GEOMETRIC DISSECTION

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A project report submitted in partial fulfilment of the requirements for the award of Bachelor of Science (Honours) Applied Mathematics With Computing

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DECLARATION OF ORIGINALITY

I hereby declare that this project report is based on my original work except for citations and quotations which have been duly acknowledged. I also declare that it has not been previously and concurrently submitted for any other degree or award at UTAR or other institutions.

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APPROVAL FOR SUBMISSION

I certify that this project report entitled "**GEOMETRIC DISSECTION**" was prepared by **LEONG YEE HANG** has met the required standard for submission in partial fulfilment of the requirements for the award of Bachelor of Science (Honours) Applied Mathematics With Computing at Universiti Tunku Abdul Rahman.

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Date : 3 September 2020

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ABSTRACT

At the beginning of this project, the dissection of some polygons were studied and analysed. One of them is the solution of Haberdasher's problem which is a four-pieces dissection from an equilateral triangle to a square given by Henry Dudeney. His original construction idea is applied to construct the dissection from a square to an equilateral triangle.

After that, equidecomposability of polygons and polyhedra are discussed. Wallace-Bolyai-Gerwien Theorem states that any polygons with same area are equidecomposable. Two proofs for this theorem are given. A stronger result tells that equidecomposable polygons have a common hinged dissection. Hilbert's Third Problem asks whether two polyhedra of equal volume are equidecomposable. Max Dehn gave an negative answer to this problem. A recent alternative solution based on Bricard's condition is studied.

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CHAPTER 1

INTRODUCTION

1-1 Background & History

Dissection puzzles such as Tangram and T-puzzle were one of the childhood toys for most people. The toy usually consists of puzzle pieces and silhouettes printed on a booklet. The rule is simple: one needs to arrange the puzzle pieces to match the silhouettes.

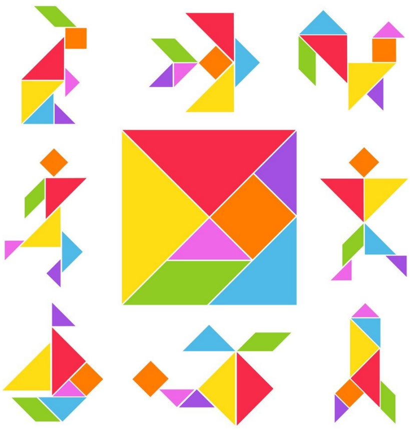

Figure 1.1: Tangram

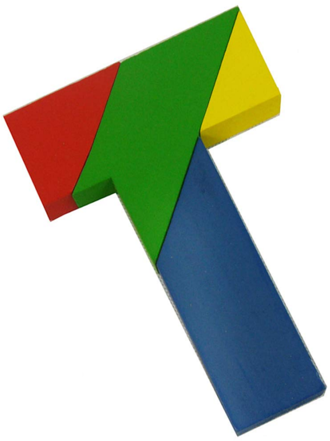

Figure 1.2: T-Puzzle

The history of dissection puzzles can be traced back to the times of Ancient Greek. Archimedes' Stomachion is a dissection puzzle similar to Tangram. It has 14 puzzle pieces which can be arranged into many different shapes like the other dissection puzzles. However, the main problem associated with Stomachion is the numbers of different ways of arranging the pieces to form a square. This problem had already been solved. There are 268 unique arrangements in which no two arrangements are congruent in terms of rotation and reflection.

Compared to solving the dissection puzzles, creating such interesting dissection puzzles is generally more challenging. This is because one needs to know how to dissect a given shape into pieces so that it can be reassembled into another shape. A mathematical study of this problem is called *geometric dissection*.

Other than puzzles, geometric dissection could be related to real-life problems.

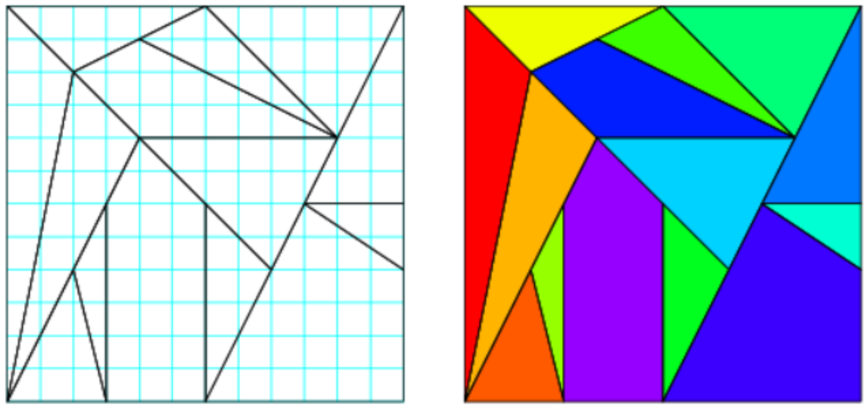

Figure 1.3: Stomachion

Gardner (1977) showed an example: a primitive man had a piece of animal skin, but it was not in desired shape. He had to find a way to cut it into pieces and sew the pieces into desired shape. This may be the first dissection problem encountered by a human. A modern version of this example would be seen when a product is designed to be in a certain shape but the material is manufactured in some other shape. The processing plant has to process the material in order to make the product. The problem is to minimize the processing cost, for instance, by optimizing the number of cuts and minimizing the waste of material.

A typical example of geometric dissection is Dudeney's solution (1908) of the Haberdasher's puzzle. The puzzle demands to dissect an equilateral triangle into four pieces such that the four pieces can form a square. The same example also illustrated *hinged dissection*. Hinged dissection is a special kind of geometric dissection such that a number of hinge points can be added to connect the pieces so that transformation into another shape can be done by rotating the pieces around the hinges. Hinged dissection was popularised by Dudeney; therefore, it was also known as *Dudeney dissection*.

An article by Abbott et al. (2012) suggested a good possible application of hinged dissection, that is building transformable nanobots. Non-hinged dissection is not preferred for this purpose as it might be difficult to control the transform when the parts are not physically connected.

Other than dealing with shapes on two-dimensional plane, geometric dissection can also be done for polyhedra in three-dimensional space. Dissection of polyhedra is sometimes known as *polyhedral dissection*. The concept of hinged dissection can be applied in polyhedral dissection. Instead of having points as hinges, hinged dissection

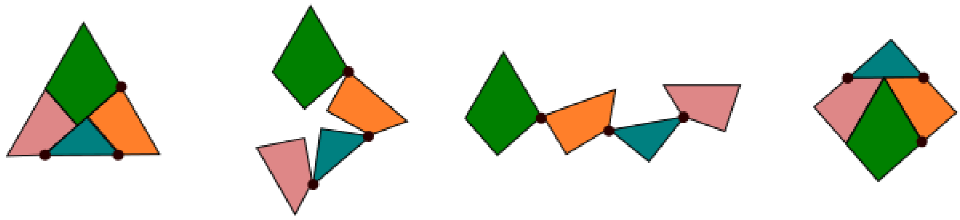

Figure 1.4: Hinged Dissection from Triangle to Square

of polyhedra uses lines as hinges.

1-2 Problem Statement

The most common type of problems in geometric dissection is to find a way to dissect a given shape into another shape. We want to find out what are the methods that could be applied to find the dissection. In terms of geometry, we want to build a procedure to construct the exact dissection from a starting polygon. For the real life purpose, we may also be interested with the numerical values of side lengths and angles of the dissected pieces.

Another interesting problem is to know how some important results are proved and whether there are different ways of proving the same result.

1-3 Objectives

The first objective of this project is to obtain general understanding in geometric dissection. A clear definition of geometric dissection for both two-dimensional and three-dimensional space has to be understood. The definition should include conditions and limitations for dissections and movement of the pieces.

Another objective is to study and research on significant or well-known results and theorems in geometric dissection. This is mainly done by reading research papers, journal articles and books. By studying the theorems and proofs, some popular methods of solving geometric dissection problems can be discovered.

Besides, the project also includes studying of some special geometric dissections. One of the famous dissections is the hinged dissection. These special dissections usually imposed some conditions upon the regular definition of geometric dissection.

The last objective is to discover new dissections or some new findings. As a challenge, some open problems related to geometric dissection may be studied. Solving the problem or part of it will be attempted.

1-4 Notation and Terminology

Since this is a project mainly based on geometry, a few geometry terms and notations will be introduced or clarified in this section.

1-4-1 Line, Ray and Line Segment

A *line* that passes through points A and B is denoted by $\overleftrightarrow{AB}$. Both ends of a line extend infinitely. $\overleftrightarrow{AB}$ is same as $\overleftrightarrow{BA}$.

Figure 1.5: Line $\overleftrightarrow{AB}$

A *ray* starting from point A that passes through point B and extends infinitely is denoted by $\overrightarrow{AB}$. Note that $\overrightarrow{AB}$ is different from $\overrightarrow{BA}$. The point that comes first is the starting endpoint of the ray.

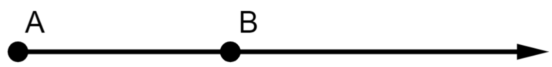

Figure 1.6: Ray $\overrightarrow{AB}$

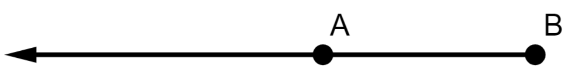

Figure 1.7: Ray $\overrightarrow{BA}$

A *line segment* $\overline{AB}$ is a straight line with endpoints A and B. $\overline{AB}$ is same as $\overline{BA}$. For convenience and tidiness, we would just write AB instead of $\overline{AB}$ to denote line segment.

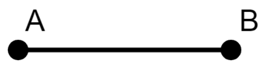

Figure 1.8: Line Segment AB

We cannot measure the length of a line or a ray since they extends infinitely but we can measure the length of a line segment. For AB, we denote it length by $|AB|$.

1-4-2 Tetrahedron

A *tetrahedron* is a polyhedron bounded by 4 triangles as illustrated in Figure 1.9. If all the 4 triangle are equivalent, we say that it is a *regular tetrahedron*.

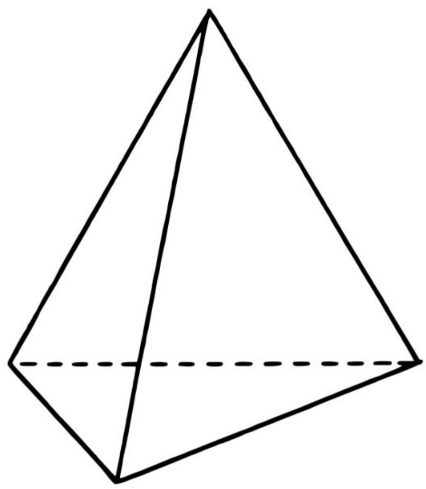

Figure 1.9: A Tetrahedron

In this report, we will use the terminologies stated to prevent confusion since some authors refer (irregular) tetrahedra as "triangular pyramids" and the regular tetrahedron as "tetrahedron".

1-5 Project Planning

This project is split into two phases: Project I and Project II. Each phase would take one semester to complete. The planning for both phases are shown below.

1-5-1 Project I

Week	Plan
6 - 10	Obtain general understanding of geometric dissection.
11 - 12	Prepare interim report and presentation of the report.

1-5-2 Project II

Week	Plan
1 - 5	Research and analyse known results and their proofs.
6 - 10	Attempt to solve problems / discover new dissection.
11 - 12	Finish final report and prepare for presentation.

CHAPTER 2

LITERATURE REVIEW

It has been more than a century since geometric dissection is studied extensively by various famous mathematician. A lot of results regarding geometric dissection have been published in books, research papers and journal articles. In this chapter, the resources are studied and some results that have been published are mentioned.

2-1 Dissection Problems

A question like "how to dissect a hexagon to form a square" is called a *dissection problem*. The solution to it is called a *dissection*. Several dissections published in some books and articles are studied.

A good book to start with is *Dissections: Plane and Fancy* written by Frederickson (2003). This book is excellently written such that some important concepts are introduced and they can be easily understood by beginners in geometric dissection. The book consists of several chapters where each chapter discusses different topics such as dissections of regular polygons, symmetrical dissections, dissections of shapes with curve lines and more. A few chapters near the end of the book are mainly about dissections of polyhedra in three-dimensional space.

One of the chapters discusses how tessellation can be used to create dissection easily. It is done first by finding ways to tessellate the given shapes into a plane or a strip. Then, by *superimposing* two different tessellations in certain manner, a dissection pattern could be found. *Superimposition* of two tessellations means putting the two tessellations on top of each other so that a combined figure can be observed. Figure 2.1 shows an example of how dissection from a Greek cross to a square using tessellation. Tessellation is indeed an elegant way to discover dissections.

The book has also included a lot of interesting dissections discovered by other authors as well as by himself. The author has included some histories and well-known results regarding geometric dissection including Wallace-Bolyai-Gerwien Theorem and Hilbert's Third Problem in some chapters. Besides, there were a number of puzzles inserted throughout the book for readers to solve. The solutions of these puzzles are

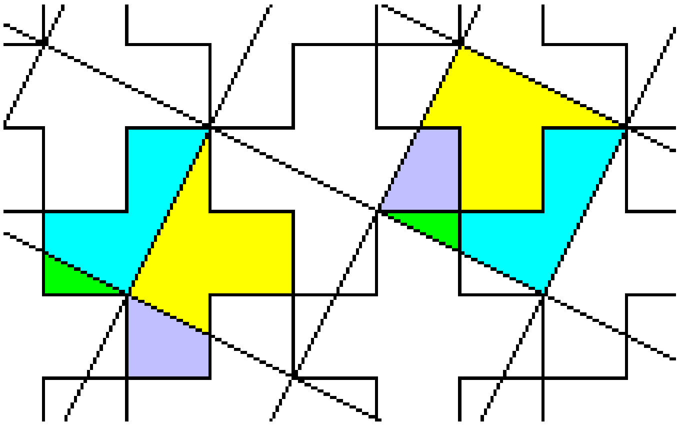

Figure 2.1: Dissection from Greek Cross to Square

printed in the last chapter.

Another book *Mathematical Recreation* authored by Kraitchik (1953) discusses about interesting problems in various areas of mathematics. A chapter in the book titled "Dissection of Plane Figures" is about dissections of polygons. This chapter starts with discussion on how to construct dissection of a rectangle into three pieces to form a square. Of course, this is impossible if the side lengths of the rectangle differ a lot. When the ratio of length of the longer side to the length of the shorter side increases, the number of pieces required increases. For the case of three pieces, two constructions are provided as solution. The first construction is limited to rectangles which have a longer side that does not exceed the double of the shorter side. Another construction is better as it only requires the longer side of the rectangle to be not exceeding four times of the shorter side. This construction will be discussed in details in Section 3-2 of this report.

The other problems discussed include dissecting a square with area 3 into five pieces to form two squares with area of 1 and 2 respectively, dissecting a square into seven pieces to form three congruent squares, and dissecting a regular hexagon into five pieces to form a square. The solutions and algorithms to construct the dissections for the problems mentioned are all stated along in the chapter.

The last problem discussed in the topic is to dissect a square into a finite number of smaller squares such that all the squares are mutually incongruent. Although this problem is so difficult that it remained unsolved for a long period since the problem was proposed, Brooks et al. (1940) successfully found a solution to it.

2-2 Polygon and Polyhedron

In some literature, the terms *polygon* and *polyhedron* are used without stating a clear definition. Krasilnikova (2015) says that there are a lot of different definitions of polyhedron and some definitions are not compatible with each other. If we take a look at some formal mathematical definitions of polygon and polyhedron, they could involve some high-level mathematics which is difficult to understand.

Referring to the book *Proofs From THE BOOK* by Aigner and Ziegler (2018), we try to informally define polygons and polyhedra in some simpler English terms. We know that convex polygons are 2D "shapes" bounded by some straight lines. A convex polygon can be represented by a system of linear inequalities in $\mathbb{R}^2$. Similarly, convex polyhedra are 3D "shapes" bounded by some planes and we can write systems of linear inequalities in $\mathbb{R}^3$ to represent them. *Convex polytope* is the term for such convex "shapes" generalised across dimensions. A convex d-polytope is a convex polytope in d dimension and we can represent the convex d-polytope using a system of linear inequalities in $\mathbb{R}^d$. Thus, convex polygons are actually convex 2-polytopes and convex polyhedra are convex 3-polytopes. *Facets* of a convex d-polytope are the convex $(d-1)$-polytopes bounding it. Facets of a polygon are *edges* while $faces$ are facets for polyhedra. For a polyhedron, the *dihedral angle* of an edge is the angle between the two faces intersecting at the edge. A general d-polytope is a union of some non-intersecting convex d-polytopes which are attached facet-to-facet.

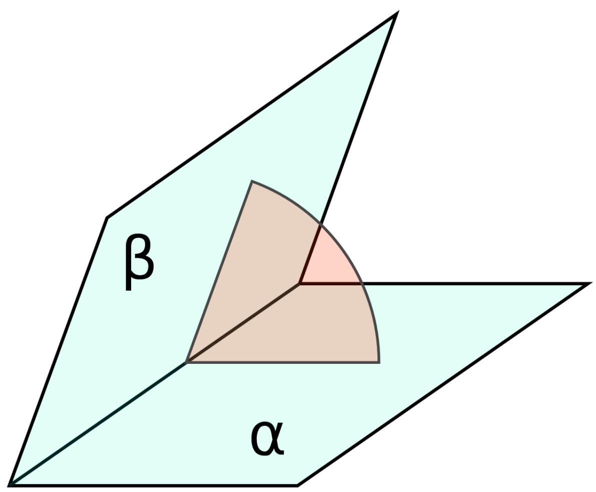

Figure 2.2: Dihedral Angle

A *simplex* in d dimension is the "smallest" d-polytope in terms of the number of facets. For example, the simplex of polygons is triangle whereas the simplex of polyhedra is tetrahedron. Simplex are said to be the generalisation of triangle into higher dimensions. According to Károlyi and Lovász (1991), a convex polytope can be decomposed into a finite number of simplices. Based on this result and how polygons and polyhedra are defined earlier, we have the following two propositions:

Proposition 1. *Any polygon can be dissected into a finite number of triangles.*

Proposition 2. *Any polyhedron can be dissected into a finite number of tetrahedra.*

These two prepositions play an important role in Wallace-Bolyai-Gerwien Theorem and Hilbert's Third Problem which are to be discussed later.

2-3 Equidecomposability

Equidecomposability is the term used to describe the possibility of cutting a polygon into smaller polygons which can be reassembled to form a second polygon. A mathematical definition of equidecomposability as in *Proofs From THE BOOK* by Aigner and Ziegler (2018) is as follows.

Definition 1 (Equidecomposability)**.** Two polygons A and B are said to be equidecomposable if A and B can be dissected into a finite number smaller polygons

$$A = A_1 \cup A_2 \cup \cdots \cup A_n$$

$$B = B_1 \cup B_2 \cup \cdots \cup B_n$$

and

$$A_i \text{ is congruent to } B_i$$

for $i = 1, 2, ..., n$.

The definition also applies in the case of three-dimension by considering A, B, A_i's and B_i's in the definition as polyhedra. Based on Propositions 1 and 2, it is always possible to further dissect the pieces A_i's and B_i's to their simplices. This gives the tweaked definition:

Definition 2 (Equidecomposability). For $d = 2$ or $d = 3$, two d-polytopes A and B are said to be equidecomposable if A and B can be dissected into a finite number of simplices (triangles if $d = 2$ and tetrahedra if $d = 3$)

$$A = A_1 \cup A_2 \cup \cdots \cup A_n$$

$$B = B_1 \cup B_2 \cup \cdots \cup B_n$$

and

$$A_i \text{ is congruent to } B_i$$

for $i = 1, 2, ..., n$.

The term equidecomposability are used in a number of books and papers including 'A problem of sallee on equidecomposable convex bodies' by Gardner (1985) and 'Hilbert's third problem (a story of threes)' by Krasilnikova (2015). *Scissors congruence* is a term which is equivalent to equidecomposability. This term is used in some papers as well such as Welsh's expository paper titled 'Scissors congruence'.

Speaking of equidecomposability, the two major questions concerned are:

1. Are any two given **polygons** of equal area are equidecomposable?

2. Are any two given **polyhedra** of equal volume equidecomposable?

The answer to the first question is "yes" which is given by the Wallace-Bolyai-Gerwien Theorem. In terms of equidecomposability, the theorem states

Theorem 2-3-1 (Wallace-Bolyai-Gerwien Theorem). *If two polygons have the same area, then they are equidecomposable.*

Note that the converse of Wallace-Bolyai-Gerwien Theorem is trivially true. Areas of the dissected pieces always sum up to be the same, and this sum must also equal to the area of the two polygon. In this case, area is called an *invariant* for equidecomposability of polygons.

The history of this theorem can be traced back to early 19th century. According to Frederickson (2003) and also the article by Ciesielska and Ciesielski (2018), William Wallace presented the problem in 1807. John Lowry was the first person who gave a proof 1814. Later, the same result was proved by Farkas Bolyai and Paul Gerwien independently in 1832 and 1833 respectively. We will further discuss Wallace-Bolyai-Gerwien Theorem in Section 4-1

As for the second question about equidecomposability of polyhedra, it is more generally referred to the Hilbert's Third Problem. Integrating the information obtained from various literature, it is found that there are quite some stories about this problem. According to Aigner and Ziegler (2018), the story began with the letters between Carl Friedrich Gauss and Christian Ludwig Gerling in 1844. There was already a simple proof based on geometric dissection that shows that two triangles with same base and same height have the same area. The idea for this proof can be seen in Lemma 4.4 in Section 4-1 of this report. However, the fact that two tetrahedra with the same base area and a same height must have equal volume were proved using calculus. In the letter to Gerling, Gauss questioned if there is a dissection between two tetrahedra of same base and same height which can be used as the proof of their equal volume.

In 1900, David Hilbert presented 23 problems which he considered important in that century that had just begun. The third problem came to our concern. The problem asks to specify two tetrahedra with same base area and same height which do not have a common dissection. From how he presented the problem, we can see that Hilbert is conjecturing that equidecomposability does not hold for some polyhedra despite having equal volume. It turned out that Hilbert's conjecture was true. Soon in the same year, Max Dehn, a student of Hilbert, solved the Hilbert's Third Problem. Other than volume, Dehn had discovered the second invariant, which is known as Dehn invariant now, for the equidecomposability of polyhedra. He showed that if two polyhedra are equidecomposable, then they have the same Dehn invariants. Then, he proved that there exists polyhedra with equal volume and different Dehn invariant. Dehn's solution is based on abstract algebra as the tensor product of modules are required to define Dehn invariant. According to Welsh (2016), Sylder successfully proved that equal volume and equal Dehn invariant are sufficient to guarantee equidecomposability of two polyhedra in 1965.

Theorem 2-3-2 (Sydler)**.** *Two polyhedra are equildecomposable if and only if they have equal volume and equal Dehn invariant.*

Some books commented that Dehn's proof was difficult to understand. Some authors had contributed to rewrite Dehn's proof. A notable simplification of the proof was given by Kagan in 1903.

Before Hilbert proposed the problem, Bricard had been working on equidecomposability of polyhedra and he published a theorem in 1896 which is currently known as "Bricard's condition". This theorem can easily show that some polyhedra of equall volume are not equidecomposable. However, the proof for Bricard's condition provided by himself was incorrect. Since then, Bricard's condition remained unproven for more than a century. In 2007, Benko published a correct proof for Bricard's condition. He used some arguments from Kagan's work which Aigner and Ziegler called "pearl lemma" and "cone lemma" to prove Bricard's condition, which becomes a second solution of Hilbert's Third Problem. We will look at the proof of this solution in Section 4-3.

In a recent paper, Ciesielska and Ciesielski (2018) claimed that Hilbert's Third Problem had already been solved in 1883, without being known by Hilbert and Dehn. They found that in 1882, an academy in Kraków, a city of Poland, held a mathematics contest where one of the questions asked was exactly the same as Hilbert's Third Problem. A mathematics teacher named Ludwik Antoni Birkenmajer submitted a correct solution for the contest. Birkenmajer's solution was independent from the ones by Dehn and Bricard. Regrettably, this result was not published globally and it was in Polish which was a language less known by the mathematics community at that time.

2-4 Hinged Dissection

According to the definition by Abbott et al. (2012), hinged dissection are dissections where every pieces are connected directly or indirectly by hinges. A hinge connects two pieces at some vertex from each of the two pieces. For a hinged dissection, when we try to move the pieces from the original polygon to form another polygon, if this is impossible to done without some pieces intersecting each other, then it is called a *wobbly hinged dissection*. They proved that non-wobbly hinged dissection exists for any two polygons, which appears to be a result stronger than Wallace-Bolyai-Gerwien Theorem. A brief discussion of this result will be discussed in Section 4-2.

Akiyama and Nakamura (1998) preferred to call hinged dissection as Dudeney dissection. In their research paper, the definition of hinged dissection is stricter than the one defined by Abbott et al.. It requires the dissected pieces to be connected into

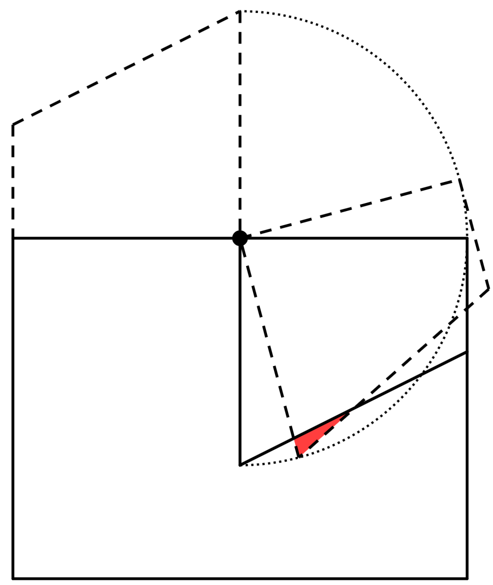

Figure 2.3: A Wobbly Hinged Dissection

a chain, that every piece has exactly two hinges, except for two pieces located at both ends of the chain which have only one hinge. Furthermore, for every pieces, the edges which are part of edges of the first polygon should be at the interior of the second polygon and vice versa. They proved that there exists hinged dissection from

- any quadrilateral to another quadrilateral,
- any quadrilateral to a parallelogram,
- any triangle to a parallelogram,
- any parallelhexagon to a trapezoid,
- any parallelhexagon to a triangle, and
- any trapezoidal pentagon to a trapezoid.

For each of the six results stated, an algorithm to construct the Dudeney dissection was provided along with the proof. All the results obtained mainly relied on tessellation of the first polygon. A downside of these algorithms was little or no control over the output polygon. In some algorithms, there were some steps which a random point needed to be chosen. This caused the output polygon to have different dimension in terms of side lengths or angles between the edges even though the algorithm was started with the same polygon. Also, the algorithms only produce polygons with similar diameter as the first polygon. In other words, “thin and long” polygon will be dissected to form another “thin and long” polygon by the algorithm.

CHAPTER 3

DISSECTION OF SOME POLYGONS

In this section, we will show and discuss a few dissections of some common 2D shapes such as rectangles and triangles. Some of the dissection will be analysed in details.

3-1 Combining Two Squares into One

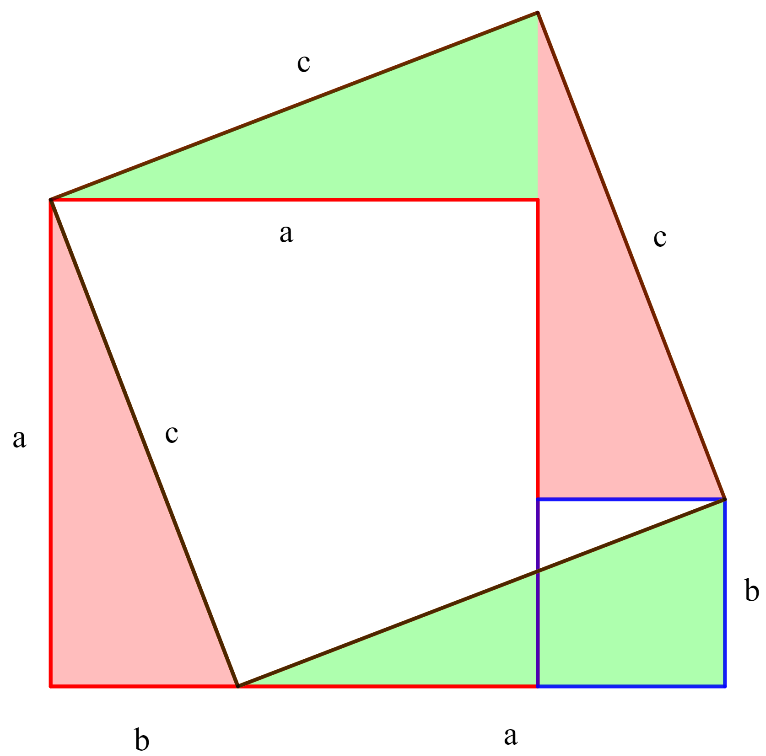

Figure 3.1: Dissect Two Squares into a Larger Square

Square is a simple one to begin with. We will first see how to dissect two given squares to make one larger square. This problem and solution are included in Chapter 2 of *Dissections: Plane and Fancy*. Let a and b be the side lengths of two squares with $a \geqslant b$. The construction of this dissection is shown in Figure 3.1. Interestingly, this dissection can be seen as proof of the Pythagoras theorem. This dissection is due to Sir George Biddle Airy. Figure 3.2 shows how this dissection can be found using tessellation.

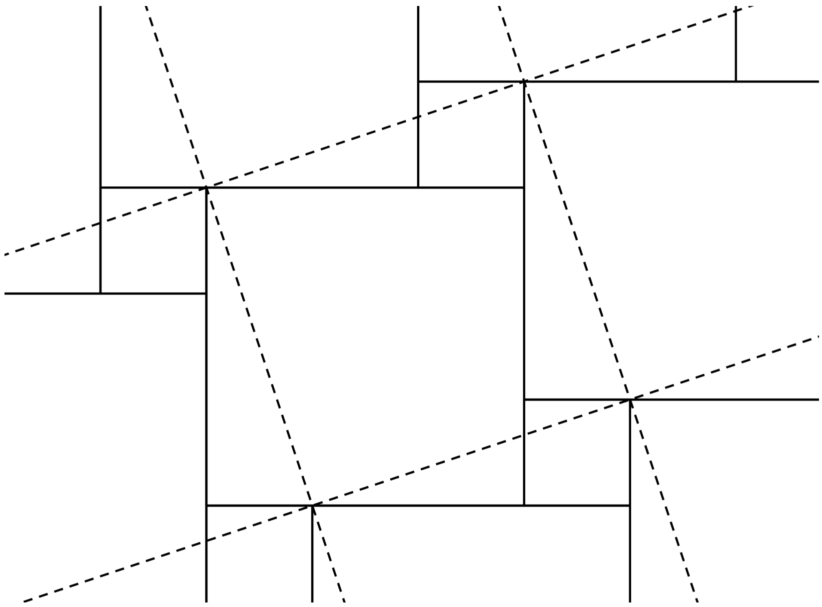

Figure 3.2: Superimposing Tessellation of Two Squares and a Large Square

3-2 Rectangle to Square

Now we will look into the construction which dissect a rectangle into a square provided that the longer side of the rectangle is not exceeding four times of the shorter side.

Construction:

1. Construct a suitable rectangle $ABCD$ as in Figure 3.3.
2. Let E be the point on $\overrightarrow{CB}$ such that $|BA| = |BE|$
3. Mark midpoint of EC as F.
4. Draw semicircle with centre F and diameter EC.
5. Let G be the intersection point between $\overrightarrow{BA}$ and the semicircle EC.
6. Mark M' on $\overrightarrow{DC}$ such that $|BG| = |M'D|$.
7. Construct square $JB'M'D$ with J on AD.
8. Let AM' intersects JB' and BC at A' and M respectively.

Now we try to show that this construction is correct. For convenience, let $|AB| = |EB| = x$ and $|BC| = y$. Note that the square that we want to construct needs to have equal area as the rectangle. Therefore, first few steps of the construction are actually

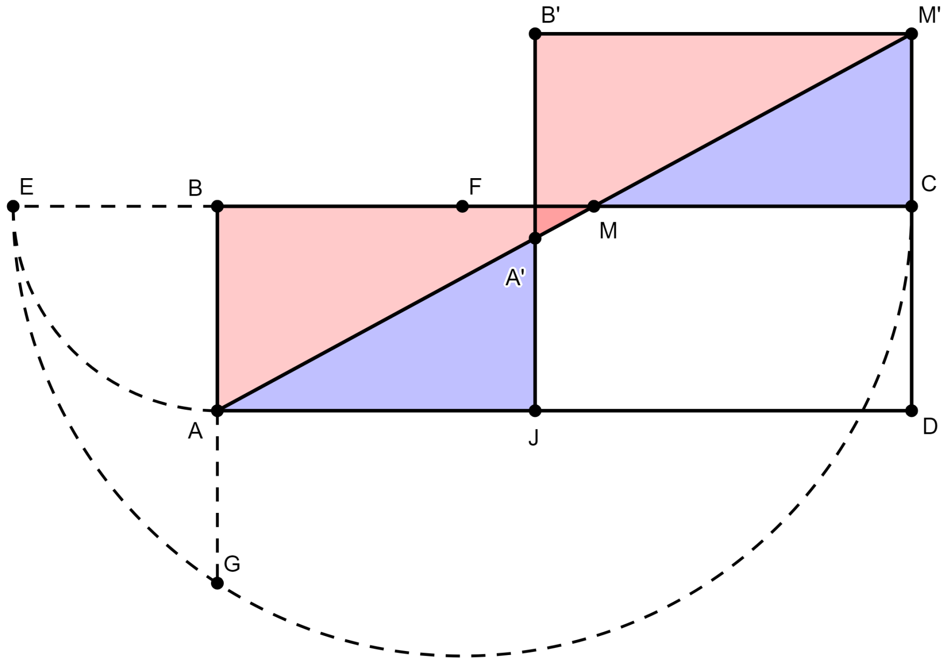

Figure 3.3: Dissection From Rectangle to Square

aimed to construct a line segment with length $\sqrt{xy}$ in order to construct the side length of the square.

We have

$$|EC| = x + y$$

$$|FG| = |FE| = |FC| = \frac{x+y}{2}$$

$$|FB| = \frac{x+y}{2} - x = \frac{y-x}{2}$$

By the Pythagoras theorem, we obtain

$$|BG| = \sqrt{\left(\frac{x+y}{2}\right)^2 - \left(\frac{y-x}{2}\right)^2} = \sqrt{xy}$$

which is used in Step 6 to construct the side of the square.

Observe that triangles ADM', AJA', MCM', MBA and $M'B'A'$ are similar to each other. Triangles ADM' and AJA' being similar implies that

$$\frac{|JA'|}{|JA|} = \frac{|DM'|}{|DA|}$$

Substituting known lengths gives

$$\frac{|JA'|}{y - \sqrt{xy}} = \frac{\sqrt{xy}}{y}$$

which simplifies to

$$|JA'| = \sqrt{xy} - x = |DM'| - |DC| = |CM'|$$

This result is sufficient to guarantee that triangle AJA' is congruent to triangle MCM'. This further implies that triangles MBA and $M'B'A'$ have equal area as rectangle $ABCD$ and square $JB'M'D$ must have equal area. Thus, the congruence between triangles MBA and $M'B'A'$ is guaranteed by their equal area and similarity. This completes the proof for the construction.

3-3 Equilateral Triangle to Square

Next, we will investigate the Haberdasher's puzzle which asks for a dissection of an equilateral triangle into four pieces to form a square. This puzzle was first solved by Dudeney (1908). He published the construction of the dissection in his book "The Canterbury Puzzles".

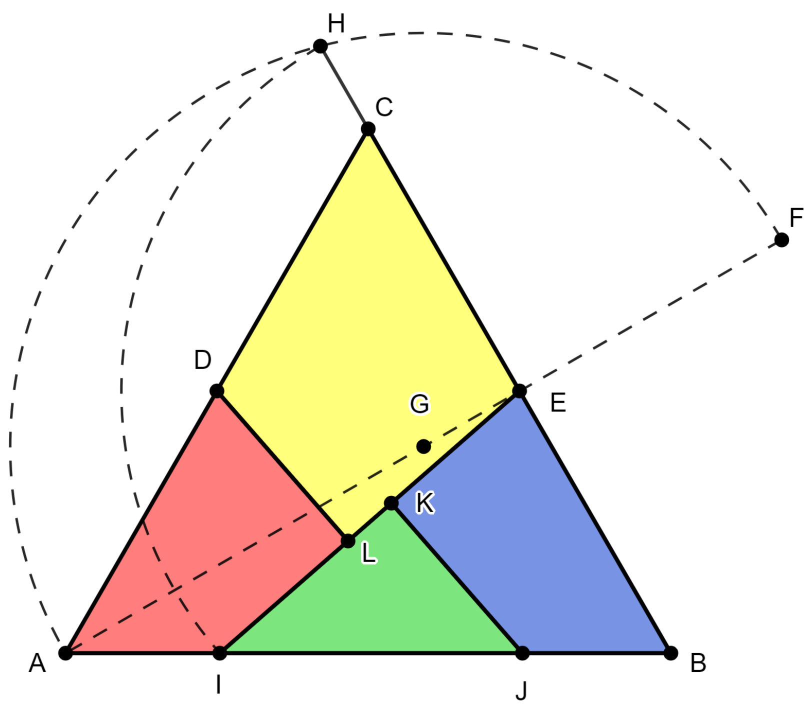

Figure 3.4: Construction of Dissection from Equilateral Triangle to Square

Construction:

1. Construct an equilateral triangle ABC as in Figure 3.4.

2. Let points D and E be the midpoints of AC and BC respectively.

3. Mark F on $\overrightarrow{AE}$ such that $|EF| = |EC|$.

4. Mark the midpoint of AF as G.

5. Let H be the point on $\overrightarrow{BC}$ such that $|GH| = |GF|$.

6. Mark I on AB such that $|EI| = |EH|$.

7. Mark J on IB such that $|IJ| = |AD|$.

8. Let K and L be on EI such that DL and JK are both perpendicular to IE.

9. The dissection is done by cutting along IE, DL and JK.

This dissection is in fact a non-wobbly hinged dissection. Referring to Figure 3.5, we can first rotate triangle IEB by $180°$ around point E. Next, rotate quadrilateral $ADLI$ by $180°$ around point D. Finally, the image of triangle IKJ formed by the rotation of triangle IEB is rotated by another $180°$ around point J' to become $I'MJ'$.

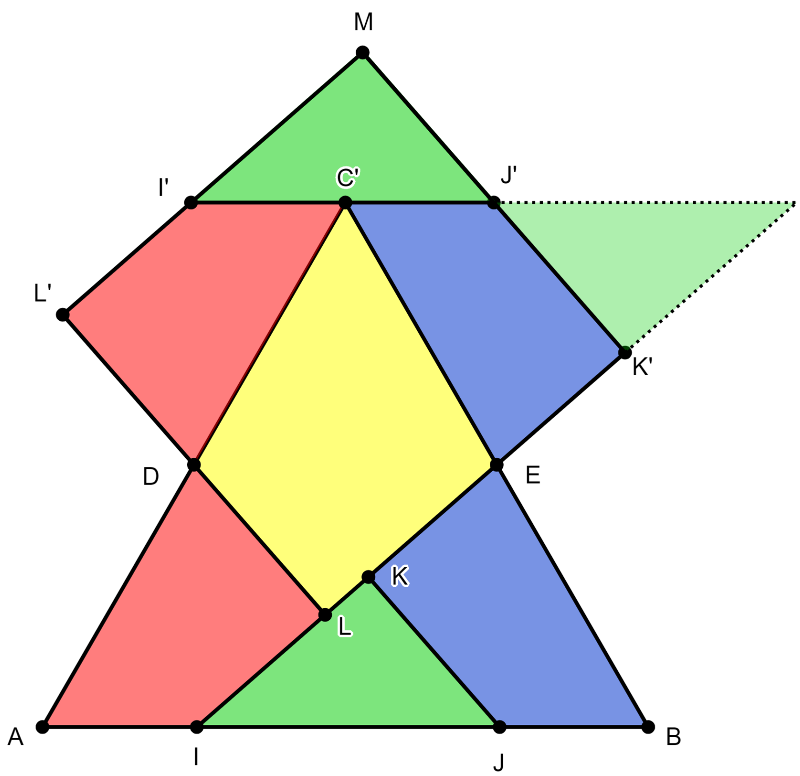

Figure 3.5: Dissection from Equilateral Triangle to Square

Since $|AD| = |DC|$, when we rotate quadrilateral $ADLI$ around point D to become $DL'I'C$, point A would coincide with point C after the rotation. With similar reasoning, point B will coincide with point C after rotating quadrilateral $JKEB$ by $180°$ around point E. Furthermore, points I', C, and J' are collinear as $\angle I'CD = \angle DCE = \angle ECJ'$. Also, $|IJ| = |AD| = 0.5|AB|$ implies that $|IJ| = |AI| + |JB| = |I'C| + |CJ'| = |I'J'|$. Next, $L'I'M$, $MJ'K'$, $K'EL$ and LDL' are all straight lines as the respective pairs of angles at point I', J', E and D are supplementary. With $\angle DL'I' = \angle I'MJ' = \angle J'K'E = \angle ELD = 90°$, so far $L'MK'L$ has been proved to be a rectangle. It can be shown that $L'MK'L$ is a square.

Assuming that triangle ABC has edges of length 2, we have

$$|AE| = \sqrt{2^2 - 1^2} = \sqrt{3}$$

$$|AF| = \sqrt{3} + 1$$
$$|GH| = |GF| = \frac{1}{2}\left(\sqrt{3} + 1\right)$$
$$|GE| = |GF| - |EF| = \frac{1}{2}\left(1 + \sqrt{3}\right) - 1 = \frac{1}{2}\left(\sqrt{3} - 1\right)$$

By Pythagoras theorem,

$$|EI| = |EH| = \sqrt{|GH|^2 - |GE|^2} = \sqrt{\left(\frac{1+\sqrt{3}}{2}\right)^2 - \left(\frac{\sqrt{3}-1}{2}\right)^2} = \sqrt{\sqrt{3}} = 3^{\frac{1}{4}}$$

The area of rectangle $L'MK'L$ is equal to the area of the triangle which is $0.5(2)(\sqrt{3}) = \sqrt{3}$. Rectangle $L'MK'L$ can be proved to be a square by showing that one of its side has length of $\sqrt{\sqrt{3}}$. In particular, this can be done by showing that $|ML'| = |EI|$.

Observe that $|ML'| = |K'L|$ and

$$\begin{aligned} 2|ML'| &= |ML'| + |K'L| \\ &= |MI'| + |I'L'| + |K'E| + |EL| \\ &= |KI| + |IL| + |KE| + |EL| \\ &= 2|EI| \end{aligned}$$

which gives $|ML'| = |EI|$ as wanted.

From the analysis, we can see that the most crucial and challenging part in this construction is to construct a line segment of length $3^{\frac{1}{4}}$.

Although we know how to construct this dissection starting from the triangle, constructing the dissection from the other way round, which is to start with a square, may not be simple. To find such a construction, we start by inspecting the dissection of square $L'MK'L$ in Figure 3.5. Given such a square, the objective is to construct the points I', J', E, D and C. The "cuts" that need to made are $I'J'$, CD and CE. Points J' and D are midpoints of MK' and $L'L$ respectively. Thus, these two points are relatively simple to construct. When the square has side length $3^{\frac{1}{4}}$, it is found that $|I'J'| = |CD| = |CE| = 1$.

Now, the main challenge is to construct the a line segment of length 1 using a line segment of length $3^{\frac{1}{4}}$ from the starting square. We can apply the technique used in the construction of line segment of length $3^{\frac{1}{4}}$ from an equilateral triangle of length 2. If we do the same construction on an equilateral triangle of length $3^{\frac{1}{4}}$ instead, we could obtain a line segment of length $3^{\frac{1}{4}}/2 \cdot 3^{\frac{1}{4}} = \sqrt{3}/2$ instead. We have already known that $\sqrt{3}$ is the height of an equilateral triangle with side length 2, thus $\sqrt{3}$ is the height of an equilateral triangle with side length 1. Using this idea, we are able to create the construction.

Construction:

1. Construct a square $ABCD$.
2. Construct an equilateral triangle DEC as in Figure 3.6.
3. Let F be midpoint of EC.
4. Mark G on $\overrightarrow{DF}$ such that $|FG| = |FE|$.
5. Let H be the midpoint of DG.
6. Mark I on $\overrightarrow{CE}$ such that $|HI| = |HG|$.
7. Mark J on $\overrightarrow{AD}$ such that $|DJ| = |FI|$.
8. Draw a line perpendicular to $\overleftrightarrow{AJ}$ at J and let K be the point where the line meets DE.
9. Mark L and M as the midpoints of DC and AB respectively.
10. Mark N on AD such that $|LN| = |DK|$.

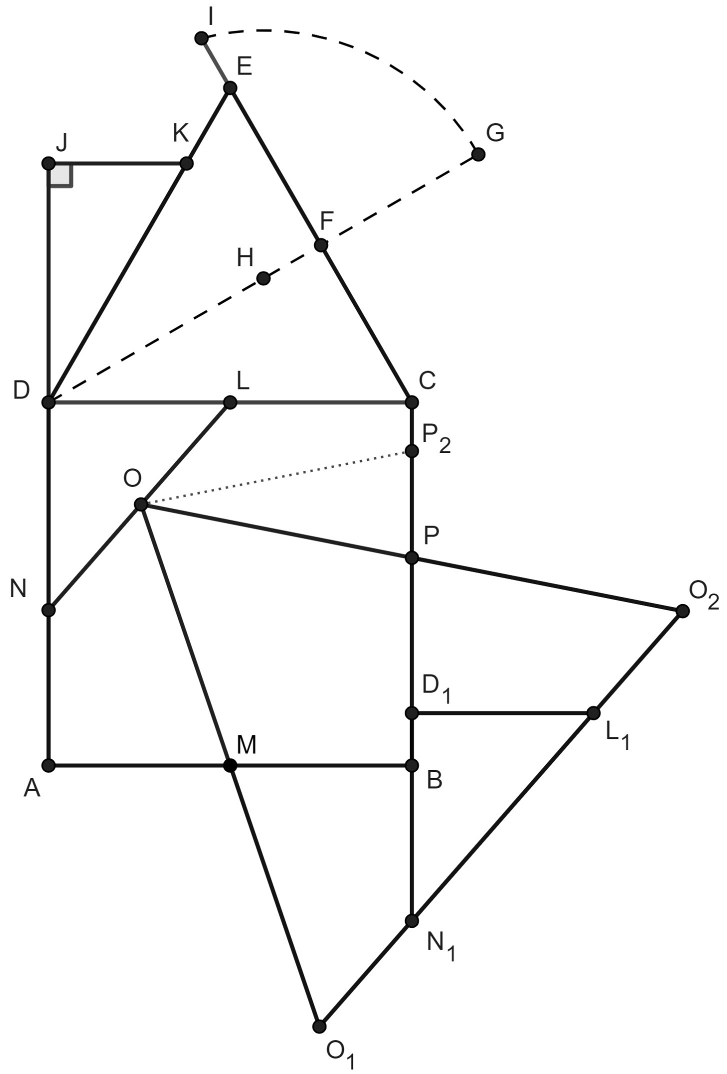

Figure 3.6: Construction of Dissection from Square to Equilateral Triangle

11. Mark O on LN such that $|MO| = |DK|$.

12. Mark P on BC such that $|OP| = |DK|$. There are two such possible P, choose the one which is nearer to B.

13. The dissection of the square is given by cutting along LN, MO and OP.

The construction steps 2 to 6 are actually the same as those in the construction from equilateral triangle to square. We shall assume $|CD| = 3^{\frac{1}{4}}$. Based of our arguments earlier on, $|DJ| = |FI|$ would be $\sqrt{3}$. It can be easily verified that $\angle JDK = 30°$ and $\angle DKJ = 60°$. Triangle DJK is a halved equilateral triangle with DJ as height and DK as an edge. We have successfully constructed a line segment of length 1 which is DK. By using DK, we can easily construct the "cuts" needed to made on the square.

CHAPTER 4

EQUIDECOMPOSABILITY

This chapter starts with discussion about some properties of equidecomposition. After that, we will look into the proofs of Wallace-Bolyai-Gerwien Theorem. Finally, we will study the solution of Hilbert's Third Problem and its partial proof.

First, let us recall the definition of equidecomposability on page 11. Equidecomposability is in fact an equivalence relation but this is considered trivial and thus not mentioned in most books and articles. The reflexive and transitive properties are obviously true, while the transitivity can be proved easily.

Suppose we have polygons A, B and C such that A is equidecomposable to B and B is equidecomposable to C. Then, there exists two dissections of polygon B, one gives the pieces that can be arranged to form polygon A and the other gives the pieces can be arranged to form polygon C as illustrated in Figure 4.1. By superimposing the two dissection figures of polygon B, that is, placing one dissection figure of polygon B on top of another, we observe a new dissection of polygon B as in Figure 4.2. The pieces produced from this new dissection are able to form polygons A or C when they are arranged accordingly. Thus, polygons A and C are also equidecomposable.

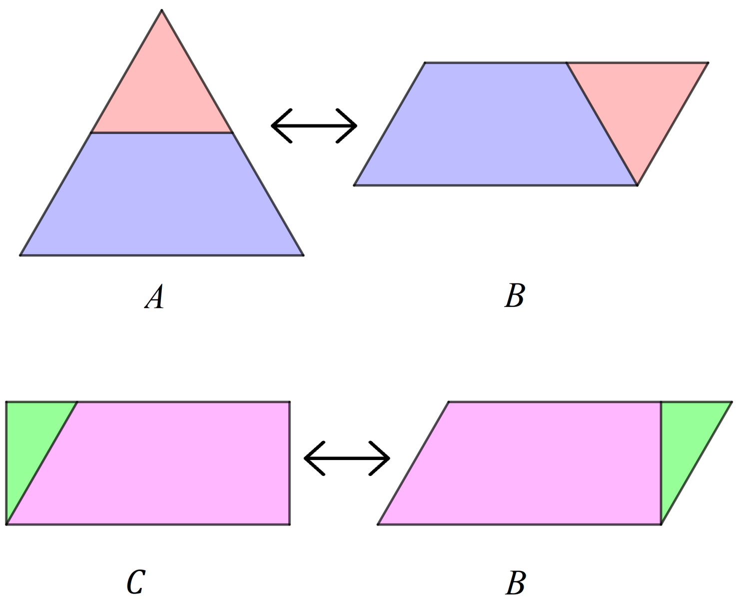

Figure 4.1: B is equidecomposable to A and C

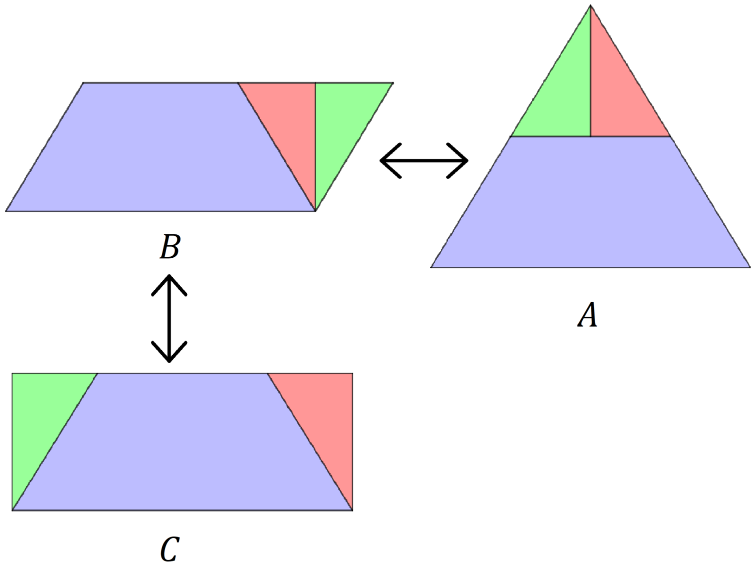

Figure 4.2: Superimposition Gives Common Dissection

Nevertheless, an issue that could arises from Definition 1 is that a polygon is congruent to its reflection. In some cases, we do not want to "flip" some of the pieces dissected from the first polygon to form the second polygon. The same problem happens for polyhedra and it is more likely to cause some trouble if any results that follow this definition are to be applied in engineering. This is because we cannot simply "flip" a polyhedron in the real world to get its reflection.

Aigner and Ziegler (2018) had mentioned that restricting reflection from congruences of the dissected pieces in Definition 2 would not cause a difference in equidecomposability of polygons or polyhedra. This result was given by Gerling in 1844 but there is no further information in the book. A more detailed explanation of Gerling's result can be found in Chapter 20 of Frederickson's *Dissections: Plane and Fancy*. We have mentioned that Gauss questioned whether two tetrehedra of same base area and same height are equidecomposable in his letter to Gerling. As a reply, Gerling was only able to find the dissection for a special case, which required one tetrahedron to be the reflection of the other. He found a way to dissect a tetrahedron into 6 pairs of smaller tetrahedral pieces such that in each pair of tetrehedra, one is the reflection of the other piece. These pieces then can be reassembled to form the reflection of the original tetrahedron. By Definition 2, if some tetrahedral pieces A_k and B_k are congruent by reflection, they can be further dissected so that all pieces are congruent only by translation and rotation.

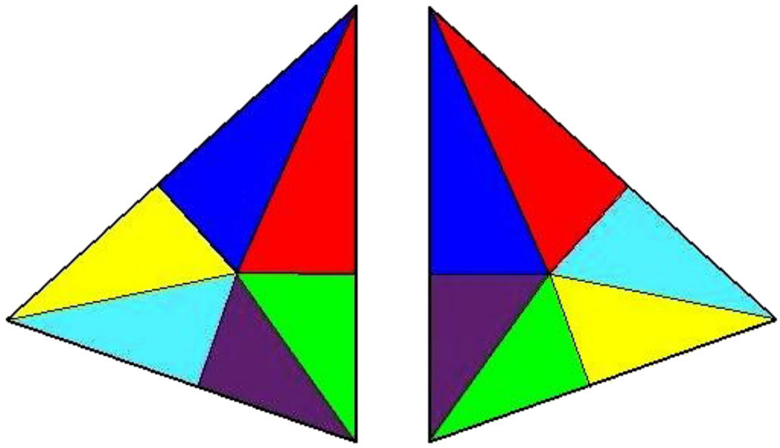

Figure 4.3: Gerling's Dissection from a Triangle to its Reflection (Ciesielska and Ciesielski, 2018)

The idea of Gerling's result can be brought into the second dimension. A triangle can be dissected into 3 pairs of triangular pieces which can be reassembled to form the reflection of the first triangle as illustrated in Figure 4.3. Such dissection can be easily constructed by drawing lines where each line passes the *incenter* and one vertex and of the triangle. The incenter of a triangle is the center of inscribed circle of the triangle. The incenter can be constructed as the point where the three angle bisectors of three vertices of the triangle intersect.

4-1 Wallace-Bolyai-Gerwien Theorem

Theorem 2-3-1 (Wallace-Bolyai-Gerwien Theorem)**.** *If two polygons have the same area, then they are equidecomposable.*

Since equidecomposability is an equivalence relation, each of the following statements is equivalent to Wallace-Bolyai-Gerwien Theorem.

1. A polygon is equidecomposable to a square.
2. A polygon with area a is equidecomposable to a $1 \times a$ rectangle.

With either statement, we can say that any two polygons of equal area are both equidecomposable to a third polygon, thus the first two polygons are also equidecomposable. We shall start by proving the first statement with the aid of the following two lemmas.

Lemma 4-1-1. *Every triangle is equidecomposable to a rectangle.*

Proof. Consider a triangle ABC with AB as its longest edge. The dissection of the triangle to rectangle can be constructed as follows:

1. Construct a line segment that passes through C and perpendicularly intersects AB at D.

2. Construct a perpendicular bisector of segment CD. Let the line intersects CD at E, CA at F and CB at G.

3. The pieces $ABGF$, FEC and EGC can be arranged into a rectangle as shown in Figure 4.4

□

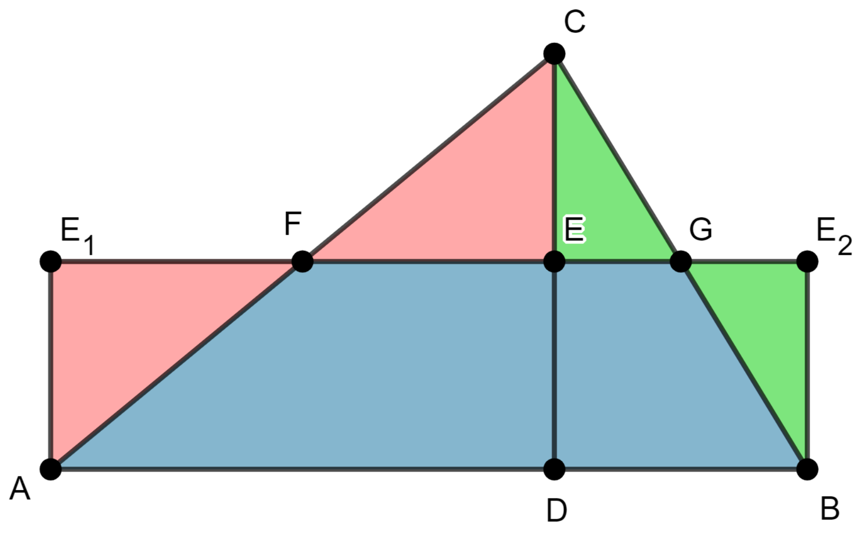

Figure 4.4: Dissection from a Triangle to Rectangle

Lemma 4-1-2. *Every rectangle is equidecomposable to a square.*

Proof. Consider a non-square rectangle with length l and height h. Without loss of generality, assume that l is greater than h.

In Section 3-2, we had shown a way to dissect a rectangle to form a square provided that $l \leqslant 4h$.

For the case of $l > 4h$, we show that the rectangle is equidecomposable with a $l' \times h'$ rectangle with $l' \leqslant 4h'$. We can cut the original rectangle into two rectangles with length $l/2$ and height h. These two pieces can be stacked to form a new rectangle with length $l/2$ and width $2h$. This process can be performed repeatedly to reduce the length and increase the height of the rectangle as illustrated in Figure 4.5.

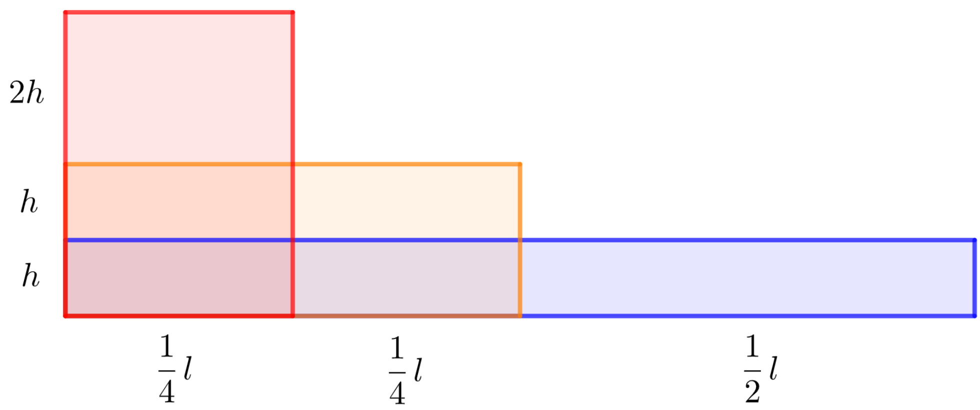

Figure 4.5: Halving and Stacking Rectangles

Let l_k denote the length and h_k denote the width of the rectangle obtained after k rounds of the process. It is easy to see that

$$l_k = \left(\frac{1}{2}\right)^k l$$

and

$$h_k = 2^k h$$

and also

$$\frac{l_k}{h_k} = \frac{1}{4^k}\left(\frac{l}{h}\right)$$

We claim that $h_n < l_n \leqslant 4h_n$ when $n = \lceil log_4(l/h) \rceil - 1$.

$$n = \left\lceil log_4 \left(\frac{l}{h} \right) \right\rceil - 1$$

implies

$$log_4 \left(\frac{l}{h} \right) \leqslant n + 1 < log_4 \left(\frac{l}{h} \right) + 1 = log_4 \left(\frac{4l}{h} \right)$$

which further implies

$$\frac{l}{h} \leqslant 4^{n+1} < \frac{4l}{h}$$

As such, we have

$$1 \leqslant 4^{n+1} \left(\frac{h}{l} \right) < 4$$

Taking the reciprocal

$$\frac{1}{4} < \frac{1}{4^{n+1}} \left(\frac{l}{h} \right) \leqslant 1$$

we have

$$1 < \frac{1}{4^n} \left(\frac{l}{h} \right) \leqslant 4$$

implying that

$$1 < \frac{l_n}{h_n} \leqslant 4$$

and we have

$$h_n < l_n \leqslant 4h_n$$

Now the rectangle with length $l' = l_n$ and height $h' = h_n$ satisfies what we want. □

Theorem 4-1-3. *A polygon is equidecomposable to a square of the same area.*

Proof. First, dissect the polygon into triangles. By Lemma 4-1-1 the triangles can be dissected to form rectangles. Next, each small rectangle is dissected to form a square by Lemma 4-1-2. By using the method mentioned in Section 3-1, we can repeatedly combine two squares into a larger square until we are left with a single square which has the same area as the original polygon. □

That is the first complete proof for the Wallace-Bolyai-Gerwien Theorem. The second proof follow the similar arguments as in the paper titled 'Scissors congruence' by Welsh (2016). The steps are the same as previous proof until the original polygon is dissected which can form some small rectangles. In the next step, Welsh tried to prove a lemma stating that any two rectangles of the same area are equidecomposable but there was an incorrect statement in his proof. This mistake was found to be caused by a wrong construction step of the dissection that comes before that statement. The construction of the 3-pieces dissection is actually very similar to the dissection from a rectangle to a square shown in Section 3-2. In fact, we can use this result to prove the lemma easily.

Lemma 4-1-4. *Any two rectangles of the same area are equidecomposable.*

Proof. By Lemma 4-1-2, both the rectangles are equidecomposable to a same square which shares the same area as the two rectangles. Therefore, they are equidecomposable. □

These results are enough to write the second proof.

Theorem 4-1-5. *A polygon with area a is equidecomposable to a $1 \times a$ rectangle.*

Proof. The polygon can be first dissected into triangles and the triangles are further dissected to form rectangles by Lemma 4-1-1. Next, by Lemma 4-1-4, each of the rectangle can be dissected to form a rectangle with height 1. Then, all these rectangles can be stacked to form a long rectangle with height 1 and base equal to sum of the bases of the rectangular pieces, which obviously has to be numerically equal to a. □

This concludes the second proof of Wallace-Bolyai-Gerwien Theorem. However, the construction of the dissection along the proofs usually creates a lot of pieces. It is generally a difficult problem to find a minimal dissection for any two given polygons into a minimum number of pieces.

Based on the nature of the proofs, we obtain the following result.

Theorem 4-1-6. *For a finite set of polygons which all have the same area, there exists a dissection which the pieces can be arranged to reassemble any polygon from the set.*

Some literature refers this stronger result as the Wallace-Bolyai-Gerwien Theorem.

4-2 Hinged Dissection between Any Polygons

Compared to proving Wallace-Bolyai-Gerwien Theorem, finding out whether any two polygons have a hinged dissection is a more difficult task. This is because the existence of hinged dissection does not have a trivial transitive property like equidecomposability. Even so, (Abbott et al., 2012) had managed to prove the result:

Theorem 4-2-1. *For a finite set of polygons which all have the same area, there exists a hinged dissection in which the pieces connected by hinges can be moved without any intersection between themselves to reassemble any polygon from the set.*

Here we only include the proof outline with brief explanation. The constructive proof extends from the result in previous section, which is Theorem 4-1-6. The first step is to add some hinges to the pieces which are from the dissection in Theorem 4-1-6. After adding those hinges, it is not necessary that the pieces with hinges can reassemble every polygon in the set.

The next step is the most crucial part of the proof. In brief, it is proven that by further dissecting the current pieces into even smaller pieces which are still connected by hinges, an original hinge can "change" its position. This idea is illustrated in Figure 4.6.

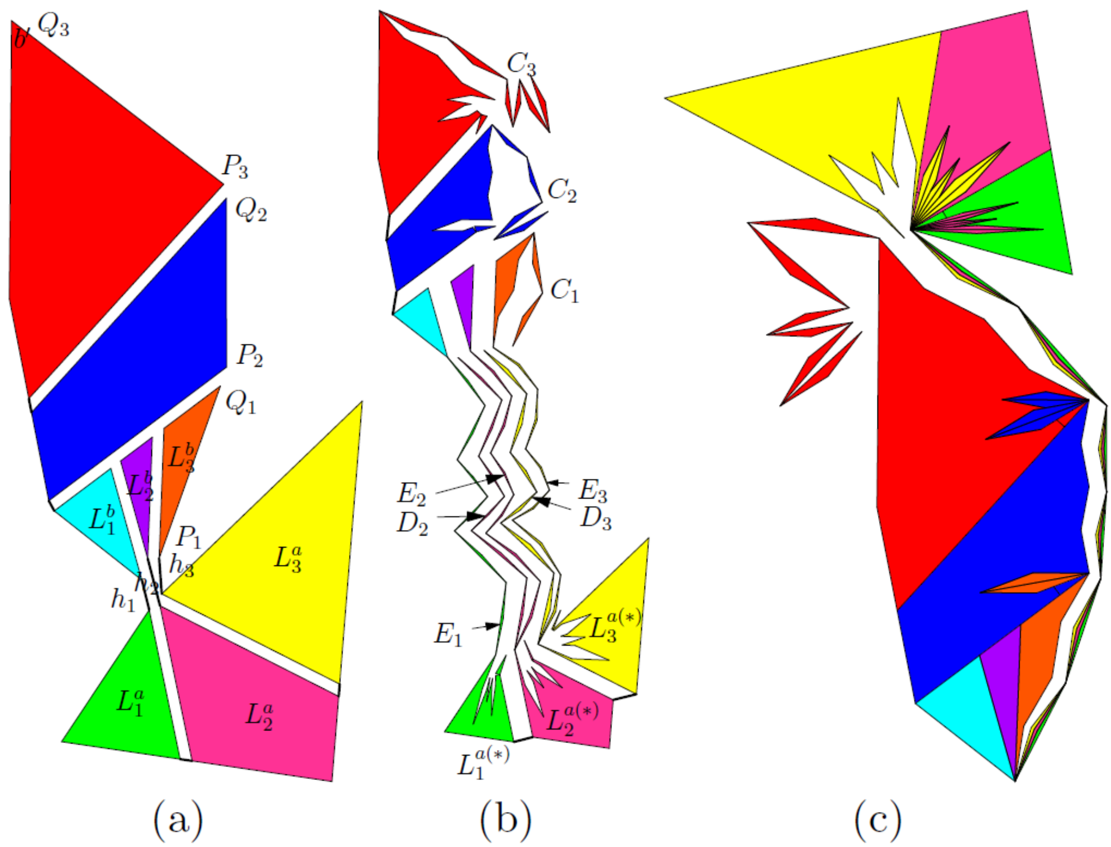

Figure 4.6: Moving a Hinge (Abbott et al., 2012)

The method allows the hinges to be "moved" freely and thus a wobbly hinged dissection that applies for all polygons in the set can be found. To obtain a non-wobbly hinged dissection, those pieces where intersection occur are further dissected so that they do not "block the way".

We can observe from Figure 4.6 that there could be a lot of tiny pieces generated. If the construction is to be built in real world, it could be a challenge for engineers.

4-3 Hilbert's Third Problem

Answering the Hilbert's Third Problem, there exists polyhedra of equal volume which are not equidecomposable. An approach to prove this answer is to use Bricard's condition which was correctly proven by Benko. This proof requires the "pearl lemma" and "cone lemma". The following proofs are reproduced following the arguments in *Proofs from THE BOOK* by (Aigner and Ziegler, 2018).

Before that, we need to understand the concept of *segment* (not to confuse with *line segment* which is defined earlier). The concept is easier to be explained in 2D dissection. A dissection of polygon produces some polygonal pieces. When the pieces are assembled to form the polygon, every piece has some edges lying at the interior of the polygon and touching other pieces. Some of these edges are observed to be subdivided into smaller part by vertices of other pieces. These subdivided part are called segments. If an edge is not subdivided, then whole edge is considered as a single segment.

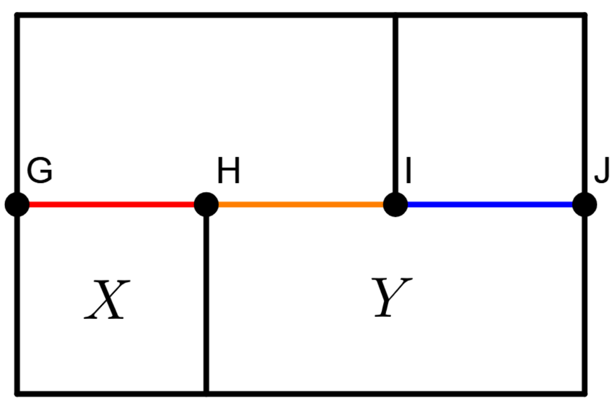

Figure 4.7: Illustrating Segments

We use the dissection in Figure 4.7 to clearly illustrate the concept of segments. Edge HJ of piece Y consists of two segments as the edge is subdivided by vertices of other pieces. Meanwhile, edge GH of piece X has the full edge as a single segment. The same concept can be extended to dissection of polyhedra. The edges of the polyhedral pieces can be subdivided into segments by vertices or edges of other pieces.

Lemma 4-3-1 (Pearl Lemma)**.** *Let P and Q be two polygons or two polyhedra which are equidecomposable. They are dissected into pieces; $P = P_1 \cup ... \cup P_k$, $Q = Q_1 \cup ... \cup Q_k$ with P_i congruent to Q_i. It is possible to put a positive real number of pearls on the segments such that for each pair of congruent pieces P_i and Q_i, the number of pearls on their corresponding edges are the same.*

Figure 4.8 shows a correct assignment of pearls for that dissection but in this lemma it is not necessary for the number of pearls assigned to be integers.

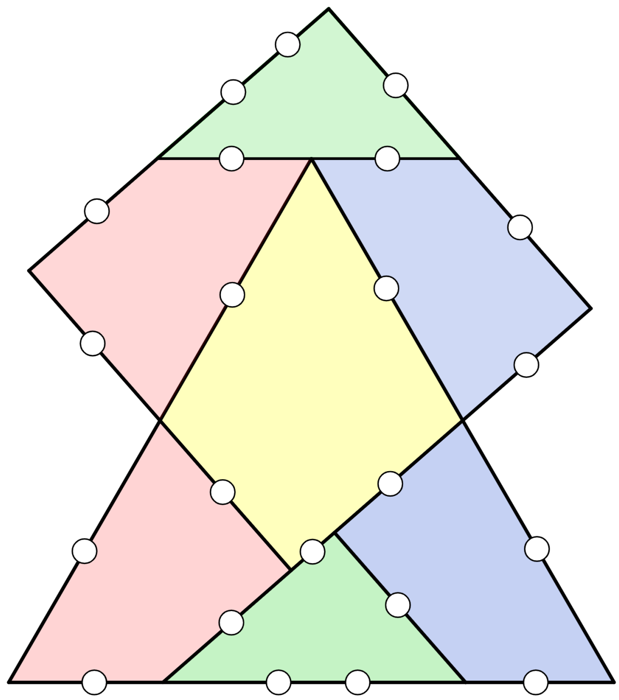

Figure 4.8: A Correct Assignment of Pearls

Proof. Suppose e is an edge of an arbitrary piece P_i and e' is the corresponding edge of Q_i. If e is subdivided into segments s_k and e' is subdivided into segments s'_l, we can write the linear equation

$$\sum_{k:s_k \subseteq e} x_k - \sum_{l:s'_l \subseteq e'} y_l = 0$$

where x_k and y_l are the numbers of pearls assigned to the segment s_k and s'_l respectively. If we write the linear equation for all edges of every pieces, we have a system of linear equation

$$A\mathbf{x} = \mathbf{0}, \quad \mathbf{x} > \mathbf{0}$$

where A is an integer matrix with entries 1, -1 or 0 and **0** is the zero vector. Obviously, a possible solution is the length of each segments and this completes the proof for pearl lemma.

□

Lemma 4-3-2 (Cone Lemma)**.** *Let A be an integer matrix such that $A\boldsymbol{x} = \boldsymbol{0}$ has a positive real solution. Then it must has a positive integer solution.*

Proof. If the system has a positive real solution, then by multiplying the solution with a suitable scalar, we can obtain a real solution of at least 1. Therefore, the system

$$A\mathbf{x} = \mathbf{0}, \quad \mathbf{x} \geqslant \mathbf{1}$$

where **1** is the vector with all 1's has a real solution. If we can show that this system has a rational solution, then we can multiply the solution with the common denominator to obtain an integer solution which is wanted. Note that we can write the system

$$A\mathbf{x} = \mathbf{0}, \quad \mathbf{x} \geqslant \mathbf{1}$$

as

$$A\mathbf{x} \geqslant \mathbf{0}, \quad -A\mathbf{x} \geqslant \mathbf{0}, \quad \mathbf{x} \geqslant \mathbf{1}$$

which is the same as

$$\begin{pmatrix} A \\ -A \end{pmatrix} \mathbf{x} \geqslant \mathbf{0}, \quad \mathbf{x} \geqslant \mathbf{1}$$

The matrix $\begin{pmatrix} A \\ -A \end{pmatrix}$ is nothing but a integer matrix just like A. We have reduced the problem to proving that

$$A\mathbf{x} \geqslant \mathbf{0}, \quad \mathbf{x} \geqslant \mathbf{1}$$

has a rational solution. We shall prove a more general result, that is,

$$A\mathbf{x} \geqslant \mathbf{b}, \quad \mathbf{x} \geqslant \mathbf{1}$$

where **b** is an integer vector, has a rational solution. Suppose A is a $m \times n$ matrix. We prove by induction on n. When $n = 1$, A becomes a vector and **x** becomes a scalar. All the inequalities in the system can be written in the form

$$\mathbf{x} \geqslant \frac{b_i}{a_i}$$

where a_i and b_i are integers. We choose **x** to be its smallest possible value which must be a rational number. This is guaranteed by existence of a real solution and the right-hand side of every inequalities being rational.

For $n > 1$, we write

$$A = \begin{pmatrix} A' & \mathbf{a}_n \end{pmatrix}$$

where A' is the $m \times (n-1)$ matrix which is the first $n-1$ columns of A and $\mathbf{a}_n$ is the nth column of A. Also, write

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}' \\ x_n \end{pmatrix}$$

where $\mathbf{x}'$ is the vector extracted from the first $n-1$ entries of **x** and x_n be the last entry of **x**. The system

$$A\mathbf{x} \geqslant \mathbf{b}, \qquad \mathbf{x} \geqslant \mathbf{1}$$

can now be written as

$$\begin{pmatrix} A' & \mathbf{a}_n \end{pmatrix} \begin{pmatrix} \mathbf{x}' \\ x_n \end{pmatrix} \geqslant \mathbf{b}, \qquad \mathbf{x}' \geqslant \mathbf{1}, \ \ x_n \geqslant 1$$

which is equivalent to

$$A'x' \geqslant \mathbf{b}, \quad \mathbf{x}' \geqslant \mathbf{1}, \qquad \mathbf{a}_n x_n \geqslant \mathbf{b}, \quad x_n \geqslant 1$$

We can see that there are two independent linear system. By inductive hypotheses, the first subsystem has a rational solution and in the second subsystem, the smallest possible x_n is chosen similar to what was argued in the base case of $n = 1$. This completes the proof.

In fact, the rational solution we obtain in this way is lexicographically smallest in the solution space. Lexicographical ordering orders the vector based on the first entries in the vectors. If there is a tie, then those tied vectors are ordered based on their second entries and so on. This ordering is also called the dictionary ordering since the ordering of words in a dictionary is similar.

□

Theorem 4-3-3 (Bricard's condition)**.** *Let P and Q be equidecomposable polyhedra. Suppose P has m edges with dihedral angles $\alpha_1, ..., \alpha_m$ and Q has n edges with dihedral angles $\beta_1, ..., \beta_n$. There exists positive integers a_i, b_j and an integer c such that*

$$a_1\alpha_1 + \cdots + a_m\alpha_m = b_1\beta_1 + \cdots + b_n\beta_n + c\pi$$

Proof. According to the pearl lemma and the cone lemma, we know that it is possible to assign a positive integer of pearls at the segments of the dissected pieces for both P and Q such that the corresponding edge for each pair of congruent pieces have the same number of pearls.

Consider a polyhedral piece P_1 dissected from P, every pearl on P_1 lies on one of the edge of P_1. We take the sum of dihedral angle of each pearl and write the sum as S_1. Suppose P is dissected into k pieces, them we can apply the calculation for the remaining pieces to obtain $S_2, S_3, ..., S_k$ and let $S = S_1 + S_2 + \cdots + S_k$.

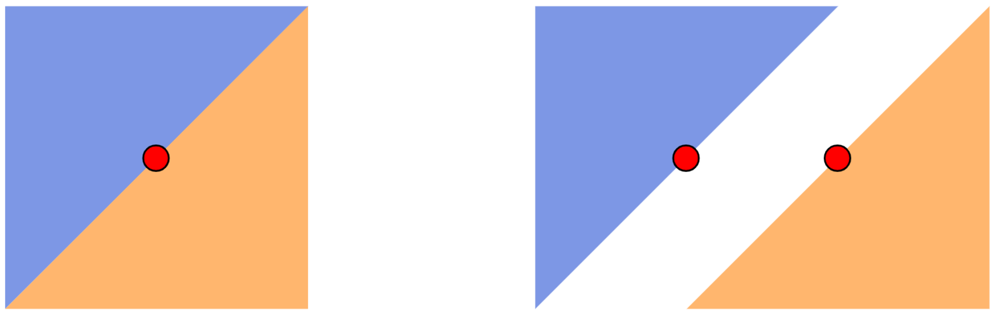

Figure 4.9: Pearls may Coincide

When the pieces are put together to reassemble P, some pearls from different pieces may coincide if they share the same segment in P. This concept is illustrated in Figure 4.9 in 2D. For polyhedra, this situation happens when the pearl lies on a face of P or at the interior of P. The pearl of the formal case contributes exactly π to S while a pearl of the latter case contributes exactly 2π to S. The remaining pearls that are not covered by these two cases must be lying on one of the m edges. Therefore, we can write

$$S = a_1\alpha_1 + \cdots + a_m\alpha_m + c_p\pi$$

where a_1,...,a_m are positive integers (there is at least one pearl on each edge) and c_p is a non-negative integer. Since the pieces can also reassemble Q, similar reasoning gives

$$S = b_1\beta_1 + \cdots + b_n\beta_n + c_q\pi$$

where b_1,...,b_n are positive integers and c_p is a non-negative integer. We finally arrive at the Bricard's condition

$$a_1\alpha_1 + \cdots + a_m\alpha_m = b_1\beta_1 + \cdots + b_n\beta_n + c\pi$$

by taking $c = c_q - c_p$.

□

With Bricard's condition, now we are able prove that some polyhedra of equal volume are not equidecomposable. Here we show an example using a square and a regular tetrahedron. Even if we know that equal volume are necessary for equidecomposability, we do not really need to care about volume since Bricard's condition only takes the dihedral angles into account.

Bricard's condition tells us that if a square and a regular tetrahedron are equidecomposable, then

$$12a\left(\frac{\pi}{2}\right) = 6b(\beta) + c\pi$$

where β is the dihedral angle that is the same for all edges of a regular tetrahedron, a, b are positive integers and c is a non-negative integer. Observe that left-hand side is a rational multiple of π. We are done if we can prove that β is not a rational multiple of π.

We need to find the value of β. Referring to Figure 4.10, let $ABCD$ be a tetrahedron. E is the center of triangle ABC and ED is to plane ABC. Observe that

$$|AE| = |BE| = 2|EF|$$

which implies

$$|DF| = |AF| = 3|EF|$$

Therefore,

$$cos(\beta) = \frac{|EF|}{|DF|} = \frac{1}{3}$$

which gives

$$\beta = cos^{-1}\left(\frac{1}{3}\right)$$

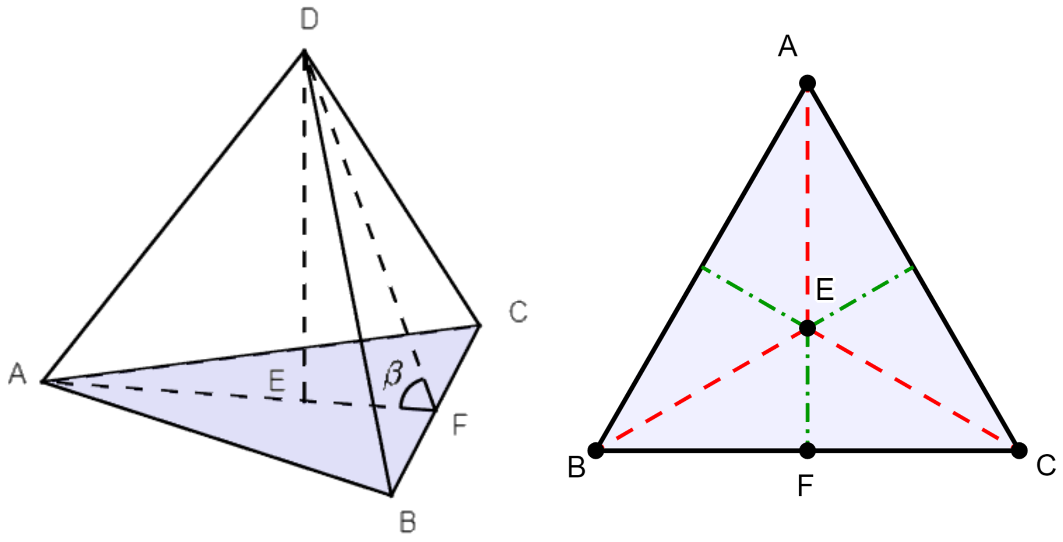

Figure 4.10: Dihedral Angle of Regular Tetrahedron

Aigner and Ziegler (2018) has proved in another chapter that for all odd integers n more than or equal to 3,

$$\frac{1}{\pi}cos^{-1}\left(\frac{1}{\sqrt{n}}\right)$$

is irrational. This result can give what we want by using $n = 9$. We use the argument from the proof to show that

$$\frac{1}{\pi}cos^{-1}\left(\frac{1}{3}\right)$$

is irrational.

We claim that

$$cos(k\beta) = \frac{N_k}{3^k}$$

where A_k is an integer not divisible by 9 for all non-negative integer k. We prove this claim by induction. For $k = 0$ and $k = 1$, we can easily obtain $N_0 = N_1 = 1$. For $k \geqslant 2$, using the identity

$$cos(A) + cos(B) = 2cos\left(\frac{A+B}{2}\right)cos\left(\frac{A-B}{2}\right)$$

with $A = (k+1)\beta$ and $B = (k-1)\beta$, we have

$$\begin{aligned} cos[(k+1)\beta] &= 2cos(k\beta)cos(\beta) - cos[(k-1)\beta] \\ &= 2 \cdot \frac{N_k}{3^k} \cdot \frac{1}{3} - \frac{N_{k-1}}{3^{k-1}} \\ &= \frac{2N_k - 9A_{k-1}}{3^{k+1}} \end{aligned}$$

We obtain $N_{k+1} = 2N_k - 9N_{k-1}$ which is also not divisible by 9 since A_k is not divisible by 9. The claim is proved.

Assume that

$$\beta = \frac{a}{b}\pi$$

for some positive integers a, b. Then

$$b\beta = a\pi$$

gives

$$cos(b\beta) = cos(a\pi) = \pm 1$$

which can be written as

$$\frac{N_b}{3^b} = \pm 1$$

or

$$N_b = \pm 3^b$$

for some integer N_b not divisible by 9. This forces $b = 1$ which implies

$$cos(\beta) = \frac{\pm 3}{3} = \pm 1$$

This is a contradiction. Thus β is not a rational multiple of π. This completes the proof that a square and a tetrahedron are not equidecomposable.

CHAPTER 5

CONCLUSION

The study of geometric dissection starts from investigating how to dissect a polygon into pieces to other polygons through geometry. In this project, the dissections between some commonly seen polygons are studied and analysed in details. Some dissections are trivial. Some examples are the dissection of two squares into one large square and from a triangle to a rectangle. Meanwhile, some dissection are relatively harder to discover. The Haberdasher's problem which asks for a dissection of a equilateral triangle that form a square is an example.

Hinged dissection or Dudeney dissection is a special kind of dissection which has an additional requirement. In this dissection, all the pieces are connected by some hinges and the movement of the pieces are restricted to rotation around the hinges. If some pieces intersect during the transformation of the pieces from one polygon to another polygon, this dissection is called a wobbly hinged dissection.

If it is possible to dissect a polygon into finite polygonal pieces and use the pieces to reassemble another polygon, then these two polygons are said to be equidecomposable or scissors congruent. This definition applies for polyhedra analogously.

Wallace-Bolyai-Gerwien Theorem states an important result that any two polygons are equidecomposable if and only if the two polygons have the same area. In other words, area is the only invariant for dissection of polygons and having equal area is sufficient and necessary to guarantee equidecomposable of polygons. The constructive proof of this theorem provided a way to dissect a polygon to reassemble another given polygon. However, the dissection produces a lot more pieces than required in most of the cases. In general, finding the minimal dissection between two given polygons remains as a difficult problem. A recent paper proved a stronger result that hinged dissection is possible between polygons of equal area. The proof of this finding extends from the Wallace-Bolyai-Gerwien Theorem.

The generalisation of Wallace-Bolyai-Gerwien Theorem into third dimension, which would state that equidecomposability of two polyhedra is decided by volume only, is false. This is generally referred as Hilbert's Third Problem. Dehn is considered the first person who solved this problem. He proved that there exists a second invariant called the Dehn invariant for dissection of polyhedra. Sylder later proved that equal volume and equal Dehn invariant are sufficient to guarantee equidecomposability between two polygons. An alternative solution of Hilbert's Third Problem is based on the Bricard's condition. Different from Dehn's solution which is based on abstract algebra, the solution by Bricard's condition requires only elementary mathematics.

5-1 Project Review & Future Study

Geometric dissection is a large branch of mathematics and a lot of mathematicians had been contributing and lots of findings were published. Due to time constrains, I am not able to study everything within these months when this project was conducted. Anyway, the objectives of this project such as understanding geometric dissection of polygons and polyhedra, and studying important results from literature are achieved. It is also fun to see and learn the interaction between different fields of mathematics such as geometry and algebra.

Some future study which can be done includes studying the Dehn's proof which is regarded as the classical solution of Hilbert's Third Problem. In this project, the main focus is dissection of polygons and polyhedra only. The dissection of shapes with curves such as circles and sphere are not studied and this topic could be researched. Another interesting yet difficult topic is dissection in the fourth and higher dimension. It will probably be challenging to define equidecomposability in higher dimension or to find out whether the number of invariants for polytope dissection are growing linearly or exponentially in the higher dimensions.

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