AN INVENTORY MODEL WITH TIME-VARYING DEMAND AND RECYCLING

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A project report submitted in partial fulfilment of the requirements for the award of Bachelor of Science (Honours) Applied Mathematics with Computing

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DECLARATION

I hereby declare that this project report is based on my original work except for citations and quotations which have been duly acknowledged. I also declare that it has not been previously and concurrently submitted for any other degree or award at UTAR or other institutions.

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APPROVAL FOR SUBMISSION

I certify that this project report entitled **"AN INVENTORY MODEL WITH TIME-VARYING DEMAND AND RECYCLING"** was prepared by **LIM XINRU** has met the required standard for submission in partial fulfilment of the requirements for the award of Bachelor of Science (Honours) Applied Mathematics with Computing at Universiti Tunku Abdul Rahman.

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ABSTRACT

Inventory models are excellent examples to use mathematical models in order to solve real world problems. They are used frequently in any business to determine the optimal level of inventories, which are the stocks, so as to minimize the total inventory cost.

In this project, the most general inventory model with time-varying demand and recycling has been built, which is the inventory model with multiple production and remanufacturing set-ups per cycle. The production set-ups produce new products from scratch, while the remanufacturing set-ups utilize returned items from the returned cycle to remanufacture them to produce products which are considered as good as new. All products are produced, remanufactured and returned at constant rates, while the demand rate is an arbitrary function of time. The goal is to formulate a total cost per unit time function to find the minimum cost of the model. Since the total cost per unit time function is a function of the acceptable returned quantity, it is plotted against the variable in order to prove the optimality of the total cost per unit time function. Other than that, comparisons between several policies with different production and remanufacturing set-ups per cycle have been done to observe the optimal policy that gives the minimum cost. Finally, sensitivity analysis has been performed to show that the inventory model built is robust.

Python is used to compute all calculations and plot all visualizations in this report. Python is a high level programming language that is easily interpreted and understood by beginner programmers. It has various data science libraries that make the process of complex computations to be done effortlessly and effectively in a short time. The optimization function in the SciPy library is used to calculate the optimal value of the total cost per unit time and the Matplotlib library is used to plot the graphs.

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LIST OF SYMBOLS / ABBREVIATIONS

P_m	production rate
R	return rate
P_c	remanufacturing rate
D	demand rate
I_m	inventory level for manufactured items
I_R	inventory level for returned items
I_c	inventory level for remanufactured items
Q	acceptable returned quantity
c_m	unit cost, which includes materials cost
s_m	unit production cost, which includes labour, machinery, etc.
h_m	unit holding cost per unit time of manufactured stock
k_m	production set-up cost per cycle
c_R	unit cost, which includes purchase cost
h_R	unit holding cost per unit time of returned stock
k_R	order cost per cycle
s_c	unit remanufacturing cost
h_c	unit holding cost per unit time of remanufactured stock
k_c	remanufacturing set-up cost per cycle
T	last point of time in the cycle
TCUT	total cost per unit time
(m,n) model	inventory model with m remanufacturing, n production set-ups per cycle

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CHAPTER 1

INTRODUCTION

1.1 General Introduction

Inventory models are excellent examples to use mathematical models in order to solve real world problems. They are used frequently in any business to determine the optimal level of inventories, which are the stocks, so as to minimize the total inventory cost. When an effective inventory management is implemented, it can improve the sales with excellence productions (Nemtajela and Mbohwa, 2016). There are various types of inventory models which have been established with different types of stocks and number of set-ups. These inventory models are mainly divided into two categories, one with time-varying demand and the other with constant demand. This report considers time-varying demand which is continuous throughout the cycle. Although discrete demand is more realistic, it is much more complicated to be analysed.

On top of that, recycling factor is also included in this research. The reason being is that nearly half of the population on Earth are starting to be conscious about the environment in this day and age, making recycling items to be a norm as well as a trend. Products to be recycled can range from small items like paper cups, to large items like refrigerators. Many manufacturing companies have policies for collecting used products and reusable parts from the consumers who purchased their product. These items collected from the consumers are reused in the production of new products. They serve as an important source for production in the supply chain, other than the procurement i.e. purchasing manufacturing materials from other parties. Therefore, recycling can be considered as a factor in inventory models and they are known as reverse logistics.

A reverse logistics model normally includes three stocks, which are the manufactured stock, remanufactured stock and returned stock. The manufactured stock contains newly produced items, remanufactured stock involves remanufactured items which uses returned items as materials and are considered as good as new and the returned stock contains reused items or parts collected from the consumers (Bouras and Tadj, 2015). This is the type of inventory model that will be built throughout this project.

1.2 Problem Statement

The main problem statement of this project is “How to build a general inventory model with time-varying demand and recycling?”. After building the general model, the question interested is “What are the best numbers of production and remanufacturing set-ups that give the minimum cost?”. Since there are many parameters that need to be set, “What is the impact of different parameters to the optimal result?”

1.3 Aim and Objectives

The ultimate goal of this project is to build a general inventory model with time-varying demand and recycling that has multiple production and remanufacturing set-ups per cycle. After successfully building one, the optimal numbers of production and remanufacturing set-ups per cycle that give the minimum cost need to be found. Other than that, with a different set of parameters, the optimal result will be observed.

1.4 Scope and Limitations of Study

When building inventory models, several assumptions are considered so as to simplify the formulation process. For example, environmental factors and back-orders will not be considered throughout this research. Other than that, all the production, remanufacturing and return rates will be treated as constants, although they can be arbitrary functions of time. Only the basic cost components will be involved, which include item costs, production cost, remanufacturing cost, holding costs and set-up costs.

After deciding on the assumptions, formulation of models may be started. The formulation process will be mentioned in the methodology chapter. When models are finalized, the function of total cost per unit time needs to be found, which is our goal to minimize it.

1.5 Work Plan

Table 3.1: Work Plan of Project I

Task	Week													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Reading and collecting research materials														
Work on proposal and interim report														
Mock presentation for proposal														
Submission of proposal														
Building simple (1,1) inventory model														
Testing and coding the model built														
Mock presentation for interim report														
Submission of interim report														
Oral presentation of Project I														

Table 3.2: Work Plan of Project II

Task	Week													
	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Reading and collecting research materials														
Prepare final report														
Building (*1*,*n*) model														
Testing (*1*,*n*) model														
Building (*m*,*1*) model														

Testing (m,1) model														
Building (m,n) model														
Testing (m,n) model														
Preparation of project poster														
Submission of project poster														
Submission of final report														
Oral presentation of Project II														

Above tables are the proposed work plans for the whole project.

CHAPTER 2

LITERATURE REVIEW

Recycling system in the supply chain or mostly known as reverse logistics, is not a new policy in the industries. It has been introduced long ago before recycling becomes a trend. Therefore, experts have also been developing different inventory models with recycling since then.

El Saadany and Jaber (2010) have proposed inventory models with the purchasing price of collected items to be a decision variable subject to the return rate of used items which follows a demand-like function. They used these assumptions to extend the model of Dobos and Richter (2003, 2004, 2006). Two models have been proposed: single remanufacturing, single production set-ups per cycle (*1,1*) and *m* remanufacturing, *n* production set-ups per cycle (*m,n*). When same values are substituted into the model of Dobos and Richter (2003) and their own model, different results are obtained. The model of Dobos and Richter (2003) shows that a pure remanufacturing approach is the best. On the other hand, for their own proposed model, they found that a mixed approach produced the least cost. This happens because high quality returned items are bought at a low price. From this paper, we can see that different assumptions in parameters can produce different optimal result. If an inventory model is to be applied to the real world, the actual system description is essential to do case study in order to construct assumptions and parameters that fit the system perfectly, so that the accurate optimal policy can be found.

Similar inventory model have been proposed by Alamri (2011), it has one remanufacturing and one production set-up per cycle which is a (*1,1*) model. The difference between the models proposed by El Saadany and Jaber (2010) and the model proposed by Alamri (2011) is that Alamri (2011) treated the purchasing price of collected items as a constant parameter but not as a decision variable subject to the return rate of used items. He considered deterioration of the inventories as a factor, which makes the model to be more realistic for inventories that may turn bad overtime. Deterioration rates of manufactured stocks, remanufactured stocks and returned stocks are taken as arbitrary functions of time. Returned items are only accepted if the item has passed a certain quality level. Therefore, he assumed that remanufactured items are as good as new. Through several numerical verifications, he found out that when return rate of used items are dependent on the acceptance quality level and purchasing price, the combination of remanufacturing and production approach is better as compared to either pure remanufacturing or pure production. Therefore, he also noted the same result as El Saadany and Jaber (2010) that a mixture of production and remanufacturing strategy is more profitable.

Furthermore, in the research paper by El Saadany, Jaber and Bonney (2013), they focused on the assumption that the number of times that an item can be repaired is unlimited. They were to find out how many times a given product should be restored in order to minimize the total cost. They developed a mathematical expression to estimate the number of times a product can be repaired. They modified two classic models, which are model of Richter (1997) and model of Teunter (2001) by changing the infinite recovery to a limited number of times. They noted that the results produced following the assumption that an item can be repaired infinitely are very distinct from the ones following a finite restoration assumption. For the model of Richter (1997), infinite restoration produces a lower total cost. However, when costs such as cost to increase the life of an item and waste disposal cost are taken into account, the finite restoration performs better. The same theory applies to the model of Teunter (2001) as well. Therefore, it can be concluded that when there is no investment cost involved, the total cost is lower for the case with infinite remanufacturing, whereas when the cost is included along with the disposal cost, infinite restoration is not a good decision.

From another point of view, Bazan, Jaber and Zanoni (2016) has written a review on the inventory models for reverse logistics in an environmental perspective. They mentioned that most of the models proposed did not involve environmental aspects like greenhouse-gas (GHG) emissions, landfill disposal, energy usage and so on. These should be taken into account as well to get an idea on the potential benefits and enhancement on existing models. By taking environmental factors into account, inventory models can be more realistic as they represent the issues of the real world. The potentiality of current models to be extended from a single objective, i.e. to minimize the total cost, to multiple objectives which include environmental objectives should be figured out. Environmental factors are one of the aspects that are ignored by many in this generation. Most of them are aware of this issue but do not wish to look into it, being worried that it will be less profitable. This should be changed as it is our responsibility as a citizen of the Earth to be involved in causing less harm to our homeland.

Other than the regular single-channel strategy where new items and remanufactured items are distributed from the retailer, Batarfi, Jaber and Aljazzar (2017) have considered a dual-channel strategy which is a norm recently, where products are distributed not only from the retailer but also through an online channel. They have adopted the online channel to offer the remanufactured products as well as customized products. The remanufactured products are produced by a third-party logistics provider. On the other hand, the retailer will only distribute the newly manufactured items. They assumed that returned items that cannot be repaired are disposed off. Hence, costs such as inventory cost, remanufacturing cost, outsourcing cost and disposal cost are considered. When comparing a single-channel strategy with a dual-channel strategy, it is obvious that the dual-channel strategy decreases the cost and increases the profit. This is because for the dual-channel strategy, there will be no need to stock the remanufactured items in the retailer's side which is the same for the returned items from the consumers. Different return policies, i.e. full refund, partial refund, or no refund, are proposed as well to find out which policy maximizes the total profit of the system. They have found that the higher the refund, the higher the profits. This is due to the fact that with a higher refund, consumers are willing to recycle the items which can be remanufactured. Therefore, more remanufactured products can be produced and sold, hence higher profits.

CHAPTER 3

METHODOLOGY

Before building an inventory model, several assumptions need to be set since not all factors of the real world problem can be considered. The variables, parameters, type of costs involved and their respective notations are required to be stated clearly to let the formulation process becomes smoother. These assumptions and parameters will be applied to all the models built in this research.

In order to build a general (m,n) model, it would be easier to combine a ($m,1$) model with a ($1,n$) model, rather than formulating the (m,n) model from scratch. The process of formulating an inventory model involves a standard procedure, i.e. the 3 inventory models go through the same procedure to be built.

Before getting into the formulation of any inventory model, the idea about the type of inventory model that is to be built needs to be clear. This is to make sure that a simple sketch on the inventory variations of the model can be done to get an overview about the model. From there, the inventory levels for manufactured, remanufactured and returned items, which are functions of time, can be obtained easily by solving the differential equations on the changes of inventory levels.

Since the goal is to minimize the total cost per unit time, a function for the total cost per unit time which includes all the cost components needs to be formulated. In order to do that, the inventory holdings for each stocks during the time period needs to be known, since holding costs are included as parameters. This is represented by the area under the curves for each stock cycles. It is obvious that integration needs to be done on the functions of all the inventory levels to find out the inventory holdings.

Furthermore, in order to ease the process of minimizing the total cost per unit time, the function obtained needs to be converted into a function of one variable. With that, the minimum total cost per unit time can be obtained by finding the optimal value of the one variable.

After the function for the total cost per unit time is formulated, the built model needs to be verified numerically to test the practicality of the model.

Since it involves many variables and equations, calculation by hand is extremely tedious. Python can be used to handle the complex computations with its existing data science libraries such as NumPy and SciPy.

Once the values of all the parameters and the demand function are set, they can be substituted into the function for total cost per unit time. The Matplotlib library in Python can be used to obtain a plot of total cost per unit time against the variable of the total cost per unit time function, so that we are able to make sure that there exists a minimum point. Optimization can then be carried out using the minimization function in Python.

After obtaining the general (m,n) model, comparisons between inventory models with different m and n can be done to observe which policy performs the best when using the same set of parameters. The comparison will be repeated by using different parameters.

Last but not least, sensitivity analysis will be performed to understand the relationship between parameters and the optimal number of production set-ups and remanufacturing set-ups per cycle.

CHAPTER 4

INVENTORY MODEL WITH SINGLE REMANUFACTURING, MULTIPLE PRODUCTION SET-UPS PER CYCLE, (*1*,*N*) MODEL

4.1 Introduction

We will start by building an inventory model with single remanufacturing and multiple production set-ups per cycle, which is the (*1*,*n*) model. It involves obtaining recycled items from the returned stock only once to be remanufactured and *n* procurements throughout the whole time period.

In order to get a clearer picture about the inventory variations of the (*1*,*n*) model, the overview of the inventory variations is plotted through Python, by taking *n* to be 2. This is shown in Figure 4.1 below.

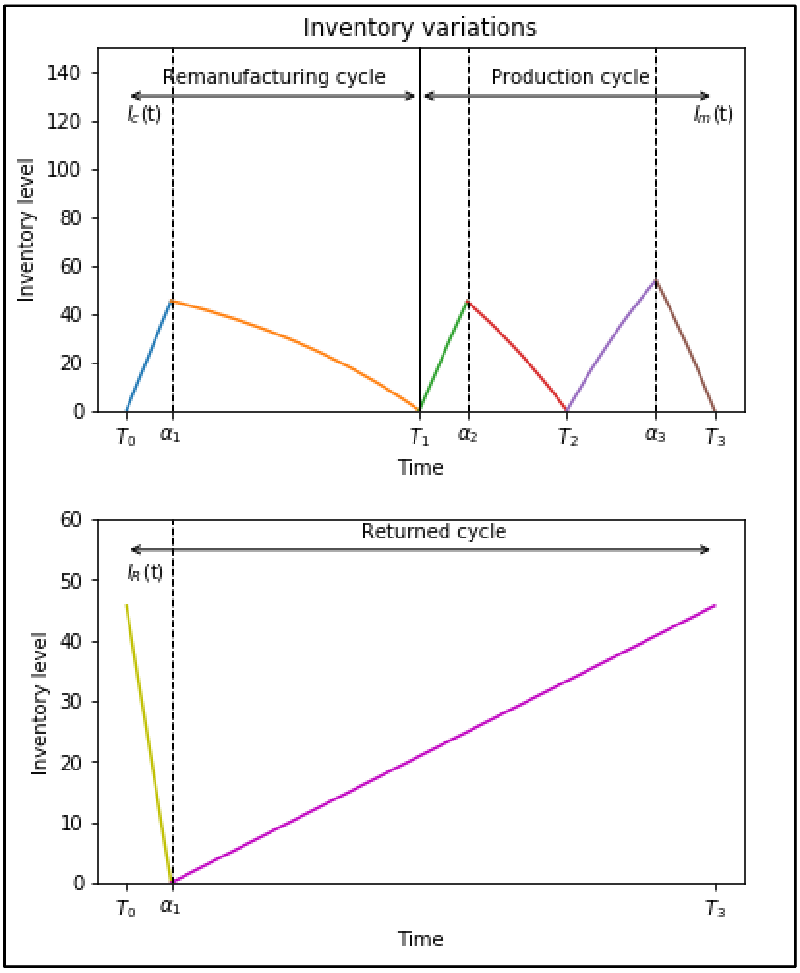

Figure 4.1: Overview of Inventory Variations of a (*1*,*2*) Policy

From above, there is only one downward curve in the remanufacturing cycle but two downward curves in the production cycle, which represents the two production set-ups per cycle. To further explain the plot, the rising blue curve from T_0 to α_1 represents the increased remanufactured stock corresponding to the yellow decreasing line in the returned cycle as the items are used to produce remanufactured products. While the remanufactured stock increases, demand is satisfied as well during T_0 to α_1, hence it is not a straight line but a slight curve. The orange plunging curve from α_1 to T_1 represents that the demand is being satisfied and the pink inclining line in the returned cycle from α_1 to T_3 represents that recycled items are being collected. The same concept is applied for the production cycle as well, where both the green (T_1 to α_2) and purple (T_2 to α_3) increasing curves represent procurements and demand at the same time, while the red (α_2 to T_2) and brown (α_3 to T_3) declining curves show that the produced items are used to satisfy demands.

After understanding the inventory variations of the (*1,2*) policy, we can use it as a reference to make a generalization to obtain the (*1,n*) model and start the formulation process.

4.2 Formulation of (*1,n*) Model

Assumptions and notations:

1. Production, remanufacturing and return rates are denoted as P_m, P_c and R respectively.
2. The demand rate $D(t)$ is satisfied by production of new items and remanufactured items which are considered as good as new.
3. The last point of time in the whole cycle is denoted as T, which is equivalent to $T_{n+1.}$
4. The demand rate is an arbitrary function of time, while production, return and remanufacturing rates are constant parameters.
5. Only recycled items which have passed a certain acceptable quality will be accepted into the returned stock.
6. The inventory levels for manufactured, remanufactured and returned items at time, t are depicted as $I_m(t)$, $I_c(t)$ and $I_R(t)$, respectively.

7. The time-weighted inventory holdings for the time period $a \leq t \leq b$ for manufactured, remanufactured and returned items are denoted as $I_m(a,b)$, $I_c(a,b)$ and $I_R(a,b)$, respectively.
8. Constraints to formulate the model are as follows:
   $P_c > D(t)$, $P_m > D(t)$, $D(t) > R$, $P_c > R$, $D(t) \neq 0$, $R \neq 0$, $\forall t \geq 0$.
9. Shortages are not allowed.
10. Each repeated set-ups in the cycle has the same time length.
11. The cost parameters for the manufactured stock are as follows:
    c_m = Unit cost, which includes materials cost.
    s_m = Unit production cost, which includes labour, machinery, etc.
    h_m = Unit holding cost per unit time.
    k_m = Set-up cost per cycle.
12. The cost parameters for the remanufactured stock are as follows:
    s_c = Unit remanufacturing cost, which includes labour, machinery, etc.
    h_c = Unit holding cost per unit time.
    k_c = Set-up cost per cycle.
13. The cost parameters for the returned stock are as follows:
    c_R = Unit cost, which includes purchase cost.
    h_R = Unit holding cost per unit time.
    k_R = Order cost per cycle.

The changes in the inventory levels are governed by the following differential equations:

$T_0 \leq t < \alpha_1$:

$$\frac{dI_c(t)}{dt} = P_c - D(t), \text{ with the initial condition } I_c(T_0) = 0, \tag{1}$$

$\alpha_1 \leq t \leq T_1$:

$$\frac{dI_c(t)}{dt} = -D(t), \text{ with the ending condition } I_c(T_1) = 0, \tag{2}$$

$T_k \leq t < \alpha_{k+1}$:

$$\left(\frac{dI_m(t)}{dt}\right)_k = P_m - D(t), \text{ with the initial condition } I_m(T_k) = 0, \tag{3}$$

$\alpha_{k+1} \leq t \leq T_{k+1}$:

$$\left(\frac{dI_m(t)}{dt}\right)_k = -D(t), \text{ with the ending condition } I_m(T_{k+1}) = 0, \tag{4}$$

where $k = 1, 2, \ldots, n$

$T_0 \le t < \alpha_1$:

$$\frac{dI_R(t)}{dt} = -P_c + R, \text{ with the ending condition } I_R(\alpha_1) = 0, \text{ and} \quad (5)$$

$\alpha_1 \le t \le T$:

$$\frac{dI_R(t)}{dt} = R, \quad \text{with the initial condition } I_R(\alpha_1) = 0. \quad (6)$$

The solutions of the above differential equations are:

$T_0 \le t < \alpha_1$:

$$I_c(t) = P_c(t - T_0) - \int_{T_0}^{t} D(u)\,du \quad (7)$$

$\alpha_1 \le t \le T_1$:

$$I_c(t) = \int_{t}^{T_1} D(u)\,du \quad (8)$$

$T_k \le t < \alpha_{k+1}$:

$$I_m(t)^{[k]} = P_m(t - T_k) - \int_{T_k}^{t} D(u)\,du \quad (9)$$

$\alpha_{k+1} \le t \le T_{k+1}$:

$$I_m(t)^{[k]} = \int_{t}^{T_{k+1}} D(u)\,du \quad (10)$$

where k = 1, 2, …, n

$T_0 \le t < \alpha_1$:

$$I_R(t) = (P_c - R)(\alpha_1 - t) \quad (11)$$

$\alpha_1 \le t \le T$:

$$I_R(t) = R(t - \alpha_1) \quad (12)$$

respectively.

In order to find the inventory holdings for each stocks, let

$$I(t_1, t_2) = \int_{t_1}^{t_2} I(u)\,du,$$

then from (7) – (12) we have:

$T_0 \le t < \alpha_1$:

$$I_c(T_0, \alpha_1) = P_c(\alpha_1 - T_0)^2 - \int_{T_0}^{\alpha_1} (\alpha_1 - u) D(u)\,du \quad (13)$$

$\alpha_1 \le t \le T_1$:

$$I_c(\alpha_1, T_1) = \int_{\alpha_1}^{T_1} (u - \alpha_1) D(u)\,du \quad (14)$$

$T_k \le t < \alpha_{k+1}$:

$$I_m(T_k, \alpha_{k+1}) = P_m(\alpha_{k+1} - T_k)^2 - \int_{T_k}^{\alpha_{k+1}} (\alpha_{k+1} - u) D(u)\, du \tag{15}$$

$\alpha_{k+1} \le t \le T_{k+1}$:

$$I_m(\alpha_{k+1}, T_{k+1}) = \int_{\alpha_{k+1}}^{T_{k+1}} (u - \alpha_{k+1}) D(u)\, du \tag{16}$$

where k = 1, 2, …, n

$T_0 \le t < \alpha_1$:

$$I_R(T_0, \alpha_1) = \frac{P_c - R}{2}(\alpha_1 - T_0)^2 \tag{17}$$

$\alpha_1 \le t \le T$:

$$I_R(\alpha_1, T) = \frac{R}{2}(T - \alpha_1)^2 \tag{18}$$

respectively.

Without loss of generality, set $T_0 = 0$. The cost components per cycle for the inventory model are as follow:

Items cost $= c_m \sum_{k=1}^{n} \int_{T_k}^{\alpha_{k+1}} P_m\, dt + c_R \int_0^T R\, dt$

$$= c_m P_m \sum_{k=1}^{n} (\alpha_{k+1} - T_k) + C_R R T \tag{19}$$

Production cost $= s_m \sum_{k=1}^{n} \int_{T_k}^{\alpha_{k+1}} P_m\, dt$

$$= s_m P_m \sum_{k=1}^{n} (\alpha_{k+1} - T_k) \tag{20}$$

Remanufacturing cost $= s_c \int_0^{\alpha_1} P_c\, dt$

$$= s_c P_c \alpha_1 \tag{21}$$

Holding cost

$= h_c[I_c(0, \alpha_1) + I_c(\alpha_1, T_1)] + h_m[\sum_{k=1}^{n} I_m(T_k, \alpha_{k+1}) + I_m(\alpha_{k+1}, T_{k+1})] + h_R[I_R(0, \alpha_1) + I_R(\alpha_1, T)]$

$$= h_c\left[P_c(\alpha_1)^2 - \int_0^{\alpha_1}(\alpha_1 - u)D(u)\, du + \int_{\alpha_1}^{T_1}(u - \alpha_1)D(u)\, du\right] + h_m\left[\sum_{k=1}^{n} P_m(\alpha_{k+1} - T_k)^2 - \int_{T_k}^{\alpha_{k+1}}(\alpha_{k+1} - u)D(u)\, du + \int_{\alpha_{k+1}}^{T_{k+1}}(u - \alpha_{k+1})D(u)\, du\right] + h_R\left[\frac{P_c - R}{2}(\alpha_1)^2 + \frac{R}{2}(T - \alpha_1)^2\right] \tag{22}$$

Thus, the total cost per unit time (TCUT) of the inventory model during the cycle [0,T], as a function of T_k and n, say $Z(T_k, n)$ where k represents integers from 1 to n+1, is given by the sum of (19) – (22) divided by T:

$Z(T_k, n) = \frac{1}{T}\{s_c P_c \alpha_1 + (c_m + s_m) P_m \sum_{k=1}^{n} (\alpha_{k+1} - T_k) + c_R R T$

$$+h_c\left[P_c(\alpha_1)^2 - \int_0^{\alpha_1}(\alpha_1 - u)D(u)\,du + \int_{\alpha_1}^{T_1}(u - \alpha_1)D(u)\,du\right]$$
$$+h_m\left[\sum_{k=1}^{n} P_m(\alpha_{k+1} - T_k)^2 - \int_{T_k}^{\alpha_{k+1}}(\alpha_{k+1} - u)D(u)\,du + \int_{\alpha_{k+1}}^{T_{k+1}}(u - \alpha_{k+1})D(u)\,du\right] + h_R\left[\frac{P_c - R}{2}(\alpha_1)^2 + \frac{R}{2}(T - \alpha_1)^2\right]$$
$$+k_c + nk_m + k_R\} \qquad (23)$$

Our goal is to find T_k that minimizes $Z(T_k, n)$ given by (23) with the constant n. In order to simplify it, we can convert it to a function of only one variable, since all the time variables T_k are related to each other through the relations:

$$T_{k-1} < T_k, \qquad k = 1, 2, \ldots, n+1 \qquad (24)$$

$$P_c(\alpha_1 - 0) - \int_0^{\alpha_1} D(u)\,du = \int_{\alpha_1}^{T_1} D(u)\,du \qquad (25)$$

$$P_m(\alpha_{k+1} - T_k) - \int_{T_k}^{\alpha_{k+1}} D(u)\,du = \int_{\alpha_{k+1}}^{T_{k+1}} D(u)\,du, \quad k = 1, 2, \ldots, n \qquad (26)$$

$$(P_c - R)(\alpha_1 - 0) = R(T - \alpha_1) \qquad (27)$$

$$\frac{T - T_1}{n} = T_{k+1} - T_k, \qquad k = 2, 3, \ldots, n \qquad (28)$$

Let Q be the acceptable returned quantity for used items in the interval [0,T], then

$$Q = \int_0^T R\,dt = RT \qquad (29)$$

From (29), we note that T is a function of Q, which is given by:

$$T = \frac{Q}{R} = g_{n+1}(Q) \qquad (30)$$

From (27), we can see that α_1 can be determined as a function of T. Hence, from (30), a function of Q:

$$\alpha_1 = \frac{Q}{P_c} = f_1(Q) \qquad (31)$$

From (25), we find that T_1 is a function of α_1, hence a function of Q, from (31), say:

$$T_1 = g_1(Q) \qquad (32)$$

From (28), we see that for all k from 2 to n, T_k is a function of T and T_1, hence:

$$T_k = g_k(Q, n), \qquad k = 2, 3, \ldots, n \qquad (33)$$

From (26), α_{k+1} can be determined as a function of T_k and T_{k+1}, for all k from 1 to n, which from (30), (32) and (33), a function of Q, say:

$$\alpha_{k+1} = f_{k+1}(Q, n), \quad k = 1, 2, \ldots, n \qquad (34)$$

Therefore, by substituting (30) – (34) into (23), we get the TCUT in terms of the variable Q and n which is a constant:

$$\begin{aligned}\mathrm{TCUT}(Q, n) = \frac{1}{g_{n+1}}\{ & s_c P_c f_1 + (c_m + s_m) P_m \textstyle\sum_{k=1}^{n} (f_{k+1} - g_k) + c_R RT \\ & + h_c \left[P_c (f_1)^2 - \int_0^{f_1} (f_1 - u) D(u)\, du + \int_{f_1}^{g_1} (u - f_1) D(u)\, du \right] \\ & + h_m \left[\textstyle\sum_{k=1}^{n} P_m (f_{k+1} - g_k)^2 - \int_{g_k}^{f_{k+1}} (f_{k+1} - u) D(u)\, du + \int_{f_{k+1}}^{g_{k+1}} (u - \right. \\ & \left. f_{k+1}) D(u)\, du \right] + h_R \left[\frac{P_c - R}{2} (f_1)^2 + \frac{R}{2} (g_{n+1} - f_1)^2 \right] \\ & + k_c + n k_m + k_R \} \end{aligned} \quad (35)$$

where $g_{n+1}(Q) = \frac{Q}{R}$ and $f_1(Q) = \frac{Q}{P_c}$.

4.3 Numerical Example

We start the computation by defining the demand function and setting all the parameters with respect to the constraints as stated in the last section.

$D(t) = e^{0.05t}$,	$P_m = 15$,	$R = 0.99$,	$P_c = 13$,
$c_m = 10$,	$s_m = 15$,	$h_m = 10$,	$k_m = 50$,
$s_c = 10$,	$h_c = 10$,	$k_c = 1600$,	
$c_R = 5$,	$h_R = 5$,	$k_R = 1200$.	

However, this is not realistic in practical since the production set-up cost is chose to be much smaller than the remanufacturing set-up cost and order cost to show that multiple production set-ups do decrease the total cost. It is hard to achieve in the real world since it is a very extreme situation to have such price difference. On a bright side, as technologies improve, the production set-up cost may go down since machines are more reliable, causing the resources used to set-up the production cycle to be less.

4.3.1 Optimality of TCUT Function

In order to show that there is a minimum point in the TCUT function shown in (35), an example of (*1,2*) policy is chosen. The TCUT function is plotted against the acceptable returned quantity (Q) within the range from 1 to 80. The range is chosen as such in order to display clearly the "U-shape curve" in the plot, where the minimum point represents the optimal value for Q to minimize the TCUT.

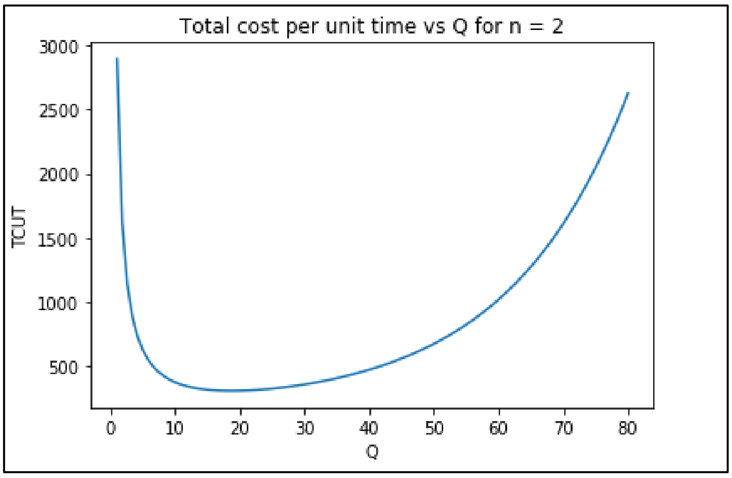

Figure 4.2: Graph of TCUT Versus Q for (*1*,*2*) Policy

The figure above shows that the TCUT plummets quickly when Q increases until the optimal point and rise gradually after that. The decrease in the TCUT is much more rapid than the increase after the optimal Q.

It is challenging to determine the optimal value from the graph above since the minimum point is not obvious. However, it shows a good sign about the inventory model where the approximated optimal Q obtained from the model is not far from the true optimal Q in the real world situation. Since when building an inventory model, the parameter values that are fitted into the model are not accurate representations of the real world situation, i.e. they are approximated values, hence it is vital to obtain the TCUT which is not sensitive to the changes in Q near the approximated optimal Q.

By displaying all TCUT values throughout Q from 1 to 80 through Python, the minimum TCUT of the (*1*,*2*) policy is found to be 310.72 when Q is 18.5556. The table of TCUT values can be found in Table A-1.

4.3.2 Optimal Number of Production Set-ups

With the parameters above, we wish to find the optimal number of production set-ups per cycle (n). In order to do that, optimal Q for all n needs to be found to compute the minimum TCUT of each n. From there, we can observe n with the smallest optimal TCUT. n ranging from 1 to 10 is tested and plotted against their respective minimum TCUT. The plot is shown below:

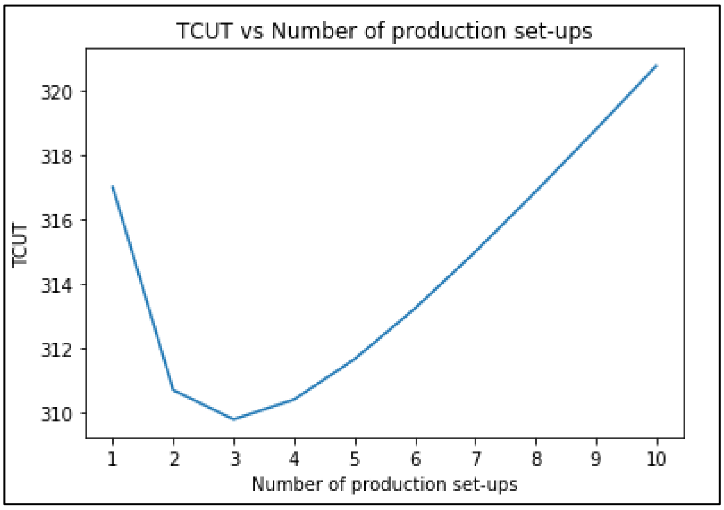

Figure 4.3: Graph of TCUT Versus Number of Production Set-ups

It is obvious that the optimal number of production set-ups per cycle is 3 with the TCUT of around 310. The TCUT increases steadily with the increment of the number of production set-ups, therefore we can say that the range from 1 to 10 is enough to show that 3 is the optimal number of production set-ups per cycle.

4.4 Sensitivity Analysis

We want to observe the relationship between the unit holding cost of the manufactured stock with the optimal number of production set-ups per cycle as well as the effect of production set-up cost on the optimal number of production set-ups per cycle.

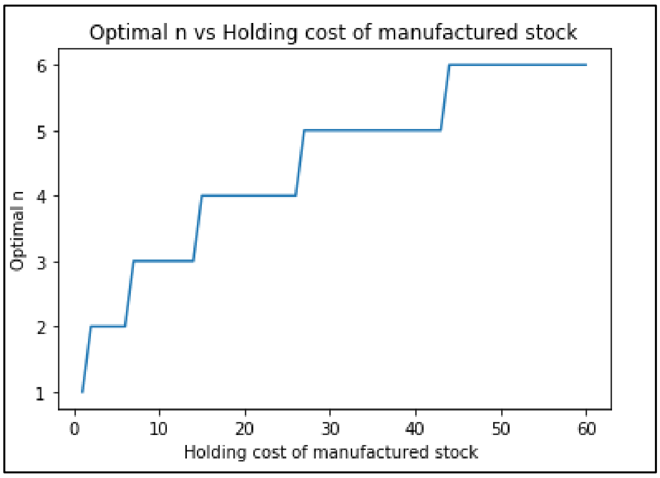

Figure 4.4: Optimal *n* Versus Unit Holding Cost of Manufactured Stock

From the plot in Figure 4.4, we observed a linear relationship between unit holding cost of manufactured stock and optimal *n*. When the unit holding cost increases, the optimal *n* increases as well. This is because when the unit holding cost is high, less stocks are wished to be held in each set-up, hence causing more production set-ups to minimize the total cost (Greeff and Ghoshal, 2004).

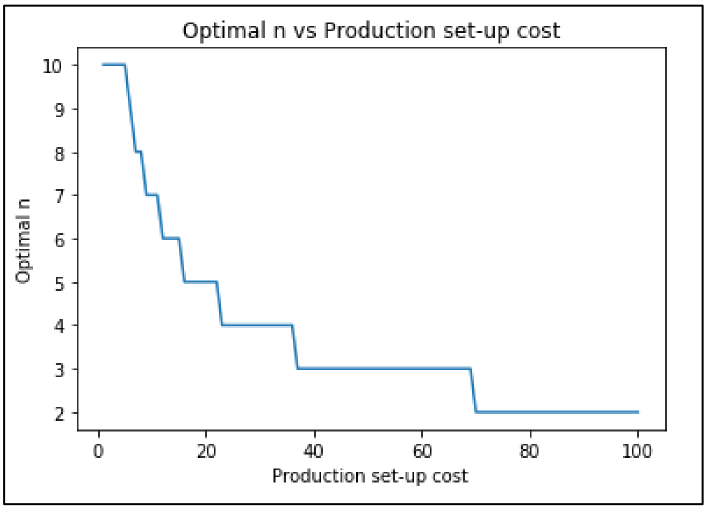

Figure 4.5: Optimal *n* Versus Production Set-up Cost

Conversely, when the set-up cost of production cycle is high, numerous production set-ups is not desired, and hence the optimal *n* will be lower. Therefore, the set-up cost of production cycle and the optimal *n* are inversely related as shown in the figure above.

CHAPTER 5

INVENTORY MODEL WITH MULTIPLE REMANUFACTURING, SINGLE PRODUCTION SET-UP PER CYCLE, (*M*,*1*) MODEL

5.1 Introduction

After looking at the (*1*,*n*) model, we will build the (*m*,*1*) model which is the inventory model with multiple remanufacturing, single production set-up per cycle. Since remanufactured items utilize returned items as materials, the remanufacturing cycle and returned cycle are well-related. *m* remanufacturing set-ups represent that returned items are passed to the remanufacturing cycle from the returned cycle *m* times to produce remanufactured items.

The overview of the inventory variations of a (*2*,*1*) policy inventory model is plotted through Python to understand the relationship mentioned above.

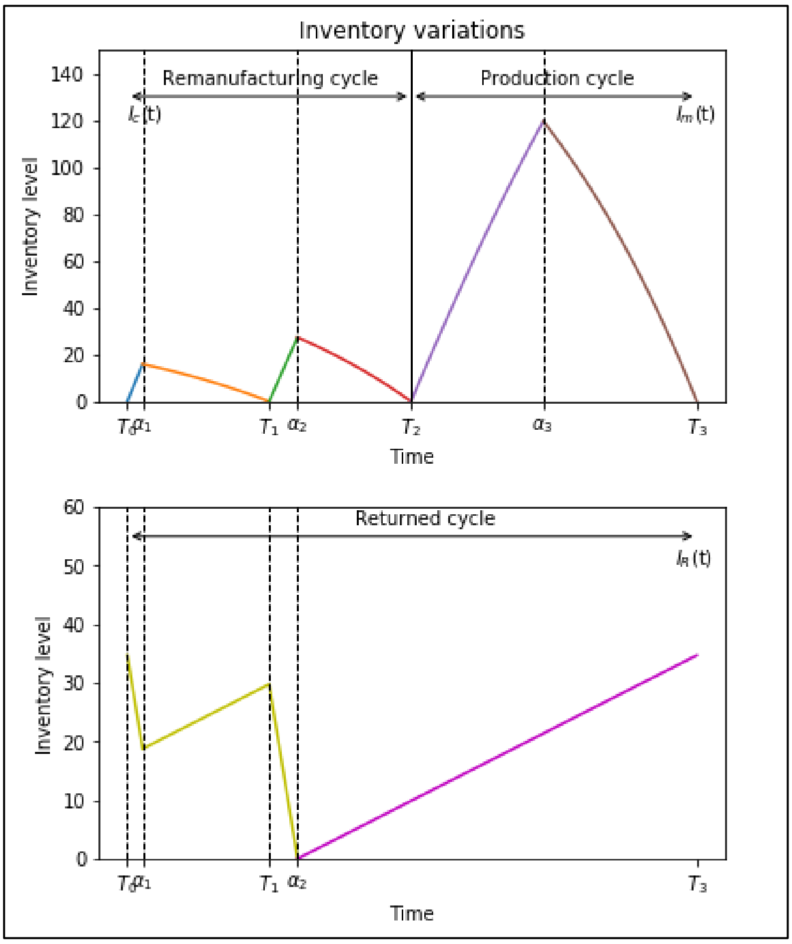

Figure 5.1: Overview of Inventory Variations of a (*2*,*1*) Policy

From Figure 5.1, the blue upward curve in the remanufacturing cycle corresponds to the decreasing line in the returned cycle from T_0 to α_1. This represents that returned items are collected but at the same time are used to satisfy demands by producing remanufactured items. The orange downward curve represents that demands are being satisfied by the remanufactured items produced. This corresponds to the increasing yellow line from α_1 to T_2, where returned items are being collected. While for the second remanufacturing set-up, all returned items will be used up as it is the last remanufacturing set-up in the cycle. After the remanufacturing cycle, there is only one production cycle with procurement that happens only once.

This (*2,1*) policy is used to generalise to the (*m*,*1*) model to start the formulation process.

5.2 Formulation of (*m*,*1*) Model

All assumptions and notations are the same as the formulation of (*1*,*n*) model in section 4.2, except that the last point of time in the whole cycle which is denoted as T is equivalent to $T_{m+1.}$

The changes in the inventory levels are governed by the following differential equations:

$T_{k-1} \leq t < \alpha_k$:

$$\left(\frac{dI_c(t)}{dt}\right)_k = P_c - D(t), \text{ with the initial condition } I_c(T_{k-1}) = 0, \qquad (1)$$

$\alpha_k \leq t \leq T_k$:

$$\left(\frac{dI_c(t)}{dt}\right)_k = -D(t), \text{ with the ending condition } I_c(T_k) = 0, \qquad (2)$$

where k = 1, 2, …, m

$T_m \leq t < \alpha_{m+1}$:

$$\frac{dI_m(t)}{dt} = P_m - D(t), \text{ with the initial condition } I_m(T_m) = 0, \qquad (3)$$

$\alpha_{m+1} \leq t \leq T$:

$$\frac{dI_m(t)}{dt} = -D(t), \text{ with the ending condition } I_m(T) = 0, \qquad (4)$$

$\alpha_m \leq t \leq T$:

$$\frac{dI_R(t)}{dt} = R, \quad \text{ with the initial condition } I_R(\alpha_m) = 0. \qquad (5)$$

However, the inventory levels of returned cycle from T_0 to α_m are not able to be represented by a differential equation since they are all related, where all returned items are accumulating throughout the cycle. Hence, we use another way to represent the inventory levels by observing the plot of inventory variations:

For all k from 1 to m-1,

$$I_R(T_0) = I_R(T) \tag{6}$$

$$I_R(\alpha_k) = I_R(T_{k-1}) - (P_c - R)(\alpha_k - T_{k-1}) \tag{7}$$

$$I_R(T_k) = I_R(\alpha_k) + R(T_k - \alpha_k) \tag{8}$$

$$I_R(\alpha_m) = 0 \tag{9}$$

The solutions of all the differential equations are:

$T_{k-1} \leq t < \alpha_k$:

$$I_c(t)^{[k]} = P_c(t - T_{k-1}) - \int_{T_{k-1}}^{t} D(u)\, du \tag{10}$$

$\alpha_k \leq t \leq T_k$:

$$I_c(t)^{[k]} = \int_{t}^{T_k} D(u)\, du \tag{11}$$

where k = 1, 2, …, m

$T_m \leq t < \alpha_{m+1}$:

$$I_m(t) = P_m(t - T_m) - \int_{T_m}^{t} D(u)\, du \tag{12}$$

$\alpha_{m+1} \leq t \leq T$:

$$I_m(t) = \int_{t}^{T} D(u)\, du \tag{13}$$

$\alpha_m \leq t \leq T$:

$$I_R(t) = R(t - \alpha_m) \tag{14}$$

respectively.

In order to find the inventory holdings for each stocks, let

$$I(t_1, t_2) = \int_{t_1}^{t_2} I(u)\, du,$$

then from (6) – (14) we have:

$T_{k-1} \leq t < \alpha_k$:

$$I_c(T_{k-1}, \alpha_k) = P_c(\alpha_k - T_{k-1})^2 - \int_{T_{k-1}}^{\alpha_k} (\alpha_k - u) D(u)\, du \tag{15}$$

$\alpha_k \leq t \leq T_k$:

$$I_c(\alpha_k, T_k) = \int_{\alpha_k}^{T_k} (u - \alpha_k) D(u)\, du \quad (16)$$

where k = 1, 2, …, m

$T_m \leq t < \alpha_{m+1}$:

$$I_m(T_m, \alpha_{m+1}) = P_m(\alpha_{m+1} - T_m)^2 - \int_{T_m}^{\alpha_{m+1}} (\alpha_{m+1} - u) D(u)\, du \quad (17)$$

$\alpha_{m+1} \leq t \leq T$:

$$I_m(\alpha_{m+1}, T) = \int_{\alpha_{m+1}}^{T} (u - \alpha_{m+1}) D(u)\, du \quad (18)$$

$T_0 \leq t < \alpha_m$:

$$I_R(T_{k-1}, \alpha_k) = \frac{1}{2}[I_R(T_{k-1}) + I_R(\alpha_k)](\alpha_k - T_{k-1}) \quad (19)$$

where k = 1, 2, …, m

$$I_R(\alpha_k, T_k) = \frac{1}{2}[I_R(\alpha_k) + I_R(T_k)](T_k - \alpha_k) \quad (20)$$

where k = 1, 2, …, m-1

$\alpha_m \leq t \leq T$:

$$I_R(\alpha_m, T) = \frac{R}{2}(T - \alpha_m)^2 \quad (21)$$

respectively.

Without loss of generality, set $T_0 = 0$. The cost components per cycle for the inventory model are as follow:

Items cost $= c_m \int_{T_m}^{\alpha_{m+1}} P_m\, dt + c_R \int_0^T R\, dt$

$$= c_m P_m(\alpha_{m+1} - T_m) + C_R RT \quad (22)$$

Production cost $= s_m \int_{T_m}^{\alpha_{m+1}} P_m\, dt$

$$= s_m P_m(\alpha_{m+1} - T_m) \quad (23)$$

Remanufacturing cost $= s_c \sum_{k=1}^{m} \int_{T_{k-1}}^{\alpha_k} P_c\, dt$

$$= s_c P_c \sum_{k=1}^{m} (\alpha_k - T_{k-1}) \quad (24)$$

Holding cost

$= h_c[\sum_{k=1}^{m} I_c(T_{k-1}, \alpha_k) + I_c(\alpha_k, T_k)] + h_m[I_m(T_m, \alpha_{m+1}) + I_m(\alpha_{m+1}, T)] + h_R[\sum_{k=1}^{m} I_R(T_{k-1}, \alpha_k) + \sum_{k=1}^{m-1} I_R(\alpha_k, T_k) + I_R(\alpha_m, T)]$

$= h_c\left[\sum_{k=1}^{m} P_c(\alpha_k - T_{k-1})^2 - \int_{T_{k-1}}^{\alpha_k} (\alpha_k - u) D(u)\, du + \int_{\alpha_k}^{T_k} (u - \alpha_k) D(u)\, du\right] + h_m\left[P_m(\alpha_{m+1} - T_m)^2 - \int_{T_m}^{\alpha_{m+1}} (\alpha_{m+1} - u) D(u)\, du + \int_{\alpha_{m+1}}^{T} (u - \alpha_{m+1}) D(u)\, du\right] + h_R[\sum_{k=1}^{m} \frac{1}{2}[I_R(T_{k-1}) + I_R(\alpha_k)](\alpha_k - T_{k-1}) + \sum_{k=1}^{m-1} \frac{1}{2}[I_R(\alpha_k) + I_R(T_k)](T_k - \alpha_k) + \frac{R}{2}(T - \alpha_m)^2]$ (25)

Thus, the total cost per unit time (TCUT) of the inventory model during the cycle [0,T], as a function of T_k and *m*, say $Z(T_k, m)$ where k represents integers from 1 to *m*+1, is given by the sum of (22) – (25) divided by T:

$$Z(T_k, m) = \frac{1}{T}\{s_c P_c \sum_{k=1}^{m}(\alpha_k - T_{k-1}) + (c_m + s_m)P_m(\alpha_{m+1} - T_m) + c_R RT + h_c\left[\sum_{k=1}^{m} P_c(\alpha_k - T_{k-1})^2 - \int_{T_{k-1}}^{\alpha_k}(\alpha_k - u)D(u)\,du + \int_{\alpha_k}^{T_k}(u-\alpha_k)D(u)\,du\right] + h_m\left[P_m(\alpha_{m+1} - T_m)^2 - \int_{T_m}^{\alpha_{m+1}}(\alpha_{m+1} - u)D(u)\,du + \int_{\alpha_{m+1}}^{T}(u - \alpha_{m+1})D(u)\,du\right] + h_R[\sum_{k=1}^{m}\frac{1}{2}[I_R(T_{k-1}) + I_R(\alpha_k)](\alpha_k - T_{k-1}) + \sum_{k=1}^{m-1}\frac{1}{2}[I_R(\alpha_k) + I_R(T_k)](T_k - \alpha_k) + \frac{R}{2}(T-\alpha_m)^2] + mk_c + k_m + mk_R\} \quad (26)$$

Our goal is to find T_k that minimizes $Z(T_k, m)$ given by (26) with the constant *m*. We simplify it by converting it into a function of only one variable, since all the time variables T_k are related to each other through the relations:

$$T_{k-1} < T_k, \qquad k = 1, 2, \ldots, m+1 \quad (27)$$

$$P_c(\alpha_k - T_{k-1}) - \int_{T_{k-1}}^{\alpha_k} D(u)\,du = \int_{\alpha_k}^{T_k} D(u)\,du, \qquad k = 1, 2, ,\ldots, m \quad (28)$$

$$P_m(\alpha_{m+1} - T_m) - \int_{T_m}^{\alpha_{m+1}} D(u)\,du = \int_{\alpha_{m+1}}^{T} D(u)\,du \quad (29)$$

$$\int_0^{T_m} D(t)\,dt = I_R(T) + \int_0^{\alpha_m} R\,dt = RT \quad (30)$$

$$T_k = \frac{kT_m}{m}, \qquad k = 0, 1, \ldots, m-1 \quad (31)$$

Let Q be the acceptable returned quantity for used items in the interval [0,T], then

$$Q = \int_0^T R\,dt = RT \quad (32)$$

From (32), we note that T is a function of Q, which is given by:

$$T = \frac{Q}{R} = f_{m+1}(Q) \quad (33)$$

From (30), we see that T_m is a function of T, therefore from (33), a function of Q:

$$T_m = f_m(Q) \quad (34)$$

From (31), we can see that for all k from 0 to *m*-1, T_k is a function of T_m. Hence, from (34), a function of Q:

$T_k = f_k(Q, m),$ k = 0, 1, …, m-1 (35)

From (28), we find that α_k is a function of T_k, hence a function of Q, from (35), say:

$\alpha_k = g_k(Q, m),$ k = 1, 2, …, m (36)

From (29), we see that α_{m+1} is a function of T_m and T, hence:

$\alpha_{m+1} = g_{m+1}(Q, m)$ (37)

Therefore, by substituting (33) – (37) into (26), we get the TCUT in terms of the variable Q and the constant m :

TCUT(Q, m) =

$$\frac{1}{f_{m+1}}\{s_c P_c \sum_{k=1}^{m}(g_k - f_{k-1}) + (c_m + s_m)P_m(g_{m+1} - f_m) + c_R RT +$$

$$h_c\left[\sum_{k=1}^{m} P_c\left(g_k - f_{k-1}\right)^2 - \int_{f_{k-1}}^{g_k}\left(g_k - u\right)D(u)\,du + \int_{g_k}^{f_k}\left(u - g_k\right)D(u)\,du\right] +$$

$$h_m\left[P_m\left(g_{m+1} - f_m\right)^2 - \int_{f_m}^{g_{m+1}}\left(g_{m+1} - u\right)D(u)\,du + \int_{g_{m+1}}^{f_{m+1}}\left(u - g_{m+1}\right)D(u)\,du\right] + h_R\Big[\sum_{k=1}^{m}\frac{1}{2}\left[I_R\left(f_{k-1}\right) + I_R\left(g_k\right)\right]\left(g_k - f_{k-1}\right) +$$

$$\sum_{k=1}^{m-1}\frac{1}{2}\left[I_R\left(g_k\right) + I_R\left(f_k\right)\right]\left(f_k - g_k\right) + \frac{R}{2}\left(f_{m+1} - g_m\right)^2\Big]$$

$$+mk_c + k_m + mk_R\} \qquad (38)$$

where $f_{m+1}(Q) = \frac{Q}{R}$.

5.3 Numerical Example

The demand function and all other parameters with respect to the constraints as stated in the assumptions are set as below:

$D(t) = e^{0.05t}$,	$P_m = 15$,	$R = 0.99$,	$P_c = 13$,
$c_m = 10$,	$s_m = 15$,	$h_m = 10$,	$k_m = 2400$,
$s_c = 10$,	$h_c = 10$,	$k_c = 100$,	
$c_R = 5$,	$h_R = 5$,	$k_R = 80$.	

We assume that remanufacturing and ordering of returned items are much easier than production with a much lower remanufacturing set-up cost and order cost.

5.3.1 Optimality of TCUT Function

Similarly as the (*1*,*n*) model, the optimality of the TCUT function in (38) needs to be tested. Choosing m = 2 to be an example for the test, the TCUT function is plotted against Q from the range of 1 to 70.

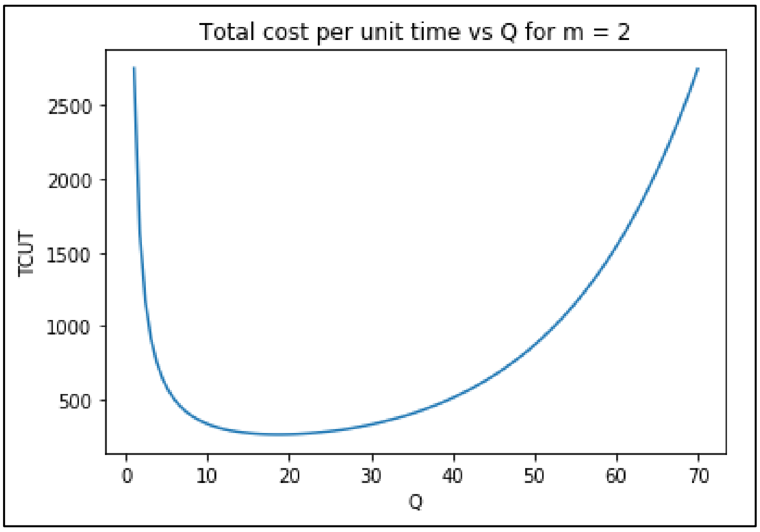

Figure 5.2: Graph of TCUT Versus Q for (*2*,*1*) Policy

From above, we see that the plot is a "U-curve" with a smooth turn at the minimum point, making it difficult to observe the exact value of the optimal Q and TCUT. However, the plot still shows that there is a minimum value for TCUT. Therefore, we display all TCUT values throughout Q from 1 to 70 to find the exact values of optimal Q and TCUT. We found that the minimum TCUT of this (*2*,*1*) policy is 263.163 corresponding to Q of 18.4242 The table of TCUT values can be found in Table A-2.

5.3.2 Optimal Number of Remanufacturing Set-ups

With the parameters above, we are interested in finding the optimal number of remanufacturing set-ups per cycle (m). Optimal Q of each m is found, followed by the minimum TCUT through the optimization function in SciPy of Python. The optimal TCUT of each m is plotted against a range of m from 1 to 10. The plot is shown below:

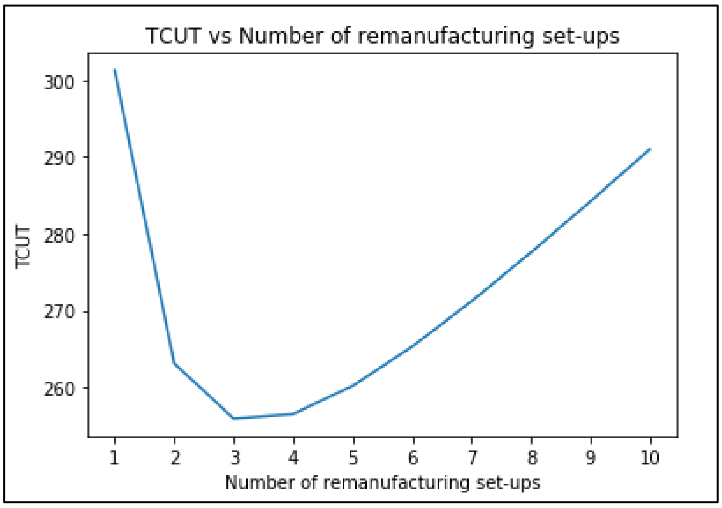

Figure 5.3: Graph of TCUT Versus Number of Remanufacturing Set-ups

We can see that the lowest point in the figure above is when m = 3 and TCUT is about 255. Therefore, we say that the optimal number of remanufacturing set-ups is 3 corresponding to TCUT about 255. However, this is just one of the case with the above parameters. Surely, the optimal number of remanufacturing set-ups will change accordingly with different sets of parameters. To show that the change in some parameters will display a uniform relationship with the optimal m, i.e. either decreases or increases the optimal m, sensitivity analysis is done is the next section.

5.4 Sensitivity Analysis

We first look at the unit holding cost of both remanufactured and returned stock. We can roughly make an inference by using the result in section 4.4, where when the holding cost of manufactured stock increases, the optimal n increases as well. The same reasoning applies to the remanufactured and returned stock, where high unit holding costs will cause more remanufacturing cycles since this will decrease the holding of the stocks. This inference can be proven by the plots in Figure 5.4.

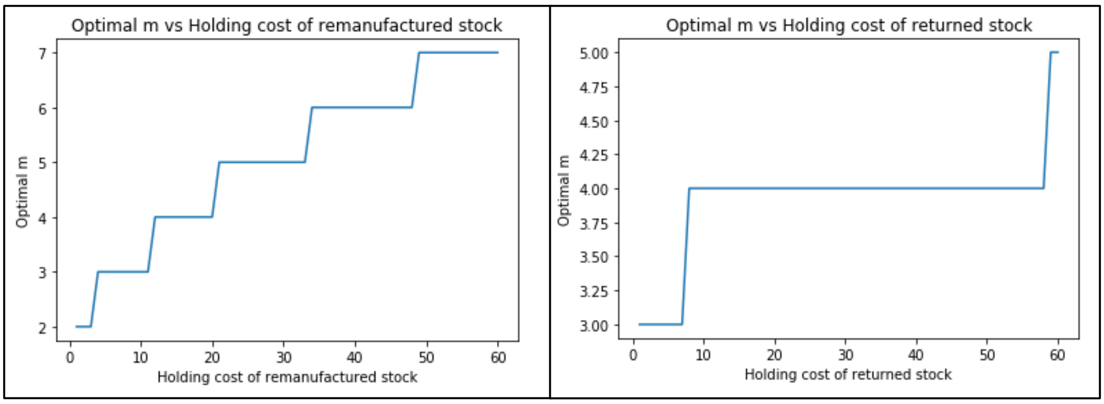

Figure 5.4: Optimal *m* Versus Unit Holding Costs of Remanufactured and Returned Stock

When the unit holding cost of remanufactured stock increases from 0 to 60, the optimal number of remanufacturing cycle increases drastically from 2 to 7. However, looking at the plot of unit holding cost of returned stock against optimal *m*, we can observe that the increment of optimal *m* is much slower compared to the unit holding cost of remanufactured stock. The optimal *m* only rises from 3 to 5 when the unit holding cost of returned stock increases from 0 to 60. From here, we can say that the effect of unit holding cost of remanufactured stock is much more significant on the optimal *m* than the unit holding cost of returned stock.

Next, we observe the relationships of remanufacturing set-up cost and order cost with the optimal number of remanufacturing cycle.

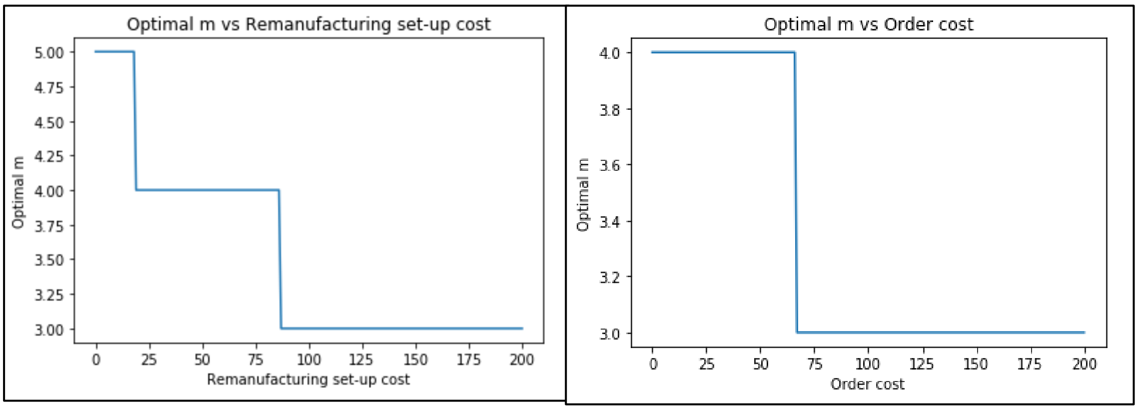

Figure 5.5: Optimal *m* Versus Remanufacturing Set-up Cost and Order Cost

Analogous to the decrease in the optimal number of production set-ups with the increment of production set-up cost, the remanufacturing set-up cost and order cost shows the same relationship with the optimal number of remanufacturing cycle as well. Furthermore, we see that the influence of remanufacturing set-up cost on optimal m is more notable than the order cost since the optimal m declines from 5 to 3 for the remanufacturing set-up cost, while for the order cost, it only decreases from 4 to 3.

CHAPTER 6

INVENTORY MODEL WITH MULTIPLE REMANUFACTURING, MULTIPLE PRODUCTION SET-UPS PER CYCLE, (*M*,*N*) MODEL

6.1 Introduction

Coming to the general model, which is the multiple remanufacturing, multiple production set-ups per cycle, (*m*,*n*) model. This is the combination of the last two models whereby returned items are ordered *m* times from the returned cycle to be remanufactured, followed by *n* procurements in the production cycle.

The overview of the inventory variations of a (*m*,*n*) model with *m* and *n* both set to be 2, is plotted out below:

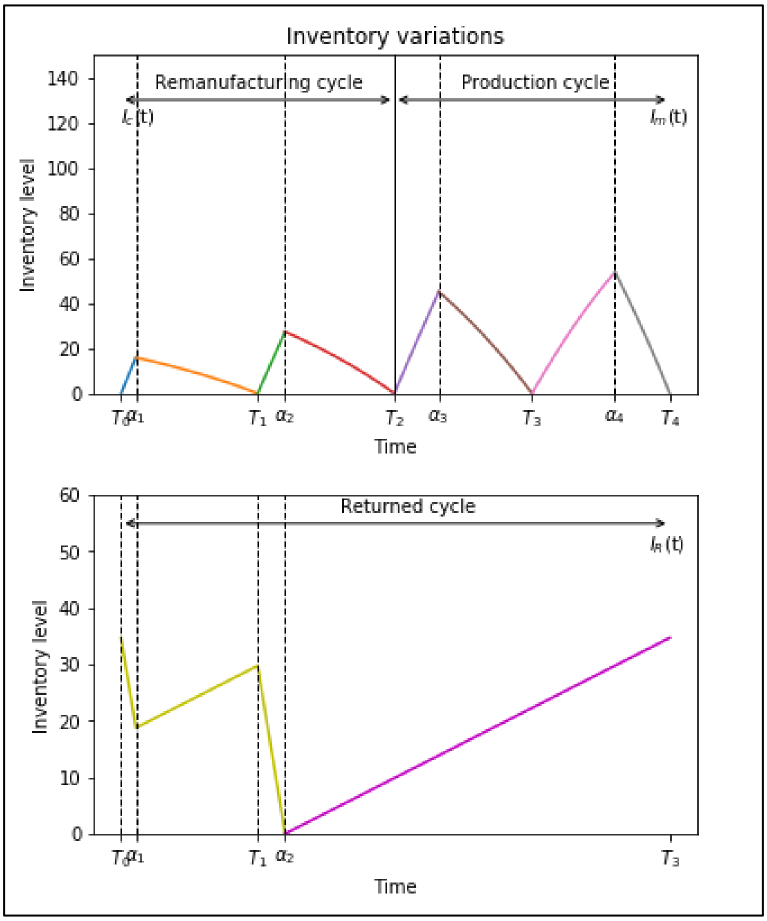

Figure 6.1: Overview of Inventory Variations of a (*2*,*2*) Policy

Figure 6.1 shows 2 remanufacturing set-ups followed by 2 production set-ups. The two yellow decreasing line from T_0 to α_1 and T_1 to α_2 correspond to the 2 remanufacturing set-ups since returned items are remanufactured to satisfy demands within that 2 periods. After all returned items are remanufactured, they will be collected again when manufacturing is going on. This is represented by the inclining pink line in the returned cycle.

The formulation of (*m,n*) model can be easily done by combining the formulation of (*1,n*) and (*m,1*) model which we have completed in the previous chapters.

6.2 Formulation of (*m,n*) Model

All assumptions and notations are the same, except that the last point of time in the whole cycle which is denoted as T is equivalent to T_{m+n}.

The changes in the inventory levels are governed by the following differential equations:

$T_{k-1} \leq t < \alpha_k$:

$$\left(\frac{dI_c(t)}{dt}\right)_k = P_c - D(t), \text{ with the initial condition } I_c(T_{k-1}) = 0, \quad (1)$$

$\alpha_k \leq t \leq T_k$:

$$\left(\frac{dI_c(t)}{dt}\right)_k = -D(t), \text{ with the ending condition } I_c(T_k) = 0, \quad (2)$$

where k = 1, 2, …, *m*

$T_i \leq t < \alpha_{i+1}$:

$$\left(\frac{dI_m(t)}{dt}\right)_i = P_m - D(t), \text{ with the initial condition } I_m(T_i) = 0, \quad (3)$$

$\alpha_{i+1} \leq t \leq T_{i+1}$:

$$\left(\frac{dI_m(t)}{dt}\right)_i = -D(t), \text{ with the ending condition } I_m(T_{i+1}) = 0, \quad (4)$$

where i = *m*, *m*+1, …, *m*+*n*-1

$\alpha_m \leq t \leq T$:

$$\frac{dI_R(t)}{dt} = R, \quad \text{with the initial condition } I_R(\alpha_m) = 0. \quad (5)$$

Similarly as the formulation of (*m,1*) model, we will represent the inventory levels of the returned cycle from T_0 to α_m to be as follow:

For all k from 1 to m-1,

$$I_R(T_0) = I_R(T) \tag{6}$$

$$I_R(\alpha_k) = I_R(T_{k-1}) - (P_c - R)(\alpha_k - T_{k-1}) \tag{7}$$

$$I_R(T_k) = I_R(\alpha_k) + R(T_k - \alpha_k) \tag{8}$$

$$I_R(\alpha_m) = 0 \tag{9}$$

The solutions of all the differential equations are:

$T_{k-1} \leq t < \alpha_k$:

$$I_c(t)^{[k]} = P_c(t - T_{k-1}) - \int_{T_{k-1}}^{t} D(u)\, du \tag{10}$$

$\alpha_k \leq t \leq T_k$:

$$I_c(t)^{[k]} = \int_t^{T_k} D(u)\, du \tag{11}$$

where k = 1, 2, …, m

$T_i \leq t < \alpha_{i+1}$:

$$I_m(t)^{[i]} = P_m(t - T_i) - \int_{T_i}^{t} D(u)\, du \tag{12}$$

$\alpha_{i+1} \leq t \leq T_{i+1}$:

$$I_m(t)^{[i]} = \int_t^{T_{i+1}} D(u)\, du \tag{13}$$

where i = m, m+1, …, m+n-1

$\alpha_m \leq t \leq T$:

$$I_R(t) = R(t - \alpha_m) \tag{14}$$

respectively.

In order to find the inventory holdings for each stocks, let

$$I(t_1, t_2) = \int_{t_1}^{t_2} I(u)\, du,$$

then from (6) – (14) we have:

$T_{k-1} \leq t < \alpha_k$:

$$I_c(T_{k-1}, \alpha_k) = P_c(\alpha_k - T_{k-1})^2 - \int_{T_{k-1}}^{\alpha_k} (\alpha_k - u) D(u)\, du \tag{15}$$

$\alpha_k \leq t \leq T_k$:

$$I_c(\alpha_k, T_k) = \int_{\alpha_k}^{T_k} (u - \alpha_k) D(u)\, du \tag{16}$$

where k = 1, 2, …, m

$T_i \leq t < \alpha_{i+1}$:

$$I_m(T_i, \alpha_{i+1}) = P_m(\alpha_{i+1} - T_i)^2 - \int_{T_i}^{\alpha_{i+1}} (\alpha_{i+1} - u) D(u)\, du \tag{17}$$

$\alpha_{i+1} \le t \le T_{i+1}$:

$$I_m(\alpha_{i+1}, T_{i+1}) = \int_{\alpha_{i+1}}^{T_{i+1}} (u - \alpha_{i+1}) D(u)\, du \tag{18}$$

where i = m, m+1, …, m+n-1

$T_0 \le t < \alpha_m$:

$$I_R(T_{k-1}, \alpha_k) = \frac{1}{2}[I_R(T_{k-1}) + I_R(\alpha_k)](\alpha_k - T_{k-1}) \tag{19}$$

where k = 1, 2, …, m

$$I_R(\alpha_k, T_k) = \frac{1}{2}[I_R(\alpha_k) + I_R(T_k)](T_k - \alpha_k) \tag{20}$$

where k = 1, 2, …, m-1

$\alpha_m \le t \le T$:

$$I_R(\alpha_m, T) = \frac{R}{2}(T - \alpha_m)^2 \tag{21}$$

respectively.

Without loss of generality, set $T_0 = 0$. The cost components per cycle for the inventory model are as follow:

Items cost $= c_m \sum_{i=m}^{m+n-1} \int_{T_i}^{\alpha_{i+1}} P_m\, dt + c_R \int_0^T R\, dt$

$$= c_m P_m \sum_{i=m}^{m+n-1} (\alpha_{i+1} - T_i) + C_R RT \tag{22}$$

Production cost $= s_m \sum_{i=m}^{m+n-1} \int_{T_i}^{\alpha_{i+1}} P_m\, dt$

$$= s_m P_m \sum_{i=m}^{m+n-1} (\alpha_{i+1} - T_i) \tag{23}$$

Remanufacturing cost $= s_c \sum_{k=1}^{m} \int_{T_{k-1}}^{\alpha_k} P_c\ dt$

$$= s_c P_c \sum_{k=1}^{m} (\alpha_k - T_{k-1}) \tag{24}$$

Holding cost

$$= h_c[\sum_{k=1}^{m} I_c(T_{k-1}, \alpha_k) + I_c(\alpha_k, T_k)] + h_m[\sum_{i=m}^{m+n-1} I_m(T_i, \alpha_{i+1}) + I_m(\alpha_{i+1}, T_{i+1})] + h_R[\sum_{k=1}^{m} I_R(T_{k-1}, \alpha_k) + \sum_{k=1}^{m-1} I_R(\alpha_k, T_k) + I_R(\alpha_m, T)]$$

$$= h_c\left[\sum_{k=1}^{m} P_c(\alpha_k - T_{k-1})^2 - \int_{T_{k-1}}^{\alpha_k} (\alpha_k - u) D(u)\, du + \int_{\alpha_k}^{T_k} (u - \alpha_k) D(u)\, du\right] + h_m\left[\sum_{i=m}^{m+n-1} P_m(\alpha_{i+1} - T_i)^2 - \int_{T_i}^{\alpha_{i+1}} (\alpha_{i+1} - u) D(u)\, du + \int_{\alpha_{i+1}}^{T_{i+1}} (u - \alpha_{i+1}) D(u)\, du\right] + h_R[\sum_{k=1}^{m} \frac{1}{2}[I_R(T_{k-1}) + I_R(\alpha_k)](\alpha_k - T_{k-1}) + \sum_{k=1}^{m-1} \frac{1}{2}[I_R(\alpha_k) + I_R(T_k)](T_k - \alpha_k) + \frac{R}{2}(T - \alpha_m)^2] \tag{25}$$

Thus, the total cost per unit time (TCUT) of the inventory model during the cycle [0,T], as a function of T_k, m and n, say $Z(T_k, m, n)$ where k represents integers from 1 to $m+n$, is given by the sum of (22) – (25) divided by T:

$$Z(T_k, m, n) = \frac{1}{T}\{s_c P_c \sum_{k=1}^{m}(\alpha_k - T_{k-1}) + (c_m + s_m)P_m \sum_{i=m}^{m+n-1}(\alpha_{i+1} - T_i) + c_R RT + h_c\left[\sum_{k=1}^{m} P_c(\alpha_k - T_{k-1})^2 - \int_{T_{k-1}}^{\alpha_k}(\alpha_k - u)D(u)\,du + \int_{\alpha_k}^{T_k}(u - \alpha_k)D(u)\,du\right] + h_m\left[\sum_{i=m}^{m+n-1} P_m(\alpha_{i+1} - T_i)^2 - \int_{T_i}^{\alpha_{i+1}}(\alpha_{i+1} - u)D(u)\,du + \int_{\alpha_{i+1}}^{T_{i+1}}(u - \alpha_{i+1})D(u)\,du\right] + h_R[\sum_{k=1}^{m}\frac{1}{2}[I_R(T_{k-1}) + I_R(\alpha_k)](\alpha_k - T_{k-1}) + \sum_{k=1}^{m-1}\frac{1}{2}[I_R(\alpha_k) + I_R(T_k)](T_k - \alpha_k) + \frac{R}{2}(T - \alpha_m)^2] + mk_c + nk_m + mk_R\} \quad (26)$$

Our goal is to find T_k that minimizes $Z(T_k, m, n)$ given by (26) with m and n as constants. We simplify it by converting it into a function of only one variable, since all the time variables T_k are related to each other through the relations:

$T_{k-1} < T_k$, k = 1, 2, …, $m+n$ (27)

$P_c(\alpha_k - T_{k-1}) - \int_{T_{k-1}}^{\alpha_k} D(u)\,du = \int_{\alpha_k}^{T_k} D(u)\,du$, k = 1, 2, ,…, m (28)

$P_m(\alpha_{i+1} - T_i) - \int_{T_i}^{\alpha_{i+1}} D(u)\,du = \int_{\alpha_{i+1}}^{T_{i+1}} D(u)\,du$, i = m, m+1, …, m+n-1 (29)

$\int_0^{T_m} D(t)\,dt = I_R(T) + \int_0^{\alpha_m} R\,dt = RT$ (30)

$T_k = \frac{kT_m}{m}$, k = 0, 1, …, m-1 (31)

$\frac{T-T_m}{n} = T_{i+1} - T_i$, i = m+1, m+2, …, m+n-1 (32)

Let Q be the acceptable returned quantity for used items in the interval [0,T], then

$Q = \int_0^T R\,dt = RT$ (33)

From (33), we note that T is a function of Q, which is given by:

$T = \frac{Q}{R} = f_{m+n}(Q)$ (34)

From (30), we see that T_m is a function of T, therefore from (34), a function of Q:

$T_m = f_m(Q)$ (35)

From (31), we can see that for all k from 0 to m-1, T_k is a function of T_m. Hence, from (35), a function of Q:

$T_k = f_k(Q, m),$ k = 0, 1, …, m-1 (36)

From (28), we find that α_k is a function of T_k, hence a function of Q, from (36), say:

$\alpha_k = g_k(Q, m),$ k = 1, 2, …, m (37)

From (32), T_i is a function of T and T_m, which is a function of Q from (34) and (35):

$T_i = f_i(Q, m, n),$ i = m+1, m+2, …, m+n-1 (38)

From (29), we see that α_{i+1} is a function of T_i, hence:

$\alpha_{i+1} = g_{i+1}(Q, m, n),$ i = m, m+1, …, m+n-1 (39)

Therefore, by substituting (34) – (39) into (26), we get the TCUT in terms of the variable Q and constants m and n :

TCUT(Q, m, n) =

$$\frac{1}{f_{m+n}}\{s_c P_c \sum_{k=1}^{m}(g_k - f_{k-1}) + (c_m + s_m)P_m \sum_{i=m}^{m+n-1}(g_{i+1} - f_i) + c_R RT +$$
$$h_c\left[\sum_{k=1}^{m} P_c\left(g_k - f_{k-1}\right)^2 - \int_{f_{k-1}}^{g_k}\left(g_k - u\right)D(u)\,du + \int_{g_k}^{f_k}\left(u - g_k\right)D(u)\,du\right] +$$
$$h_m\left[\sum_{i=m}^{m+n-1} P_m(g_{i+1} - f_i)^2 - \int_{f_i}^{g_{i+1}}(g_{i+1} - u)D(u)\,du + \int_{g_{i+1}}^{f_{i+1}}(u -$$
$$g_{i+1})D(u)\,du\right] + h_R[\sum_{k=1}^{m}\frac{1}{2}\left[I_R\left(f_{k-1}\right) + I_R\left(g_k\right)\right]\left(g_k - f_{k-1}\right) +$$
$$\sum_{k=1}^{m-1}\frac{1}{2}\left[I_R\left(g_k\right) + I_R\left(f_k\right)\right]\left(f_k - g_k\right) + \frac{R}{2}\left(f_{m+n} - g_m\right)^2]$$
$$+mk_c + nk_m + mk_R\} \quad (40)$$

where $f_{m+n}(Q) = \frac{Q}{R}$.

6.3 Numerical Example

The demand function and all other parameters with respect to the constraints as stated in the assumptions are set as below:

$D(t) = e^{0.05t}$,	$P_m = 15$,	$R = 0.99$,	$P_c = 13$,
$c_m = 10$,	$s_m = 15$,	$h_m = 10$,	$k_m = 300$,
$s_c = 10$,	$h_c = 10$,	$k_c = 100$,	
$c_R = 5$,	$h_R = 5$,	$k_R = 75$.	

6.3.1 Optimality of TCUT Function

We test the optimality of the TCUT function constructed to show that the function can be optimized through the variable Q. Since m and n are constants, we set both of them to be 2 in order to proceed with the testing of optimality. The TCUT function in (40) with m and n equal to 2 is plotted against Q ranging from 0 to 60.

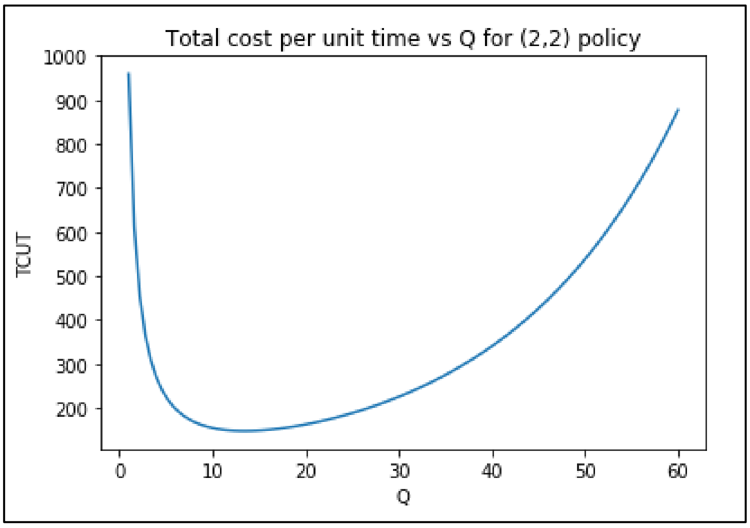

Figure 6.2: Graph of TCUT Versus Q for (*2*,*2*) Policy

From the figure above, we conclude that TCUT function constructed in (40) can be optimized with respect to the variable Q, by looking at the existence of a minimum point. Through listing out all TCUT values in Table A-3, we find the optimal TCUT of the (2,2) policy to be 146.998 with Q = 13.5152.

6.3.2 Optimal Number of Remanufacturing and Production Set-ups

In order to find the optimal number of remanufacturing set-ups (m) and production set-ups (n) with the parameters mentioned above, we set m and n ranging from 1 to 5 and calculated the optimal TCUT for all 25 sets of duple. The result is shown in the table below:

Table 6.1: TCUT of (m,n) Model with Different m and n

TCUT		*m*				
		1	2	3	4	5
n	1	134.968	125.232	128.674	135.549	143.647
	2	166.828	146.993	145.869	149.702	155.424
	3	192.786	166.047	161.983	163.932	168.227
	4	215.205	183.124	176.868	177.44	180.701
	5	235.212	198.741	190.735	190.221	192.675

The minimum TCUT in the table above which is highlighted in yellow is 125.232 which is the optimal TCUT of the $(2,1)$ policy. This result is with respect to the parameters mentioned earlier. In order to see the effect of some parameters on the optimal result, sensitivity analysis is carried out for the unit holding costs of all 3 stocks as well as the set-up costs of each stock.

6.4 Sensitivity Analysis

We first look at the effect of unit holding costs of all 3 stocks on the optimal number of remanufacturing and production cycles.

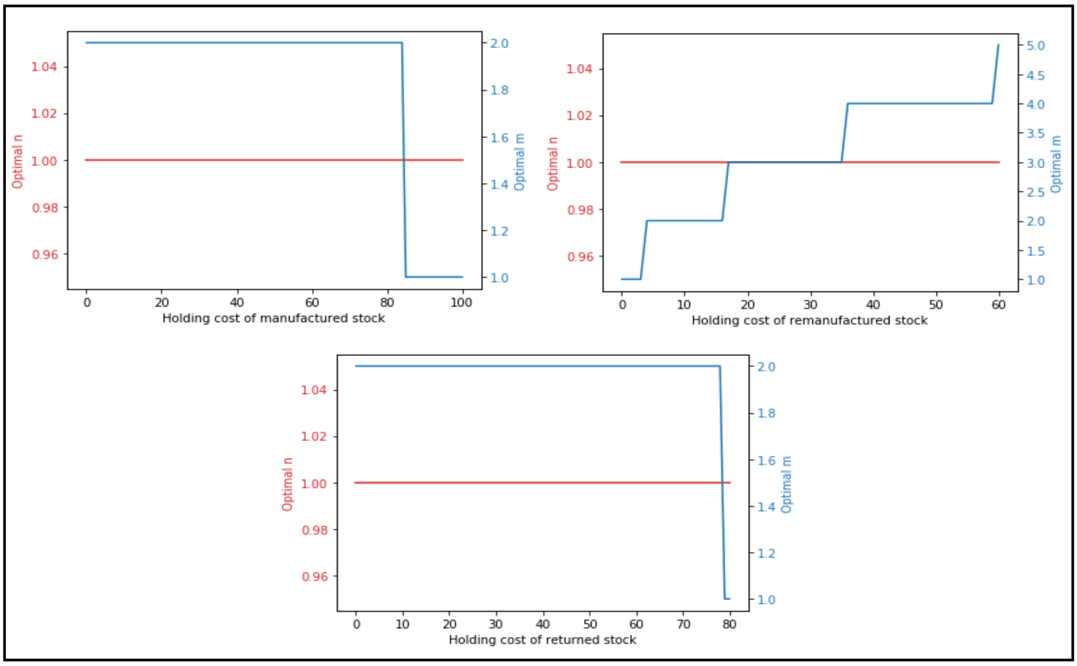

Figure 6.3: Optimal m and n Versus Unit Holding Costs of All 3 Stocks

The sensitivity analysis above is carried out using parameters in the last section. From the plot of unit holding cost of manufactured stock against optimal *m* and *n*, we see that the optimal *n* remains at 1 throughout the plot and the optimal *m* decreases from 2 to 1 with the increase of unit holding cost of manufactured stock from 0 to 100. The reason being that when the unit holding cost of manufactured stock increases, the unit holding cost of remanufactured and returned stock which are constants at 10 and 5 respectively, are relatively small compared to the increasing unit holding cost of manufactured stock, hence we can hold more stocks in a set-up, causing less number of remanufacturing set-up.

However, the optimal *m* has a linear relationship with the unit holding cost of remanufactured stock with the optimal *n* being stagnant at 1. When the unit holding cost of remanufactured stock increases, we do not wish to stock many remanufactured items, causing more remanufacturing set-ups to happen, hence higher optimal *m*.

On the other hand, the optimal *m* decreases with the increase of unit holding cost of returned stock. It has a different behaviour as in the (*m,1*) model, since the difference between the production and remanufacturing set-up cost is small with the parameters set in this chapter, their impact on the result is cancelled off by each other. This leaves only the constant unit holding cost of remanufactured stock and the increasing unit holding cost of returned stock to affect the optimal result. Observing the decrease of optimal *m*, we can say that the low unit holding cost of remanufactured stock is much more significant than the unit holding cost of returned stock.

In fact, from the plot of unit holding cost of remanufactured stock against optimal *m* and *n*, we can see a rapid increase in the optimal *m*, showing that the unit holding cost of remanufactured stock has a more significant effect on the optimal result.

Next, we look at production and remanufacturing set-up costs and order cost.

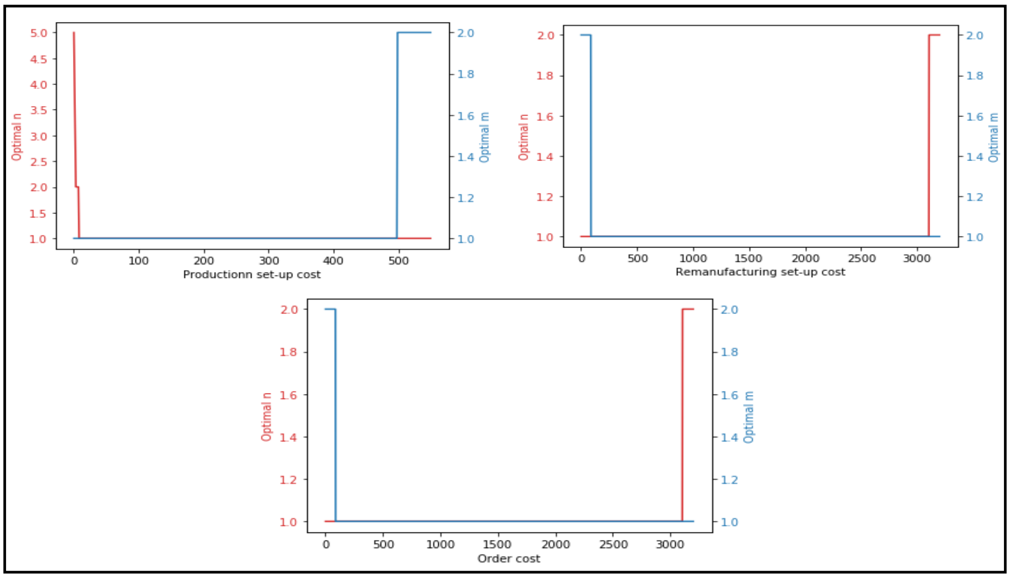

Figure 6.4: Optimal *m* and *n* Versus Set-up Costs of All 3 Stocks

This sensitivity analysis is carried out with the set-up costs of the other 2 stocks set to be equal at 250 when examining one of the set-up cost. This is to prevent any stock to be more outstanding than the others, so that we can see the impact of that set-up cost on the optimal result without the influence of the other 2 set-up costs.

For the production set-up cost plot, we see that the optimal n drops promptly from n = 5 to 1 when the optimal m stays at 1. The optimal model remains to be the (*1*,*1*) policy until when the production set-up cost increases to about 500, the optimal m rises to 2. This is because when the production set-up cost is about double the set-up costs of the other 2 stocks, it becomes cheaper to have more remanufacturing set-ups with a lower set-up cost.

The optimal m and n display the same relationship for remanufacturing set-up cost and order cost. They start off with the optimal policy of (*2*,*1*) when the set-up cost is much lower, followed by the stagnant (*1*,*1*) policy until the set-up costs passed 3000, where the optimal n escalates to 2. The remanufacturing set-up cost and order cost need to be about 12 times higher than the production set-up cost to have such escalation.

When looking at all the plots in Figure 6.3 and 6.4, we see that the (*m*,*1*) and (*1*,*n*) policies seem to dominate the (*m*,*n*) policy. It is sufficient to conclude that the (*m*,*n*) policy with m and $n > 1$ will not be optimal in most settings.

CHAPTER 7

CONCLUSIONS AND RECOMMENDATIONS

7.1 Conclusions

We have built 3 inventory models in this research, which are the (*1,n*) model, (*m,1*) model and (*m,n*) model. The (*m,n*) inventory model is the general case of the other 2 models. In the (*1,n*) model, we see that the optimal inventory policy using the parameters in section 4.3 is the (*1,3*) policy. The optimal *n* increases with the unit holding cost of manufactured stock but decreases with the increase of production set-up cost. On the other hand, the (*3,1*) policy is the optimal inventory policy for the (*m,1*) model with the parameters in section 5.3. Similarly, the optimal *m* has a linear relationship with the unit holding cost of remanufactured and returned stock, but an inverse relationship with the remanufacturing set-up cost and order cost.

Generally, we find that the (*m,n*) model will not be the optimal inventory model. The possible optimal inventory models are either the (*1,1*), (*1,n*) or (*m,1*) policies, according to the sensitivity analysis of the (*m,n*) model. This might be due to the characteristic of the model that we have built.

7.2 Recommendations for Future Work

Through this research, we found that we are unable to obtain (*m,n*) policy as the optimal inventory model. Research in the future may look into this and search for ways that will obtain a (*m,n*) policy as the optimal model. More parameters can also be added to make the model more realistic since there are countless factors that can be accounted for when building an inventory model. All parameters can be changed including the type of demand, from a continuous demand into a discrete demand which is more practical in the real world, since the continuous demand that we applied through this research is quite impossible in reality, but is easier to be examined.

REFERENCES

Alamri, A.A., 2011. Theory and methodology on the global optimal solution to a General Reverse Logistics Inventory Model for deteriorating items and time-varying rates. *Computers and Industrial Engineering*, 60(2), pp.236–247.

Batarfi, R., Jaber, M.Y. and Aljazzar, S.M., 2017. A profit maximization for a reverse logistics dual-channel supply chain with a return policy. *Computers & Industrial Engineering*, 106, pp.58–82.

Bazan, E., Jaber, M.Y. and Zanoni, S., 2016. A review of mathematical inventory models for reverse logistics and the future of its modeling: An environmental perspective. *Applied Mathematical Modelling*, 40(5–6), pp.4151–4178.

Bouras, A. and Tadj, L., 2015. Production planning in a three-stock reverse-logistics system with deteriorating items under a continuous review policy. *Journal of Industrial and Management Optimization*, 11(4), pp.1041–1058.

Greeff, G. and Ghoshal, R., 2004. *Practical E-Manufacturing and Supply Chain Management*. *IDC Technologies and Gerhard Greeff*. London: IDC Technologies and Gerhard Greeff.

Nemtajela, N. and Mbohwa, C., 2016. Inventory management models and their effects on uncertain demand. In: *IEEE International Conference on Industrial Engineering and Engineering Management*. IEEE Computer Society.pp.1046–1048.

El Saadany, A.M.A. and Jaber, M.Y., 2010. A production/remanufacturing inventory model with price and quality dependant return rate. *Computers & Industrial Engineering*, 58(3), pp.352–362.

El Saadany, A.M.A., Jaber, M.Y. and Bonney, M., 2013. How many times to remanufacture? *International Journal of Production Economics*, 143(2), pp.598–604.

APPENDICES

APPENDIX A: Tables on Values of TCUT with Respect to *Q*

Table A-1: Values of TCUT for *Q* from 1 to 80 for (*1*,*2*) Policy

Q	TCUT	*Q*	TCUT	*Q*	TCUT	*Q*	TCUT
1	2893.92	21.7475	315.059	42.4949	515.498	63.2424	1179.45
1.79798	1625.87	22.5455	317.47	43.2929	529.684	64.0404	1222.72
2.59596	1141.12	23.3434	320.349	44.0909	544.499	64.8384	1267.89
3.39394	887.126	24.1414	323.672	44.8889	559.971	65.6364	1315.04
4.19192	732.083	24.9394	327.425	45.6869	576.125	66.4343	1364.26
4.9899	628.505	25.7374	331.591	46.4848	592.99	67.2323	1415.65
5.78788	555.098	26.5354	336.163	47.2828	610.596	68.0303	1469.29
6.58586	500.893	27.3333	341.132	48.0808	628.974	68.8283	1525.3
7.38384	459.664	28.1313	346.493	48.8788	648.156	69.6263	1583.77
8.18182	427.619	28.9293	352.245	49.6768	668.178	70.4242	1644.81
8.9798	402.316	29.7273	358.387	50.4747	689.074	71.2222	1708.54
9.77778	382.111	30.5253	364.92	51.2727	710.884	72.0202	1775.07
10.5758	365.861	31.3232	371.847	52.0707	733.646	72.8182	1844.53
11.3737	352.743	32.1212	379.174	52.8687	757.403	73.6162	1917.05
12.1717	342.154	32.9192	386.907	53.6667	782.197	74.4141	1992.77
12.9697	333.64	33.7172	395.054	54.4646	808.073	75.2121	2071.82
13.7677	326.855	34.5152	403.623	55.2626	835.08	76.0101	2154.36
14.5657	321.529	35.3131	412.626	56.0606	863.267	76.8081	2240.53
15.3636	317.452	36.1111	422.074	56.8586	892.686	77.6061	2330.5
16.1616	314.455	36.9091	431.98	57.6566	923.392	78.404	2424.44
16.9596	312.403	37.7071	442.359	58.4545	955.44	79.202	2522.52
17.7576	311.188	38.5051	453.225	59.2525	988.891	80	2624.92
18.5556	310.72	39.303	464.596	60.0505	1023.81		
19.3535	310.928	40.101	476.489	60.8485	1060.25		
20.1515	311.752	40.899	488.923	61.6465	1098.29		
20.9495	313.142	41.697	501.919	62.4444	1138		

Table A-2: Values of TCUT for *Q* from 1 to 70 for (*2*,*1*) Policy

Q	TCUT	*Q*	TCUT	*Q*	TCUT	*Q*	TCUT
1	2751.78	19.1212	263.231	37.2424	447.907	55.3636	1177.53
1.69697	1632.55	19.8182	263.882	37.9394	462.883	56.0606	1225.38
2.39394	1166.79	20.5152	265.09	38.6364	478.612	56.7576	1275.28
3.09091	912.465	21.2121	266.833	39.3333	495.12	57.4545	1327.32
3.78788	752.887	21.9091	269.091	40.0303	512.437	58.1515	1381.58
4.48485	643.903	22.6061	271.851	40.7273	530.594	58.8485	1438.15
5.18182	565.119	23.303	275.104	41.4242	549.623	59.5455	1497.12
5.87879	505.815	24	278.841	42.1212	569.556	60.2424	1558.59
6.57576	459.819	24.697	283.06	42.8182	590.431	60.9394	1622.65
7.27273	423.328	25.3939	287.757	43.5152	612.282	61.6364	1689.42
7.9697	393.869	26.0909	292.934	44.2121	635.149	62.3333	1759
8.66667	369.769	26.7879	298.593	44.9091	659.071	63.0303	1831.49
9.36364	349.855	27.4848	304.737	45.6061	684.09	63.7273	1907.03
10.0606	333.28	28.1818	311.374	46.303	710.25	64.4242	1985.72
10.7576	319.418	28.8788	318.511	47	737.595	65.1212	2067.69
11.4545	307.799	29.5758	326.157	47.697	766.174	65.8182	2153.08
12.1515	298.061	30.2727	334.323	48.3939	796.035	66.5152	2242.01
12.8485	289.922	30.9697	343.021	49.0909	827.229	67.2121	2334.63
13.5455	283.162	31.6667	352.265	49.7879	859.81	67.9091	2431.08
14.2424	277.604	32.3636	362.069	50.4848	893.833	68.6061	2531.52
14.9394	273.107	33.0606	372.45	51.1818	929.356	69.303	2636.11
15.6364	269.554	33.7576	383.424	51.8788	966.439	70	2744.99
16.3333	266.853	34.4545	395.012	52.5758	1005.14		
17.0303	264.928	35.1515	407.232	53.2727	1045.54		
17.7273	263.715	35.8485	420.106	53.9697	1087.69		
18.4242	263.163	36.5455	433.656	54.6667	1131.66		

Table A-3: Values of TCUT for *Q* from 1 to 60 for (*2*,*2*) Policy

Q	TCUT	*Q*	TCUT	*Q*	TCUT	*Q*	TCUT
1	959.88	16.4949	150.712	31.9899	242.904	47.4848	477.585
1.59596	611.239	17.0909	152.159	32.5859	248.738	48.0808	491.126
2.19192	453.581	17.6869	153.805	33.1818	254.77	48.6768	505.109
2.78788	364.435	18.2828	155.641	33.7778	261.007	49.2727	519.549
3.38384	307.609	18.8788	157.659	34.3737	267.452	49.8687	534.461
3.9798	268.591	19.4747	159.851	34.9697	274.114	50.4646	549.861
4.57576	240.431	20.0707	162.211	35.5657	280.997	51.0606	565.764
5.17172	219.382	20.6667	164.735	36.1616	288.109	51.6566	582.188
5.76768	203.247	21.2626	167.42	36.7576	295.456	52.2525	599.149
6.36364	190.653	21.8586	170.263	37.3535	303.045	52.8485	616.666
6.9596	180.698	22.4545	173.261	37.9495	310.884	53.4444	634.757
7.55556	172.765	23.0505	176.414	38.5455	318.981	54.0404	653.44
8.15152	166.418	23.6465	179.721	39.1414	327.343	54.6364	672.736
8.74747	161.34	24.2424	183.182	39.7374	335.979	55.2323	692.665
9.34343	157.297	24.8384	186.797	40.3333	344.898	55.8283	713.248
9.93939	154.11	25.4343	190.567	40.9293	354.108	56.4242	734.506
10.5354	151.642	26.0303	194.493	41.5253	363.619	57.0202	756.462
11.1313	149.785	26.6263	198.578	42.1212	373.441	57.6162	779.139
11.7273	148.454	27.2222	202.822	42.7172	383.583	58.2121	802.56
12.3232	147.581	27.8182	207.229	43.3131	394.056	58.8081	826.752
12.9192	147.11	28.4141	211.802	43.9091	404.871	59.404	851.738
13.5152	146.998	29.0101	216.543	44.5051	416.038	60	877.546
14.1111	147.206	29.6061	221.456	45.101	427.57		
14.7071	147.705	30.202	226.544	45.697	439.478		
15.303	148.469	30.798	231.812	46.2929	451.775		
15.899	149.477	31.3939	237.264	46.8889	464.473		

APPENDIX B: Detailed Calculations on Formulation of Models

Taking formulation of (*1*,*n*) model as an example.

Solve $\frac{dI_c(t)}{dt} = P_c - D(t)$, with the initial condition $I_c(T_0) = 0$, for $T_0 \le t < \alpha_1$:

$I_c(t) = \int P_c - D(t)dt + C$

$Let\ t = T_0$: $\quad I_c(T_0) = [\int P_c - D(t)dt]_{t=T_0} + C$

$$0 = [\int P_c - D(t)dt]_{t=T_0} + C$$

$$C = -[\int P_c - D(t)dt]_{t=T_0}$$

Substitute C into $I_c(t)$:

$$\begin{aligned} I_c(t) &= \int P_c - D(t)dt - [\int P_c - D(t)dt]_{t=T_0} \\ &= \int_{T_0}^{t} P_c - D(t)dt \\ &= \int_{T_0}^{t} P_c\ dt - \int_{T_0}^{t} D(t)\ dt \\ &= P_c(t - T_0) - \int_{T_0}^{t} D(t)\ dt \end{aligned}$$

All differential equations in this research are solved in the same way as above.

To find the inventory holdings,

$$\begin{aligned} I_c(T_0, \alpha_1) &= \int_{T_0}^{\alpha_1} [P_c(t - T_0) - \int_{T_0}^{t} D(u)\ du]\ dt \\ &= \int_{T_0}^{\alpha_1} P_c(t - T_0)\ dt - \int_{T_0}^{\alpha_1} (\int_{T_0}^{t} D(u)\ du)\ dt \end{aligned}$$

Solve them separately,

$$\begin{aligned} \int_{T_0}^{\alpha_1} P_c(t - T_0)\ dt &= P_c \int_{T_0}^{\alpha_1} (t - T_0)\ dt \\ &= P_c \left[\frac{t^2}{2} - T_0 t\right]_{T_0}^{\alpha_1} \\ &= P_c(\frac{\alpha_1^2}{2} - T_0\alpha_1 - \frac{T_0^2}{2} + T_0^2) \\ &= P_c(\frac{\alpha_1^2 + T_0^2}{2} - T_0\alpha_1) \\ &= \frac{P_c}{2}(\alpha_1^2 + T_0^2 - 2T_0\alpha_1) \\ &= P_c(\alpha_1 - T_0)^2 \end{aligned}$$

Solve $\int_{T_0}^{\alpha_1} (\int_{T_0}^{t} D(u)\ du)\ dt$ through integration by parts,

$Let\ w = \int_{T_0}^{t} D(u)\ du \qquad\qquad dv = dt$

$dw = D(t)\ dt \qquad\qquad v = t$

$$\int_{T_0}^{\alpha_1}\left(\int_{T_0}^{t} D(u)\,du\right)dt = t\int_{T_0}^{t} D(u)\,du\,\Big|_{t=T_0}^{t=\alpha_1} - \int_{T_0}^{\alpha_1} t\,D(t)\,dt$$
$$= \alpha_1\int_{T_0}^{\alpha_1} D(u)\,du - \int_{T_0}^{\alpha_1} u\,D(u)\,du$$
$$= \int_{T_0}^{\alpha_1}(\alpha_1 - u)\,D(u)\,du$$

Therefore, substituting the results of the two integrations into $I_c(T_0, \alpha_1)$, we obtain:

$$I_c(T_0, \alpha_1) = P_c(\alpha_1 - T_0)^2 - \int_{T_0}^{\alpha_1}(\alpha_1 - u)\,D(u)\,du$$

Similarly, $I_m(T_k, \alpha_{k+1})$ can be obtained using the above procedure, for k = 1, 2, …, *n*.

For $I_c(\alpha_1, T_1)$ and $I_m(\alpha_{k+1}, T_{k+1})$, where k = 1, 2, …, *n*, they are similar to solving $\int_{T_0}^{\alpha_1}\left(\int_{T_0}^{t} D(u)\,du\right)dt$ using integration by parts.

$I_R(T_0, \alpha_1)$ and $I_R(\alpha_1, T)$ can be solved the same way as $\int_{T_0}^{\alpha_1} P_c(t - T_0)\,dt$.