SKEWNESS OF GRAPHS

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A project report submitted in partial fulfilment of the requirements for the award of Bachelor of Science (Honours) Applied Mathematics with Computing

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April 2020

DECLARATION

I hereby declare that this project report is based on my original work except for citations and quotations which have been duly acknowledged. I also declare that it has not been previously and concurrently submitted for any other degree or award at UTAR or other institutions.

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APPROVAL FOR SUBMISSION

I certify that this project report entitled **"SKEWNESS OF GRAPHS"** was prepared by **TAN CHUNG YUEH** has met the required standard for submission in partial fulfilment of the requirements for the award of Bachelor of Science (Honours) Applied Mathematics with Computing at Universiti Tunku Abdul Rahman.

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Last but not least, I would like to my family members for their unconditional and emotional support. I was able to carry out this project without much stress thanks to their moral support.

ABSTRACT

Graph Theory is the study of graphs and its properties. In graph theory, a *graph* is a drawing represented by vertices (points) and edges (lines). A *plana*r graph is a graph which can be redrawn on plane so that its edges do not cross each other. In this project, we look into the *skewness* of graph, which is the minimum number of edges whose deletion results in a planar graph. In particular, we look into the skewness of a family of cubic graphs, denoted $H(n, k)$.

The problem of determining the skewness of a graph is known to be NP-complete. In past research papers, the skewness of some families of graphs have been determined which includes the complete graph, complete bipartite graph, complete multipartite graph, n-cube and the generalized Petersen graph. The method used for determining the skewness mostly involve using graph theory related concepts and properties on a case-by-case basis. A well-known and recurring theorem used in this research is the Euler's formula. Other useful theorem for finding the skewness includes the Handshaking lemma for planar graphs and Kuratowski's theorem.

Some of the important preliminaries and definitions regarding graphs are given. The mathematical definitions of various families of graphs are also clearly defined together with a survey on their respective skewness. The main results of this research is presented in Chapter 4. For the graph $H(n, k)$, we first start with the simplest form of the graph which is $H(7,5)$, also known as the Heawood graph. This project presents a bound for the skewness of the graphs $H(n, 5)$ in which the relevant logic and methods used are explained in detail and also a conjecture for the skewness of $H(n, 5)$.

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CHAPTER 1

INTRODUCTION

1.1 Introduction

Graph theory is one of many areas of study in mathematics. It focuses on the field of graphs in which the techniques of drawing graphs, finding attributes of graphs and its application are studied. These graphs are different from the statistical graphs in which there are x-axis and y-axis.

In graph theory, a *graph* G consists of vertices and edges joining these vertices. The vertices are points on the plane and the edges connect vertices by lines.

In this project, we will be focusing on one of the branches of graph theory which is the *skewness* of graphs. Before we go into the skewness of graphs, we need to know a fundamental concept in graph theory which is *planarity*. A graph is called *planar* if it can be redrawn in the plane such that there is no crossings of edges.

Figure 1.1: Planar (left) and Non-planar (Right) Drawings of K_4

From there, the *skewness* of a graph G is the minimum number of edges to be deleted in order to obtain a planar graph, and is denoted by *sk(G)* (Chia and Lee, 2005). It is also a way of graph planarization. Another closely related concept is the crossing number of graphs. The *crossing number* of a graph is the minimum number of crossings (intersections) of its edges among all the drawings of a graph (Chia and Lee, 2005).

1.2 Importance of the Study

A graph can be a representation of some real world situations and information can be obtained depending on how we interpret it.

For instance, a graph can be used to model a road map by treating its vertices as locations and edges as roads. The crossing of edges indicates that there will be a junction between the roads and hence there will be a disruption of traffic flow (assuming multi-level roads are not an option). In this case, finding the crossing number would mean finding the road design in which there are minimum number of junctions and finding the skewness would mean finding to the number of roads to be removed so that there can be no junctions.

The skewness of graphs also has applications in computer science field such as printed circuit board layout and VLSI (Very Large Scale Integration) circuit routing (Cimikowski, 1992). Generally, in many applications of graphs, it is often important to find a drawing that is easily readable (Czap and Hudak, 2013). Obviously, a graph with no crossings would be a better view than those with crossings.

1.3 Problem Statement

The problem of planarizing graphs and finding skewness had begun decades ago, but until now the problem is known to be NP-complete (Liu and Geldmacher, 1979).

The problem of determining the skewness of graphs can involve the use of various techniques and graph operations. Even so, the complexity of the problem still depends on the complexity of the graphs in which the skewness we want to determine. In this project, we consider the family of cubic graphs which contain the Heawood graph as a special case which is denoted as $H(n, k)$.

1.4 Objectives

The objectives of this project is:

1. To study on and understand some known results regarding the skewness of graphs
2. To survey on the techniques used by others in finding the skewness of graphs
3. To attempt to obtain some new results regarding the skewness of the graph $H(n, k)$

1.5 Scope

This project revolves around the *planarization* and determining the *skewness* of families of graphs. The final goal is to determine the skewness for the generalized case. The skewness of the graph that we want to find is the family of cubic graphs $H(n, k)$. We will first start with the graphs of $H(n, 5)$ which is the smallest possible form of this family.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

This section provides some of about the results obtained from past papers regarding the skewness of some families of graphs.

2.2 Literature Review

In the process of finding the skewness of graphs, Chia and Lee (2009) proved several theorems which are used for the determination of skewness of some of graphs. One of it is the formula defined as

$$\pi(G) \coloneqq \left\lceil q - \frac{g}{g-2}(p-2) \right\rceil$$

where p and q denote the numbers of vertices and edges in G respectively and g is the girth of G.

Another theorem related to this is for any connected graph G, $sk(G) \geq \pi(G)$. Also, Chia and Lee (2009) stated that if a planar subgraph can be obtained from G by deleting s edges, then $sk(G) \leq s$. These two theorems are frequently used in this problem.

2.2.1 Complete Graph and Complete Bipartite Graph

A *complete graph* is a graph where every vertex is connected all other vertices in the graph, and it is denoted by K_n with if it has n vertices.

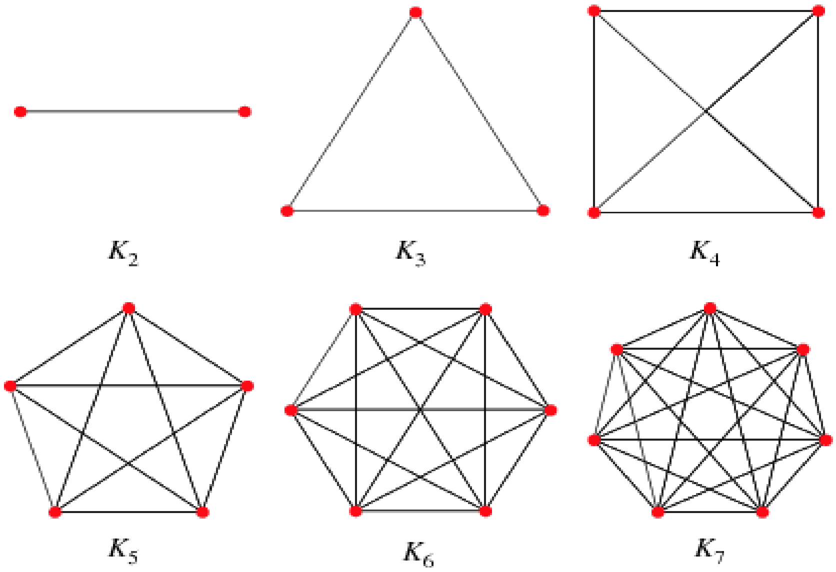

Figure 2.1: Complete Graph, K_n

A *bipartite graph* is a graph whose vertex set can be partitioned into two independents sets and that vertices from the same set are never connected to each other. A *complete bipartite graph* is a bipartite graph such that a vertex is connected to every other vertices which are not from the same set, and is denoted by $K_{m,n}$ with the two vertex sets having m and n vertices respectively (Sim, 2014).

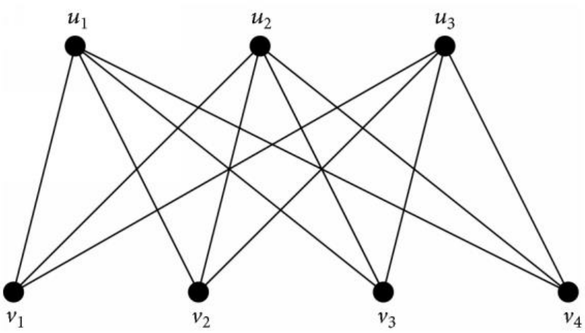

Figure 2.2: Complete Bipartite Graph, $K_{3,4}$

Chia and Lee (2009) stated that for both of these graphs, the skewness is $sk(G) = \pi(G)$. Note that by substituting q in the formula $\pi(G)$ with the respective formula for number of edges, $\pi(G) = \binom{n-3}{2}$ for the complete graph and $\pi(G) = (m-2)(n-2)$ for the complete bipartite graph.

2.2.2 The *n*-cube

According to Chia and Lee (2009), the definition of n-cube, denoted Q_n is the complete graph K_2 if $n = 1$, while for $n \geq 2$ it is $Q_{n-1} \times K_2$. The operation $Q_{n-1} \times K_2$ is defined as the Cartesian product between Q_{n-1} and K_2. The skewness of the n-cube, $sk(Q_n) = \pi(Q_n)$ for $n \geq 2$, with $\pi(Q_n) = 2^{n-1}(n - 4) + 4 \; for \; n \geq 2$.

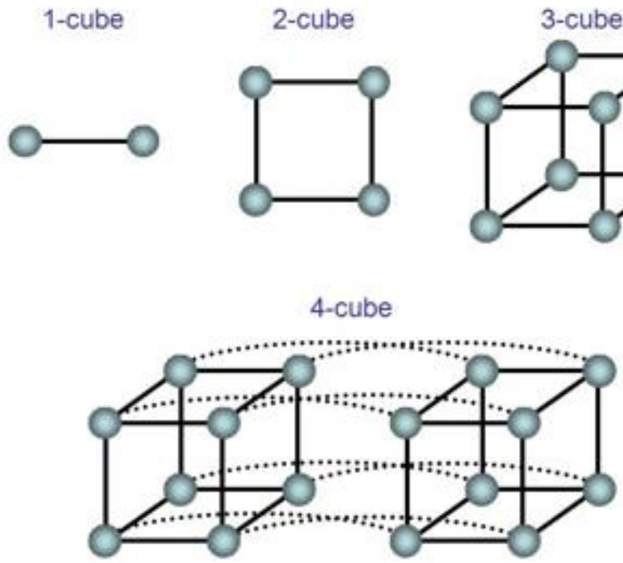

Figure 2.3: The n-cube, Q_n

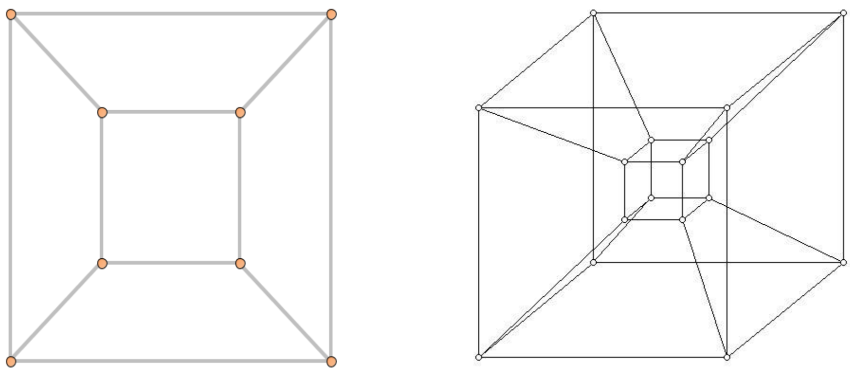

Figure 2.4: 3-cube (left) and 4-cube (right)

2.2.3 Complete Multipartite Graph (*k*-partite graph)

The multipartite graph or *k*-partite graph is a graph whose vertices can be partition in *k* different independent sets. Previous section has shown the skewness for the complete bipartite graphs, where *k* = 2. Here, we will consider a higher number *k*.

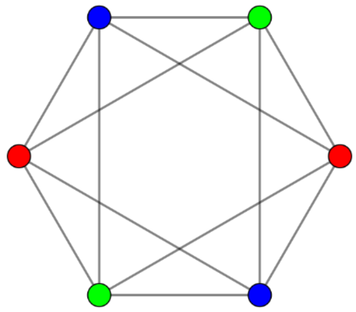

Figure 2.5: The complete tripartite graph, $K_{2,2,2}$

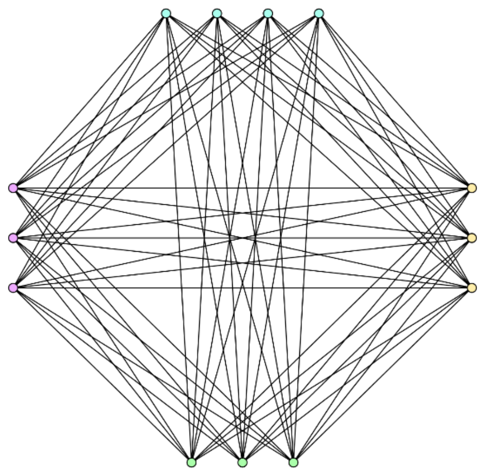

Figure 2.6: The complete 4-partite graph, $K_{3,3,3,4}$

The skewness of the complete tripartite graphs, $K_{m,n,r}$ and complete 4-partite graphs, $K_{m,n,r,s}$ has been completely obtained by Chia and Sim (2013). The steps for obtaining these results are long and tedious and involved the usage of multiple previously defined theorems. For k ≥ 5, the skewness has yet to be determined as it involved more cases and large amount of computation, and different techniques may need to be considered for such process (Chia and Sim, 2013).

2.2.4 Generalized Petersen Graph

The *generalized Petersen graph* P(n, k) is defined as a graph having the vertex set

$$\{u_i, v_i : i = 0, 1, \ldots, n - 1\}$$

and edge set

$$\{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : i = 0, 1, \ldots, n - 1 \; with\; subscripts\; reduced\; modulo\; n\}$$

where *n* and *k* are integers and $1 \leq k \leq n - 1$ (Chia and Lee, 2005). There are quite a few papers that delved into the *crossing numbers* and *skewness* of the generalized Petersen graphs.

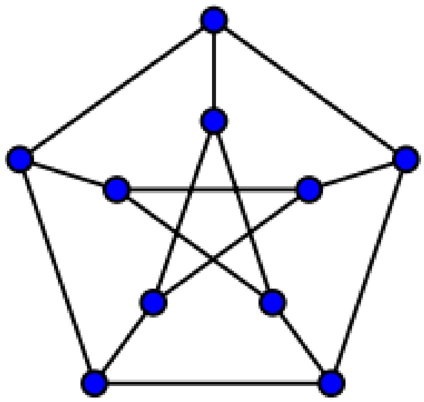

Figure 2.7: Petersen Graph, P(5, 2)

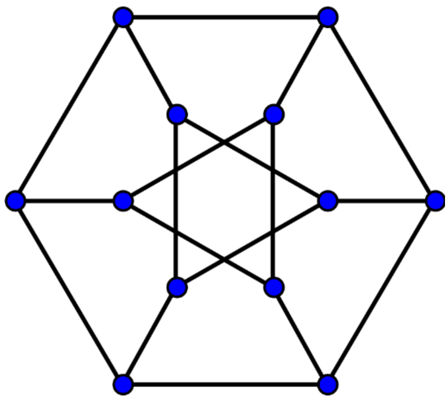

Figure 2.8: Generalized Petersen graph P(6, 2)

The skewness of some cases of the generalized Petersen graphs such as $P(3k, k)$ and $P(4k, k)$ have been determined by Chia and Lee (2005, 2009, 2012), and Chia, Lee and Ling (2019). Similarly, the proofs for the generalized

Petersen graphs involved long and tedious steps as the graph become very complex as the order of the graph increases. The skewness for the generalized case $P(sk,k)$ has not been determined and it likely involve more complicated steps.

2.3 Summary

The skewness of some families of graphs have been explored. However, there are still a lot of families of graphs in which their skewness are yet to be determined. Even the skewness for the generalized Petersen graph has not yet been completely found.

As of now, there are still no exact method that can be used to find the skewness of an arbitrary non-planar graph (NP-completeness). Naturally, we instead explore the families of graphs so that a formula can be derived. However, depending on the complexity of the graph, some conventional methods may not be reliable and different approach need to be considered.

CHAPTER 3

METHODOLOGY

3.1 Introduction

This section introduces the methods used in finding the skewness of graphs.

3.2 Requirement

As mentioned previously, the *skewness* of graphs is a subtopic of graph theory. Various concepts derived in graph theory are needed in order to explored this topic. Hence, the fundamental knowledge of graph theory is compulsory.

3.3 Methodology

The following section shows the proof for some of the theorems in section 2.

3.3.1 π(G)

First, from Euler's polyhedron formula we have

$$p - q + f \ \geq 2 \tag{3.1}$$

where *p*, *q* and *f* denote the numbers of vertices, edges and faces in *G* respectively and *g* is the girth of *G*.

Let G be a plane graph with girth g. Note that the girth $g \geq 3$ (because G is a simple graph). Let f_k denote the number of k-faces in G. Then $f \ = \sum_{k \geq g} f_k$ and $2q \ = \sum_{k \geq g} kf_k$ and we have

$$2q \ \geq gf \tag{3.2}$$

By eliminating f in (1) and (2), we have the inequality

$$q \ \leq \frac{g}{g-2}(p-2). \tag{3.3}$$

Hence we defined the formula π(G) to be

$$\pi(G) \coloneqq \left\lceil q - \frac{g}{g-2}(p-2) \right\rceil.$$

Next, let H be a planar graph obtained by removing sk(G) edges from G. Then H has $q - sk(G)$ edges with girth g'. Obviously, $g' \geq g$ and by the previous theorem, $q - sk(G) \leq \frac{g'(p-2)}{g'-2} \leq \frac{g(p-2)}{g-2}$. Since $sk(G)$ is an integer, we have $sk(G) \geq \left\lceil q - \frac{g(p-2)}{g-2} \right\rceil$ and this leads to the result

$$sk(G) \geq \pi(G)$$

(Chia and Lee, 2009).

3.3.2 Complete Graph and Complete Bipartite Graph

Recall that for complete graph, $\pi(G) = \binom{n-3}{2}$ and $\pi(G) = (m-2)(n-2)$ if G is the complete bipartite graph. It is known from previous theorem that $sk(G) \geq \pi(G)$. In order for $sk(G) = \pi(G)$, an upper bound $sk(G) \leq \pi(G)$ is needed, which means we need to obtain a planar subgraph H by removing π(G) edges from G.

Since these two families of graph are not very complicated, H can be obtained simply by inspection. For the complete graph K_n, H can be obtained by taking two non-adjacent vertices and connecting them with every vertex of an (n – 2)-cycle. This can be done by placing one vertex inside the cycle and another outside the cycle.

For the complete bipartite graph $K_{m,n}$, if we label the two partite sets of vertices as $u_0, u_1, \dots, u_{m-1}$ and $v_0, v_1, \dots, v_{n-1}$ and remove $(m-2)(n-2)$ edges $u_i v_j, where\ i = 1,2,\dots,m-2\ and\ j = 1,2,\dots,n-2$, a planar subgraph can be obtained (Chia and Lee, 2009).

3.3.3 The n-cube

In order to prove $sk(Q_n) = \pi(Q_n)$ for $n \geq 2$, we first need the following theorem (due to Chia and Lee (2009)) concerning the skewness of Cartesian product of graphs. This states that, for a graph G with $p \geq 4$ vertices and having girth 4 and suppose $sk(G) = \pi(G)$, then $sk(G \times K_2) = 2sk(G) + p - 4$ and hence $sk(G \times K_2) = \pi(G \times K_2)$ (Chia and Lee, 2009).

Clearly, $sk(Q_2) = 0 = \pi(Q_2)$ and $sk(Q_3) = 0 = \pi(Q_3)$. By induction, assume $sk(Q_{n-1}) = \pi(Q_{n-1})$. Then $sk(Q_n) = sk(Q_{n-1} \times K_2) = \pi(Q_{n-1} \times K_2) = \pi(Q_n)$ (Chia and Lee, 2009).

3.4 Summary

The proof for the *generalized Petersen graph* and *multipartite graph* is different and tedious, but the general idea for the method in finding the skewness is shown. Since the lower bound for the skewness, $sk(G) \geq \pi(G)$ has been proven, we then attempt to find the upper bound. Finally, we will attempt to find an equation for the skewness using the bound.

This is usually done by starting with inspection on the graph. After the calculation of $\pi(G)$, we will do trial and error on the graph. For simple graphs (such as the complete graphs or complete bipartite graphs), perhaps the skewness can be obtained from there. For more difficult graphs, we usually do some inspection first before moving on to a more systematic way, which may involve tedious steps as shown in the proofs for the multipartite graphs and the generalized Petersen graph.

CHAPTER 4

RESULTS AND DISCUSSION

4.1 Introduction

This section introduces the graph of interest, which is a family of cubic graphs which contain the Heawood graph as a special case, and is denoted *H(n, k)*.

4.2 H(n, k)

The graph *H(n, k)* is defined as a graph consisting of a $2n$-cycle $x_0x_1 \dots x_{2n-1}x_0$, together with n edges of the form $x_{2i}x_{2i+k}$, $i = 0,1, \dots, n-1$ with subscript reduced to modulo $2n$ and k is odd. It is also a bipartite graph as its vertices can be partitioned into two sets.

4.3 H(n, 5)

The Heawood graph is the graph $H(7,5)$ and is also the simplest form of the family of graph *H(n, k)*.

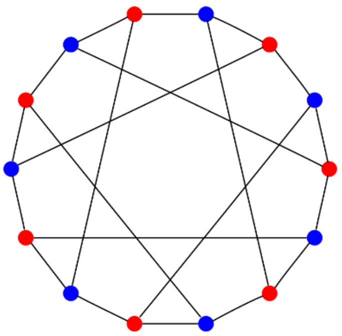

Figure 4.1: Heawood graph

It is easy to see that the number of edges of a $H(n,k)$ graph is $3n$. Note that for any graph $H(n,5)$, the girth is 6. By substituting in the formula, we get $\pi\big(H(n,5)\big) = 3$ and hence $sk\big(H(n,5)\big) \geq 3$. We are left with determining the upper bound for the skewness of Heawood graph.

4.3.1 H(7, 5)

The following figure shows a subgraph of the Heawood graph by deleting 3 edges, namely x_1x_{10}, x_3x_{12} *and* x_4x_9. The deleted edges are illustrated with red line on the original graph and dotted line on the resulting graph.

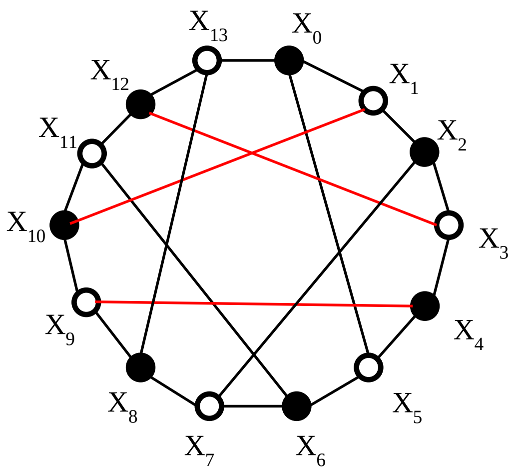

Figure 4.2: H(7, 5)

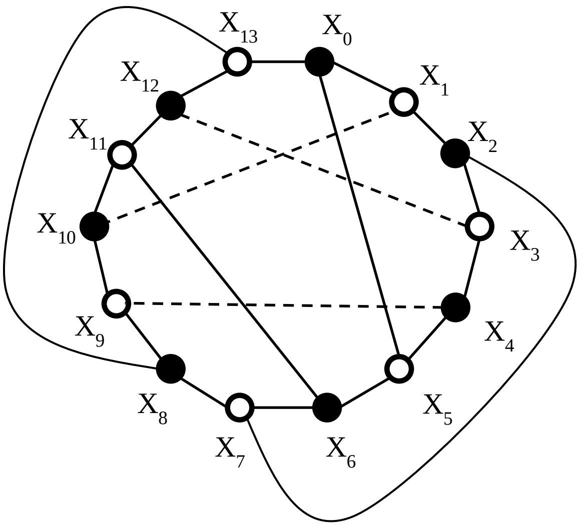

Figure 4.3: Planar subgraph of H(7, 5)

Since a planar subgraph can be obtained by deleting 3 edges, thus $sk\big(H(7,5)\big) \leq 3$. Since we obtained the same number from the lower and upper bound, we can directly conclude that $sk\big(H(7,5)\big) = 3$.

From the Euler's formula, we know that if we delete r edges from $H(n,5)$, then the number of faces of the resulting subgraph is

$$f = n + 2 - r. \tag{4.1}$$

Since the girth of $H(n,5)$ is 6, then we also have

$$f = f_6 + f_8 + f_{10} + f_{12} + \cdots . \tag{4.2}$$

where f_i is the number of i faces. Eliminating f in (4.1) and (4.2), we will get

$$n + 2 - r = f_6 + f_8 + f_{10} + f_{12} + \cdots . \tag{4.3}$$

Also, from the Handshaking lemma for planar graph, we know that the sum of face degrees for a connected planar graph with f faces and q edges is

$$2q = \sum_{i\geq 3} 2if_{2i} \tag{4.4}$$

Expanding (4.3) and substituting $q = 3n - r$ will give us

$$6n - 2r = 6(f_6 + f_8 + f_{10} + f_{12} + \cdots) + 2f_8 + 4f_{10} + 6f_{12} + \cdots . \tag{4.5}$$

Finally, combining (4.3) and (4.5) will result in

$$r = 3 + \frac{f_8}{2} + f_{10} + \frac{3f_{12}}{2} + \cdots . \tag{4.6}$$

Equation (4.6) tells us the number of non 6-faces of the resulting subgraph must have if we delete r edges from $H(n,5)$. If $r = 3$, then the subgraph must have all faces with degree 6. This can be used to verify the validity of the subgraph when the skewness is equal to 3.

4.3.2 H(8, 5)

Next for the graph $H(8,5)$, a planar subgraph is obtained by deleting 3 edges. Hence, we can conclude that $sk(H(8,5)) = 3$. The edges deleted are different from $H(7,5)$ where we only delete the inner edge while for $H(8,5)$ we delete 2 inner edges and 1 edge from the 2n-cycle. The edges deleted in the illustration below are $x_4x_9,\ x_{10}x_{15}\ and\ x_{13}x_{14}$.

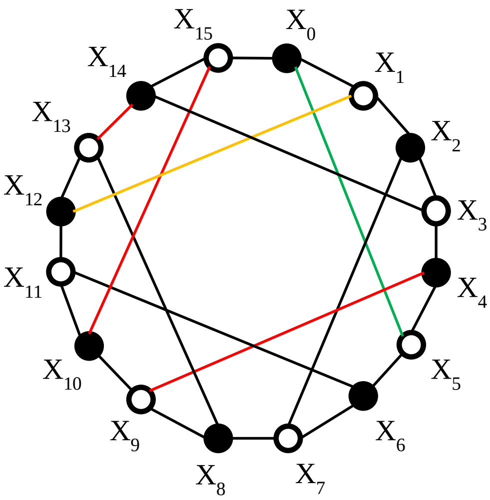

Figure 4.4: H(8, 5)

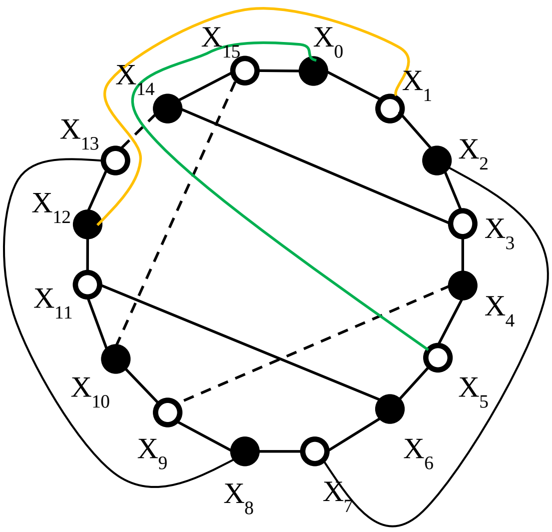

Figure 4.5: Planar subgraph of H(8, 5)

4.3.3 H(9, 5)

Similarly for the graph $H(9,5)$, a planar subgraph can obtained by deleting 3 edges, namely x_4x_9, x_3x_{16} *and* $x_{10}x_{15}$ and $sk\big(H(9,5)\big) = 3$.

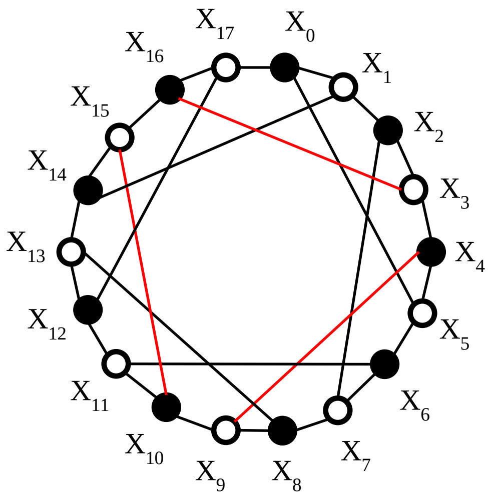

Figure 4.6: H(9, 5)

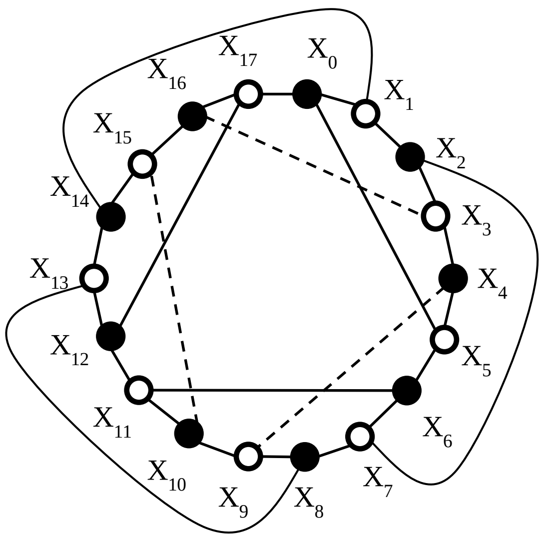

Figure 4.7: Planar subgraph of H(9, 5)

4.3.4 Upper bound for sk(H(n, 5))

Using similar method that is done for the graphs $H(7,5)$, $H(8,5)$ and $H(9,5)$, the number of edges to be removed is 4 for the graphs $H(10,5)$, $H(11,5)$ and $H(12,5)$.

By close inspection, it is found that for this particular subgraph method, there is a recurring pattern after every increment of *n* by 3. With this, we can separate the graphs of $H(n,5)$ into three groups. Let *n* be in one of the following form:

i) $n = 3m - 2$
ii) $n = 3m - 1$
iii) $n = 3m$

where $m \geq 3$. Then, we can obtain a planar subgraph by deleting a set of *m* edges depending on which form *n* belongs to:

i) $n = 3m - 2$
Delete the set of edges $\{(x_{4+6i}, x_{9+6i}) \mid i = 0,1,2,\dots,m-2\}$ and (x_{6m-6}, x_{6m-1}) with subscripts reduced modulo $2n$

ii) $n = 3m - 1$
Delete the set of edges $\{(x_{4+6i}, x_{9+6i}) \mid i = 0,1,2,\dots,m-2\}$ and (x_{6m-1}, x_{6m}) with subscripts reduced modulo $2n$

iii) $n = 3m - 2$
Delete the set of edges $\{(x_{4+6i}, x_{9+6i}) \mid i = 0,1,2,\dots,m-1\}$ with subscripts reduced modulo $2n$

Theorem 1 *Suppose G is a graph of the family $H(n,5)$ where $n \geq 7$, then*

$$3 \leq sk(G) \leq \left\lceil \frac{n}{3} \right\rceil.$$

4.3.5 H(10, 5) to H(20, 5)

Consider the graph $H(10,5)$, from Theorem 1 the skewness of $H(10,5)$ is either 3 or 4. However, it is found that $sk\big(H(10,5)\big) = 3$. The way of deleting edges differs from the previous 3 graphs.

By trial and error, it is found that a planar subgraph can also be obtain for $H(11,5)\ to\ H(20,5)$ by deleting the 3 edges which are x_0x_{2n-1}, x_1x_{2n-4} and x_3x_{2n-2}. Hence, we can also conclude that their skewness is also 3. However, the way that the subgraph is drawn does not mirror the original graph anymore.

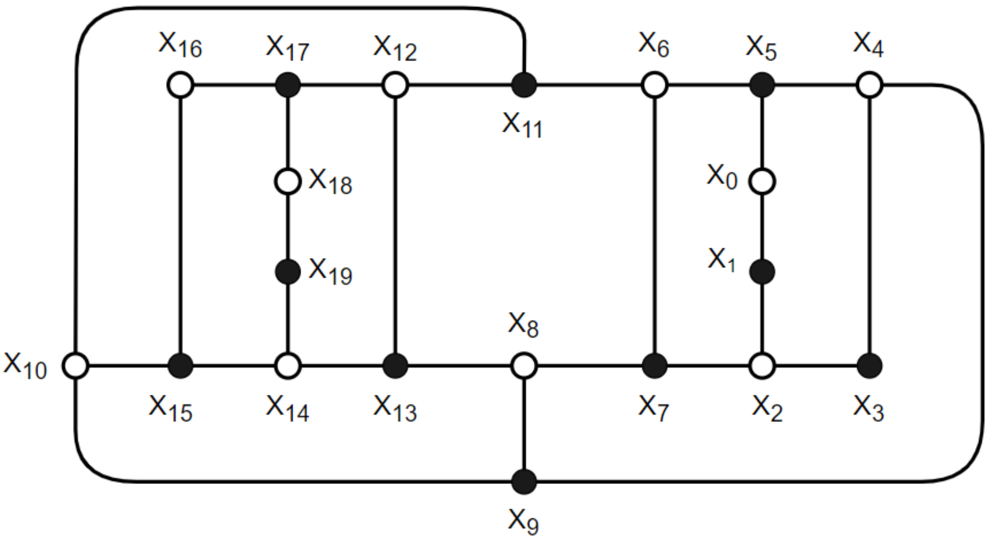

Figure 4.8: Planar subgraph of H(10, 5)

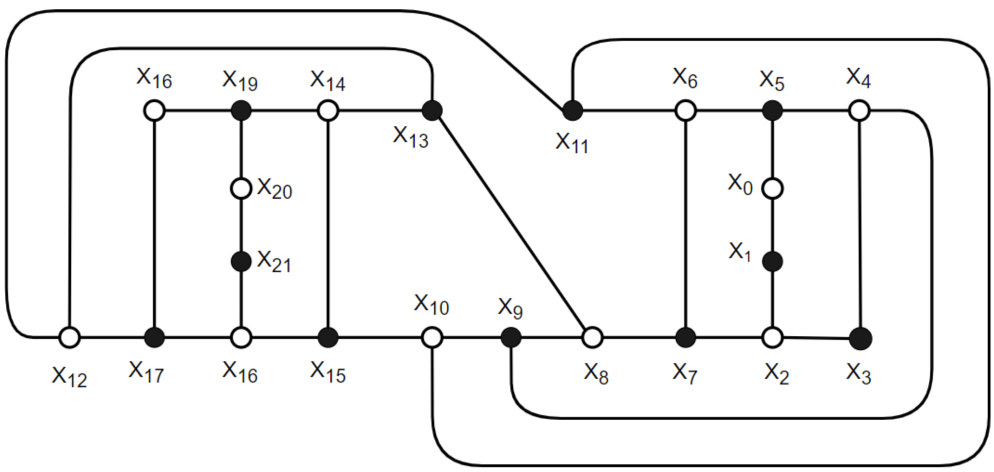

Figure 4.9: Planar subgraph of H(11, 5)

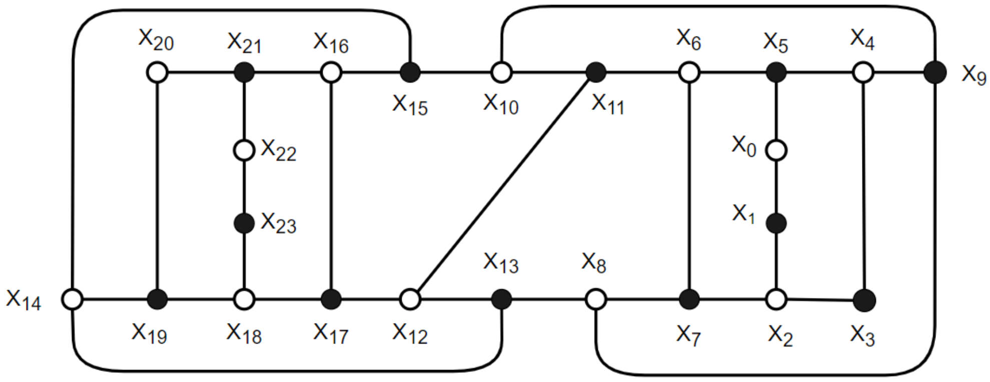

Figure 4.10: Planar subgraph of H(12, 5)

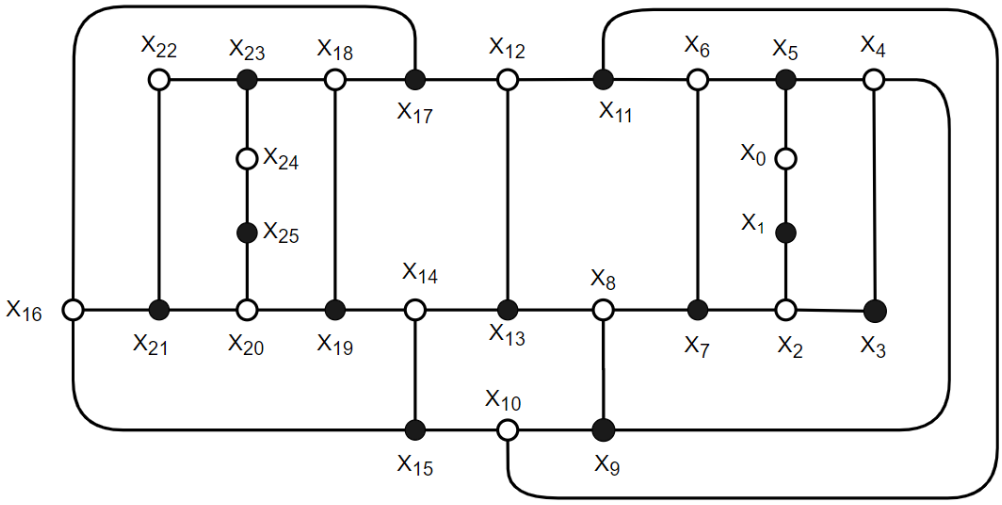

Figure 4.11: Planar subgraph of H(13, 5)

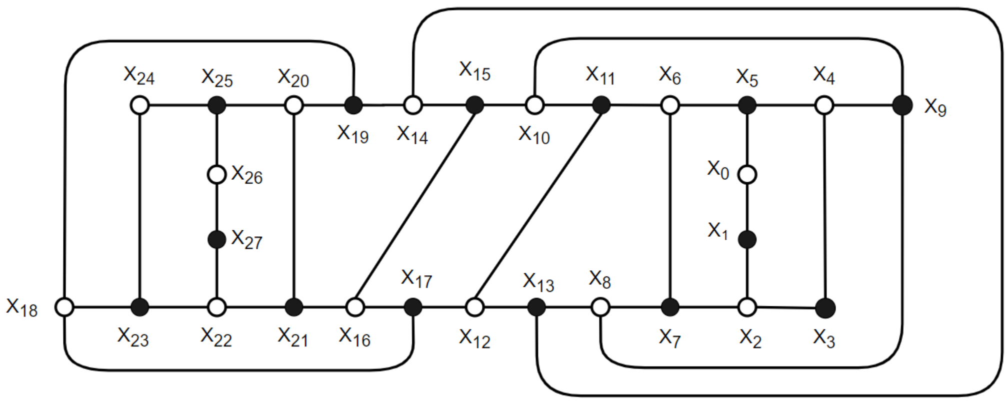

Figure 4.12: Planar subgraph of H(14, 5)

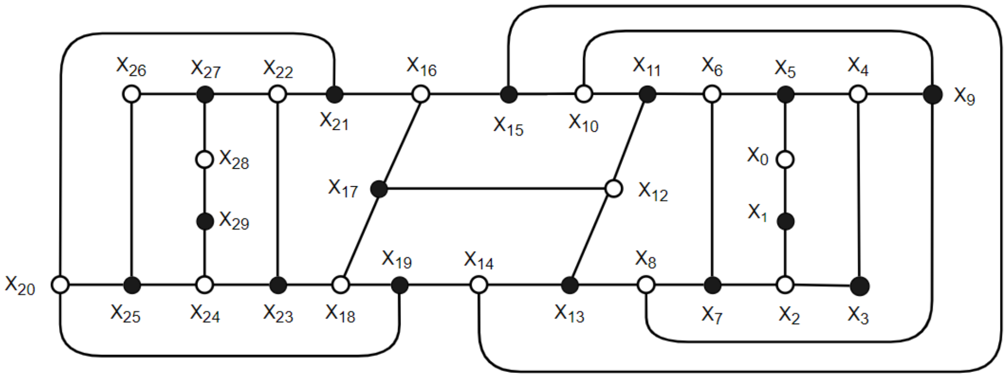

Figure 4.13: Planar subgraph of H(15, 5)

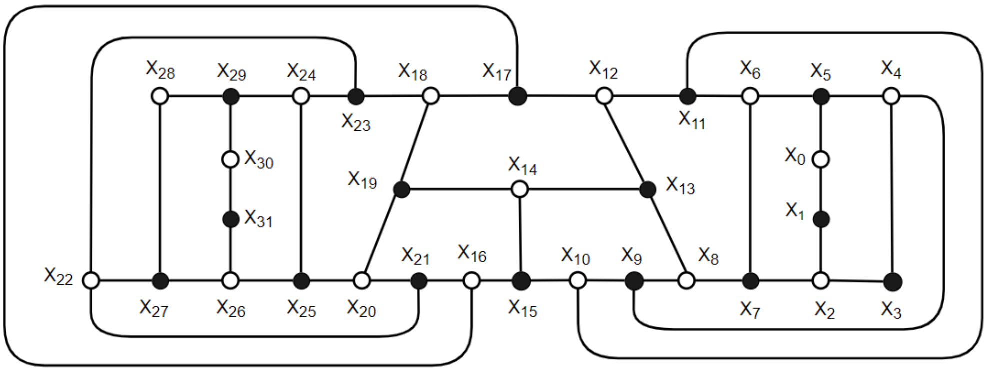

Figure 4.14: Planar subgraph of H(16, 5)

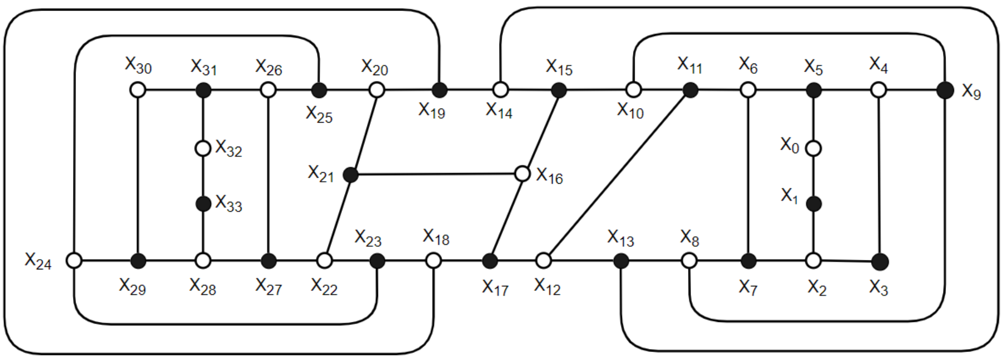

Figure 4.15: Planar subgraph of H(17, 5)

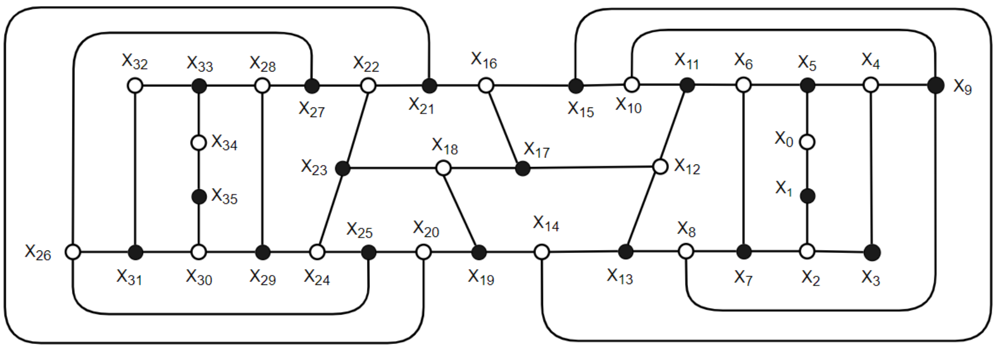

Figure 4.16; Planar subgraph of H(18, 5)

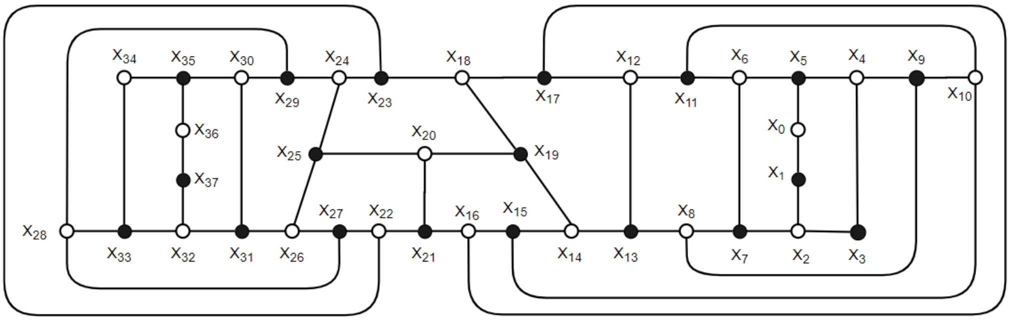

Figure 4.17: Planar subgraph of H(19, 5)

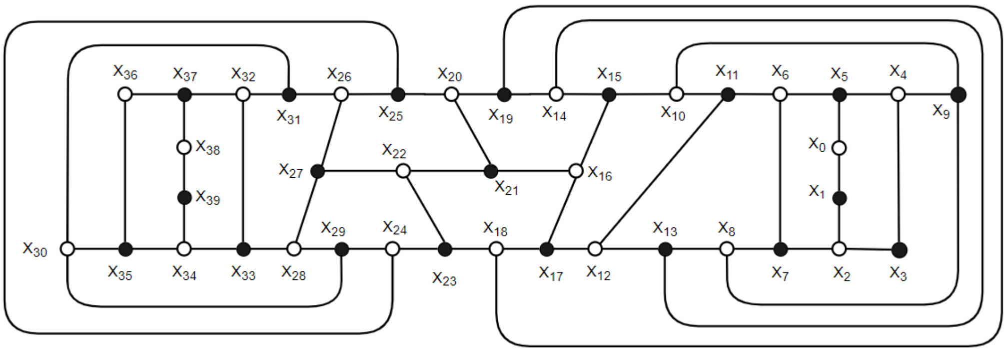

Figure 4.18: Planar subgraph of H(20, 5)

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusion

The skewness of $H(n,5)$ for $n = 7,8,9,\ldots,20$ has been found. Even when n increases to up to 20, the skewness remains to be 3. It may be true that $sk\big(H(n,5)\big) = 3$ for $n \geq 7$. In that case, we may need to derive a general way to draw the planar subgraph as the ones shown in this project are simply done by exhaustion.

5.2 Recommendations for future work

One may need to look into the faces of graphs to find the general case of $H(n,5)$. Since we know that the degree of all the faces must be equal to 6 if the skewness is 3, we may be able to find the general way of drawing from there. Also, the skewness of $H(n,k)\ for\ k \geq 7$ may also be found using the same method shown for $H(10,5)$ to $H(20,5)$.

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