(t) \\
&= \int_0^{\infty} e^{-(\mu+\gamma)t(1-z)} dF(t) \\
&= F^*[(\mu+\gamma)(1-z)]. \qquad (5.54)
\end{aligned}
$$

where $F^*(z)$ is the Laplace-Stieltjes transform (LST) of the positive customers interarrival times CDF. The characteristic equation $z = F^*[(\mu+\gamma)(1-z)]$ can be solved to obtain solutions that are used in evaluating $\{q_n\}$ has been proved by Gross and Harris (2008). Equation (5.53) has exactly one real root in (0,1) with the assumption that the utilization factor, $\rho < 1$. By denoting r as the root for $P(z)$, the following is the solution to Equation (5.53).

$$q_i = Cr^i. \qquad (5.55)$$

where C is an arbitrary constant.

The constant value of $C = 1 - r$ can be figured out by summing $\{q_i\}$ for $i = 0, 1, 2, \ldots$, and equating the sum to 1. Then, the steady-state arrival point distribution can be expressed in term of the real root r as follows:

$$q_n = (1-r)r^n, \quad n \geq 0, \quad \rho < 1. \qquad (5.56)$$

Next is to solve the root r by successive substitution. The guaranteed convergence numerical procedure given in Gross and Harris (2008) is as follows:

$$r^{(k+1)} = F^*[(\mu+\gamma)(1-r^k)], \quad k = 0, 1, 2, \ldots, \quad 0 < r^0 < 1. \qquad (5.57)$$

By solving the root r, the stationary probabilities $\{q_n\}$ can be obtained. For $n > 0$, the following has been concluded and expressed in Gross and Harris (2008) as

$$p_n = \sum_{i=n-1}^{\infty} q_n \int_0^{\infty} Pr\{i-n+1 \text{ departures in } t\}\lambda[1-F(t)]dt \qquad (5.58)$$

The departures in Equation (5.58) indicate that the positive customers are departing the system by either service completion or removal by negative customers, and λ can be found by taking reciprocal to the mean interarrival time of positive arrivals. It is readily known that p_0 must be equal to $1-\rho$ where ρ can be obtained as the fraction of the mean service and removal time to the mean interarrival time of customers. Hence,

$$p_n = \lambda \sum_{i=n-1}^{\infty} (1-r)r^i \int_0^{\infty} \frac{e^{-(\mu+\gamma)t}((\mu+\gamma)t)^{i-n+1}}{(i-n+1)!}[1-F(t)]dt, \quad n > 0$$

Letting $j = i-n+1$, we get

$$\begin{aligned}
p_n &= \lambda(1-r)r^{n-1}\int_0^{\infty} e^{-(\mu+\gamma)t}[1-F(t)]\sum_{j=0}^{\infty}\frac{((\mu+\gamma)rt)^j}{j!}dt \\
&= \lambda(1-r)r^{n-1}\int_0^{\infty} e^{-(\mu+\gamma)t(1-r)}[1-F(t)]dt \\
&= \lambda(1-r)r^{n-1}\left[-\frac{e^{-(\mu+\gamma)t(1-r)}}{(1-r)(\mu+\gamma)}\bigg|_0^{\infty} - \int_0^{\infty} e^{-(\mu+\gamma)t(1-r)}F(t)dt\right] \\
&= \frac{\lambda}{\mu+\gamma}r^{n-1}\left[1-\int_0^{\infty} e^{-(\mu+\gamma)t(1-r)}F(t)dt\right], \quad n > 0
\end{aligned}$$

and from Equations (5.45) and (5.46),

$$\int_0^{\infty} e^{-(\mu+\gamma)t(1-r)}F(t)dt = F^*[(\mu+\gamma)(1-r)] = P(r) = r.$$

Thus for $n > 0$, the stationary queue length distribution can be obtained as

$$p_n = \frac{\lambda}{\mu+\gamma}r^{n-1}(1-r) = \frac{\lambda q_{n-1}}{\mu+\gamma}, \quad n > 0. \qquad (5.59)$$

Once the Equation (5.59) is obtained, the mean queue length can be computed using the following equation.

$$\begin{aligned} E(N) &= \sum_{n=0}^{\infty} n p_n \\ &= \sum_{n=0}^{\infty} n r^{n-1}(1-r) \\ &= \frac{\lambda(1-r)}{\mu+\gamma} \sum_{n=0}^{\infty} n r^{n-1} \\ &= \frac{\lambda(1-r)}{\mu+\gamma} \frac{d}{dr} \sum_{n=0}^{\infty} r^n \\ &= \frac{\lambda}{(\mu+\gamma)(1-r)} \end{aligned} \tag{5.60}$$

By Little's formulas, $E(N) = \lambda E(W)$, then the mean waiting time of this queueing model is

$$E(W) = \frac{1}{(\mu+\gamma)(1-r)} \tag{5.61}$$

When both the Equations (5.60) and (5.61) are analyzed closely, a similar conclusion can be obtained where a *GI*/*M*/1 queue with negative customer is exactly equal to a ordinary *GI*/*M*/1 without negative customer if the service rate in ordinary *GI*/M/1 queue is modified to the sum of service rate and negative arrival rate.

5.3 Numerical Example

Consider a queueing system in which the interarrival time of the positive customers has a gamma distribution with parameter (κ, θ) $= (2.6, 0.5)$ where κ is the shape parameter and θ is the scale parameters of gamma distribution. While the service time of positive customers and the interarrival time of negative customers follow exponential distribution with rate $\mu = 0.5$ and $\gamma = 0.\overline{4}$ respectively. The utilization factor of the ordinary queue without negative customers is greater than 1 and with the arrival of negative customers, the utilization factor of the queue will be reduced to $\rho \approx 0.82 < 1$.

By using the backward recursive substitution method introduced in Section 5.1.2 and the extrapolation method in Section 4.3, the stationary queue length distribution of *CAR/M/*1 queue with negative customers is found. On the other hand, Equation (5.51) is used to obtain the stationary queue length distribution for *GI/M/*1 queue with negative customers. Simulation procedure in Appendix A is also performed with a running step of 10^8.

Table 5.1: Stationary Queue Length Distribution Computed from the Proposed Numerical Method, Generating Function Method and Simulation Procedure.
$[(\boldsymbol{\kappa}, \boldsymbol{\theta}) = (\mathbf{2.6}, \mathbf{0.5}), \boldsymbol{\mu} = \mathbf{0.5}, \boldsymbol{\gamma} = \mathbf{0.\overline{4}}]$

Queue Length, n	Stationary Queue Length Distribution, p_n			
	Numerical Method		Analytical Method	Simulation Procedure
	2-GLE	LE		
0	0.185518	0.185541	0.185520	0.185721
1	0.211733	0.211795	0.211733	0.212092
2	0.156691	0.156750	0.156691	0.156910
3	0.115957	0.116003	0.115957	0.116106
4	0.085812	0.085843	0.085813	0.085800
5	0.063504	0.063520	0.063505	0.063434
6	0.046996	0.046999	0.046996	0.046892
7	0.034778	0.034772	0.034779	0.034696
8	0.025737	0.025724	0.025738	0.025625
9	0.019047	0.019029	0.019047	0.018924
10	0.014095	0.014076	0.014095	0.014008
⋮	⋮	⋮	⋮	⋮
35	7.53E-06	7.22E-06	7.59E-06	3.22E-05
Mean Queue Length	3.133019	3.128586	3.133019	3.135264

Table 5.1 shows that the results computer by the numerical approach introduced in this dissertation are closed to the one obtained by the analytical method. The analytical method in this table is the one that has been derived in the previous section using Equation (5.59). In this queueing model, it is much

more simple to compute the stationary queue length distribution using closed-form formula in Equation (5.59). While the computation using 2-GLE will be lengthy when compare to Equation (5.59).

Other than that, the linear extrapolation method in this numerical example does not give close approximation to the analytical result because the stationary queue length distribution obtained using various Δt are not linear at one. While the K-GLE process gives better approximation. Besides, it can also be concluded that the simulation procedure may overestimated the stationary probability for larger n. From the perspective of mean queue length

A similar verification process is also performed using statistical testing procedure that has been introduced in Section 4.4.

Table 5.2: Wilcoxon Signed-Rank Test on the Stationary Queue Length Distributions Obtained Using *CAR*/*M*/1, *GI*/*M*/1 and Simulation Procedure.

Wilcoxon Signed-Rank Test	2-GLE vs Analytical Method	2-GLE vs Simulation
Test Statistics	20	187
P-value	0.442206	0.058865
Conclusion	Do not reject null hypothesis	Do not reject null hypothesis

Table 5.2 above shows the test statistics and p-value obtained from the Wilcoxon signed-rank test that applied to the stationary queue length distributions obtained in Table 5.1. With $\alpha = 0.05$, the critical value is 13. Comparing to Table 5.2, it is obvious that there is no significance evidence to show that the stationary queue length distribution obtained from the three methods are from different distribution. However, if a significance level of $\alpha = 0.1$ is used, the null hypothesis for the case where 2-GLE method is compare to simulation pro-

cedure can be rejected as the p-value$= 0.058$ is less than the significance level of $\alpha = 0.1$.

In the next chapter, a queueing model with negative customers is introduced in which both the interarrival time and service time of the positive customers follow *CAR* distribution.

CHAPTER 6

STATIONARY QUEUE LENGTH DISTRIBUTION OF *CAR/CAR/*1 G-QUEUE USING RCH REMOVING DISCIPLINE

In Chapter 3, it has been proven analytically that the proposed numerical method can be used to find the stationary queue length distribution for the simplest model $M/M/1$ queue with negative customers. The same method is then used to find the stationary queue length distribution for the $M/G/1$ queue and $GI/M/1$ queue in Chapter 4 and 5 respectively. The numerical examples given in the two chapters have also shown that the results obtained by the proposed method are closed to those computed by the analytical method and simulation procedure. In this chapter, the study is extended to a single-server infinite capacity $CAR/CAR/1$ queue with negative customers. The same removal discipline RCH is considered. However, the results obtained in this chapter can only be compared to those computed by the simulation procedure since to the best of my knowledge, no paper published on finding the stationary queue length distribution of a $GI/G/1$ queue with negative customers. In Section 6.1, derivation of equations for stationary probabilities is presented. The set of equations derived in Section 6.1 is solved in Section 6.2 using the numerical method. A numerical example is shown in Section 6.3 and are compared to results computed by the simulation procedure as shown in Appendix A.

6.1 Derivation of Equations for Stationary Probabilities

6.1.1 Transition Probabilities and Events

In this chapter, the following assumptions are made:

- interarrival time distribution of positive customers has *CAR*.
- Service time distribution has *CAR*
- Negative customers arrive to the queue follows Poisson Process with the rate γ.
- The single-server queue has infinite capacity with first-come first serve (FCFS) queueing discipline.

Similar to Section 3.1, the time-axis is first segmented into equal length of interval, Δt. Next, let

- $f(t)$ be the probability density function (PDF) of the interarrival time of positive customers; and
- $\bar{F}(t)$ be the survival function (SF) of the interarrival time of positive customers; and
- $g(t)$ be the probability density function (PDF) of the service time of positive customers; and
- $\bar{G}(t)$ be the survival function (SF) of the service time of positive customers.

Then, the hazard rate λ_k and μ_k in each interval, τ_k are expressed as follows:

$$\lambda_k = \frac{f(k\Delta t)}{\bar{F}(k\Delta t)}, \quad 1 \leq k \leq U \tag{6.1}$$

$$\mu_k = \frac{g(k\Delta t)}{\bar{G}(k\Delta t)}, \quad 1 \leq k \leq U \tag{6.2}$$

where U is an integer that is large enough such that

$$\lambda_U \simeq \lim_{k \to \infty} \lambda_k, \quad \mu_U \simeq \lim_{k \to \infty} \mu_k \tag{6.3}$$

In Figure 6.1, t_a and t_s are the points of time in which the hazard rate of interarrival time and service time distributions start to converge respectively. Let

$$I = \frac{t_a}{\Delta t}, \quad J = \frac{t_s}{\Delta t} \tag{6.4}$$

where U can be obtained as follows:

$$U = \max(I, J) \tag{6.5}$$

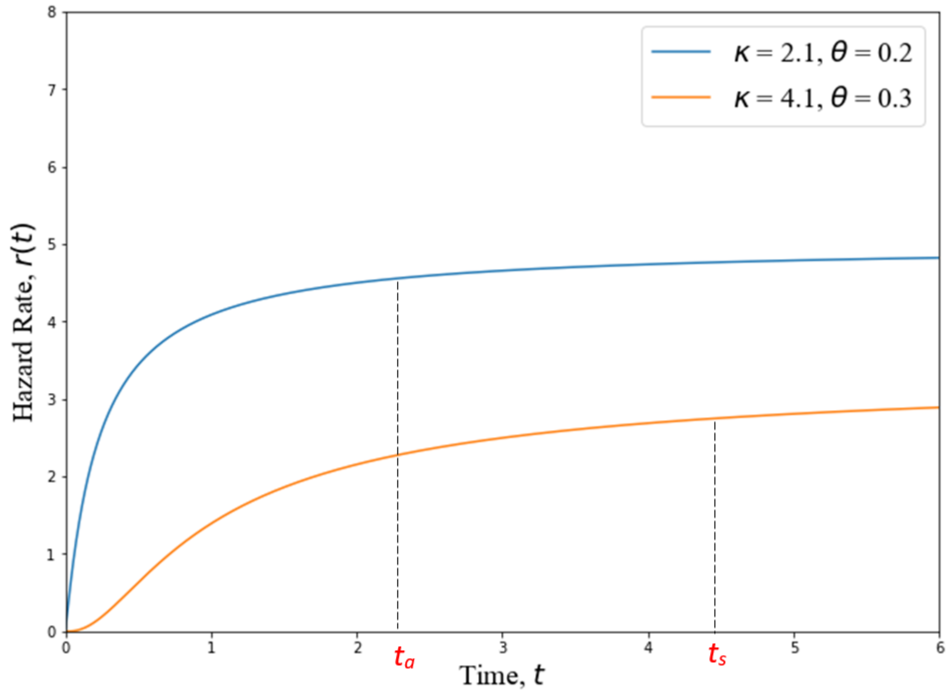

Figure 6.1: Hazard Rate Functions Graph for Two Gamma Distribution with Different Parameters.

With these, the transition probabilities can be written and interpreted as follows:

$\lambda_k \Delta t$: the conditional probability that a positive customer arrives in the interval τ_k given there is no positive arrival in $\tau_1, \tau_2, \cdots, \tau_{k-1}$.

$\gamma \Delta t$: the probability that a negative customer arrives in any interval.

$\mu_k \Delta t$: the conditional probability that a service is completed in the interval τ_k given there is no service completion in $\tau_1, \tau_2, \cdots, \tau_{k-1}$.

$\lambda_k \Delta t$ and $\mu_k \Delta t$ are defined as a conditional probability because their distributions does not possess the memoryless property.

6.1.2 State Spaces

The same notations defined in Equations (5.4), (4.4) and (3.4) are used to denote the state spaces for positive arrival process, service process and negative arrival process respectively.

Assume that at the end of the interval τ_{k-1}, the number of positive customers in the system is $n_{k-1} \geq 1$, the state of the positive arrival process is $i-1$, the negative arrival process is in state 0 and the state of the service process is $j-1$. Then, the state vector at the end of the interval τ_{k-1} is $Z_{k-1} = \{n_{k-1}, \Lambda_{k-1}, \Gamma_{k-1}, M_{k-1}\}$ where $\Lambda_{k-1} = i-1, \Gamma_{k-1} = 0$ and $M_{k-1} = j-1$. Let $i^* = \max(i, U)$ and $j^* = \max(j, U)$, by taking into consideration the events that could possibly occurs in the interval τ_k, the state vector Z_k could be obtained as follows:

1. A positive customer arrives to the system and $Z_k = \{n_{k-1}+1, 0, 0, j^*\}$.

2. A negative customer arrives to the system and $Z_k = \{n_{k-1}-1, i^*, 1, 0\}$.

3. A completion of service occurs in the system and $Z_k = \{n_{k-1}-1, i^*, 0, 0\}$.

4. None of the above events occurs and $Z_k = \{n_{k-1}, i^*, 0, j^*\}$.

When the queue is empty at the end of of the interval τ_{k-1} where $Z_{k-1} = \{0, i-1, 1, 0\}$, the following are the possible events that could occur in the next interval to obtain the state vector Z_k at the end of the interval τ_k.

1. A positive customer arrives to the system and $Z_k = \{1, 0, 0, j^*\}$.

2. A negative customer arrives to the system and $Z_k = \{0, i^*, 1, 0\}$.

3. None of the above events occurs and $Z_k = \{0, i^*, 0, 0\}$.

It is obvious that the system state space is more now complicated since both the interarrival time and service time have *CAR* distributions. The dimension of the set of stationary probabilities equations is escalated compared to the previous few queueing models.

6.1.3 Stationary Probabilities

The stationary probabilities can be obtained as discussed in Equation (3.7). As mentioned in the Section 6.1.2, the size of the set of stationary probabilities equations is increased and depends on the integer U. The following is an example to find the probabilities $p_{nirj}^{(k)}$ at the end of the interval τ_k for $k = 7$:

Let $Z_6 = \{3, i-1, 0, j-1\}$ be the state vector at the end of the interval τ_6, where the probabilities at the end of this interval is $p_{3(i-1)0(j-1)}^{(6)}$. By taking into consideration the events that could possible occur in the interval τ_7, the following equations to find the probabilities $p_{nirj}^{(7)}$ are found:

1. A positive customer arrives to the system, $p_{400j^*}^{(7)} = p_{3(i-1)0(j-1)}^{(6)}(\lambda_i \Delta t)$.

2. A negative customer arrives to the system, $p^{(7)}_{2i^*10} = p^{(6)}_{3(i-1)0(j-1)}(\gamma\Delta t)$.

3. A service completion occurs in the system, $p^{(7)}_{2i*00} = p^{(6)}_{3(i-1)0(j-1)}(\mu_j\Delta t)$.

4. None of the above events occurs, $p^{(7)}_{3i^*0j^*} = p^{(6)}_{3(i-1)0(j-1)}(1-\lambda_i\Delta t-\gamma\Delta t-\mu_j\Delta t)$

When the queue is empty at the end of the interval τ_2, only event number 3 above is not considered since no service is provided when queue length equal to 0. When $k \to \infty$, the following stationary probabilities p_{nirj} are obtained:

$$n=0, \quad p_{n100} \cong p_{(n+1)001}\mu_2\Delta t \tag{6.6}$$

$$\begin{aligned} n=0, \quad p_{ni00} &\cong \sum_{r=0}^{1} p_{n(i-1)r0}(1-\lambda_i\Delta t-\gamma\Delta t) + P_{(n+1)(i-1)10}\mu_1\Delta t \\ &+ \sum_{\substack{j=0 \\ j\neq i-1}}^{i} P_{(n+1)(i-1)0j}\mu_{j+1}\Delta t, \quad \text{for } 2 \le i \le U-1 \end{aligned} \tag{6.7}$$

$$\begin{aligned} n=0, \quad p_{nU00} &\cong \left(\sum_{j=0}^{U} P_{(n+1)U0j} + \sum_{\substack{j=0 \\ j\neq U-1}}^{U} P_{(n+1)(U-1)0j}\right)\mu_{min(j+1,U)}\Delta t \\ &+ \sum_{r=0}^{1}(P_{n(U-1)r0} + P_{nUr0})(1-\lambda_U\Delta t-\gamma\Delta t) \\ &+ (P_{(n+1)(U-1)10} + P_{(n+1)U10})\mu_1\Delta t \end{aligned} \tag{6.8}$$

$$n=0, \quad p_{n110} \cong p_{(n+1)001}\gamma\Delta t \tag{6.9}$$

$$\begin{aligned} n=0, \quad p_{ni10} &\cong \left(\sum_{r=0}^{1} p_{n(i-1)r0} + P_{(n+1)(i-1)10} + \sum_{\substack{j=0 \\ j\neq i-1}}^{i} P_{(n+1)(i-1)0j}\right)\gamma\Delta t, \\ &\text{for } 2 \le i \le U-1 \end{aligned} \tag{6.10}$$

$$\begin{aligned} n=0, \quad p_{nU10} &\cong \Bigg(\sum_{j=0}^{U} P_{(n+1)U0j} + \sum_{\substack{j=0 \\ j\neq U-1}}^{U} P_{(n+1)(U-1)0j} + P_{(n+1)U10} \\ &+\sum_{r=0}^{1}(P_{n(U-1)r0} + P_{nUr0}) + P_{(n+1)(U-1)10}\Bigg)\gamma\Delta t \end{aligned} \tag{6.11}$$

$$\begin{aligned} n=1, \quad p_{ni0j} &\cong p_{n(i-1)0(j-1)}(1-\lambda_i\Delta t-\gamma\Delta t-\mu_j\Delta t), \\ &\text{for } 1 \le i \le U-1, \\ &\text{for } 2 \le j \le \min(i+1,U), \quad i \neq j \end{aligned} \tag{6.12}$$

$$n \geq 1, \quad p_{n001} \cong \sum_{i=1}^{U} (p_{(n-1)i00} + p_{(n-1)i10}) \lambda_{\min(i+1,U)} \Delta t \tag{6.13}$$

$$n \geq 1, \quad p_{ni01} \cong (p_{n(i-1)00} + p_{(n-1)i10})(1 - \lambda_i \Delta t - \gamma \Delta t - \mu_1 \Delta t), \text{ for } 2 \leq i \leq U-1 \tag{6.14}$$

$$n \geq 1, \quad p_{nU01} \cong (p_{n(U-1)00} + p_{n(U-1)10} + p_{nU00} + p_{nU10})(1 - \lambda_U \Delta t - \gamma \Delta t - \mu_1 \Delta t) \tag{6.15}$$

$$n \geq 1, \quad p_{nU0j} \cong (p_{n(U-1)0(j-1)} + p_{nU0(j-1)})(1 - \lambda_U \Delta t - \gamma \Delta t - \mu_j \Delta t), \text{ for } 2 \leq j \leq U-1 \tag{6.16}$$

$$n \geq 1, \quad p_{nU0U} \cong (p_{n(U-1)0U} + p_{nU0(U-1)} + p_{nU0U})(1 - \lambda_U \Delta t - \gamma \Delta t - \mu_U \Delta t) \tag{6.17}$$

$$n \geq 1, \quad p_{n100} \cong \sum_{j=1}^{U} p_{(n+1)00j} \mu_{\min(j+1,U)} \Delta t \tag{6.18}$$

$$n \geq 1, \quad p_{ni00} \cong \sum_{\substack{j=0 \\ j \neq i-1}}^{U} p_{(n+1)(i-1)0j} \mu_{\min(j+1,U)} \Delta t + p_{(n+1)(i-1)10} \mu_1 \Delta t, \text{ for } 2 \leq i \leq U-1 \tag{6.19}$$

$$n \geq 1, \quad p_{nU00} \cong \left(\sum_{j=0}^{U} p_{(n+1)U0j} + \sum_{\substack{j=0 \\ j \neq U-1}}^{U} p_{(n+1)(U-1)0j} \right) \mu_{\min(j+1,U)} \Delta t + (p_{(n+1)(U-1)10} + p_{(n+1)U10}) \mu_1 \Delta t \tag{6.20}$$

$$n \geq 1, \quad p_{n110} \cong \sum_{j=1}^{U} p_{(n+1)00j} \gamma \Delta t \tag{6.21}$$

$$n \geq 1, \quad p_{ni10} \cong \left(\sum_{\substack{j=0 \\ j \neq i-1}}^{U} p_{(n+1)(i-1)0j} + p_{(n+1)(i-1)10} \right) \gamma \Delta t, \text{ for } 2 \leq i \leq U-1 \tag{6.22}$$

$$n \geq 1, \quad p_{nU10} \cong \left(\sum_{j=0}^{U} p_{(n+1)U0j} + \sum_{\substack{j=0 \\ j \neq U-1}}^{U} p_{(n+1)(U-1)0j} + p_{(n+1)(U-1)10} + p_{(n+1)U10} \right) \gamma \Delta t \tag{6.23}$$

$$n = 2, \quad p_{n00j} \cong \sum_{\substack{i=j-2 \\ i \neq j-1}}^{U} p_{(n-1)i0(j-1)} \lambda_{\min(i+1,U)} \Delta t, \text{ for } 2 \leq j \leq U-1 \tag{6.24}$$

$$
\begin{aligned}
n = 2, \quad p_{n00U} &\cong \sum_{i=U-1}^{U} p_{(n-1)i0U}\lambda_{\min(i+1,U)}\Delta t \\
&+ \sum_{\substack{i=j-2 \\ i\neq j-1}}^{U} p_{(n-1)i0(U-1)}\lambda_{\min(i+1,U)}\Delta t \qquad (6.25)\\
n \geq 2, \quad p_{ni0j} &\cong p_{n(i-1)0(j-1)}(1-\lambda_i\Delta t - \gamma\Delta t - \mu_j\Delta t), \\
&\text{for } 1 \leq i \leq U-1, \\
&\text{for } 2 \leq j \leq U-1, \quad i \neq j \qquad (6.26)\\
n \geq 2, \quad p_{ni0U} &\cong (p_{n(i-1)0(U-1)} + p_{n(i-1)0U})(1-\lambda_i\Delta t - \gamma\Delta t - \mu_U\Delta t), \\
&\text{for } 1 \leq i \leq U-1 \qquad (6.27)\\
n \geq 3, \quad p_{n00j} &\cong \sum_{\substack{i=0 \\ i\neq j-1}}^{U} p_{(n-1)i0(j-1)}\lambda_{\min(i+1,U)}\Delta t, \\
&\text{for } 2 \leq j \leq U-1 \qquad (6.28)\\
n \geq 3, \quad p_{n00U} &\cong \sum_{i=0}^{U} p_{(n-1)i0U}\lambda_{\min(i+1,U)}\Delta t \\
&+ \sum_{\substack{i=0 \\ i\neq j-1}}^{U} p_{(n-1)i0(U-1)}\lambda_{\min(i+1,U)}\Delta t \qquad (6.29)
\end{aligned}
$$

6.2 Stationary Queue Length Distribution

The set of equations derived in the previous two chapters are solved using two different recursive methods. However, these method may not be helpful in solving *CAR*/*CAR*/1 queueing model where the dimension is extremely huge. In the matrix recursive method, matrix inversion is required to perform along the process of recursion and this takes up a lot of temporary memory of the calculator which leads to memory failure. On the other hand, the recursive method which has been introduced in Chapter 5 is complicated to perform when each of the stationary equation are related to more stationary probabilities in *CAR*/*CAR*/1 queueing model.

First, Equations (6.6) to (6.29) are expressed in the form of sparse matrix using the notations $\mathbf{P}_n^*$ and $\mathbf{Q}_{n,m}$ that have been introduced in Section 4.2. For example, by definition, when $n = 0$, $\mathbf{P}_n^*$ can be expressed as follows:

$$\mathbf{P}_0^* = \begin{pmatrix} p_{0100} \\ p_{0110} \\ p_{0200} \\ \vdots \\ p_{0U00} \\ p_{0U10} \end{pmatrix} \tag{6.30}$$

By using the relationship of $\mathbf{P}_n^*$ and $\mathbf{Q}_{n,m}$ which is given in Equation (4.25), the following sparse matrix is formed from Equations (6.6) to (6.29).

$$\mathbf{I}\begin{pmatrix} \mathbf{P}_0^* \\ \mathbf{P}_1^* \\ \mathbf{P}_2^* \\ \vdots \\ \mathbf{P}_{N-1}^* \\ \mathbf{P}_N^* \end{pmatrix} = \begin{pmatrix} \mathbf{Q}_{0,0} & \mathbf{Q}_{0,1} & & & & \\ \mathbf{Q}_{1,0} & \mathbf{Q}_{1,1} & \mathbf{Q}_{1,2} & & & \\ & \mathbf{Q}_{2,1} & \mathbf{Q}_{2,2} & \mathbf{Q}_{2,3} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mathbf{Q}_{N-1,N-2} & \mathbf{Q}_{N-1,N-1} & \mathbf{Q}_{N-1,N} \\ & & & & \mathbf{Q}_{N,N-1} & \mathbf{Q}_{N,N} \end{pmatrix} \tag{6.31}$$

where identity matrix $\mathbf{I}$ has the same dimension as the sparse matrix and $N + 1$ is an integer that is large enough such that $\mathbf{P}_{N+1}^* \approx 0$.

There are two ways to solve the above sparse matrix: direct methods or numerical methods where direct methods may be more complicated than numerical methods. For direct method, we may refer to Duff et al. (2017) while Saad (2003) provided numerical iterative methods. In this research, an iterative method is used since the direct method takes considerable time and requires a larger memory to solve the sparse matrix. Different iterative methods could be used to solve for different forms of sparse matrices. The selection of an iterative method depends on the computational cost and the properties of the matrix that are required to apply the chosen method. The properties considered in a matrix are symmetry, transpose matrix, initial value, positive define etc. Comparison of commonly used iterative methods for sparse matrices can be found in Sickel et al. (2005)

To solve the sparse matrix in Equation (6.31), the biconjugate gradient stabilized method (BICGSTAB) is used since the result converges rapidly using this method. By using the K-GLE method introduced in Section 4.3, the stationary probabilities are obtained when $\Delta t \approx 0$. Hence, the stationary queue length distribution is found and the mean queue length can be obtained using Equation (4.34) as well.

6.3 Numerical Example

Consider a single-server queueing system in which both the interarrival time and service time of the positive customers follow gamma distribution with parameters $(\kappa_1, \theta_1) = (2.1, 0.3)$ and $(\kappa_2, \theta_2) = (1.2, 0.4)$ respectively. The interarrival time of negative customers is assumed to have exponential distribution with rate $\gamma = 2.5$. The time that the hazard rate of the interarrival time and service time tend to constant distributions are $t_a = 1$ and $t_s = 2.5$ respectively.

By using the combination of numerical method and K-GLE method, stationary queue length distribution of *CAR*/*CAR*/1 queieng model with negative customers is obtained. The stationary queue length distribution may also be computed using simulation procedure. There is no existing method on finding the same results for *GI*/*G*/1 queue with negative customers. Comparison of results is tabulated in Table 6.1 and the simulation procedure is carried out with a running step of 10^8. The results in Table 6.1 once again show that K-GLE method is better than linear extrapolation method. Some analysis on the results obtained using different Δt from 0.01 to 0.06 are presented graphically and tabulated for several stationary queue length probabilities.

Table 6.1: Stationary Queue Length Distribution Computed from the Proposed Numerical Method and Simulation Procedure
$[(\kappa_1, \theta_1) = (2.1, 0.3), (\kappa_2, \theta_2) = (1.2, 0.4), \gamma = 2.5, t_a = 1, t_s = 2.5]$

Queue Length, n	Stationary Queue Length Distribution, p_n		
	Numerical Method		Simulation Procedure
	2-GLE	LE	
0	0.353511	0.362772	0.352593
1	0.309303	0.314748	0.309786
2	0.163594	0.166261	0.164005
3	0.084434	0.084268	0.084740
4	0.043317	0.041365	0.043473
5	0.022181	0.019500	0.022216
6	0.011351	0.008638	0.011354
7	0.005808	0.003418	0.005794
8	0.002971	0.001031	0.002964
9	0.001520	0.000027	0.001514
10	0.000778	-0.000330	0.000773
⋮	⋮	⋮	⋮
25	1.50E-08	-7.90E-07	0
Mean Queue Length	1.337774	1.221142	1.340704

From Table 6.1, it can be observed that the normal linear extrapolation method obtains invalid results where some of the stationary probabilities are negative. This show that the linear extrapolation are not reliable. However, the stationary queue length distribution obtained using the proposed method with the combination of 2-GLE is closed to the simulation procedure. In term of mean queue length, the linear extrapolation method underestimated the value a lot compare to both 2-GLE and simulation procedure. A further verification can be done by carrying out the Wilcoxon signed-rank test that is mentioned in Section 4.4 to the results obtained in Table 6.1. From Table 6.2, with $\alpha = 0.05$, the null hypothesis that the stationary queue length distribution obtained using 2-GLE method and LE method have same distribution is rejected while there is no strong evidence that the results yielded by simulation procedure and 2-GLE method respectively are from different distribution. The conclusion obtained from this test is same as what we can observed as linear extrapolation provides invalid result in which stationary probabilities is negative.

Table 6.2: Wilcoxon Signed-Rank Test on the Stationary Queue Length Distributions Obtained Using *CAR/CAR/*1 (2-GLE), *CAR/CAR/*1 (LE) and Simulation Procedure.

Wilcoxon Signed-Rank Test	2-GLE vs LE	2-GLE vs Simulation
Test Statistics	73	166
P-value	0.009233	0.809337
Conclusion	Reject null hypothesis	Do not reject null hypothesis

Next, the results that are obtained using different Δt in proposed method are non-linear and shown in Figures 6.2 to 6.10 and Table 6.3 below.

Table 6.3: Stationary Queue Length Distribution Obtained Using Various Values of Δt.
$[(\kappa_1, \theta_1) = (2.1, 0.3), (\kappa_2, \theta_2) = (1.2, 0.4), \gamma = 2.5, t_a = 1, t_s = 2.5]$

n	Δt					
	0.01	0.02	0.03	0.04	0.05	0.06
0	0.343747	0.332362	0.319594	0.305579	0.290374	0.273980
1	0.303427	0.297053	0.290044	0.282235	0.273443	0.263482
2	0.165400	0.167086	0.168527	0.169570	0.170041	0.169745
3	0.088160	0.092068	0.096089	0.100130	0.104073	0.107774
8	0.003654	0.004519	0.005609	0.006981	0.008701	0.010849
19	3.25E-06	5.82E-06	1.05E-05	1.93E-05	3.56E-05	6.62E-05
20	1.70E-06	3.14E-06	5.87E-06	1.11E-05	2.12E-05	4.07E-05
24	9.46E-08	1.95E-07	4.09E-07	8.65E-07	1.85E-06	3.97E-06
25	3.10E-08	6.53E-08	1.40E-07	3.02E-07	6.58E-07	1.45E-06

Then, the estimation of the stationary probability of different n at $\Delta t = 0$ using different methods are also plotted in the Figures 6.2 to 6.10. In particular, these figures display that the resultant red trend line produced by the numerical method (blue line) is in line with the stationary probabilities generated by 2-GLE method (yellow square). These figures also tell that the LE method (brown circle) does not always fit into the resultant red trend line while the simulation (grey triangle) is closed to the trend line for most of the p_n except for the case when the queue length n is large. Besides, it can also be observed that LE method would overestimate the stationary probabilities for Figures 6.2 to 6.4 while underestimating the stationary probabilities for Figures 6.6 to 6.10. A such, it can be inferred that LE method is not appropriate to be used generally to extrapolate results in this research.

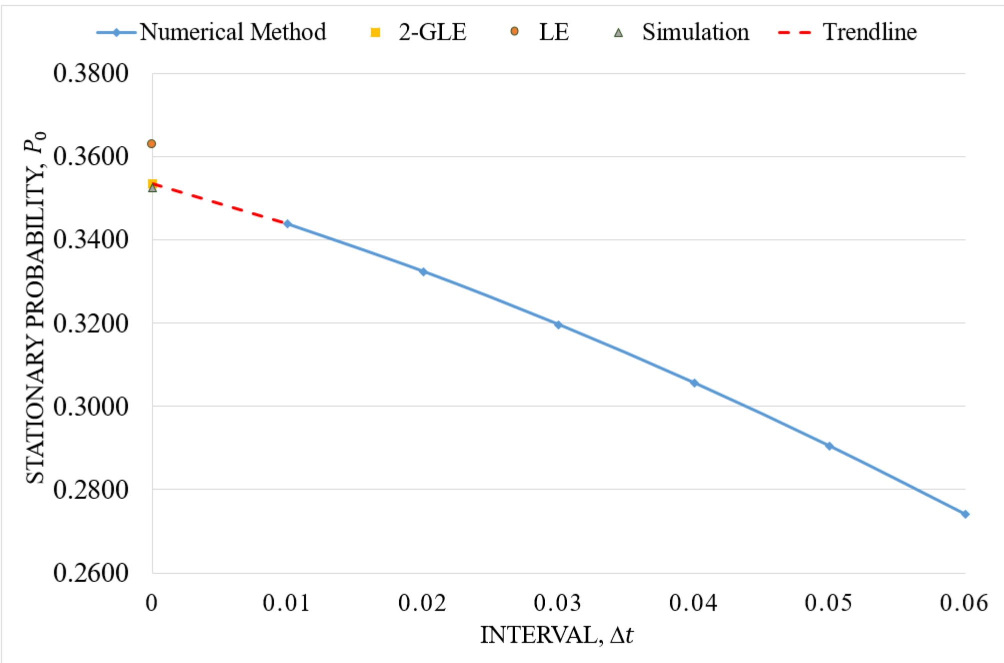

Figure 6.2: Stationary Probability P_0.$[(\kappa_1, \theta_1) = (2.1, 0.3), (\kappa_2, \theta_2) = (1.2, 0.4), \gamma = 2.5, t_a = 1, t_s = 2.5]$

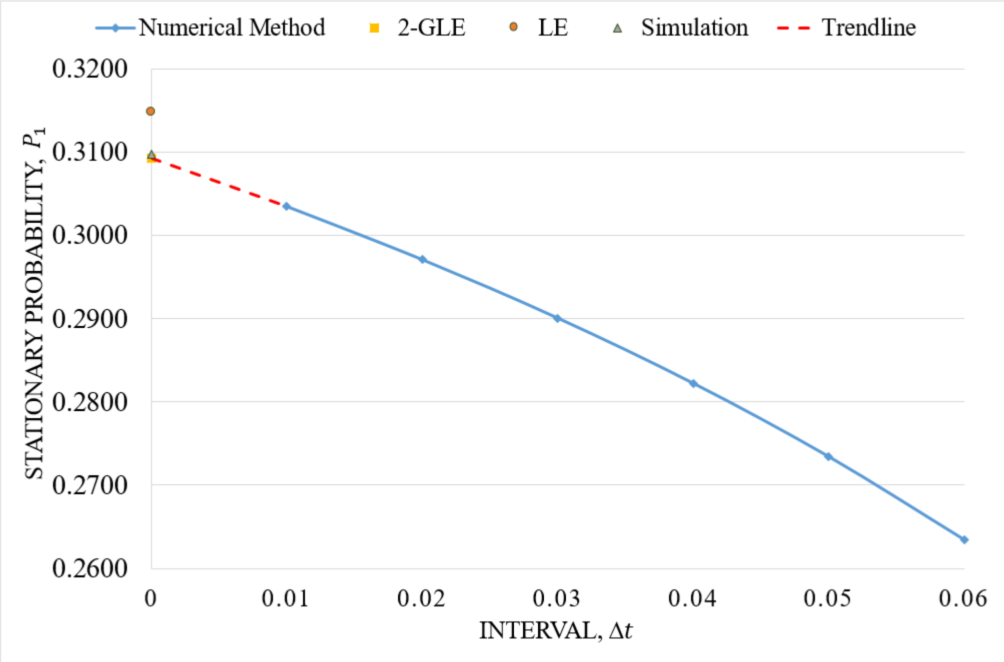

Figure 6.3: Stationary Probability $P_1.[(\kappa_1, \theta_1) = (2.1, 0.3), (\kappa_2, \theta_2) = (1.2, 0.4), \gamma = 2.5, t_a = 1, t_s = 2.5]$

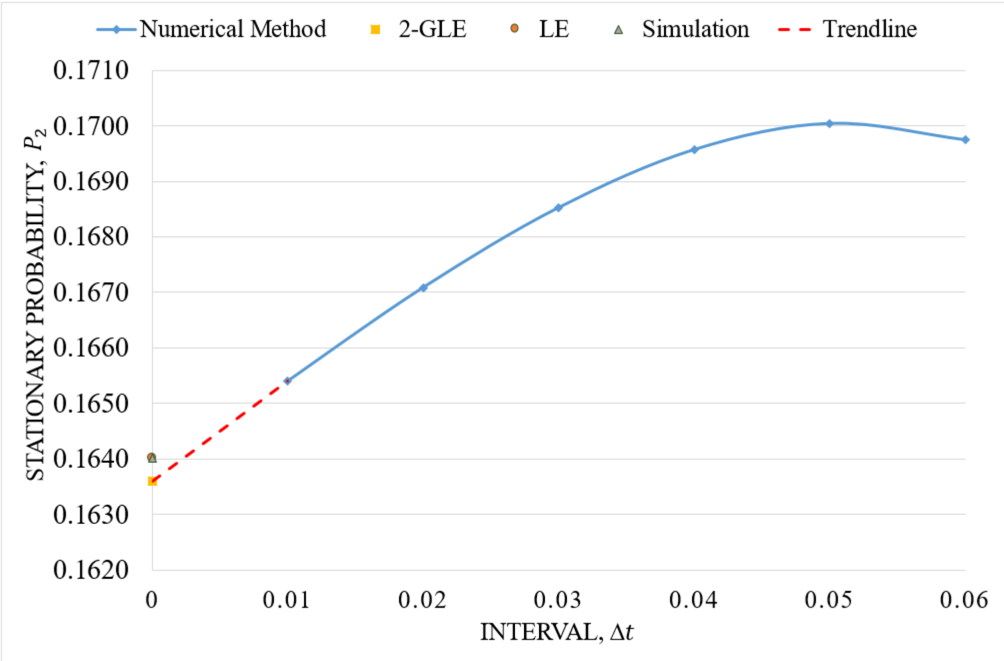

Figure 6.4: Stationary Probability $P_2.[(\kappa_1, \theta_1) = (2.1, 0.3), (\kappa_2, \theta_2) = (1.2, 0.4), \gamma = 2.5, t_a = 1, t_s = 2.5]$

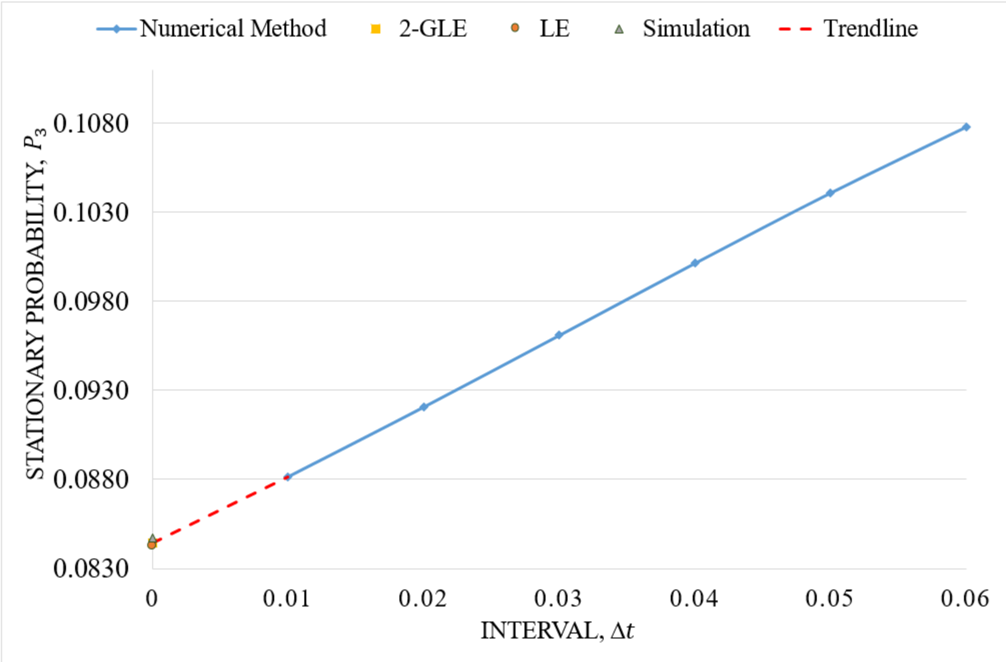

Figure 6.5: Stationary Probability P_3.$[(\kappa_1, \theta_1) = (2.1, 0.3), (\kappa_2, \theta_2) = (1.2, 0.4), \gamma = 2.5, t_a = 1, t_s = 2.5]$

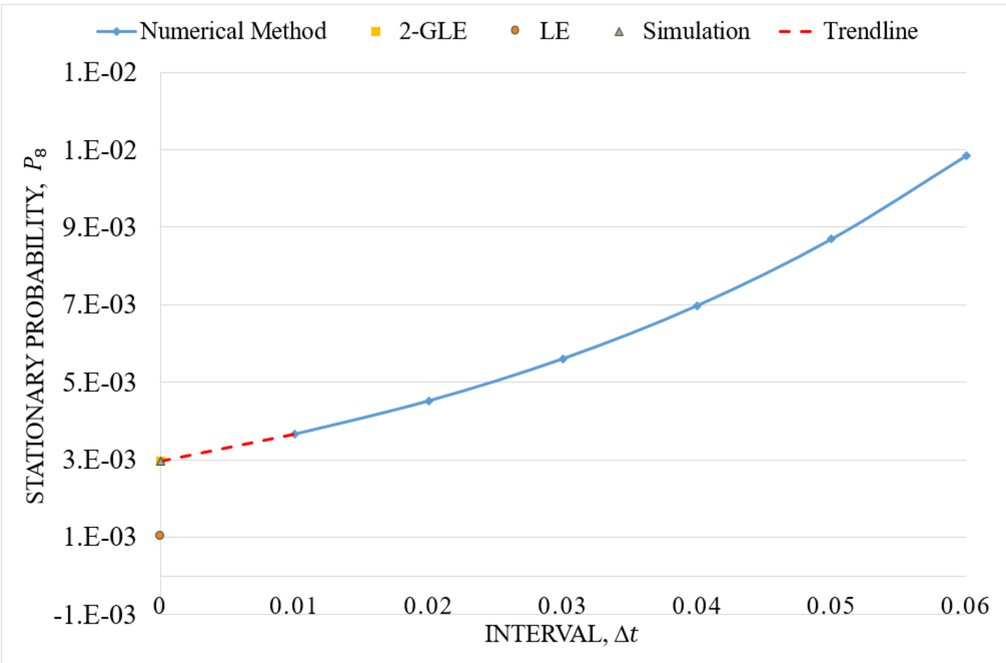

Figure 6.6: Stationary Probability P_8.$[(\kappa_1, \theta_1) = (2.1, 0.3), (\kappa_2, \theta_2) = (1.2, 0.4), \gamma = 2.5, t_a = 1, t_s = 2.5]$

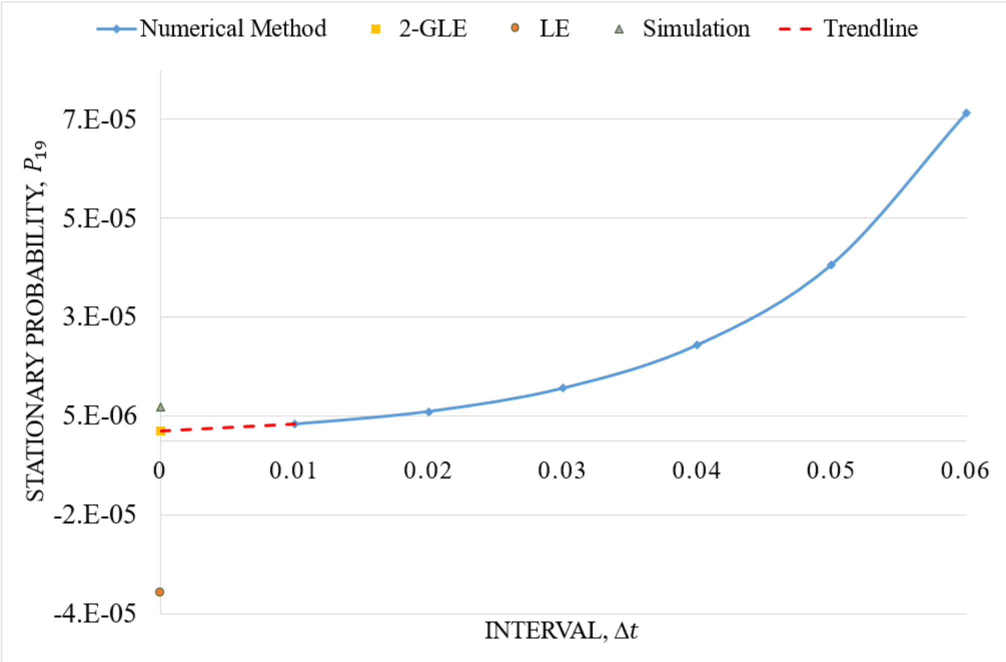

Figure 6.7: Stationary Probability P_{19}.$[(\kappa_1, \theta_1) = (2.1, 0.3), (\kappa_2, \theta_2) = (1.2, 0.4), \gamma = 2.5, t_a = 1, t_s = 2.5]$

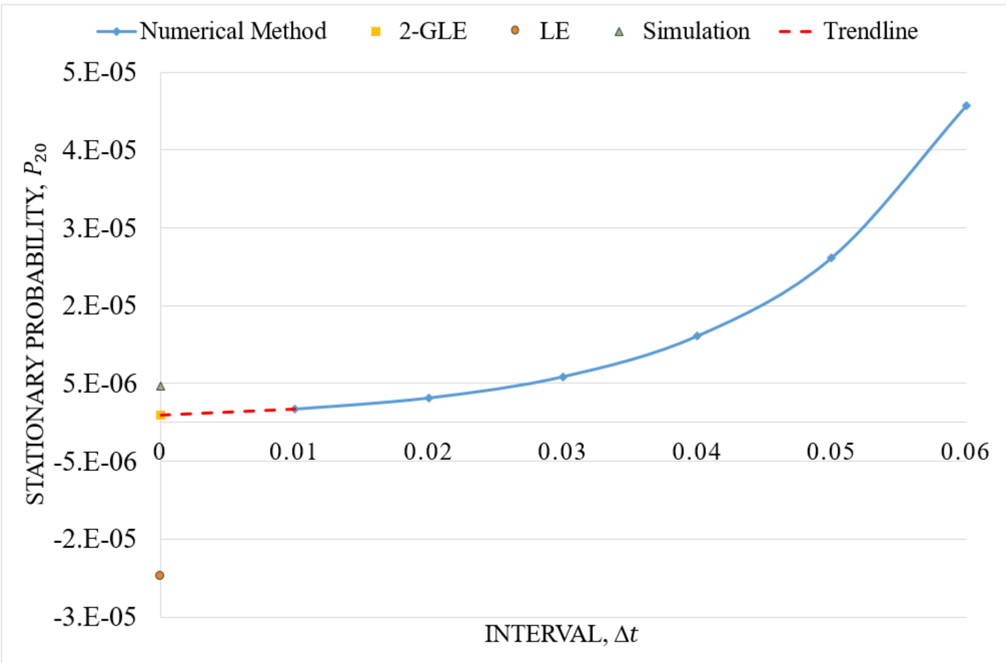

Figure 6.8: Stationary Probability P_{20}.$[(\kappa_1, \theta_1) = (2.1, 0.3), (\kappa_2, \theta_2) = (1.2, 0.4), \gamma = 2.5, t_a = 1, t_s = 2.5]$

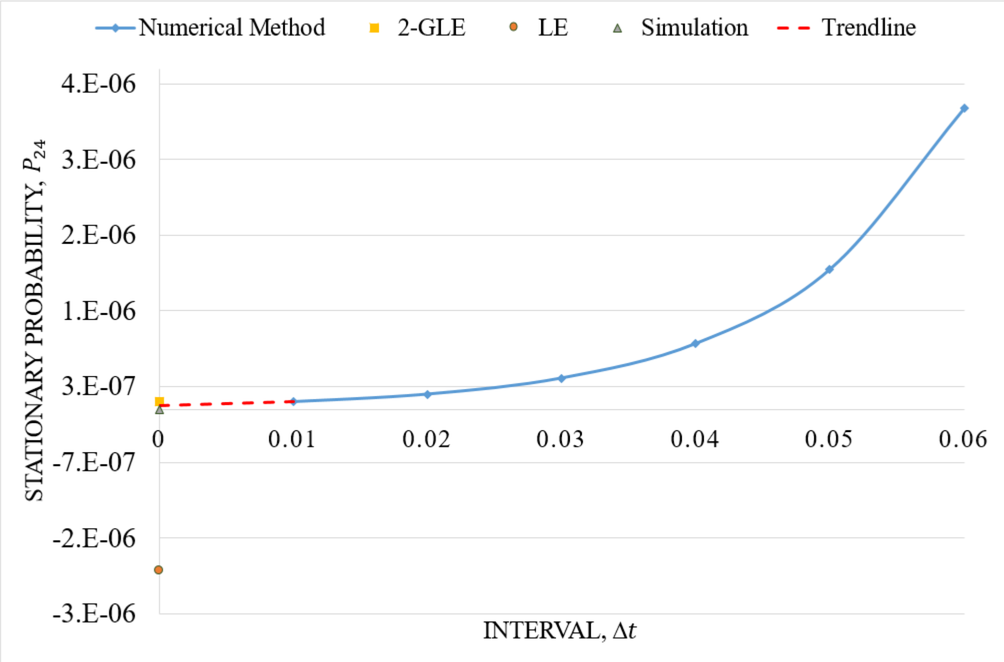

Figure 6.9: Stationary Probability P_{24}.$[(\kappa_1, \theta_1) = (2.1, 0.3), (\kappa_2, \theta_2) = (1.2, 0.4), \gamma = 2.5, t_a = 1, t_s = 2.5]$

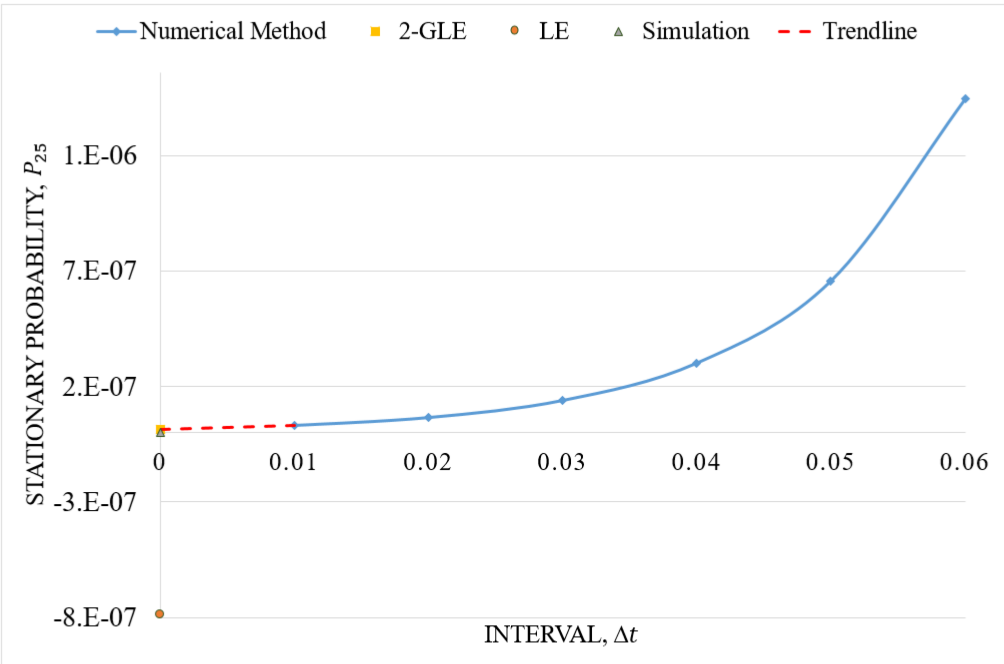

Figure 6.10: Stationary Probability P_{25}.$[(\kappa_1, \theta_1) = (2.1, 0.3), (\kappa_2, \theta_2) = (1.2, 0.4), \gamma = 2.5, t_a = 1, t_s = 2.5]$

In conclusion, the alternative method with the new extrapolation method may result a convincing outcome. This outcome has been verified with statistical testing to the simulation procedure that has also been justified in previous chapters. As mentioned in the literature of this thesis, there is no study of *GI*/*G*/1 queue with negative customers is published. In the existing study, a queueing model of MAP/*G*/1 queue with negative customer has been published by Li and Zhao (2004). While Liu and Wu (2009) extended the model to G-queue with preemptive resume and multiple vacations where the author consider priority service and server vacation that is not considered in this dissertation. Meanwhile, they also provided the stationary queue length distribution of the queueing system during busy period. While the current *CAR*$/$*CAR*$/1$ queue with negative customers can be recognized as a model that is fairly closed to *GI*/*G*/1 queue with negative customers compare to MAP/*G*/1 queue with negative customer but the *CAR*/*CAR*/1 G-queue does not include the study of busy period and other features. In the next chapter, the waiting time distribution of *CAR*/*CAR*/1 queue with negative customers is found by using the stationary probabilities obtained from Chapters 3 to 6.

CHAPTER 7

WAITING TIME DISTRIBUTION FOR CAR/CAR/1 G-QUEUE USING RCH REMOVING PRINCIPLE

This chapter illustrates the derivation of the waiting time distribution from stationary probabilities obtained in all the model mentioned in the previous chapters. The waiting time distribution formula is found in Section 7.1. Simulation procedure for waiting time distribution is introduced in Appendix B. Section 7.2 contains the numerical results obtained using the newly derived formula and compare to simulation results and existing analytic method if any.

7.1 Derivation of Waiting Time Distribution

Firstly, the notations that are used in derivation of waiting time distribution are denoted as follows:

q_n	:	the probability that upon arrival instant of a positive customer will find n customers in the system.
T_q	:	the random varible for time spend in the queue
W_q	:	cumulative probability distribution (CDF) of T_q
T_b	:	the random variable for service time of positive customers.
T_c	:	the random varible for intearrival time of negative customers

When the queue is empty, it makes intuitive sense that the waiting time for the arriving positive customers is 0. Therefore,

$$
\begin{aligned}
W_q(0) &= Pr(T_q = 0) \\
&= Pr(\{\text{Arriving positive customer find 0 in the system}\}) \\
&= q_0
\end{aligned} \tag{7.1}
$$

However, when the system is not empty, q_n is an important element to find the waiting time distribution, $W_q(t)$ and it can be derived using Bayes' Theorem. From this theorem, the probability of having n positive customers in the system given that the next positive arrival will take place when the process is in the steady state $Z_k = \{n_k, \Lambda_k, \Gamma_k, M_k\}$ when k goes to infinity is derived as

$$
\begin{aligned}
q_n &= Pr(n_k = n | \{\text{arrival about to occur in the steady state } Z_k\}) \\
&= \frac{Pr(n_k = n \cap \{\text{arrival about to occur in the steady state } Z_k\})}{Pr(\{\text{arrival about to occur in the steady state } Z_k\})}
\end{aligned}
$$

If there is a positive arrival, by substituting the statement $\{$arrival about to occur in the steady state $Z_k\}$ is with $\Lambda_k = i$ which indicates the current state of the positive arrival process upon a new arrival, we get

$$
\begin{aligned}
q_n &= \frac{Pr(n_k = n \cap \Lambda_k = i)}{Pr(\Lambda_k = i)} \\
&= \frac{Pr(n_k = n) \bullet Pr(\Lambda_k = i | n_k = n)}{\sum_{n_k=0}^{N} Pr(n_k = n) \bullet Pr(\Lambda_k = i | n_k = n)} \\
&= \frac{\sum_i \sum_r \sum_j p_{nirj} \lambda_{\min(i+1,U)} \Delta t}{\sum_{n=0}^{N} \sum_i \sum_r \sum_j p_{nirj} \lambda_{\min(i+1,U)} \Delta t} \\
&= \frac{\sum_i \sum_r \sum_j p_{nirj} \lambda_{\min(i+1,U)}}{\sum_{n=0}^{N} \sum_i \sum_r \sum_j p_{nirj} \lambda_{\min(i+1,U)}}
\end{aligned} \tag{7.2}
$$

A positive customer will leave the system by either service completion or removed by negative customer whichever come first. The time until a positive customer leave the queue since the beginning of its service can be denoted as $T_q = \min(T_b, T_c)$. Let $G(t)$ and $H(t)$ be the CDF of the service time and the interarrival time of negative customer respectively while $g(t)$ and $h(t)$ be their PDF respectively. Then,

$$Pr(T_q \leq t) = 1 - (1 - G(t))(1 - H(t)) \tag{7.3}$$

$$Pr(T_q = t) = h(t)(1 - G(t)) + g(t)(1 - H(t)) \tag{7.4}$$

Supposed that upon the arrival of a positive customer, there are n customers in the queue, the distribution of this n customers to leave the system is the n-fold convolution of T_q distribution which is $T_q^{(n)}$. Then, the probability for the new arrival to wait at most time t is,

$$Pr(\{n \text{ service completion \& removal} \leq t\}) = Pr(T_q^{(n)} \leq t) \tag{7.5}$$

Combining Equations (7.1), (7.2) and (7.5), the waiting time distribution for *CAR*/*CAR*/1 queueing model with negative customer can be found by:

$$\begin{aligned} W_q(t) &= q_0 + \sum_{n=1}^{\infty} q_n Pr(T_q^{(n)} \leq t) \\ &= q_0 + \sum_{n=1}^{\infty} q_n \int_0^t Pr(T_q^{(n)} = u)du, \quad t > 0 \end{aligned} \tag{7.6}$$

The component in the summation part of Equation (7.6) can be interpreted as the probability that n customers leave the system by time t given there are n positive customers in the system upon its arrival.

Let's consider a simple model *M*/*M*/1 queue with negative customer. Supposed that the interarrival time and service time distributions of positive customers are exponential with rates λ and μ respectively. While negative customer arrives to the system according to the Poisson process with the rate γ. Then Equation (7.2) can be simplified as

$$\begin{aligned} q_n &= \frac{\sum_i \sum_r \sum_j p_{nirj}\lambda}{\sum_{n=0}^{N} \sum_i \sum_r \sum_j p_{nirj}\lambda} \\ &= \frac{\sum_i \sum_r \sum_j p_{nirj}}{\sum_{n=0}^{N} \sum_i \sum_r \sum_j p_{nirj}} \\ &= p_n \end{aligned} \tag{7.7}$$

From Equation (7.7), it can be concluded that $q_0 = p_0$ and substituting Equation (7.7) into Equation (7.6) yields the following.

$$W_q(t) = p_0 + \sum_{n=1}^{\infty} p_n \int_0^t Pr(T_q^{(n)} = u)du \tag{7.8}$$

On the other hand, the exponential assumptions allow Equation (7.4) to be simplified as follow:

$$\begin{aligned} Pr(T_q = t) &= \gamma e^{-\gamma t}(e^{-\mu t}) + \mu e^{-\mu t}(e^{-\gamma t}) \\ &= (\mu + \gamma)e^{-(\mu+\gamma)} \end{aligned} \tag{7.9}$$

Equation (7.9) is an exponential distribution with rates $(\mu+\gamma)$ and the convolution of n exponential distributions is an Erlang distribution with shape parameter of n and scale parameter of $1/(\mu+\gamma)$. By performing n-fold convolution on Equation (7.9) and substituting Equation (3.22) into Equation (7.8) will obtain:

$$W_q(t) = 1 - \rho + \rho \int_0^t (\mu+\gamma)(1-\rho)e^{-(\mu+\gamma)(1-\rho)u}du$$

where the expression inside the integral part in the derivation above is the PDF of an exponential distribution with rate $(\mu+\gamma)(1-\rho)$, implying that

$$\int_0^t (\mu+\gamma)(1-\rho)e^{-(\mu+\gamma)(1-\rho)u}du = 1 - e^{-(\mu+\gamma)(1-\rho)t}$$

Thus,

$$W_q(t) \;=\; 1-\rho+\rho\big(1-e^{-(\mu+\gamma)(1-\rho)t}\big)$$

or simply,

$$W_q(t) \;=\; 1-\rho e^{-(\mu+\gamma)(1-\rho)t}, \quad t \geq 0 \tag{7.10}$$

By replacing the parameters $\mu+\gamma$ in Equation (7.10) with service rate, an equation to find the waiting time distribution of an ordinary *M*/*M*/1 queue without negative customers is formed and this is once again agreed with the conclusion made in Chapter 3. Hence, Equation (7.6) is proven to able to generate results of waiting time distribution for an *M*/*M*/1 queue with negative customers.

Similarly, when an *GI*/*M*/1 queue with negative customers is considered, Equation (5.48) is used to substitute the term q_n in Equation (7.6). As it is similar to Equation (3.21) where ρ is replaced by r, it can be concluded that the waiting time distribution of *GI*/*M*/1 queue with negatives customers can be expressed as the following equation.

$$W_q(t) \;=\; 1-re^{-(\mu+\gamma)(1-r)t}, \quad t \geq 0. \tag{7.11}$$

where r is the root mentioned in Section 5.2.

In conclusion, Equation (7.6) can be utilized to generate the waiting time distribution for all models considered in Chapters 3 to 6 and the mean waiting time of each model can be calculated using Little's formula Equation (4.34) as shown below.

$$E(W)=\frac{E(N)}{\lambda}=\frac{\sum_n\sum_i\sum_r\sum_j np_{nirj}}{\lambda} \tag{7.12}$$

7.2 Numerical Examples

In this section, two numerical examples are presented. From Equation (7.2), q_n can be calculated from p_{nirj} and $\lambda_{\min(i+1,U)}$.

The waiting time distributions of the *CAR*/*M*/1 queue and *CAR*/*CAR*/1 queue with negative customers are found by Equation (7.6). The result for *CAR*/*M*/1 queue is verified with the results obtained by Equation (7.11) where the values of r in Equation (7.11) are obtained using analytical method in Section 5.2. However, the results for *CAR*/*CAR*/1 queue will only be compared to the simulation procedure in Appendix B since no existing analytical method is found.

For the *GI*/*M*/1 queue with negative customers, the interarrival time of the positive customers follows gamma distribution with parameters $(\kappa, \theta) = (2.1, 0.3)$, the service time of positive customer and interarrival time of negative customers are exponentially distributed with rates $\mu = 1$ and $\gamma = 2$ where integer $N = 100$ is chosen to find the stationary probabilities. The time point when the hazard rate of the interarrival time starts to converge to an approximately constant value is $t_a = 10$.

In Table 7.1, the linear extrapolation method is not considered since it had been shown that this method may give invalid results in the previous chapter. From the table, the results for the waiting time distribution obtained using several methods are closed. It can also be observed that the waiting time distribution obtained using K-GLE method is converging closer to the analytical method when the value of K increases. Other than that, the mean waiting time obtained by the simulation procedure is underestimated. Hence, a conclusion that the proposed numerical method with K-GLE method outperforms the simulation procedure as shown in Appendix B is made.

Table 7.1: Waiting Time Distribution Computed from Proposed the Numerical Method, Analytical Method and Simulation Procedure. $[(\kappa, \theta) = (2.1, 0.3), \mu = 1, \gamma = 2, N = 100, t_a = 10]$

Waiting Time, t	Waiting Time Distribution, $W_q(t)$			
	Numerical Method		Analytical Method	Simulation Procedure
	3-GLE	2-GLE		
0.00	0.592342	0.592344	0.592339	0.592359
0.04	0.620313	0.620315	0.620310	0.620286
0.08	0.646365	0.646366	0.646362	0.646375
0.12	0.670629	0.670630	0.670626	0.670610
0.16	0.693228	0.693230	0.693226	0.693244
0.20	0.714277	0.714278	0.714275	0.714251
0.24	0.733881	0.733883	0.733879	0.733790
0.28	0.752141	0.752142	0.752139	0.752047
0.32	0.769147	0.769149	0.769145	0.769062
0.36	0.784987	0.784988	0.784985	0.784913
0.40	0.799740	0.799741	0.799738	0.799663
⋮	⋮	⋮	⋮	⋮
3.00	0.998027	0.998033	0.998027	0.997914
Mean Waiting Time	0.562736	0.562733	0.562740	0.562090

As mentioned previously, since the waiting time distribution and the mean waiting time is depended on the stationary probabilities as shown in Equations (7.6) and (7.12), if the Wilcoxon signed-rank test shows that the stationary queue length distribution obtained from different method are significantly closed, then the hypothesis testing on waiting time distribution and mean waiting time can obtain the same conclusion as well.

For the *GI*/*G*/1 queue with negative customers is considered where both the interarrival time and service time distributions are assumed to follow gamma distribution with parameters $(\kappa_1, \theta_1) = (1.9, 0.4)$ and $(\kappa_2, \theta_2) = (2.3, 0.2)$ respectively and the interarrival time distribution of negative customer is exponentially distributed with $\gamma = 0.5$. The time that the hazard rate functions for both interarrival time and service time distribution tend to constant $t_a = 4$ and $t_s = 2$ respectively. By choosing $N = 20$, Table 7.2 is tabulated. In Table 7.2, the results obtained using the proposed method with either 2-GLE and 3-GLE are closed to those obtained by the simulation procedure. In conclusion, Equations (7.2) and Equations (7.6) are shown to be able to generate waiting time distribution successfully when the distributions for both the interarrival time and service times are non-exponential.

Table 7.2: Waiting Time Distribution Computed from the Proposed Numerical Method and Simulation Procedure. $[(\kappa_1, \theta_1) = (1.9, 0.4), (\kappa_2, \theta_2) = (2.3, 0.2), \gamma = 0.5, t_a = 4, t_s = 2, N = 20]$

Waiting Time, t	Waiting Time Distribution, $W_q(t)$		
	Numerical Method		Simulation Procedure
	3-GLE	2-GLE	
0.00	0.610683	0.610656	0.610647
0.04	0.639265	0.639252	0.639145
0.08	0.666764	0.666746	0.666732
0.12	0.693247	0.693223	0.693116
0.16	0.718229	0.718240	0.718060
0.20	0.741748	0.741728	0.741625
0.24	0.763684	0.763639	0.763570
0.28	0.784069	0.784051	0.783986
0.32	0.802973	0.802919	0.802887
0.36	0.820384	0.820326	0.820315
0.40	0.836438	0.836439	0.836427
⋮	⋮	⋮	⋮
2.00	0.996676	0.996658	0.996725
Mean Waiting Time	0.567752	0.567746	0.566979

Compare to the waiting time distribution that is derived in Li and Zhao (2004), the new method proposed in this chapter seem to be simple and direct as the calculation can be done directly using stationary probabilities. However, they provided some analysis on the busy period of the queueing system with different killing discipline. On the other hand, Liu and Wu (2009) studied the MAP/*G*/1 G-queue without further investigation on the waiting time distribution. In the next chapter, a conclusion on the works done in this research are summarized and the possible future works are also discussed.

CHAPTER 8

CONCLUSION AND FUTURE WORKS

8.1 Conclusion

The dissertation introduces a modified method of Koh (2013) with a new numerical method K-GLE for finding the stationary queue length distribution for the single-server continuous-time queue with negative arrivals. In this proposed numerical method, the interarrival time and service time distributions are assumed to have a fairly general distribution called the *CAR* distribution. Sets of equations for governing the system are derived to find the stationary queue length distributions. The proposed numerical method has been proven mathematically for an *M*/*M*/1 G-queue in Chapter 3. While the numerical results obtained for the stationary queue length distribution for *GI*/*M*/1 G-queue and *M*/*G*/1 G-queue are verified by the analytical results and those computed from the simulation procedure. For the *CAR*/*CAR*/1 G-queue, since no papers have been published on finding the stationary queue length for a continuous-time *GI*/*G*/1 G-queue, the numerical results are only compared to those obtained by the simulation procedure. The stationary probabilities found by the numerical method are then used to calculate the waiting time distribution, which is another new numerical method introduced in this dissertation. All the comparisons show that the numerical methods are successfully applied in finding the basic performance measures of a queueing system with negative arrivals. The drawback of using the alternative numerical method in finding the stationary queue length distribution is the dimensionality problem when I, J or U is a large number and hence producing large system of equations. To solve for the dimensionality problem, three methods of solving the sets of equations derived are given in each chapters of 4, 5 and 6. The selection of methods on solving for the values of the stationary probabilities depends on the dimensionality and computational time.

Besides introducing the alternative numerical approach in finding the stationary queue length and waiting time distributions, an analytical approach to find the stationary queue length distribution for *GI*/*M*/1 queue with negative customers is also presented in Chapter 5. This dissertation also provide simulation algorithms in finding the basic performance measures. Apart from the proving in Chapter 3, the close results obtained from all the approaches have further validate the proposed numerical methods introduced in this dissertation.

It is confidence that the proposed numerical method could be applied to other queueing systems with other features like a repairable queue, negative customers with different removal discipline, vacation etc. which have been studied extensively in a recent decade by other researchers.

8.2 Future Works

One of the future works that will be considered is to reduce the dimensionality of the system of equations by combining the state spaces of the service process and negative arrival processes since these two processes cannot occur concurrently. By substituting Equations (7.3) and (7.4) into Equation (4.1) and using the same definitions of $g(t), G(t), h(t)$ and $H(t)$ in Section 7.1 , the newly modified service rate, σ_k, can be expressed as follow:

$$\begin{aligned} \sigma_k &= \frac{h(k\Delta t)(1-G(k\Delta t))+g(k\Delta t)(1-H(k\Delta t))}{(1-G(k\Delta t))(1-H(k\Delta t))}, \quad 1 \le k \le J \\ &= \frac{h(k\Delta t)}{(1-G(k\Delta t))}+\frac{g(k\Delta t)}{(1-H(k\Delta t))} \\ &= \gamma_k+\mu_k \end{aligned} \tag{8.1}$$

Equation (8.1) can also be used for the case when the negative arrival process is non-exponentially distributed and fulfill the assumption of *CAR* distribution. It is believed that this equation could be used to yield a system of equations with lower dimensionality.

Another future work that could be conducted is to analyze the quality of service with the existence of negative customer since some customers are not able to complete their service before leaving the system. Suppose that a data file is transferring from server to local computer and once in a while the processes are interrupted. This causes some of the information is lost and a new transferring process for this data file may need to be carried out again. Study on how negative customer represent the real-life application has been conducted in the previous few decades and it is the time to look into the quality of the service. What is the appropriate parameter of the distribution for negative customers should be used? What are the solutions that can be used to minimize the loss during the removal process? Is the scarification of a few customers from the queue can satisfy or increase the utility of those who are waiting behind the queue? Yet, this is an interesting topic that needs to be studied.

Similar queueing models can be performed by replacing the RCH removing discipline to RCE, DST or a random removal amount and extends the work by including features such as retrial queue, priority service, vacations etc. For example, Wang et al. (2015) considered a retrial queue and study the Nash equilibrium joining strategies with optimization on the social net benefit. As compared to this dissertation, this paper studied an $M/M/1$ G-queue with more features are included. Besides, Sun and Wang (2018) lately also studied this equilibrium joining strategies in an $M/M/1$ G-queue by considering observable and unobservable case. They derived optimal strategies in these two case and presented the results numerically with different parameters.

LIST OF REFERENCES

Abbas, K. and Assani, D., 2010. Strong stability of the embedded Markov chain in an GI/M/1 queue with negative customers. *Applied Mathematical Modelling*, 34(10), pp. 2806 – 2812.

Alfa, A.S., 2010. *Queueing theory for telecommunications: discrete time modelling of a single node system.* New York: Springer Science & Business Media.

Altiok, T., 2012. *Performance analysis of manufacturing systems*. New York: Springer Science & Business Media.

Artalejo, J.R., 2000. G-networks: A versatile approach for work removal in queueing networks. *European Journal of Operational Research*, 126(2), pp. 233 – 249.

Atencia, I. and Moreno, P., 2004. The discrete-time $Geo/Geo/1$ queue with negative customers and disasters. *Computers & Operations Research*, 31(9), pp. 1537 – 1548.

Atencia, I. and Moreno, P., 2006. A discrete-time $Geo/G/1$ retrial queue with the server subject to starting failures. *Annals of Operations Research*, 141(1), pp. 85 – 107.

Bayer, N. and Boxma, O.J., 1996. Wiener-Hopf analysis of an $M/G/1$ queue with negative customers and of a related class of random walks. *Queueing Systems*, 23(1 – 4), pp. 301 – 316.

Bailey, N.T., 1954. A continuous time treatment of a simple queue using generating functions. *Journal of the Royal Statistical Society. Series B (Methodological)*, pp. 288 – 291.

Benson, F. and Cox, D.R., 1951. The productivity of machines requiring attention at random intervals. *Journal of the Royal Statistical Society. Series B (Methodological)*, pp. 65 – 82.

Bhat, U.N., 1969. Sixty years of queueing theory. *Management Science*, 15(6), pp. B-280.

Bhat, U.N., Shalaby, M. and Fischer, M.J., 1979. Approximation techniques in the solution of queueing problems. *Naval Research Logistics Quarterly*, 26(2), pp. 311 – 326.

Bi, H., 2016. Evacuee Flow Optimisation Using G-network with Multiple Classes of Positive Customers. *In Modeling, Analysis and Simulation of Computer and Telecommunication Systems (MASCOTS), 19 -21 September 2016, IEEE 24th International Symposium*, Imperial College, London. Imperial College: IEEE, pp.135 – 143.

Bolch, G., Greiner, S., De Meer, H. and Trivedi, K.S., 2006. *Queueing networks and Markov chains: modeling and performance evaluation with computer science applications*. Hoboken, New Jersey: John Wiley & Sons.

Boucherie, R.J. and Boxma, O.J., 1996. The workload in the $M/G/1$ queue with work removal. *Probability in the Engineering and Informational Sciences*, 10(2), pp. 261 – 277.

Boxma, O.J., Perry, D. and Stadje, W., 2001. Clearing models for $M/G/1$ queues. *Queueing Systems*, 38(3), pp. 287 – 306.

Buzen, J. P., 1973. Computational algorithms for closed queueing networks with exponential servers. *Communications of the ACM*, 16(9), pp. 527 - 531.

Chaudhry, M.L. and Templeton, J.G., 1983. *A first course in bulk queues*. New York; Toronto: J. Wiley.

Coffman, E.G. and Hofri, M., 1986. Queueing models of secondary storage devices. *Queueing Systems*, 1(2), pp. 129 – 168.

Do, T.V., 2011. Bibliography on G-networks, negative customers and applications. *Mathematical and Computer Modelling: An International Journal*, 53(1 – 2), pp. 205 – 212.

Do, T.V., Papp, D., Chakka, R., Sztrik, J. and Wang, J., 2014. M/M/1 retrial queue with working vacations and negative customer arrivals. *International Journal of Advanced Intelligence Paradigms*, 6(1), pp. 52 – 65.

Dudin, A.N. and Karolik, A.V., 2001. $BMAP/SM/1$ queue with Markovian input of disasters and non-instantaneous recovery. *Performance Evaluation*, 45(1), pp. 19 – 32.

Dudin, A. and Nishimura, S., 1999. A $BMAP/SM/1$ queueing system with Markovian arrival input of disasters. *Journal of Applied Probability*, 36(3), pp. 868 – 881.

Duff, I.S., Erisman, A.M. and Reid, J.K., 2017. *Direct methods for sparse matrices*. New York: Oxford University Press.

Erlang, A.K., 1909. The theory of probabilities and telephone conversations. *Nyt Tidsskrift for Matematik B*, 20, pp. 33 – 39.

Fabens, A.J., 1961. The solution of queueing and inventory models by semi-Markov processes. *Journal of the Royal Statistical Society. Series B (Methodological)*, pp. 113 – 127.

Fourneau, J.M. and Gelenbe, E., 2017. G-networks with adders. *Future Internet*, 9(3), p. 34.

Fry, T.C. and Bell Telephone Laboratories., 1928. *Probability and Its Engineering Uses*, p. 183. New York: Van Nostrand.

Gao, S. and Wang, J., 2014. Performance and reliability analysis of an M/G/1-G retrial queue with orbital search and non-persistent customers. *European Journal of Operational Research*, 236(2), pp. 561 – 572.

Gelenbe, E., 1994. G-networks: a unifying model for neural and queueing networks. *Annals of Operations Research*, 48(5), pp. 433 – 461.

Gelenbe, E., 1993a. G-networks by triggered customer movement. *Journal of Applied Probability*, 30(3), pp. 742 – 748.

Gelenbe, E., 1993b. G-networks with signals and batch removal. *Probability in the Engineering and Informational Sciences*, 7(3), pp. 335 – 342.

Gelenbe, E. 1993c. Learning in the recurrent random neural network. *Neural Computation*, 5(1), pp. 154 – 164.

Gelenbe, E., 1991. Product-form queueing networks with negative and positive customers. *Journal of Applied Probability*, 28(3), pp. 656 – 663.

Gelenbe, E., 1989. Random neural networks with negative and positive signals and product form solution. *Neural Computation*, 1, pp. 502 – 510.

Gelenbe, E., Glynn, P. and Sigman, K., 1991. Queues with negative arrivals. *Journal of Applied Probability*, 28(1), pp. 245 – 250.

Goyal, S. K., & Gunasekaran, A. (1990). Multi-stage production-inventory systems. *European Journal of Operational Research*, 46(1), 1 – 20.

Graves, S. C. (1982). The application of queueing theory to continuous perishable inventory systems. *Management Science*, 28(4), 400 – 406.

Gross, D. and Harris, C.M., 2008. *Fundamentals of queueing theory*. New York: John Wiley and Sons. Inc.

Haghighinejad, H. A., Kharazmi, E., Hatam, N., Yousefi, S., Hesami, S. A., Danaei, M., & Askarian, M. (2016). Using queuing theory and simulation modelling to reduce waiting times in an Iranian emergency department. *International Journal of Community Based Nursing and Midwifery*, 4(1), 11.

Harrison, P.G. and Pitel, E., 1993. Sojourn times in single-server queues by negative customers. *Journal of Applied Probability*, 30(4), pp. 943 – 963.

Harrison, P.G. and Pitel, E., 1995. Response time distributions in tandem G-networks. *Journal of Applied Probability*, 32(1), pp. 224 – 246.

Harrison, P.G. and Pitel, E., 1996. The *M*/*G*/1 queue with negative customers. *Advances in Applied Probability*, 28, pp. 540 – 566.

Jain, G. and Sigman, K., 1996. A PollaczekKhintchine formula for $M/G/1$ queues with disasters. *Journal of Applied Probability*, 33(4), pp. 1191 – 1200.

Jiang, T. and Liu, L., 2017. Analysis of a GI/M/1 queue in a multi-phase service environment with disasters. *RAIRO-Operations Research*, 51(1), pp. 79 – 100.

Karol, M., Hluchyj, M. and Morgan, S., 1987. Input versus output queueing on a space-division packet switch. *IEEE Transactions on Communications*, 35(12), pp. 1347 – 1356.

Kendall, D.G., 1953. Stochastic processes occurring in the theory of queues and their analysis by the method of the imbedded Markov chain. *The Annals of Mathematical Statistics*, pp. 338 – 354.

Koh, S.K., 2013. *Maintenance of deteriorating non-exponential single server queue*. PhD thesis), University of Malaya, Malaysia.

Kumar, B.K., Arivudainambi, D. and Krishnamoorthy, A., 2006. Some results on a generalized $M/G/1$ feedback queue with negative customers. *Annals of Operations Research*, 143(1), pp. 277 – 296.

Lakshmi, C., & Iyer, S. A. (2013). Application of queueing theory in health care: A literature review. *Operations Research for Health Care*, 2(1 – 2), 25 – 39.

Lee, A.M., 1966. *Applied Queueing Theory*. New York: Macmillan.

Lee, D.H. and Kim, K., 2014. Analysis of repairable $Geo/G/1$ queues with negative customers. *Journal of Applied Mathematics, 2014*.

Li, J.H. and Tian, N.S., 2007. The discrete-time $GI/Geo/1$ queue with working vacations and vacation interruption. *Applied Mathematics and Computation*, 185(1), pp. 1 – 10.

Li, T. and Zhang, L., 2017. An M/G/1 Retrial G-Queue with General Retrial Times and Working Breakdowns. *Mathematical and Computational Applications*, 22(1), p. 15.

Li, Q.L. and Zhao, Y.Q., 2004. A $MAP/G/1$ queue with negative customers. *Queueing Systems*, 47(1 – 2), pp. 5 – 43.

Lindley, D.V., 1952. The theory of queues with a single server. *Mathematical Proceedings of the Cambridge Philosophical Society*, April 1952 Cambridge. Cambridge: Cambridge University Press, 48(2), pp. 277 – 289.

Liu, Z., Wu, J. and Yang, G., 2009. An $M/G/1$ retrial G-queue with preemptive resume and feedback under N-policy subject to the server breakdowns and repairs. *Computers & Mathematics with Applications*, 58(9), pp. 1792 – 1807.

Liu, Z. and Wu, J., 2009. An MAP/*G*/1 G-queues with preemptive resume and multiple vacations. *Applied Mathematical Modelling*, 33(3), pp. 1739 – 1748.

Ledermann, W. and Reuter, G.E.H., 1954. Spectral theory for the differential equations of simple birth and death processes. *Phil. Trans. R. Soc. Lond. A*, 246(914), pp. 321 – 369.

Molina, E.C., 1927. Application of the theory of probability to telephone trunking problems. *Bell System Technical Journal*, 6(3), pp. 461 – 494.

Moss, G. (1987). Hospital pharmacy staffing levels and outpatient waiting times. *The Pharmaceutical Journal*, 239, 69 – 70.

Newell, C., 2013. *Applications of queueing theory (Vol. 4)*. New York: Springer Science & Business Media.

Palm, C., 1947. The distribution of repairmen in servicing automatic machines. *Industritidningen Norden*, 75, pp. 75 – 80.

Park, H.M., Yang, W.S. and Chae, K.C., 2009. The $Geo/G/1$ queue with negative customers and disasters. *Stochastic Models*, 25(4), pp. 673 – 688.

Papadopoulos, H.T. and Heavey, C., 1996. Queueing theory in manufacturing systems analysis and design: A classification of models for production and transfer lines. *European Journal of Operational Research*, 92(1), pp. 1 – 27.

Pollaczek, F., 1934. Über das warteproblem. *Mathematische Zeitschrift*, 38(1), pp. 492 – 537.

Pollaczek, F., 1965. Concerning an analytic method for the treatment of queueing problems. *Congestion Theory*, pp. 1 – 42.

Saad, Y., 2003. *Iterative methods for sparse linear systems (Vol. 82)*. Philadelphia: siam.

Saaty, T.L., 1961. Elements of queueing theory: with applications (Vol. 34203). New York: McGraw-Hill.

Saaty, T.L., 1966. Seven more years of queues. A lament and a bibliography. *Naval Research Logistics Quarterly*, 13(4), pp. 447 – 476.

Schwarz, M., & Daduna, H. (2006). Queueing systems with inventory management with random lead times and with backordering. *Mathematical Methods of Operations Research*, 64(3), 383 – 414.

Schwarz, M., Sauer, C., Daduna, H., Kulik, R., & Szekli, R. (2006). *M/M/*1 queueing systems with inventory. *Queueing Systems*, 54(1), 55 – 78.

Sickel, S., Yeung, M.C. and Held, M.J., 2005. A comparison of some iterative methods in scientific computing. *Summer Research Apprentice Program.*

Shin, Y.W., 1998. Sojourn time distributions for $M/M/c$ G-queue. *Communications of the Korean Mathematical Society*, 13(2), pp. 405 – 434.

Soujanya, M.L., 2015. Perishable inventory system with service interruptions, retrial demands and negative customers. *Applied Mathematics and Computation*, 262, pp. 102 – 110.

Syski, R., 1960. *Introduction to Congestion Theory in Telephone Systems*. London: Oliver and Boyd.

Sun, K. and Wang, J., 2018. Equilibrium joining strategies in the single server queues with negative customers. *International Journal of Computer Mathematics*, pp. 1 – 23.

Tamir, Y. and Frazier, G.L., 1992. Dynamically-allocated multi-queue buffers for VLSI communication switches. *IEEE Transactions on Computers*, 41(6), pp. 725 – 737.

Tian, N. and Zhang, Z.G., 2002. The discrete-time $GI/Geo/1$ queue with multiple vacations. *Queueing Systems*, 40(3), pp. 283 – 294.

Wang, J., Huang, Y. and Van Do, T., 2013. A single-server discrete-time queue with correlated positive and negative customer arrivals. *Applied Mathematical Modelling*. 37(9), pp. 6212 – 6224

Wang, F., Wang, J. and Zhang, F., 2015. Strategic behavior in the single-server constant retrial queue with individual removal. *Quality Technology & Quantitative Management*, 12(3), pp. 325 – 342.

Ward, D. S., & Rubin, I. (1977). Applications of queuing theory to some problems in neuronal circuity. *Applied Mathematics and Computation*. 3, 227 – 245

Wu, J. and Lian, Z., 2013a. A single-server retrial G-queue with priority and unreliable server under Bernoulli vacation schedule. *Computers & Industrial Engineering*, 64(1), pp. 84 – 93.

Wu, J. and Lian, Z., 2013b. Analysis of the $M_1, M_2/G/1$ G-queueing system with retrial customers. *Nonlinear Analysis: Real World Applications*, 14(1), pp. 365 – 382.

Yang, W.S. and Chae, K.C., 2001. A note on the *GI*/*M*/1 queue with Poisson negative arrivals. *Journal of Applied Probability*, 38(4), pp. 1081 – 1085.

Zhang, M. and Liu, Q., 2015. An $M/G/1$ G-queue with server breakdown, working vacations and vacation interruption. *Opsearch*, 52(2), pp. 256 – 270.

Zheng, Y. S., & Zipkin, P. (1990). A queueing model to analyze the value of centralized inventory information. *Operations Research*, 38(2), 296 - 307.

Zhou, W.H., 2005. Performance analysis of discrete-time queue *GI*/*G*/1 with negative arrivals. *Applied Mathematics and Computation*, 170, pp. 1349 - 1355.

Zhu, Y., 2003. $M/GI/1$ models with negative arrivals to be served. *Journal of Systems Science and Complexity*, 16(4), pp. 533 – 540.

Zhu, Y. and Zhang, Z.G., 2004. $M/GI/1$ queues with services of both positive and negative customers. *Journal of Applied Probability*, 41(4), pp. 1157-1170.

LIST OF PUBLICATIONS

Koh, S.K., Chin, C.H., Tan, Y.F., Pooi, A.H., Goh, Y.K., Lee, M.C. and Ng, T.C., 2018. Repairable queue with non-exponential interarrival time and variable breakdown rates. *International Journal of Engineering & Technology*, 7 (2.15), pp. 76 – 80.

Koh, S.K., Chin, C.H., Tan, Y.F., Pooi, A.H., Goh, Y.K. and Teoh, L.E., 2018. Stationary queue length of a single-server queue with negative arrivals and non-exponential service time distributions. *In MATEC Web of Conferences*, 189, p. 02006. EDP Sciences.

Chin. C.H., Koh, S.K., Tan, Y.F., Pooi, A.H. and Goh, Y.K., 2018. Stationary queue length distribution of a continuous-time queueing system with negative arrival. *Accepted, waiting to be published*.

APPENDIX A

SIMULATION PROCEDURE OF STATIONARY QUEUE LENGTH DISTRIBUTION

The simulation procedure to obtain the stationary queue length distribution applied the concept about fraction of time where the stationary probability p_n can be interpreted as "the fraction of time that there are n customers in the system where $n = 0, 1, 2, \cdots$" that is mentioned in Gross and Harris (2008).

Let $F(t_1)$ be the cumulative distribution function of the interarrival time of positive customers, $G(t_2)$ be the cumulative distribution function of service time and $H(t_3)$ be the cumulative distribution function of the interarrival time of negative customers. Then the complete algorithm of the simulation procedure can be presented as follows:

1. Initialization:
   - Three variables $clock\_time$, $previous\_event\_time$ and $system\_size$ are set to be 0.
   - Generate a uniform random number u_1 on [0,1].
     (a) Let $u_1 = F(t_1)$ and $t_1 = F^{-1}(u_1)$.
     (b) Let $t_2 = \infty$.
     (c) Let $t_3 = \infty$.
2. For each iteration i:
   - Set $previous\_event\_time = clock\_time$.
   - Precedence operation:

(a) if $(t_1 < t_2)$ and $(t_1 < t_3)$, then

   i. Update $clock\_time$ by adding t_1;

   ii. Update the total duration $Time_n$ for current $system\_size$ by adding $clock\_time - previous\_event\_time$.

   iii. Update $system\_size$.

   iv. if $system\_size = 1$, then

      – Generate two uniform random numbers u_2 and u_3 on [0,1].

      – Let $u_2 = G(t_2)$ and $t_2 = G^{-1}(u_2)$ to represent service times.

      – Let $u_3 = H(t_3)$ and $t_3 = H^{-1}(u_3)$ to represent next negative interarrival time.

   v. Generate a uniform random number u_1 on [0,1], let $u_1 = F(t_1)$ and $t_1 = F^{-1}(u_1)$ to represent the next interarrival time.

(b) if $(t_2 \leq t_1)$ and $(t_2 \leq t_3)$, then

   i. Update $clock\_time$ by adding t_2;

   ii. Update the total duration $Time_n$ for current $system\_size$ by adding $clock\_time - previous\_event\_time$.

   iii. Update $system\_size$.

   iv. if $system\_size \geq 1$, then

      – Generate a uniform random number u_2 on [0,1], let $u_2 = G(t_2)$ and $t_2 = G^{-1}(u_2)$ to represent next service time.

      else:

      – Let $t_2 = \infty$.

      – Let $t_3 = \infty$.

(c) if $(t_3 < t_2)$ and $(t_3 < t_1)$, then

   i. Update $clock\_time$ by adding t_3;

ii. Update the total duration $Time_n$ for current $system\_size$ by adding $clock\_time - previous\_event\_time$.

iii. Update $system\_size$.

iv. if $system\_size \geq 1$, then

   – Generate two uniform random numbers u_2 and u_3 on [0,1].

   – Let $u_2 = G(t_2)$ and $t_2 = G^{-1}(u_2)$.

   – Let $u_3 = H(t_3)$ and $t_3 = H^{-1}(u_3)$.

   else:

   – Let $t_2 = \infty$.

   – Let $t_3 = \infty$.

3. Computation of p_n:

   • Compute the proportion of the total duration $Time_n$ when $system\_size = n$ to the final record $clock\_time$.

$$p_n = \frac{\text{The total duration } Time_n \text{ with respect to } n}{\text{Final } clock\_time}$$

APPENDIX B

SIMULATION PROCEDURE OF WAITING TIME DISTRIBUTION

The simulation procedure to find the waiting time distribution almost the same as the one in Appendix A except that it's calculation does not based on the fraction of time but the count of the simulated waiting time that is less than time t divided but the total count of simulated waiting time.

Let the time spent in the system by nth positive customer be $W_{q,n}$, T_n be the interarrival time between the $(n+1)$th positive customer and nth positive customer and S_n be the service time of the nth positive customer. While B_n is the time/duration between the departure time of the $(n-1)$th customer and the nth customer. A simple illustration of the concept is shown in Figure B.1 below.

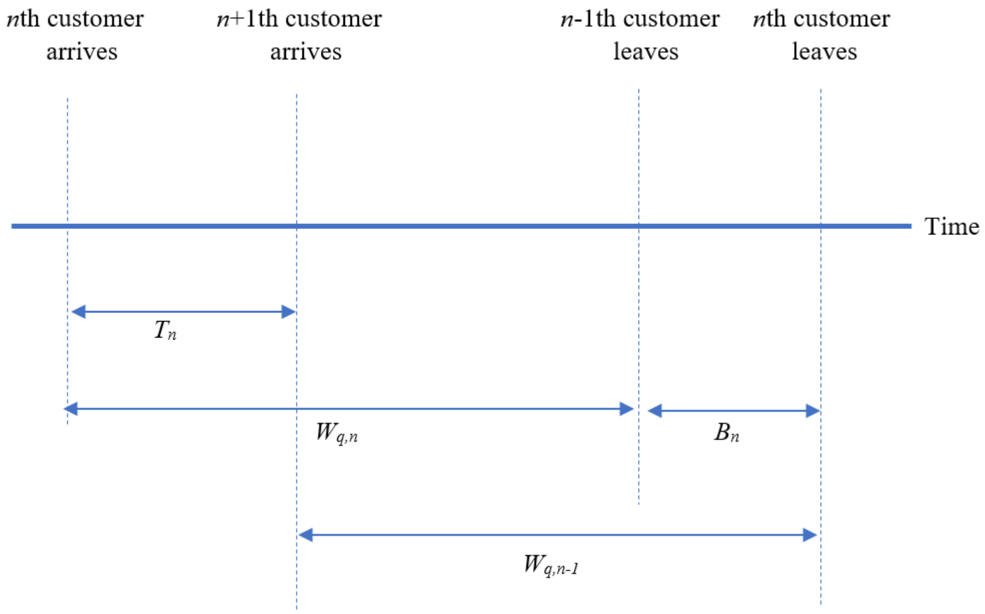

Figure B.1: Successive *GI/G/1* Waiting Times with Negative Customers.

With the aid of Figure B.1, it can be seen that there is a simple recurrence relation between $W_{q,n}$ and $W_{q,n-1}$ and expressed as the following:

$$W_{q,n+1} = \begin{cases} 0, & \text{if } W_{q,n} \leq T_n - B_n, \\ W_{q,n} + B_n - T_n, & \text{if } W_{q,n} > T_n - B_n. \end{cases} \tag{B.1}$$

The term B_n in Equation (B.1) is the minimum time between the service time, S_n and the first interarrival time of negative arrivals that occurs during a service process, R, $B_n = \min(S_n, R)$. While $R = \infty$ if no negative customer arrives during a service process. A positive customer that is undergoing service can leave the system by either removed by a negative customer or complete a service whichever come first. If no negative customer arrives during a service process, the positive customer will depart from the system by service completion. Figure B.2 below represents the actual condition the time B_n that occurs in the last segment of Figure B.1.

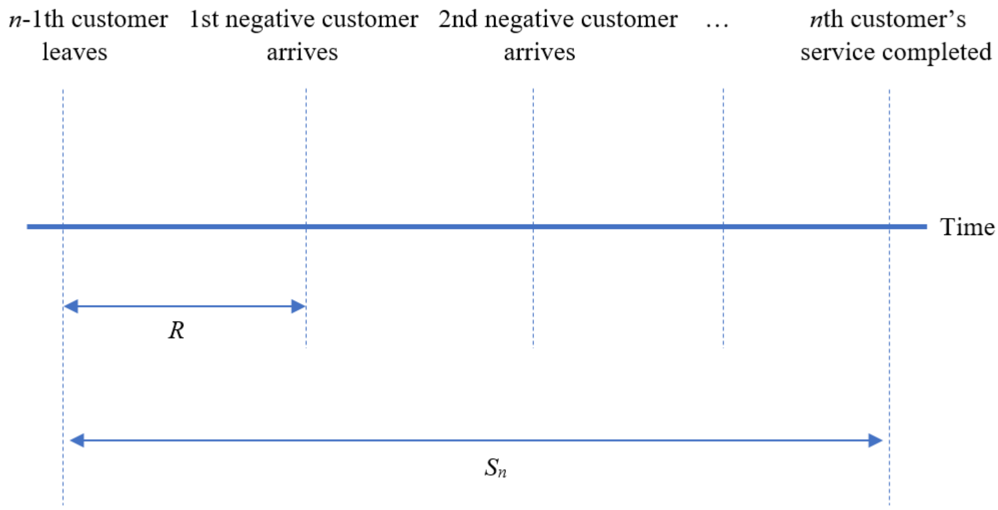

Figure B.2: Events That Affects the Leaving Point of Positive Customers.

Bookkeeping is done by collecting the frequency of $W_{q,n}$ in each step of the simulation procedure. Once the simulation procedure are completed, the waiting time distribution $W_q(t)$ can be calculated as follows:

$$W_q(t) = \frac{\text{Counts of } W_{q,n} \text{ less than or equal to } t}{\text{Total number of } W_{q,n}} \tag{B.2}$$