AN EMPPIRICAL STUDY ON ASYMMETRIC JUMP DIFFUSION FOR OPTION AND ANNUITY PRICING

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By

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ABSTRACT

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Lau Kein Joe

In this research, we are presenting a method for estimation of market parameters modeled by jump diffusion process. As we are concerned about the current pricing model with geometric Brownian motion is not sufficient to capture the events of jump spikes. The method proposed is based on the Gibbs sampling method, while the market parameters are the drift, the volatility, the jump intensity and its rate of occurrence.

We have demonstrated that Kou's jump diffusion model is insufficient to observe and to identify the effect on jump spike event onto the market indexes as it assumes jumps are symmetrical to each other for both directions. Asymmetric double normal jump diffusion model is introduced, where the jump component is modified into two different directions instead of fusing as one.

The empirical method is used to estimate the parameters of asymmetric double normal jump diffusion model from real market history data. Demonstration on how to use these parameters to estimate the fair price of European call option and annuity will be shown, for the situation where the market is modeled by jump diffusion process with different intensity and occurrence. The results are

compared to conventional options to observe the impact of jump effects.

In conclusion, the proposed asymmetric double normal jump diffusion model able to capture the jump distribution of underlying assets in two directions. It can be applied into the pricing model of both European call option and annuity.

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APPROVAL SHEET

This thesis entitled "**AN EMPIRICAL STUDY ON ASYMMETRIC JUMP DIFFUSION FOR OPTION AND ANNUITY PRICING**" was prepared by LAU KEIN JOE and submitted as partial fulfillment of the requirements for the degree of Master of Science at Universiti Tunku Abdul Rahman.

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SUBMISSION OF DISSERTATION

It is hereby certified that **Lau Kein Joe** (ID No: **16UEM01201**) has completed this dissertation entitled "**AN EMPIRICAL STUDY ON ASYMMETRIC JUMP DIFFUSION FOR OPTION AND ANNUITY PRICING**" under the supervision of **Dr. Goh Yong Kheng** (Supervisor) from the Department of Mathematical and Actuarial Sciences, Lee Kong Chian Faculty of Engineering and Science, and **Dr. Lai An-Chow** (Co-Supervisor) from the Department of Mathematical and Actuarial Sciences, Lee Kong Chian Faculty of Engineering and Science.

I understand that University will upload softcopy of my dissertation in pdf format into UTAR Institutional Repository, which may be made accessible to UTAR community and public.

Yours truly,

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(*Lau Kein Joe*)

DECLARATION

I Lau Kein Joe hereby declare that the dissertation is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UTAR or other institutions.

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LIST OF ABBREVIATIONS

GBM	Geometric Brownian motion
BS model	Black-Scholes model
MCMC	Markov Chain Monte Carlo
VA	Variable Annuity
SPVA	Single Premium Variable Annuity
FPVA	Flexible Premium Variable Annuity
GBDB	Guaranteed Minimum Death Benefit
GMLB	Guaranteed Minimum Living Benefit
GMAB	Guaranteed Minimum Accumulation Benefit
GMWB	Guaranteed Minimum Withdrawal Benefit
GLWB	Guaranteed Lifetime Withdrawal Benefit
M&E	Mortality and Expense risk charge
SIR	Stochastic interest rate
SV	Stochastic volatility

CHAPTER 1

INTRODUCTION

The financial market is well known to be volatile and can be difficult to predict. Despite its characteristics, investors are still trying to learn and forecast the financial market. The most commonly used method in modeling stock price movement is Brownian motion. An example of the application of Brownian motion in pricing option is using the Black-Scholes model.

In the early 90s, Black-Scholes model (Black, 1973) is considered to be one of the most favorable methods in calculating option prices. The model contains an implied volatility and is used to model future prices of the financial assets, where many investors use it to calculate the fair price of an option. However, during the 1997 financial crisis, most investors are suffering from major losses due to the drastic jumps in prices, including experienced market investors. This had shown that the Black-Sholes model might be useful to a certain extent, however, it could not handle the extreme events where there are large market movements.

Here we propose a model aimed to extend the usage of Black-Scholes model with jumps. Incorporating jumps in the Black-Scholes model is not new. However, currently available models assume symmetric jump distributions in both upward and downward directions. In our model, we will treat the jumps in the two directions separately.

1.1 Problem Statements

Are the jump events in the markets following symmetric jump distributions in both upward and downward directions like what Kou's and Merton's models assumed?

Initially, Kou proposed that jump event follows a normal distribution (Kou, 2002) and followed by a double exponential later in the year 2007 (Kou, 2007). While other researchers mostly based on these two models where both models imply that the occurrence of the jump is symmetric in both directions. This assumption allocates the jump distribution around zero, where positive indicates upward spike and negative indicates a downward spike. This will cause the average jump intensity to occur around zero frequently and can hardly to be detected.

The next problem statement we concern about is whether the current pricing method with geometric Brownian motion (GBM) sufficiently enough to determine the fair price of market securities? The impact of jump event is never considered in pricing method. Hence, whenever the market changes drastically, the pricing of an underlying asset often breach the expectation and caused massive losses for investors and risk managers.

Last problem that we need to find out is whether our jump diffusion model able to fit into the pricing of securities, such as European call options and annuity. We like to have a model that can generalize to multiple types of financial derivatives but not limited to one. Hence, we will use the modified jump model onto the pricing model of European call options and annuity.

1.2 Research Objectives

The aim of this project is to study the volatility-volume interactions on multiple financial instruments. Mathematical models, as well as technical models, will be used to achieve more accuracy in predicting the market reversals. Below are the objectives of this research.

Firstly, we will modify the existing Kou model to suit both asymmetric upward and downward jump distribution. As later in our research, we discover that the distribution of jump spike is asymmetric within each other.

Next, we would like to make a comparison between GBM and modified jump diffusion model. This is to show whether the modified jump diffusion model able to reproduce similar results when there is no jump occurs. At the same time, find out a better projection of the underlying asset in the future.

Lastly, we will perform some simulations and observe the effect of jump diffusion model on fair prices of pricing on European call option and annuity. This objective aim to determine how each jump spike parameters able to impact the price on different financial derivatives.

1.3 Overview

In this project, the aim is to investigate the impact of jump diffusion in pricing models of volatile markets. By using the Black-Scholes model (BS model) as the foundation to calculate the theoretical price of European options and annuity, a comparison of price between the jump diffusion model and the GBM will be made. The details will be further discussed in Chapter 3.

In Chapter 2 we will briefly explain the important terminologies and concepts needed in this research. Discussion of similar papers, studies and researches will be included in this chapter with brief explanations on the fundamental ideas of GBM, Black-Scholes model and jump diffusion model. To further investigate the occurrence and characteristic of jump event, we would need the knowledge regarding Bayesian theorem, Markov chain Monte Carlo and Gibbs sampling method.

Chapter 3 will discuss the method used to carry out the research. Investors often try to analyze the trends of the market. The intensity, the force of the trends or the drift is denoted with the parameter, μ; while the volatility, σ are the deviation from the expected prices. The research aims to obtain and retrieve these crucial variables or parameters that are affecting the market prices so that it is possible to project the market price with better accuracy in the near future. The higher the accuracy of projection, the better the estimation of the risk that investors might be facing soon.

In the current research, we choose European options and annuity products. The crucial parameter will be retrieved with the Gibbs sampling method. Limitation of Gibbs sampling method will be discussed and hence introduce the application of an empirical method. Results are studied and tested with jump diffusion model, to see whether the market are closely related to jump diffusion simulated data. The detail methods are further discussed Section 3.1 to Section 3.6. The jump factor will be tested in various intensity to test the effect on future market price. The relation between both will be studied and shown in results in Section 4.

Chapter 4 will show the results and data analysis for the applications of jump diffusion model. Chapter 5 will include the discussion and conclusion of the research. The importance of modification to jump diffusion model and its applications will be highlighted in this chapter.

CHAPTER 2

LITERATURE REVIEW

The market is where people buy and purchase their financial products for different purposes, such as hedging, make arbitrages, investing for the future, and much more. However, the movement of the market is so unpredictable such that there is no one could consistently beat the market in an efficient market. The market is usually compared with geometric Brownian motion, as they possess some similarities in their randomness.

However, reality shows that the market is far more different than the GBM. For example, the market crashed event that happened in real life such as the 1997 crisis and the 2008 sub-prime crisis are happen with an extremely low probability under the log-normal hypothesis. Yet, such an event almost happens once every decade. Hence, this research wishes to seek a way of forecasting a market instrument by with historical data and prices which are attainable from the market, that could include the predictions of jump event.

Regarding the market, one of the famous models being widely used is the Black-Scholes model. Its ability that able to price the option premium. However, from the research paper by Steven R. Dunbar (Dunbar, 2016), he stated the limitations of the Black-Scholes model. Steven R.D. stated that a few assumptions of the Black-Scholes model are not so practical in real life predictions for option prices (will be discussed in Section 2.3).

2.1 Financial Products

Financial products are products available on the market which provide convenience to everyday lives. For example, insurance, bank services, investment products or retirement schemes. Besides those, there are derivatives which act as a contract between two parties who agree to have a flow of cash now and in the future. Normally with a strike price, K at the specific day in the future, time t. There are many kinds of derivatives exist currently, and we will introduce some of them here.

Options are known as a deal with the right to buy or to sell an asset in the future, with the price set today for the options buyer, while the writer (seller) is obliged to deal is it is exercised. This type of derivatives is one of the common one existing in the market.

Futures is a deal with an obligation to buy or to sell an asset in the future, with the price set today. It is similar to options, but instead of having the right, both parties are obliged to exercise the deal during the set period. Besides, there are swaps, where both parties exchanging one stream of future payments for another one, possibly in two different country currencies.

Lastly, there are annuity contracts between the buyer and the insurance company. An annuity can be referred to as a contract or agreement by which one receives fixed payments on an investment for a lifetime or for a specified number of years. Usually, an annuity contract is treated as a retirement plan that rewards the buyer while they are still living.

2.2 Brownian Motion

We will discuss the Brownian motion and geometric Brownian motion (GBM), while the latter being one of the most popular models in financial theory. Brownian motion, also known as Wiener process. First, we introduce a random walk model denoted by

$$W(t_{k+1}) = W(t_k) + dW(t), \tag{2.1}$$

where,

- $dW(t) \sim \epsilon_{t_k}\sqrt{\Delta t}$;
- $t_{k+1} - t_k = \Delta t$, and $k = 0, ..., N$ with $t_0 = 0$;
- $\epsilon_{t_k} \sim N(0,1)$ are identical and independent distributed (i.i.d.) random variables;
- assume that $W(t_0) = 0$;

The model above is known as a discrete random walk model. When $j < k$, it could be rearranged into

$$W(t_k) - W(t_j) = \sum_{i=j}^{k-1} \epsilon_{t_i}\sqrt{\Delta t}. \tag{2.2}$$

When the random walk model takes the limit of $\Delta t \to \infty$, this random walk model will approach to Brownian motion. Equation 2.2 shows the difference between two periods. It has the characteristics as below:

- $W(t_k) - W(t_j)$ is normally distributed as the right hand side is a sum of i.i.d. random variables that follows normal distribution;
- The expectation is $\mathbf{E}(W(t_k) - W(t_j)) = 0$; while
- the variance is $\mathbf{Var}(W(t_k) - W(t_j)) = \mathbf{E}[\sum_{i=j}^{k-1} \epsilon_{t_i}\sqrt{\Delta t}]^2 = (k-j)\Delta t = t_k - t_j$; and
- $W(t_4) - W(t_3)$ is uncorrelated with $W(t_2) - W(t_1)$, for $t_1 < t_2 < t_3 < t_4$.

From the previous Equation 2.2, by taking limits as Δt approaches to zero, it will form a Wiener process (Brownian motion),

$$dW(t) = \epsilon(t)\sqrt{dt},$$

where $\epsilon(t)$ is i.i.d. standard normal random variable. The process can be interpreted as a continuous-time approximation of the random walk model.

Definition 2.1. *Wiener process, $W(t)$ is defined with following properties.*

1. *For $t < s, W(s) - W(t)$ is a normally distributed random variable, with zero mean and $s - t$ variance.*
2. *For $0 \leq t_1 < t_2 \leq t_3 < t_4, W(t_4) - W(t_3)$ is uncorrelated with $W(t_2) - W(t_1)$. This is independent increment property.*
3. *$W(t_0) = 0$ with probability 1.*
4. *$W(t)$ is continuous with respect to t.*

Definition above could further deduce into important properties of Brownian motion.

1. *Martingale* property of Brownian motion, where $\mathbf{E}(W(s) \mid W(t)) = \mathbf{E}(W(s) - W(t) + W(t) \mid W(t)) = W(t)$, for $t < s$.
2. Wiener process, $W(t)$ is nowhere differentiable. The term

$$\mathbf{E}\left(\left(\frac{W(s) - W(t)}{s - t}\right)^2\right) = \frac{1}{s - t}$$

approaches to ∞ as $s - t$ close approaches to zero.

3. Brownian motion is everywhere continuous even nowhere differentiable.

2.2.1 Itô Process and Geometric Brownian Motion

A stochastic process is called Itô process when it could solves the following:

$$X_t = X_0 + \int_0^t \mu(X_s, s)ds + \int_0^t \sigma(X_s, s)dW_s. \qquad t \geq 0. \tag{2.3}$$

The shorthand, stochastic differential equation (SDE) for Itô differential dX_t is

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t, \tag{2.4}$$

where,

- X_0 is a scalar starting point;
- $\{\mu(X_t, t) : t \geq 0\}$ and $\{\sigma(X_t, t) : t \geq 0\}$ are stochastic processes satisfying regularity condition.
- $\{\mu(X_t, t)\}$ and $\{\sigma(X_t, t)\}$ are called the drift and volatility respectively.

Before we look into GBM, we briefly discuss about continuous-time version of Brownian motion, which is

$$d \log S(t) = vdt + \sigma dW(t).$$

The right hand side is normally distributed with the drift of vdt and volatility of $\sigma^2 dt$. Integrating both sides will yield the following model:

$$\log S(t) = \log S(0) + vt + \sigma W(t). \tag{2.5}$$

The expected log return is $\mathbf{E}(\log S(t)) = \log S(0) + vt$. The process $S(t)$ will be following GBM, when the expected log prices grow linearly with t , and define as below.

Definition 2.2. *Let $X(t)$ be a Brownian motion with drift v and variance σ^2,*

$$dX(t) = vdt + \sigma W(t).$$

The process $S(t) = e^{X(t)}$ *is a GBM with drift* μ*, where* $\mu = v + \frac{1}{2}\sigma^2$*. The process* $S(t)$ *satisfy the followings*

$$d \log S(t) = \big(\mu - \frac{1}{2}\sigma^2\big)dt + \sigma dW(t).$$

This is mentioned by Samuelson (Samuelson, 1952) the GBM is shown, where the stochastic process of the prices of an asset could be in a form of the following stochastic differential equation (SDE):

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t), \tag{2.6}$$

where,

- μ is the drift rate or the rate of return;
- σ is the volatility of the asset;
- $W(t)$ is a Brownian motion or Wiener process,and
- $S(t)$ is the spot price of the underlying assets.

In brief, we can say that the change of price denoted as $dS(t)$, is controlled by a drift μ where is can say to be the trend for the asset. While the volatility, σ determine its fluctuation or deviation.

The work done by Adeosun and his colleague (Adeosun, M. E., Edeki, S.O. and Ugbebor, O. O., 2015) explains that as forecasting or anticipating a market takes much more than a normal GBM, where only when the volatility and drift of an asset are considered. GBM solely is not sufficient enough to accommodate the market prices. There are several times that the asset prices changed significantly greater than its drift in reality.

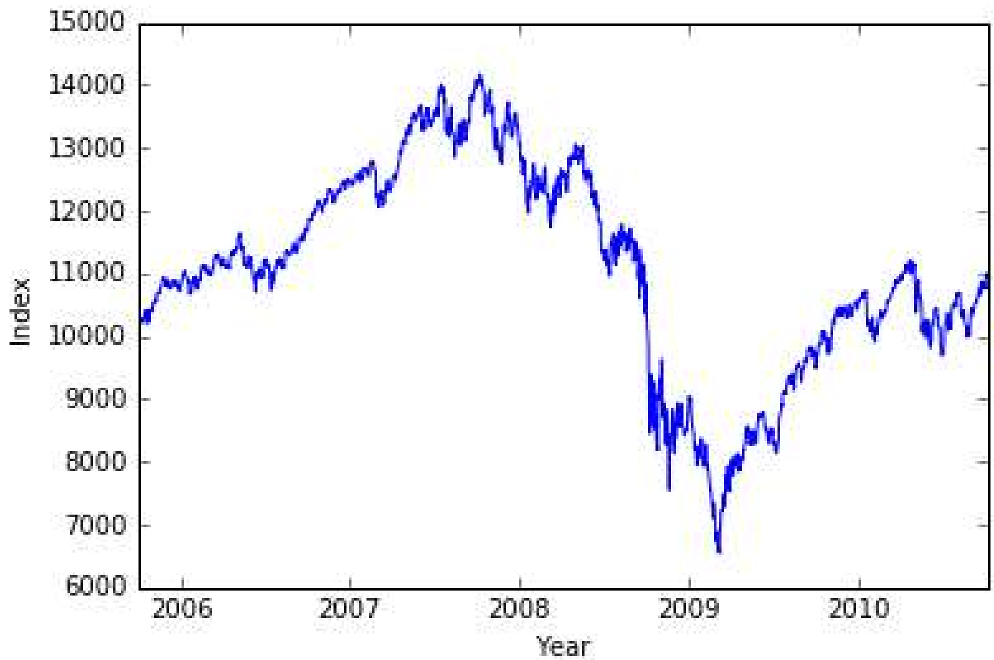


Figure 2.1: DJI index level from 1, October 2005 to 10, October 2010 (extracted from market)

From Figure 2.1, the period between the year 2008 to June 2009 are parts and regions that the jump parts are more than normal GBM. This is one of the financial crises that the world undergoes during 1985, 1997 and 2008 that had drastically changed the prices of assets and indexes all over the world. The history shows that jump events actually happened in the market. Even though in normal GBM simulations without the jump, the chances of these changes are almost negligible or so close to impossible. Hence, a question is raised, ”Is geometric Brownian motion sufficient?”

2.3 Black-Scholes Model

2.3.1 Introduction

Black-Scholes model plays an important role in calculating options pricing and was introduced in 1973 (Black, 1973). This model serve as a guideline to calculate a price on an option, for a stock or index. Hence, we will have a brief introduction and explanation for Black-Scholes model.

Recall from Section 2.1, we had introduced options with a brief definition. An option is a financial derivative that gives the holder the right (without obligation) to sell or to buy an asset for a strike price (K) before it expires within a certain date (T). A call option gives the holder the right to purchase while put options provide the holder the right to sell. Options holder are required to pay an option premium for their given right.

Next, we will introduce 4 basic types of option position.

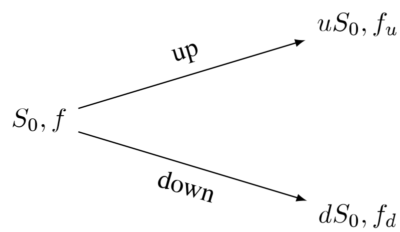
- Long position in a call option. Payoff = max$(S_t - K, 0)$.
- Long position in a put option. Payoff = max$(K - S_t, 0)$.
- Short position in a call option. Payoff = -max$(S_t - K, 0)$.
- Short position in a put option. Payoff = -max$(K - S_t, 0)$.

An American option can be exercised at any time up to the expiration, while European option can only be exercised on the expiration date.

2.3.2 One Period Binomial Tree and Risk-Neutral Probability

Considering a single period binomial model, where a price of a share and option are S_0 and f respectively. After a period, share price will either increase to uS_0

or decrease to dS_0, where $u > 1$ and $d < 1$. At the same time, option price, f will becomes either f_u or f_d depending on the direction of the stock.



Suppose that we long Δ shares of the stock and short a call European option. Our payoff would be

$$\Delta uS_0 - f_u, \quad \text{if stock moves up,}$$

and

$$\Delta dS_0 - f_d, \quad \text{if stock moves down.}$$

Assume that we need to construct a hedging portfolio, such that we would be risk-free when Δ is chosen, regardless the stock goes up or down. Hence, we would have the following where

$$\Delta uS_0 - f_u = \Delta dS_0 - f_d.$$

Rearranging for Δ, we can get

$$\Delta = \frac{f_u - f_d}{uS_0 - dS_0}.$$

No arbitrage assumption is made here, considering we are at an efficient market, no one could take advantage of an arbitrage opportunity. As the consequences, the portfolio above is risk-free regardless of the outcome of the stock.

Hence, the present value of this portfolio must equal to $(\Delta u S_0 - f_u)\exp^{(-rT)}$. By denoting the present value of option as f, and present value of portfolio as $S_0\Delta - f$, we have

$$S_0\Delta - f = (\Delta u S_0 - f_u)\exp^{(-rT)}.$$

By rearranging the equation, we have

$$\begin{aligned} f &= S_0\Delta - (\Delta u S_0 - f_u)\exp^{(-rT)} \\ &= \frac{f_u - f_d}{u-d}\left(1 - u\exp^{(-rT)}\right) + f_u\exp^{(-rT)} \\ &= \exp^{(-rT)}\left[\exp^{(rT)}\frac{f_u - f_d}{u-d}(1 - u\exp^{(-rT)}) + f_u\right] \\ &= \exp^{(-rT)}\left(f_u\frac{\exp^{(rT)} - d}{u-d} + f_d\frac{u - \exp^{(rT)}}{u-d}\right) \\ &= \exp^{(-rT)}\Big[pf_u + (1-p)f_d\Big], \qquad \text{where} \qquad p = \frac{\exp^{(rT)} - d}{u-d}. \end{aligned} \tag{2.7}$$

Here value p is defined as the probability of the underlying stock, moving up in a risk-neutral world, resulting the present value of the option equals to the expected value of the option in one period discounted by the risk-free rate,

$$\begin{aligned} f &= \exp^{(-rT)}\Big[pf_u + (1-p)f_d\Big] \\ &= \exp^{(-rT)}\mathbf{E}(f). \end{aligned} \tag{2.8}$$

In a risk-neutral world, stock grows as a risk-free rate under the risk-neutral probability, which is denoted as p previously. This risk-neutral assumption is important for the following Black-Scholes model. The following section will be discussing the other assumption that is needed for the model.

2.3.3 Assumption

The research by Ermogenous (Ermogenous, 2006) had clearly listed out the basic assumptions of the Black-Scholes model. Black-Scholes model assumed that an asset had the interest rates remain constant and known, but it is not practical. However, some of the assumptions are crucial for the Black-Scholes model to work. For example, the expected returns are log-normally distributed, is reasonable else it would be tedious to figure out the outcome and risk. The following are the basic assumptions that are needed for the Black-Scholes model.

- Options used are European options.
- No dividend is paid out throughout the life of the option.
- Returns on the underlying asset are normally distributed.
- Risk-free rate and volatility of the underlying asset are constant and known.
- Efficient market.
- No commissions and transaction cost.

Every assumption is important as they could affect the accuracy of the model in different ways. European options are used compare to American options as the latter usually will not be exercised until the expiration. No dividend payout can be a limiting assumption as, in the real world, most of the company provides dividends. This can be overcome by subtracting the discounted future dividend from the underlying price. Efficient market suggesting that people could not constantly predict the direction of the market.

2.3.4 Black-Scholes Model

Black-Scholes model was worked out and developed by three economists - Fisher Black, Myron Scholes and Robert Merton.

$$C = S\mathrm{N}(d_1) - K\exp^{(-rt)}\mathrm{N}(d_2), \tag{2.9}$$

where

- C = Call premium at time zero,
- S = Current stock price,
- K = Strike price,
- t = time to expire,
- r = risk-free rate,
- N = Cumulative standard normal distribution,
- $d_1 = \frac{\ln\frac{S}{K}+(r+\frac{\sigma^2}{2})t}{\sigma\sqrt{t}}$,
- $d_2 = d_1 - \sigma\sqrt{t}$,
- σ = standard deviation of the underlying return, and

Dividing the model into two parts, where the first part $S\mathrm{N}(d_1)$, the current stock price (S) is multiplied by the change in the call premium with respect to a change in underlying stock price ($\mathrm{N}(d_1)$). The second part of the Black-Scholes model, $K\exp^{(-rt)}\mathrm{N}(d_2)$ provides the present value of paying the exercise price upon the day it expires. The fair value of the option should be the difference between these two different parts.

2.4 Jump Diffusion model

In the past 40 years, various models had been proposed to reflect the discontinuity and jump in an asset's returns including Merton (Kou, 2002; Bates, 1996; Merton, 1976).However, Merton came out with a more specific explanation and idea for jump diffusion using the Black-Scholes model as the foundation.

Merton added a jump component into the Black-Sholes formula by using the compound Poisson model (Press, 1967),

$$S(t) = S(0)e^{(\mu - \frac{1}{2}\sigma^2) + \sigma W(t)} \prod_{i=1}^{N(t)} e^{Y_i}, \tag{2.10}$$

where

- $N(t)$ is Poisson process, where its probability distribution function is

$$\mathrm{P}(X = x) = \frac{\lambda^x \exp^{-\lambda}}{x!}$$

- Y_i is standard normal distribution with mean of zero and standard deviation of 1,
- μ is the drift of GBM,
- σ is volatility of GBM.

Assuming the market follows a GBM, the Poisson part is the arrival of jump event. The jump event has its individual drift and variance that differ from the GBM. Kou modified the above model, with Y_i changing to double exponential distribution (Kou, 2002). He claimed that this enables the user to get analytic solutions for most path-dependent options, including barrier options and analytic approximations for American options.

In Kou's paper, he used a jump diffusion model as below:

$$S(t) = S(0)\exp\left[(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)\right]\prod_{i=1}^{N(t)}\exp(Y_i), \tag{2.11}$$

with a SDE of

$$\mathrm{d}S(t) = \mu S(t)\mathrm{d}t + \sigma S(t)\mathrm{d}W(t). \tag{2.12}$$

Take note that in the Merton model, Y has a normal distribution (Merton, 1976) while Kou has an asymmetric double exponential distribution for Y. In 2002, Kou explained that the distribution is:

$$f_Y(y) = p\cdot\eta_1\exp^{-\eta_1 y}1_{\{y\geq 0\}} + q\cdot\eta_2\exp^{\eta_2 y}1_{\{y<0\}}, \tag{2.13}$$

where

- $\eta_1 > 1, \quad \eta_2 > 0;$
- *p, q* ≥ 0 , *p* + *q* = 1, representing the upward and downward jumps;
- $\eta_1 > 1$ is required so that $\mathbf{E}(e^y) < \infty$;
- $\mathbf{E}(S(t)) < \infty$.

Kou had pointed out two properties of double exponential distribution which are important for the model. The first property is the leptokurtic feature of the jump size where it inherits the return distribution (Balakrishnan, N., Johnson, N. L. and Kotz, S., 1995).

This property is making sense as the jump of an instrument is not totally random, but depends on the characteristic of the instrument itself. For example, a low-price instrument would not have a significant jump that is too high to attain in real life. For instance, $0.40 asset would not jump to $100.

The second crucial property of the double exponential distribution has is martingale property. This unique property allows closed-form solutions (or approximations) for option pricing problems become feasible under the double exponential jump-diffusion model.

This jump diffusion model could be useful in the research as it meets a few important aspects of modeling. The first aspect is that the model needs to be internally *self-consistent*. The model needs to be free of arbitrage in an equilibrium setting, so that no one could abuse and consistently beating the market.

Next, it needs to be able to capture some important empirical phenomena. The empirical test always favors models with more parameters, but the more parameters, the calibration becomes more difficult as it involves high-dimensional calculations. In year 2007, Zeng and Ramezan showed that the double exponential distribution jump model is sufficiently better than the Black-Scholes model.

However, modeler always struggles between over-fitting and underfitting when try to compute a sufficiently good model for most cases.

Lastly, a closed-form solution is needed. The computation of the model can be carried out, and able to yield a closed-form solution like Black-Scholes model so that a clearer picture could be seen.

In short, jump diffusion model is closer to the scenario where stocks experience a sudden change in prices, which cannot be captured by GBM. There are many other models that satisfy the criteria mentioned above, but it could be troublesome and difficult while the results might not be certain.

Jump diffusion model attempts to improve the empirical implications of the Black-Scholes model while still its analytical tractability is remained and retained. Hence, the model simplicity made it a better option than the others.

Research made by R Chen (Chen et al., 2017) concludes that option pricing under the double exponential jump diffusion has high time efficiency, while the goodness of fit and pricing accuracy is significantly higher than Black Scholes and the Kou's model (Kou, 2002). However, his double exponential model is not able to accommodate two directions jump distribution.

An option pricing model proposed by Zhang and Wang (Zhang, S. M. and Wang, L. H., 2013) integrate a stochastic interest rate (SIR), stochastic volatility (SV), and double exponential jumps that are limited to determining the price of a European option. This limits the model unable to be generalized to other financial derivatives that are important to the market.

Our research aims to come up with a jump diffusion model that can be used in various types of derivatives. Therefore, we had formulated an asymmetric double normal jump diffusion model, that can determine the price of European option and annuity using an empirical method.

2.5 Markov Chain Monte Carlo

Markov Chain Monte Carlo (MCMC) is a technique for numerical integration using random numbers. MCMC draws samples from the required distribution, and forms sample averages to approximate expectations. MCMC could draw desired samples by running a well-constructed Markov Chain. There are a lot of ways in constructing the Markov Chains, including the Gibbs sampling method (Geman and Geman, 1984), which are the special case of the framework built by Metropolis (1953) and Hastings (1970). In this chapter, we will briefly introduce MCMC and Metropolis Hastings framework, as we will be using Gibbs sampling method in this research.

2.5.1 Bayesian Inference

Most applications of MCMC to date are oriented towards Bayesian inference and has the ability to incorporate uncertainties for the unknown parameters.

From the Bayesian perspective, there is no fundamental difference between the observable and the parameters of the statistical model, as all are considered as random quantities.

We denote observed data as $\mathbf{X} = \{x_1, x_2, ..., x_n\}$ while $\boldsymbol{\theta} = \{\theta_1, \theta_2, ..., \theta_n\}$ as the model parameters. We are required to set up a joint probability distribution, $\mathbf{P}(\mathbf{X}, \boldsymbol{\theta})$ over all random quantities. The joint distribution consists of two parts: the prior distribution $\mathbf{P}(\theta)$ and the likelihood $\mathbf{P}(\mathbf{X} \mid \boldsymbol{\theta})$. Specifying both distributions provide a full probability model, through

$$\mathbf{P}(\mathbf{X}, \boldsymbol{\theta}) = \mathbf{P}(\mathbf{X} \mid \boldsymbol{\theta})\mathbf{P}(\boldsymbol{\theta})$$

.

With observed data $\mathbf{X}$, Bayes theorem will be used to determine the distribution of $\boldsymbol{\theta}$ conditional on $\mathbf{X}$:

$$\mathbf{P}(\boldsymbol{\theta} \mid \mathbf{X}) = \frac{\mathbf{P}(\boldsymbol{\theta})\mathbf{P}(\mathbf{X} \mid \boldsymbol{\theta})}{\int \mathbf{P}(\boldsymbol{\theta})\mathbf{P}(\mathbf{X} \mid \boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta}}.$$

The obtained distribution is called posterior distribution of $\boldsymbol{\theta}$, and it is the object of Bayesian inference.

2.6 Metropolis-Hasting Model

In this research, we wish to find out how the market behaved and its distribution so that we could prepare for the risk we are undertaking. To do so, we need underlying asset's parameters such as the drift μ, variance σ^2, jump's intensity and jump's variance. However, it is difficult to identify merely from the market data, and we found out the method introduced by Nicholas Metropolis along with other authors in 1953 (Metropolis, 1953). Metropolis Hastings algorithm allow us to sample out the distribution when direct approach is not available. Next, we will introduce how it works.

2.6.1 Setup and Goal

We begin with a probability mass function (pmf), π on a countable set of states, X that has discrete distribution, and a function $f(X)$ on X, where $X \in \mathbb{R}$.

The goal and purpose of the Metropolis-Hasting algorithm are to sample from π approximately, or to approximate the expected value, $\mathbf{E}[f(X)]$ where $X \sim \pi$ and distributed according to π.

Both π and $f(X)$ maybe so complicated such that computing the value exactly is intractable and sampling exactly is impossible.

2.6.2 Approach

The approach of the Metropolis-Hasting algorithm starts off by constructing a Markov Chain, with π that has a stationary distribution. The ergodic theorem is applied here, where when running the Markov Chain sufficiently long enough, the last state is approximately distributed according to π, which would provide us the samples of expected parameters, θ. As long as the Markov chain is ergodic, irreducible and has a stationary distribution, π, it is possible to approximate the expected value by using the sample mean over all the states of the Markov chain takes.

2.6.3 Algorithm

The steps of Gibbs sampling algorithm are as follows:

1. We introduce the proposal matrix **Q**. **Q** is a stochastic matrix, where all its element is positive, and sum of each row is equal to 1. ($Q_{ab} = Q_{ba} \quad \forall a, b \in X$)

2. Initialize $X_0 \in X$. Where X_0 is a random sample from **X**.

3. For iteration where i = 0, 1, 2, ..., n-1:

   (a) Sample x from $\mathbf{Q}(x_i, x)$, such that x_i is a fixed, known variable and x is the sample that range over all possible state, (or can be say as $\mathbf{P}(x \mid x_i) = Q(x_i, x)$.)

   (b) Sample a u from uniform(0,1), where u represent the constant for the rate to accept or not accept in the next step.

   (c) If $u < \frac{\tilde{\pi}(x)}{\tilde{\pi}(x_i)}$ (the probability of x), then $x_{i+1} = x$, else we reject the newly drawn sample and $x_{i+1} = x_i$.

4. The output will be sequence of $\{x_0, x_1, x_2, ..., x_n\}$ as i changes from 1 to $n - 1$.

2.6.4 Correctness of Metropolis-Hasting Algorithm and Proving

In order to check the Markov Chain constructed by the algorithm satisfied the property that is necessary for the ergodic theorem, the following requirements must be satisfied.

- Irreducibility,
- stationary distribution for π, and
- aperiodicity for sampling.

The transition matrix of the Markov Chain is determined as below. Suppose that $x_{i+1} \neq x_i$,

$$\begin{aligned} \mathbf{T}(x_i, x) = \mathbf{P}(x_{i+1} = x \mid x_i = x_i) = \mathbf{P}(x \mid x_i)\mathbf{P}(\text{accept } x \mid x, x_i) \\ = Q(x_i, x) \text{ min}\left(1, \frac{\tilde{\pi}(x)}{\tilde{\pi}(x_i)}\right) \\ \mathbf{T}(x_i, x_i) = 1 - \sum_{x \neq x_i} \mathbf{T}(x_i, x) \\ \text{(property of stochastic matrix)} \end{aligned} \tag{2.14}$$

Now, we check the stationary distribution for π by detailed balance. For more information on detailed balance, can refer to examples shown by Wong and Chan (Chan, 2006). We claimed that the Markov Chain (X_i) constructed by the algorithm has the stationary distribution of π. From detailed balance theorem, if π satisfy the detailed balance with respect to the transition matrix, $\mathbf{T}$,then any Markov Chain with transition matrix, $\mathbf{T}$ has stationary distribution π.

$$\pi_a \mathbf{T}_{ab} = \pi_b \mathbf{T}_{ba}, \quad \forall a, b \in X$$

If $a = b$ then the equation above is true, since $\pi_a \mathbf{T}_{aa} = \pi_a \mathbf{T}_{aa}$.

If $a \neq b$, then

$$\begin{aligned} \pi_a \mathbf{T}_{ab} = \pi(a)\mathbf{T}(a, b) &= \pi(a)Q(a, b)\text{min}(1, \frac{\pi(b)}{\pi(a)}) \\ &= Q(a, b)\text{min}(\pi(a), \frac{\pi(b)\pi(a)}{\pi(a)}) \\ &= Q(a, b)\text{min}(\pi(a), \pi(b)) \quad \text{where both are symmetry.} \end{aligned} \tag{2.15}$$

These show that X_i has the distribution π.

Irreducibility and aperiodicity are needed for the ergodic theorem to hold. Hence, we need to make another claim:

Assume that $\pi(x) > 0 \qquad \forall x \in X$.

- If $\mathbf{Q}$ is reducible, then the Markov chain with transition matrix, $\mathbf{T}$ is irreducible. For a homogeneous Markov chain, its irreducibility does not depend on the initial distribution of the Markov chain but depends on transition matrix, $\mathbf{T}$. In order to check $\mathbf{T}$ is irreducibility, check the proposal matrix, $\mathbf{Q}$.
- If $\mathbf{Q}$ is aperiodic, then the transition matrix, T is aperiodic.

2.6.5 Result of Metropolis Hasting Algorithm

If probability of X, $\pi(x) > 0 \quad \forall x \in X$ and proposal matrix $\mathbf{Q}$ is irreducible, then the sample mean converges almost surely to the true mean.

$$\frac{1}{n}\sum_{i=1}^{n} f(X_i) \xrightarrow[n\to\infty]{\text{almost surely}} \mathbf{E}[f(X)], \quad X \sim \pi.$$

If further, $\mathbf{Q}$ is aperiodic, then regardless the initial state, the last state will approximate to $\pi(x)$ after running the algorithm sufficiently long.

$$P(X_n = x \mid X_0 = x_0) \xrightarrow[n\to\infty]{} \pi(x) \quad \text{for } \forall x \in X.$$

The goal is met by using the Metropolis-Hasting algorithm method. Next, we investigate Gibbs sampling method, which is similar to Metropolis Hasting method while we change the acceptance rate from $u < \frac{\tilde{\pi}(x)}{\tilde{\pi}(x_i)}$ to 1, where we accept every single time and update in each run of the algorithm.

2.7 Gibbs Sampling Method

There are many ways to retrieve the parameters needed (μ, σ, λ, μ_{jump} and σ_{jump}) to model GBM process and jump diffusion process. In this research, we chose to use the Gibbs sampling method. As we mentioned in the previous section, the Gibbs sampling method is a special case of the Metropolis-Hasting method, where it accepts every time when the sample changes. The main reason we choose the Gibbs Sampling method but not Metropolis Hasting is because the usefulness of Gibbs sampling method increases greatly as the dimension of a problem increases. As Metropolis-Hasting does not accept every sample, it will need a longer algorithm to converge, especially if the dimension is high. Besides, the accept-reject criteria are objective and hard to define the correctness. Hence, we will introduce and use the Gibbs sampling method and show some basic examples.

Similar to the Metropolis-Hasting algorithm, Gibbs sampling method is an iterative process too, where we could sample out the marginal density given the joint distribution of the needed variable or parameter. So, for a simple example, let us begin with a multivariate model, where we want to sample out the mean of the variable, x while variable y; is the other variables that exists in the model we yet to discover.

Given a joint density:

$$f(x, y_1, y_2, y_3, .., y_k).$$

We are interested in obtaining the marginal density of x,

$$f(x) = \int ... \int f(x, y_1, ..., y_k) dy_1 ... dy_k$$

and the mean of variable X, $E(X)$, where

$$E(X) = \int x f(x) dx.$$

It would be tedious and difficult to determine the marginal density of X by using integration directly from the huge joint distribution. Therefore, the usefulness of the Gibbs sampling method comes in here, where we can simplify the work down to a few steps of iterations.

Assuming that is possible to sample $k+1$ number of univariate conditional densities:

$$f(X \mid y_1, y_2, ...y_k)$$

$$f(Y_1 \mid x, y_2, y_3, y_4, ...y_k)$$

$$f(Y_2 \mid x, y_1, y_3, y_4, ...y_k)$$

$$f(Y_3 \mid x, y_1, y_2, y_4, ...y_k)$$

$$...$$

$$f(Y_k \mid x, y_1, y_2, y_3, y_4, ...y_{k-1}).$$

Firstly, we need to initialize k values for $Y^1 = y_1^0, Y^2 = y_2^0, Y^3 = y_3^0, ..., Y^k = y_k^0$. Let Y_m^n be the n^{th} element of Y in m^{th} iteration. Then the first iteration cycle will be created as follows:

x^1 by a draw from $f(X \mid y_1^0, y_2^0, y_3^0, ..., y_k^0)$

y_1^1 by a draw from $f(Y_1 \mid x^1, y_2^0, y_3^0, ..., y_k^0)$

y_2^1 by a draw from $f(Y_1 \mid x^1, y_1^1, y_3^0, ..., y_k^0)$

...

y_k^1 by a draw from $f(Y_1 \mid x^1, y_1^1, y_2^1, ..., y_{k-1}^0)$

x^1 here will be the updated value of parameter y_1^0 and will be used throughout the first iteration cycle. On the second step of the first iteration, we will use Y^1 and x^1 to sample and update the third parameter. The first cycle will repeatedly sample and update a parameter until all the parameters are updated. These updates values will be used for the second iteration cycle, and similarly, the second set's values update the third iteration and so on until the iteration is completed. George and George (George C. and Edward, I., 1992) illustrated that the Gibbs sampling method could converge to the true value given that the conditional is fulfilled, and the iteration is large enough with sufficiently large amount of data. To be more specific, without computing $f(x)$ directly, we can generate samples $X_1, ..., X_m \sim f(x)$ by using the Gibbs sampling method.

Gibbs sampling method can be thought of as a practical implementation of the fact that knowledge of the conditional distributions is sufficient to determine a joint distribution.

Hence the Gibbs sampling method is used as a foundation for this research. A list of codes which could be used in Python was done, where it can converge the market data into the crucial parameters we are seeking, in this case, they are the drift, μ volatility, σ the jump intensity, μ_{jump} and jump arrival, λ. The method and usage would be further discussed in Chapter 3, while a clear explanation is shown by George and George in their paper (George C. and Edward, I., 1992).

Take note that Gibbs sampling method requires a formation of posterior that consist of the likelihood function and prior distribution, however, there are many ways to specify a prior distribution in the Bayesian setting. Some researchers prefer noninformative priors, and others prefer priors that are analytically tractable. We choose conjugate priors as they are adopted to address the latter preference.

According to Chan and Wong (Chan and Wong, 2006) the conjugate priors of some likelihood functions are as shown as the Table 2.1.

Table 2.1: Conjugate Prior for each likelihood function.

Likelihood $L(\theta)$	**Conjugate Prior** $p(\theta)$
Poisson $\theta = \lambda$	Gamma(α, β)
Binomial $\theta = p$	Beta(α, β)
Normal $\theta = \mu, \sigma^2$ known	Normal(m, τ^2)
Normal $\theta = \sigma^2, \mu$ known	Inverse Gamma(α, β)

The likelihood function is the density function for the required parameter as shown in Table 2.1. For instance, the likelihood function of the jump arrival event follows a Poisson distribution with the conjugate prior distribution of Gamma(α, β).

Gibbs sampling method, however, has its own limitations, where it could not converge fast enough when the dimensionality problems occurred. In this research, we would not be encounter problems with high dimension, we could use the Gibbs sampling method's ability to reduce a multidimensional problem to an iteration of low-dimensional problems.

In short, the main reason to learn Gibbs sampling method is to estimate these unknown parameters, which are μ, σ, Y_i and $N(t)$ from a joint model (jump model) which is the price. From the samples, we can estimate the parameters by taking its mean. When it is possible to attain the crucial parameters and indicators from the asset's prices, investors could estimate the risk and hence formulate a strategy for the risk they are undertaking. However, we need try out more simulations and model modification in order to have a better and broad calculation accuracy.

2.8 Heston Stochastic Volatility Model for Annuity Products

In order to incorporate the application of jump diffusion model and Gibbs sampling, we need to check the possible effects of jump event on annuity products, and hence how it affects the pricing method. Here we introduce a comparison model, Heston model for annuity pricing and compare with jump diffusion model.

2.8.1 Heston Model

A stochastic volatility model is introduced by Heston in 1993. Heston model is chosen because it incorporates the relationship between asset returns and its volatility and it is easily tractable compared to other stochastic volatility models available (Heston, 1993).

The variance assigned in the Heston stochastic volatility model follows a stochastic differential equation (SDE) instead of a constant variance. At a different point in time, the Heston model will have a different variance, to accommodate the volatile option or financial instrument that do not possess a constant variance. First, we specify the Heston stochastic volatility model and provide some details of the model.

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sqrt{V_t} S_t \mathrm{d}W_t^s, \tag{2.16}$$

where V_t is the instantaneous variance, which follows a CIR process given by the following SDE:

$$\mathrm{d}V_t = \kappa(\theta - V_t)\mathrm{d}t + \sigma_V \sqrt{V_t}\mathrm{d}W_t^V, \tag{2.17}$$

- S_t is the asset spot price,
- V_t is variance of asset price,
- μ is the rate of return from an asset,
- κ is the mean reversion rate,
- θ is long run variance,
- W_t^s and W_t^V are the standard Brownian movement for price and volatility respectively, and correlated with ρ, where $\mathrm{d}W_t^s \cdot \mathrm{d}W_t^V = \rho \mathrm{d}t$.
- σ_V is the volatility of variance.

The process V_t is strictly positive if the parameters obey Feller condition such that

$$2\kappa\theta > \sigma_V^2.$$

Here, the κ, θ and σ_V are predefined variables for variance's SDE. Therefore, it indicates that the Heston model is a GBM with a variance that follows its own SDE.

Heston Model is tested to be a good model when calculating or formulating high volatile options, such as bonds and currency options. In a highly volatile market, the assumption of constant variance is not applicable. Therefore, a model with a non-constant variance is needed to accommodate such assets. Besides, volatility changes over the long run, hence a volatility updating

Heston model is more preferred than the Black-Scholes model. Hence, simulation and comparison of the results of the Heston Model together along with jump diffusion is included in the research area.

2.8.2 Annuity

An annuity is a long-term agreement between two parties: the provider and annuitant. Annuity provider agrees and obliges to make periodical fixed payment to the annuitant for a set period or for a lifetime. There are few types of annuity: variable annuity, fixed annuity equity-indexed annuity, immediate annuity, and longevity annuity. This research focuses on the pricing of variable annuity only, as the other types of annuity do not include the performance of investment are not chosen.

The agreement between the life insurance company and the annuitant for the writer to pay periodical income in exchange for premium payment from the holder is called as a variable annuity (VA). They are called variable because their value will fluctuate based on the performance of the underlying stock, bond and money market investments that are chosen. Two types of VA existing in the market is known as the single premium variable annuity (SPVA), which is purchased with one payment and the flexible premium variable annuity (FPVA), purchased with multiple payments, which may be regular or occasional.

Section 2.8.3 will discuss the concept of VA pricing and some terminologies needed to understand.

2.8.3 Terminology and Concept of Variable Annuity

Regardless of what type of VA, the holder needs to pay down a premium, whether is SPVA or FPVA, according to the asset he desired. The premium paying phase also is known as accumulation phase, while payout phase is let the holder receives payments in either a lump sum, periodic withdrawals or through the process of annuitization, which converts the assets into an ongoing income stream. Therefore, if the holder wishes for a larger the income stream in the future, the premium needed will increase accordingly as well.

The payout amount mostly depends on two variable which is the death benefits and living benefits. A death benefit is a sum where the beneficiaries will receive when the holder passes away. Depending on the contract, the beneficiaries will be provided a guaranteed minimum death benefit (GMDB) upon the death of the holder, or some contracts allow their spouse to become the new owner of the contract while still receiving a death benefit.

A guaranteed minimum living benefit (GMLB) will be provided to the holder of the annuity. There exist various types of benefits including Guaranteed Minimum Income Benefit (GMIB), Guaranteed Minimum Accumulation Benefit (GMAB), Guaranteed Minimum Withdrawal Benefit (GMWB) and Guaranteed Lifetime Withdrawal Benefit (GLWB). To find out more details on each benefit, can refer to studies done by Stone, R. (Stone, 2003). In short, these are the benefits that could be claimed by holder when they are alive.

In order to cover the cost of administration, distribution, insurance benefits (living and death benefits) and the investment management, fees and charges will be imposed upon the annuity holder. These include mortality and expense risk charge (M&E) too.

The fees and charges should be fair else, the insurance company may suffer a loss when it is undercharged while holder might surrender the annuity if they are overcharged. Hence, we need to identify the fair price of M&E and study the impact of jump event on the expected fair price.

In Chapter 3, the research methodology will demonstrate how we retrieve the data, carry out simulations and come out with a modified jump diffusion model.

CHAPTER 3

RESEARCH METHODOLOGY

3.1 Data Collection

In this research, we had chosen Dow Jones industrial (DJI), NASDAQ Composite 100 (NASDAQ 100), Financial Times Stock Exchange 100 Index (FTSE 100), Standard & Poor's 500 (S&P 500) and NYSE ARCA oil & gas Index (OilGas). We retrieve those market data from Yahoo Finance. The python script below provides an example of attaining the S&P 500 prices.

```python
from pandas.io.data import DataReader as DR
from datetime import datetime as dt
import pandas as pd

start     = dt(1995, 8, 8)
end      = dt(20015, 8, 7)
data      = DR("SPY", 'yahoo', start, end)
```

Listing 3.1: Data Retrieving

The selection on indexes is based on a few factors. Firstly, we choose those that are highly volatile. As the purpose of research is to observe the market pattern of unpredictable market assets, non-volatile assets are needless to investigate.

Next factor that we considered is the chosen assets need to be diversified. This is to eliminate the market spikes that are caused by the unsystematic risk of the specific company. Choosing a diversified basket of assets would tell us more about the economic failure.

Lastly, we would like to build a model that is not limited to a single sector of the market or a single type of market instrument. Hence, we had chosen composite indexes and some highly volatile indexes such as NYSE ARCA Oil & Gas index, that cover the market as an index. This would allow our pricing model less biased towards a single instrument.

The data collected covering period throughout the 1, January 2005 to 1, January 2015. We separate the extracted data into two different periods: (a) between 2005 to 2010 and (b) between 2011 to 2015. The first period covers the year 2007 and 2008, which is known as the economic crisis period. The second period is used as the control period. By doing this we could compare the parameters for both sets of data.

3.2 Jump Diffusion Simulation

From Equation 2.10, a simulation is conducted using Python. The jump diffusion model proposed by Kou is defined as Equation 3.1 and parameters are preset to a fix value.

$$S_t = S_{t-1}\Bigg[\Big[\exp((\mu - \frac{\sigma^2}{2})dt + \sigma W_{t-1})\Big] + \Big[\exp(\mu_{\text{jump}} + \delta_{\text{jump}} N_{t-1}) - 1\Big] Y_{t-1}\Bigg], \tag{3.1}$$

where

- W_t is a Wiener process, it is substituted with $\xi_t \times \sqrt{dt}$ in simulation, where ξ_i is a i.i.d. random variables with mean 0 and variance 1.
- Y_t is a Poisson process, while the arrival of jump, λ of the process will be further investigate in Section 3.4. In Listing 3.3, λ is preset as zero to test the result with Black-Scholes model.

- N_t is normal distributed with zero mean and unit variance.

A jump diffusion model is a combined model of geometric Brownian motion (GBM) and jumps. Thus, a jump process without a jump event should produce GBM.

In order to let both simulations carried out to be identical, all the required parameters are initialized to a constant. For example, the initial price, S_0 is set to be \$100 in both cases. While the drift, μ of the Brownian motion is 0.09 and has a volatility, σ of 0.4.

```
n=n_partitions=2265
mu=0.09
sigma=0.4
S0=100
n_path=500

#create path
t=p.linspace(0,1,n+1)
dB=p.randn(n_path,n+1)/p.sqrt(n)
dB[:,0]=0
B=dB.cumsum(axis=1)

#calculate stock price
nu=mu-sigma**2/2.0
#create zero matrix with same size as B
S=p.zeros_like(B)
S[:,0]=S0    #assign 1st input as S0
# generate the equation
S[:,1:]=S0*p.exp(nu*t[1:]+sigma*B[:,1:])
S_line=S[:] #simulate the first run and show on plot
```

Listing 3.2: GBM model

Listing 3.2 shows the simulation for GBM. By setting the arrival of the jump to zero, the expected price for the Kou jump model's simulation should be close to GBM. Listing 3.3 shows the example of code to compare both GBM and jump, while the results are demonstrated in Table 3.1.

```python
mu1 = np.linspace(0.04,0.2,20)
s1  = np.linspace(0.01,0.1,20)

for i in range(20):
    mu2 = mu1[i] ; s2 = s1[i];
    gbm = mf.GBMmodel(mu=mu2,sigma=s2)
    jump = mf.jump(SIGMAY=s2,LAMBDA=0,MUY=mu2)
    #lambda is set to zero
    print(i,np.mean(gbm[-1]),np.mean(jump[-1]))
```

Listing 3.3: Comparison of parameters between jump model and GBM

The comparison in Table 3.1 used 20 sets of different drift, μ ranging from 0.04 to 0.2 and volatility, σ ranging from 0.01 to 0.1 respectively. Both models share the same value for drift and volatility in each simulation.

Table 3.1 shows that by setting the arrival of the jump to zero, it could produce similar results with the data produced by GBM. Hence, we conclude that the jump diffusion model can reproduce identical result as GBM given the jump parameters (frequency arrival) is set to zero.

3.3 Forming Posterior Distribution for the 5 Parameters in Gibbs Sampling Method

As we mentioned previously from Section 2.5.1 until Section 2.6.3, in order to sample out the parameters of Kou's jump diffusion model, the distributions

Table 3.1: Comparison of GBM model and jump diffusion model

Number of Set	GBM	Jump Diffusion
1	104.07664	104.08376
2	104.95692	104.94972
3	105.81053	105.79469
4	106.69594	106.72289
5	107.58215	107.64197
6	108.50163	108.46564
7	109.41966	109.38033
8	110.29786	110.35610
9	111.33659	111.14443
10	112.11270	112.17157
11	112.98148	113.02250
12	113.92696	113.88695
13	114.85349	114.94359
14	115.89334	115.85538
15	116.68048	116.80766
16	117.62269	117.79220
17	118.75309	118.76939
18	119.53128	119.56619
19	120.57232	120.56695
20	121.67333	121.31453

of these parameters are needed. By multiplying their conjugate prior distributions and likelihood functions respectively, we attain a posterior distribution for each parameter. The distribution of parameters is proportional to that posterior distribution that is formed.

From Equation 3.1, we observed five important parameters, which are the distribution of the asset's drift, asset's volatility, the number of jump arrival in a period, and jump intensity and jump volatility. (μ, σ, λ, μ_{jump}, σ_{jump}). The likelihood function of each parameter is determined as follows:

- Asset's drift and jump intensity follow a normal distribution with known mean, σ.
- Frequency of jump arrival in a period follows a Poisson distribution.

- Asset's volatility and jump volatility follow a normal distribution with known mean μ.

Section 3.3.1 shows the working on how to get the posterior distribution of each parameter. The relationship of conjugate prior distribution and likelihood function can refer to Table 3.2.

Table 3.2: Conjugate Prior for each likelihood function.

Likelihood $L(\theta)$	Conjugate Prior $p(\theta)$
Poisson $\theta = \lambda$	$G(\alpha, \beta)$
Binomial $\theta = p$	$Be(\alpha, \beta)$
Normal $\theta = \mu, \sigma^2$ known	$\mathrm{N}(m, \tau^2)$
Normal $\theta = \sigma^2, \mu$ known	$IG(\alpha, \beta)$

3.3.1 Asset Drift, μ (normal distribution)

Both underlying asset drift, μ and the intensity of jump, μ_{jump} have likelihood functions that follow the normal distributions with known mean μ.

Likelihood function of a normal distribution is proportional to:

$$\exp\left[-\frac{n}{2\sigma^2}\left(\mu - \sum_{i=1}^{n}(x_i - Y_i \Delta N_i)\right)^2\right]. \tag{3.2}$$

Prior distribution $P(\mu_1)$ follows:

$$P(\mu_1) \sim N(m, \tau^2) = \frac{1}{\sqrt{2\pi\tau^2}} \exp\left[-\frac{n}{2\sigma^2}(\mu - m)^2\right]. \tag{3.3}$$

Posterior distribution, $P(\mu_1 \mid m, \tau^2)$ is proportional to Prior multiplying the Likelihood function:

$$\begin{aligned}
\text{Posterior} &= \text{Likelihood} \times \text{Prior} \\
&= P(\mu_1 \mid m, \tau^2) \\
&= \exp\left[-\frac{n}{2\sigma^2}\left(\mu - \sum_{i=1}^{n}(x_i - Y_i\Delta N_i)\right)^2\right] \\
&\quad \cdot \frac{1}{\sqrt{2\pi\tau^2}}\exp\left[-\frac{1}{2\tau^2}\left(\mu - m\right)^2\right].
\end{aligned}$$

Let $\sum_{i=1}^{n}(x_i - Y_i\Delta N_i)$ be α, then

$$\begin{aligned}
P(\mu_1 \mid m, \tau^2) &= \exp\left[-\frac{n}{2\sigma^2}\left(\mu - \alpha\right)^2\right] \cdot \frac{1}{\sqrt{2\pi\tau^2}}\exp\left[-\frac{1}{2\tau^2}\left(\mu - m\right)^2\right] \\
&\propto \exp\left[-\frac{n}{2\sigma^2}(\mu-\alpha)^2 - \frac{1}{2\tau^2}(\mu - m)^2\right] \\
&\propto \exp\left[-\frac{1}{2}\left(\frac{n}{\sigma^2}(\mu-\alpha)^2 + \frac{1}{\tau^2}(\mu - m)^2\right)\right] \\
&\propto \exp\left[-\frac{1}{2}\left(\frac{n}{\sigma^2}(\mu^2 - 2\mu\alpha + \alpha^2) + \frac{1}{\tau^2}(\mu^2 - 2\mu m + m^2)\right)\right] \\
&\propto \exp\left[-\frac{1}{2}\left(\mu^2(\frac{n}{\sigma^2} + \frac{1}{\tau^2}) - 2\mu(\frac{\alpha n}{\sigma^2} + \frac{m}{\tau^2}) + (\frac{\alpha^2 n}{\sigma^2} + \frac{m^2}{\tau^2})\right)\right].
\end{aligned}$$

Let $(\frac{n}{\sigma^2} + \frac{1}{\tau^2})$ be A, $(\frac{\alpha n}{\sigma^2} + \frac{m}{\tau^2})$ be B, and $(\frac{\alpha^2 n}{\sigma^2} + \frac{m^2}{\tau^2})$ be a constant, C, we got

$$\begin{aligned}
P(\mu_1 \mid m, \tau^2) &\propto \exp\left[-\frac{1}{2}\left(\mu^2 A - 2\mu B + C\right)\right] \\
&\propto \exp\left[-\frac{1}{2}A\left(\mu^2 - 2\mu\frac{B}{A} + C\right)\right].
\end{aligned}$$

By completing the square, we have

$$\begin{aligned}
P(\mu_1 \mid m, \tau^2) &\propto \exp\left[-\frac{1}{2}A\left(\mu - \frac{B}{A}\right)^2 + C\right] \\
&\propto \exp\left[-\frac{1}{2(A^{-1})}\left(\mu - \frac{B}{A}\right)^2\right].
\end{aligned}$$

Finally. putting back A,B and α

$$
\begin{aligned}
P(\mu_1 \mid m, \tau^2) &\propto \exp\Bigg[-\frac{1}{2(\frac{n}{\sigma^2}+\frac{1}{\tau^2})^{-1}}\Bigg(\mu-\frac{\frac{\alpha n}{\sigma^2}+\frac{m}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2})}\Bigg)^2\Bigg] \\
&\propto \exp\Bigg[-\frac{1}{2(\frac{n}{\sigma^2}+\frac{1}{\tau^2})^{-1}}\Bigg(\mu-\frac{\frac{\sum_{i=1}^{n}(x_i-Y_i\Delta N_i)n}{\sigma^2}+\frac{m}{\tau^2}}{(\frac{n}{\sigma^2}+\frac{1}{\tau^2})}\Bigg)^2\Bigg].
\end{aligned}
$$

3.3.2 Asset Volatility, σ

The volatility of the underlying asset, σ follows an inverse gamma distribution.

Likelihood function of asset's volatility, σ is proportional to:

$$
(\sigma^2)^{-\frac{n}{2}} \exp\Bigg[-\frac{1}{2\sigma^2\Delta t}\sum_{i=1}^{n}\Bigg(x_i-\mu\Delta t-Y_i\Delta N_i\Bigg)^2\Bigg]. \tag{3.4}
$$

Prior distribution $P(\mu)$ follows:

$$
P(\mu_2) \sim IG(\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{-\alpha-1}\exp\big(\frac{-\beta}{x}\big). \tag{3.5}
$$

Posterior distribution, $P(\mu_2 \mid \alpha,\beta)$ is proportional to Prior distribution multiplying the Likelihood function:

$$
\begin{aligned}
&\text{Posterior} = \text{Likelihood} \times \text{Prior} \\
&\qquad = \frac{1}{(\sigma^2)^{\frac{n}{2}}}\frac{\beta^\alpha}{\Gamma(\alpha)}\exp\Bigg[\frac{-\beta}{x}-\frac{1}{2\sigma^2\Delta t}\sum_{i=1}^{n}\Bigg(x_i-\mu\Delta t-Y_i\Delta N_i\Bigg)^2\Bigg].
\end{aligned}
$$

By letting $x = \sigma^2$, we can get

$$
\begin{aligned}
P(\mu_2 \mid \alpha, \beta) &= \frac{1}{(\sigma^2)^{\frac{n}{2}}} \frac{\beta^\alpha}{\Gamma(\alpha)} \exp\left[\frac{-\beta}{\sigma^2} - \frac{1}{2\sigma^2 \Delta t} \sum_{i=1}^{n} \left(x_i - \mu\Delta t - Y_i \Delta N_i\right)^2\right] \\
&= \frac{\beta^\alpha}{\Gamma(\alpha)} (\sigma^2)^{-(\alpha+\frac{n}{2})-1} \\
&\quad \exp\left[- \frac{1}{\sigma^2}\left(\beta + \frac{\sum_{i=1}^{n}(x_i - \mu\Delta t - Y_i \Delta N_i)^2}{2\Delta t}\right)\right] \\
\text{P}(\sigma^2) &= \text{IG}\left(\alpha + n/2 \quad , \quad \beta + \frac{\sum_{i=1}^{n}\left(x_i - \mu\Delta t - Y_i \delta N_i\right)^2}{2\Delta t}\right).
\end{aligned}
$$

3.3.3 Frequency of Jump Arrival in a Period, λ

Lastly, the arrival of jump distribution follows a Poisson distribution. Likelihood function of the arrival of jump, λ is proportional to:

$$
(\lambda \Delta t)^N (1 - \lambda \Delta t)^{n-N}. \tag{3.6}
$$

Prior distribution $P(\mu_3)$ follows:

$$
P(\mu_3) \sim Be(\alpha, \beta) = \frac{\lambda^{\alpha-1}(1-\lambda)^{\beta-1}}{B(\alpha,\beta)}, \qquad \text{where } B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}. \tag{3.7}
$$

Posterior distribution, $P(\mu_3 \mid \lambda)$ is proportional to Prior multiplying the Likelihood function:

$$
\begin{aligned}
\text{Posterior} &= \text{Likelihood} \times \text{Prior} \\
&= (\lambda \Delta t)^N (1 - \lambda \Delta t)^{n-N} \cdot \frac{\lambda^{\alpha-1}(1-\lambda)^{\beta-1}}{B(\alpha,\beta)} \\
&= -\frac{(\lambda)^{(\alpha+N)-1}(1-\lambda)^{(\beta+n-N)-1}}{B(\alpha,\beta)}, \qquad \text{where } B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\
\text{P}(\lambda) &= \text{Beta}\left(\alpha + N \quad , \quad \beta + n - N\right).
\end{aligned} \tag{3.8}
$$

The posterior distributions formed will be used in Section 3.4 where Gibbs sampling method used to retrieve all five parameters (μ, σ, μ_{jump}, σ_{jump} and λ_{jump}).

3.4 Gibbs Sampling Method and Jump Diffusion Model Parameters

As mentioned in previous section, we start off using the Gibbs sampling method to estimate the values of the parameters (μ, σ, μ_{jump}, σ_{jump} and λ_{jump}) in the jump diffusion model (Chan, 2006 and Gibbs, 1992).

Recall from Section 2.7, the fundamental idea of the Gibbs sampling method is using the Bayesian inference from Markov Chain Monte Carlo (MCMC) with Metropolis-Hasting model. Metropolis-Hasting model is an iterating process that updates initialize values towards the distribution with an accept-reject method. While the Gibbs sampling technique is a special case where we accept every update for the iterations.

The Metropolis-Hasting model (MH model) begins with a probability mass function (pmf), π on a countable set of states, X and a real-valued function, $f(X)$. Here, both π and $f(X)$ are assumed to be complicated and computing their values exactly is intractable and sampling exactly is impossible. Hence, we use the MH model to draw samples from π approximately or to approximate the expected value $E[f(X)]$ where X follows the distribution of π.

The Gibbs sampling algorithm is as shown in Section 2.6.3. The iteration process will be used to sample out the five parameters. Each different parameter has its own prior distribution and likelihood function that forms the posterior distribution. By iterating the Gibbs sampling method algorithm with their posterior distribution, we can sample out and calculate the expected value of parameters.

The parameters that we interested are the drift, μ, volatility of the underlying asset, σ, the arrival of jump event, λ the intensity of jump, μ_{jump} and jumps volatility, σ_{jump} for each jump. These parameters will show the behavior and movement of the underlying asset, and hence knowing these characteristics will provide an insight for investors when managing risks.

In the context of parameter sampling with the Gibbs sampling method, the proposal matrix Q would be the posterior distributions of parameters. We will initialize the initial parameter as X_0, and repeat the sampling process until it is converged. The sampling results of the parameter are recorded for 500 iterations.

Figure 3.1 shows the results produced by the algorithm with a simulated price. Results show that the values from the Gibbs sampling method from first to the last iteration. Each graph from left to right, top to bottom are in order of μ, σ of asset, λ_{jump}, μ_{jump}, and σ_{jump}.

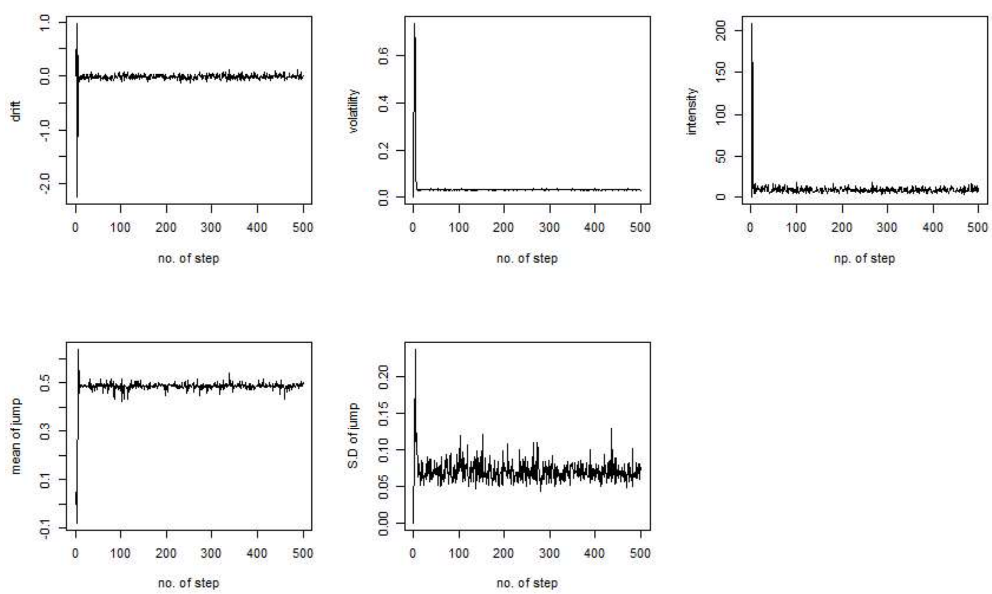


Figure 3.1: The converging path of the 5 parameters, over 500 iterations.

We can observe that $\sigma_{\text{jump}}, \lambda_{\text{jump}}$ are deviated largely from initial iteration and converged to a more stable range of values, while the other 3 parameters are swinging around initial point. Hence, we take the average values for each parameter after 5th iteration, to avoid outliers and get a better estimation.

Next, we simulate a jump diffusion model with preset values as shown in Table 3.3 for each parameter. Using the simulated data formed from preset values, we recheck the parameters with the Gibbs sampling method and recorded in Table 3.3. It shows that the values of the parameters from the Gibbs sampling method are different from the initial value of the simulated data.

Table 3.3: Testing Gibbs Sampling Method Algorithm.

Parameters	Preset	Mean	Standard Deviation
Drift of Asset	0.1	-0.02066	0.43901
Volatility of Asset	0.5	0.04132	0.02244
Arrival of Jump	10.0	10.51186	3.77489
Intensity of Jump	0.5	0.48472	0.08975
Volatility of Jump	0.025	0.07128	0.03999

The results show a few limitations of the Gibbs sampling method in converging for the jump diffusion model's parameters. Firstly, the mean of drift of asset, μ deviate from the preset 0.1 and had a large standard deviation of 0.43. Secondly, the volatility of the asset is lower than our expectation as well. On the other hand, the frequency of jump arrival, λ the jump's drift, μ_{jump} and standard deviation of jump shows a good result with the value we set.

However, the last limitation is that the jump diffusion model proposed by Kou assumes the intensity of jump event is symmetric around zero. Therefore, we realized that by using Kou's jump model, the results of Gibbs sampling will provide a mean jump intensity close to zero, whilst the asset's volatility deviates largely from our initial value.

As this situation contradicts our definition where a jump spike should be an event that causes a vigorous change in an asset, and it should not close to zero. Hence, we proposed a modified jump diffusion model in the next section.

3.5 Modified Double Normal Jump Diffusion Models

In year 2007, Kou had suggested a jump diffusion model using a double exponential model in Equation 2.13. The usage of double exponential requires more parameters for Kou's model. Besides the drift and volatility of the assets, μ and σ, Kou model requires η_1 and η_2 that replace the drift of jump, μ_{jump} with normal distribution. η_1 and η_2 are the mean intensity of upward and downward jumps that are needed for the double exponential model. The arrival of jump events, λ_{jump} includes both upward and downward jump event in Kou model.

Throughout the research, a quick check on the correlation between the arrival of an upward jump and arrival of downward jump shows that they are not dependent on each other. Hence, we propose a modified method that splits the parameter of jump arrival into two different parameters. This will ensure the retractability of upward and downward jump arrivals.

$$S(t) = S(0)e^{(\mu-\frac{1}{2}\sigma^2)+\sigma W(t)} \prod_{i=1}^{N_1(t)} e^{Y_{1,i}} \prod_{i=1}^{N_2(t)} e^{Y_{2,i}}, \tag{3.9}$$

where $N_i(t)$ is a Poisson process for i = 1 and 2; $Y_{1,i}, Y_{2,i}$ are the standard normal distributed random variables; μ is the drift of underlying GBM; and σ is the volatility of the underlying GBM.

$$\begin{aligned} f_{Y_1}(y) &= p\cdot \eta_1 e^{-\eta_1 y}, \\ f_{Y_2}(y) &= q\cdot \eta_2 e^{\eta_2 y}. \end{aligned} \tag{3.10}$$

- $\eta_1 > 1, \quad \eta_2 > 0$;
- *p, q* ≥ 0 , *p* + *q* = 1, representing the upward and downward jumps;
- $\eta_1 > 1$ is required so that $\mathbf{E}(e^y) < \infty$;
- $\mathbf{E}(S(t)) < \infty$.

As Gibbs sampling method had limitations against noise and drift of assets, and the process of forming both likelihood function and posterior function is tedious, we propose a different method to attain the values of parameters, which is through empirical method.

3.5.1 Empirical Method for Parameter Extraction

First of all, we take the difference in price for consecutive days and normalize them to the initial price. This step is to normalize the data into changes in percentage rather than the exact amount, that might differ largely between different types of assets.

As a market spike does not occur through a single night, and most of the time, it acts as a trend. Where these consecutive days changes the price more vigorously than normal days. Hence, the following step is to combine the similar direction price changes on consecutive days into a small period. The sequence formed will be in alternating sign after fusion, acting as periods of bear and boom.

On the next step, we arrange according to ascending order and separate them into positive and negative period sequence. By doing so, we can obtain a median value for each sequence. We choose arbitrarily 4 times median as a threshold. A value with four times larger than the positive median (and four times smaller than negative median), this would act as a threshold line.

The scale of 4 is chosen after we tried out other scales of the median. Table 3.4 shows how our scale factor affects the detected jump arrival in two different directions. Table 3.4 shows that too many arrivals of jump events when the scale is below 4. An averaging amount of 200 jumps are too much over a period of 20 years. The scale of 4 and 5 are much more realistic compared to previous two scales. Considering the scale of 5 medians, an average of 3 jumps per year can hardly provide sufficient data and point for analysis. Hence, we choose 4 medians in this research for a better analysis of the impact of jumps on financial instruments.

Table 3.4: Relation between scale median and arrival of jump spike from year 1995 to 2014.

Scale	Index	Up [1]	Up jump/ year	Down [2]	Down jump/ year
2	SnP	276.0	13.8	306.0	15.3
2	DJI	271.0	13.55	314.0	15.7
2	NASDAQ	242.0	12.1	282.0	14.1
2	FTSE	254.0	12.7	296.0	14.8
2	OilGas	253.0	12.65	268.0	13.4
3	SnP	99.0	4.95	158.0	7.9
3	DJI	103.0	5.15	166.0	8.3
3	NASDAQ	103.0	5.15	137.0	6.85
3	FTSE	88.0	4.4	153.0	7.65
3	OilGas	91.0	4.55	144.0	7.2
4	SnP	32.0	1.6	86.0	4.3
4	DJI	42.0	2.1	88.0	4.4
4	NASDAQ	36.0	1.8	73.0	3.65
4	FTSE	38.0	1.9	79.0	3.95
4	OilGas	29.0	1.45	78.0	3.9
5	SnP	16.0	0.8	41.0	2.05
5	DJI	14.0	0.7	52.0	2.6
5	NASDAQ	9.0	0.45	37.0	1.85
5	FTSE	19.0	0.95	42.0	2.1
5	OilGas	15.0	0.75	39.0	1.95

[1] [2]

After we choose the scale of the median, we proceed into searching the arrival of jump events occurs in Dow Jones industrial during the year 2007. We

[1] Up represents the number of upward spikes over 20 years.
[2] Down represents the number of downward spikes over 20 years.

use this set of data as an initial observation. The occurrence of jump events that exceeded the threshold line in Dow Jones 2007 is shown in Figure 3.2.

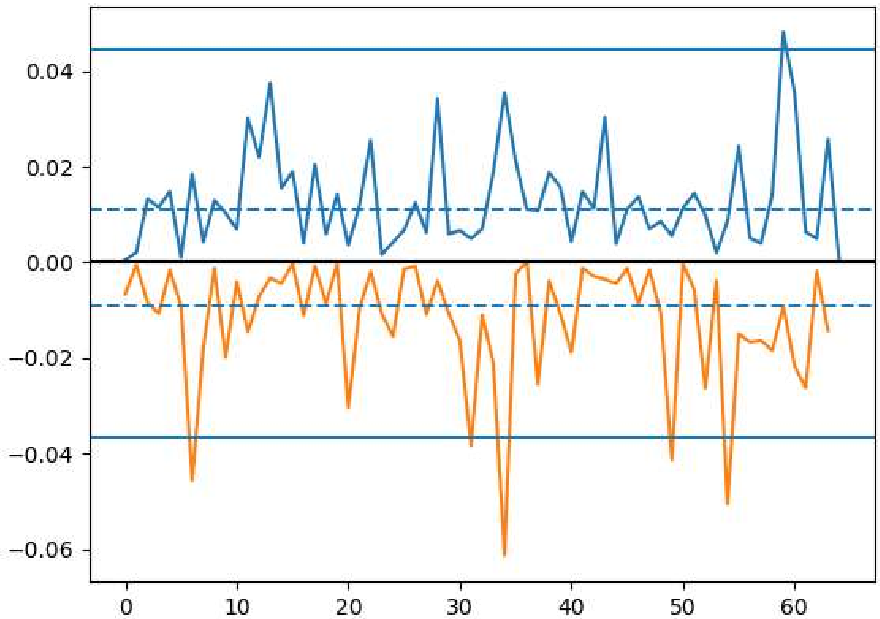


Figure 3.2: An illustration jump event identification where the spikes exceeded the lines are considered as jump events.

Figure 3.2, the blue line is the positive price changes of trends, whilst the yellow line represents negative price changes. Dotted lines are the median for both positive and negative trends. The highest and lowest horizontal dotted lines are the threshold lines we made to identify jump spikes. Figure 3.2 shows that there is one jump in positive changed trend, and five jumps in negatively changed trends.

Median is a better choice as standard deviation or mean will include extreme values, thus magnifying the value, and reduced the observations of spike occurrence.

Table 3.5: Relation between 4x median and standard deviation. NYSE ARCA Oil and Gas index from year 1995 to 2014.

Year	4* Median in terms of Std Dev	Scaled Median	Std Deviation
1995	3.26811	0.03155	0.00965
1996	4.36665	0.05231	0.01198
1997	5.35989	0.06985	0.01303
1998	4.25975	0.07773	0.01824
1999	2.34041	0.05946	0.02540
2000	4.20935	0.09014	0.02141
2001	3.84901	0.08182	0.02125
2002	3.35986	0.06371	0.01896
2003	3.86838	0.05510	0.01424
2004	4.82452	0.06562	0.01360
2005	4.03862	0.07856	0.01945
2006	3.18046	0.06807	0.02140
2007	4.83167	0.08069	0.01670
2008	2.64681	0.12467	0.04710
2009	3.75362	0.09883	0.02633
2010	4.01966	0.07688	0.01912
2011	3.92576	0.08806	0.02243
2012	3.33338	0.05179	0.01553
2013	3.79365	0.04523	0.01192
2014	2.70471	0.04504	0.01665

Table 3.5 shows that how much is quadruple median is in terms of standard deviation. Taking an average, a quadruple median is roughly having a standard deviation of 3.7967σ. Table 3.6 shows the expected probability and approximated frequency for an event with a different range of standard deviation.

Table 3.6: Relation between standard deviation and chance of occurrence

Range	Expected probability inside range	Approximated frequency outside range	Approximated frequency for an event
$\mu \pm 2\sigma$	0.95449973	1 in 21	Every three weeks
$\mu \pm 2.5\sigma$	0.98758066	1 in 80	Quarterly
$\mu \pm 3\sigma$	0.99730020	1 in 371	Yearly
$\mu \pm 3.5\sigma$	0.99953474	1 in 2148	Every six
$\mu \pm 4\sigma$	0.99993665	1 in 15788	Every 43 years
$\mu \pm 4.5\sigma$	0.99999320	1 in 147161	Every 403 years
$\mu \pm 5\sigma$	0.99999942	1 in 1744279	Every 4776 years

From Table 3.6, we can observe that an event with a standard deviation of 3.8σ has a rate of occurrence less than 0.0004%, marking it a rare event (spike event), which is supposed to occur one every six years. This is to show that, using the scaling of four of the median is sufficiently enough to determine a jump event.

Any value that is larger or equal to this value in positive period sequence would be classified as an upward jump arrival. Vice versa for the negative period sequence, values that equal or smaller than the quadruple scale negative median would be considered as a negative jump spike.

Values that exceeded the support line will be input as the jump intensity, μ_{jump} in the double exponential part of the modified jump model. The number of times of such spike occurred is treated as the arrival of jump events, λ and will be used in the Poisson distribution of jump event.

For example, referring to Table 3.7 we use the data from Dow Jones in the year 2007. We will get an expected upward spike and five downward spikes, with an intensity of 0.012 and 0.0107 respectively. This can be understood as, there is an upward spike with 0.012 intensity, and five downward spikes with an average -0.0107.

For Table 3.7 until Table 3.10 are retrieved using the empirical methods on S&P 500, DJI, NASDAQ, and NYSE ARCA Oil and Gas indexes from the year

1995 until 2014.

From the results in both Table 3.7 and Table 3.8, we can see that the arrival of an upward jump and downward jump is not equivalent. Not only the arrival of the downward jump is more frequent, but the intensities of upward and downward jumps are also different from each other too. This observation shows that the jump model proposed by Kou need to be modified. Upward and downward jumps are not symmetrical and their intensity of jump, μ_{jump} should not be zero unless there is no jumps occurrence in that direction.

Table 3.10 and Table 3.9 show that NASDAQ and NYSE ARCA Oil & Gas index behave slightly different to S&P 500 and Dow Jones industry. They had more years that consist of either no upward jumps or downwards jump events. However, they still show the same observations where the year 2008 had the largest downward jump intensity.

Table 3.7: Jump parameters for DJI from year 1995 to 2014.

Year	Up	Up Intensity	Down	Down Intensity
1995	0.0	0.000000	4.0	0.006452
1996	5.0	0.006385	9.0	0.010148
1997	0.0	0.000000	1.0	0.029571
1998	3.0	0.010684	3.0	0.025415
1999	1.0	0.009298	5.0	0.012175
2000	1.0	0.040104	3.0	0.016139
2001	4.0	0.016356	5.0	0.020903
2002	5.0	0.028068	2.0	0.018651
2003	1.0	0.015568	2.0	0.015361
2004	2.0	0.007285	5.0	0.008304
2005	1.0	0.008557	2.0	0.007423
2006	0.0	0.000000	5.0	0.009115
2007	1.0	0.012031	5.0	0.010785
2008	3.0	0.052077	4.0	0.039357
2009	2.0	0.017149	3.0	0.017408
2010	1.0	0.009696	7.0	0.012784
2011	2.0	0.023121	3.0	0.017075
2012	3.0	0.007883	6.0	0.007448
2013	3.0	0.005309	5.0	0.008180
2014	4.0	0.008574	9.0	0.008321

Table 3.8: Jump parameters for S&P 500 from year 1995 to 2014.

Year	Up	Up Intensity	Down	Down Intensity
1995	1.0	0.003583	5.0	0.007322
1996	2.0	0.009151	11.0	0.008307
1997	1.0	0.006960	1.0	0.025787
1998	2.0	0.013205	8.0	0.017721
1999	0.0	0.000000	1.0	0.013725
2000	5.0	0.015921	2.0	0.018840
2001	1.0	0.020283	3.0	0.021277
2002	3.0	0.024552	2.0	0.030803
2003	1.0	0.014023	2.0	0.014982
2004	1.0	0.007017	3.0	0.010429
2005	0.0	0.000000	2.0	0.009941
2006	0.0	0.000000	5.0	0.008411
2007	2.0	0.013596	3.0	0.012453
2008	4.0	0.052090	7.0	0.048799
2009	1.0	0.027948	5.0	0.022862
2010	1.0	0.011456	10.0	0.013855
2011	2.0	0.024247	4.0	0.016401
2012	0.0	0.000000	4.0	0.010550
2013	1.0	0.005189	3.0	0.011769
2014	4.0	0.008849	5.0	0.013490

Table 3.9: Jump parameters for NASDAQ from year 1995 to 2014.

Year	Up	Up Intensity	Down	Down Intensity
1995	2.0	0.008506	4.0	0.013369
1996	2.0	0.007055	3.0	0.014014
1997	2.0	0.012279	5.0	0.014220
1998	0.0	0.0	5.0	0.027389
1999	3.0	0.016421	1.0	0.019825
2000	0.0	0.0	3.0	0.041728
2001	3.0	0.049639	4.0	0.030443
2002	5.0	0.023689	0.0	0.0
2003	0.0	0.0	1.0	0.014402
2004	1.0	0.008292	3.0	0.012336
2005	1.0	0.006071	4.0	0.011428
2006	1.0	0.010068	3.0	0.009760
2007	1.0	0.009435	2.0	0.015314
2008	3.0	0.039349	4.0	0.040370
2009	3.0	0.016006	3.0	0.018405
2010	1.0	0.01040	6.0	0.017369
2011	1.0	0.009948	4.0	0.022343
2012	0.0	0.0	7.0	0.010448
2013	3.0	0.006620	6.0	0.011208
2014	4.0	0.010051	5.0	0.015984

Table 3.10: Jump parameters for NYSE ARCA Oil & Gas Index from year 1995 to 2014.

Year	Up	Up Intensity	Down	Down Intensity
1995	3.0	0.005473	4.0	0.008726
1996	1.0	0.007275	5.0	0.009440
1997	0.0	0.0	4.0	0.016136
1998	0.0	0.0	2.0	0.016658
1999	7.0	0.018213	0.0	0.0
2000	0.0	0.0	0.0	0.0
2001	0.0	0.0	2.0	0.025206
2002	3.0	0.026054	5.0	0.023634
2003	0.0	0.0	3.0	0.015523
2004	0.0	0.0	3.0	0.015207
2005	1.0	0.024491	5.0	0.016034
2006	1.0	0.012630	7.0	0.017549
2007	0.0	0.0	0.0	0.0
2008	4.0	0.061320	6.0	0.057834
2009	2.0	0.012249	1.0	0.017197
2010	1.0	0.015945	7.0	0.0153836
2011	1.0	0.033673	7.0	0.020615
2012	1.0	0.014218	6.0	0.011326
2013	1.0	0.007706	2.0	0.023314

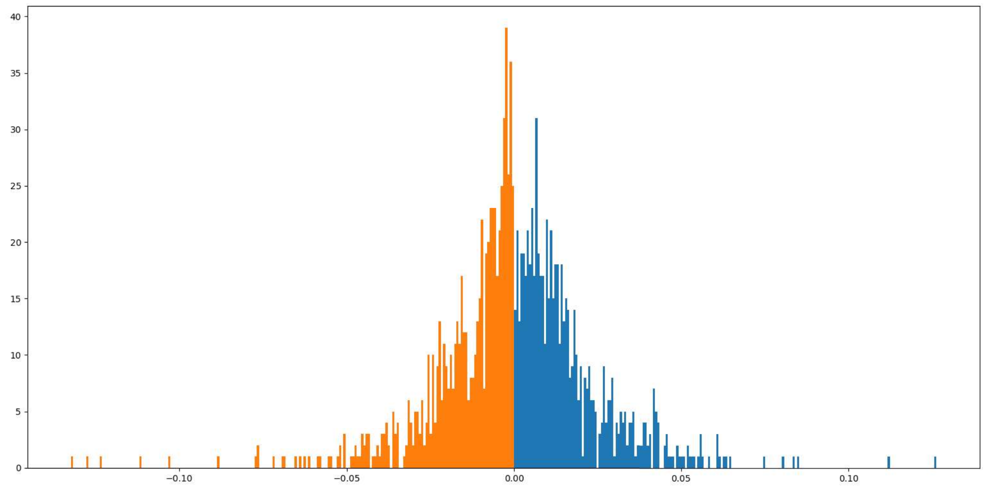


Figure 3.3: Distribution of S&P 500 daily log return from 1995 to 2014.

From Figure 3.3, the blue part is the upward changes while the orange part represents downward changes. Results show that their peak is not allocated at the center, but slightly away from the origin. This resembles a capital 'M' shape, and it does not support the models of Kou and Merton where they formulate their model using double exponential and normal that centered its peak, which similar to a normal distribution with mean around the zero.

Hence, we can conclude that it is important to modify the jump diffusion models proposed by Kou and Merton, in order to accommodate two different way of jumps. Therefore, Equation 3.10 consists of two different jump arrivals (upward and downward) and intensity of spikes (upward and downward). This is different from most of the others research that had been done. Most of the researchers used Equation 2.11 that assume the mean of the jump is equal to zero and captures both asymmetry jump event of upwards and downwards as a single event (Kou, 2002). This two assumptions caused the model fails to capture that the peaks of jumps are not at the center. The mean of the jump should not be zero.

This modification changed the equation from a double exponential to double normal distribution. A double normal distribution should similar to the Figure 3.4. Figure 3.4 is an illustration with a combination of two normal distribution where their volatility (wideness) is 1 while the mean of normal distributions is +0.05 and -0.02 respectively.

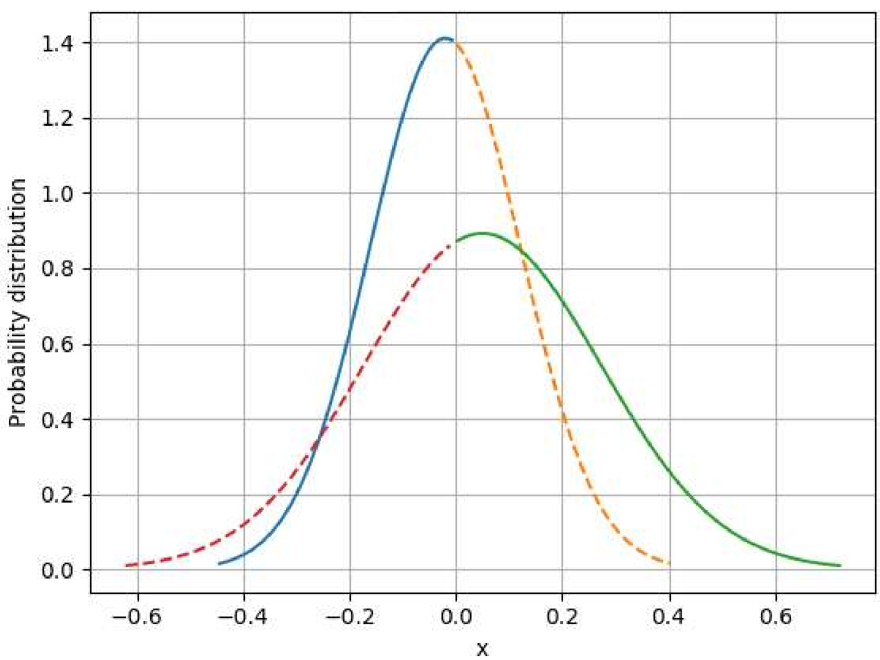


Figure 3.4: An illustration of double normal distribution for S&P 500 with mean of -0.02 and +0.05.

In our modified double normal jump diffusion equation, only a single direction (either upwards or downwards) at most will be triggered in order to avoid the spike effect cancel off each other. There are four different parameters needed for both normal distributions, the mean intensity of jump $\mu_{\text{up jump}}$, $\mu_{\text{down jump}}$ and number of occurrences for each distribution, $\lambda_{\text{up jump}}$ and $\lambda_{\text{down jump}}$.

After we obtain four parameters of jump distribution, by taking the average log of the rate of return of the underlying asset, we would get the drift and volatility of the underlying asset. The volatility of jump spikes is treated as one (similar to a standard deviation of a normal distribution). These six parameters allow us to simulate out the modified jump diffusion model with the historical price of underlying assets or indexes. The modified double normal jump diffusion model will be used in the Section 3.6.

3.6 Market Instrument with Modified Jump Diffusion Model

The aim of this section is to observe how will jump event affects the normal GBM in market prices. The jump event has two important factors which are the intensity of jump and number of jumps. The intensity of jump is said to be the impact of changes in prices each time a jump spike occurred. As different asset yields a different response when the market had an impact, hence the intensity acts differently for each instrument.

While the number of jump events is different for each market instrument too. Generally, there is a belief where a market crisis will occur once every ten years. However, jump does not mean only a drop in prices, an upward jump is also possible. The human behavior might cause heavy fluctuations in prices as well.

Our research aims to express the impact jump spikes into numeric parameters. The modified jump diffusion model is able to extract and present the yearly jump spikes into upwards and downwards direction along with their respective intensities. This modification extends the ability of jump diffusion model to acknowledge the degree of intensity of jump, rather than assuming the jump intensity is symmetrical around zero.

How should the investors group and identify the jump aside from daily normal fluctuation, will be a critical element to consider, we believe this research will provide better insight on handling market risk.

3.6.1 European Call with Jump Diffusion Model

Here we will explain how we calculate and compare the price of a European call simulated from the jump diffusion model and GBM. We calculate the expected price of GBM using the Black-Scholes model as a benchmark. For modified jump diffusion model, we use the same strike price, K and the initial price of the underlying assets and project the expected price. We can compare the expected price from jump model with the price from GBM.

Since the values of the jump parameters are different for different instruments. We will setting the range of $0 < \lambda < 4$, $0 < \mu_{\text{jump}} < 0.08$, while the initial price, S_0 and strike price, K equal to \$100. The drift and volatility of the asset, μ and σ were fixed at 0.08 and 0.4. We will use this to simulate the jump diffusion process and we calculate the expectation of the price for the European call option.

Preset parameters will be used to generate multiple simulations with different intensity of jump and number of jump event. For each level of parameters, the projected final price will be recorded and will then be included in a table. This will be used to compile and to be plotted. Next, we will input different level of parameters and re-simulate a different path again.

In fact, the initial payoff projection by Kou's jump model is almost similar to GBM. This result has shown that Kou's jump model is underestimating the payoff. As the jump intensity is normalized with mean zero, the expected option price from jump spike would be zero. This would contradict our initial idea where the expected jump event occurrence is calculated from the real market price.

Hence, we propose that a modification is needed for the jump diffusion model. Instead of a single jump parameter, λ we expand to two different jump components accommodating an upward spike and a downward spike.

Therefore, we had prepared simulation script for both modified jump diffusion model and GBM. After the simulations, we can proceed to calculate the price of European call option in the next Section 3.6.2.

3.6.2 Pricing European Call Option with Different Jump Intensity and Occurrence

This section is aiming to determine how will jump event affects the normal GBM for financial instrument prices. We start from the simplest instrument on a toy model. For instance, we will investigate the influence of jumps on the European call option.

The jump event has two important factors which are the intensity of jump and number of jumps. The intensity of jump is said to be the height of jump in prices each time a jump occurred. As different asset yields a different response when the market had an impact, hence the intensity is tuned differently for each instrument.

While the number of jump events is different for each market instrument too. Generally, there is a belief where a market crisis will occur once every 10 years. However, jump does not mean a drop in prices, an upward jump is also possible. How should the investors group and identify the jump aside from daily normal fluctuation, will be a critical element to consider.

Here in this research, the preset parameters in Table 3.3 from Section 3.4 will be used to generate multiple simulations with different intensity of jump and number of jump event. The final prices will be recorded in a table and will be plotted for each different level of jump intensity and frequency of jump arrival.

We have written a python script for the above simulations and the results are shown in Section 4.3.2. The effect of different intensity of jump and number of jumps will be compared and analyzed so that it is possible to investigate which parameters have the most impact on the prices. In the latter part of this research, we wish to determine the possible jump factor for S&P 500, for its intensity and occurrence of each jump event.

The code fragment below shows the modified jump model we used to compare the price of European call.We use the modified jump model with the different arrival of jump (`lambdas`) and jump intensities (`intensities`). The variable s would be the projected price, with variable v is the price after a year.

```
lambdas     = np.arange(1, 3,0.025)
intensities = np.arange(0,0.04,0.0005)
for lambd in lambdas :
    for intensity in intensities :
        s   = mf.jumpmodel(lamb=lambd, mujump=intensity)
        v   =((s[-1]-K > 0 )*(s[-1]-K)).mean()
        outputlist.append(v)
```

Listing 3.4: Expected price of European call with modified jump model

Results of the simulation will be shown in the next chapter in Section 4.3. Next, we proceed to the methodology of jump diffusion model application on an annuity.

Take note thatthe 'jumpmodel' represents the modified jump diffusion model. The 'lambdas' represents the arrival of jump in a period range from 1 to 3, while intensities are the range of jump intensities from 0 to 0.1 respectively.

3.6.3 Annuity with Jump Diffusion Model

An annuity is a contract where a fixed sum of money paid to a party each year (could be in other periods), for a long time frame. Usually, it is in a form of insurance or investments. The insured or investor is entitled to a series of payments.

The values of annuity vary on the different requested requirement by the annuitant. A larger stream of income and insured guarantee in the future would cost higher premium (or a constant annuity payment) and vice versa. Under the Black-Scholes model assumptions, the dynamics of annuity account value are as follows:

$$dA_t = (\mu - c)A_t dt + \sigma A_t dB_t + k dt, \tag{3.11}$$

where A_t and c are denoted as the sub-account value at time t and the mortality and expense (M&E) fee payable continuously respectively. While k is the subsequent contributions to the sub-account. Principle account is the account where the investor or insured pay their premium, while part of the amount is reinvested in this sub-account. Hence, the investing sub-account would be affected by market changes.

Under the Black-Scholes model, the combined GBM (GBM) with jump event (jump model in Equation (2.10)) is given by the following SDE:

$$dS_t = \mu S_t dt + \sigma S_t dB_t + J S_t dN_t. \tag{3.12}$$

With consideration of extreme event, the sub-account value will be modified with jump-diffusion model assumption and becomes:

$$\begin{aligned} dA_t &= A_t(dS_t)/S_t - cA_t dt + k dt \\ &= A_t(dt + dB_t + J dN_t) - cA_t dt + k dt \\ &= (-c)A_t dt + A_t dB_t + J A_t dN_t + k dt. \end{aligned} \tag{3.13}$$

An additional jump term, $JA_t dN_t$ is added into consideration. The J here represents the arrival (occurrence) of jump spike that follows a Poisson distribution, multiply by the intensity of jump that is normalized dN_t.

As we mentioned in Section3.5.1, although jump can occur in both directions, at one time only one direction of jump occurs. Hence, we drew the jump occurrence and spikes before adding jump term into the Heston model equation, assuming that the jump element and GBM are independent to each other.

We consider annuitant has guaranteed benefits with roll-up premium, consisting both guarantee minimum death benefits and guarantee minimum accumulation benefits (GMDB & GMAB). The guarantee benefits had a pre-agreed guaranteed interest rate $g \geq 0$, which is chosen such that $g < r$ (the feature of GMAB). Hence the guarantee benefits are given as follows:

$$G_t = \begin{cases} A_t(dS_t)/S_t - cA_t dt + kdt \\ A_t(\mu dt + \sigma dB_t + JdN_t) - cA_t dt + kdt, \end{cases} \tag{3.14}$$

where the J includes either a positive upward jump, J^+ or a negative downward jump, J^-. A quick recall from Equation 3.5, we are using two sets of parameters for each direction of the jump. The frequency of jump arrival and its jump intensity are different for both directions

From the Equation 3.14 above, there is only a single jump parameter that is included. As we mentioned that the jump and GBM are independent, hence we draw a random sample from a direction of jump before we include in the guarantee benefits (Equation 3.14).

This guarantee benefits resembling an Asian put option where the sub account value, A_t becomes the underlying asset. The payoff function, $P(t)$ will be given as following:

$$P(t) = [G(t) - A_t]_+ = \max(G_t - A_t, 0), \qquad \text{for } t \leq T \tag{3.15}$$

Annuity price is determined by Heston model most of the time. Hence in the next section, we will be calculating and simulating the price of the annuity with jump model with Heston model for both S&P 500 and Dow Jones index.

3.6.4 Pricing Annuity with Jump in Heston Model

This section aims to calculate an annuity with jump events in a Heston model. Using the Black Scholes model on annuity products requires a constant volatility when calculating the "fair" price. This would cause a mispricing for the "fair" price in the long run as volatility would change accordingly to the market. Therefore, the Black Scholes model could not accommodate when the annuity is volatile and requires a non-constant variance.

Heston model could attain such requirement by assigning a stochastic differential equation (SDE) to its variance, at the same time, it is retractable. Hence the Heston model would perform better compared to the Black Scholes model when pricing a long run annuity.

Given that the value of a unit asset, S_t at time *t* is assumed to follow the Heston model process given by the following stochastic differential equation (SDE):

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sqrt{V_t} S_t \mathrm{d}W_t^s, \qquad S(0) = S_0, \tag{3.16}$$

where μ is the rate of return from an asset and $S(0) = S_0$ is the initial boundary condition. In this research, we assume the risk-free interest rate, *r* as the rate of return, hence we rewrite Equation 3.16 as:

$$\mathrm{d}S_t = rS_t \mathrm{d}t + \sqrt{V_t} S_t \mathrm{d}W_t^s, \qquad S(0) = S_0, \tag{3.17}$$

where V_t is the instantaneous variance, which follows a CIR (Cox Ingersoll Ross) process given by the following SDE:

$$\mathrm{d}V_t = \kappa(\theta - V_t)\mathrm{d}t + \sigma_V \sqrt{V_t}\mathrm{d}W_t^V, \qquad V(0) = V_0, \tag{3.18}$$

where the parameter κ is the speed of adjustment, with θ equal to the mean, and σ_V be the volatility. The $\kappa(\theta - V_t)$ is the drift factor that ensures the mean reversion of interest rate is towards θ in the long run.

We will further modify the Heston model to incorporate the jump criteria, the model would become as below:

$$\mathrm{d}V_t = \kappa(\theta - V_t)\mathrm{d}t + \sigma_V \sqrt{V_t}\mathrm{d}W_t^V + J_t dW_t, \qquad V(0) = V_0. \tag{3.19}$$

Here, J_t is a compound Poisson process such that the *i*-th jump is equal to $e^{Y_i} - 1$. Y_i is the distribution of the jump. For instance, if Y_i have the Gaussian distribution, S_t will have log-normally distributed jump.

```
def A(r=r,c=0,sigma=sigma,k=k,A0=A0, t0=0,tN=T,
     N=N,M=M,lamb=lamb,UI_mean=0.008,UI_sigma = 0.0363687):
    t  = np.linspace(t0, tN, N+1); dt = t[1]-t[0]
    At = np.zeros([M, N+1]); At[:,0]=A0
    Bt = np.zeros([M, N+1]); Bt[:,0]=A0
    for i in range(N):
        dW = np.random.randn(M) * np.sqrt(dt)
        J  = sigmaJ*np.random.randn(M,N+1)+muJ
        dN = np.random.poisson(lamb * dt, (M,N+1))
        J_up= np.random.normal(UI_mean,UI_sigma,(M,N))
        At[:,i+1] = At[:,i] + (r-c)*At[:,i]*dt + sigma*At[:,i]*dW
                    + k*dt +J_up[:,i]*dN[:,i]
        Bt[:,i+1] = Bt[:,i] + (r-c)*Bt[:,i]*dt - sigma*Bt[:,i]*dW
                    + k*dt  +J_up[:,i]*dN[:,i]
```

```
    return t, np.concatenate([At,Bt])
```

Listing 3.5: Heston model calculation

CHAPTER 4

RESULTS AND DATA ANALYSIS

4.1 Gibbs Sampling Method for Market Indexes

In this section, we try to verify the parameters obtained from the Gibbs sampling method. Here we supposed the parameters are the drift, μ and volatility, σ of the underlying asset, the frequency of jump arrival, λ the jump intensities, μ_{jump} and jump's volatility, σ_{jump}. To verify those parameters, we re-simulate Kou's jump model by using some preset values for parameters.

In methodology, Section 3.4 we had mentioned that by using the Gibbs sampling method we could retrieve the drift, μ and volatility, σ of GBM along with jump parameters for a jump model from raw price data.

Hence, we tested with simulated data with preset values. The methodology can be referred from Section 3.4 and results are as shown in Table 4.1. The "Mean" in Table 4.1 is the results from Gibbs sampling method based on the simulated data produced with "Preset" parameters for each different set.

Table 4.1: Gibbs sampling method algorithm on different sets of simulated jump diffusion model.

Set	Set1	Set1	Set2	Set2	Set3	Set3
Parameters	**Preset**	**Mean**	**Preset**	**Mean**	**Preset**	**Mean**
Drift	0.1	-0.02066	0.01	0.02107	0.5	0.16016
Volatility	0.5	0.04132	0.07	0.02449	1	0.08843
Jump Arrival	10.0	10.51186	10	11.99715	5	4.92009
Jump Intensity	0.5	0.48472	0.5	0.48244	1	2.62712
S.D of jump	0.025	0.07128	0.05	0.07209	2	0.30517

In Table 4.1, results from "Set1" and "Set2" show that, when the 'Drift' and 'Volatility' of an asset is relatively small, the Gibbs sampling method could provide a closer result to the input preset value. Else, the converging of "Preset"s' parameters, “Drift” and “Volatility” of assets will fail as shown in “Set3”.

In “Set3”, jump intensity, μ_{jump} shows a value of 2.6 instead of 1. The standard deviation of the jump, σ_{jump} seems to maintain it’s converging around the value of 0.07 when the input is small, could not get back the “Preset” value of 2 in “Set3”. Gibbs sampling method can converge ’Jump intensity’ when it is smaller than 1 as shown in “Set1” and “Set2”, while it fails when the intensity raises beyond one as in “Set3”. Only the ’Frequency of jump’ are able to converge back closer to the initial input value throughout the three different cases.

Next, we tried to apply these on market indexes. We picked Dow Jones industrial (DJI), NASDAQ Composite 100 (NASDAQ 100), FTSE 100, S&P 500 and NYSE ARCA OIL & GAS INDEX (OilGas) with two different periods. In Table 4.2 we show the results from Gibbs sampling method using data from the year 2005 October to the year 2010 December, while Table 4.3 shows the results for the period between October 2010 and December 2015.

Table 4.2: Comparison between extracted parameters of different indexes between year 2005 and 2010.

Parameters	**DJI**	**S&P 500**	**NASDAQ 100**	**FTSE 100**	**OilGas**
Drift	0.10245	0.11345	0.10625	0.11171	0.22821
Volatility	0.12574	0.14119	0.17082	0.14919	0.22557
Jump Arrival	19.33768	19.24939	14.06721	16.59051	15.91333
Jump Intensity	-0.00407	-0.00519	-0.00388	-0.00475	-0.01131
S.D of jump	0.04524	0.04711	0.05277	0.04554	0.06655

Table 4.3: Comparison between extracted parameters of different indexes between year 2010 and 2015.

Parameters	**DJI**	**S&P 500**	**NASDAQ 100**	**FTSE 100**	**OilGas**
Drift	0.13043	0.14949	0.17544	0.06069	0.11527
Volatility	0.12365	0.12797	0.14913	0.14301	0.18886
Jump Arrival	5.23150	6.83121	6.07712	6.08239	12.74748
Jump Intensity	-0.00592	-0.00478	-0.00568	-0.00446	-0.00615
S.D of jump	0.05644	0.05294	0.05739	0.05095	0.04405

Table 4.2 and Table 4.3 are calculated by the Gibbs sampling method on market prices, hence the results depend on the market performance. Table 4.1 shows that convergence of 'Drift', 'Volatility', 'Jump Intensity', and 'S.D of jump' are not very reliable in converging. However, we can make use of the 'Jump Arrival' values to get some insights on the frequency of jump that happen across both periods over 10 years for each index.

By looking at the 'Jump Arrival' in Table 4.2 and Table 4.3, we show that the period between 2005 and 2010 contains more arrival of jump relative to the period between 2010 and 2015. In the first period, Table 4.2 shows that a minimum arrival of 14 jumps across the indexes. The intensity of jump, however, had a smaller scale than what we expected. Hence, we believe that the parameters possess clustering effect between jump arrival,λ jump intensity, μ_{jump} and its volatility, σ_{jump}.

As we compare the values in Table 4.2 to Table 4.3, the values for "jump intensity" and "S.D of jump" does not differ much. We suspect that using Kou's jump model with Gibbs sampling method could not provide good insights in observations of other variables besides "jump arrival". Whilst the arrival of jump shows an obvious difference in Table 4.3 where most of the indexes had an average arrival of jump of 6, besides NYSE ARCA OIL & GAS INDEX that remains high at 12.7.

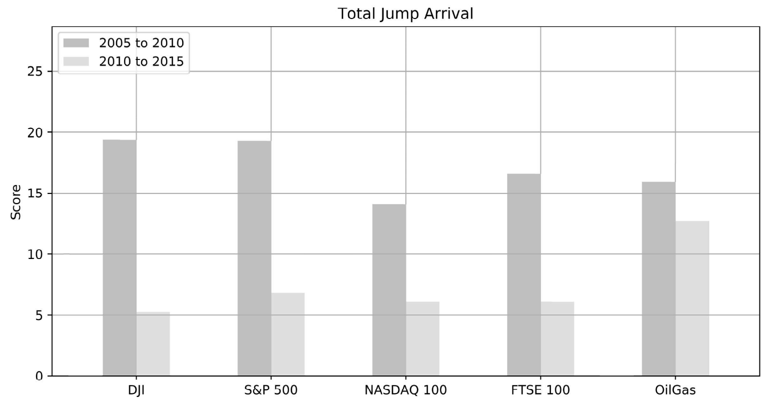


Figure 4.1: Jump arrival of different market index for 2 different period (2005 to 2010 and 2010 to 2015)

Figure 4.1 shows the jump arrival of each index for two different periods. In Figure 4.1, we can see that the market had a high jump event between Oct 2005 to Dec 2010 compared to the next 5 years period. Recall that, we had a world economic recession during the year 2007 and 2008. The huge drop in prices caused the frequency of jump arrivals to shoot up to nearly 20. Whilst, the market is more stable after 2010, and hence the jump occurrence is dropped to 5 and 6.

The jumps arrival for NYSE ARCA oil & gas Index during the period of 2010 to 2015 are higher compared to the other four indexes. As their average arrival of jump had lower to 6, the NYSE ARCA Oil & Gas index remain high at 13 arrivals. Hence, we check on the historical index prices throughout Oct 1, 2005, till Sept 30, 2015, in Figure 4.2.

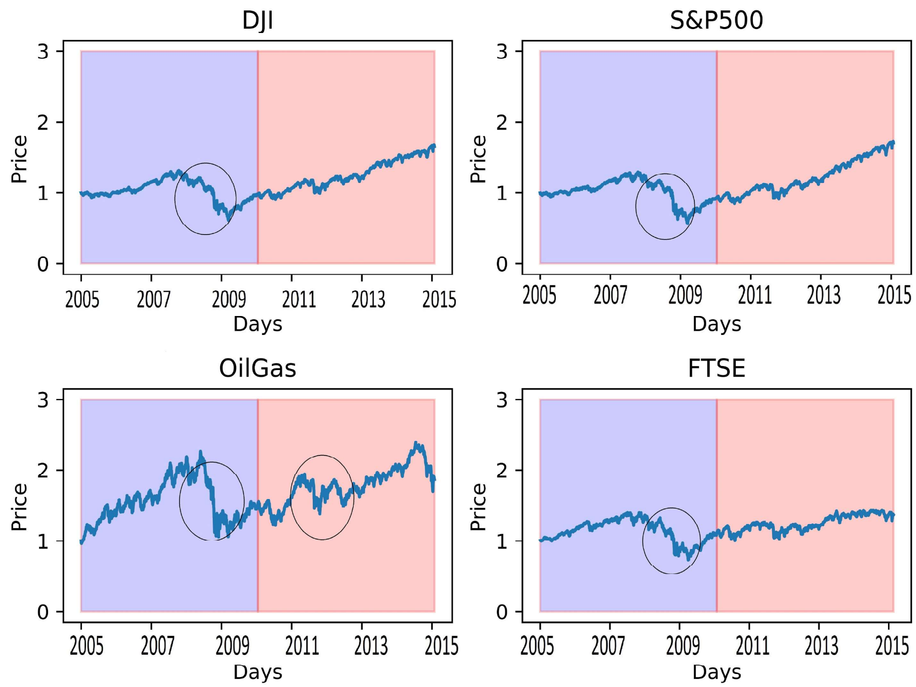


Figure 4.2: Comparison of price behavior for 4 indexes. (Data retrieved from Yahoo Finance)

Figure 4.2 shows that there are some high spike changes between 2008 to 2009, those regions are in the circles. We can see that the NYSE ARCA OIL & GAS INDEX remains high amount of jump arrivals due to the sudden drop in prices on the second period (around 2012), while the other indexes had started to be more stable. Until the year 2015, the oil and petrol prices remain highly volatile from time to time, hence the arrival of the jump event remains at a higher level.

This shows that the Gibbs sampling method could provide good expectation on the number of jump arrival occurred over the period, however it might suffer from clustering effects between the parameters. We need to aware that the number of jump arrival contains both upward and downward jump. Understanding this frequency of jump arrival solely is not enough for research purpose as the direction of jump directly affects the pricing of financial products. Therefore, we introduced the modified jump diffusion model and the empirical method in Section 3.5.

4.2 Modification of Upward and Downward Jump Parameters in Jump Diffusion Model

In Section 3.5.1, the empirical method shows that S&P 500 does not fit the distribution used by Kou's model see Figure 3.3. Kou's jump model proposed that the jump model is distributed with a double exponential distribution centered around zero. However, S&P 500 shows that the distribution is asymmetric for positive and negative changes.

We further investigate the distribution of DJI indexes and NASDAQ to check whether their distribution for different directions of changes is symmetry.

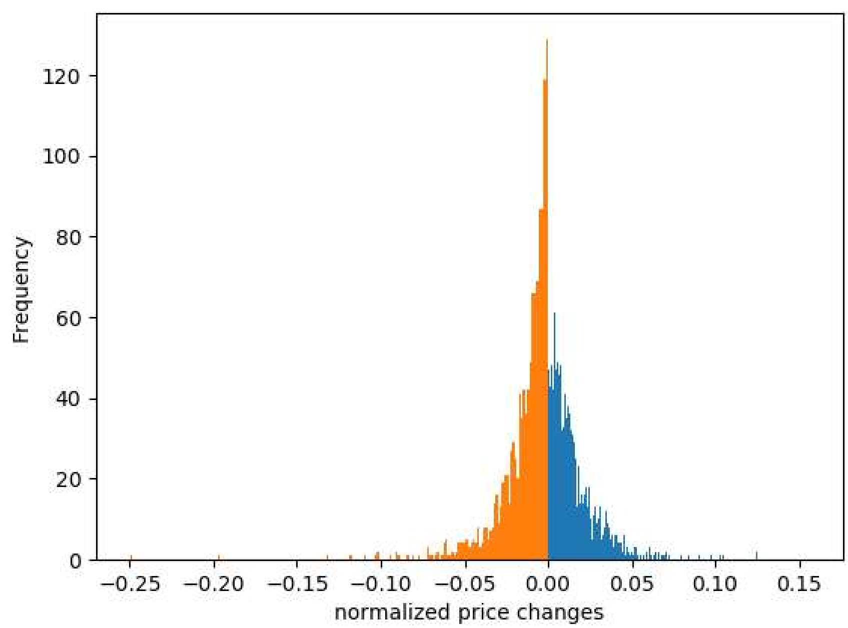


Figure 4.3: Distribution of DJI daily log return from 1995 to 2014.

Figure 4.3 shows that the distribution of positive and negative directions are asymmetry to each other. The positive distribution (blue right region) is slightly lower than the negative (orange left region), and they show a further distance from zero (heavy-tailed). Whilst the negative changes seem to have a higher frequency and larger range of spike as the largest negative change is up to -0.25 while the largest positive change around +0.13.

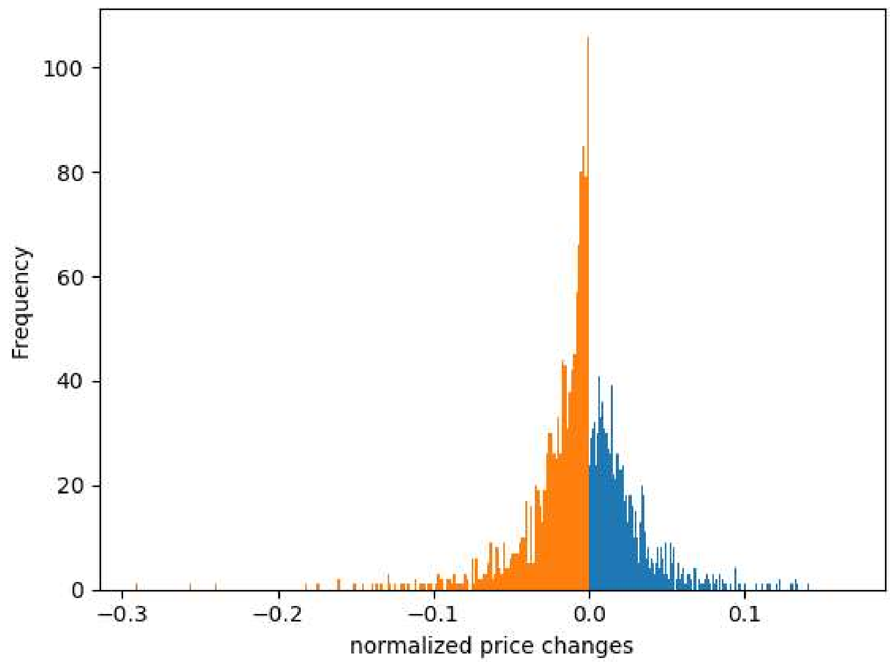


Figure 4.4: Distribution of NASDAQ daily log return from 1995 to 2014.

Figure 4.4 shows a similar characteristic with DJI, where the distribution of positive and negative directions are asymmetry to each other. However, compared to DJI, NASDAQ has a much heavier distribution on the left side (negative side) of the figure. On it's right, the positive distribution resembles a normal distribution rather an exponential distribution. The empirical method was applied to other indexes, which are FTSE and NYSE ARCA Oil & Gas index.

The results from Figure 4.5 and Figure 4.6 show that they resembled double exponential distribution where both the negative and positive parts spread broader and higher. However, Figure 4.5 shows that the positive part has a relatively lower distribution compared to the negative part. While Figure 4.6 had a steep drop in distribution for positive distribution when it approaches zero.

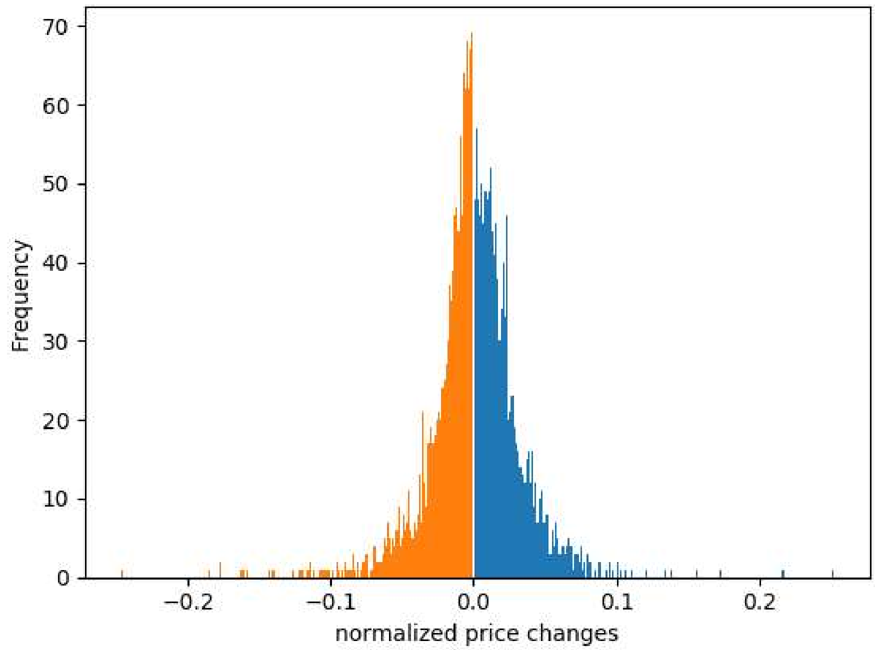


Figure 4.5: Distribution of NYSE ARCA oil and gas index index normalized price changes from 1995 to 2014.

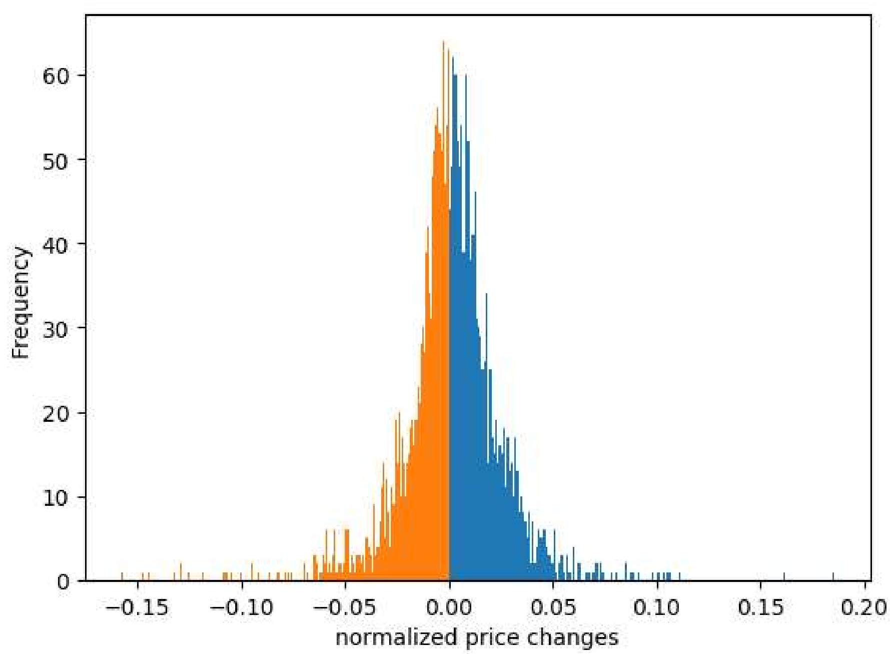


Figure 4.6: Distribution of FTSE normalized price changes from 1995 to 2014.

The results suggest us to modify the Kou's model such that it could capture a better distribution for the jump in two different directions. As we observe from the four different figures, the distribution for the positive tail could be different for different underlying assets. The usage of double exponential distribution will limit the distribution of FTSE and NYSE ARCA oil and gas index. We had introduced an empirical method in Section 3.5.1, where we propose a "piece-wise" double normal distribution.

The results in Table 3.7 and Table 3.8 show that the frequency of jump arrival is different for indexes. By referring to Table 3.9 and Table 3.10, we can observe that the total downwards jump is much more frequent than upwards jump too.

Table 4.4: Comparison total number of jumps captured between different indexes over 20 years.

Index	Total upward jump	Total downward jump
S&P 500	32	86
DJI	42	88
NASDAQ	36	73
Oil & Gas	36	92

For a clearer comparison, we compare the total number of jumps in both directions for each index we had in Table 4.4. We can observe that the 'Total downward jump' almost occur twice likely compare to 'Total upward jump'. The number of jumps here is determined using the empirical method, with a support line of the quadruple median refer to Section 3.5.1.

From Table 3.7, Table 3.8, Table 3.9 and Table 3.10, we observe that the year with highest jump intensity was at year 2008. The average value of these indexes mostly between -0.02 to 0.02. While in year 2008, the value of intensity is as shown in Table 4.5.

Obtaining the frequency of jump arrival and intensity of jump for both upward and downward directions allow us to modify the model to fill the gap where

Table 4.5: Comparison of jumps intensity between different indexes in year 2008.

Index	Up intensity	Down intensity
S&P 500	0.05209	-0.048799
DJI	0.052077	-0.039357
NASDAQ	0.039349	-0.040370
Oil & Gas	0.061320	-0.057834

previous researchers fail to capture.

The modified model uses the value of parameters we attain from the empirical method to simulate out an expected price in the next year, by using the previous year price data. Using the modified jump diffusion model allow us to calculate the fair price of financial products more precisely as we included the expected jump arrival.

In Section 4.3 and Section 4.4, we observe how modified jump diffusion model changes European call options and the annuity from geometric Brownian motion (GBM), and check its impact on the pricing model of both instruments. This is to show that whether the modified jump model can remain its characteristic among two different types of instruments.

4.3 Jump Diffusion Models and Geometric Brownian Motion Model on European Call Option Pricing

4.3.1 Convert Risk-neutral Measures to Market Measures

By Black-Scholes model, the options pricing could be calculated, to determine the potential risk and return of the option. As jump event included in the jump model, hence the risk and payoff shall be different from ordinary Black-Scholes model.

The purpose here is to determine how much and how big the impact of jump spike would affect the payoff of the call option. Here, we had built in a modified jump diffusion model and the GBM model function, where we can simulate their samples. We fix the value for the initial price, S_0 to \$100, with a strike price, $K = \$100$ too. The drift μ is set to a constant following the underlying asset, where it is -0.005 for DJI index. While the volatility σ is constructed with Dupire's formula under a risk-neutral measure in order to reduce arbitrage opportunities. Dupires formula is the local volatility, expressed entirely in terms of the volatility surface $C(T, K)$:

$$\hat{\sigma}(T, K) = \frac{1}{K}\sqrt{\frac{2\delta C/\delta T(T, K)}{\delta^2 C/\delta x^2(T, K)}} \tag{4.1}$$

where, $C(T, K)$ is the price of the call option, C with exercise time, T and strike price, $\$K$. Variable δx would be a small step change in strike price. By assembling a matrix of quoted option price and yield curve, we formed a smooth volatility curve. The constructed volatility with Dupire's formula in Equation (4.1) for DJI index and S&P 500 over 20 years is similarly close to 0.00555.

4.3.2 The Price Comparison of European Call Option with Jump Diffusion Models and Geometric Brownian Motion Model

The payoff of the European call option using GBM is calculated using the Black-Scholes model. We will exercise the option when the expected price at time t, S_t is higher than the strike price, K. We will not exercise when it is less than that. Hence the payoff at time, t is equivalent to $\max(S_t - K, 0)$.

Whilst, the payoff of the European call option using jump diffusion model will be calculated using Black-Scholes model too. However, the jump diffusion model has different ranges for its parameters. For its arrival of extreme events, λ or jump arrival in Figure 4.7, it is set from zero (no jump event, or normal GBM) until four, which indicating four spikes per year. While another parameter is the intensity for each jump is ranging from 0 to 0.08. The results are shown in Figure 4.7.

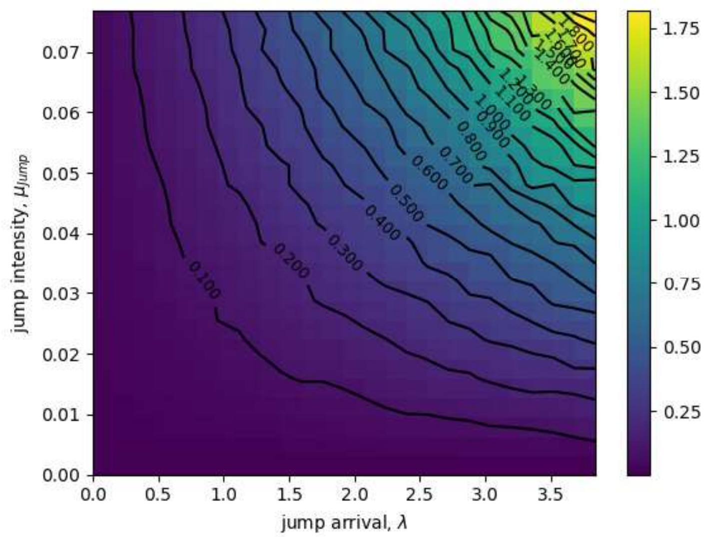


Figure 4.7: The expected payoff for jump intensity.

The origin point in Figure 4.7 is the payoff of the European call option with GBM. Along the vertical axis when jump had zero arrival, we can observe that the call option's payoff fluctuates around zero.

When the jump event occurs, the payoff surpasses the expected payoff of the Brownian motion at the white dot and escalates proportionally to the increment of intensity. From the result, as the intensity of jump increase, the payoff increases more. The highest peak occurs when the jump intensity and arrival are at their highest level. This results only consider if there is a positive jump spike, there would be a negative jump spike where will induce larger negative payoff for an asset. The negative payoff is neglected as this is a call option, where we would not exercise if the underlying asset is out of the market. This shows that with the Black-Scholes model, GBM could not expect the larger risk behind each jump event that will occur. This will cause mispricing for the call option in the market.

We take S&P 500 index and Dow Jones index and compare with the European call option. Figure 4.8 shows the pricing for S&P 500 index for 20 years and while the Figure 4.9 shows the expected pricing of the Dow Jones index.

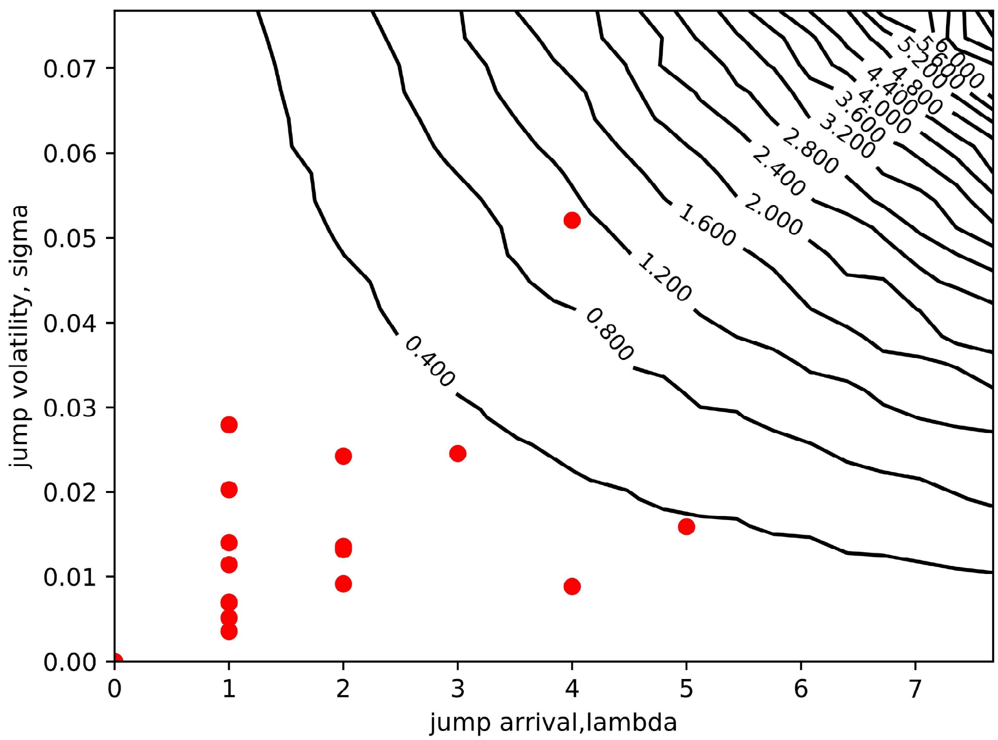


Figure 4.8: The expected pricing for 20 years of S&P 500 index.

From Figure 4.8, the red dots show the expected positive jump arrivals and intensities for S&P 500 over the past 20 years from 1995 to 2014. We could observe that the expected payoff in pricing differs from the European call that did not accommodate jump event. To visualize the price for simulated European call with S&P 500, we take a look on Table 4.6.

We can observe that the highest price occurs during the year 2008 with four jumps with intensities at least 0.05. The highest attainable payoff exceeded 1.0 for a log return on that year. The fair price of GBM (when no jump is included) is 0.00049 in this simulated European call.

For some other years such as 2000, 2002 and 2014, there exist high number jump arrival of three to five. Even though their intensities of the jump are not

Table 4.6: Pricing of S&P 500 in European call with modified jump diffusion model.

Year	Frequency of jump	Jump intensity	Price, $S_t - K$
1995	1	0.00358	0.01744
1996	2	0.00915	0.06892
1997	1	0.00696	0.03098
1998	2	0.01320	0.10249
1999	0	0.00000	0.00049
2000	5	0.01592	0.34634
2001	1	0.02028	0.08377
2002	3	0.02455	0.30534
2003	1	0.01402	0.06104
2004	1	0.00702	0.03122
2005	0	0.00000	0.00049
2006	0	0.00000	0.00049
2007	2	0.01360	0.10505
2008	4	0.05209	1.07665
2010	1	0.01146	0.05134
2011	2	0.02425	0.19231
2012	0	0.00000	0.00049
2013	1	0.00519	0.02381
2014	4	0.00885	0.13973

as high as 2008, their high price should be an alarm for fluctuation in S&P 500 index. Next part, we will take a look on simulation of European call option using the Dow Jones Index as underlying assets.

From Figure 4.9, the asterisks show the expected positive jump arrivals and intensities for Dow Jones(DJI) over the past 20 years from 1995 to 2014. We could observe that the expected payoff in pricing not only differ from the European call but different as S&P 500 indexes in Figure 4.8. Using the pricing calculation with GBM where we did not consider jump at all, the expected payoff if 0.0005. However, almost every pricing of each year is different from each other when jump events are considered.

In Table 4.7, the highest attainable payoff could even reach approximately 0.73 for a log return during the year 2008. This year yield the highest price compared to other years, which is exactly the same as S& P 500 index. However,

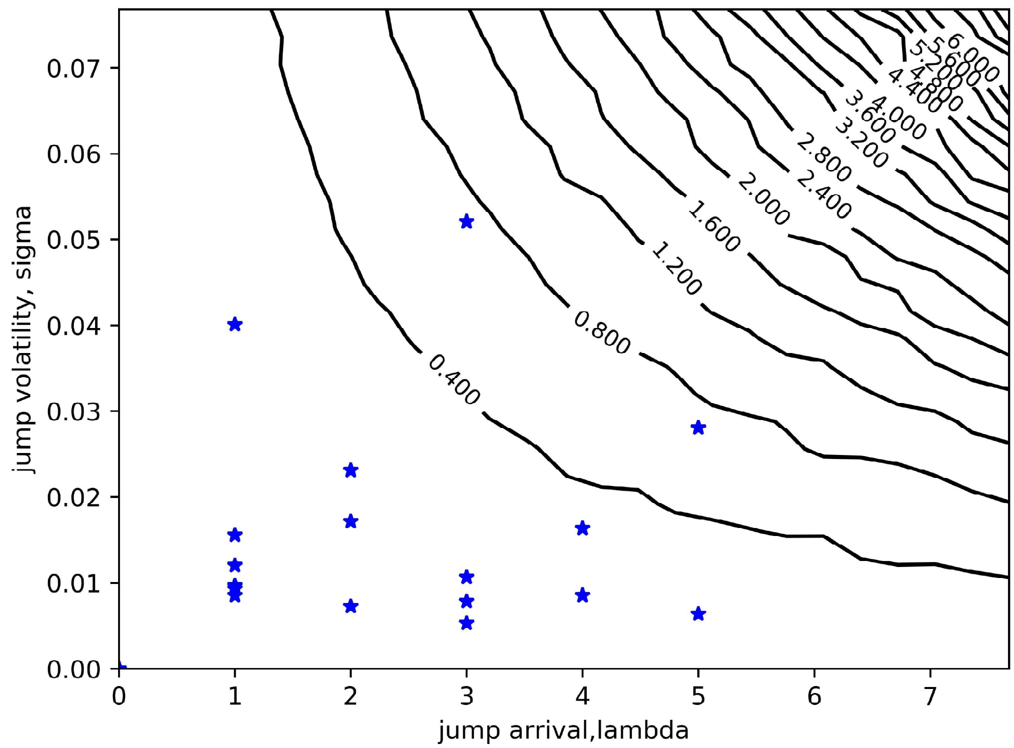


Figure 4.9: The expected pricing for 20 years of Dow Jones index.

the price is much lower than S& P 500. Considering the price of European call without an upward jump event is 0.00051, jump events are heavily affecting the price of European call.

Besides, it is rather interesting that Down Jones indexes had more years that possessed a higher frequency of jump arrival. There are about forty percent over 20 years had a jump frequency that is higher than three. This shows that the Dow Jones industry is slightly volatile compared to S&P 500 index, but the fluctuation is not as big as S&P 500. These two simulations are based on positive upwards jump event only.

As we are considering a simulated European call, a sharp drop in price that is due to negative downwards jump event would not trigger the call option to be exercised. This observation is shown in Figure 4.10. In Figure 4.10, we considered both positive and negative jump events, by multiplying the intensity and arrival of jumps, we attain the impact of jump and thus able to form the

Table 4.7: Pricing of DJI in European call with modified jump diffusion model.

Year	Frequency of jump	Jump intensity	Price, $S_t - K$
1995	0.0	0.0	0.000512
1996	5.0	0.006384	0.130372
1997	0.0	0.0	0.000512
1998	3.0	0.010684	0.127682
1999	1.0	0.009297	0.040946
2000	1.0	0.040103	0.159591
2001	4.0	0.016355	0.272328
2002	5.0	0.028068	0.667495
2003	1.0	0.015567	0.064986
2004	2.0	0.007284	0.056885
2005	1.0	0.008556	0.037806
2006	0.0	0.0	0.000512
2007	1.0	0.012031	0.054281
2008	3.0	0.052076	0.736973
2009	2.0	0.017149	0.129034
2010	1.0	0.009695	0.042700
2011	2.0	0.023120	0.180512
2012	3.0	0.007883	0.094637
2013	3.0	0.005308	0.062119
2014	4.0	0.008573	0.135558

contour of fair price by using interpolation method.

Figure 4.10 shows that the price of European call option increases in price as the arrival of positive jump events increases. The events of negative jumps would increase the fluctuation of price as well, and hence it also increases the price of options when it occurs a positive fluctuation.

Vice versa, if we are considering a European put option, the negative downwards jump events would impact the price of the European put option more compared to positive upwards jump events. Next, in Section 4.4, we would try out the application of a modified jump diffusion model in annuity pricing. The simulation would base on S&P 500 and DJI index too.

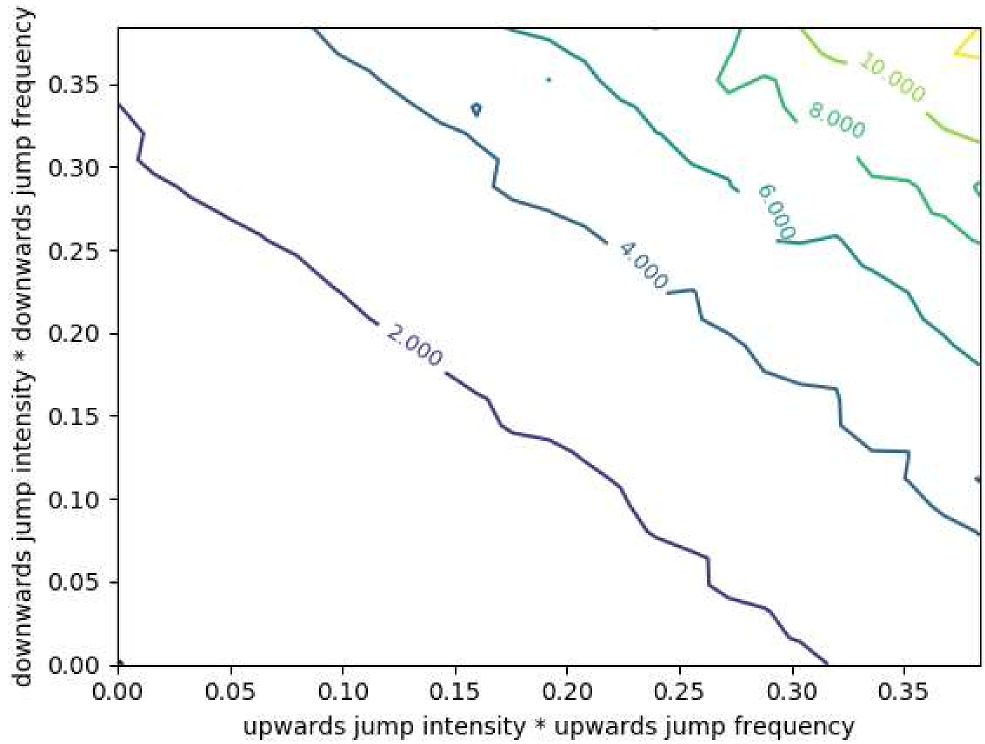


Figure 4.10: Price of European call option base on Dow Jones index with two directions of jumps.

4.4 Annuity Pricing with Jump Diffusion Model

As we mentioned in the Section 3.6.3, the pricing of annuity is indicated by Equation (3.15). Even though pricing company is aware that the expected fair price of annuity is affected by the extreme events, but they did not include the chances of risk in pricing.

Figure 4.11 shows that the price of an annuity with jump diffusion model during negative downwards jump event. The figure shows the pricing payoff that is based on the Dow Jones index. The red dots are historical yearly pricing from 1995 to 2015. The contour shows different layers of payoff based on different numbers of jump arrival and intensity. The jump intensity ranged from 0 to 0.08 and the frequency of jump arrival ranged from 0 to 8.

Here we observed that the contour is different from European call which

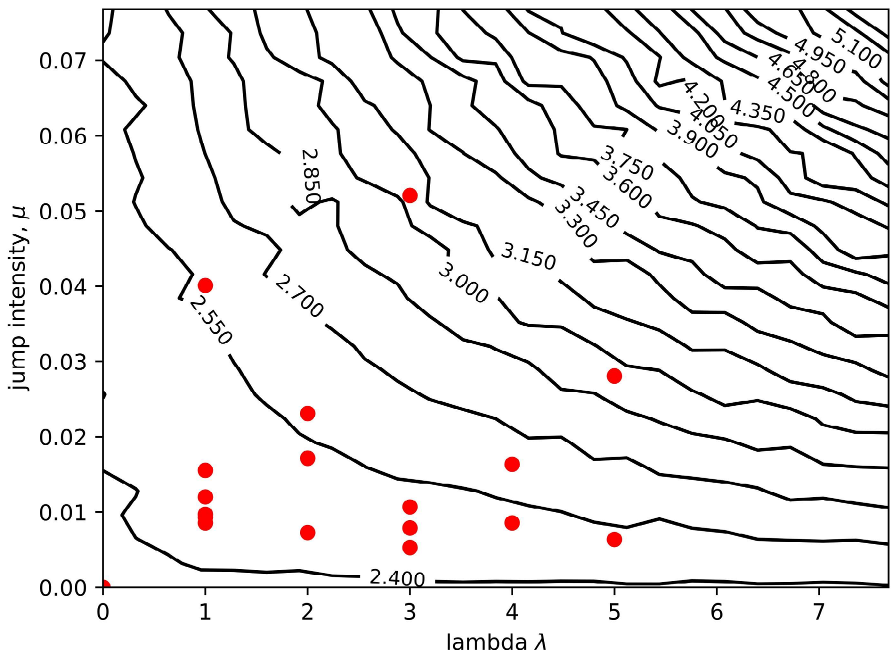


Figure 4.11: The expected rewards function from the annuity with jump model based on DJI.

increases exponentially as the activity of jump events increased. For an annuity, the price seems to be increased slowly and linear when the jump event occurs. This shows that jump events provide relatively less impact on annuity compared to European call option. However, the effects are not negligible.

By using Heston model with normal GBM, the price of annuity is located at the origin, where the price of annuity is around 2.403. This can be seen from Table 4.8, when the frequency of jump is zero.

Table 4.8 shows that the Dow Jones index had its two highest prices during the year 2002 and 2008 with the price of 2.96 and 3.08 respectively. Considering the year 1995 and 1997 that did not have a jump event, the price of annuity should be around 2.403. Annuity of the year 2002 and 2008 are priced 20% higher compared to the year 1995 and 1997 which did not have any jump events.

Nonetheless, the intensity of jump event plays a crucial part here in the

Table 4.8: Pricing of DJI in Annuity with modified jump diffusion model.

Year	Frequency of jump	Jump intensity	Price of Annuity
1995	0.0	0.0	2.4036566
1996	5.0	0.0063849	2.5340034
1997	0.0	0.0	2.4036566
1998	3.0	0.0106843	2.5190019
1999	1.0	0.0092978	2.4342046
2000	1.0	0.0401039	2.5828067
2001	4.0	0.0163555	2.6409887
2002	5.0	0.0280681	2.9621931
2003	1.0	0.0155677	2.4612259
2004	2.0	0.0072845	2.4568861
2005	1.0	0.0085566	2.4350260
2006	0.0	0.0	2.4036566
2007	1.0	0.0120313	2.4503885
2008	3.0	0.0520768	3.0836130
2009	2.0	0.0171492	2.5361915
2010	1.0	0.0096956	2.4345195
2011	2.0	0.0231206	2.5922390
2012	3.0	0.0078834	2.4864413
2013	3.0	0.0053087	2.4600295
2014	4.0	0.0085735	2.5253195

pricing model as well. As in the year 1996, with a whopping five jump arrivals, the price merely increased by 5% to 2.53 from 2.403 due to the low intensity of jump of 0.006. While in the year 2005, a single jump with high intensity of 0.04 is enough to make the annuity being priced at 2.58.

Figure 4.12 is based on S&P 500 index and same year period with DJI which is 1995 to 2014. They are represented by blue asterisks point. The figure shows that most of the frequency of jump is around one, while only a few years had a frequency of jump arrival more than 3.

In Table 4.9, the pricing of the annuity without considering jump events are allocated at the origin with a payoff less than 2.403. This can be observed in the year 1996, 2005, 2006 and 2012.

From Table 4.9, during the year 2008, the annuity is priced at 3.307. The

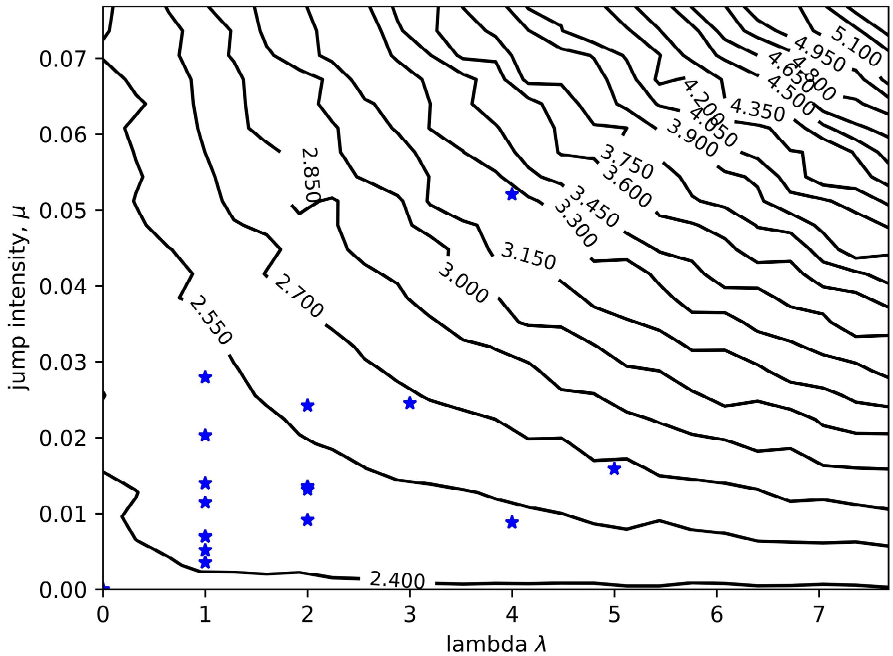


Figure 4.12: The expected rewards function from the annuity with jump model based on S&P 500.

arrival of jump events caused the price of annuity to increase 37% higher than the normal. The year 2000 also had an increase in the price of 8%. In these simulations, only the year 2008 made a change in price higher than 10%. Therefore, we can say that S&P 500 index is slightly more stable than Dow Jones Indexes. Yet, precaution and consideration of the jump events should not neglect.

Without considering the chances of negative extreme events, the annuity is undervalued. In Figure 4.13, we combined the intensity of jump with the arrival of jump in each direction. We observe that as the investment risk is higher (more negative jump), the price of the annuity will be higher. On the other hand, the price may be lowered than expected price (from \$2.4 to \$1.6) if the market is expected to be good.

The observation from Figure 4.13 supports the characteristic of annuity where the annuitant receives the same flow of return regardless of the market

Table 4.9: Pricing of S&P 500 in Annuity with modified jump diffusion model.

Year	Frequency of jump	Jump intensity	Price of Annuity
1995	1.0	0.003582	2.421296
1996	2.0	0.009150	2.469661
1997	1.0	0.006960	2.436795
1998	2.0	0.013204	2.507621
1999	0.0	0.0	2.403656
2000	5.0	0.015920	2.690801
2001	1.0	0.020282	2.499405
2002	3.0	0.024551	2.688754
2003	1.0	0.014022	2.458091
2004	1.0	0.007017	2.436732
2005	0.0	0.0	2.403656
2006	0.0	0.0	2.403656
2007	2.0	0.013596	2.509773
2008	4.0	0.052090	3.307822
2009	1.0	0.027947	2.527172
2010	1.0	0.011455	2.446479
2011	2.0	0.024247	2.596834
2012	0.0	0.0	2.403656
2013	1.0	0.005189	2.430489
2014	4.0	0.008849	2.529361

performance. When the market is expected to be bullish, the writer of the annuity will gain from the market performance, hence the price of annuity can be lowered to attract more annuitant. When the market is expected to be bearish, the price of the annuity will be increased to cover the risk exposure.

Considering the chances of extreme events, the annuity could be either overvalued or undervalued. The expected price of the annuity would be expecting a surged in price during a negatively drifted jump event, causing the annuity policy to be overvalued.

We had shown that the European call option can be priced higher when we expect a positive drifting jump event, but annuity should be priced lowered instead. This concludes that without knowing the directions of jump and the characteristic of underlying assets, mispricing would occur.

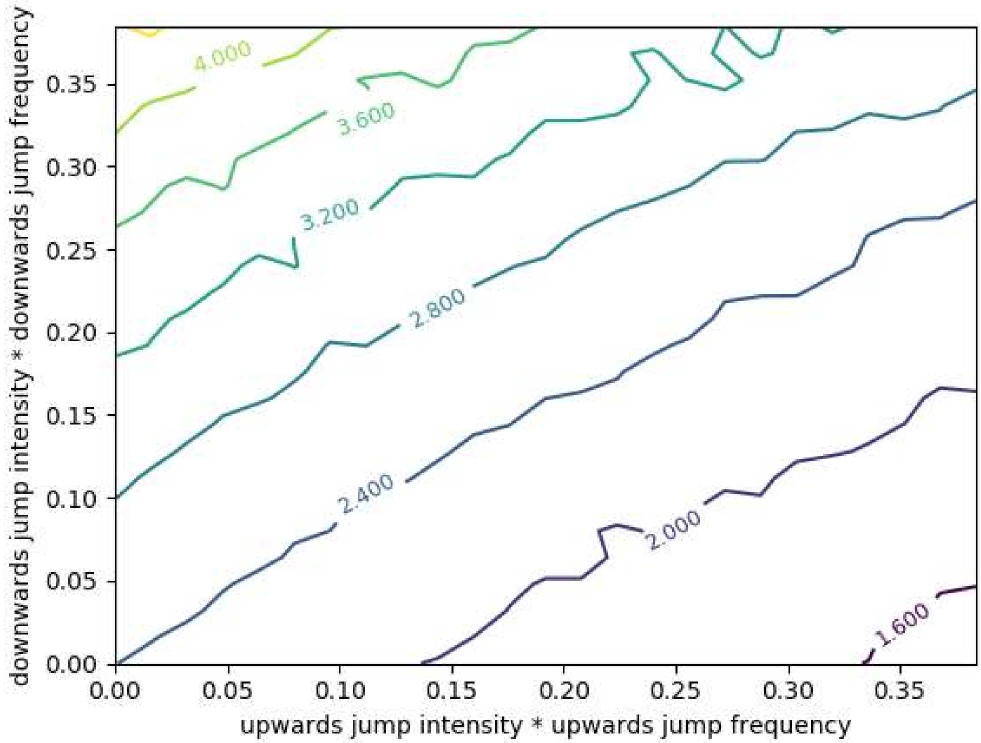


Figure 4.13: The expected price from the annuity with jump model based on DJI and S&P 500 when considering 2 different direction of jumps.

Either drift will be resulting in a fail in fair pricing. The need of consideration on jump event in the pricing model is important as the pricing company would lose its competitiveness to the other company when they failed to provide a fair price.

CHAPTER 5

CONCLUSION AND DISCUSSION

Both economic downturn in 1998 and the sub-prime crisis in the year 2007 are the most noticeable extreme events happens in the recent years. The occurrence of such extreme events had surpassed the limits of pricing model that used geometric Brownian motion (GBM) as fundamental. The impact of the extreme event can be tremendous if it is left neglected by investors. The inability of the Black Scholes model to capture the extreme jump event would lead the investor into a riskier situation. Hence, the ability to quantify its signal before it happens is significant and important as it would reduce the risk of being bearded by investors.

The jump diffusion model suggested by Kou is more realistic in capturing such signal compare to GBM. GBM propose that the underlying asset changes according to its drift and volatility. While the jump diffusion model added a component of jump into GBM. This allows the underlying asset to model extreme event. Jump diffusion model is capable to infuse distribution of jump event into the pricing model, which allow investors to aware of the occurrence of jump event.

Kou model proposed that the jump event should be distributed as normal distribution or double exponential distribution. Both distributions are distributed heavily around the zero. In Kou research studies (Kou, 2008), he used data of S&P 500 from 1980 until 2005 and plotted a histogram based on daily log returns (see Figure 5.1).

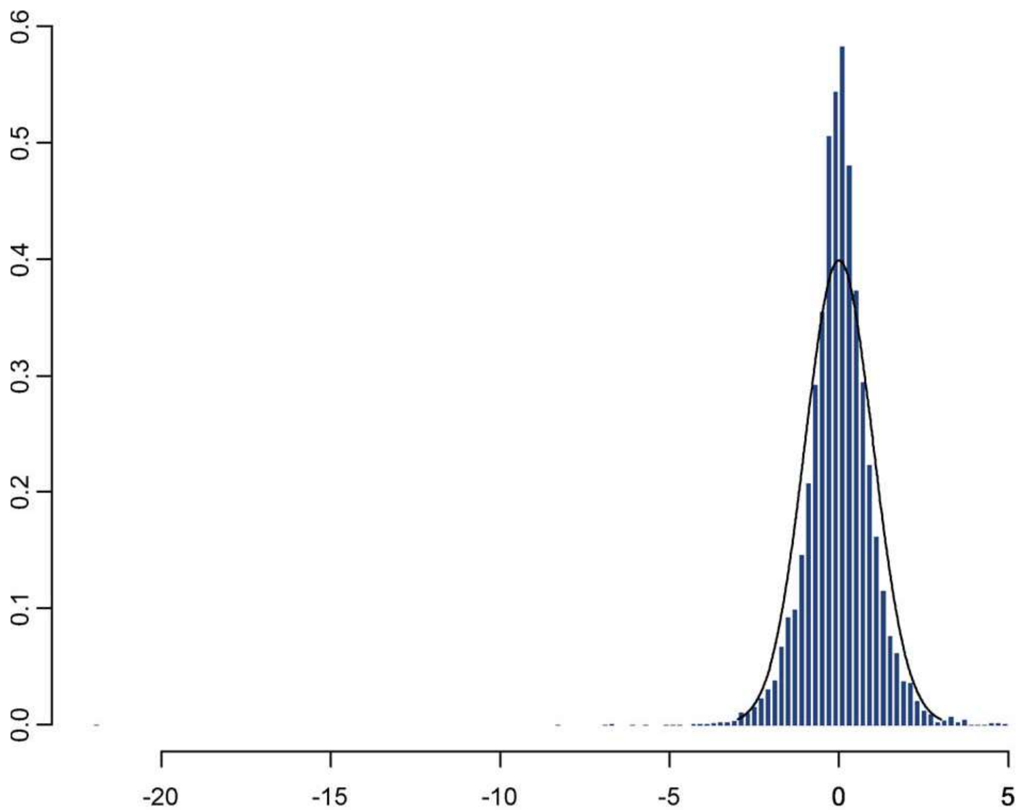


Figure 5.1: Distribution of S&P 500 daily log return from 1980 to 2005 by Kou.(source from *Kou, 2008*)

Kou claimed that the Figure 5.1 is normally distributed. However, in our research where we used more bins in the histogram and more updated data from 1995 to 2014, our findings in Figure 3.3 shows differently compared to Kou's findings.

The result seems to be less symmetry when more bins in the histogram in Figure 3.3. On the other hand, Kou's observations show that the distribution of the negative part had a longer tail than the positive part. This minor difference will impact pricing in simulations in the long run. The difference occurred between our result and Kou's result may due to the changes in the time period. However, our goal is to find and determine a jump model that can be generalized over various time periods and even across different types of financial instrument.

So, the jump model should be able to accommodate the difference between time and instruments.

If the jump event distribution is centered at zero as proposed by Kou, we tend to underprice the financial product due to the jump event is always close to zero, causing the "expected" spike in price is negligible. In order to get higher spike intensity, the arrival of a jump needs to be higher such that the jump components get a larger value that is not around zero. By doing this, the frequency of jump will be high and cause contradiction to our initial definition for jump event. A jump event is a rare, large fluctuation in price, but not a frequently minor changes event like volatility.

Initially, Gibbs sampling technique is used to calibrate out the values of parameters from the market data, which is the drift and volatility of the model itself, the jump arrival and the intensity and volatility of the jump. The variables used to represent each parameter are μ, σ, λ, μ_{jump}, and σ_{jump}.

The paper from Juliet G. D'Cunha, (D'Cunha, J. G., and Rao, K. A., 2014) shows the approach to determine the volatility of stock price can be done with Bayesian inference. While his results show it is best to use inverse gamma for volatility. The distribution of jump is determined as normal distribution here and the arrival of a jump is Poisson distribution. Table 2.1 from Section 2.7 shows the conjugate prior used for different parameters.

Chan and Wong (Chan and Wong, 2006) show the possibility to simulate and retrieve Dow Jones indexes parameters needed for jump diffusion model simulations. However, the Gibbs sampling method shows its limitation in converging of parameters. According to Table 4.1 from Section 4.1, we show that Gibbs sampling method could handle the "jump arrival" well but fails to differentiate the clustering effects between "volatility" of underlying assets and the "jump intensity" from jump event.

We observed that the same path and pattern of a stock price could be simulated using different sets of parameters. For example, a configuration of high jump's intensity and low asset's volatility can simulate a stocks path. This path might be able to attain by a slightly lower jump's intensity but higher asset's volatility.

The results attained by Gibbs sampling method is not constant with every calculation and it causes Gibbs sampling method fails to choose the best solution from the possible solutions. Therefore, we improvised into an empirical method that enables us to distinguish noise and jump intensity.

Isolation of stochastic terms is difficult by means of Gibbs sampling method, and hence we choose a simpler way to separate the clustering effects by using the empirical method. A brief explanation will be shown, while the details can be retrieved in Section 3.5.1.

For further research in the future, we suggest that this method of isolation can be further improved by using stochastic ordering method. Klenke and Mattner (Klenke, A. and Mattner, L., 2010) shows five different methods on stochastic ordering, including the application in the Markov Chain. This might be an alternative way for isolation of stochastic terms.

The empirical method we introduced focus on separation of the intensity of jump, jump arrival with the asset's volatility. In order to capture each jump spikes precisely with our empirical method, we had separated jump spike intensity and volatility with two threshold lines. Daily log return that is between the threshold lines would be classified under asset's volatility. If it exceeded the threshold lines, it is identified as a jump spike.

The threshold line is based on the median of daily log return of the year. Therefore, it is different for each asset and each year. Mean of daily log return is not used here to avoid jump spike being included within the threshold lines calculation. We had shown the results from empirical method in Table 3.7 and

3.8. They show that the occurrence and intensity of jump are distinct for both different directions. Hence, we had solved our first question in problem statement where the jump distribution is not symmetry. We proceed to first objective and modify the model that incorporates piece-wise “double” normal jump into the jump diffusion model.

We modified the distribution of jump event into piece-wise “double” normal distribution for the jump diffusion model as shown in Equation 3.5. This model retains its leptokurtic feature of the underlying assets and we are able to recover and reproduce the parameter value, similar to both Kou’s jump model and GBM. The reason we choose piece-wise “double” normal distribution over double exponential distribution is based on the results we get from Figure 3.3. Both positive and negative directions fit with double normal distribution as shown in Figure 3.4 rather than double exponential which with the peak allocated in the middle.

With piece-wise “double” normal distribution, the height, width and local of each positive and negative normal distribution can be modified easily compare to the exponential distribution. The flexibility of piece-wise “double” normal distribution allow us to fit the model into different types of assets with different types of jump events.

For example, Figure 4.3 and Figure 4.4 show that DJI index and NASDAQ had their positive daily log return’s distribution denser when further from zero. While Figure 4.5 and Figure 4.6 show that Oil & Gas index and the FTSE index had their positive daily log return higher and closer to zero.

In Section 3.6, this research had demonstrated the effect of jump on European call option, on different jump intensity and occurrence. In Section 4.3.2, the results from Figure 4.7 shows that the drift of jump, μ_{jump}, arrival of jump, λ and its direction will affect the option payoff. Higher drift resulting in an elevated payoff. An increase in the arrival of an upward jump will increase the option payoff as well.

During the economic recession, the arrival of a downward jump event largely increases relative to upwards, causing the prices of underlying assets or indexes greatly reduced. This means that, whenever we are expecting an increase of downwards jump from the stock or market, a larger risk and fluctuation of price should be expected.

In this research, we investigate the impact of an upward positive jump to the European call options based on indexes. The findings show a similar behavior to the previous simulation, where an increase in the arrival of an upward jump or jump intensity will increase the option price significantly. In Table 4.6 and Table 4.7, we have shown that the arrival of jump events will affect the pricing model of the European call option. For the European call option, both simulated data created with Dow Jones index and S&P 500 changes drastically when there are jump events. The changes could be from fifty percent and exceed even a hundred percent compared with the price without jump event.

The modified jump diffusion model does not solely fit in European call options only. It can be used in the pricing model of other financial securities as well. The next financial security we use is the annuity. The price of annuity is calculated with the Heston model. The reward function of the Heston model is depending on market performance as well. This shows that the price of the annuity will be affected by extreme events.

Under the application of piece-wise "double" normal jump diffusion model, the price of annuity is shown in results in Section 4.4. The reward function is shown to be higher when the underlying assets are undergoing a negative jump event. The price of an annuity would be undervalued if we did not account for downward jump events. For example, we refer back to Figure 4.11 and Table 4.8 from the previous section. During the year 2008, the maximum price of annuity should be around 3.08 rather than 2.403 which without considering the risk of the jump. The price was undervalued with 20%.

On the other hand, the price of annuity should be expecting a drop when it is undergoing a positive jump event. The price of an annuity would be higher if we calculate using GBM. In this case, we can say that the annuity is overvalued, as the risk of lost being undertaking is far larger than what they are expecting.

The price of a financial security should be measured and considered thoroughly, whether that the underlying assets it used are under the risk of jump events. Hence, it is important to consider piece-wise "double" normal jump model rather than GBM when dealing with market derivatives.

Besides European call option and annuity, there are other derivatives that the modified jump diffusion model able to fit in. However, the type of instrument used needed to be determined carefully for better risk managing and lost preventing. A research was done by Dante and Steven (Lomibao, D., and Zhu, S., 2005) shows that path-dependent instrument is sensitive to extreme jump events. Hence whether it is a path-dependent instrument or not will affect the outcome of "fair" price heavily.

Examples of the path-dependent instrument are American options and barrier options. American options can be exercised during any time within the expiration period, unlike European that can only be exercised upon expiration. Barrier options can be either knock-in or knock-out. A knock-in barrier option contains no value until the underlying reaches a certain price. Knock-out barrier option will expire worthless if the underlying asset exceeds a certain price. This will limit both the profits for the holder and limit losses for the writer.

Now we had compared modified jump diffusion model and GBM for European call option and annuity. If we apply the modified jump diffusion model to the path-dependent instrument, what would be the expected impact? Taking barrier options as an example, what should we expect when we are using GBM to simulate the price. Will the price touch the barriers as frequent as the price that is simulated by the jump diffusion model?

Barrier options with jump diffusion model would have larger fluctuation in price with the arrival of extreme jump event. If the underlying asset is highly volatile, the price will jump up and down rapidly and the frequency of extreme jump event increase as well. As the intensity of jump goes larger, the chance of surpassing the barrier will be higher.

If we are using the same set of data for GBM simulation, the fluctuation will not be as big as the jump diffusion model. The chance that the price will hit the barrier will be relatively lower. In the future, we will investigate the effects of jump diffusion model on path dependent instrument.

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