A Study on Compound-Commuting Mappings

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DECLARATION OF ORIGINALITY

I hereby declare that this project report is based on my original work except for citations and quotations which have been duly acknowledged. I also declare that it has not been previously and concurrently submitted for any other degree or award at UTAR or other institutions.

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APPROVAL FOR SUBMISSION

I certify that this project report entitled "**A Study on Compound-Commuting Mappings**" was prepared by **CHAN TAI CHONG** has met the required standard for submission in partial fulfilment of the requirements for the award of Bachelor of Science (Honours) Applied Mathematics With Computing at Universiti Tunku Abdul Rahman.

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ABSTRACT

Let $\mathbb{F}$ be a field carrying an involution $-$ and $\mathbb{F}^-$ be a fixed field of $\mathbb{F}$ corresponding to the involution $-$ where $\mathbb{F}^- = \{\alpha \in \mathbb{F} \mid \overline{\alpha} = \alpha\}$. Let m, n be positive integers with $m, n > 2$. We denote the set of all Hermitian matrices of order n underlying the field $\mathbb{F}$ by $\mathbb{H}_n(\mathbb{F})$. Furthermore, the $(n-1)$-th compound of a matrix A and the rank of the matrix A, we denote them by $C_{n-1}(A)$ and $\mathrm{rk}(A)$, respectively. In our study, we characterise a mapping $\Upsilon\colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ that satisfies one of the following conditions:

[P1] $\Upsilon(C_{n-1}(A-B)) = C_{m-1}(\Upsilon(A) - \Upsilon(B))$ for any $A, B \in \mathbb{H}_n(\mathbb{F})$;

[P2] $\Upsilon(C_{n-1}(A+\alpha B)) = C_{m-1}(\Upsilon(A) + \alpha\Upsilon(B))$ for any $A, B \in \mathbb{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$.

In order to obtain a general form of a mapping Υ satisfying [P1] or [P2], we need to impose some assumptions on Υ. If Υ satisfies [P1] with $\Upsilon(I_n) \neq 0_m$, then Υ satisfies $\mathrm{rk}(A-B) = n$ if and only if $\mathrm{rk}(\Upsilon(A) - \Upsilon(B)) = m$ for any $A, B \in \mathbb{H}_n(\mathbb{F})$. Also, if Υ satisfies [P2] with $\Upsilon(I_n) \neq 0_m$, then Υ is a rank-one non-increasing additive mapping. In case of Υ satisfies [P2] with $\Upsilon(I_n) = 0_m$, we have $\Upsilon(A) = 0_m$ for any $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) \leqslant 1$, $\Upsilon(C_{n-1}(A)) = 0_m$ for any $A \in \mathbb{H}_n(\mathbb{F})$ and $\mathrm{rk}(\Upsilon(A)) \leqslant m-2$ for any $A \in \mathbb{H}_n(\mathbb{F})$. Some examples of non-zero mapping Υ satisfying [P2] with $\Upsilon(I_n) = 0_m$ are constructed.

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CHAPTER 1

INTRODUCTION

1.1 Preserver Problems

There are a lot of topics that are studied in matrix theory. One of the famous topics that are studied in matrix theory is Linear Preserver Problems. Linear Preserver Problems focus on several types of linear operators on matrix spaces or linear mappings from a matrix space to another matrix space that preserve certain functions, subsets, relations, etc., invariant.

Remarkably, over the last few decades, there was much academic work on Linear Preserver Problems. Although there are plenty of fascinating results, many unanswered questions still exist. There are several reasons why Linear Preserver Problems is attractive. The results obtained from Linear Preserver Problems are often very clean, simple, elegant, and possess some nice properties. Here, we give some applications of Linear Preserver Problems.

Solving the systems of differential equations can also touch upon the Linear Preserver Problems. Before we solve it, we may apply some transformations to the system. This is to simplify the problems. The transformation we apply should be clean, simple, elegant, and possesses some nice properties. In particular, we apply a linear transformation on a linear differential system so that the system's stability or the eigenmodes can be preserved.

Apart from this, in system theory, we are interested in the linear operator which preserves the observable systems or controllable systems. If we are able to construct or find such linear operators, the complicated system would become a simpler system in which the system's nature is not influenced. For more details on this problem, refer to Fung (1996). On the other hand, in quantum system, we are interested in linear operator that transforms systems without influencing their entropy.

A few Linear Preserver Problems are treated as special cases for some mathematical problems. For instance, in Banach space, we desire to understand the structure of the linear isometries on them. If we treat the matrix spaces as special cases for Banach space, then this problem is considered as a Linear Preserver Problem.

1.2 Objectives

Until now, a lot of linear preserver results are extended to non-linear mappings by considering additive preserver problems and multiplicative preserver problems. Furthermore, some research has been carried without any assumptions of additivity or homogeneity.

This project's main objective is to study the classification of compound-commuting mappings on Hermitian matrices and symmetric matrices. Besides, new mathematical techniques in Linear Algebra used in preserver problems are to be studied and to be established in this project.

1.3 Problem Statements

First of all, we state ~~should understand~~ the definition of compound-commuting mapping. Let $m, n \in \mathbb{N}$ with $m, n > 2$. Let V_1 and V_2 be matrix spaces underlying the same field $\mathbb{F}$. Υ is a compound-commuting mappings if $\Upsilon\colon V_1 \to V_2$ satisfies

$$\Upsilon(C_{n-1}(A)) = C_{m-1}(\Upsilon(A)) \text{ for any } A \in V_1$$

where $C_{n-1}(A)$ is $(n-1)$-th compound of a matrix A.

The compound-commuting additive mappings on Hermitian matrices and symmetric matrices were researched by Chooi (2011). In the paper of Chooi and Ng (2010), they are studied adjoint-commuting mappings on square matrices without imposing additivity and homogeneity condition on Υ. Inspired by their work, we continue to study the compound-commuting mappings on Hermitian matrices and symmetric matrices in this project.

1.4 Methodology

In order to achieve the objectives, books, journals and articles related to preserver problems on space of matrices and compound-commuting mappings are to be collected and reviewed. Besides that, the intensive mathematical analysis of the collected literature is to be carried out in this project. Moreover, for typesetting, LaTeX is used.

1.5 Project Planning

The following tables show our project planning.

Table 1.1: Plan for Project I

Week	Plans
1	Registration of project title
2-3	Strengthen Linear Algebra background by extensive reading
4-6	Draft the proposal and discuss it with the supervisor
7	Mock proposal presentation and submit the proposal
8-9	Study different types of preserver problems
10-11	Draft the interim report
12	Submit interim report
13	Present the project I result

Table 1.2: Plan for Project II

Week	Plans
1-6	Collect and analyse research materials
7-8	Draft the final report
9	Prepare FYP poster and video recorded for poster presentation
10	Submit FYP poster, video recorded for poster presentation and a complete draft of the final report to the supervisor
11-12	Do correction for the report and submit all necessary documents
13	Present the project II result and attend the poster competition

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

2.1.1 Definitions and Notations

In this project, we obey the following notations unless otherwise specified. Let m, n be positive integers and $\mathbb{F}$ be a field. Likewise, the set of all positive integers, real numbers and complex numbers are denoted as $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$, respectively.

Definition 2.1.1.1. Let $\mathbb{F}$ be a field and ϕ be a mapping from $\mathbb{F}$ to $\mathbb{F}$. If $\phi(\alpha + \beta) = \phi(\alpha) + \phi(\beta)$ and $\phi(\alpha\beta) = \phi(\alpha)\phi(\beta)$ for any $\alpha, \beta \in \mathbb{F}$, then ϕ is a field homomorphism of $\mathbb{F}$. Moreover, ϕ is a field monomorphism of $\mathbb{F}$ if ϕ is a field homomorphism of $\mathbb{F}$ with ϕ is one-to-one and ϕ is a field automorphism of $\mathbb{F}$ if ϕ is a field homomorphism of $\mathbb{F}$ with ϕ is bijective.

The following remarks show us some simple deduction for **Definition 2.1.1.1**.

Remark. If ϕ is a field homomorphism of $\mathbb{F}$, then ϕ is a field monomorphism of $\mathbb{F}$. This is because all field homomorphism of $\mathbb{F}$ is one-to-one.

Remark. If ϕ is a field homomorphism of $\mathbb{F}$, then $\phi(0_{\mathbb{F}}) = 0_{\mathbb{F}}, \phi(1_{\mathbb{F}}) = 1_{\mathbb{F}}, \phi(-\alpha) = -\phi(\alpha)$, and $\phi(\alpha^{-1}) = \phi(\alpha)^{-1}$ for all $\alpha \in \mathbb{F}$ where $0_{\mathbb{F}}$ and $1_{\mathbb{F}}$ are additive identity and multiplicative identity of $\mathbb{F}$.

Definition 2.1.1.2. Let $\mathbb{F}$ be a field and let $-$ be a mapping from $\mathbb{F}$ to $\mathbb{F}$. If $-$ is a bijective mapping and for any $\alpha, \beta \in \mathbb{F}, \overline{\overline{\alpha}} = \alpha, \overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}$, and $\overline{\alpha\beta} = \overline{\beta}\overline{\alpha}$, then $-$ is an involution of $\mathbb{F}$. Also, if $-$ is an involution of $\mathbb{F}$, then $\mathbb{F}$ carries an involution $-$. Besides, the fixed field of $\mathbb{F}$ corresponding to the involution $-$, is denoted as $\mathbb{F}^-$ where $\mathbb{F}^- = \{\alpha \in \mathbb{F} \mid \overline{\alpha} = \alpha\}$.

Generally, the additive identity and the multiplicative identity of any field are denoted by 0 and 1, respectively. Since $0, 1 \in \mathbb{F}^-$ and $\alpha - \beta, \alpha\beta^{-1} \in \mathbb{F}^-$ for any $\alpha, \beta \in \mathbb{F}^-$, then $\mathbb{F}^-$ is a subfield of $\mathbb{F}$. If $\mathbb{F} = \mathbb{F}^-$, then $-$ is identity. Otherwise, $-$

is said to be proper. If it is not mentioned whether $-$ is identity or proper, $-$ can be identity or proper.

The set of all $m \times n$ matrices underlying the field $\mathbb{F}$ is denoted by $\mathbb{M}_{m\times n}(\mathbb{F})$. Let $A \in \mathbb{M}_{m\times n}(\mathbb{F})$. We write a_{ij} or A_{ij} to denote the entry of A in the i-th row and j-th column of A (in short, (i,j)-th entry of A), where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$. In some cases, this notation may lead to confusion. For instance, use a_{m-1n-1} or A_{m-1n-1} to represent the $(m-1, n-1)$-th entry of A. To avoid confusion, a comma is includes such as $a_{m-1,n-1}$ or $A_{m-1,n-1}$. Further, when $m = n$ happens, we denote $\mathbb{M}_{n\times n}(\mathbb{F})$ by $\mathbb{M}_n(\mathbb{F})$ and A is a square matrix of order n underlying the field $\mathbb{F}$.

The zero matrix and the identity matrix of the set of all $m \times n$ matrices are denoted by $0_{m\times n}$ and $I_{m\times n}$, respectively. In the case of $m = n$, we denote them by 0_n and I_n. Let $A \in \mathbb{M}_{m\times n}(\mathbb{F})$, $A[i \mid j]$ stands for a matrix obtained by eliminating i-th row and j-th column from A and $A[i \mid j] \in \mathbb{M}_{(m-1)\times(n-1)}(\mathbb{F})$.

Moreover, a matrix obtained from A by applying ϕ entrywise is denoted by A^{ϕ}. Furthermore, we use $|A|$, $\mathrm{rk}(A)$ and $\mathrm{char}(\mathbb{F})$ to represent the determinant of the matrix A, the rank of the matrix A and the characteristic of the field $\mathbb{F}$, respectively. To avoid misunderstanding with the notation of determinant, we use $||S||$ to represent the number of elements in a set S.

Let Galois field of order 2 be $GF(2)$. $GF(2)$ is a finite field with two elements which are the additive identity and the multiplicative identity. In other words, $GF(2) = \{0, 1\}$. Obviously, $\mathrm{char}(GF(2)) = 2$. This also tells us if $GF(2)$ is a subfield of $\mathbb{F}$, then $\mathrm{char}(\mathbb{F}) = 2$.

Consider a matrix in $\mathbb{M}_{m\times n}(\mathbb{F})$, if all entries of the matrix are equal to 0 except (i,j)-th entry and the (i,j)-th entry is 1 where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, then the matrix is denoted by E_{ij}. Besides that, we use E_{ij} to introduce some square matrix of order n that is

$$W_n = \sum_{i=1}^{n} (-1)^{i+1} E_{ii} \qquad \text{and} \qquad J_n = \sum_{i=1}^{n} E_{n+1-i,i}.$$

Now we would like to introduce an operator that is less often used in matrix spaces, which is a direct sum of matrices. Let $A_i \in \mathbb{M}_{m_i\times n_i}(\mathbb{F})$ for each $i \in \{1, 2, \cdots, k\}$.

$$A_1 \oplus A_2 = \begin{bmatrix} A_1 & 0_{m_1\times n_2} \\ 0_{m_2\times n_1} & A_2 \end{bmatrix}$$

where $A_1 \oplus A_2 \in \mathbb{M}_{(m_1+m_2)\times(n_1+n_2)}(\mathbb{F})$.

In general,

$$\begin{aligned}\bigoplus_{i=1}^{k} A_i &= A_1 \oplus A_2 \oplus \cdots \oplus A_k \\ &= \begin{bmatrix} A_1 & 0_{m_1 \times n_2} & \cdots & 0_{m_1 \times n_k} \\ 0_{m_2 \times n_1} & A_2 & \cdots & 0_{m_2 \times n_k} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{m_k \times n_1} & 0_{m_k \times n_2} & \cdots & A_k \end{bmatrix}\end{aligned}$$

where $\bigoplus_{i=1}^{k} A_i \in \mathbb{M}_{(m_1+m_2+\cdots+m_k)\times(n_1+n_2+\cdots+n_k)}(\mathbb{F})$. In case $m_i = n_i = n$ for each $i \in \{1, 2, \cdots, k\}$, then

$$\begin{aligned}\bigoplus_{i=1}^{k} A_i &= A_1 \oplus A_2 \oplus \cdots \oplus A_k \\ &= \begin{bmatrix} A_1 & 0_n & \cdots & 0_n \\ 0_n & A_2 & \cdots & 0_n \\ \vdots & \vdots & \ddots & \vdots \\ 0_n & 0_n & \cdots & A_k \end{bmatrix}\end{aligned}$$

where $\bigoplus_{i=1}^{k} A_i$ is a square matrix of order kn underlying the field $\mathbb{F}$.

Definition 2.1.1.3. Let V_1 and V_2 be matrix spaces underlying the same field $\mathbb{F}$ and Υ be a mapping from V_1 to V_2. For any $A, B \in V_1$ and $\alpha \in \mathbb{F}$,

1. Υ is additive if $\Upsilon(A + B) = \Upsilon(A) + \Upsilon(B)$.
2. Υ is homogeneous if $\Upsilon(\alpha A) = \alpha\Upsilon(A)$.
3. Υ is linear if Υ is additive and homogeneous.
4. Υ is a linear operator on V_1 if Υ is a linear mapping from V_1 to V_1.

Definition 2.1.1.4. Let $\mathbb{F}$ be a field carrying an involution $-$. Let V_1 and V_2 be matrix spaces underlying the same field $\mathbb{F}$ and Υ be a mapping from V_1 to V_2. Υ is $\mathbb{F}^-$-homogeneous if $\Upsilon(\alpha A) = \alpha\Upsilon(A)$ for any $\alpha \in \mathbb{F}^-$ and $A \in V_1$.

Definition 2.1.1.5. Let V be a matrix space underlying the field $\mathbb{F}$. Υ is called a functional on V if Υ is a mapping from V to $\mathbb{F}$.

2.1.2 Symmetric Matrices

Let $A \in \mathbb{M}_{m\times n}(\mathbb{F})$. The transpose of a matrix A is a matrix whose rows are the columns of A in the same order. We denote the transpose of a matrix A by A^T. All entries of A^T satisfy $(A^T)_{ij} = A_{ji}$ and $A^T \in \mathbb{M}_{n\times m}(\mathbb{F})$.

Example 2.1.2.1. Let $A = \begin{bmatrix} 3 & 8 & 9 & 12 & 7 \\ 7 & 19 & 3 & 9 & 23 \\ 12 & 2 & 4 & 11 & 8 \\ 5 & 0 & 9 & 2 & 4 \end{bmatrix} \in \mathbb{M}_{4\times 5}(\mathbb{R})$.

Then $A^T = \begin{bmatrix} 3 & 7 & 12 & 5 \\ 8 & 19 & 2 & 0 \\ 9 & 3 & 4 & 9 \\ 12 & 9 & 11 & 2 \\ 7 & 23 & 8 & 4 \end{bmatrix}$.

When $m = n$, $A \in \mathbb{M}_n(\mathbb{F})$. A is equal to its transpose only if A is a symmetric matrix of order n underlying the field $\mathbb{F}$. This tells us that $A = A^T$ and all entries of A satisfy $a_{ij} = a_{ji}$. We denote the set of all symmetric matrices of order n underlying the field $\mathbb{F}$ by $\mathbb{S}_n(\mathbb{F})$.

Example 2.1.2.2. Let $A = \begin{bmatrix} -5 & 2 & 0 & -14 & 9 \\ 2 & 3 & 15 & -7 & 23 \\ 0 & 15 & -12 & 1 & 4 \\ -14 & -7 & 1 & 2 & 11 \\ 9 & 23 & 4 & 11 & 5 \end{bmatrix} \in \mathbb{M}_5(\mathbb{R})$.

Then $A^T = \begin{bmatrix} -5 & 2 & 0 & -14 & 9 \\ 2 & 3 & 15 & -7 & 23 \\ 0 & 15 & -12 & 1 & 4 \\ -14 & -7 & 1 & 2 & 11 \\ 9 & 23 & 4 & 11 & 5 \end{bmatrix}$. Since $A = A^T$, then A is a symmetric matrix of order 5 underlying the field $\mathbb{R}$.

Now, we would like to introduce a matrix related to the transpose of a matrix. Let $A \in \mathbb{M}_{m\times n}(\mathbb{F})$, $A^{\sim} = J_n A^T J_m$ and $A^{\sim} \in \mathbb{M}_{n\times m}(\mathbb{F})$. Here, we proof that (i, j)-th entry of $A^{\sim}$ is $a_{m+1-j,n+1-i}$.

Let $A \in \mathbb{M}_{m\times n}(\mathbb{F})$ and $B = A^T$. Let $e_i \in \mathbb{M}_{n\times 1}(\mathbb{F})$ with i-th entry is equal to 1 and the other entries are equal to 0 for all $i \in \{1, 2, \cdots, n\}$ and $f_j \in \mathbb{M}_{m\times 1}(\mathbb{F})$ with j-th entry is equal to 1 and the other entries are equal to 0 for all $j \in \{1, 2, \cdots, m\}$.

$$\begin{aligned}(A^{\sim})_{ij} &= e_{n+1-i}^T A^T f_{m+1-j} \\ &= e_{n+1-i}^T \begin{bmatrix} b_{1,m+1-j} \\ b_{2,m+1-j} \\ \vdots \\ b_{n,m+1-j} \end{bmatrix} \\ &= b_{n+1-i,m+1-j} \\ &= a_{m+1-j,n+1-i}.\end{aligned}$$

2.1.3 Hermitian Matrices

When we discuss Hermitian matrices, the field $\mathbb{F}$ must carry an involution $-$. Let $A \in \mathbb{M}_{m\times n}(\mathbb{F})$. We extend the transpose of a matrix A to the Hermitian transpose of a matrix A. We denote the Hermitian transpose of a matrix A by A^H. By taking the transpose of a matrix A first, then apply the involution $-$ on all the entries in A^T, we obtain A^H. Thus $A^H = \overline{A^T}$, all entries in A^H satisfy $(A^H)_{ij} = \overline{A_{ji}}$ and $A^H \in \mathbb{M}_{n\times m}(\mathbb{F})$.

Example 2.1.3.1. Define a mapping $- : \mathbb{C} \to \mathbb{C}$ satisfying $\overline{a_1 + a_2 i} = a_1 - a_2 i$ for any $a_1 + a_2 i \in \mathbb{C}$ where $i = \sqrt{-1}$.
For any $a_1 + a_2 i, a_3 + a_4 i \in \mathbb{C}$,

1. $\overline{a_1 + a_2 i} = \overline{a_3 + a_4 i}$ implies $a_1 + a_2 i = a_3 + a_4 i$ (i.e., $-$ is one-to-one).

2. $\overline{a_1 - a_2 i} = a_1 + a_2 i$ (i.e., $-$ is onto).

3. $\overline{\overline{a_1 + a_2 i}} = \overline{a_1 - a_2 i} = a_1 + a_2 i$.

4. $\overline{(a_1 + a_2 i) + (a_3 + a_4 i)} = \overline{(a_1 + a_3) + (a_2 + a_4) i} = (a_1 + a_3) - (a_2 + a_4) i = (a_1 - a_2 i) + (a_3 - a_4 i) = \overline{a_1 + a_2 i} + \overline{a_3 + a_4 i}$.

5. $\overline{(a_1 + a_2 i)(a_3 + a_4 i)} = \overline{(a_1 a_3 - a_2 a_4) + (a_1 a_4 + a_2 a_3) i} = (a_1 a_3 - a_2 a_4) - (a_1 a_4 + a_2 a_3) i = (a_1 a_3 - (-a_2)(-a_4)) + (a_1(-a_4) + (-a_2) a_3) i = (a_1 - a_2 i)(a_3 - a_4 i) = (\overline{a_1 + a_2 i})(\overline{a_3 + a_4 i}) = (\overline{a_3 + a_4 i})(\overline{a_1 + a_2 i})$.

These show us that $-$ is an involution of $\mathbb{C}$. Thus $\mathbb{C}$ carries an involution $-$. We see that $-$ is actually the complex conjugate of $\mathbb{C}$.

Example 2.1.3.2. Let $A = \begin{bmatrix} 2+i & 3 & 9 & -i \\ 0 & 7+5i & -2-3i & 15 \\ -5 & 12 & 2i & -4 \\ 9 & 5 & 3+8i & 10-9i \\ -2+3i & 6+13i & 7 & 12 \end{bmatrix} \in \mathbb{M}_{5\times 4}(\mathbb{C})$.

Then $A^T = \begin{bmatrix} 2+i & 0 & -5 & 9 & -2+3i \\ 3 & 7+5i & 12 & 5 & 6+13i \\ 9 & -2-3i & 2i & 3+8i & 7 \\ -i & 15 & -4 & 10-9i & 12 \end{bmatrix}$.

So $A^H = \overline{A^T} = \begin{bmatrix} 2-i & 0 & -5 & 9 & -2-3i \\ 3 & 7-5i & 12 & 5 & 6-13i \\ 9 & -2+3i & -2i & 3-8i & 7 \\ i & 15 & -4 & 10+9i & 12 \end{bmatrix}$.

When $m = n$, $A \in \mathbb{M}_n(\mathbb{F})$. A is a Hermitian matrix of order n underlying the field $\mathbb{F}$ when A is equal to its Hermitian transpose, A^H. This implies $A = A^H$ and all entries in A satisfy $a_{ij} = \overline{a_{ji}}$. It is clear that for all entries in the main diagonal, a_{ii} must be equal to $\overline{a_{ii}}$. We denote the set of all Hermitian matrices of order n underlying the field $\mathbb{F}$ by $\mathbb{H}_n(\mathbb{F})$ and the Hermitian matrix is originated by Charles Hermite (Simmons (1992)).

Example 2.1.3.3. Let

$$A = \begin{bmatrix} 2 & 0 & -2+5i & 1+7i & 8 \\ 0 & 3 & 16-7i & 6 & -3+9i \\ -2-5i & 16+7i & -2 & 7+12i & -3 \\ 1-7i & 6 & 7-12i & 5 & 1 \\ 8 & -3-9i & -3 & 1 & 7 \end{bmatrix} \in \mathbb{M}_5(\mathbb{C}).$$

Then $A^T = \begin{bmatrix} 2 & 0 & -2-5i & 1-7i & 8 \\ 0 & 3 & 16+7i & 6 & -3-9i \\ -2+5i & 16-7i & -2 & 7-12i & -3 \\ 1+7i & 6 & 7+12i & 5 & 1 \\ 8 & -3+9i & -3 & 1 & 7 \end{bmatrix}$.

So $A^H = \overline{A^T} = \begin{bmatrix} 2 & 0 & -2+5i & 1+7i & 8 \\ 0 & 3 & 16-7i & 6 & -3+9i \\ -2-5i & 16+7i & -2 & 7+12i & -3 \\ 1-7i & 6 & 7-12i & 5 & 1 \\ 8 & -3-9i & -3 & 1 & 7 \end{bmatrix}$. Since $A = A^H$, then A is a Hermitian matrix of order 5 underlying the field $\mathbb{C}$.

Remark. Let $\mathbb{F}$ be a field carrying an involution $-$. It is clear that if $-$ is identity, then $\mathbb{H}_n(\mathbb{F}) = \mathbb{S}_n(\mathbb{F})$.

2.1.4 Adjoint Matrices

Muir (1960) gives the early background of the adjoint matrix. Let $A \in \mathbb{M}_n(\mathbb{F})$. By taking the transpose of a cofactor matrix of a matrix A, we attain an adjoint matrix of A. Adjoint of A also belongs to $\mathbb{M}_n(\mathbb{F})$. We denote the adjoint matrix of A by $\mathrm{adj}(A)$ and (i,j)-th entry of $\mathrm{adj}(A)$ is defined as

$$(\mathrm{adj}(A))_{ij} = (-1)^{i+j}|A[j \mid i]|.$$

Example 2.1.4.1. Let $A \in \mathbb{M}_5(\mathbb{F})$.

Then $\mathrm{adj}(A) = \begin{bmatrix} |A[1\mid1]| & -|A[2\mid1]| & |A[3\mid1]| & -|A[4\mid1]| & |A[5\mid1]| \\ -|A[1\mid2]| & |A[2\mid2]| & -|A[3\mid2]| & |A[4\mid2]| & -|A[5\mid2]| \\ |A[1\mid3]| & -|A[2\mid3]| & |A[3\mid3]| & -|A[4\mid3]| & |A[5\mid3]| \\ -|A[1\mid4]| & |A[2\mid4]| & -|A[3\mid4]| & |A[4\mid4]| & -|A[5\mid4]| \\ |A[1\mid5]| & -|A[2\mid5]| & |A[3\mid5]| & -|A[4\mid5]| & |A[5\mid5]| \end{bmatrix}$.

Example 2.1.4.2. Let $A = \begin{bmatrix} 3+5i & -2 & 1-3i \\ 9 & 7 & -4 \\ 4 & 2+i & -6+8i \end{bmatrix} \in \mathbb{M}_3(\mathbb{C})$. Then

$$\mathrm{adj}(A) = \begin{bmatrix} |A[1 \mid 1]| & -|A[2 \mid 1]| & |A[3 \mid 1]| \\ -|A[1 \mid 2]| & |A[2 \mid 2]| & -|A[3 \mid 2]| \\ |A[1 \mid 3]| & -|A[2 \mid 3]| & |A[3 \mid 3]| \end{bmatrix}$$

$$= \begin{bmatrix} \begin{vmatrix} 7 & -4 \\ 2+i & -6+8i \end{vmatrix} & -\begin{vmatrix} -2 & 1-3i \\ 2+i & -6+8i \end{vmatrix} & \begin{vmatrix} -2 & 1-3i \\ 7 & -4 \end{vmatrix} \\ -\begin{vmatrix} 9 & -4 \\ 4 & -6+8i \end{vmatrix} & \begin{vmatrix} 3+5i & 1-3i \\ 4 & -6+8i \end{vmatrix} & -\begin{vmatrix} 3+5i & 1-3i \\ 9 & -4 \end{vmatrix} \\ \begin{vmatrix} 9 & 7 \\ 4 & 2+i \end{vmatrix} & -\begin{vmatrix} 3+5i & -2 \\ 4 & 2+i \end{vmatrix} & \begin{vmatrix} 3+5i & -2 \\ 9 & 7 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -34+60i & -7+11i & 1+21i \\ 38-72i & -62+6i & 21-7i \\ -10+9i & -9-13i & 39+35i \end{bmatrix}.$$

2.1.5 Compound Matrices

A compound matrix is a special kind of matrix. All the entries in a compound matrix are determinants of submatrices (Aitken (1949)). Let $A \in \mathbb{M}_{m\times n}(\mathbb{F})$. Given that $I \subseteq \{1, 2, ..., m\}$ and $J \subseteq \{1, 2, ..., n\}$, where I, J both are not empty-set and the ways to arrange the elements in I and J must be in dictionary order. The submatrix of A whose rows and columns are indexed by I and J is denoted by $A(I|J)$.

Let $k \in \mathbb{N}$ for which $k \leqslant \min\{m, n\}$. $C_k(A)$ means k-th compound of a matrix A. The size of the matrix $C_k(A)$ is $\binom{m}{k} \times \binom{n}{k}$ and

$$(C_k(A))_{ij} = |A(I_i|J_j)|.$$

Also the number of elements in I_i and J_i both are equal to k. Besides that the subsets $I_1, I_2, \cdots, I_{\binom{m}{k}}$ and the subsets $J_1, J_2, \cdots, J_{\binom{n}{k}}$ must be arranged in dictionary order. Hence $I_1 < I_2 < \cdots < I_{\binom{m}{k}}$ and $J_1 < J_2 < \cdots < J_{\binom{n}{k}}$.

In this project, we use the $(n-1)$-th compound of a matrix A, that is $C_{n-1}(A)$. Besides, the matrix A we consider is a square matrix of order n underlying the field

$\mathbb{F}$. $C_{n-1}(A)$ belongs to $\mathbb{M}_{\binom{n}{n-1}\times\binom{n}{n-1}}(\mathbb{F}) = \mathbb{M}_{n\times n}(\mathbb{F}) = \mathbb{M}_n(\mathbb{F})$. Further, we represent the (i,j)-th entry of $C_{n-1}(A)$ in a more straightforward form, that is,

$$(C_{n-1}(A))_{ij} = |A[n+1-i \mid n+1-j]|.$$

Example 2.1.5.1. Let $A \in \mathbb{M}_{5\times 4}(\mathbb{F})$. Then $C_3(A) \in \mathbb{M}_{\binom{5}{3}\times\binom{4}{3}}(\mathbb{F}) = \mathbb{M}_{10\times 4}(\mathbb{F})$ and

$$C_3(A) = \begin{bmatrix} |A(\{1,2,3\}|\{1,2,3\})| & |A(\{1,2,3\}|\{1,2,4\})| & |A(\{1,2,3\}|\{1,3,4\})| & |A(\{1,2,3\}|\{2,3,4\})| \\ |A(\{1,2,4\}|\{1,2,3\})| & |A(\{1,2,4\}|\{1,2,4\})| & |A(\{1,2,4\}|\{1,3,4\})| & |A(\{1,2,4\}|\{2,3,4\})| \\ |A(\{1,2,5\}|\{1,2,3\})| & |A(\{1,2,5\}|\{1,2,4\})| & |A(\{1,2,5\}|\{1,3,4\})| & |A(\{1,2,5\}|\{2,3,4\})| \\ |A(\{1,3,4\}|\{1,2,3\})| & |A(\{1,3,4\}|\{1,2,4\})| & |A(\{1,3,4\}|\{1,3,4\})| & |A(\{1,3,4\}|\{2,3,4\})| \\ |A(\{1,3,5\}|\{1,2,3\})| & |A(\{1,3,5\}|\{1,2,4\})| & |A(\{1,3,5\}|\{1,3,4\})| & |A(\{1,3,5\}|\{2,3,4\})| \\ |A(\{1,4,5\}|\{1,2,3\})| & |A(\{1,4,5\}|\{1,2,4\})| & |A(\{1,4,5\}|\{1,3,4\})| & |A(\{1,4,5\}|\{2,3,4\})| \\ |A(\{2,3,4\}|\{1,2,3\})| & |A(\{2,3,4\}|\{1,2,4\})| & |A(\{2,3,4\}|\{1,3,4\})| & |A(\{2,3,4\}|\{2,3,4\})| \\ |A(\{2,3,5\}|\{1,2,3\})| & |A(\{2,3,5\}|\{1,2,4\})| & |A(\{2,3,5\}|\{1,3,4\})| & |A(\{2,3,5\}|\{2,3,4\})| \\ |A(\{2,4,5\}|\{1,2,3\})| & |A(\{2,4,5\}|\{1,2,4\})| & |A(\{2,4,5\}|\{1,3,4\})| & |A(\{2,4,5\}|\{2,3,4\})| \\ |A(\{3,4,5\}|\{1,2,3\})| & |A(\{3,4,5\}|\{1,2,4\})| & |A(\{3,4,5\}|\{1,3,4\})| & |A(\{3,4,5\}|\{2,3,4\})| \end{bmatrix}.$$

Example 2.1.5.2. Let $A = \begin{bmatrix} 3 & 7+i & 9-2i \\ -12+7i & 5-3i & 2+4i \\ 6+13i & -4 & 9+5i \end{bmatrix} \in \mathbb{M}_3(\mathbb{C})$, then

$$\begin{aligned} C_2(A) &= \begin{bmatrix} |A[3 \mid 3]| & |A[3 \mid 2]| & |A[3 \mid 1]| \\ |A[2 \mid 3]| & |A[2 \mid 2]| & |A[2 \mid 1]| \\ |A[1 \mid 3]| & |A[1 \mid 2]| & |A[1 \mid 1]| \end{bmatrix} \\ &= \begin{bmatrix} \begin{vmatrix} 3 & 7+i \\ -12+7i & 5-3i \end{vmatrix} & \begin{vmatrix} 3 & 9-2i \\ -12+7i & 2+4i \end{vmatrix} & \begin{vmatrix} 7+i & 9-2i \\ 5-3i & 2+4i \end{vmatrix} \\ \begin{vmatrix} 3 & 7+i \\ 6+13i & -4 \end{vmatrix} & \begin{vmatrix} 3 & 9-2i \\ 6+13i & 9+5i \end{vmatrix} & \begin{vmatrix} 7+i & 9-2i \\ -4 & 9+5i \end{vmatrix} \\ \begin{vmatrix} -12+7i & 5-3i \\ 6+13i & -4 \end{vmatrix} & \begin{vmatrix} -12+7i & 2+4i \\ 6+13i & 9+5i \end{vmatrix} & \begin{vmatrix} 5-3i & 2+4i \\ -4 & 9+5i \end{vmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 106-46i & 100-75i & -29+67i \\ -41-97i & -53-90i & 94+36i \\ -21-75i & -103-47i & 68+14i \end{bmatrix}. \end{aligned}$$

Example 2.1.5.3. Let $A = \begin{bmatrix} 10 & 29 & 28+4i & -25 \\ 2-5i & 8+12i & -9 & 1+13i \\ -27 & 9i & -1-i & -7+9i \\ 7-2i & i & -10 & -11-10i \\ -4-12i & 0 & 22 & 18i \end{bmatrix} \in \mathbb{M}_{5\times 4}(\mathbb{C})$.

Then $C_2(A) \in \mathbb{M}_{\binom{5}{2}\times\binom{4}{2}}(\mathbb{C}) = \mathbb{M}_{10\times 6}(\mathbb{C})$ and

$$
\begin{aligned}
&C_2(A)\\
&= \begin{bmatrix}
|A(\{1,2\}|\{1,2\})| & |A(\{1,2\}|\{1,3\})| & |A(\{1,2\}|\{1,4\})| & |A(\{1,2\}|\{2,3\})| & |A(\{1,2\}|\{2,4\})| & |A(\{1,2\}|\{3,4\})| \\
|A(\{1,3\}|\{1,2\})| & |A(\{1,3\}|\{1,3\})| & |A(\{1,3\}|\{1,4\})| & |A(\{1,3\}|\{2,3\})| & |A(\{1,3\}|\{2,4\})| & |A(\{1,3\}|\{3,4\})| \\
|A(\{1,4\}|\{1,2\})| & |A(\{1,4\}|\{1,3\})| & |A(\{1,4\}|\{1,4\})| & |A(\{1,4\}|\{2,3\})| & |A(\{1,4\}|\{2,4\})| & |A(\{1,4\}|\{3,4\})| \\
|A(\{1,5\}|\{1,2\})| & |A(\{1,5\}|\{1,3\})| & |A(\{1,5\}|\{1,4\})| & |A(\{1,5\}|\{2,3\})| & |A(\{1,5\}|\{2,4\})| & |A(\{1,5\}|\{3,4\})| \\
|A(\{2,3\}|\{1,2\})| & |A(\{2,3\}|\{1,3\})| & |A(\{2,3\}|\{1,4\})| & |A(\{2,3\}|\{2,3\})| & |A(\{2,3\}|\{2,4\})| & |A(\{2,3\}|\{3,4\})| \\
|A(\{2,4\}|\{1,2\})| & |A(\{2,4\}|\{1,3\})| & |A(\{2,4\}|\{1,4\})| & |A(\{2,4\}|\{2,3\})| & |A(\{2,4\}|\{2,4\})| & |A(\{2,4\}|\{3,4\})| \\
|A(\{2,5\}|\{1,2\})| & |A(\{2,5\}|\{1,3\})| & |A(\{2,5\}|\{1,4\})| & |A(\{2,5\}|\{2,3\})| & |A(\{2,5\}|\{2,4\})| & |A(\{2,5\}|\{3,4\})| \\
|A(\{3,4\}|\{1,2\})| & |A(\{3,4\}|\{1,3\})| & |A(\{3,4\}|\{1,4\})| & |A(\{3,4\}|\{2,3\})| & |A(\{3,4\}|\{2,4\})| & |A(\{3,4\}|\{3,4\})| \\
|A(\{3,5\}|\{1,2\})| & |A(\{3,5\}|\{1,3\})| & |A(\{3,5\}|\{1,4\})| & |A(\{3,5\}|\{2,3\})| & |A(\{3,5\}|\{2,4\})| & |A(\{3,5\}|\{3,4\})| \\
|A(\{4,5\}|\{1,2\})| & |A(\{4,5\}|\{1,3\})| & |A(\{4,5\}|\{1,4\})| & |A(\{4,5\}|\{2,3\})| & |A(\{4,5\}|\{2,4\})| & |A(\{4,5\}|\{3,4\})|
\end{bmatrix}\\
&= \begin{bmatrix}
\begin{vmatrix} 10 & 29 \\ 2-5i & 8+12i \end{vmatrix} & \begin{vmatrix} 10 & 28+4i \\ 2-5i & -9 \end{vmatrix} & \begin{vmatrix} 10 & -25 \\ 2-5i & 1+13i \end{vmatrix} & \begin{vmatrix} 29 & 28+4i \\ 8+12i & -9 \end{vmatrix} & \begin{vmatrix} 29 & -25 \\ 8+12i & 1+13i \end{vmatrix} & \begin{vmatrix} 28+4i & -25 \\ -9 & 1+13i \end{vmatrix} \\
\begin{vmatrix} 10 & 29 \\ -27 & 9i \end{vmatrix} & \begin{vmatrix} 10 & 28+4i \\ -27 & -1-i \end{vmatrix} & \begin{vmatrix} 10 & -25 \\ -27 & -7+9i \end{vmatrix} & \begin{vmatrix} 29 & 28+4i \\ 9i & -1-i \end{vmatrix} & \begin{vmatrix} 29 & -25 \\ 9i & -7+9i \end{vmatrix} & \begin{vmatrix} 28+4i & -25 \\ -1-i & -7+9i \end{vmatrix} \\
\begin{vmatrix} 10 & 29 \\ 7-2i & i \end{vmatrix} & \begin{vmatrix} 10 & 28+4i \\ 7-2i & -10 \end{vmatrix} & \begin{vmatrix} 10 & -25 \\ 7-2i & -11-10i \end{vmatrix} & \begin{vmatrix} 29 & 28+4i \\ i & -10 \end{vmatrix} & \begin{vmatrix} 29 & -25 \\ i & -11-10i \end{vmatrix} & \begin{vmatrix} 28+4i & -25 \\ -10 & -11-10i \end{vmatrix} \\
\begin{vmatrix} 10 & 29 \\ -4-12i & 0 \end{vmatrix} & \begin{vmatrix} 10 & 28+4i \\ -4-12i & 22 \end{vmatrix} & \begin{vmatrix} 10 & -25 \\ -4-12i & 18i \end{vmatrix} & \begin{vmatrix} 29 & 28+4i \\ 0 & 22 \end{vmatrix} & \begin{vmatrix} 29 & -25 \\ 0 & 18i \end{vmatrix} & \begin{vmatrix} 28+4i & -25 \\ 22 & 18i \end{vmatrix} \\
\begin{vmatrix} 2-5i & 8+12i \\ -27 & 9i \end{vmatrix} & \begin{vmatrix} 2-5i & -9 \\ -27 & -1-i \end{vmatrix} & \begin{vmatrix} 2-5i & 1+13i \\ -27 & -7+9i \end{vmatrix} & \begin{vmatrix} 8+12i & -9 \\ 9i & -1-i \end{vmatrix} & \begin{vmatrix} 8+12i & 1+13i \\ 9i & -7+9i \end{vmatrix} & \begin{vmatrix} -9 & 1+13i \\ -1-i & -7+9i \end{vmatrix} \\
\begin{vmatrix} 2-5i & 8+12i \\ 7-2i & i \end{vmatrix} & \begin{vmatrix} 2-5i & -9 \\ 7-2i & -10 \end{vmatrix} & \begin{vmatrix} 2-5i & 1+13i \\ 7-2i & -11-10i \end{vmatrix} & \begin{vmatrix} 8+12i & -9 \\ i & -10 \end{vmatrix} & \begin{vmatrix} 8+12i & 1+13i \\ i & -11-10i \end{vmatrix} & \begin{vmatrix} -9 & 1+13i \\ -10 & -11-10i \end{vmatrix} \\
\begin{vmatrix} 2-5i & 8+12i \\ -4-12i & 0 \end{vmatrix} & \begin{vmatrix} 2-5i & -9 \\ -4-12i & 22 \end{vmatrix} & \begin{vmatrix} 2-5i & 1+13i \\ -4-12i & 18i \end{vmatrix} & \begin{vmatrix} 8+12i & -9 \\ 0 & 22 \end{vmatrix} & \begin{vmatrix} 8+12i & 1+13i \\ 0 & 18i \end{vmatrix} & \begin{vmatrix} -9 & 1+13i \\ 22 & 18i \end{vmatrix} \\
\begin{vmatrix} -27 & 9i \\ 7-2i & i \end{vmatrix} & \begin{vmatrix} -27 & -1-i \\ 7-2i & -10 \end{vmatrix} & \begin{vmatrix} -27 & -7+9i \\ 7-2i & -11-10i \end{vmatrix} & \begin{vmatrix} 9i & -1-i \\ i & -10 \end{vmatrix} & \begin{vmatrix} 9i & -7+9i \\ i & -11-10i \end{vmatrix} & \begin{vmatrix} -1-i & -7+9i \\ -10 & -11-10i \end{vmatrix} \\
\begin{vmatrix} -27 & 9i \\ -4-12i & 0 \end{vmatrix} & \begin{vmatrix} -27 & -1-i \\ -4-12i & 22 \end{vmatrix} & \begin{vmatrix} -27 & -7+9i \\ -4-12i & 18i \end{vmatrix} & \begin{vmatrix} 9i & -1-i \\ 0 & 22 \end{vmatrix} & \begin{vmatrix} 9i & -7+9i \\ 0 & 18i \end{vmatrix} & \begin{vmatrix} -1-i & -7+9i \\ 22 & 18i \end{vmatrix} \\
\begin{vmatrix} 7-2i & i \\ -4-12i & 0 \end{vmatrix} & \begin{vmatrix} 7-2i & -10 \\ -4-12i & 22 \end{vmatrix} & \begin{vmatrix} 7-2i & -11-10i \\ -4-12i & 18i \end{vmatrix} & \begin{vmatrix} i & -10 \\ 0 & 22 \end{vmatrix} & \begin{vmatrix} i & -11-10i \\ 0 & 18i \end{vmatrix} & \begin{vmatrix} -10 & -11-10i \\ 22 & 18i \end{vmatrix}
\end{bmatrix}\\
&= \begin{bmatrix}
22+265i & -166+132i & 60+5i & -437-368i & 229+677i & -249+368i \\
783+90i & 746+98i & -745+90i & 7-281i & -203+486i & -257+199i \\
-203+68i & -304+28i & 65-150i & -286-28i & -319-265i & -518-324i \\
116+348i & 284+352i & -100-120i & 638 & 522i & 478+504i \\
261+342i & -250+3i & 58+404i & 4+61i & -47-21i & 51-67i \\
-75-66i & 43+32i & -105-54i & -80-111i & 45-213i & 109+220i \\
-112+144i & 8-218i & -62+100i & 176+264i & -216+144i & -22-448i \\
-18-90i & 279+5i & 328+193i & -1-89i & 99-92i & -69+111i \\
-108+36i & -586-16i & -136-534i & 198i & -162 & 172-216i \\
-12+4i & 114-164i & 112-46i & 22i & -18 & 242+40i
\end{bmatrix}.
\end{aligned}
$$

We observe that (i, j)-th entry of $\mathrm{adj}(A)$ and $C_{n-1}(A)$ both are determinants of a submatrix where the submatrix is obtained by eliminating one of the rows and one of the columns from A. So, there is a relationship between $\mathrm{adj}(A)$ and $C_{n-1}(A)$. The relationship is shown in the following lemma.

Lemma 2.1.5.4. (Chooi (2011) Lemma 2.3). Let $\mathbb{F}$ be a field and $n \in \mathbb{N}$ with $n > 2$. For all $A \in \mathbb{M}_n(\mathbb{F}), C_{n-1}(A) = W_n \mathrm{adj}(A)^{\sim} W_n$.

Proof.

Let $A \in \mathbb{M}_n(\mathbb{F})$. Let $B = \mathrm{adj}(A)$ and $G = \mathrm{adj}(A)^{\sim}$. Let $e_i \in \mathbb{M}_{n\times 1}(\mathbb{F})$ with i-th entry is equal to 1 and the other entries are equal to 0 for all $i \in \{1, 2, \cdots, n\}$.

$$\begin{aligned}
&(W_n(\operatorname{adj}(A))^{\sim}W_n)_{ij}\\
=&(-1)^{i+1}e_i^T(\operatorname{adj}(A))^{\sim}(-1)^{j+1}e_j\\
=&(-1)^{i+j}e_i^T\begin{bmatrix} g_{1,j}\\ g_{2,j}\\ \vdots\\ g_{n,j}\end{bmatrix}\\
=&(-1)^{i+j}g_{i,j}\\
=&(-1)^{i+j}b_{n+1-j,n+1-i}\\
=&(-1)^{i+j}(-1)^{(n+1-j)+(n+1-i)}|A[n+1-i\mid n+1-j]|\\
=&|A[n+1-i\mid n+1-j]|\\
=&(C_{n-1}(A))_{ij}.
\end{aligned}$$

□

Example 2.1.5.5. Let $A = \begin{bmatrix} 3 & 7+i & 9-2i\\ -12+7i & 5-3i & 2+4i\\ 6+13i & -4 & 9+5i\end{bmatrix} \in \mathbb{M}_3(\mathbb{C})$. By **Example 2.1.5.2**, we have $C_2(A) = \begin{bmatrix} 106-46i & 100-75i & -29+67i\\ -41-97i & -53-90i & 94+36i\\ -21-75i & -103-47i & 68+14i\end{bmatrix}$. Because

$$\begin{aligned}
\operatorname{adj}(A) &= \begin{bmatrix} |A[1\mid 1]| & -|A[2\mid 1]| & |A[3\mid 1]|\\ -|A[1\mid 2]| & |A[2\mid 2]| & -|A[3\mid 2]|\\ |A[1\mid 3]| & -|A[2\mid 3]| & |A[3\mid 3]|\end{bmatrix}\\
&= \begin{bmatrix}
\begin{vmatrix} 5-3i & 2+4i\\ -4 & 9+5i\end{vmatrix} & -\begin{vmatrix} 7+i & 9-2i\\ -4 & 9+5i\end{vmatrix} & \begin{vmatrix} 7+i & 9-2i\\ 5-3i & 2+4i\end{vmatrix}\\
-\begin{vmatrix} -12+7i & 2+4i\\ 6+13i & 9+5i\end{vmatrix} & \begin{vmatrix} 3 & 9-2i\\ 6+13i & 9+5i\end{vmatrix} & -\begin{vmatrix} 3 & 9-2i\\ -12+7i & 2+4i\end{vmatrix}\\
\begin{vmatrix} -12+7i & 5-3i\\ 6+13i & -4\end{vmatrix} & -\begin{vmatrix} 3 & 7+i\\ 6+13i & -4\end{vmatrix} & \begin{vmatrix} 3 & 7+i\\ -12+7i & 5-3i\end{vmatrix}
\end{bmatrix}\\
&= \begin{bmatrix} 68+14i & -94-36i & -29+67i\\ 103+47i & -53-90i & -100+75i\\ -21-75i & 41+97i & 106-46i\end{bmatrix},
\end{aligned}$$

then we obtain

$$\begin{aligned}
W_n \mathrm{adj}(A)^{\sim} W_n &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 106-46i & -100+75i & -29+67i \\ 41+97i & -53-90i & -94-36i \\ -21-75i & 103+47i & 68+14i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 106-46i & -100+75i & -29+67i \\ -41-97i & 53+90i & 94+36i \\ -21-75i & 103+47i & 68+14i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 106-46i & 100-75i & -29+67i \\ -41-97i & -53-90i & 94+36i \\ -21-75i & -103-47i & 68+14i \end{bmatrix} \\
&= C_2(A).
\end{aligned}$$

2.1.6 Alternate Matrices

Let $A \in \mathbb{M}_n(\mathbb{F})$. If $v^T A v = 0$ for all $v \in \mathbb{M}_{n\times 1}(\mathbb{F})$, then A is an alternate matrix of order n underlying the field $\mathbb{F}$. The set of all alternate matrices of order n underlying the field $\mathbb{F}$ we denote it by $\mathcal{L}_n(\mathbb{F})$. Here, we proof that if all the diagonal elements in A are 0 and $A = -A^T$, then A is belonged to $\mathcal{L}_n(\mathbb{F})$.

Let $A \in \mathbb{M}_n(\mathbb{F})$ and suppose that $v^T A v = 0$ for all $v \in \mathbb{M}_{n\times 1}(\mathbb{F})$. Let $e_i \in \mathbb{M}_n(\mathbb{F})$ with i-th entry is equal to 1 and the other entries are equal to 0 for all $i \in \{1, 2, \cdots, n\}$. Thus we have

$$\begin{aligned}
& e_i^T A e_i = 0 \\
\Rightarrow\ & e_i^T \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{bmatrix} = 0 \\
\Rightarrow\ & a_{ii} = 0
\end{aligned}$$

and for all $1 \leqslant i < j \leqslant n$,

$$
\begin{aligned}
&(e_i + e_j)^T A(e_i + e_j) = 0 \\
&\Rightarrow (e_i + e_j)^T \begin{bmatrix} a_{1i} + a_{1j} \\ a_{2i} + a_{2j} \\ \vdots \\ a_{ni} + a_{nj} \end{bmatrix} = 0 \\
&\Rightarrow a_{ii} + a_{ij} + a_{ji} + a_{jj} = 0 \\
&\Rightarrow a_{ij} = -a_{ji}.
\end{aligned}
$$

This proof shows us that our statement is true. Next we want to show that if $B \in \mathbb{S}_n(\mathbb{F})$ and $B \in \mathcal{L}_n(\mathbb{F})$ where $B \neq 0_n$, then $\mathrm{char}(\mathbb{F})$ must be equal to 2.

Suppose that there exists some non-zero matrix $B \in \mathbb{M}_n(\mathbb{F})$ such that $B \in \mathbb{S}_n(\mathbb{F})$ and $B \in \mathcal{L}_n(\mathbb{F})$ with $\mathrm{char}(\mathbb{F}) \neq 2$. Thus we know that all the diagonal elements in B are 0, $B = B^T$ and $B = -B^T$. Hence $2B = 0_n$. Since $\mathrm{char}(\mathbb{F}) \neq 2$, then $B = 0_n$. This contradicts to the facts that $B \neq 0_n$. This contradiction shows that our supposition is false. So we conclude that the given statement is true.

2.1.7 Some Elementary Properties

In previous section of this chapter, we have introduced many types of matrices. Now, we would like to introduce some elementary properties which are needed in this project.

Lemma 2.1.7.1. (Mirsky (1955)). Let $\mathbb{F}$ be a field and $n \in \mathbb{N}$. For any $A, B \in \mathbb{M}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}$,

(a) $\mathrm{rk}(A) = 0$ if and only if $A = 0_n$.

(b) $\mathrm{rk}(A + B) \leqslant \mathrm{rk}(A) + \mathrm{rk}(B)$.

(c) $\mathrm{rk}(AB) = \mathrm{rk}(A) = \mathrm{rk}(BA)$ if $\mathrm{rk}(B) = n$.

(d) $\mathrm{rk}(\alpha A) = \mathrm{rk}(A)$ if $\alpha \neq 0$.

Lemma 2.1.7.2. (Chooi (2011), Lemma 2.2 and Lemma 2.3). Let $\mathbb{F}$ be a field carrying an involution $-$ and $n \in \mathbb{N}$ with $n > 2$. Let ϕ be a field monomorphism of $\mathbb{F}$. For any $A, B \in \mathbb{M}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}$,

(a) $C_{n-1}(0_n) = 0_n$.

(b) $C_{n-1}(I_n) = I_n$.

(c) $C_{n-1}(\alpha A) = \alpha^{n-1} C_{n-1}(A)$.

(d) $C_{n-1}(AB) = C_{n-1}(A) C_{n-1}(B)$.

(e) $C_{n-1}(A^{-1}) = C_{n-1}(A)^{-1}$ when $\mathrm{rk}(A) = n$.

(f) $C_{n-1}(A^T) = C_{n-1}(A)^T$.

(g) $C_{n-1}(\overline{A}) = \overline{C_{n-1}(A)}$.

(h) $C_{n-1}(A^H) = C_{n-1}(A)^H$.

(i) $C_{n-1}(A^\sim) = C_{n-1}(A)^\sim$.

(j) $C_{n-1}(A^\phi) = C_{n-1}(A)^\phi$.

(k) $C_{n-1}(A) = \begin{cases} -A & \text{if } n \equiv 0, 3 \pmod 4, \\ A & \text{otherwise} \end{cases}$
when $A = W_n$ or $A = J_n$.

Lemma 2.1.7.3. Let $\mathbb{F}$ be a field and $n \in \mathbb{N}$ with $n > 2$. For all invertible matrix $\mathcal{Q} \in \mathbb{M}_n(\mathbb{F})$, $C_{n-1}(C_{n-1}(\mathcal{Q})) = |\mathcal{Q}|^{n-2}\mathcal{Q}$.

Proof.

$$\begin{aligned} C_{n-1}(C_{n-1}(\mathcal{Q})) &= C_{n-1}(W_n \mathrm{adj}(\mathcal{Q})^\sim W_n) \\ &= C_{n-1}(W_n) C_{n-1}(\mathrm{adj}(\mathcal{Q})^\sim) C_{n-1}(W_n) \\ &= \begin{cases} [-W_n] C_{n-1}(\mathrm{adj}(\mathcal{Q})^\sim)[-W_n] & \text{if } n \equiv 0, 3 \pmod 4, \\ W_n C_{n-1}(\mathrm{adj}(\mathcal{Q})^\sim) W_n & \text{otherwise} \end{cases} \\ &= W_n C_{n-1}(\mathrm{adj}(\mathcal{Q})^\sim) W_n \\ &= W_n W_n \mathrm{adj}(\mathrm{adj}(\mathcal{Q})^\sim)^\sim W_n W_n \\ &= I_n \mathrm{adj}(\mathrm{adj}(\mathcal{Q})^\sim)^\sim I_n \\ &= \mathrm{adj}(\mathrm{adj}(\mathcal{Q})^\sim)^\sim. \end{aligned}$$

By the fact that $\mathrm{adj}(K^\sim) = \mathrm{adj}(K)^\sim$ and $(K^\sim)^\sim = K$ for every $K \in \mathbb{M}_n(\mathbb{F})$. So we have

$$\begin{aligned}
C_{n-1}(C_{n-1}(\mathcal{Q})) &= \mathrm{adj}(\mathrm{adj}(\mathcal{Q})^{\sim})^{\sim} \\
&= (\mathrm{adj}(\mathrm{adj}(\mathcal{Q}))^{\sim})^{\sim} \\
&= \mathrm{adj}(\mathrm{adj}(\mathcal{Q})) \\
&= \mathrm{adj}(|\mathcal{Q}|\mathcal{Q}^{-1}) \\
&= |\mathcal{Q}|^{n-1}\mathrm{adj}(\mathcal{Q}^{-1}) \\
&= |\mathcal{Q}|^{n-1}|\mathcal{Q}^{-1}|\mathcal{Q} \\
&= |\mathcal{Q}|^{n-2}\mathcal{Q}.
\end{aligned}$$

□

Lemma 2.1.7.4. (Chooi (2011), Lemma 2.2). Let $\mathbb{F}$ be a field carrying an involution $-$ and $n \in \mathbb{N}$ with $n > 2$.

(a) For all $\alpha \in \mathbb{F}$ and $i, j \in \mathbb{N}$ with $i, j \leqslant n$ and $j \neq n+1-i$, $C_{n-1}(I_n - E_{n+1-i,n+1-i} - E_{jj} + \alpha E_{jj}) = \alpha E_{ii}$.

(b) For all $\alpha \in \mathbb{F}$ and $i, j \in \mathbb{N}$ with $1 \leqslant i < j \leqslant n$, $C_{n-1}(I_n - E_{n+1-i,n+1-i} - E_{n+1-j,n+1-j} + (-1)^{i+j+1}(\alpha E_{n+1-j,n+1-i} + \overline{\alpha}E_{n+1-i,n+1-j})) = \alpha E_{ij} + \overline{\alpha}E_{ji} - \alpha\overline{\alpha}(I_n - E_{ii} - E_{jj})$.

Proof.

(a) Let B be $I_n - E_{n+1-i,n+1-i} - E_{jj} + \alpha E_{jj}$. When $j \neq n+1-i$, $(n+1-i)$-th row and $(n+1-i)$-th column of B are a row of zeros and a column of zeros, respectively. Let D be a submatrix of B where D is obtained by eliminating x-th row and y-th column from B. We observe that if D is obtained by not eliminating $(n+1-i)$-th row or $(n+1-i)$-th column from B, then there exists a row or a column of zeros in D. It follows that $|D| = 0$. Thus $(C_{n-1}(B))_{uv} = |B[n+1-u \mid n+1-v]| = |D| = 0$ for all $u \neq i$ and $v \neq i$. On the other hand, if D is obtained by eliminating $(n+1-i)$-th row and $(n+1-i)$-th column from B, then D is a diagonal matrix with $D_{uu} = \alpha$ for some $1 \leqslant u \leqslant n-1$. Other than D_{uu}, all the diagonal entries of D are 1. Therefore $|D| = \underbrace{1(1)\cdots(1)}_{n-2 \text{ times of } 1}\alpha = \alpha$. Consequently, $(C_{n-1}(B))_{ii} = |B[n+1-i \mid n+1-i]| = |D| = \alpha$. By combining these two cases, we are done.

(b) Let B be $I_n - E_{n+1-i,n+1-i} - E_{n+1-j,n+1-j} + (-1)^{i+j+1}(\alpha E_{n+1-j,n+1-i} + \overline{\alpha}E_{n+1-i,n+1-j})$. Let D be a submatrix of B where D is obtained by eliminating x-th row and y-th column from B. Now we want to separate our proof into the following four cases.

Case 1: $x \notin \{n+1-i, n+1-j\}$ with $y \neq x$. Obviously, there exists a column of zeros in D. This makes $|D|= 0$. Therefore $(C_{n-1}(B))_{uv} = |B[n+1-u \mid n+1-v]|= |D|= 0$ for all $u \notin \{i, j\}$ with $v \neq u$.

Case 2: $x = n+1-i$ (respectively, $x = n+1-j$) with $y \neq n+1-j$ (respectively, $y \neq n+1-i$). It is easy to see that there exists a column of zeros in D. Plainly, $|D|= 0$. Hence $(C_{n-1}(B))_{iv} = |B[n+1-i \mid n+1-v]|= |D|= 0$ for all $v \neq j$ and $(C_{n-1}(B))_{jv} = |B[n+1-j \mid n+1-v]|= |D|= 0$ for all $v \neq i$.

We know that $1 \leqslant i < j \leqslant n$, therefore $1 \leqslant n+1-j < n+1-i \leqslant n$.

Case 3: $x \notin \{n+1-i, n+1-j\}$ with $y = x$. There are only three possibilities for x. Now we want to divide the x into three subcases.

Subcase 1: $1 \leqslant x < n+1-j < n+1-i \leqslant n$. Then

$$D = \sum_{\substack{k=1,\\ k\neq n-j,n-i}}^{n-1} E_{kk} + (-1)^{i+j+1}\alpha E_{n-j,n-i} + (-1)^{i+j+1}\overline{\alpha} E_{n-i,n-j}.$$

Subcase 2: $1 \leqslant n+1-j < x < n+1-i \leqslant n$. Hence

$$D = \sum_{\substack{k=1,\\ k\neq n+1-j,n-i}}^{n-1} E_{kk} + (-1)^{i+j+1}\alpha E_{n+1-j,n-i} + (-1)^{i+j+1}\overline{\alpha} E_{n-i,n+1-j}.$$

Subcase 3: $1 \leqslant n+1-j < n+1-i < x \leqslant n$. Thence

$$D = \sum_{\substack{k=1,\\ k\neq n+1-j,n+1-i}}^{n-1} E_{kk} + (-1)^{i+j+1}\alpha E_{n+1-j,n+1-i} + (-1)^{i+j+1}\overline{\alpha} E_{n+1-i,n+1-j}.$$

Without loss of generality, D can be expressed as

$$D = \sum_{\substack{k=1,\\ k\neq p,q}}^{n-1} E_{kk} + (-1)^{i+j+1}\alpha E_{pq} + (-1)^{i+j+1}\overline{\alpha} E_{qp} \text{ where } 1 \leqslant p < q \leqslant n-1.$$

We notice that when we interchange p-th column and q-th column in D, we acquire a diagonal matrix, we call it D',

$$D' = \sum_{\substack{k=1,\\ k\neq p,q}}^{n-1} E_{kk} + (-1)^{i+j+1}\alpha E_{pp} + (-1)^{i+j+1}\overline{\alpha} E_{qq} \text{ where } 1 \leqslant p < q \leqslant n-1.$$

Consequently, $|D| = -|D'| = -\underbrace{1(1)\cdots(1)(1)}_{n-3 \text{ times of } 1}((-1)^{i+j+1}\alpha)((-1)^{i+j+1}\overline{\alpha}) = -\alpha\overline{\alpha}$. This implies $C_{n-1}(B)_{uu} = |B[n+1-u \mid n+1-u]| = |D| = -\alpha\overline{\alpha}$ for all $u \notin \{i, j\}$.

Case 4: $x = n+1-i$ (respectively, $x = n+1-j$) with $y = n+1-j$ (respectively, $y = n+1-i$). Let D'' be a submatrix of B where D'' is obtained by eliminating $(n+1-i)$-th row & $(n+1-j)$-th column and $(n+1-j)$-th row & $(n+1-i)$-th column from B. Apparently, $D'' = I_{n-2}$. If D is obtained by eliminating $(n+1-i)$-th row and $(n+1-j)$-th column, then the coordinates of an element $(-1)^{i+j+1}\alpha$ in D is $D_{n+1-j,n-i}$. This tells us that

$$\begin{aligned}
&\text{cofactor of an element } (-1)^{i+j+1}\alpha \text{ in } D \\
&= (-1)^{(n+1-j)+(n-i)}[\text{Minor of } (-1)^{i+j+1}\alpha \text{ in } D] \\
&= (-1)^{i+j-1}[\text{Minor of } (-1)^{i+j+1}\alpha \text{ in } D] \\
&= (-1)^{i+j-1}|D[\text{row of } D \text{ contains } \alpha \mid \text{column of } D \text{ contains } \alpha]| \\
&= (-1)^{i+j-1}|D''| \\
&= (-1)^{i+j-1}|I_{n-2}| \\
&= (-1)^{i+j-1}.
\end{aligned}$$

It follows that $|D| = (-1)^{i+j-1}((-1)^{i+j+1}\alpha) + \underbrace{0+0+\cdots+0}_{n-2 \text{ times of } 0} = \alpha$. Thereby $C_{n-1}(B)_{ij} = |B[n+1-i \mid n+1-j]| = |D| = \alpha$. By using a similar reasoning as above, we get $C_{n-1}(B)_{ji} = \overline{\alpha}$.

By combining all the cases, we attain the result we want.

□

Lemma 2.1.7.5. Let $\mathbb{F}$ be a field carrying an involution $-$ and $n \in \mathbb{N}$ with $n > 2$. For any A belongs to $\mathbb{H}_n(\mathbb{F})$, $C_{n-1}(A)$ also belongs to $\mathbb{H}_n(\mathbb{F})$.

Proof.

Let A belongs to $\mathbb{H}_n(\mathbb{F})$. Thus $A = \overline{A^T}$. When we take the $(n-1)$-th compound matrix on both sides, $C_{n-1}(A) = C_{n-1}(\overline{A^T}) = \overline{C_{n-1}(A)^T}$. This means that $C_{n-1}(A)$ also belongs to $\mathbb{H}_n(\mathbb{F})$.

□

As an immediate consequence of **Lemma 2.1.7.5**, we have the following corollary when $-$ is identity.

Corollary 2.1.7.6. Let $\mathbb{F}$ be a field and $n \in \mathbb{N}$ with $n > 2$. For any A belongs to $\mathbb{S}_n(\mathbb{F})$, $C_{n-1}(A)$ also belongs to $\mathbb{S}_n(\mathbb{F})$.

2.1.8 Decomposition of Hermitian Matrices

From Huang (2006), we have learned that the Hermitian matrices can be decomposed into the following form. These results are to be used when we prove our lemmas and theorems.

Lemma 2.1.8.1. (Huang (2006), Theorem 2.5.1). Let $\mathbb{F}$ be a field carrying an involution $-$. Let $n, k \in \mathbb{N}$ with $n > 1$ and $k \leqslant n$. For all $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) = k$, there exist some $G \in \mathbb{M}_n(\mathbb{F})$ with $\mathrm{rk}(G) = n$ and diagonal matrix $D \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(D) = k$ such that $A = GDG^H$ if $-$ is proper or $A \notin \mathcal{L}_n(\mathbb{F})$ when $-$ is identity.

Let $w_1, w_2 \cdots, w_k \in \mathbb{N}$ with $1 \leqslant w_1 < w_2 < \cdots < w_k \leqslant n$. Here, we intend to prove that D is in the form $\gamma_1 E_{w_1,w_1} + \gamma_2 E_{w_2,w_2} + \cdots + \gamma_k E_{w_k,w_k}$ where $\gamma_1, \gamma_2, \cdots, \gamma_k$ are non-zero elements in $\mathbb{F}^-$.

Given $G \in \mathbb{M}_n(\mathbb{F})$ with $\mathrm{rk}(G) = n$ and $D \in \mathbb{H}_n(\mathbb{F})$ is a diagonal matrix with $\mathrm{rk}(D) = k$. Suppose that B is a permutation matrix obtained by interchanging the p-th column and q-th column of I_n where $1 \leqslant p, q \leqslant n$. Thus we have $B_{kk} = 1$ for all $k \in \{1, 2, \cdots, n\} - \{p, q\}$, $B_{pq} = B_{qp} = 1$ and the rest of the entries in B are equal to 0. We claim that BDB is obtained by interchange the D_{pp} and D_{qq} of D. In case $p = q$, we are done. Now we consider for $p \neq q$,

$$\begin{aligned}
&(BDB)_{ij} \\
&= \sum_{k=1}^{n} B_{ik}(DB)_{kj} \\
&= \sum_{k=1}^{n} B_{ik} \sum_{\alpha=1}^{n} D_{k\alpha} B_{\alpha j} \\
&= \sum_{k=1}^{n} \sum_{\alpha=1}^{n} B_{ik} D_{k\alpha} B_{\alpha j}.
\end{aligned}$$

Since D is a diagonal matrix, then we know that $D_{k\alpha} = 0$ for all $1 \leqslant k \neq \alpha \leqslant n$. Therefore

$$\begin{aligned}
&(BDB)_{ij}\\
=&\sum_{k=1}^{n} B_{ik}D_{kk}B_{kj}\\
=&\sum_{\substack{k=1,\\k\neq p,q}}^{n} B_{ik}D_{kk}B_{kj}+B_{ip}D_{pp}B_{pj}+B_{iq}D_{qq}B_{qj}\\
=&\begin{cases}0 & \text{if } i\neq j,\\ D_{ii} & \text{if } i=j \text{ with } i\neq p,q,\\ D_{pp} & \text{if } i=j=q,\\ D_{qq} & \text{if } i=j=p.\end{cases}
\end{aligned}$$

Let B_x be a permutation matrix obtained by interchange the p_x-th column and q_x-th column in I_n where $1 \leqslant p_x, q_x \leqslant n$ for all $x \in \mathbb{N}$. Certainly $B_x = B_x^H = B_x^{-1}$. Without loss of generality, there exists some $y \in \mathbb{N}$ such that

$$D' = B_y \cdots B_1 D B_1 \cdots B_y = \gamma_1 E_{w_1,w_1} + \gamma_2 E_{w_2,w_2} + \cdots + \gamma_k E_{w_k,w_k}$$

where $w_i \in \mathbb{N}$ with $1 \leqslant w_1 < w_2 < \cdots < w_k \leqslant n$ and $\gamma_i \in \mathbb{F}^-$ with $\gamma_i \neq 0$ for all $i \in \{1, 2, \cdots, k\}$. Consequently, we obtain

$$\begin{aligned}
A &= GDG^H\\
&= GB_1 \cdots B_y B_y \cdots B_1 D B_1 \cdots B_y B_y \cdots B_1 G^H\\
&= (GB_1 \cdots B_y)(B_y \cdots B_1 D B_1 \cdots B_y)(B_y \cdots B_1 G^H)\\
&= (GB_1 \cdots B_y) D' (GB_1 \cdots B_y)^H.
\end{aligned}$$

Clearly, $GB_1 \cdots B_y \in \mathbb{M}_n(\mathbb{F})$ with $\mathrm{rk}(GB_1 \cdots B_y) = n$.

Corollary 2.1.8.2. Let $\mathbb{F}$ be a field carrying an involution $-$. Let $n, k \in \mathbb{N}$ with $n > 1$ and $k \leqslant n$. Let $w_1, w_2 \cdots, w_k \in \mathbb{N}$ with $1 \leqslant w_1 < w_2 < \cdots < w_k \leqslant n$. For all $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) = k$, there exist some non-zero elements $\gamma_1, \gamma_2, \cdots, \gamma_k \in \mathbb{F}^-$ and $G \in \mathbb{M}_n(\mathbb{F})$ with $\mathrm{rk}(G) = n$ such that $A = G(\gamma_1 E_{w_1,w_1} + \gamma_2 E_{w_2,w_2} + \cdots + \gamma_k E_{w_k,w_k})G^H$ if $-$ is proper or $A \notin \mathcal{L}_n(\mathbb{F})$ when $-$ is identity.

We notice that if $A \in \mathbb{H}_n(\mathbb{F})$ and $A \in \mathcal{L}_n(\mathbb{F})$ when $-$ is identity, then $\mathrm{char}(\mathbb{F})$ must be equal to 2. By referring to **Lemma 2.1.8.3**, A can be decomposed.

Lemma 2.1.8.3. (Huang (2006), Theorem 2.5.3). Let $\mathbb{F}$ be a field carrying an identity involution $-$ with $\mathrm{char}(\mathbb{F}) = 2$. Let $n, k \in \mathbb{N}$ with $n > 1$ and $k \leqslant n$. For all $A \in \mathbb{H}_n(\mathbb{F})$ with $A \in \mathcal{L}_n(\mathbb{F})$, definitely $\mathrm{rk}(A)$ is even. Moreover if $\mathrm{rk}(A) = k$, then

$$A = G\left(\bigoplus_{i=1}^{\frac{k}{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus 0_{n-k}\right) G^H$$

where $G \in \mathbb{M}_n(\mathbb{F})$ with $\mathrm{rk}(G) = n$.

By combining the **Corollary 2.1.8.2** and **Lemma 2.1.8.3**, we have the following corollary.

Corollary 2.1.8.4. Let $\mathbb{F}$ be a field carrying an involution $-$. Let $n, k \in \mathbb{N}$ with $n > 1$ and $k \leqslant n$. Let $w_1, w_2 \cdots, w_k \in \mathbb{N}$ with $1 \leqslant w_1 < w_2 < \cdots < w_k \leqslant n$. For all $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) = k$, there exists some $G \in \mathbb{M}_n(\mathbb{F})$ with $\mathrm{rk}(G) = n$ such that either

$$A = G(\gamma_1 E_{w_1,w_1} + \gamma_2 E_{w_2,w_2} + \cdots + \gamma_k E_{w_k,w_k})G^H \tag{2.1}$$

where $\gamma_i \in \mathbb{F}^-$ with $\gamma_i \neq 0$ for all $i \in \{1, 2, \cdots, k\}$ or

$$A = G\left(\bigoplus_{i=1}^{\frac{k}{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus 0_{n-k}\right) G^H \tag{2.2}$$

when $A \in \mathcal{L}_n(\mathbb{F})$ and $-$ is identity. If A is of the form (2.2), absolutely $\mathrm{char}(\mathbb{F}) = 2$ and $\mathrm{rk}(A)$ is even.

Lemma 2.1.8.5. (Chooi and Ng (2011), Lemma 2.3). Let $\mathbb{F}$ be a field carrying an involution $-$. Let $n, k, t \in \mathbb{N}$ with $n > 1$ and $k \leqslant n$. For all $A \in \mathbb{H}_n(\mathbb{F})$, there exist at least one $A_1, \cdots, A_t \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A_1) = \cdots = \mathrm{rk}(A_t) = 1$ such that $A = A_1 + \cdots + A_t$ where $t = k + 1$ if $A \in \mathcal{L}_n(\mathbb{F})$ and $-$ is identity, otherwise $t = k$.

Proof.

Let $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) = k$. From **Corollary 2.1.8.4**, we know that A is either of the form (2.1) or (2.2). In case A is of the form (2.1), then $A = G(\gamma_1 E_{11} + \gamma_2 E_{22} + \cdots + \gamma_k E_{kk})G^H$ where $G \in \mathbb{M}_n(\mathbb{F})$ with $\mathrm{rk}(G) = n$ and $\gamma_i \in \mathbb{F}^-$ with $\gamma_i \neq 0$ for all $i \in \{1, 2 \cdots, k\}$. Since $\gamma_i G E_{ii} G^H \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(\gamma_i G E_{ii} G^H) = \mathrm{rk}(\gamma_i E_{ii}) = 1$ for all $i \in \{1, 2, \cdots, k\}$ and $A = (\gamma_1 G E_{11} G^H) + \cdots + (\gamma_k G E_{kk} G^H)$. Thus we take

$A_i = \gamma_i G E_{ii} G^H$ for all $i \in \{1, 2, \cdots, k\}$. However, if A is of the form (2.2), we realize that $A \in \mathcal{L}_n(\mathbb{F})$, $-$ is identity, $\mathrm{rk}(A) = k$ is even and $\mathrm{char}(\mathbb{F}) = 2$. At the same time, $A = \mathcal{Q}\left(\bigoplus_{i=1}^{\frac{k}{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus 0_{n-k}\right) \mathcal{Q}^H$ where $\mathcal{Q} \in \mathbb{M}_n(\mathbb{F})$ with $\mathrm{rk}(\mathcal{Q}) = n$. Now, we consider a matrix $B = A + \mathcal{Q}E_{11}\mathcal{Q}^H + \mathcal{Q}E_{22}\mathcal{Q}^H$. It is clear that $B \in \mathbb{H}_n(\mathbb{F})$ as $A, \mathcal{Q}E_{11}\mathcal{Q}^H, \mathcal{Q}E_{22}\mathcal{Q}^H \in \mathbb{H}_n(\mathbb{F})$. Hence,

$$\begin{aligned}
B &= A + \mathcal{Q}E_{11}\mathcal{Q}^H + \mathcal{Q}E_{22}\mathcal{Q}^H \\
&= \mathcal{Q}\left(\bigoplus_{i=1}^{\frac{k}{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus 0_{n-k}\right) \mathcal{Q}^H + \mathcal{Q}E_{11}\mathcal{Q}^H + \mathcal{Q}E_{22}\mathcal{Q}^H \\
&= \mathcal{Q}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \bigoplus_{i=2}^{\frac{k}{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus 0_{n-k}\right) \mathcal{Q}^H + \mathcal{Q}E_{11}\mathcal{Q}^H + \mathcal{Q}E_{22}\mathcal{Q}^H \\
&= \mathcal{Q}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \oplus \bigoplus_{i=2}^{\frac{k}{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus 0_{n-k}\right) \mathcal{Q}^H.
\end{aligned}$$

This shows that $\mathrm{rk}(B) = k - 1$. By the fact that $B \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(B) = k - 1$ where $k - 1$ is odd, $B = P(\lambda_1 E_{11} + \cdots + \lambda_{k-1} E_{k-1,k-1})P^H$ where $P \in \mathbb{M}_n(\mathbb{F})$ with $\mathrm{rk}(P) = n$ and $\lambda_i \in \mathbb{F}^-$ with $\lambda_i \neq 0$ for all $i \in \{1, \cdots, k-1\}$. Hence

$$A + \mathcal{Q}E_{11}\mathcal{Q}^H + \mathcal{Q}E_{22}\mathcal{Q}^H = P(\lambda_1 E_{11} + \cdots + \lambda_{k-1} E_{k-1,k-1})P^H$$

which implies

$$A = (\lambda_1 P E_{11} P^H) + \cdots + (\lambda_{k-1} P E_{k-1,k-1} P^H) + (\mathcal{Q}E_{11}\mathcal{Q}^H) + (\mathcal{Q}E_{22}\mathcal{Q}^H).$$

Because $\lambda_i P E_{ii} P^H \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(\lambda_i P E_{ii} P^H) = \mathrm{rk}(\lambda_i E_{ii}) = 1$ for every $i \in \{1, 2, \cdots, k-1\}$ and $\mathcal{Q}E_{ii}\mathcal{Q}^H \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(\mathcal{Q}E_{ii}\mathcal{Q}^H) = \mathrm{rk}(E_{ii}) = 1$ for all $i \in \{1, 2\}$, we take $A_i = \lambda_i P E_{ii} P^H$ for all $i \in \{1, 2, \cdots, k-1\}$, $A_k = \mathcal{Q}E_{11}\mathcal{Q}^H$ and $A_{k+1} = \mathcal{Q}E_{22}\mathcal{Q}^H$.

□

2.2 Types of Linear Preserver Problems

By far, there exist many types of Linear Preserver Problems. Actually, the Linear Preserver Problems is to find the characterisation of mappings that preserve certain properties. Here, we show four classic types of Linear Preserver Problems outlined by Li and Tsing (1992). In this section, we denote a matrix space by V and linear operator on V by Υ unless otherwise specified.

2.2.1 Linear Preserver of Functions

Problem 1. Let f be a (scalar-valued, set-valued or vector-valued) function on V. Characterise Υ that satisfies

$$f(\Upsilon(A)) = f(A) \text{ for every } A \in V.$$

Frobenius (1897) was possibly the first person who did research about **Problem 1**. He characterised a linear operator Υ on $\mathbb{M}_n(\mathbb{C})$ which preserves the determinant. The following theorem was proved by Frobenius (1897).

Theorem 2.2.1.1. (Frobenius (1897)). Let $n \in \mathbb{N}$ and Υ be a linear operator on $\mathbb{M}_n(\mathbb{C})$. Υ preserves the determinant, i.e.,

$$|\Upsilon(A)| = |A| \text{ for every } A \in \mathbb{M}_n(\mathbb{C}),$$

if and only if there exist some non-singular matrices $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathbb{M}_n(\mathbb{C})$ with $|\mathcal{Q}_1\mathcal{Q}_2| = 1$ such that

$$\Upsilon(A) = \mathcal{Q}_1 A \mathcal{Q}_2 \text{ for every } A \in \mathbb{M}_n(\mathbb{C}) \tag{2.3}$$

or

$$\Upsilon(A) = \mathcal{Q}_1 A^T \mathcal{Q}_2 \text{ for every } A \in \mathbb{M}_n(\mathbb{C}). \tag{2.4}$$

Before we continue our discussion, we would like to give a remark.

Remark. If we say Υ has the form (2.3), this means

$$\Upsilon(A) = \mathcal{Q}_1 A \mathcal{Q}_2 \text{ for every } A \in V$$

where V is dependent on what problems we are discussing.

Moreover, Frobenius (1897) continued to characterise a linear operator Υ on the matrix space other than $\mathbb{M}_n(\mathbb{C})$ that preservers the determinant. The matrix spaces he considered are $V_1 = \mathbb{S}_n(\mathbb{R})$ and $V_2 = \{P \mid P \in \mathbb{M}_n(\mathbb{C}) \text{ and trace of } P = 0\}$. For both cases, Υ has the form (2.3) or (2.4) with some new assumption on the matrices $\mathcal{Q}_1$ and $\mathcal{Q}_2$. For V_1, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are non-singular matrices in $\mathbb{M}_n(\mathbb{R})$ where $\mathcal{Q}_2 = \alpha\mathcal{Q}_1^T$ for some scalar $\alpha \in \mathbb{R}$ and $|\mathcal{Q}_1\mathcal{Q}_2|= 1$ while for V_2, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are non-singular matrices in $\mathbb{M}_n(\mathbb{C})$ where $\mathcal{Q}_2 = \alpha\mathcal{Q}_1^{-1}$ for some scalar $\alpha \in \mathbb{C}$ and $|\mathcal{Q}_1\mathcal{Q}_2|= 1$.

On a more advanced level, we can extend **Problem 1** to two (scalar-valued, set-valued or vector-valued) functions on V which is stated as **Problem 2**.

Problem 2. Let f and g be a (scalar-valued, set-valued or vector-valued) function on V. Characterise Υ that satisfies

$$g(\Upsilon(A)) = f(A) \text{ for every } A \in V.$$

Pólya (1913) once studied this kind of problem. He was interested in whether there exists a linear operator Υ on $\mathbb{M}_n(\mathbb{R})$ that satisfies

$$\text{per}(\Upsilon(A)) = |A| \text{ for every } A \in \mathbb{M}_n(\mathbb{R})$$

where per represents the permanent of the matrix. If there exists such a linear operator, the matrix's permanent can be obtained more quickly. This is because when the size of the matrix is large, the permanent of the matrix is more difficult to calculate than the determinant of the matrix. This problem was solved by Marcus and Minc (1961). They showed that there does not exist such a linear operator.

2.2.2 Linear Preserver of Subsets

Problem 3. Let S be a subset of V. Characterise Υ that satisfies

$$\Upsilon(A) \in S \text{ for every } A \in S.$$

Marcus (1959) studied this type of problem. Let $\mathfrak{U}_{m\times n}(\mathbb{C})$ be the set of all unitary matrices in $\mathbb{M}_{m\times n}(\mathbb{C})$. In short, $\mathfrak{U}_{n\times n}(\mathbb{C}) = \mathfrak{U}_n(\mathbb{C})$ when $m = n$. He studied the characterisation of linear operator Υ on $\mathbb{M}_n(\mathbb{C})$ that preservers unitary matrices. The following theorem is discovered by Marcus (1959).

Theorem 2.2.2.1. (Marcus (1959)). Let $n \in \mathbb{N}$ and Υ be a linear operator on $\mathbb{M}_n(\mathbb{C})$. Υ preserves unitary matrices, i.e.,

$$\Upsilon(A) \in \mathfrak{U}_n(\mathbb{C}) \text{ for every } A \in \mathfrak{U}_n(\mathbb{C})$$

if and only if there exist some $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathfrak{U}_n(\mathbb{C})$ such that

$$\Upsilon(A) = \mathcal{Q}_1 A \mathcal{Q}_2 \text{ for every } A \in \mathfrak{U}_n(\mathbb{C})$$

or

$$\Upsilon(A) = \mathcal{Q}_1 A^T \mathcal{Q}_2 \text{ for every } A \in \mathfrak{U}_n(\mathbb{C}).$$

Further, the result~~s~~ in **Theorem 2.2.2.1** was extended to the $\mathfrak{U}_{m\times n}(\mathbb{C})$ with $m \leqslant n$ by Grone (1976) in his Ph.D. thesis. He introduced a linear operator Υ on $\mathbb{M}_{m\times n}(\mathbb{C})$ that preserves unitary matrices. He concluded that Υ has the form (2.3) or (2.4) where $m, n \in \mathbb{N}$ with $m \leqslant n$, $\mathcal{Q}_1 \in \mathfrak{U}_m(\mathbb{C})$ and $\mathcal{Q}_2 \in \mathfrak{U}_n(\mathbb{C})$.

Besides, Dieudonné (1948) further evolved **Theorem 2.2.1.1** by characterising a linear operator Υ on $\mathbb{M}_n(\mathbb{F})$ that preserves singular matrices where $\mathbb{F}$ is any field. We denote the set of all singular matrices in $\mathbb{M}_n(\mathbb{F})$ by $\mathbb{MS}_n(\mathbb{F})$. Dieudonné (1948) proved the following results.

Theorem 2.2.2.2. (Dieudonné (1948)). Let $\mathbb{F}$ be a field and $n \in \mathbb{N}$. Let Υ be a linear operator on $\mathbb{M}_n(\mathbb{F}).\Upsilon$ preserves singular matrices, i.e.,

$$\Upsilon(A) \in \mathbb{MS}_n(\mathbb{F}) \text{ for every } A \in \mathbb{MS}_n(\mathbb{F})$$

if and only if there exist some non-singular matrices $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathbb{M}_n(\mathbb{F})$ such that either

$$\Upsilon(A) = \mathcal{Q}_1 A \mathcal{Q}_2 \text{ for every } A \in \mathbb{MS}_n(\mathbb{F})$$

or

$$\Upsilon(A) = \mathcal{Q}_1 A^T \mathcal{Q}_2 \text{ for every } A \in \mathbb{MS}_n(\mathbb{F}).$$

Furthermore, **Theorem 2.2.2.2** was continued to explore by Marcus and Purves (1959). Marcus and Purves (1959) generalized **Theorem 2.2.2.2** by characterising a linear operator Υ on $\mathbb{M}_n(\mathbb{F})$ that preserves the non-singular matrices. They showed that Υ has the form (2.3) or (2.4) with $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are non-singular matrices in $\mathbb{M}_n(\mathbb{F})$.

Apart from this, the characterisation of linear operator Υ on $\mathbb{M}_{m\times n}(\mathbb{C})$ that preserves the rank k matrices was found by Beasley (1988). Next, we illustrate the theorem done by Beasley (1988).

Theorem 2.2.2.3. Let $m, n, k \in \mathbb{N}$ with $k \leqslant \min\{m, n\}$ and Υ be a linear operator on $\mathbb{M}_{m\times n}(\mathbb{C})$. Υ preserves rank k matrices, i.e.,

$$\mathrm{rk}(\Upsilon(A)) = k \text{ for every } A \in \mathbb{M}_{m\times n}(\mathbb{C}) \text{ with } \mathrm{rk}(A) = k$$

if and only if there exist some non-singular matrices $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathbb{M}_n(\mathbb{C})$ such that

$$\Upsilon(A) = \mathcal{Q}_1 A \mathcal{Q}_2 \text{ for every } A \in \mathbb{M}_{m\times n}(\mathbb{C}) \text{ with } \mathrm{rk}(A) = k$$

or

$$\Upsilon(A) = \mathcal{Q}_1 A^T \mathcal{Q}_2 \text{ for every } A \in \mathbb{M}_{m\times n}(\mathbb{C}) \text{ with } \mathrm{rk}(A) = k \text{ when } m = n.$$

By using the transfer principle, the results in **Theorem 2.2.2.3** can be extended to any algebraically closed field with characteristic 0. The method of transfer principle was introduced by Guterman et al. (2000). For more details on the method of transfer principle, Guterman et al. (2000) can be referred. Actually **Problem 3** can be extended to **Problem 4** as stated below.

Problem 4. Let S_1, S_2 be subsets of V. Characterise Υ that satisfies

$$\Upsilon(A) \in S_2 \text{ for every } A \in S_1.$$

For problems like **Problem 4**, Li and Tsing (1988), Li and Tsing (1991) and Li and Tsing (1993) can be referred.

We observed that the characterisation of Υ has the form (2.3) or (2.4) although the types of problems and matrix spaces we are discussed are totally different. It is pretty interesting. Even though the results are pretty similar, but the proving and complexity can be totally different. Of course, not all the Υ has the form (2.3) or (2.4).

2.2.3 Linear Preserver of Relations

Problem 5. Let $\sim$ be a relation on V. Characterise Υ that satisfies

$$\Upsilon(A) \sim \Upsilon(B) \text{ if } A \sim B \text{ for every } A, B \in V.$$

In fact, Hiai (1987) completed characterising a linear operator Υ on $\mathbb{M}_n(\mathbb{C})$ which preserves similarity. The following theorem was shown by Hiai (1987).

Theorem 2.2.3.1. (Hiai (1987)). Let $n \in \mathbb{N}$ and Υ be a linear operator on $\mathbb{M}_n(\mathbb{C})$. Υ preserves similarity, i.e.,

$$\Upsilon(A) \text{ is similar to } \Upsilon(B) \text{ if } A \text{ is similar to } B \text{ for every } A, B \in \mathbb{M}_n(\mathbb{C})$$

if and only if there exists some $\mathcal{Q}_0 \in \mathbb{M}_n(\mathbb{C})$ such that

$$\Upsilon(A) = \mathrm{tr}(A)\mathcal{Q}_0 \text{ for every } A \in \mathbb{M}_n(\mathbb{C}), \tag{2.5}$$

or there exist some $a, b \in \mathbb{C}$ and non-singular matrix $\mathcal{Q} \in \mathbb{M}_n(\mathbb{C})$ such that

$$\Upsilon(A) = a\mathcal{Q}A\mathcal{Q}^{-1} + b\mathrm{tr}(A)I_n \text{ for every } A \in \mathbb{M}_n(\mathbb{C}) \tag{2.6}$$

or

$$\Upsilon(A) = a\mathcal{Q}A^T\mathcal{Q}^{-1} + b\mathrm{tr}(A)I_n \text{ for every } A \in \mathbb{M}_n(\mathbb{C}) \tag{2.7}$$

where tr stands for the trace of the matrix.

At the same time, he also completed characterising a linear operator Υ_1 on $\mathbb{H}_n(\mathbb{C})$ which preserves unitary equivalence and linear operator Υ_2 on $\mathbb{S}_n(\mathbb{R})$ which preserves orthogonal equivalence. He illustrated Υ_1 and Υ_2 have the form (2.5) or (2.6) with $a, b \in \mathbb{R}$ and other assumption on the matrices $\mathcal{Q}_0$ and $\mathcal{Q}$. For Υ_1, $\mathcal{Q}_0 \in \mathbb{H}_n(\mathbb{C})$ and $\mathcal{Q}$ is a unitary matrix in $\mathbb{M}_n(\mathbb{C})$ while for Υ_2, $\mathcal{Q}_0 \in \mathbb{S}_n(\mathbb{R})$ and $\mathcal{Q}$ is an orthogonal matrix in $\mathbb{M}_n(\mathbb{R})$.

Besides that, this kind of problem was also studied by Pierce and Watkins (1978). The characterisation of a linear operator on $\mathbb{M}_n(\mathbb{F})$ that preserves commutative relation ($n > 2$) was found by them. The following theorem was established by Pierce and Watkins (1978).

Theorem 2.2.3.2. (Pierce and Watkins (1978)). Let $\mathbb{F}$ be a field and $n \in \mathbb{N}$ with $n > 2$. Let Υ be a linear operator on $\mathbb{M}_n(\mathbb{F})$. Υ preserves commutative relation, i.e.,

$$\Upsilon(A)\Upsilon(B) = \Upsilon(B)\Upsilon(A) \text{ if } AB = BA \text{ for every } A, B \in \mathbb{M}_n(\mathbb{F})$$

if and only if there exist some $a \in \mathbb{F}$, non-singular matrix $\mathcal{Q} \in \mathbb{M}_n(\mathbb{F})$ and linear functional Υ' on $\mathbb{M}_n(\mathbb{F})$ such that

$$\Upsilon(A) = a\mathcal{Q}^{-1}A\mathcal{Q} + \Upsilon'(A)I_n \text{ for every } A \in \mathbb{M}_n(\mathbb{F}) \tag{2.8}$$

or

$$\Upsilon(A) = a\mathcal{Q}^{-1}A^T\mathcal{Q} + \Upsilon'(A)I_n \text{ for every } A \in \mathbb{M}_n(\mathbb{F}). \tag{2.9}$$

The same type of problems has been studied by Chan and Lim (1982). They characterised a linear operator Υ on $\mathbb{S}_n(\mathbb{R})$ that preserves commutative relation $(n > 1)$. In their work, they verified that for $n > 2$, Υ has the form (2.8) with $a \in \mathbb{R}$, $\mathcal{Q}$ is an orthogonal matrix in $\mathbb{M}_n(\mathbb{R})$ and Υ' is a linear functional on $\mathbb{S}_n(\mathbb{R})$. Simultaneously, for $n = 2$, there exists a counterexample of Υ that preserves commutative relation but not in the form (2.8). The following theorem is for the case $n = 2$.

Theorem 2.2.3.3. (Chan and Lim (1982)). Let Υ be a linear operator on $\mathbb{S}_2(\mathbb{R})$. Υ preserves commutative relation if and only if there exists an orthogonal matrix $\mathcal{Q} \in \mathbb{M}_2(\mathbb{R})$ such that

$$\mathcal{Q}\Upsilon(E_{11})\mathcal{Q}^T = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \quad \text{and} \quad \mathcal{Q}\Upsilon(E_{22})\mathcal{Q}^T = \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix}$$

with $\alpha_1 + \beta_1 = \alpha_2 + \beta_2$.

2.2.4 Linear Preserver of Transformations

Problem 6. Let $\mathcal{F}$ be a transformation from V to V. Characterise the Υ that satisfies

$$\Upsilon(\mathcal{F}(A)) = \mathcal{F}(\Upsilon(A)) \text{ for every } A \in V.$$

Some researchers call Υ as $\mathcal{F}$-commuting mapping. The first person who discussed this type of problem is Sinkhorn (1982). The problem he studied was adjoint-commuting mappings on $\mathbb{M}_n(\mathbb{C})$. When $n = 1$, it is trivial because Υ is adjoint-commuting mappings as $\mathrm{adj}(A) = A$ for all $A \in \mathbb{M}_1(\mathbb{F})$ and for all field $\mathbb{F}$. By using **Theorem 2.2.1.1**, Sinkhorn (1982) established the following theorem.

Theorem 2.2.4.1. (Sinkhorn (1982)). Let $n \in \mathbb{N}$ with $n > 1$ and Υ be a linear operator on $\mathbb{M}_n(\mathbb{C})$. Υ is an adjoint-commuting mapping, i.e.,

$$\Upsilon(\mathrm{adj}(A)) = \mathrm{adj}(\Upsilon(A)) \text{ for every } A \in M_n(\mathbb{C})$$

if and only if there exist some non-singular matrix $\mathcal{Q} \in \mathbb{M}_n(\mathbb{C})$ and $\varkappa \in \mathbb{C}$ with $\varkappa^{n-1} = \varkappa$ such that either

$$\Upsilon(A) = \varkappa \mathcal{Q} A \mathcal{Q}^{-1} \text{ for every } A \in \mathbb{M}_n(\mathbb{C}) \tag{2.10}$$

or

$$\Upsilon(A) = \varkappa \mathcal{Q} A^T \mathcal{Q}^{-1} \text{ for every } A \in \mathbb{M}_n(\mathbb{C}) \tag{2.11}$$

for all $n > 2$. Moreover, there exist certain $k_1, k_2 \in \mathbb{N}$ such that

$$\Upsilon(A) = \sum_{i=1}^{k_1} \alpha_i X_i A \mathrm{adj}(X_i) + \sum_{i=1}^{k_2} \beta_i Y_i A^T \mathrm{adj}(Y_i)$$

where $\alpha_1, \cdots, \alpha_{k_1}, \beta_1, \cdots, \beta_{k_2} \in \mathbb{C}$ and $X_1, \cdots, X_{k_1}, Y_1, \cdots, Y_{k_2} \in \mathbb{M}_2(\mathbb{C})$.

The work done by Sinkhorn (1982) was continued by Chan et al. (1987). Chan et al. (1987) extended the field $\mathbb{C}$ to any infinite field $\mathbb{F}$. They also simplified the result when $n = 2$. They showed that if $n = 2$, then there exists certain $k \in \mathbb{N}$ such that

$$\Upsilon(A) = \sum_{i=1}^{k} \alpha_i \mathcal{Q}_i A \mathrm{adj}(\mathcal{Q}_i) \tag{2.12}$$

where $\alpha_1, \cdots, \alpha_k \in \mathbb{F}$ and $\mathcal{Q}_1, \cdots, \mathcal{Q}_k \in \mathbb{M}_2(\mathbb{F})$.

We denote the set of all skew-symmetric matrices of order n underlying the field $\mathbb{F}$ by $\mathbb{SK}_n(\mathbb{F})$. Chan et al. (1987) also studied the adjoint-commuting mappings on $\mathbb{S}_n(\mathbb{F})$ and $\mathbb{SK}_n(\mathbb{F})$ in the same paper. For these two cases, the field $\mathbb{F}$ must be infinite and $\mathrm{char}(\mathbb{F}) \neq 2$. They proved that adjoint-commuting mappings on $\mathbb{S}_n(\mathbb{F})$ $(n > 2)$ and $\mathbb{SK}_n(\mathbb{F})$ (n is even positive integer except $n = 4$) has the form (2.10) with $\varkappa \in \mathbb{F}$ and $\mathcal{Q}$ is a non-singular matrix in $\mathbb{M}_n(\mathbb{F})$ for which $\varkappa^{n-1} = \varkappa$ and $\mathcal{Q}^{-1} = \alpha \mathcal{Q}^T$ for some $\alpha \in \mathbb{F}$. Furthermore, for the adjoint-commuting mappings on $\mathbb{S}_2(\mathbb{F})$, it has the form (2.12) with $\alpha_1, \cdots, \alpha_k \in \mathbb{F}$ and $\mathcal{Q}_1, \cdots, \mathcal{Q}_k$ are non-singular matrices in $\mathbb{M}_2(\mathbb{F})$ for which $\mathrm{adj}(\mathcal{Q}_i) = \pm \mathcal{Q}_i^T$ for all $i \in \{1, \cdots, k\}$. Also, for the adjoint-commuting mappings on $\mathbb{SK}_4(\mathbb{F})$, Υ may possibly have the following form

$$\Upsilon(A) = \varkappa \mathcal{Q} \begin{bmatrix} 0 & a_{34} & a_{24} & a_{23} \\ -a_{34} & 0 & a_{14} & a_{13} \\ -a_{24} & -a_{14} & 0 & a_{12} \\ -a_{23} & -a_{13} & -a_{12} & 0 \end{bmatrix} \mathcal{Q}^{-1}$$

where $\varkappa^2 = 1$.

Again, Chan et al. (1987) found that the characterisation of linear operator Υ on V that satisfies $\Upsilon(e^A) = e^{\Upsilon(A)}$ for all $A \in V$ where $V \in \{\mathbb{S}_n(\mathbb{R}), \mathbb{S}_n(\mathbb{C}), \mathbb{M}_n(\mathbb{R}), \mathbb{M}_n(\mathbb{C})\}$ in the same paper. They concluded that Υ has the form (2.10) or (2.11) with $\varkappa = 1$

and $\mathcal{Q}$ is a non-singular matrix in V when $V \in \{\mathbb{M}_n(\mathbb{R}), \mathbb{M}_n(\mathbb{C})\}$. In case $V \in \{\mathbb{S}_n(\mathbb{R}), \mathbb{S}_n(\mathbb{C})\}$, Υ has the form (2.10) with $\varkappa = 1$ and $\mathcal{Q}$ is a orthogonal matrix in $\mathbb{S}_n(\mathbb{R})$.

Later on, **Problem 6** was also studied by Chan and Lim (1992). They showed that a linear operator Υ on $\mathbb{M}_n(\mathbb{F})$ ($\mathrm{char}(\mathbb{F}) = 0$ or $\mathrm{char}(\mathbb{F}) > k$) satisfying $\Upsilon(A^k) = \Upsilon(A)^k$ for certain fixed positive integers $k > 1$ has the form (2.10) or (2.11) with $\varkappa^n = \varkappa$ and $\mathcal{Q}$ is a non-singular matrix in $\mathbb{M}_n(\mathbb{F})$. They also studied a similar problem by considering $\mathbb{S}_n(\mathbb{F})$. If $\mathbb{M}_n(\mathbb{F})$ is replaced by $\mathbb{S}_n(\mathbb{F})$ ($\mathbb{F}$ is an algebraically closed field with $\mathrm{char}(\mathbb{F}) = 0$ or $\mathrm{char}(\mathbb{F}) > k$), Υ has the form (2.10) with $\varkappa^n = \varkappa$ and $\mathcal{Q}$ is an orthogonal matrix in $\mathbb{M}_n(\mathbb{F})$.

Let U and V be two matrix spaces. The results we discussed above are all ~~very~~ classic linear preserver problems. These problems have one thing in common, that is, Υ is a linear operator. This means Υ is a linear mapping from V to itself. Nowadays, many researchers have extended the linear preserver problems by not emphasizing Υ must be a mapping from V to itself. Instead, Υ as an additive or a homogeneous mapping from U to V is considered. When Υ is additive, we called it additive preserver problems. Besides that, when Υ is homogeneous, we called it multiplicative preserver problems. For example, adjoint-commuting additive mapping on square matrices and symmetric matrices, triangular matrices and complex Hermitian matrices were studied by (Tang and Zhang (2006)), (Chooi (2010) and Tang and Zhang (2006)) and (Tang (2005)), respectively.

In more general cases, the adjoint-commuting mappings on Hermitian matrices and symmetric matrices, and alternate matrices and skew-Hermitian matrices which satisfy certain conditions (without imposing additivity and homogeneity condition on Υ) were studied by (Chooi and Ng (2011)) and (Chooi and Ng (2014)), respectively.

CHAPTER 3

PRELIMINARY RESULTS

3.1 Introduction

Definition 3.1.1. Let $m, n \in \mathbb{N}$ with $m, n > 2$. Let V_1 and V_2 be matrix spaces underlying the same field $\mathbb{F}$. Υ is a compound-commuting mapping if $\Upsilon\colon V_1 \to V_2$ satisfies

$$\Upsilon(C_{n-1}(A)) = C_{m-1}(\Upsilon(A)) \text{ for any } A \in V_1.$$

The compound-commuting additive mappings on Hermitian matrices and symmetric matrices were researched by Chooi (2011). In the paper of Chooi and Ng (2010), they are studied adjoint-commuting mappings on square matrices without imposing additivity and homogeneity condition on Υ. They characterise a mapping $\Upsilon\colon \mathbb{M}_n(\mathbb{F}) \to \mathbb{M}_m(\mathbb{F})$ $(m, n > 2)$ satisfies one of the following conditions:

[B1] $\Upsilon(\mathrm{adj}(A - B)) = \mathrm{adj}(\Upsilon(A) - \Upsilon(B))$ for any $A, B \in \mathbb{M}_n(\mathbb{F})$;

[B2] $\Upsilon(\mathrm{adj}(A + \alpha B)) = \mathrm{adj}(\Upsilon(A) + \alpha\Upsilon(B))$ for any $A, B \in \mathbb{M}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}$.

They shows that if Υ satisfies [B1] or [B2], then Υ is an adjoint-commuting mapping as $\Upsilon(0_n) = 0_m$.

Inspired by their work, we continue to study the compound-commuting mappings on Hermitian matrices and symmetric matrices in this project. Now we list the problems that we want to solve.

Problem 7. Let $\mathbb{F}$ be a field carrying an involution $-$ and $m, n \in \mathbb{N}$ with $m, n > 2$. Characterise the $\Upsilon\colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ that satisfies one of the following conditions:

[P1] $\Upsilon(C_{n-1}(A - B)) = C_{m-1}(\Upsilon(A) - \Upsilon(B))$ for any $A, B \in \mathbb{H}_n(\mathbb{F})$;

[P2] $\Upsilon(C_{n-1}(A+\alpha B)) = C_{m-1}(\Upsilon(A)+\alpha\Upsilon(B))$ for any $A, B \in \mathbb{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$.

The following lemma shows us that Υ is a compound-commuting mappings if Υ satisfies [P1] or [P2].

Lemma 3.1.2. Let $\mathbb{F}$ be a field carrying an involution $-$ and $m, n \in \mathbb{N}$ with $m, n > 2$. If a mapping $\Upsilon\colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ satisfies [P1], then

(a) $\Upsilon(0_n) = 0_m$.

(b) $\Upsilon(C_{n-1}(A)) = C_{m-1}(\Upsilon(A))$ for any $A \in \mathbb{H}_n(\mathbb{F})$.

Proof.

(a) Let $A \in \mathbb{H}_n(\mathbb{F})$, $\Upsilon(0_n) = \Upsilon(C_{n-1}(0_n)) = \Upsilon(C_{n-1}(A - A)) = C_{m-1}(\Upsilon(A) - \Upsilon(A)) = C_{m-1}(0_m) = 0_m$.

(b) For any $A \in \mathbb{H}_n(\mathbb{F})$, $\Upsilon(C_{n-1}(A)) = \Upsilon(C_{n-1}(A - 0_n)) = C_{m-1}(\Upsilon(A) - \Upsilon(0_n)) = C_{m-1}(\Upsilon(A) - 0_m) = C_{m-1}(\Upsilon(A))$.

$\square$

Remark. Let $\mathbb{F}$ be a field carrying an involution $-$ and $m, n \in \mathbb{N}$ with $m, n > 2$. We see that if a mapping $\Upsilon\colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ satisfies [P1], then Υ satisfies [P2] immediately. In other words, if the results hold true for [P1], then the results hold true for [P2] automatically. Recall that, $\mathbb{H}_n(\mathbb{F}) = \mathbb{S}_n(\mathbb{F})$ if $-$ is identity.

3.2 Fundamental Theorem of Geometry of Hermitian Matrices

In this section, we introduce some helpful theorem that help us to characterise the compound-commuting mappings on Hermitian matrices and symmetric matrices.

Definition 3.2.1. Let $\mathbb{F}$ be a field carrying an involution $-$ and $n \in \mathbb{N}$ with $n > 2$. Let $A, B \in \mathbb{H}_n(\mathbb{F})$. If $\operatorname{rk}(A - B) = 1$, then A and B are adjacent.

Theorem 3.2.2. (Huang and Havlicek (2008)). Let $\mathbb{F}$ be a field carrying an involution $-$ with $||\mathbb{F}^-|| > 3$ and $\operatorname{char}(\mathbb{F}) \neq 2$ when $-$ is identity. Let $m, n \in \mathbb{N}$ with $m, n > 2$. If $\Upsilon\colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ satisfies $\operatorname{rk}(A - B) = n$ if and only if $\operatorname{rk}(\Upsilon(A) - \Upsilon(B)) = m$ for every $A, B \in \mathbb{H}_n(\mathbb{F})$ with Υ is onto, then Υ is bijective and preserves the adjacency (i.e., $\operatorname{rk}(\Upsilon(A) - \Upsilon(B)) = 1$ if $\operatorname{rk}(A - B) = 1$ for every $A, B \in \mathbb{H}_n(\mathbb{F})$) and $m = n$.

Theorem 3.2.3. (Huang et al. (2004)). Let $\mathbb{F}$ be a field carrying an involution $-$ with $\mathrm{char}(\mathbb{F}) \neq 2$ when $-$ is identity. Let $n \in \mathbb{N}$ with $n > 2$. If $\Upsilon\colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_n(\mathbb{F})$ preserves the adjacency with Υ is bijective, then Υ has the form

$$\Upsilon(A) = \xi Z A^{\phi} Z^{H} + \mathcal{R}_0 \text{ for all } A \in \mathbb{H}_n(\mathbb{F})$$

where ξ is a non-zero element in $\mathbb{F}^{-}$, Z is a non-singular matrix in $\mathbb{M}_n(\mathbb{F})$, ϕ is a field automorphism of $\mathbb{F}$ with $\phi(\overline{\rho}) = \overline{\phi(\rho)}$ for all $\rho \in \mathbb{F}$ and $\mathcal{R}_0 \in \mathbb{H}_n(\mathbb{F})$.

3.3 Characterisation of Rank-One Non-Increasing Additive Maps

Apart from **Section 3.2**, in this section we state some important theorems that help to accomplish our main results.

Definition 3.3.1. Let $\mathbb{F}$ be a field carrying an involution $-$ and $m, n \in \mathbb{N}$ with $m, n > 2$. Let Υ be an additive mapping from $\mathbb{H}_n(\mathbb{F})$ to $\mathbb{H}_m(\mathbb{F})$. Υ is a rank-one non-increasing additive map if $\mathrm{rk}(\Upsilon(A)) \leqslant 1$ for all $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) = 1$.

Theorem 3.3.2. (Orel and Kuzma (2007)). Let Galois field of order 4 be $GF(4)$. Let $\mathbb{F}$ be a field carrying a proper involution $-$ with $\mathbb{F} \neq GF(4)$. Let $m, n \in \mathbb{N}$ with $m, n > 2$. If $\Upsilon\colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ is a rank-one non-increasing additive map and there exists at least one $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(\Upsilon(A)) = m$, then Υ has the form

$$\Upsilon(A) = \xi Z A^{\phi} Z^{H} \text{ for all } A \in \mathbb{H}_n(\mathbb{F})$$

where ξ is a non-zero element in $\mathbb{F}^{-}$, Z is a matrix in $\mathbb{M}_{m\times n}(\mathbb{F})$($Z$ is a non-singular matrix in $\mathbb{M}_n(\mathbb{F})$ if $m = n$) and ϕ is a non-zero field monomorphism of $\mathbb{F}$ with $\phi(\overline{\rho}) = \overline{\phi(\rho)}$ for all $\rho \in \mathbb{F}$.

Theorem 3.3.3. (Orel and Kuzma (2009)). Let Galois field of order 4 be $GF(4)$ and $m, n \in \mathbb{N}$ with $m, n > 2$. Let $-$ be an involution of $GF(4)$. If $\Upsilon\colon \mathbb{H}_n(GF(4)) \to \mathbb{H}_m(GF(4))$ is a rank-one non-increasing additive map and there exists at least one $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(\Upsilon(A)) = m$, then Υ has the form

$$\Upsilon(A) = \xi Z A^{\phi} Z^{H} \text{ for all } A \in \mathbb{H}_n(\mathbb{F})$$

where ξ is a non-zero element in $GF(4)$ with $\overline{\xi} = \xi$, Z is a matrix in $\mathbb{M}_{m\times n}(GF(4))(Z$ is a non-singular matrix in $\mathbb{M}_n(GF(4))$ if $m = n)$ and ϕ is a non-zero field monomorphism of $GF(4)$ with $\phi(\overline{\rho}) = \overline{\phi(\rho)}$ for all $\rho \in GF(4)$.

Theorem 3.3.4. (Orel and Kuzma (2006)). Let $\mathbb{F}$ be a field and $m, n \in \mathbb{N}$ with $m, n > 2$. If $\Upsilon \colon \mathbb{S}_n(\mathbb{F}) \to \mathbb{S}_m(\mathbb{F})$ is a rank-one non-increasing additive maps and there exists at least one $A \in \mathbb{S}_n(\mathbb{F})$ with $\mathrm{rk}(\Upsilon(A)) = m$, then Υ has the form

$$\Upsilon(A) = \xi Z A^{\phi} Z^T \text{ for all } A \in \mathbb{S}_n(\mathbb{F}),$$

or

$$\Upsilon(A) = \mathcal{Q}\Upsilon'(A)\mathcal{Q}^T \text{ for all } A \in \mathbb{S}_n(\mathbb{F}) \text{ when } m = 3 \text{ and } \mathbb{F} = GF(2)$$

where ξ is a non-zero element in $\mathbb{F}$, Z is a matrix in $\mathbb{M}_{m\times n}(\mathbb{F})(Z$ is a non-singular matrix in $\mathbb{M}_n(\mathbb{F})$ if $m = n)$, ϕ is a non-zero field monomorphism of $\mathbb{F}$, $\mathcal{Q}$ is a non-singular matrix in $\mathbb{M}_3(GF(2))$ and Υ' is an additive mapping from $\mathbb{S}_n(GF(2))$ to $\mathbb{S}_3(GF(2))$.

3.4 Some Requirements

Lemma 3.4.1. (Chooi (2011), Lemma 2.3). Let $\mathbb{F}$ be a field carrying an involution $-$ and $n \in \mathbb{N}$ with $n > 2$. For all $A \in \mathbb{H}_n(\mathbb{F})$,

(a) $\mathrm{rk}(A) = n$ if and only if $\mathrm{rk}(C_{n-1}(A)) = n$.

(b) $\mathrm{rk}(A) = n - 1$ if and only if $\mathrm{rk}(C_{n-1}(A)) = 1$.

(c) $\mathrm{rk}(A) \leqslant n - 2$ if and only if $\mathrm{rk}(C_{n-1}(A)) = 0$.

Proof.

We begin our proof with the necessity part.

(a) Suppose that $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) = n$. Since $\mathrm{rk}(A) = n$, then A is a non-singular matrix. Thus A^{-1} exists, so $C_{n-1}(A^{-1}) = C_{n-1}(A)^{-1}$ also exists. Therefore $C_{n-1}(A)$ is an invertible matrix, this implies that $\mathrm{rk}(C_{n-1}(A)) = n$.

(b) & (c) Let $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) = k$. By referring to **Corollary 2.1.8.4**, we see that there exists some $G \in \mathbb{M}_n(\mathbb{F})$ with $\mathrm{rk}(G) = n$ such that either

$$A = G(\gamma_1 E_{11} + \gamma_2 E_{22} + \cdots + \gamma_k E_{kk})G^H$$

where $\gamma_i \in \mathbb{F}^-$ with $\gamma_i \neq 0$ for all $i \in \{1, 2, \cdots, k\}$ or

$$A = G\left(\bigoplus_{i=1}^{\frac{k}{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus 0_{n-k}\right) G^H$$

when $A \in \mathcal{L}_n(\mathbb{F})$ and $-$ is identity. When we take the $(n-1)$-th compound matrix on both sides, we get $C_{n-1}(A) = C_{n-1}(GXG^H) = C_{n-1}(G)C_{n-1}(X)C_{n-1}(G)^H$ where either $X = \gamma_1 E_{11} + \gamma_2 E_{22} + \cdots + \gamma_k E_{kk}$ or $X = \bigoplus_{i=1}^{\frac{k}{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus 0_{n-k}$. Because $\operatorname{rk}(G) = n$, then $\operatorname{rk}(C_{n-1}(G)) = n$. This implies $\operatorname{rk}(C_{n-1}(A)) = \operatorname{rk}(C_{n-1}(X))$.

If $k = n-1$, then $(C_{n-1}(X))_{11} = |X[n \mid n]| \neq 0$. At the same time, for all $i \neq 1$ and $1 \leqslant j \leqslant n$, $(C_{n-1}(X))_{ij} = |X[n+1-i \mid n+1-j]| = 0$. This is because the last row of $X[n+1-i \mid n+1-j]$ is row of zeros. This leads to $|X[n+1-i \mid n+1-j]| = 0$. This shows us that there only exists one non-zero column in $C_{n-1}(X)$. So we conclude that $\operatorname{rk}(C_{n-1}(A)) = \operatorname{rk}(C_{n-1}(X)) = 1$.

Now, we consider for $k = n-2$. For all $1 \leqslant i, j \leqslant n$, $(C_{n-1}(X))_{ij} = |X[n+1-i \mid n+1-j]| = 0$. This is because there exists at least one row of zeros in $X[n+1-i \mid n+1-j]$. This implies $|X[n+1-i \mid n+1-j]| = 0$. Consequently, $C_{n-1}(X) = 0_n$. Thereupon we assert that $\operatorname{rk}(C_{n-1}(A)) = \operatorname{rk}(C_{n-1}(X)) = 0$.

Since we have proved that the necessity part is true, the sufficiency part follows immediately by using proof by contradiction.

$\square$

Lemma 3.4.2. (Chooi (2011), Lemma 2.4). Let $\mathbb{F}$ be a field carrying an involution $-$ and $n \in \mathbb{N}$ with $n > 2$. For all $A \in \mathbb{H}_n(\mathbb{F})$, if $\operatorname{rk}(A) = 1$, then there exists some matrix $\mathcal{Q} \in \mathbb{H}_n(\mathbb{F})$ such that $A = C_{n-1}(\mathcal{Q})$ where $\operatorname{rk}(\mathcal{Q}) = n-1$.

Proof.

Let $A \in \mathbb{H}_n(\mathbb{F})$ with $\operatorname{rk}(A) = 1$. By **Corollary 2.1.8.4**, there exist some non-zero element $\gamma \in \mathbb{F}^-$ and an invertible matrix $G \in \mathbb{M}_n(\mathbb{F})$ such that $A = \gamma G E_{11} G^H$. Besides that, according to **Lemma 2.1.7.3**, we have $C_{n-1}(C_{n-1}(G)) = |G|^{n-2}G$. Thus

$$
\begin{aligned}
A &= \gamma G E_{11} G^H \\
&= \left[|G|^{n-2} G\right] \left[\gamma |G|^{2-n} |G^H|^{2-n} E_{11}\right] \left[|G^H|^{n-2} G^H\right] \\
&= C_{n-1}(C_{n-1}(G)) \left[\gamma |G|^{2-n} \left|\overline{G}\right|^{2-n} E_{11}\right] C_{n-1}(C_{n-1}(G^H)) \\
&= C_{n-1}(C_{n-1}(G)) \left[\gamma \left|G\overline{G}\right|^{2-n} E_{11}\right] C_{n-1}(C_{n-1}(G^H)) \\
&= C_{n-1}(C_{n-1}(G)) C_{n-1}(I_n - E_{nn} - E_{jj} + \gamma \left|G\overline{G}\right|^{2-n} E_{jj}) C_{n-1}(C_{n-1}(G^H)) \\
&= C_{n-1}(C_{n-1}(G)(I_n - E_{nn} - E_{jj} + \gamma |G\overline{G}|^{2-n} E_{jj}) C_{n-1}(G)^H)
\end{aligned}
$$

where $1 \leqslant j \leqslant n$ and $j \neq n$. We let $D = I_n - E_{nn} - E_{jj} + \gamma |G\overline{G}|^{2-n} E_{jj}$. Obviously, D is a diagonal matrix. Because G is an invertible matrix, we have $\mathrm{rk}((C_{n-1}(G)) = n$. It follows that $\mathrm{rk}(C_{n-1}(G) D C_{n-1}(G)^H) = \mathrm{rk}(D)$. As $\gamma \left|G\overline{G}\right|^{2-n} \neq 0$, we see that there exist $n-1$ non-zero rows in D. Thus $\mathrm{rk}(C_{n-1}(G) D C_{n-1}(G)^H) = \mathrm{rk}(D) = n-1$. Since $\gamma \in \mathbb{F}^-$ and $\overline{\left|G\overline{G}\right|^{2-n}} = \left|\overline{G\overline{G}}\right|^{2-n} = \left|G\overline{G}\right|^{2-n} \in \mathbb{F}^-$, we obtain $\gamma |G\overline{G}|^{2-n}$ is also belonged to $\mathbb{F}^-$. Hence all the diagonal entries of D are belonged to $\mathbb{F}^-$. Consequently, $D = D^H$. As a result, $C_{n-1}(G) D C_{n-1}(G)^H \in \mathbb{H}_n(\mathbb{F})$.

$\square$

Lemma 3.4.3. Let $\mathbb{F}$ be a field carrying an involution $-$ and $m, n \in \mathbb{N}$ with $m, n > 2$. If a mapping $\Upsilon \colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ satisfies [P1], then for all $A \in \mathbb{H}_n(\mathbb{F})$,

(a) $\mathrm{rk}(\Upsilon(A)) \leqslant m-1$ when $\mathrm{rk}(A) = n-1$.

(b) $\mathrm{rk}(\Upsilon(A)) \leqslant m-2$ when $\mathrm{rk}(A) \leqslant n-2$.

(c) $\mathrm{rk}(\Upsilon(A)) \leqslant 1$ when $\mathrm{rk}(A) = 1$.

Proof.

(a) Assume that $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) = n-1$. Thereupon $\mathrm{rk}(C_{n-1}(A)) = 1$. Thus $\mathrm{rk}(C_{n-1}(C_{n-1}(A))) = 0$. This implies $C_{n-1}(C_{n-1}(A)) = 0_n$. Consequently, $\Upsilon(C_{n-1}(C_{n-1}(A))) = \Upsilon(0_n)$. Hence $C_{m-1}(C_{m-1}(\Upsilon(A))) = 0_m$. It follows that $\mathrm{rk}(C_{m-1}(C_{m-1}(\Upsilon(A)))) = 0$ and then we obtain $\mathrm{rk}(C_{m-1}(\Upsilon(A))) \leqslant m-2$. Therefore $\mathrm{rk}(\Upsilon(A)) \leqslant m-1$.

(b) Suppose that $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) \leqslant n-2$. Then $\mathrm{rk}(C_{n-1}(A)) = 0$. This implies $C_{n-1}(A) = 0_n$ and hence $\Upsilon(C_{n-1}(A)) = \Upsilon(0_n)$. This leads to $C_{m-1}(\Upsilon(A)) = 0_m$. Therefore $\mathrm{rk}(C_{m-1}(\Upsilon(A))) = 0$. So $\mathrm{rk}(\Upsilon(A)) \leqslant m-2$.

(c) Suppose that $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) = 1$. By **Lemma 3.4.2**, there exists a matrix $\mathcal{Q} \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(\mathcal{Q}) = n-1$ such that $A = C_{n-1}(\mathcal{Q})$. This implies $\Upsilon(A) = \Upsilon(C_{n-1}(\mathcal{Q})) = C_{m-1}(\Upsilon(\mathcal{Q}))$. Since $\mathrm{rk}(\mathcal{Q}) = n-1$, we attain $\mathrm{rk}(\Upsilon(\mathcal{Q})) \leqslant m-1$. Thus $\mathrm{rk}(\Upsilon(A)) = \mathrm{rk}(C_{m-1}(\Upsilon(\mathcal{Q}))) \leqslant 1$.

□

Lemma 3.4.4. (Chooi and Ng (2011), Lemma 2.4). Let $\mathbb{F}$ be a field carrying an involution $-$. Let $n \in \mathbb{N}$ with $n > 2$. Let r be a non-negative integer with $r \leqslant n$. For every $A, B \in \mathbb{H}_n(\mathbb{F})$,

(a) $\mathrm{rk}(A) = r$ implies there exists a matrix $\mathcal{Q} \in \mathbb{H}_n(\mathbb{F})$ such that $\mathrm{rk}(A + \mathcal{Q}) = n$ where $\mathrm{rk}(\mathcal{Q}) = n - r$.

(b) There exists a matrix $\mathcal{Q} \in \mathbb{H}_n(\mathbb{F})$ such that $\mathrm{rk}(A + \mathcal{Q}) = \mathrm{rk}(B + \mathcal{Q}) = n$.

(c) There exists a matrix $\mathcal{Q} \in \mathbb{H}_n(\mathbb{F})$ for which $\mathcal{Q} \neq 0_n$ such that $\mathrm{rk}(A + \mathcal{Q}) = n$ where either A or $\mathcal{Q}$ is an invertible matrix but not both.

(d) If $A \neq 0_n$, then there exists at least one matrix $\mathcal{Q} \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(\mathcal{Q}) \leqslant n-2$ such that $\mathrm{rk}(A + \mathcal{Q}) = n - 1$.

(e) If the number of elements in $\mathbb{F}^-$ is more than $n+1$ and $\mathrm{rk}(A + B) = n$. Then there exists some λ which belongs to $\mathbb{F}^-$ such that $\mathrm{rk}(A + \lambda B) = n$ where λ is non-trivial, that is $\lambda \neq 1$.

Lemma 3.4.5. Let $\mathbb{F}$ be a field carrying an involution $-$ and $m, n \in \mathbb{N}$ with $m, n > 2$. Let $\mathcal{K}$ be an arbitrary constant invertible matrix in $\mathbb{M}_n(\mathbb{F})$. If a mapping $\Upsilon \colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ satisfies [P1], then the following statements are equivalent.

(a) $\mathrm{rk}(\Upsilon(\mathcal{K}\mathcal{K}^H)) \neq m$.

(b) $\Upsilon(|\mathcal{K}\mathcal{K}^H|^{n-2}\mathcal{K}\mathcal{K}^H) = 0_m$.

(c) $\Upsilon(\mathcal{K}A\mathcal{K}^H) = 0_m$ for all $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) \leqslant 1$.

(d) $\mathrm{rk}(\Upsilon(\mathcal{K}A\mathcal{K}^H)) \leqslant m - 2$ for all $A \in \mathbb{H}_n(\mathbb{F})$.

(e) $\Upsilon(C_{n-1}(\mathcal{K}A\mathcal{K}^H)) = 0_m$ for all $A \in \mathbb{H}_n(\mathbb{F})$.

Proof.

It is quite easy to see that $\mathcal{K}A\mathcal{K}^H \in \mathbb{H}_n(\mathbb{F})$ for all $A \in \mathbb{H}_n(\mathbb{F})$. However, because $\mathcal{K}$ is an invertible matrix, $\text{rk}(\mathcal{K}A\mathcal{K}^H) = \text{rk}(A)$ for all $A \in \mathbb{H}_n(\mathbb{F})$. For simplicity, we let $\Psi_{\mathcal{K}}(A) = \Upsilon(\mathcal{K}A\mathcal{K}^H)$ for all $A \in \mathbb{H}_n(\mathbb{F})$. According to the **Lemma 3.4.3**, we deduce that

- $\text{rk}(\Psi_{\mathcal{K}}(A)) = \text{rk}(\Upsilon(\mathcal{K}A\mathcal{K}^H)) \leqslant m-1$ when $\text{rk}(\mathcal{K}A\mathcal{K}^H) = \text{rk}(A) = n-1$.
- $\text{rk}(\Psi_{\mathcal{K}}(A)) = \text{rk}(\Upsilon(\mathcal{K}A\mathcal{K}^H)) \leqslant m-2$ when $\text{rk}(\mathcal{K}A\mathcal{K}^H) = \text{rk}(A) \leqslant n-2$.
- $\text{rk}(\Psi_{\mathcal{K}}(A)) = \text{rk}(\Upsilon(\mathcal{K}A\mathcal{K}^H)) \leqslant 1$ when $\text{rk}(\mathcal{K}A\mathcal{K}^H) = \text{rk}(A) \leqslant 1$.

Clearly, for all $A, B \in \mathbb{H}_n(\mathbb{F})$,

- $C_{m-1}(\Upsilon(A-B)) = C_{m-1}(\Upsilon(A) - \Upsilon(B))$.
- $C_{m-1}(\Psi_{\mathcal{K}}(A-B)) = C_{m-1}(\Psi_{\mathcal{K}}(A) - \Psi_{\mathcal{K}}(B))$.

This is because

$$\begin{aligned} C_{m-1}(\Upsilon(A-B)) &= \Upsilon(C_{n-1}(A-B)) \\ &= C_{m-1}(\Upsilon(A) - \Upsilon(B)) \end{aligned}$$

and

$$\begin{aligned} C_{m-1}(\Psi_{\mathcal{K}}(A-B)) &= C_{m-1}(\Upsilon(\mathcal{K}(A-B)\mathcal{K}^H)) \\ &= C_{m-1}(\Upsilon(\mathcal{K}A\mathcal{K}^H - \mathcal{K}B\mathcal{K}^H)) \\ &= C_{m-1}(\Upsilon(\mathcal{K}A\mathcal{K}^H) - \Upsilon(\mathcal{K}B\mathcal{K}^H)) \\ &= C_{m-1}(\Psi_{\mathcal{K}}(A) - \Psi_{\mathcal{K}}(B)). \end{aligned}$$

In order to complete the proof, it is inevitable that we need to put forward many claims and need to prove each claim one by one. Moreover, for the reasons of clarity, we use λ and R to represent $|\mathcal{K}\mathcal{K}^H|^{n-2}$ and $C_{n-1}(\mathcal{K})$, respectively. Surely $\lambda \neq 0$ as $\mathcal{K}$ is invertible.

(a) $\Rightarrow$ (b)

It follows from **Lemma 2.1.7.3** that $C_{n-1}(C_{n-1}(\mathcal{K}\mathcal{K}^H)) = |\mathcal{K}\mathcal{K}^H|^{n-2}\mathcal{K}\mathcal{K}^H = \lambda\mathcal{K}\mathcal{K}^H$. Then

$$\begin{aligned}\Psi_{\mathcal{K}}(\lambda I_n) &= \Upsilon(\lambda \mathcal{K}\mathcal{K}^H)\\ &= \Upsilon(C_{n-1}(C_{n-1}(\mathcal{K}\mathcal{K}^H)))\\ &= C_{m-1}(C_{m-1}(\Upsilon(\mathcal{K}\mathcal{K}^H))).\end{aligned}$$

Owing to $\mathrm{rk}(\Upsilon(\mathcal{K}\mathcal{K}^H)) \neq m$, we obtain $\mathrm{rk}(C_{m-1}(\Upsilon(\mathcal{K}\mathcal{K}^H))) \leqslant 1$ and hence $\Psi_{\mathcal{K}}(\lambda I_n) = 0_m$ as $\mathrm{rk}(\Psi_{\mathcal{K}}(\lambda I_n)) = \mathrm{rk}(C_{m-1}(C_{m-1}(\Upsilon(\mathcal{K}\mathcal{K}^H)))) = 0$.

(b) $\Rightarrow$ (c)

Claim 1. $\Upsilon(\lambda^{n-1}RR^H) = 0_m$.

$$\begin{aligned}\Upsilon(\lambda^{n-1}RR^H) &= \Upsilon(\lambda^{n-1}C_{n-1}(\mathcal{K})C_{n-1}(\mathcal{K}^H))\\ &= \Upsilon(\lambda^{n-1}C_{n-1}(\mathcal{K}\mathcal{K}^H))\\ &= \Upsilon(C_{n-1}(\lambda\mathcal{K}\mathcal{K}^H))\\ &= C_{m-1}(\Upsilon(\lambda\mathcal{K}\mathcal{K}^H))\\ &= C_{m-1}(\Psi_{\mathcal{K}}(\lambda I_n))\\ &= C_{m-1}(0_m)\\ &= 0_m.\end{aligned}$$

Claim 2. $\Upsilon(\lambda^{n-1}RE_{ii}R^H) = 0_m$ for all $i \in \{1, 2, \cdots, n\}$.

Because $C_{n-1}(I_n - E_{n+1-i,n+1-i}) = E_{ii}$, we attain

$$\begin{aligned}\Upsilon(\lambda^{n-1}RE_{ii}R^H) &= \Upsilon(\lambda^{n-1}C_{n-1}(\mathcal{K})C_{n-1}(I_n - E_{n+1-i,n+1-i})C_{n-1}(\mathcal{K}^H))\\ &= \Upsilon(\lambda^{n-1}C_{n-1}(\mathcal{K}(I_n - E_{n+1-i,n+1-i})\mathcal{K}^H))\\ &= \Upsilon(C_{n-1}(\mathcal{K}(\lambda I_n - \lambda E_{n+1-i,n+1-i})\mathcal{K}^H))\\ &= C_{m-1}(\Upsilon(\mathcal{K}(\lambda I_n - \lambda E_{n+1-i,n+1-i})\mathcal{K}^H))\\ &= C_{m-1}(\Psi_{\mathcal{K}}(\lambda I_n - \lambda E_{n+1-i,n+1-i}))\\ &= C_{m-1}(\Psi_{\mathcal{K}}(\lambda I_n) - \Psi_{\mathcal{K}}(\lambda E_{n+1-i,n+1-i}))\\ &= C_{m-1}(-\Psi_{\mathcal{K}}(\lambda E_{n+1-i,n+1-i})).\end{aligned}$$

Due to $\mathrm{rk}(\lambda E_{n+1-i,n+1-i}) = 1, \mathrm{rk}(-\Psi_{\mathcal{K}}(\lambda E_{n+1-i,n+1-i})) = \mathrm{rk}(\Psi_{\mathcal{K}}(\lambda E_{n+1-i,n+1-i})) \leqslant 1$. So, $\Upsilon(\lambda^{n-1}RE_{ii}R^H) = 0_m$ as $\mathrm{rk}(\Upsilon(\lambda^{n-1}RE_{ii}R^H)) = \mathrm{rk}(C_{m-1}(-\Psi_{\mathcal{K}}(\lambda E_{n+1-i,n+1-i})))) = 0$.

Claim 3. $\Psi_{\mathcal{K}}(\alpha E_{ii}) = 0_m$ for all $\alpha \in \mathbb{F}^-$ and $i \in \{1, 2, \cdots, n\}$.

When $\alpha = 0$, $\Psi_{\mathcal{K}}(\alpha E_{ii}) = \Psi_{\mathcal{K}}(0E_{ii}) = \Psi_{\mathcal{K}}(0_n) = \Upsilon(0_n) = 0_m$. Now we consider that $\alpha \neq 0$. We know that

$$\begin{aligned}
&C_{n-1}(I_n - E_{n+1-i,n+1-i} - E_{jj} + \alpha\lambda^{-(n-1)^2-1}E_{jj}) = \alpha\lambda^{-(n-1)^2-1}E_{ii} \\
\Rightarrow\ &\lambda^{(n-1)^2}C_{n-1}(I_n - E_{n+1-i,n+1-i} - E_{jj} + \alpha\lambda^{-(n-1)^2-1}E_{jj}) = \alpha\lambda^{-1}E_{ii} \\
\Rightarrow\ &C_{n-1}(\lambda^{n-1}I_n - \lambda^{n-1}E_{n+1-i,n+1-i} - \lambda^{n-1}E_{jj} + \alpha\lambda^{-(n-1)^2-1+(n-1)}E_{jj}) = \alpha\lambda^{-1}E_{ii} \\
\Rightarrow\ &C_{n-1}(\lambda^{n-1}I_n - \lambda^{n-1}E_{n+1-i,n+1-i} - \lambda^{n-1}E_{jj} + \alpha\lambda^{-n^2+3n-3}E_{jj}) = \alpha\lambda^{-1}E_{ii}
\end{aligned}$$

where $1 \leqslant j \leqslant n$ and $j \neq n+1-i$. Besides that $|\mathcal{K}|^{n-2}\mathcal{K} = C_{n-1}(C_{n-1}(\mathcal{K})) = C_{n-1}(R)$. It follows that

$$\begin{aligned}
&\Psi_{\mathcal{K}}(\alpha E_{ii}) \\
&= \Upsilon(\mathcal{K}(\alpha E_{ii})\mathcal{K}^H) \\
&= \Upsilon(\mathcal{K}(\lambda\lambda^{-1}\alpha E_{ii})\mathcal{K}^H) \\
&= \Upsilon((|\mathcal{K}|^{n-2}\mathcal{K})(\alpha\lambda^{-1}E_{ii})(|\mathcal{K}^H|^{n-2}\mathcal{K}^H)) \\
&= \Upsilon(C_{n-1}(R)C_{n-1}(\lambda^{n-1}I_n - \lambda^{n-1}E_{n+1-i,n+1-i} - \lambda^{n-1}E_{jj} + \alpha\lambda^{-n^2+3n-3}E_{jj})C_{n-1}(R^H)) \\
&= \Upsilon(C_{n-1}(R(\lambda^{n-1}I_n - \lambda^{n-1}E_{n+1-i,n+1-i} - \lambda^{n-1}E_{jj} + \alpha\lambda^{-n^2+3n-3}E_{jj})R^H)) \\
&= \Upsilon(C_{n-1}(\lambda^{n-1}RR^H - \lambda^{n-1}RE_{n+1-i,n+1-i}R^H - \lambda^{n-1}RE_{jj}R^H + \alpha\lambda^{-n^2+3n-3}RE_{jj}R^H)).
\end{aligned}$$

Thus we have

$$\begin{aligned}
&\Psi_{\mathcal{K}}(\alpha E_{ii}) \\
&= C_{m-1}(\Upsilon(\lambda^{n-1}RR^H - \lambda^{n-1}RE_{n+1-i,n+1-i}R^H - \lambda^{n-1}RE_{jj}R^H + \alpha\lambda^{-n^2+3n-3}RE_{jj}R^H)) \\
&= C_{m-1}(\Upsilon(\lambda^{n-1}RR^H - \lambda^{n-1}RE_{n+1-i,n+1-i}R^H + \alpha\lambda^{-n^2+3n-3}RE_{jj}R^H) - \Upsilon(\lambda^{n-1}RE_{jj}R^H)) \\
&= C_{m-1}(\Upsilon(\lambda^{n-1}RR^H - \lambda^{n-1}RE_{n+1-i,n+1-i}R^H + \alpha\lambda^{-n^2+3n-3}RE_{jj}R^H)) \\
&= C_{m-1}(\Upsilon(\lambda^{n-1}RR^H + \alpha\lambda^{-n^2+3n-3}RE_{jj}R^H) - \Upsilon(\lambda^{n-1}RE_{n+1-i,n+1-i}R^H)) \\
&= C_{m-1}(\Upsilon(\lambda^{n-1}RR^H + \alpha\lambda^{-n^2+3n-3}RE_{jj}R^H)) \\
&= C_{m-1}(\Upsilon(\lambda^{n-1}RR^H - (-\alpha\lambda^{-n^2+3n-3}RE_{jj}R^H))) \\
&= C_{m-1}(\Upsilon(\lambda^{n-1}RR^H) - \Upsilon(-\alpha\lambda^{-n^2+3n-3}RE_{jj}R^H)) \\
&= C_{m-1}(-\Upsilon(-\alpha\lambda^{-n^2+3n-3}RE_{jj}R^H)).
\end{aligned}$$

Because $\mathrm{rk}(-\alpha\lambda^{-n^2+3n-3}RE_{jj}R^H) = \mathrm{rk}(E_{jj}) = 1$. Therefore $\mathrm{rk}(-\Upsilon(-\alpha\lambda^{-n^2+3n-3} RE_{jj}R^H)) = \mathrm{rk}(\Upsilon(-\alpha\lambda^{-n^2+3n-3}RE_{jj}R^H)) \leqslant 1$. Consequently, $\Psi_{\mathcal{K}}(\alpha E_{ii}) = 0_m$ as $\mathrm{rk}(\Psi_{\mathcal{K}}(\alpha E_{ii})) = \mathrm{rk}(C_{m-1}(-\Upsilon(-\alpha\lambda^{-n^2+3n-3}RE_{jj}R^H))) = 0$.

Claim 4. $C_{m-1}(\Psi_{\mathcal{K}}(A) - \Psi_{\mathcal{K}}(D)) = C_{m-1}(\Psi_{\mathcal{K}}(A))$ for all $A \in \mathbb{H}_n(\mathbb{F})$ and diagonal matrix $D \in \mathbb{H}_n(\mathbb{F})$.

Let $D = \alpha_1 E_{11} + \alpha_2 E_{22} + \cdots + \alpha_{n-1} E_{n-1,n-1} + \alpha_n E_{nn}$ where $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}, \alpha_n \in \mathbb{F}^-$. Hence

$$\begin{aligned}
&C_{m-1}(\Psi_{\mathcal{K}}(A) - \Psi_{\mathcal{K}}(D)) \\
&= C_{m-1}(\Psi_{\mathcal{K}}(A) - \Psi_{\mathcal{K}}(\alpha_1 E_{11} + \alpha_2 E_{22} + \cdots + \alpha_{n-1} E_{n-1,n-1} + \alpha_n E_{nn})) \\
&= C_{m-1}(\Psi_{\mathcal{K}}(A - \alpha_1 E_{11} - \alpha_2 E_{22} - \cdots - \alpha_{n-1} E_{n-1,n-1} - \alpha_n E_{nn})) \\
&= C_{m-1}(\Psi_{\mathcal{K}}(A - \alpha_1 E_{11} - \alpha_2 E_{22} - \cdots - \alpha_{n-1} E_{n-1,n-1}) - \Psi_{\mathcal{K}}(\alpha_n E_{nn})) \\
&= C_{m-1}(\Psi_{\mathcal{K}}(A - \alpha_1 E_{11} - \alpha_2 E_{22} - \cdots - \alpha_{n-1} E_{n-1,n-1})).
\end{aligned}$$

By using the same arguments, we obtain

$$\begin{aligned}
&C_{m-1}(\Psi_{\mathcal{K}}(A) - \Psi_{\mathcal{K}}(D)) \\
&= C_{m-1}(\Psi_{\mathcal{K}}(A - \alpha_1 E_{11} - \alpha_2 E_{22})) \\
&= C_{m-1}(\Psi_{\mathcal{K}}(A - \alpha_1 E_{11}) - \Psi_{\mathcal{K}}(\alpha_2 E_{22})) \\
&= C_{m-1}(\Psi_{\mathcal{K}}(A - \alpha_1 E_{11})) \\
&= C_{m-1}(\Psi_{\mathcal{K}}(A) - \Psi_{\mathcal{K}}(\alpha_1 E_{11})) \\
&= C_{m-1}(\Psi_{\mathcal{K}}(A)).
\end{aligned}$$

Claim 5. $\Upsilon(R(\alpha E_{ii})R^H) = 0_m$ for all $\alpha \in \mathbb{F}^-$ and $i \in \{1, 2, \cdots, n\}$.

Because $C_{n-1}(I_n - E_{n+1-i,n+1-i} - E_{jj} + \alpha E_{jj}) = \alpha E_{ii}$ with $1 \leqslant j \leqslant n$ and $j \neq n+1-i$, thereupon

$$\begin{aligned}
\Upsilon(R(\alpha E_{ii})R^H) &= \Upsilon(C_{n-1}(\mathcal{K})C_{n-1}(I_n - E_{n+1-i,n+1-i} - E_{jj} + \alpha E_{jj})C_{n-1}(\mathcal{K}^H)) \\
&= \Upsilon(C_{n-1}(\mathcal{K}(I_n - E_{n+1-i,n+1-i} - E_{jj} + \alpha E_{jj})\mathcal{K}^H)) \\
&= C_{m-1}(\Upsilon(\mathcal{K}(I_n - E_{n+1-i,n+1-i} - E_{jj} + \alpha E_{jj})\mathcal{K}^H)) \\
&= C_{m-1}(\Psi_{\mathcal{K}}(I_n - E_{n+1-i,n+1-i} - E_{jj} + \alpha E_{jj})) \\
&= C_{m-1}(\Psi_{\mathcal{K}}(0_n - (-I_n + E_{n+1-i,n+1-i} + E_{jj} - \alpha E_{jj}))) \\
&= C_{m-1}(\Psi_{\mathcal{K}}(0_n) - \Psi_{\mathcal{K}}(-I_n + E_{n+1-i,n+1-i} + E_{jj} - \alpha E_{jj})) \\
&= C_{m-1}(\Psi_{\mathcal{K}}(0_n)) \\
&= C_{m-1}(\Upsilon(0_n)) \\
&= C_{m-1}(0_m) \\
&= 0_m.
\end{aligned}$$

Claim 6. $C_{m-1}(\Upsilon(A) - \Upsilon(RDR^H)) = C_{m-1}(\Upsilon(A))$ for all $A \in \mathbb{H}_n(\mathbb{F})$ and for all diagonal matrix $D \in \mathbb{H}_n(\mathbb{F})$.

Let $D = \alpha_1 E_{11} + \alpha_2 E_{22} + \cdots + \alpha_{n-1} E_{n-1,n-1} + \alpha_n E_{nn}$ where $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}, \alpha_n \in \mathbb{F}^-$. So

$$\begin{aligned}
&C_{m-1}(\Upsilon(A) - \Upsilon(RDR^H)) \\
&= C_{m-1}(\Upsilon(A - RDR^H)) \\
&= C_{m-1}(\Upsilon(A - R(\alpha_1 E_{11} + \alpha_2 E_{22} + \cdots + \alpha_{n-1} E_{n-1,n-1} + \alpha_n E_{nn})R^H)) \\
&= C_{m-1}(\Upsilon(A - R(\alpha_1 E_{11})R^H - R(\alpha_2 E_{22})R^H - \cdots - R(\alpha_{n-1} E_{n-1,n-1})R^H - R(\alpha_n E_{nn})R^H)) \\
&= C_{m-1}(\Upsilon(A - R(\alpha_1 E_{11})R^H - R(\alpha_2 E_{22})R^H - \cdots - R(\alpha_{n-1} E_{n-1,n-1})R^H) - \Upsilon(R(\alpha_n E_{nn})R^H)) \\
&= C_{m-1}(\Upsilon(A - R(\alpha_1 E_{11})R^H - R(\alpha_2 E_{22})R^H - \cdots - R(\alpha_{n-1} E_{n-1,n-1})R^H)).
\end{aligned}$$

By using the same arguments, we have

$$\begin{aligned}
C_{m-1}(\Upsilon(A) - \Upsilon(RDR^H)) &= C_{m-1}(\Upsilon(A - R(\alpha_1 E_{11})R^H - R(\alpha_2 E_{22})R^H)) \\
&= C_{m-1}(\Upsilon(A - R(\alpha_1 E_{11})R^H) - \Upsilon(R(\alpha_2 E_{22})R^H)) \\
&= C_{m-1}(\Upsilon(A - R(\alpha_1 E_{11})R^H)) \\
&= C_{m-1}(\Upsilon(A) - \Upsilon(R(\alpha_1 E_{11})R^H)) \\
&= C_{m-1}(\Upsilon(A)).
\end{aligned}$$

For the purpose of convenience, we denote $I_n - E_{ii} - E_{jj}$ by Λ_{ij}. Now we continue our proof.

Claim 7. For all $\alpha \in \mathbb{F}$ and $i, j \in \mathbb{N}$ with $1 \leqslant i < j \leqslant n$, $\Upsilon(R(\alpha E_{ij} + \overline{\alpha} E_{ji} - \alpha\overline{\alpha}\Lambda_{ij})R^H) = 0_m$.

When $\alpha = 0$, $\Upsilon(R(\alpha E_{ij} + \overline{\alpha} E_{ji} - \alpha\overline{\alpha}\Lambda_{ij})R^H) = \Upsilon(R(0E_{ij} + \overline{0}E_{ji} - 0\overline{0}\Lambda_{ij})R^H) = \Upsilon(0_n) = 0_m$. Now we consider that $\alpha \neq 0$. By the fact that $C_{n-1}(\Lambda_{n+1-i,n+1-j} + (-1)^{i+j+1}(\alpha E_{n+1-j,n+1-i} + \overline{\alpha} E_{n+1-i,n+1-j})) = \alpha E_{ij} + \overline{\alpha} E_{ji} - \alpha\overline{\alpha}\Lambda_{ij}$ where $1 \leqslant i < j \leqslant n$. Therefore

$$\begin{aligned}
&\Upsilon(R(\alpha E_{ij} + \overline{\alpha} E_{ji} - \alpha\overline{\alpha}\Lambda_{ij})R^H) \\
&= \Upsilon(C_{n-1}(\mathcal{K})C_{n-1}(\Lambda_{n+1-i,n+1-j} + (-1)^{i+j+1}(\alpha E_{n+1-j,n+1-i} + \overline{\alpha} E_{n+1-i,n+1-j}))C_{n-1}(\mathcal{K}^H)) \\
&= \Upsilon(C_{n-1}(\mathcal{K}(\Lambda_{n+1-i,n+1-j} + (-1)^{i+j+1}(\alpha E_{n+1-j,n+1-i} + \overline{\alpha} E_{n+1-i,n+1-j}))\mathcal{K}^H)) \\
&= C_{m-1}(\Upsilon(\mathcal{K}(\Lambda_{n+1-i,n+1-j} + (-1)^{i+j+1}(\alpha E_{n+1-j,n+1-i} + \overline{\alpha} E_{n+1-i,n+1-j}))\mathcal{K}^H)) \\
&= C_{m-1}(\Psi_{\mathcal{K}}(\Lambda_{n+1-i,n+1-j} + (-1)^{i+j+1}(\alpha E_{n+1-j,n+1-i} + \overline{\alpha} E_{n+1-i,n+1-j}))) \\
&= C_{m-1}(\Psi_{\mathcal{K}}((-1)^{i+j+1}(\alpha E_{n+1-j,n+1-i} + \overline{\alpha} E_{n+1-i,n+1-j}) - (-\Lambda_{n+1-i,n+1-j}))).
\end{aligned}$$

So we have

$$\begin{aligned}
&\Upsilon(R(\alpha E_{ij} + \overline{\alpha} E_{ji} - \alpha\overline{\alpha}\Lambda_{ij})R^H) \\
&= C_{m-1}(\Psi_{\mathcal{K}}((-1)^{i+j+1}(\alpha E_{n+1-j,n+1-i} + \overline{\alpha} E_{n+1-i,n+1-j})) - \Psi_{\mathcal{K}}(-\Lambda_{n+1-i,n+1-j})) \\
&= C_{m-1}(\Psi_{\mathcal{K}}((-1)^{i+j+1}(\alpha E_{n+1-j,n+1-i} + \overline{\alpha} E_{n+1-i,n+1-j}))).
\end{aligned}$$

For the reason that $\mathrm{rk}((-1)^{i+j+1}(\alpha E_{n+1-j,n+1-i}+\overline{\alpha}E_{n+1-i,n+1-j})) = 2$, it follows that $\mathrm{rk}(\Psi_{\mathcal{K}}((-1)^{i+j+1}(\alpha E_{n+1-j,n+1-i}+\overline{\alpha}E_{n+1-i,n+1-j}))) \leqslant m-2$ for all $n > 3$. Evidently for all $n > 3$, $\Upsilon(R(\alpha E_{ij}+\overline{\alpha}E_{ji}-\alpha\overline{\alpha}\Lambda_{ij})R^H) = C_{m-1}(\Psi_{\mathcal{K}}((-1)^{i+j+1}(\alpha E_{n+1-j,n+1-i}+ \overline{\alpha}E_{n+1-i,n+1-j}))) = 0_m$ as $\mathrm{rk}(C_{m-1}(\Psi_{\mathcal{K}}((-1)^{i+j+1}(\alpha E_{n+1-j,n+1-i}+\overline{\alpha}E_{n+1-i,n+1-j})))$ $) = 0$. Now we consider for $n = 3$. Now we list down all the three possible of $\alpha E_{4-j,4-i} + \overline{\alpha} E_{4-i,4-j}$ which are

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & \overline{\alpha} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ \overline{\alpha} & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & \alpha & 0 \\ \overline{\alpha} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

From these three matrices, we get a simple result that is $C_2(\alpha E_{4-j,4-i} + \overline{\alpha} E_{4-i,4-j}) = -\alpha\overline{\alpha} E_{kk}$ where $k \neq i, j$. Apparently $-\alpha\overline{\alpha} \in \mathbb{F}^-$. This leads to

$$\begin{aligned}
&\Upsilon(R(\alpha E_{ij} + \overline{\alpha} E_{ji} - \alpha\overline{\alpha}\Lambda_{ij})R^H) \\
&= C_{m-1}(\Psi_{\mathcal{K}}((-1)^{i+j+1}(\alpha E_{4-j,4-i} + \overline{\alpha} E_{4-i,4-j}))) \\
&= C_{m-1}(\Upsilon(\mathcal{K}((-1)^{i+j+1}(\alpha E_{4-j,4-i} + \overline{\alpha} E_{4-i,4-j}))\mathcal{K}^H)) \\
&= \Upsilon(C_2(\mathcal{K}((-1)^{i+j+1}(\alpha E_{4-j,4-i} + \overline{\alpha} E_{4-i,4-j}))\mathcal{K}^H)) \\
&= \Upsilon((-1)^{(i+j+1)(2)} C_2(\mathcal{K}(\alpha E_{4-j,4-i} + \overline{\alpha} E_{4-i,4-j})\mathcal{K}^H)) \\
&= \Upsilon(C_2(\mathcal{K}(\alpha E_{4-j,4-i} + \overline{\alpha} E_{4-i,4-j})\mathcal{K}^H)) \\
&= \Upsilon(C_2(\mathcal{K})C_2(\alpha E_{4-j,4-i} + \overline{\alpha} E_{4-i,4-j})C_2(\mathcal{K}^H)) \\
&= \Upsilon(R(-\alpha\overline{\alpha}E_{kk})R^H)) \\
&= 0_m.
\end{aligned}$$

Claim 8. For all $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) \leqslant 1$, $\Psi_{\mathcal{K}}(A) = 0_m$.

When $\mathrm{rk}(A) = 0$, $A = 0_n$. Thus $\Psi_{\mathcal{K}}(A) = \Psi_{\mathcal{K}}(0_n) = \Upsilon(0_n) = 0_m$. Now we assume that $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) = 1$. Since $\lambda^{-1}A \in \mathbb{H}_n(\mathbb{F})$ and $\mathrm{rk}(\lambda^{-1}A) = 1$, according to **Lemma 3.4.2**, there exists a matrix $\mathcal{Q} \in \mathbb{H}_n(\mathbb{F})$ such that $\lambda^{-1}A =$

$C_{n-1}(\mathcal{Q})$ where $\mathrm{rk}(\mathcal{Q}) = n-1$. Also, we observe that $|\mathcal{K}|^{n-2}\mathcal{K} = C_{n-1}(R)$ as $C_{n-1}(C_{n-1}(\mathcal{K})) = C_{n-1}(R)$. Thus we know that

$$\begin{aligned}
&\Psi_{\mathcal{K}}(A)\\
&= \Upsilon(\mathcal{K}A\mathcal{K}^H)\\
&= \Upsilon(\lambda\lambda^{-1}\mathcal{K}A\mathcal{K}^H)\\
&= \Upsilon((|\mathcal{K}|^{n-2}\mathcal{K})(\lambda^{-1}A)(|\mathcal{K}^H|^{n-2}\mathcal{K}^H))\\
&= \Upsilon(C_{n-1}(R)C_{n-1}(\mathcal{Q})C_{n-1}(R^H))\\
&= \Upsilon(C_{n-1}(R\mathcal{Q}R^H))\\
&= C_{m-1}(\Upsilon(R\mathcal{Q}R^H)).
\end{aligned}$$

Furthermore, by the definition of Hermitian matrices, for all $1 \leqslant i < j \leqslant n$ and $1 \leqslant i \leqslant n$, $\mathcal{Q}_{ij} = \overline{\mathcal{Q}_{ji}}$ and $\mathcal{Q}_{ii} \in \mathbb{F}^-$. So we have

$$\begin{aligned}
&\Psi_{\mathcal{K}}(A)\\
&= C_{m-1}(\Upsilon(R(\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\mathcal{Q}_{ij}E_{ij} + \mathcal{Q}_{ji}E_{ji}) + \sum_{i=1}^{n}\mathcal{Q}_{ii}E_{ii})R^H))\\
&= C_{m-1}(\Upsilon(R(\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\mathcal{Q}_{ij}E_{ij} + \overline{\mathcal{Q}_{ij}}E_{ji}))R^H - R(-\sum_{i=1}^{n}\mathcal{Q}_{ii}E_{ii})R^H))\\
&= C_{m-1}(\Upsilon(R(\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\mathcal{Q}_{ij}E_{ij} + \overline{\mathcal{Q}_{ij}}E_{ji}))R^H) - \Upsilon(R(-\sum_{i=1}^{n}\mathcal{Q}_{ii}E_{ii})R^H))\\
&= C_{m-1}(\Upsilon(R(\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(\mathcal{Q}_{ij}E_{ij} + \overline{\mathcal{Q}_{ij}}E_{ji}))R^H))\\
&= C_{m-1}(\Upsilon(R(\sum_{i=1}^{n-1}\sum_{\substack{j=i+1,\\ j\neq 2\\ \text{when } i=1}}^{n}(\mathcal{Q}_{ij}E_{ij} + \overline{\mathcal{Q}_{ij}}E_{ji}) + \mathcal{Q}_{12}E_{12} + \overline{\mathcal{Q}_{12}}E_{21})R^H))\\
&= C_{m-1}(\Upsilon(R(\sum_{i=1}^{n-1}\sum_{\substack{j=i+1,\\ j\neq 2\\ \text{when } i=1}}^{n}(\mathcal{Q}_{ij}E_{ij} + \overline{\mathcal{Q}_{ij}}E_{ji}) + (\mathcal{Q}_{12}E_{12} + \overline{\mathcal{Q}_{12}}E_{21} + \mathcal{Q}_{12}\overline{\mathcal{Q}_{12}}\Lambda_{12})\\
&\quad - \mathcal{Q}_{12}\overline{\mathcal{Q}_{12}}\Lambda_{12})R^H))\\
&= C_{m-1}(\Upsilon(R(\sum_{i=1}^{n-1}\sum_{\substack{j=i+1,\\ j\neq 2\\ \text{when } i=1}}^{n}(\mathcal{Q}_{ij}E_{ij} + \overline{\mathcal{Q}_{ij}}E_{ji}) + (\mathcal{Q}_{12}E_{12} + \overline{\mathcal{Q}_{12}}E_{21} + \mathcal{Q}_{12}\overline{\mathcal{Q}_{12}}\Lambda_{12}))R^H\\
&\quad - R(\mathcal{Q}_{12}\overline{\mathcal{Q}_{12}}\Lambda_{12})R^H)).
\end{aligned}$$

It follows that

$$
\begin{aligned}
&\Psi_{\mathcal{K}}(A)\\
&= C_{m-1}(\Upsilon(R(\sum_{i=1}^{n-1}\sum_{\substack{j=i+1,\\ j\neq 2\\ \text{when } i=1}}^{n}(\mathcal{Q}_{ij}E_{ij}+\overline{\mathcal{Q}_{ij}}E_{ji})+(\mathcal{Q}_{12}E_{12}+\overline{\mathcal{Q}_{12}}E_{21}+\mathcal{Q}_{12}\overline{\mathcal{Q}_{12}}\Lambda_{12}))R^{H})\\
&\quad -\Upsilon(R(\mathcal{Q}_{12}\overline{\mathcal{Q}_{12}}\Lambda_{12})R^{H}))\\
&= C_{m-1}(\Upsilon(R(\sum_{i=1}^{n-1}\sum_{\substack{j=i+1,\\ j\neq 2\\ \text{when } i=1}}^{n}(\mathcal{Q}_{ij}E_{ij}+\overline{\mathcal{Q}_{ij}}E_{ji})+(\mathcal{Q}_{12}E_{12}+\overline{\mathcal{Q}_{12}}E_{21}+\mathcal{Q}_{12}\overline{\mathcal{Q}_{12}}\Lambda_{12}))R^{H}))\\
&= C_{m-1}(\Upsilon(R(\sum_{i=1}^{n-1}\sum_{\substack{j=i+1,\\ j\neq 2\\ \text{when } i=1}}^{n}(\mathcal{Q}_{ij}E_{ij}+\overline{\mathcal{Q}_{ij}}E_{ji}))R^{H}\\
&\quad -R((-\mathcal{Q}_{12}E_{12})+(-\overline{\mathcal{Q}_{12}}E_{21})-(-\mathcal{Q}_{12})(-\overline{\mathcal{Q}_{12}})\Lambda_{12})R^{H}))\\
&= C_{m-1}(\Upsilon(R(\sum_{i=1}^{n-1}\sum_{\substack{j=i+1,\\ j\neq 2\\ \text{when } i=1}}^{n}(\mathcal{Q}_{ij}E_{ij}+\overline{\mathcal{Q}_{ij}}E_{ji}))R^{H})\\
&\quad -\Upsilon(R((-\mathcal{Q}_{12}E_{12})+(-\overline{\mathcal{Q}_{12}}E_{21})-(-\mathcal{Q}_{12})(-\overline{\mathcal{Q}_{12}})\Lambda_{12})R^{H}))\\
&= C_{m-1}(\Upsilon(R(\sum_{i=1}^{n-1}\sum_{\substack{j=i+1,\\ j\neq 2\\ \text{when } i=1}}^{n}(\mathcal{Q}_{ij}E_{ij}+\overline{\mathcal{Q}_{ij}}E_{ji}))R^{H})).
\end{aligned}
$$

By using the same arguments, we have

$$\Upsilon(A)=C_{m-1}\left(\Upsilon\left(R\left(\sum_{i=1}^{n-1}\sum_{\substack{j=i+1,\\ j\neq 2,3\\ \text{when } i=1}}^{n}\left(\mathcal{Q}_{ij}E_{ij}+\overline{\mathcal{Q}_{ij}}E_{ji}\right)\right)R^{H}\right)\right).$$

By using the same arguments again, we obtain

$$\Upsilon(A)=C_{m-1}\left(\Upsilon\left(R\left(\sum_{i=1}^{n-1}\sum_{\substack{j=i+1,\\ j\neq 2,3,\cdots,n\\ \text{when } i=1}}^{n}\left(\mathcal{Q}_{ij}E_{ij}+\overline{\mathcal{Q}_{ij}}E_{ji}\right)\right)R^{H}\right)\right).$$

This implies

$$\Upsilon(A)=C_{m-1}\left(\Upsilon\left(R\left(\sum_{i=2}^{n-1}\sum_{j=i+1}^{n}\left(\mathcal{Q}_{ij}E_{ij}+\overline{\mathcal{Q}_{ij}}E_{ji}\right)\right)R^{H}\right)\right).$$

By repeating the same process, we acquire

$$\begin{aligned}\Upsilon(A) &= C_{m-1}\left(\Upsilon\left(R\left(\sum_{i=2}^{n-1}\sum_{\substack{j=i+1,\\ j\neq 3,4,\cdots,n\\ \text{when } i=2}}^{n}\left(\mathcal{Q}_{ij}E_{ij}+\overline{\mathcal{Q}_{ij}}E_{ji}\right)\right)R^H\right)\right)\\ &= C_{m-1}\left(\Upsilon\left(R\left(\sum_{i=3}^{n-1}\sum_{j=i+1}^{n}\left(\mathcal{Q}_{ij}E_{ij}+\overline{\mathcal{Q}_{ij}}E_{ji}\right)\right)R^H\right)\right).\end{aligned}$$

Continue in this fashion, finally we arrive at

$$\begin{aligned}&\Upsilon(A)\\ &= C_{m-1}(\Upsilon(R(\mathcal{Q}_{n-1,n}E_{n-1,n}+\overline{\mathcal{Q}_{n-1,n}}E_{n,n-1})R^H))\\ &= C_{m-1}(\Upsilon(R(\mathcal{Q}_{n-1,n}E_{n-1,n}+\overline{\mathcal{Q}_{n-1,n}}E_{n,n-1}+\mathcal{Q}_{n-1,n}\overline{\mathcal{Q}_{n-1,n}}\Lambda_{n-1,n}\\ &\quad -\mathcal{Q}_{n-1,n}\overline{\mathcal{Q}_{n-1,n}}\Lambda_{n-1,n})R^H))\\ &= C_{m-1}(\Upsilon(R(\mathcal{Q}_{n-1,n}E_{n-1,n}+\overline{\mathcal{Q}_{n-1,n}}E_{n,n-1}+\mathcal{Q}_{n-1,n}\overline{\mathcal{Q}_{n-1,n}}\Lambda_{n-1,n})R^H\\ &\quad -R(\mathcal{Q}_{n-1,n}\overline{\mathcal{Q}_{n-1,n}}\Lambda_{n-1,n})R^H))\\ &= C_{m-1}(\Upsilon(R(\mathcal{Q}_{n-1,n}E_{n-1,n}+\overline{\mathcal{Q}_{n-1,n}}E_{n,n-1}+\mathcal{Q}_{n-1,n}\overline{\mathcal{Q}_{n-1,n}}\Lambda_{n-1,n})R^H)\\ &\quad -\Upsilon(R(\mathcal{Q}_{n-1,n}\overline{\mathcal{Q}_{n-1,n}}\Lambda_{n-1,n})R^H))\\ &= C_{m-1}(\Upsilon(R(\mathcal{Q}_{n-1,n}E_{n-1,n}+\overline{\mathcal{Q}_{n-1,n}}E_{n,n-1}+\mathcal{Q}_{n-1,n}\overline{\mathcal{Q}_{n-1,n}}\Lambda_{n-1,n})R^H))\\ &= C_{m-1}(\Upsilon(0_n-R((-\mathcal{Q}_{n-1,n}E_{n-1,n})+(-\overline{\mathcal{Q}_{n-1,n}}E_{n,n-1})\\ &\quad -(-\mathcal{Q}_{n-1,n})(-\overline{\mathcal{Q}_{n-1,n}})\Lambda_{n-1,n})R^H))\\ &= C_{m-1}(\Upsilon(0_n)-\Upsilon(R((-\mathcal{Q}_{n-1,n}E_{n-1,n})+(-\overline{\mathcal{Q}_{n-1,n}}E_{n,n-1})\\ &\quad -(-\mathcal{Q}_{n-1,n})(-\overline{\mathcal{Q}_{n-1,n}})\Lambda_{n-1,n})R^H))\\ &= C_{m-1}(\Upsilon(0_n))\\ &= C_{m-1}(0_m)\\ &= 0_m.\end{aligned}$$

(c) $\Rightarrow$ (d)

Now we have the last claim.

Claim 9. For all $A\in\mathbb{H}_n(\mathbb{F})$, $\mathrm{rk}(\Psi_{\mathcal{K}}(A))\leqslant m-2$.

Presume that $\mathrm{rk}(A)=r$. Together with the fact that $\Psi_{\mathcal{K}}(A)=0_m$ when $r\in\{0,1\}$. Undoubtedly $\mathrm{rk}(\Psi_{\mathcal{K}}(A))\leqslant m-2$ when $r\in\{0,1\}$. In case $2\leqslant r\leqslant n$, from **Lemma 2.1.8.5**, there exist at least one $A_1,A_2,\cdots,A_t\in\mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A_1)=$

$\mathrm{rk}(A_2) = \cdots = \mathrm{rk}(A_t) = 1$ such that $A = A_1 + A_2 + \cdots + A_t$ where $t \in \{r, r+1\}$. This leads to

$$\begin{aligned}
&C_{m-1}(\Psi_{\mathcal{K}}(A)) \\
&= C_{m-1}(\Psi_{\mathcal{K}}(A_1 + A_2 + \cdots + A_{t-1} + A_t)) \\
&= C_{m-1}(\Psi_{\mathcal{K}}(A_1 - (-A_2) - \cdots - (-A_{t-1}) - (-A_t))) \\
&= C_{m-1}(\Psi_{\mathcal{K}}(A_1 - (-A_2) - \cdots - (-A_{t-1})) - \Psi_{\mathcal{K}}(-A_t)) \\
&= C_{m-1}(\Psi_{\mathcal{K}}(A_1 - (-A_2) - \cdots - (-A_{t-1}))).
\end{aligned}$$

By repeating the same process, we attain

$$\begin{aligned}
&C_{m-1}(\Psi_{\mathcal{K}}(A)) \\
&= C_{m-1}(\Psi_{\mathcal{K}}(A_1 - (-A_2))) \\
&= C_{m-1}(\Psi_{\mathcal{K}}(A_1) - \Psi_{\mathcal{K}}(-A_2)) \\
&= C_{m-1}(0_m) \\
&= 0_m.
\end{aligned}$$

Hence $\mathrm{rk}(\Psi_{\mathcal{K}}(A)) \leqslant m-2$ as $\mathrm{rk}(C_{m-1}(\Psi_{\mathcal{K}}(A))) = 0$. Consequently, $\mathrm{rk}(\Psi_{\mathcal{K}}(A)) \leqslant m-2$ for all $A \in \mathbb{H}_n(\mathbb{F})$.

(d) $\Rightarrow$ (e)

For all $A \in \mathbb{H}_n(\mathbb{F})$, $\Upsilon(C_{n-1}(\mathcal{K}A\mathcal{K}^H)) = C_{m-1}(\Upsilon(\mathcal{K}A\mathcal{K}^H)) = C_{m-1}(\Psi_{\mathcal{K}}(A)) = 0_m$ as $\mathrm{rk}(\Psi_{\mathcal{K}}(A)) \leqslant m-2$.

(e) $\Rightarrow$ (a)

When $A = I_n$, then $\Upsilon(C_{n-1}(\mathcal{K}I_n\mathcal{K}^H)) = \Upsilon(C_{n-1}(\mathcal{K}\mathcal{K}^H)) = C_{m-1}(\Upsilon(\mathcal{K}\mathcal{K}^H)) = 0_m$. It follows that $\mathrm{rk}(\Upsilon(\mathcal{K}\mathcal{K}^H)) \leqslant m-2$. So we get $\mathrm{rk}(\Upsilon(\mathcal{K}\mathcal{K}^H)) \neq m$. We are done. $\square$

The **Corollary 3.4.6** is direct follow from **Lemma 3.4.5**. This corollary shows us that the result if $\Upsilon(I_n) = 0_m$.

Corollary 3.4.6. Let $\mathbb{F}$ be a field carrying an involution $-$ and $m, n \in \mathbb{N}$ with $m, n > 2$. If a mapping $\Upsilon\colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ satisfies [P1], then the following statements are equivalent.

(a) $\Upsilon(I_n) = 0_m$.

(b) $\Upsilon(A) = 0_m$ for all $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) \leqslant 1$.

(c) $\mathrm{rk}(\Upsilon(A)) \leqslant m-2$ for all $A \in \mathbb{H}_n(\mathbb{F})$.

(d) $\Upsilon(C_{n-1}(A)) = 0_m$ for all $A \in \mathbb{H}_n(\mathbb{F})$.

Proof.

When we take $\mathcal{K} = I_n$ in **Lemma 3.4.5**, we attain (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d). Furthermore, (d) $\Rightarrow$ (a) is also true as $\Upsilon(C_{n-1}(I_n)) = \Upsilon(I_n) = 0_m$.

$\square$

However, we are also very interested to know what properties ~~can~~ we have if $\Upsilon(I_n) \neq 0_m$.

Lemma 3.4.7. Let $\mathbb{F}$ be a field carrying an involution $-$ and $m, n \in \mathbb{N}$ with $m, n > 2$. If a mapping $\Upsilon\colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ satisfies [P1], then the following statements are equivalent.

(a) $\Upsilon(I_n) \neq 0_m$.

(b) $\ker(\Upsilon) = \{0_n\}$.

(c) $\mathrm{rk}(A) = n$ if and only if $\mathrm{rk}(\Upsilon(A)) = m$.

(d) $\mathrm{rk}(A - B) = n$ if and only if $\mathrm{rk}(\Upsilon(A) - \Upsilon(B)) = m$.

(e) Υ is one-to-one.

Proof.

(a) $\Rightarrow$ (b)

Since $C_{n-1}(I_n) = I_n$, then $\Upsilon(I_n) = \Upsilon(C_{n-1}(I_n)) = \Upsilon(C_{n-1}(C_{n-1}(I_n))) = C_{m-1}(C_{m-1}(\Upsilon(I_n)))$. Suppose that $\mathrm{rk}(\Upsilon(I_n)) \neq m$, thus $\mathrm{rk}(C_{m-1}(\Upsilon(I_n))) \leqslant 1$. Hence $\Upsilon(I_n) = 0_m$ as $\mathrm{rk}(\Upsilon(I_n)) = \mathrm{rk}(C_{m-1}(C_{m-1}(\Upsilon(I_n)))) = 0$. It is impossible due to $\Upsilon(I_n) \neq 0_m$. So, we ~~are forced to~~ conclude that $\mathrm{rk}(\Upsilon(I_n)) = m$. Now we let $\mathcal{K}$ be an arbitrary constant invertible matrix in $\mathbb{M}_n(\mathbb{F})$ and $\Psi_{\mathcal{K}}(A) = \Upsilon(\mathcal{K}A\mathcal{K}^H)$ for all $A \in \mathbb{H}_n(\mathbb{F})$. Under **Lemma 3.4.5**, we see that if $\mathrm{rk}(\Psi_{\mathcal{K}}(I_n)) = \mathrm{rk}(\Upsilon(\mathcal{K}\mathcal{K}^H)) \neq m$ happens, then $\mathrm{rk}(\Upsilon(I_n)) = \mathrm{rk}(\Upsilon(\mathcal{K}(\mathcal{K}^H\mathcal{K})^{-1}\mathcal{K}^H)) \leqslant m-2$, it is impossible because we get $m \leqslant m-2$. Thus $\mathrm{rk}(\Psi_{\mathcal{K}}(I_n))$ must be equal to m. Of course, $\mathrm{rk}(C_{m-1}(\Psi_{\mathcal{K}}(I_n)))$

is also equal to m. From the process of **Lemma 3.4.5** proof, we already know that for all $A, B \in \mathbb{H}_n(\mathbb{F})$, the following statements are true.

- $\mathrm{rk}(\Psi_{\mathcal{K}}(A)) \leqslant m-1$ when $\mathrm{rk}(A) = n-1$.
- $\mathrm{rk}(\Psi_{\mathcal{K}}(A)) \leqslant m-2$ when $\mathrm{rk}(A) \leqslant n-2$.
- $\mathrm{rk}(\Psi_{\mathcal{K}}(A)) \leqslant 1$ when $\mathrm{rk}(A) \leqslant 1$.
- $C_{m-1}(\Upsilon(A-B)) = C_{m-1}(\Upsilon(A) - \Upsilon(B))$.
- $C_{m-1}(\Psi_{\mathcal{K}}(A-B)) = C_{m-1}(\Psi_{\mathcal{K}}(A) - \Psi_{\mathcal{K}}(B))$.

In order to prove ~~proof~~ this lemma, we need to claim the statement one by one and prove ~~proof~~ all the statement we claim is true. For the sake of simplicity, we let $R = C_{n-1}(\mathcal{K})$.

Claim 1. For all $i \in \{1, 2, \cdots, n\}, \mathrm{rk}(\Psi_{\mathcal{K}}(E_{ii})) = 1$.

$$\begin{aligned}
m &= \mathrm{rk}(C_{m-1}(\Psi_{\mathcal{K}}(I_n))) \\
&= \mathrm{rk}\left(C_{m-1}\left(\Psi_{\mathcal{K}}\left(E_{ii} + \sum_{\substack{k=1,\\ k\neq i}}^{n} E_{kk}\right)\right)\right) \\
&= \mathrm{rk}\left(C_{m-1}\left(\Psi_{\mathcal{K}}\left(E_{ii} - \sum_{\substack{k=1,\\ k\neq i}}^{n} (-E_{kk})\right)\right)\right) \\
&= \mathrm{rk}\left(C_{m-1}\left(\Psi_{\mathcal{K}}\left(E_{ii}\right) - \Psi_{\mathcal{K}}\left(\sum_{\substack{k=1,\\ k\neq i}}^{n} (-E_{kk})\right)\right)\right).
\end{aligned}$$

This leads to $\mathrm{rk}\left(\Psi_{\mathcal{K}}\left(E_{ii}\right) - \Psi_{\mathcal{K}}\left(\sum_{\substack{k=1,\\ k\neq i}}^{n} (-E_{kk})\right)\right) = m$. Thus

$$\begin{aligned}
m &= \mathrm{rk}\left(\Psi_{\mathcal{K}}\left(E_{ii}\right) - \Psi_{\mathcal{K}}\left(\sum_{\substack{k=1,\\ k\neq i}}^{n} (-E_{kk})\right)\right) \\
&= \mathrm{rk}\left(\Psi_{\mathcal{K}}\left(E_{ii}\right) + \left(-\Psi_{\mathcal{K}}\left(\sum_{\substack{k=1,\\ k\neq i}}^{n} (-E_{kk})\right)\right)\right).
\end{aligned}$$

It follows that

$$\begin{aligned} m &\leqslant \mathrm{rk}\left(\Psi_{\mathcal{K}}\left(E_{ii}\right)\right) + \mathrm{rk}\left(-\Psi_{\mathcal{K}}\left(\sum_{\substack{k=1,\\ k\neq i}}^{n}(-E_{kk})\right)\right) \\ &= \mathrm{rk}\left(\Psi_{\mathcal{K}}\left(E_{ii}\right)\right) + \mathrm{rk}\left(\Psi_{\mathcal{K}}\left(\sum_{\substack{k=1,\\ k\neq i}}^{n}(-E_{kk})\right)\right). \end{aligned}$$

This implies $\mathrm{rk}\left(\Psi_{\mathcal{K}}\left(E_{ii}\right)\right) + \mathrm{rk}\left(\Psi_{\mathcal{K}}\left(\sum_{\substack{k=1,\\ k\neq i}}^{n}(-E_{kk})\right)\right) \geqslant m$. Together with the fact that $\mathrm{rk}\left(\Psi_{\mathcal{K}}\left(E_{ii}\right)\right) \leqslant 1$ and $\mathrm{rk}\left(\Psi_{\mathcal{K}}\left(\sum_{\substack{k=1,\\ k\neq i}}^{n}(-E_{kk})\right)\right) \leqslant m-1$ as $\mathrm{rk}(E_{ii}) = 1$ and $\mathrm{rk}\left(\sum_{\substack{k=1,\\ k\neq i}}^{n}(-E_{kk})\right) = n-1$, we are obliged to conclude that $\mathrm{rk}\left(\Psi_{\mathcal{K}}\left(E_{ii}\right)\right) = 1$.

Claim 2. For all $\beta \in \mathbb{F}$ with $\beta \neq 0, C_{n-1}(I_n - E_{n+1-i,n+1-i} - E_{jj} - E_{kk} - \beta E_{jj} + \beta^{-1}E_{kk}) = -E_{ii}$ where $1 \leqslant i,j,k \leqslant n$ and $n+1-i,j,k$ are three different integers.

To simplify writing, we let $G = C_{n-1}(I_n - E_{n+1-i,n+1-i} - E_{jj} - E_{kk} - \beta E_{jj} + \beta^{-1}E_{kk})$. Since $n+1-i,j,k$ are three different integers, then $(n+1-i)$-th row and $(n+1-i)$-th column of G are a row of zeros and a column of zeros, respectively. Let D be a submatrix of G where D is obtained by eliminating x-th row and y-th column from G. We see that if D is obtained by not eliminating $(n+1-i)$-th row or $(n+1-i)$-th column from G, then there exist a row or a column of zeros in D. Thus $|D|= 0$. This tells us $(C_{n-1}(G))_{st} = |G[n+1-s \mid n+1-t]|= |D|= 0$ for all $s \neq i$ and $t \neq i$. But if D is obtained by eliminating $(n+1-i)$-th row and $(n+1-i)$-th column from G, hence D is a diagonal matrix with $D_{uu} = -\beta$ and $D_{vv} = \beta^{-1}$ for some $1 \leqslant u \neq v \leqslant n-1$. Except for D_{uu} and D_{vv}, all the diagonal entries of D are 1. Consequently, $|D|= \underbrace{1(1)\cdots(1)}_{n-3 \text{ times of } 1}(-\beta)(\beta^{-1}) = -1$. This leads to $(C_{n-1}(G))_{ii} = |G[n+1-i \mid n+1-i]|= |D|= -1$. By combining these two cases, we get the result we desired.

Claim 3. $\mathrm{rk}(\Psi_{\mathcal{K}}(\alpha E_{ii})) = 1$ for every non-zero element $\alpha \in \mathbb{F}^-$ and $i \in \{1,2,\cdots,n\}$.

$\mathrm{rk}(\Psi_{\mathcal{K}}(\alpha E_{ii})) \leqslant 1$ as $\mathrm{rk}(\alpha E_{ii}) = 1$. Now we assume that there exist some $\beta \in \mathbb{F}^-$ with $\beta \neq 0$ and $q \in \{1, 2, \cdots, n\}$ such that $\mathrm{rk}(\Psi_{\mathcal{K}}(\beta E_{qq})) = 0$. In other words, $\Psi_{\mathcal{K}}(\beta E_{qq}) = 0_m$. By the fact that $C_{n-1}(I_n - E_{n+1-p,n+1-p} - E_{qq} - E_{rr} - \beta E_{qq} + \beta^{-1}E_{rr}) = -E_{pp}$ where $1 \leqslant p, q, r \leqslant n$ and $n+1-p, q, r$ are three different integers. It follows that

$$\begin{aligned}
&\Upsilon(R(-E_{pp})R^H)\\
&= \Upsilon(C_{n-1}(\mathcal{K})C_{n-1}(I_n - E_{n+1-p,n+1-p} - E_{qq} - E_{rr} - \beta E_{qq} + \beta^{-1}E_{rr})C_{n-1}(\mathcal{K}^H))\\
&= \Upsilon(C_{n-1}(\mathcal{K}(I_n - E_{n+1-p,n+1-p} - E_{qq} - E_{rr} - \beta E_{qq} + \beta^{-1}E_{rr})\mathcal{K}^H))\\
&= C_{m-1}(\Upsilon(\mathcal{K}(I_n - E_{n+1-p,n+1-p} - E_{qq} - E_{rr} - \beta E_{qq} + \beta^{-1}E_{rr})\mathcal{K}^H))\\
&= C_{m-1}(\Psi_{\mathcal{K}}(I_n - E_{n+1-p,n+1-p} - E_{qq} - E_{rr} - \beta E_{qq} + \beta^{-1}E_{rr}))\\
&= C_{m-1}(\Psi_{\mathcal{K}}(I_n - E_{n+1-p,n+1-p} - E_{qq} - E_{rr} + \beta^{-1}E_{rr}) - \Psi_{\mathcal{K}}(\beta E_{qq}))\\
&= C_{m-1}(\Psi_{\mathcal{K}}(I_n - E_{n+1-p,n+1-p} - E_{qq} - E_{rr} + \beta^{-1}E_{rr}))\\
&= C_{m-1}(\Psi_{\mathcal{K}}(I_n - E_{n+1-p,n+1-p} - E_{qq} + (\beta^{-1} - 1)E_{rr}))\\
&= C_{m-1}(\Upsilon(\mathcal{K}(I_n - E_{n+1-p,n+1-p} - E_{qq} + (\beta^{-1} - 1)E_{rr})\mathcal{K}^H))\\
&= \Upsilon(C_{n-1}(\mathcal{K}(I_n - E_{n+1-p,n+1-p} - E_{qq} + (\beta^{-1} - 1)E_{rr})\mathcal{K}^H))\\
&= \Upsilon(C_{n-1}(\mathcal{K})C_{n-1}(I_n - E_{n+1-p,n+1-p} - E_{qq} + (\beta^{-1} - 1)E_{rr})C_{n-1}(\mathcal{K}^H)).
\end{aligned}$$

$C_{n-1}(I_n - E_{n+1-p,n+1-p} - E_{qq} + (\beta^{-1} - 1)E_{rr}) = 0_n$ as $\mathrm{rk}(I_n - E_{n+1-p,n+1-p} - E_{qq} + (\beta^{-1} - 1)E_{rr}) = n - 2$. So we obtain $\Upsilon(R(-E_{pp})R^H) = \Upsilon(0_n) = 0_m$. By using the same arguments, we have $\Upsilon(R(-E_{ww})R^H) = 0_m$ where $1 \leqslant w \leqslant n$ and p, w are two different integers. This implies

$$\begin{aligned}
&C_{m-1}(\Upsilon(RR^H))\\
&= C_{m-1}(\Upsilon(RI_nR^H))\\
&= C_{m-1}(\Upsilon(R(I_n - E_{pp} - E_{ww} + E_{pp} + E_{ww}))R^H)\\
&= C_{m-1}(\Upsilon(R(I_n - E_{pp} - E_{ww} + E_{pp})R^H - R(-E_{ww})R^H))\\
&= C_{m-1}(\Upsilon(R(I_n - E_{pp} - E_{ww} + E_{pp})R^H) - \Upsilon(R(-E_{ww})R^H))\\
&= C_{m-1}(\Upsilon(R(I_n - E_{pp} - E_{ww} + E_{pp})R^H))\\
&= C_{m-1}(\Upsilon(R(I_n - E_{pp} - E_{ww})R^H - R(-E_{pp})R^H))\\
&= C_{m-1}(\Upsilon(R(I_n - E_{pp} - E_{ww})R^H - \Upsilon(R(-E_{pp})R^H)))\\
&= C_{m-1}(\Upsilon(R(I_n - E_{pp} - E_{ww})R^H)).
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&C_{m-1}(\Upsilon(RR^H)) \\
&= \Upsilon(C_{n-1}(R(I_n - E_{pp} - E_{ww})R^H)) \\
&= \Upsilon(C_{n-1}(R)C_{n-1}(I_n - E_{pp} - E_{ww})C_{n-1}(R^H)).
\end{aligned}$$

As a reason that $\mathrm{rk}(I_n - E_{pp} - E_{ww}) = m-2$, then $C_{n-1}(I_n - E_{pp} - E_{ww}) = 0_n$ as $\mathrm{rk}(C_{n-1}(I_n - E_{pp} - E_{ww})) = 0$. Therefore $C_{m-1}(\Upsilon(RR^H)) = \Upsilon(0_n) = 0_m$. Thus $\mathrm{rk}(\Upsilon(RR^H)) \leqslant m-2$ as $\mathrm{rk}(C_{m-1}(\Upsilon(RR^H))) = 0$. But it is impossible $\mathrm{rk}(\Upsilon(RR^H)) \leqslant m-2$. This is because

$$\begin{aligned}
\mathrm{rk}(\Upsilon(RR^H)) &= \mathrm{rk}(\Upsilon(C_{n-1}(\mathcal{K})C_{n-1}(\mathcal{K}^H))) \\
&= \mathrm{rk}(\Upsilon(C_{n-1}(\mathcal{K}\mathcal{K}^H))) \\
&= \mathrm{rk}(C_{m-1}(\Upsilon(\mathcal{K}\mathcal{K}^H))) \\
&= \mathrm{rk}(C_{m-1}(\Psi_{\mathcal{K}}(I_n))) \\
&= m.
\end{aligned}$$

This contradiction shows that our supposition is false. So we assert that **Claim 3** is true.

Claim 4. For any $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) = 1$, $\mathrm{rk}(\Upsilon(A)) = 1$.

Now we presume that $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) = 1$. According to the **Corollary 2.1.8.4**, there exist a non-zero element $\chi \in \mathbb{F}^-$ and invertible matrix $\mathcal{Q} \in \mathbb{M}_n(\mathbb{F})$ such that $A = \mathcal{Q}(\chi E_{11})\mathcal{Q}^H$. Since $\mathcal{K}$ is an arbitrary constant invertible matrix in $\mathbb{M}_n(\mathbb{F})$, thereupon we can take any invertible matrix in $\mathbb{M}_n(\mathbb{F})$ to replace $\mathcal{K}$. Then we have $\mathrm{rk}(\Upsilon(A)) = \mathrm{rk}(\Upsilon(\mathcal{Q}(\chi E_{11})\mathcal{Q}^H)) = \mathrm{rk}(\Psi_{\mathcal{Q}}(\chi E_{11})) = 1$. Now we only left one more claim need to ~~proof~~. prove

Claim 5. $\ker(\Upsilon) = \{0_n\}$.

Suppose that $\ker(\Upsilon) \neq \{0_n\}$. Accordingly there exists some $A \in \ker(\Upsilon)$ with $A \neq 0_n$ such that $\Upsilon(A) = 0_m$. From **Lemma 3.4.4**(d), we know that there exists at least one matrix $-\mathcal{Q} \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(-\mathcal{Q}) \leqslant n-2$ such that $\mathrm{rk}(A - Q) = n-1$. So we have $\mathrm{rk}(\Upsilon(C_{n-1}(A-Q))) = 1$ as $\mathrm{rk}(C_{n-1}(A-Q)) = 1$. At the same time, we observe that

$$
\begin{aligned}
&\operatorname{rk}(\Upsilon(C_{n-1}(A-Q))) \\
&= \operatorname{rk}(C_{m-1}(\Upsilon(A)-\Upsilon(Q))) \\
&= \operatorname{rk}(C_{m-1}(0_m-\Upsilon(Q))) \\
&= \operatorname{rk}(C_{m-1}(\Upsilon(0_n)-\Upsilon(Q))) \\
&= \operatorname{rk}(\Upsilon(C_{n-1}(0_n-Q))) \\
&= \operatorname{rk}(\Upsilon(C_{n-1}(-Q))) \\
&= 0
\end{aligned}
$$

as $\operatorname{rk}(-Q) \leqslant n-2$. This contradicts to the fact that $\operatorname{rk}(\Upsilon(C_{n-1}(A-Q))) = 1$. This contradiction shows that our assumption is false. So we conclude that $\ker(\Upsilon) = \{0_n\}$.

(b) $\Rightarrow$ (c)

($\Rightarrow$) Assume that $A \in \mathbb{H}_n(\mathbb{F})$ with $\operatorname{rk}(A) = n$. We also suppose that $\operatorname{rk}(\Upsilon(A)) \neq m$, thus $\operatorname{rk}(\Upsilon(A)) \leqslant m-1$, then $\operatorname{rk}(C_{m-1}(\Upsilon(A))) \leqslant 1$. Therefore $\operatorname{rk}(C_{m-1}(C_{m-1}(\Upsilon(A)))) = 0$. It follows that $C_{m-1}(C_{m-1}(\Upsilon(A))) = 0_m$. Then we get $\Upsilon(C_{n-1}(C_{n-1}(A))) = 0_m$. Since $\ker(\Upsilon) = \{0_n\}$, we obtain $C_{n-1}(C_{n-1}(A)) = 0_n$. This means $\operatorname{rk}(C_{n-1}(C_{n-1}(A))) = 0$, hence $\operatorname{rk}(C_{n-1}(A)) \leqslant n-2$. So $\operatorname{rk}(A) \leqslant n-1$. The contradiction occurs because $\operatorname{rk}(A) = n$. This contradiction shows that our supposition is false. So we conclude that $\operatorname{rk}(\Upsilon(A)) = m$.

($\Leftarrow$) Let $A \in \mathbb{H}_n(\mathbb{F})$. By the contrapositive of **Lemma 3.4.3**(a), we see that if $\operatorname{rk}(\Upsilon(A)) = m$, then $\operatorname{rk}(A) \neq n-1$. Suppose that $\operatorname{rk}(A) \leqslant n-2$, then $\operatorname{rk}(\Upsilon(A)) \leqslant m-2$. It is impossible because $\operatorname{rk}(\Upsilon(A)) = m$. This contradiction shows that our supposition is false. So we are forced to conclude that $\operatorname{rk}(A) = n$.

(c) $\Rightarrow$ (d)

Suppose that $A, B \in \mathbb{H}_n(\mathbb{F})$ with $\operatorname{rk}(A-B) = n$. Therefore

$$
\begin{aligned}
\operatorname{rk}(A-B) = n &\Leftrightarrow \operatorname{rk}(C_{n-1}(A-B)) = n \\
&\Leftrightarrow \operatorname{rk}(\Upsilon(C_{n-1}(A-B))) = m \\
&\Leftrightarrow \operatorname{rk}(C_{m-1}(\Upsilon(A)-\Upsilon(B))) = m \\
&\Leftrightarrow \operatorname{rk}(\Upsilon(A)-\Upsilon(B)) = m.
\end{aligned}
$$

(d) $\Rightarrow$ (e)

Suppose that Υ is no one-to-one. Thus there exist certain A, B belong to $\mathbb{H}_n(\mathbb{F})$ such that $\Upsilon(A) = \Upsilon(B)$ where $A \neq B$. We presume that $\operatorname{rk}(B-A) = r$. By

Lemma 3.4.4(a), there exists a matrix $\mathcal{Q} \in \mathbb{H}_n(\mathbb{F})$ such that $\mathrm{rk}((B - A) + \mathcal{Q}) = \mathrm{rk}(B - A + \mathcal{Q}) = n$ where $\mathrm{rk}(\mathcal{Q}) = n - r$. Now we would like to express $\mathcal{Q}$ in terms of A and $\mathcal{Q}$.

$$\mathcal{Q} = 0_n + \mathcal{Q} = A - A + \mathcal{Q} = A - (A - \mathcal{Q}).$$

Consequently, $C_{n-1}(\mathcal{Q}) = C_{n-1}(A - (A - \mathcal{Q}))$. Thus we obtain $\Upsilon(C_{n-1}(\mathcal{Q})) = \Upsilon(C_{n-1}(A - (A - \mathcal{Q})))$. This tell us $C_{m-1}(\Upsilon(\mathcal{Q})) = C_{m-1}(\Upsilon(A) - \Upsilon(A - \mathcal{Q}))$. Since $\Upsilon(A) = \Upsilon(B)$, then we get $C_{m-1}(\Upsilon(\mathcal{Q})) = C_{m-1}(\Upsilon(B) - \Upsilon(A - \mathcal{Q})) = \Upsilon(C_{n-1}(B-(A-\mathcal{Q}))) = \Upsilon(C_{n-1}(B-A+\mathcal{Q})) = C_{m-1}(\Upsilon(B-A+\mathcal{Q}))$. By the fact that $\mathrm{rk}(B-A+\mathcal{Q}) = n$ implies $\mathrm{rk}(\Upsilon(B-A+\mathcal{Q})) = m$. So we get $\mathrm{rk}(C_{m-1}(\Upsilon(\mathcal{Q}))) = \mathrm{rk}(C_{m-1}(\Upsilon(B - A + \mathcal{Q}))) = m$. Evidently $\mathrm{rk}(\mathcal{Q}) = n$ as $\mathrm{rk}(\Upsilon(\mathcal{Q})) = m$. Because $\mathrm{rk}(\mathcal{Q}) = n = n - 0$, this implies $\mathrm{rk}(B - A) = 0$. Consequently, $B - A = 0_n$ and inevitable $A = B$. This contradicts to the fact that $A \neq B$. This contradiction shows that our supposition is false. So we conclude that Υ is one-to-one.

(e) $\Rightarrow$ (a)

Together with the fact that Υ is one-to-one and $\Upsilon(0_n) = 0_m$, thus $\Upsilon(I_n) \neq 0_m$. Finally, we are accomplished the proof.

□

Corollary 3.4.8. Let $\mathbb{F}$ be a field carrying an involution $-$ and $m, n \in \mathbb{N}$ with $m, n > 2$. If a mapping $\Upsilon\colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ satisfies [P2], then the following statements are equivalent.

(a) $\Upsilon(I_n) \neq 0_m$.

(b) $\ker(\Upsilon) = \{0_n\}$.

(c) $\mathrm{rk}(A) = n$ if and only if $\mathrm{rk}(\Upsilon(A)) = m$.

(d) $\mathrm{rk}(A + \alpha B) = n$ if and only if $\mathrm{rk}(\Upsilon(A) + \alpha\Upsilon(B)) = m$.

(e) Υ is one-to-one.

Proof.

(a) $\Rightarrow$ (b) & (b) $\Rightarrow$ (c) & (d) $\Rightarrow$ (e) & (e) $\Rightarrow$ (a)

It follows immediately from **Lemma 3.4.7**.

(c) $\Rightarrow$ (d)

Let $A, B \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A+\alpha B) = n$. Thus

$$\begin{aligned}\mathrm{rk}(A+\alpha B) = n &\Leftrightarrow \mathrm{rk}(C_{n-1}(A+\alpha B)) = n\\ &\Leftrightarrow \mathrm{rk}(\Upsilon(C_{n-1}(A+\alpha B))) = m\\ &\Leftrightarrow \mathrm{rk}(C_{m-1}(\Upsilon(A)+\alpha\Upsilon(B))) = m\\ &\Leftrightarrow \mathrm{rk}(\Upsilon(A)+\alpha\Upsilon(B)) = m.\end{aligned}$$

□

Lemma 3.4.9. Let $\mathbb{F}$ be a field carrying an involution $-$ with $||\mathbb{F}^-|| > n+1$ and $m, n \in \mathbb{N}$ with $m, n > 2$. If $\Upsilon\colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ satisfies [P2] with $\Upsilon(I_n) \neq 0_m$, then Υ is additive and $\mathbb{F}^-$-homogeneous.

Proof.

Because $\Upsilon(I_n) \neq 0_m$, thus for any $A, B \in \mathbb{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$ with $\mathrm{rk}(A+\alpha B) = n$, we have

$$\begin{aligned}\mathrm{rk}(A+\alpha B) = n &\Rightarrow \mathrm{rk}(\Upsilon(A+\alpha B)) = m,\\ \mathrm{rk}(A+\alpha B) = n &\Rightarrow \mathrm{rk}(\Upsilon(A)+\alpha\Upsilon(B)) = m,\\ \mathrm{rk}(A+\alpha B) = n &\Rightarrow \mathrm{rk}(\Upsilon(A)+\Upsilon(\alpha B)) = m.\end{aligned}$$

We see that

$$\begin{aligned}\Upsilon(C_{n-1}(A+\alpha B)) &= C_{m-1}(\Upsilon(A+\alpha B)),\\ \Upsilon(C_{n-1}(A+\alpha B)) &= C_{m-1}(\Upsilon(A)+\alpha\Upsilon(B)),\\ \Upsilon(C_{n-1}(A+\alpha B)) &= C_{m-1}(\Upsilon(A)+\Upsilon(\alpha B)).\end{aligned}$$

This shows us that $C_{m-1}(\Upsilon(A+\alpha B)) = C_{m-1}(\Upsilon(A)+\alpha\Upsilon(B))$ and $C_{m-1}(\Upsilon(A+\alpha B)) = C_{m-1}(\Upsilon(A)+\Upsilon(\alpha B))$. Now we express the compound matrix in terms of adjoint matrix. From **Lemma 2.1.5.4**, we obtain

$$W_m \mathrm{adj}(\Upsilon(A+\alpha B))^{\sim} W_m = W_m \mathrm{adj}(\Upsilon(A)+\alpha\Upsilon(B))^{\sim} W_m. \tag{3.1}$$

Since the determinant of W_m is either 1 or -1, then the inverse of W_m exists. By cancellation law, we get

$$\mathrm{adj}(\Upsilon(A+\alpha B))^{\sim} = \mathrm{adj}(\Upsilon(A)+\alpha\Upsilon(B))^{\sim}. \tag{3.2}$$

By the fact that $(X^{\sim})^{\sim} = X$ for all $X \in \mathbb{M}_n(\mathbb{F})$,

$$\operatorname{adj}(\Upsilon(A+\alpha B)) = \operatorname{adj}(\Upsilon(A)+\alpha\Upsilon(B)). \tag{3.3}$$

Given $\operatorname{rk}(\Upsilon(A+\alpha B)) = \operatorname{rk}(\Upsilon(A)+\alpha\Upsilon(B)) = m$, we know the inverse of $\Upsilon(A+\alpha B)$ and $\Upsilon(A)+\alpha\Upsilon(B)$ both exist. By the definition of adjoint matrix, we have

$$|\Upsilon(A+\alpha B)|(\Upsilon(A+\alpha B))^{-1} = |\Upsilon(A)+\alpha\Upsilon(B)|(\Upsilon(A)+\alpha\Upsilon(B))^{-1}. \tag{3.4}$$

It is quite clear that

$$\frac{\Upsilon(A+\alpha B)}{|\Upsilon(A+\alpha B)|} = \frac{\Upsilon(A)+\alpha\Upsilon(B)}{|\Upsilon(A)+\alpha\Upsilon(B)|}. \tag{3.5}$$

By using the same arguments, we acquire

$$\frac{\Upsilon(A+\alpha B)}{|\Upsilon(A+\alpha B)|} = \frac{\Upsilon(A)+\Upsilon(\alpha B)}{|\Upsilon(A)+\Upsilon(\alpha B)|}. \tag{3.6}$$

If we set $A = 0$ in (3.5), then

$$\frac{\Upsilon(\alpha B)}{|\Upsilon(\alpha B)|} = \frac{\alpha\Upsilon(B)}{|\alpha\Upsilon(B)|} \tag{3.7}$$

where $\operatorname{rk}(\alpha B) = n$.

When $\alpha = 1$ happens in (3.5), then

$$\frac{\Upsilon(A+B)}{|\Upsilon(A+B)|} = \frac{\Upsilon(A)+\Upsilon(B)}{|\Upsilon(A)+\Upsilon(B)|} \tag{3.8}$$

where $\operatorname{rk}(A+B) = n$.

Before we continue our work, we recall that for every $X \in \mathbb{H}_n(\mathbb{F})$, $\overline{|X|} = \overline{|X^H|} = \overline{|\overline{X}|} = \left|\overline{\overline{X}}\right| = |X| \in \mathbb{F}^-$. Now we claim some statements as follows.

Claim 1. For all $\alpha \in \mathbb{F}^-$ and $A \in \mathbb{H}_n(\mathbb{F})$, if $\operatorname{rk}(A) = n$, then Υ satisfies $\Upsilon(\alpha A) = \alpha\Upsilon(A)$.

When $\alpha = 0, \Upsilon(\alpha A) = \Upsilon(0A) = \Upsilon(0_n) = 0_m = 0\Upsilon(A) = \alpha\Upsilon(A)$. Now we consider that $\alpha \neq 0$, from **Lemma 3.4.4**(c), we found that there exists a $\mathcal{Q} \in \mathbb{H}_n(\mathbb{F})$ such that $\operatorname{rk}(\mathcal{Q}+\alpha A) = n$ where $\mathcal{Q} \neq 0_n$ and $\mathcal{Q}$ is a non-invertible matrix. By (3.5) and (3.6), we get

$$\frac{\Upsilon(\mathcal{Q})+\alpha\Upsilon(A)}{|\Upsilon(\mathcal{Q})+\alpha\Upsilon(A)|} = \frac{\Upsilon(\mathcal{Q})+\Upsilon(\alpha A)}{|\Upsilon(\mathcal{Q})+\Upsilon(\alpha A)|}.$$

We let $\gamma = \dfrac{|\Upsilon(\mathcal{Q}) + \alpha\Upsilon(A)|}{|\Upsilon(\mathcal{Q}) + \Upsilon(\alpha A)|} \in \mathbb{F}^-$. Thus we obtain

$$\begin{aligned} \Upsilon(\mathcal{Q}) + \alpha\Upsilon(A) &= \gamma[\Upsilon(\mathcal{Q}) + \Upsilon(\alpha A)] \\ \Rightarrow \Upsilon(\mathcal{Q})(1-\gamma) &= \gamma\Upsilon(\alpha A) - \alpha\Upsilon(A). \end{aligned} \tag{3.9}$$

Suppose that there exist some non-zero element $\alpha \in \mathbb{F}^-$ and $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) = n$ such that $\Upsilon(\alpha A) \neq \alpha\Upsilon(A)$. Therefore $\gamma \neq \dfrac{|\Upsilon(\mathcal{Q}) + \alpha\Upsilon(A)|}{|\Upsilon(\mathcal{Q}) + \alpha\Upsilon(A)|} = 1$. Because $\mathrm{rk}(A) = n$, thereby $\mathrm{rk}(\alpha A)$ is also equal to n. By (3.7), we have

$$\frac{\Upsilon(\alpha A)}{|\Upsilon(\alpha A)|} = \frac{\alpha\Upsilon(A)}{|\alpha\Upsilon(A)|} \Rightarrow \Upsilon(\alpha A) = \alpha\mu\Upsilon(A) \tag{3.10}$$

where $\mu = \dfrac{|\Upsilon(\alpha A)|}{|\alpha\Upsilon(A)|} \in \mathbb{F}^-$. When we substitute (3.10) into (3.9), we get

$$\begin{aligned} &\Upsilon(\mathcal{Q})(1-\gamma) = \gamma[\alpha\mu\Upsilon(A)] - \alpha\Upsilon(A) \\ &\Rightarrow \Upsilon(\mathcal{Q})(1-\gamma) = (\gamma\mu - 1)\alpha\Upsilon(A) \\ &\Rightarrow \Upsilon(\mathcal{Q}) = \left[\frac{(\gamma\mu-1)\alpha}{1-\gamma}\right]\Upsilon(A). \end{aligned}$$

Due to $\gamma \neq 1, \alpha \neq 0$ and $\gamma\mu \neq 1$, we have $\dfrac{(\gamma\mu-1)\alpha}{1-\gamma} \neq 0$ and $\dfrac{(\gamma\mu-1)\alpha}{1-\gamma} \in \mathbb{F}^-$. Then $\mathrm{rk}(\Upsilon(\mathcal{Q})) = \mathrm{rk}\left(\left[\frac{(\gamma\mu-1)\alpha}{1-\gamma}\right]\Upsilon(A)\right) = \mathrm{rk}(\Upsilon(A))$. Given $\mathrm{rk}(A) = n$, consequently $\mathrm{rk}(\Upsilon(\mathcal{Q})) = \mathrm{rk}(\Upsilon(A)) = m$. In view of the fact that $\mathcal{Q}$ is a non-invertible matrix, this means $\mathrm{rk}(\mathcal{Q}) \neq n$ and this leads to $\mathrm{rank}(\Upsilon(\mathcal{Q})) \neq m$. From our results, we acquire $\mathrm{rk}(\Upsilon(\mathcal{Q})) = m$. This contradicts to the fact that $\mathcal{Q}$ is a non-invertible matrix. This contradiction shows that our supposition is false. So we conclude that **Claim 1** is true.

Claim 2. For all $A, B \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A+B) = n, \Upsilon(A)$ and $\Upsilon(B)$ are linearly independent which is followed from A and B are linearly independent.

Assume that there exist some $A, B \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A+B) = n$ such that A and B are linearly independent with $\Upsilon(A)$ and $\Upsilon(B)$ are not linearly independent. Definitely, there exists some $c_0 \in \mathbb{F}^-$ such that $\Upsilon(B) = c_0\Upsilon(A)$. Because $\mathrm{rk}(A+B) = n$, thus $\mathrm{rk}(\Upsilon(A) + \Upsilon(B)) = m$ and from **Lemma 3.4.4**(e), we know that there exists some $c_1 \in \mathbb{F}^-$ with $c_1 \neq 1$ such that $\mathrm{rk}(\Upsilon(A) + c_1\Upsilon(B)) = \mathrm{rk}(\Upsilon(A) + c_1c_0\Upsilon(A)) = \mathrm{rk}((1 + c_1c_0)\Upsilon(A)) = m$. This implies $\mathrm{rk}(\Upsilon(A)) = m$ and hence $\mathrm{rk}(A) = n$. From **Claim 1**, $\Upsilon(B) = c_0\Upsilon(A) = \Upsilon(c_0A)$. Since Υ is one-to-one, this means $B = c_0A$. This shows us that A and B are linearly dependent. This contradicts to the fact that A

and B are linearly independent. This contradiction shows that our supposition is false. So we conclude that **Claim 2** is true.

Claim 3. For any $A, B \in \mathbb{H}_n(\mathbb{F})$ with A and B are linearly independent and $\mathrm{rk}(B) = n$, $\Upsilon(A+B) = \Upsilon(A) + \Upsilon(B)$ when $\mathrm{rk}(A+B) = n$.

By **Lemma 3.4.4**(e), we know that there exists some $c_2 \in \mathbb{F}^-$ with $c_2 \neq 1$ such that $\mathrm{rk}(A + c_2 B) = \mathrm{rk}((A+B) + (c_2 - 1)B) = n$. According to (3.5) and **Claim 1**, we have

$$\begin{aligned}\frac{\Upsilon(A+c_2B)}{|\Upsilon(A+c_2B)|} &= \frac{\Upsilon(A)+c_2\Upsilon(B)}{|\Upsilon(A)+c_2\Upsilon(B)|}\\ &= \frac{\Upsilon(A)+\Upsilon(B)+(c_2-1)\Upsilon(B)}{|\Upsilon(A)+\Upsilon(B)+(c_2-1)\Upsilon(B)|}\\ &= \frac{\Upsilon(A)+\Upsilon(B)+\Upsilon((c_2-1)B)}{|\Upsilon(A)+\Upsilon(B)+\Upsilon((c_2-1)B)|}.\end{aligned} \tag{3.11}$$

From (3.6), we also have

$$\begin{aligned}\frac{\Upsilon(A+c_2B)}{|\Upsilon(A+c_2B)|} &= \frac{\Upsilon((A+B)+(c_2-1)B)}{|\Upsilon((A+B)+(c_2-1)B)|}\\ &= \frac{\Upsilon(A+B)+\Upsilon((c_2-1)B)}{|\Upsilon(A+B)+\Upsilon((c_2-1)B)|}.\end{aligned} \tag{3.12}$$

By combining (3.11) and (3.12), we see that

$$\begin{aligned}&\frac{\Upsilon(A)+\Upsilon(B)+\Upsilon((c_2-1)B)}{|\Upsilon(A)+\Upsilon(B)+\Upsilon((c_2-1)B)|} = \frac{\Upsilon(A+B)+\Upsilon((c_2-1)B)}{|\Upsilon(A+B)+\Upsilon((c_2-1)B)|}\\ \Rightarrow\ &\Upsilon(A)+\Upsilon(B)+\Upsilon((c_2-1)B) = c_3[\Upsilon(A+B)+\Upsilon((c_2-1)B)]\end{aligned}$$

where $c_3 = \dfrac{|\Upsilon(A)+\Upsilon(B)+\Upsilon((c_2-1)B)|}{|\Upsilon(A+B)+\Upsilon((c_2-1)B)|} \in \mathbb{F}^-$. It follows from (3.8) that there exists some $c_4 \in \mathbb{F}^-$ such that $\Upsilon(A)+\Upsilon(B) = c_4\Upsilon(A+B)$. So we have

$$\begin{aligned}&c_4\Upsilon(A+B)+\Upsilon((c_2-1)B) = c_3[\Upsilon(A+B)+\Upsilon((c_2-1)B)]\\ \Rightarrow\ &(c_4-c_3)\Upsilon(A+B) = (c_3-1)\Upsilon((c_2-1)B).\end{aligned}$$

Because A and B are linearly independent, $A+B$ and $(c_2-1)B$ are also linearly independent. By applying **Claim 2**, we acquire that $\Upsilon(A+B)$ and $\Upsilon((c_2-1)B)$ are also linearly independent. Consequently, $c_4 - c_3 = 0 = c_3 - 1 \Rightarrow c_3 = c_4 = 1$. So we acquire $\Upsilon(A+B) = \Upsilon(A) + \Upsilon(B)$.

Claim 4. For every $\alpha \in \mathbb{F}^-$ and $A \in \mathbb{H}_n(\mathbb{F})$, if A is a singular matrix, then $\Upsilon(\alpha A) = \alpha\Upsilon(A)$.

When $A = 0_n$, $\Upsilon(\alpha A) = \Upsilon(\alpha 0_n) = \Upsilon(0_n) = 0_m = \alpha\Upsilon(0_n) = \alpha\Upsilon(A)$ and when $\alpha = 0$, $\Upsilon(\alpha A) = \Upsilon(0A) = \Upsilon(0_n) = 0_m = 0\Upsilon(A) = \alpha\Upsilon(A)$. Now we consider that α is a non-zero element and A is a non-zero singular matrix. Hence αA is also a non-zero singular matrix. By referring to **Lemma 3.4.4**(c), we notice that there exists a $G \in \mathbb{H}_n(\mathbb{F})$ such that $\mathrm{rk}(\alpha A + G) = n$ where G is an invertible matrix. Thus $\mathrm{rk}(\alpha A + G) = \mathrm{rk}(\alpha(A + \alpha^{-1}G)) = \mathrm{rk}(A + \alpha^{-1}G) = n$ and $\alpha^{-1}G$ is invertible. By applying **Claim 1** and **Claim 3**, we have $\Upsilon(\alpha A + G) = \Upsilon(\alpha A) + \Upsilon(G)$ and $\Upsilon(\alpha A + G) = \Upsilon(\alpha(A + \alpha^{-1}G)) = \alpha\Upsilon(A + \alpha^{-1}G) = \alpha[\Upsilon(A) + \Upsilon(\alpha^{-1}G)] = \alpha\Upsilon(A) + \alpha\Upsilon(\alpha^{-1}G) = \alpha\Upsilon(A) + \alpha\alpha^{-1}\Upsilon(G) = \alpha\Upsilon(A) + \Upsilon(G)$. Inevitably, we get $\Upsilon(\alpha A) = \alpha\Upsilon(A)$.

By combining the **Claim 1** and **Claim 4**, we attain the results that for all $\alpha \in \mathbb{F}^-$ and $A \in \mathbb{H}_n(\mathbb{F})$, $\Upsilon(\alpha A) = \alpha\Upsilon(A)$. This shows us that Υ is $\mathbb{F}^-$-homogeneous. Now we need to claim some statements in order to prove that Υ is additive.

Claim 5. For all $A, B \in \mathbb{H}_n(\mathbb{F})$, if $\mathrm{rk}(A + B) = n$, then $\Upsilon(A + B) = \Upsilon(A) + \Upsilon(B)$.

For all $A, B \in \mathbb{H}_n(\mathbb{F})$, undoubtedly A and B are either linearly dependent or linearly independent. We ~~can~~ prove by considering two cases. Assume that A and B are linearly dependent. It follows that there exists some $c_5 \in \mathbb{F}^-$ such that $B = c_5 A$. Therefore $\Upsilon(A + B) = \Upsilon(A + c_5 A) = \Upsilon((1 + c_5)A) = (1 + c_5)\Upsilon(A) = \Upsilon(A) + c_5\Upsilon(A) = \Upsilon(A) + \Upsilon(c_5 A) = \Upsilon(A) + \Upsilon(B)$. Now we suppose that A and B are linearly independent. We refer to the arguments from **Claim 3**. The reason we restrict the rank of B where $\mathrm{rk}(B) = n$ because we want to apply **Claim 1**. Since we know that $\Upsilon(\alpha A) = \alpha\Upsilon(A)$ for all $\alpha \in \mathbb{F}^-$ and $A \in \mathbb{H}_n(\mathbb{F})$, it follows that we are not required to apply **Claim 1**. This means we are not required to restrict the rank of B as $\mathrm{rk}(B) = n$. By referring to the arguments from **Claim 3**, we deduce that if A and B are linearly independent, then $\Upsilon(A + B) = \Upsilon(A) + \Upsilon(B)$.

Claim 6. For all $A, B \in \mathbb{H}_n(\mathbb{F})$, $\Upsilon(A + B) = \Upsilon(A) + \Upsilon(B)$.

According to **Lemma 3.4.4**(b), there exists a $C \in \mathbb{H}_n(\mathbb{F})$ such that $\mathrm{rk}(A + C) = \mathrm{rk}((A + B) + C) = \mathrm{rk}((A + C) + B) = n$. Applying to **Claim 5**, we have

$$\Upsilon((A + B) + C) = \Upsilon(A + B) + \Upsilon(C),$$
$$\Upsilon((A + C) + B) = \Upsilon(A + C) + \Upsilon(B) = \Upsilon(A) + \Upsilon(C) + \Upsilon(B).$$

Hence we know that

$$\begin{aligned}
&\Upsilon((A+B)+C)=\Upsilon((A+C)+B)\\
\Rightarrow\ &\Upsilon(A+B)+\Upsilon(C)=\Upsilon(A)+\Upsilon(C)+\Upsilon(B)\\
\Rightarrow\ &\Upsilon(A+B)=\Upsilon(A)+\Upsilon(B).
\end{aligned}$$

We obtain the results that for all $A, B \in \mathbb{H}_n(\mathbb{F})$, $\Upsilon(A+B) = \Upsilon(A)+\Upsilon(B)$. So we conclude that Υ is additive. We are done.

□

Lemma 3.4.10. Let $\mathbb{F}$ be a field carrying an involution $-$ with $\mathbb{F}^- = GF(2)$ and $m, n \in \mathbb{N}$ with $m, n > 2$. If $\Upsilon\colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ satisfies [P1] with $\Upsilon(I_n) \neq 0_m$, then Υ is additive and $\mathbb{F}^-$-homogeneous.

Proof.

Since $\Upsilon(I_n) \neq 0_m$, then for any $A, B \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A-B) = n$. Hence

$$\begin{aligned}
&\mathrm{rk}(A-B)=n \Rightarrow \mathrm{rk}(\Upsilon(A-B))=m,\\
&\mathrm{rk}(A-B)=n \Rightarrow \mathrm{rk}(\Upsilon(A)-\Upsilon(B))=m.
\end{aligned}$$

Besides that,

$$\begin{aligned}
&\Upsilon(C_{n-1}(A-B))=C_{m-1}(\Upsilon(A-B)),\\
&\Upsilon(C_{n-1}(A-B))=C_{m-1}(\Upsilon(A)-\Upsilon(B)).
\end{aligned}$$

So we have $C_{m-1}(\Upsilon(A-B)) = C_{m-1}(\Upsilon(A)-\Upsilon(B))$. By using the same arguments from (3.1) to (3.5), we obtain

$$\frac{\Upsilon(A-B)}{|\Upsilon(A-B)|}=\frac{\Upsilon(A)-\Upsilon(B)}{|\Upsilon(A)-\Upsilon(B)|}.$$

Since $\mathrm{rk}(\Upsilon(A-B)) = \mathrm{rk}(\Upsilon(A)-\Upsilon(B)) = m$, then $|\Upsilon(A-B)|$ and $|\Upsilon(A)-\Upsilon(B)|$ are not equal to 0. Recall that $|X| \in \mathbb{F}^-$ for all $X \in \mathbb{H}_n(\mathbb{F})$. This implies $|\Upsilon(A-B)| = |\Upsilon(A)-\Upsilon(B)| = 1$. This leads to

$$\Upsilon(A-B)=\Upsilon(A)-\Upsilon(B).$$

By the fact that $GF(2)$ is a subfield of $\mathbb{F}$ and $\mathrm{char}(GF(2)) = 2$, then $\mathrm{char}(\mathbb{F}) = 2$. Therefore we acquire

$$\Upsilon(A+B)=\Upsilon(A)+\Upsilon(B).$$

Consequently, Υ is additive. Moreover, because $\Upsilon(0A) = \Upsilon(0_n) = 0_m = 0\Upsilon(A)$ and $\Upsilon(1A) = \Upsilon(A) = 1\Upsilon(A)$. So Υ is also $\mathbb{F}^-$-homogeneous.

$\square$

CHAPTER 4

COMPOUND-COMMUTING MAPPINGS ON HERMITIAN AND SYMMETRIC MATRICES

4.1 Some Non-Zero Mappings with $\Upsilon(I_n) = 0_m$

In this section, we construct some examples of non-zero mapping $\Upsilon\colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ satisfying [P2] with $\Upsilon(I_n) = 0_m$.

Let $\mathbb{F}$ be a field carrying an involution $-$ and let $m, n > 2$. From **Corollary 3.4.6**, we see that $\Upsilon(A) = 0_m$ for any $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) \leqslant 1$ and $\mathrm{rk}(\Upsilon(A)) \leqslant m-2$ for any $A \in \mathbb{H}_n(\mathbb{F})$. Also, $\Upsilon(C_{n-1}(A)) = 0_m$ for any $A \in \mathbb{H}_n(\mathbb{F})$. Let $B \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(B) = n$. Thus

$$\Upsilon(B) = \begin{cases} 0_m & \text{if } B = C_{n-1}(G) \text{ for some } G \in \mathbb{H}_n(\mathbb{F}) \text{ with } \mathrm{rk}(G) = n, \\ \Upsilon(B) \text{ with } \mathrm{rk}(\Upsilon(B)) \leqslant m-2 & \text{otherwise.} \end{cases}$$

It is very difficult to verify whether or not there exists a $G \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(G) = n$ such that $B = C_{n-1}(G)$. So in our construction, we construct $\Upsilon(A) = 0_m$ for any $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) = n$.

Example 4.1.1. Let $\mathbb{F}$ be a field carrying an involution $-$. Let $m, n \in \mathbb{N}$ with $m, n > 2$.

[E1] Let $p \in \mathbb{N}$ with $1 \leqslant p \leqslant m-2$. Let $w_1, w_2, \cdots, w_p \in \mathbb{N}$ with $1 \leqslant w_1 < w_2 < \cdots < w_p \leqslant m$. Let $t \in \mathbb{N}$ with $1 \leqslant t \leqslant n$. Let $v_1, v_2, \cdots, v_t \in \mathbb{N}$ with $1 \leqslant v_1 < v_2 < \cdots < v_t \leqslant n$. Let ζ be a non-zero mapping from $\mathbb{F}^-$ to $\mathbb{F}^-$. We construct the mapping $\Upsilon_1\colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ by

$$\Upsilon_1(A) = \left(\sum_{i=1}^{t} \zeta\left(a_{v_i,v_i}\right)\right)\left(\sum_{i=1}^{p} E_{w_i,w_i}\right)$$

if $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) \neq 0, 1, n$, otherwise $\Upsilon_1(A) = 0_m$.

[E2] Let $n \geqslant m$. Let $p \in \mathbb{N}$ with $1 \leqslant p \leqslant m-2$. Let $w_1, w_2, \cdots, w_p \in \mathbb{N}$ with $1 \leqslant w_1 < w_2 < \cdots < w_p \leqslant m$. Let ζ be a non-zero mapping from $\mathbb{F}^-$ to $\mathbb{F}^-$. We construct the mapping $\Upsilon_2\colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ by

$$\Upsilon_2(A) = \sum_{i=1}^{p} \zeta\left(a_{w_i,w_i}\right) E_{w_i,w_i}$$

if $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) \neq 0, 1, n$, otherwise $\Upsilon_2(A) = 0_m$.

[E3] Let $m, n > 3$. Let $t \in \mathbb{N}$ with $1 \leqslant t \leqslant m$. Let $p \in \mathbb{N}$ with $1 \leqslant p \leqslant m-1$. Let $w_1, w_2, \cdots, w_p \in \mathbb{N}$ with $1 \leqslant w_1 < w_2 < \cdots < w_p \leqslant m$ and $w_1, w_2, \cdots, w_p$ are not equal to t. Let $r \in \mathbb{N}$ with $1 \leqslant r \leqslant n$. Let ζ be a non-zero field monomorphism of $\mathbb{F}$ with $\zeta(\overline{\alpha}) = \overline{\zeta(\alpha)}$ for all $\alpha \in \mathbb{F}$. We construct the mapping $\Upsilon_3 \colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ by

$$\Upsilon_3(A) = \sum_{k=1}^{p} \left(\left(\sum_{1 \leqslant i < j \leqslant r} (\zeta(a_{ij})) \right) E_{t,w_k} + \left(\sum_{1 \leqslant i < j \leqslant r} (\zeta(a_{ji})) \right) E_{w_k,t} \right)$$

if $A \in \mathbb{H}_n(\mathbb{F})$ with $\operatorname{rk}(A) \neq 0, 1, n$, otherwise $\Upsilon_3(A) = 0_m$.

[E4] Let $m, n > 3$ with $n \geqslant m$. Let $t \in \mathbb{N}$ with $1 \leqslant t \leqslant m$. Let $p \in \mathbb{N}$ with $1 \leqslant p \leqslant m-1$. Let $w_1, w_2, w_p \in \mathbb{N}$ with $1 \leqslant w_1 < w_2 < \cdots < w_p \leqslant m$ and $w_1, w_2, \cdots, w_p$ are not equal to t. Let ζ be a non-zero field monomorphism of $\mathbb{F}$ with $\zeta(\overline{\alpha}) = \overline{\zeta(\alpha)}$ for all $\alpha \in \mathbb{F}$. We construct the mapping $\Upsilon_4 \colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ by

$$\Upsilon_4(A) = \sum_{i=1}^{p} \left(\zeta(a_{t,w_i}) E_{t,w_i} + \zeta(a_{w_i,t}) E_{w_i,t} \right)$$

if $A \in \mathbb{H}_n(\mathbb{F})$ with $\operatorname{rk}(A) \neq 0, 1, n$, otherwise $\Upsilon_4(A) = 0_m$.

[E5] Let $k \in \mathbb{N}$. Let $m, n > 3 + k$. Let $w_1, w_2, \cdots, w_k \in \mathbb{N}$ with $1 \leqslant w_1 < w_2 < \cdots < w_k \leqslant m$. Let $p, q \in \mathbb{N}$ with $1 \leqslant p, q \leqslant m$ and p, q are not equal to $w_1, w_2, \cdots, w_k$. Let $t \in \mathbb{N}$ with $1 \leqslant t \leqslant n$. Let $v_1, v_2, \cdots, v_t \in \mathbb{N}$ with $1 \leqslant v_1 < v_2 < \cdots < v_t \leqslant n$. Let $r \in \mathbb{N}$ with $1 \leqslant r \leqslant n$. Let ζ be a non-zero field monomorphism of $\mathbb{F}$ with $\zeta(\overline{\alpha}) = \overline{\zeta(\alpha)}$ for all $\alpha \in \mathbb{F}$. Let ϕ be a non-zero mapping from $\mathbb{F}^-$ to $\mathbb{F}^-$. We construct the mapping $\Upsilon_5 \colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ by

$$\Upsilon_5(A) = \left(\sum_{i=1}^{t} \phi(a_{v_i,v_i}) \right) \left(\sum_{i=1}^{k} E_{w_i,w_i} \right) + \left(\sum_{1 \leqslant i < j \leqslant r} (\zeta(a_{ij})) \right) E_{pq} + \left(\sum_{1 \leqslant i < j \leqslant r} (\zeta(a_{ji})) \right) E_{qp}$$

if $A \in \mathbb{H}_n(\mathbb{F})$ with $\operatorname{rk}(A) \neq 0, 1, n$, otherwise $\Upsilon_5(A) = 0_m$.

[E6] Let $k \in \mathbb{N}$. Let $m, n > 3 + k$ with $n \geqslant m$. Let $w_1, w_2, \cdots, w_k \in \mathbb{N}$ with $1 \leqslant w_1 < w_2 < \cdots < w_k \leqslant m$. Let $p, q \in \mathbb{N}$ with $1 \leqslant p, q \leqslant m$ and p, q are not equal to $w_1, w_2, \cdots, w_k$. Let ζ be a non-zero field monomorphism of $\mathbb{F}$ with $\zeta(\overline{\alpha}) = \overline{\zeta(\alpha)}$ for all $\alpha \in \mathbb{F}$. Let ϕ be a non-zero mapping from $\mathbb{F}^-$ to $\mathbb{F}^-$. We construct the mapping $\Upsilon_6 \colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ by

$$\Upsilon_6(A) = \sum_{i=1}^{k} \phi(a_{w_i,w_i}) E_{w_i,w_i} + \zeta(a_{pq}) E_{pq} + \zeta(a_{qp}) E_{qp}$$

if $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) \neq 0, 1, n$, otherwise $\Upsilon_6(A) = 0_m$.

[E7] Let $k \in \mathbb{N}$. Let $m > n+k$. Let Υ' be a non-zero mapping from $\mathbb{H}_n(\mathbb{F})$ to $\mathbb{H}_n(\mathbb{F})$. We construct the mapping $\Upsilon_7 \colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ by

$$\Upsilon_7(A) = \Upsilon'(A) \oplus 0_{m-n}$$

if $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) \neq 0, 1, n$, otherwise $\Upsilon_7(A) = 0_m$.

[E8] Let $n \geqslant m$. Let $t \in \mathbb{N}$ with $1 \leqslant \left\lfloor \dfrac{m-2}{t} \right\rfloor \leqslant m-2$ where $\lfloor\ \rfloor$ stands for the floor function. Let ζ be a non-zero field monomorphism of $\mathbb{F}$ with $\zeta(\overline{\alpha}) = \overline{\zeta(\alpha)}$ for all $\alpha \in \mathbb{F}$. Let ϕ be a non-zero mapping from $\mathbb{F}^-$ to $\mathbb{F}^-$. We construct the mapping $\Upsilon_8 \colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ by

$$\Upsilon_8(A) = \bigoplus_{k=1}^{\left\lfloor \frac{m-2}{t} \right\rfloor} \left(\sum_{1+(k-1)t \leqslant i < j \leqslant kt} (\zeta(a_{ij})E_{ij} + \zeta(a_{ji})E_{ji}) + \sum_{i=1+(k-1)t}^{kt} \phi(a_{ii})\, E_{ii} \right) \oplus 0_u$$

if $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) \neq 0, 1, n$, where $u = m - t \left\lfloor \dfrac{m-2}{t} \right\rfloor$ and E_{ii}, E_{ij}, E_{ji} $\in \mathbb{H}_t(\mathbb{F})$. Otherwise $\Upsilon_8(A) = 0_m$.

It is easy to inspect that Υ_i satisfies [P2] with $\Upsilon_i(I_n) = 0_m$ for all $i \in \{1, 2, \cdots, 8\}$. Furthermore, we realise that Υ_i are neither one-to-one nor onto and it is not necessary m and n are forced to be same for all $i \in \{1, 2, \cdots, 8\}$.

4.2 Characterisation of Compound-Commuting Mappings on Hermitian Matrices

Lemma 4.2.1. Let $\mathbb{F}$ be a field carrying an involution $-$ and $m, n \in \mathbb{N}$ with $m, n > 2$. $\Upsilon \colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ satisfies [P2] with $\Upsilon(I_n) = 0_m$ if and only if

$$\Upsilon(A) = 0_m \text{ for any } A \in \mathbb{H}_n(\mathbb{F}) \text{ with } \mathrm{rk}(A) \leqslant 1,$$

$$\Upsilon(C_{n-1}(A + \alpha B)) = 0_m \text{ for any } A, B \in \mathbb{H}_n(\mathbb{F}) \text{ and } \alpha \in \mathbb{F}^-$$

and

$$\mathrm{rk}(\Upsilon(A) + \alpha\Upsilon(B)) \leqslant m-2 \text{ for any } A, B \in \mathbb{H}_n(\mathbb{F}) \text{ and } \alpha \in \mathbb{F}^-.$$

Proof.

($\Rightarrow$) According to the **Corollary 3.4.6**, we know that $\Upsilon(A) = 0_m$ for any $A \in \mathbb{H}_n(\mathbb{F})$ with $\operatorname{rk}(A) \leqslant 1$. Besides that, we observe that $\Upsilon(C_{n-1}(A+\alpha B)) = 0_m$ for any $A, B \in \mathbb{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$. This leads to $C_{m-1}(\Upsilon(A) + \alpha\Upsilon(B)) = 0_m$. Thereupon, we obtain $\operatorname{rk}(\Upsilon(A) + \alpha\Upsilon(B)) \leqslant m-2$ for any $A, B \in \mathbb{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$.

($\Leftarrow$) Let Υ be a mapping from $\mathbb{H}_n(\mathbb{F})$ to $\mathbb{H}_m(\mathbb{F})$. Since $\operatorname{rk}(\Upsilon(A)+\alpha\Upsilon(B)) \leqslant m-2$, then $\Upsilon(C_{n-1}(A+\alpha B)) = 0_m = C_{m-1}(\Upsilon(A) + \alpha\Upsilon(B))$ for any $A, B \in \mathbb{H}_n(\mathbb{F})$ and $\Upsilon(C_{n-1}(I_n + 0B)) = \Upsilon(C_{n-1}(I_n)) = \Upsilon(I_n) = 0_m$.

$\square$

Lemma 4.2.2. (Fošner and Šemrl (2005), Lemma 2.1). Let $\mathbb{F}$ be a field and $m, n \in \mathbb{N}$ with $m > n$. For all $A_1, A_2, \cdots, A_{m-1}, A_m \in \mathbb{M}_n(\mathbb{F})$ with $\operatorname{rk}(A_1 + A_2 + \cdots + A_{m-1} + A_m) = n$, there exists certain non-empty proper subset $S \subset \{1, 2, \cdots, m-1, m\}$ such that $\operatorname{rk}\left(\sum_{k\in S} A_k\right) = n$.

Lemma 4.2.3. Let $\mathbb{F}$ be a field carrying an involution $-$ and Υ be a mapping from $\mathbb{H}_n(\mathbb{F})$ to $\mathbb{H}_m(\mathbb{F})$. Let $m, n \in \mathbb{N}$ with $m, n > 2$. If $\Upsilon(I_n) \neq 0_m$ and either $\mathbb{F}^- = GF(2)$ with Υ satisfying [P1] or $||\mathbb{F}^-|| > n+1$ with Υ satisfying [P2], then $m = n$.

Proof.

It follows from **Lemma 3.4.9** and **Lemma 3.4.10** that Υ is additive. By **Lemma 3.4.7**(c), we notice that $\operatorname{rk}(\Upsilon(I_n)) = m$. Then

$$\begin{aligned}
\operatorname{rk}(\Upsilon(I_n)) &= \operatorname{rk}(\Upsilon(E_{11} + E_{22} + \cdots + E_{n-1,n-1} + E_{nn})) \\
&= \operatorname{rk}(\Upsilon(E_{11}) + \Upsilon(E_{22}) + \cdots + \Upsilon(E_{n-1,n-1}) + \Upsilon(E_{nn})) \\
&\leqslant \operatorname{rk}(\Upsilon(E_{11})) + \operatorname{rk}(\Upsilon(E_{22})) + \cdots + \operatorname{rk}(\Upsilon(E_{n-1,n-1})) + \operatorname{rk}(\Upsilon(E_{nn})).
\end{aligned}$$

Since $\operatorname{rk}(\Upsilon(E_{ii})) \leqslant 1$ for all $i \in \{1, 2, \cdots, n-1, n\}$. So we have

$$\operatorname{rk}(\Upsilon(I_n)) \leqslant \underbrace{1 + 1 + \cdots + 1 + 1}_{n \text{ times of } 1} = n.$$

Thus we obtain $m \leqslant n$. Now we consider for $n > m$. Obviously, $\Upsilon(E_{11}), \Upsilon(E_{22}), \cdots, \Upsilon(E_{n-1,n-1}), \Upsilon(E_{nn}) \in \mathbb{H}_m(\mathbb{F}) \subset \mathbb{M}_m(\mathbb{F})$ and $\operatorname{rk}(\Upsilon(E_{11})+\Upsilon(E_{22})+\cdots+\Upsilon(E_{n-1,n-1}) + \Upsilon(E_{nn})) = \operatorname{rk}(\Upsilon(I_n)) = m$. By referring to **Lemma 4.2.2**, there exists certain non-empty proper subset $S \subset \{1, 2, \cdots, n-1, n\}$ such that $\mathrm{rk}\left(\sum_{k\in S} \Upsilon(E_{kk})\right) = m$. Consequently, $\mathrm{rk}\left(C_{m-1}\left(\sum_{k\in S} \Upsilon(E_{kk})\right)\right) = m$ and

$$\begin{aligned}
\mathrm{rk}\left(C_{m-1}\left(\sum_{k\in S} \Upsilon(E_{kk})\right)\right) &= \mathrm{rk}\left(C_{m-1}\left(\Upsilon\left(\sum_{k\in S} E_{kk}\right)\right)\right) \\
&= \mathrm{rk}\left(\Upsilon\left(C_{n-1}\left(\sum_{k\in S} E_{kk}\right)\right)\right).
\end{aligned}$$

For the reason that S is a proper subset of $\{1, 2, \cdots, n-1, n\}$, then $||S|| < n$. Therefore $\mathrm{rk}\left(\sum_{k\in S} E_{kk}\right) < n$. Hence $\mathrm{rk}\left(C_{n-1}\left(\sum_{k\in S} E_{kk}\right)\right) \leqslant 1$. It follows that $\mathrm{rk}\left(\Upsilon\left(C_{n-1}\left(\sum_{k\in S} E_{kk}\right)\right)\right) \leqslant 1$. This leads to $\mathrm{rk}\left(C_{m-1}\left(\sum_{k\in S} \Upsilon(E_{kk})\right)\right) \leqslant 1$. Now we acquire $m \leqslant 1$. This contradicts to the fact that $m > 2$. This contradiction shows that $n \not> m$. So we ~~are forced to~~ conclude that $m = n$.

□

Lemma 4.2.4. Let $\mathbb{F}$ be a field carrying an involution $-$ and $n \in \mathbb{N}$ with $n > 2$. If $\Upsilon \colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_n(\mathbb{F})$ satisfies [P1] with $\Upsilon(I_n) \neq 0_n$ and

$$\Upsilon(A) = \xi Z A^{\phi} Z^H \text{ for all } A \in \mathbb{H}_n(\mathbb{F})$$

where ξ is a non-zero element in $\mathbb{F}^-$, Z is a non-singular matrix in $\mathbb{M}_n(\mathbb{F})$ and ϕ is a non-zero field monomorphism of $\mathbb{F}$ with $\phi(\overline{\rho}) = \overline{\phi(\rho)}$ for all $\rho \in \mathbb{F}$, then $C_{n-1}(Z) = \omega Z$ for some non-zero element $\omega \in \mathbb{F}$ with $\xi^{n-2}\omega\overline{\omega} = 1$.

Proof.

Now we want to show that $C_{n-1}(Z) = \omega Z$ for some non-zero element $\omega \in \mathbb{F}$ with $\xi^{n-2}\omega\overline{\omega} = 1$. By the fact that $\Upsilon(C_{n-1}(A)) = C_{n-1}(\Upsilon(A))$ for all $A \in \mathbb{H}_n(\mathbb{F})$, we attain

$$\begin{aligned}
&\xi Z C_{n-1}(A)^{\phi} Z^H = C_{n-1}(\xi Z A^{\phi} Z^H) \\
\Rightarrow\ &\xi Z C_{n-1}(A)^{\phi} Z^H = \xi^{n-1} C_{n-1}(Z) C_{n-1}(A)^{\phi} C_{n-1}(Z)^H \\
\Rightarrow\ &C_{n-1}(A)^{\phi} Z^H (C_{n-1}(Z)^H)^{-1} = \xi^{n-2} Z^{-1} C_{n-1}(Z) C_{n-1}(A)^{\phi} \\
\Rightarrow\ &C_{n-1}(A)^{\phi} ((Z^{-1} C_{n-1}(Z))^H)^{-1} = \xi^{n-2} Z^{-1} C_{n-1}(Z) C_{n-1}(A)^{\phi}.
\end{aligned}$$

For simplification, we let $X = ((Z^{-1}C_{n-1}(Z))^H)^{-1}$ and $Y = \xi^{n-2}Z^{-1}C_{n-1}(Z)$. Now we set $A = I_n - E_{n+1-i,n+1-1}$ where $i \in \mathbb{N}$ with $1 \leqslant i \leqslant n$, we have

$$E_{ii}^{\phi}X = YE_{ii}^{\phi} \text{ for all } 1 \leqslant i \leqslant n.$$

Recall that, since ϕ is field monomorphism of $\mathbb{F}$, then $\phi(0) = 0$, $\phi(1) = 1$ and $\phi(-1) = -\phi(1) = -1$. So we have $E_{ii}X = YE_{ii}$. Definitely,

$$(E_{ii}X)_{ij} = (YE_{ii})_{ij} \text{ for all } 1 \leqslant i, j \leqslant n.$$

For all $j \neq i$, we get $X_{ij} = 0$. This means X is a diagonal matrix. Evidently, $Y = \xi^{n-2}(X^{-1})^H$ is also a diagonal matrix. Moreover, for $j = i$, we obtain $X_{ii} = Y_{ii}$. It follows that $X = Y$. Next, we set $A = I_n - E_{n+1-i,n+1-i} - E_{n+1-j,n+1-j} + (-1)^{i+j+1}(E_{n+1-j,n+1-i} + E_{n+1-i,n+1-j})$ where $i, j \in \mathbb{N}$ with $1 \leqslant i < j \leqslant n$, we get

$$(E_{ij} + E_{ji} - (I_n - E_{ii} - E_{jj}))^{\phi}X = X(E_{ij} + E_{ji} - (I_n - E_{ii} - E_{jj}))^{\phi}$$

for all $1 \leqslant i < j \leqslant n$. This implies

$$\begin{aligned}
&(E_{ij} + E_{ji} - (I_n - E_{ii} - E_{jj}))X = X(E_{ij} + E_{ji} - (I_n - E_{ii} - E_{jj})) \\
\Rightarrow\ &(E_{ij} + E_{ji})X + E_{ii}X + E_{jj}X - X = X(E_{ij} + E_{ji}) + XE_{ii} + XE_{jj} - X \\
\Rightarrow\ &(E_{ij} + E_{ji})X = X(E_{ij} + E_{ji}).
\end{aligned}$$

Undoubtedly,

$$((E_{ij} + E_{ji})X)_{ij} = (X(E_{ij} + E_{ji}))_{ij} \text{ for all } 1 \leqslant i < j \leqslant n.$$

It is clear that $X_{jj} = X_{ii}$. Consequently, X is a diagonal matrix with all diagonal elements of X are equal. Without loss of generality, $X = Y$ can be expressed as $X = Y = \kappa I_n$ for some non-zero element $\kappa \in \mathbb{F}$. So we acquire

$$\begin{aligned}
&((Z^{-1}C_{n-1}(Z))^H)^{-1} = \kappa I_n \quad \text{and} \quad \xi^{n-2}Z^{-1}C_{n-1}(Z) = \kappa I_n \\
\Rightarrow\ &C_{n-1}(Z) = \overline{\kappa}^{-1}Z \quad \text{and} \quad C_{n-1}(Z) = (\xi^{n-2})^{-1}\kappa Z \\
\Rightarrow\ &\overline{\kappa}^{-1} = (\xi^{n-2})^{-1}\kappa \\
\Rightarrow\ &\xi^{n-2}\overline{\kappa}^{-1}\kappa^{-1} = 1.
\end{aligned}$$

In order to make our results look more elegant, we replace $\overline{\kappa}^{-1}$ by ω. Hence $C_{n-1}(Z) = \omega Z$ for some non-zero element $\omega \in \mathbb{F}$ with $\xi^{n-2}\omega\overline{\omega} = 1$.

$\square$

Lemma 4.2.5. Let $\mathbb{F}$ be a field carrying an involution $-$ and $n \in \mathbb{N}$ with $n > 2$. If $\Upsilon\colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_n(\mathbb{F})$ satisfies [P2] with $\Upsilon(I_n) \neq 0_m$ and also

$$\Upsilon(A) = \xi Z A^{\phi} Z^H \text{ for all } A \in \mathbb{H}_n(\mathbb{F})$$

where ξ is a non-zero element in $\mathbb{F}^-$, Z is a non-singular matrix in $\mathbb{M}_n(\mathbb{F})$ with $C_{n-1}(Z) = \omega Z$ for some non-zero element $\omega \in \mathbb{F}$ for which $\xi^{n-2}\omega\overline{\omega} = 1$ and ϕ is a non-zero field monomorphism of $\mathbb{F}$ with $\phi(\overline{\rho}) = \overline{\phi(\rho)}$ for all $\rho \in \mathbb{F}$, then ϕ is identity or $\phi = -$.

Proof.

For the reason that Υ is $\mathbb{F}^-$-homogeneous, thus $\Upsilon(\rho I_n) = \rho\Upsilon(I_n)$ for any $\rho \in \mathbb{F}^-$. Hence we have

$$\begin{aligned}
&\Upsilon(\rho I_n) = \rho\Upsilon(I_n)\\
\Rightarrow&\xi Z(\rho I_n)^{\phi} Z^H = \rho\xi Z I_n^{\phi} Z^H\\
\Rightarrow&(\rho I_n)^{\phi} = \rho I_n^{\phi}\\
\Rightarrow&\phi(\rho) I_n^{\phi} = \rho I_n^{\phi}\\
\Rightarrow&\phi(\rho) I_n = \rho I_n\\
\Rightarrow&\phi(\rho) = \rho.
\end{aligned}$$

Therefore we know that for all $\rho \in \mathbb{F}$ with $\rho \neq 0$,

$$\phi(\rho + \overline{\rho}) = \rho + \overline{\rho} \quad \text{and} \quad \phi(\rho\overline{\rho}) = \rho\overline{\rho}$$

as $\rho + \overline{\rho}, \rho\overline{\rho} \in \mathbb{F}^-$. It follows that

$$\begin{aligned}
&\phi(\rho) + \phi(\overline{\rho}) = \rho + \overline{\rho} \quad \text{and} \quad \phi(\rho)\phi(\overline{\rho}) = \rho\overline{\rho}\\
\Rightarrow\ &\phi(\rho) + \phi(\overline{\rho}) = \rho + \overline{\rho} \quad \text{and} \quad \phi(\overline{\rho}) = \phi(\rho)^{-1}\rho\overline{\rho}\\
\Rightarrow\ &\phi(\rho) + \phi(\rho)^{-1}\rho\overline{\rho} = \rho + \overline{\rho}\\
\Rightarrow\ &\phi(\rho)^2 - (\rho + \overline{\rho})\phi(\rho) + \rho\overline{\rho} = 0\\
\Rightarrow\ &(\phi(\rho) - \rho)(\phi(\rho) - \overline{\rho}) = 0\\
\Rightarrow\ &\phi(\rho) = \rho \quad \text{or} \quad \phi(\rho) = \overline{\rho}.
\end{aligned}$$

So we conclude that ϕ is identity or $\phi = -$.

$\square$

Lemma 4.2.6. Let $\mathbb{F}$ be a field carrying an involution $-$ and $m, n \in \mathbb{N}$ with $m, n > 2$. Let Υ be a mapping from $\mathbb{H}_n(\mathbb{F})$ to $\mathbb{H}_m(\mathbb{F})$. If $m = n$ and

$$\Upsilon(A) = \xi Z A^{\phi} Z^{H} \text{ for all } A \in \mathbb{H}_n(\mathbb{F})$$

where ξ is a non-zero element in $\mathbb{F}^{-}$, Z is a non-singular matrix in $\mathbb{M}_n(\mathbb{F})$ with $C_{n-1}(Z) = \omega Z$ for some non-zero element $\omega \in \mathbb{F}$ for which $\xi^{n-2}\omega\overline{\omega} = 1$ and ϕ is identity or $\phi = -$. Then $\Upsilon\colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ satisfies [P2] with $\Upsilon(I_n) \neq 0_m$.

Proof.

By the fact given, it is clear that $\Upsilon(I_n) \neq 0_m$. Besides that, for any $A, B \in \mathbb{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^{-}$,

$$\begin{aligned}
\Upsilon(C_{n-1}(A+\alpha B)) &= \xi Z C_{n-1}(A+\alpha B)^{\phi} Z^{H} \\
&= \xi(\omega^{-1}C_{n-1}(Z))C_{n-1}(A+\alpha B)^{\phi}(\overline{\omega}^{-1}C_{n-1}(Z^{H})) \\
&= \xi\overline{\omega}^{-1}\omega^{-1}C_{n-1}(Z)C_{n-1}(A+\alpha B)^{\phi}C_{n-1}(Z^{H}) \\
&= \xi\xi^{n-2}C_{n-1}(Z(A+\alpha B)^{\phi}Z^{H}) \\
&= \xi^{n-1}C_{n-1}(Z(A+\alpha B)^{\phi}Z^{H}) \\
&= C_{n-1}(\xi Z(A+\alpha B)^{\phi}Z^{H}) \\
&= C_{n-1}(\xi Z A^{\phi} Z^{H} + \phi(\alpha)(\xi Z B^{\phi} Z^{H})) \\
&= C_{n-1}(\xi Z A^{\phi} Z^{H} + \alpha(\xi Z B^{\phi} Z^{H})) \\
&= C_{n-1}(\Upsilon(A) + \alpha\Upsilon(B)) \\
&= C_{m-1}(\Upsilon(A) + \alpha\Upsilon(B)).
\end{aligned}$$

$\square$

Theorem 4.2.7. Let $\mathbb{F}$ be a field carrying a proper involution $-$ with either $\mathbb{F}^{-} = GF(2)$ or $||\mathbb{F}^{-}|| > n+1$. Let $m, n \in \mathbb{N}$ with $m, n > 2$. If $\Upsilon\colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ satisfies $\Upsilon(C_{n-1}(A+\alpha B)) = C_{m-1}(\Upsilon(A)+\alpha\Upsilon(B))$ for any $A, B \in \mathbb{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^{-}$ if and only if

$$\Upsilon(A) = 0_m \text{ for any } A \in \mathbb{H}_n(\mathbb{F}) \text{ with } \mathrm{rk}(A) \leqslant 1,$$

$$\Upsilon(C_{n-1}(A+\alpha B)) = 0_m \text{ for any } A, B \in \mathbb{H}_n(\mathbb{F}) \text{ and } \alpha \in \mathbb{F}^{-}$$

and

$$\mathrm{rk}(\Upsilon(A) + \alpha\Upsilon(B)) \leqslant m - 2 \text{ for any } A, B \in \mathbb{H}_n(\mathbb{F}) \text{ and } \alpha \in \mathbb{F}^-,$$

or $m = n$ and there exist some non-zero element $\xi \in \mathbb{F}^-$, non-singular matrix $Z \in \mathbb{M}_n(\mathbb{F})$ with $C_{n-1}(Z) = \omega Z$ for some non-zero element $\omega \in \mathbb{F}$ for which $\xi^{n-2}\omega\overline{\omega} = 1$ such that

$$\Upsilon(A) = \xi Z A Z^H \text{ for all } A \in \mathbb{H}_n(\mathbb{F})$$

or

$$\Upsilon(A) = \xi Z \overline{A} Z^H \text{ for all } A \in \mathbb{H}_n(\mathbb{F}).$$

Proof.

We start our proof from the necessity part. In case $\Upsilon(I_n) = 0_m$, the results follows immediately from **Lemma 4.2.1**. Next, we consider for $\Upsilon(I_n) \neq 0_m$. By **Lemma 4.2.3**, we have $m = n$. Besides that, from **Lemma 3.4.3**, **Lemma 3.4.9** and **Lemma 3.4.10**, we see that Υ is a rank-one non-increasing additive mapping. According to **Theorem 3.3.2** and **Theorem 3.3.3**, we know that Υ has the form $\Upsilon(A) = \xi Z A^{\phi} Z^H$ for all $A \in \mathbb{H}_n(\mathbb{F})$ where ξ is a non-zero element in $\mathbb{F}^-$, Z is a non-singular matrix in $\mathbb{M}_n(\mathbb{F})$ and ϕ is a non-zero field monomorphism of $\mathbb{F}$ with $\phi(\overline{\rho}) = \overline{\phi(\rho)}$ for all $\rho \in \mathbb{F}$. As an immediate consequence of **Lemma 4.2.4**, we have $C_{n-1}(Z) = \omega Z$ for some non-zero element $\omega \in \mathbb{F}$ with $\xi^{n-2}\omega\overline{\omega} = 1$ Moreover, according to **Lemma 4.2.5**, we obtain ϕ is identity or $\phi = -$. So we get the desired conclusion that is

$$\Upsilon(A) = \xi Z A Z^H \text{ for all } A \in \mathbb{H}_n(\mathbb{F})$$

or

$$\Upsilon(A) = \xi Z \overline{A} Z^H \text{ for all } A \in \mathbb{H}_n(\mathbb{F}).$$

Further, the sufficiency part follows from **Lemma 4.2.1** and **Lemma 4.2.6** immediately. We are done.

□

Theorem 4.2.8. Let $\mathbb{F}$ be a field carrying an involution $-$ with either $\mathbb{F}^- = GF(2)$, or $||\mathbb{F}^-|| > 3$ and $\mathrm{char}(\mathbb{F}) \neq 2$ when $-$ is identity. Let $m, n \in \mathbb{N}$ with $m, n > 2$. If $\Upsilon \colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ satisfies $\Upsilon(C_{n-1}(A - B)) = C_{m-1}(\Upsilon(A) - \Upsilon(B))$ for any $A, B \in \mathbb{H}_n(\mathbb{F})$ with Υ is onto if and only if $m = n$ and there exist some non-zero element $\xi \in \mathbb{F}^-$, non-singular matrix $Z \in \mathbb{M}_n(\mathbb{F})$ with $C_{n-1}(Z) = \omega Z$ for some non-zero element $\omega \in \mathbb{F}$ for which $\xi^{n-2}\omega\overline{\omega} = 1$ and non-zero field automorphism of $\mathbb{F}$, ϕ with $\phi(\overline{\rho}) = \overline{\phi(\rho)}$ for all $\rho \in \mathbb{F}$ such that

$$\Upsilon(A) = \xi Z A^{\phi} Z^H \text{ for all } A \in \mathbb{H}_n(\mathbb{F}).$$

Proof.

We start our proof from the necessity part. Given the fact that Υ is onto, thus $\Upsilon(I_n) \neq 0_m$. This is because if $\Upsilon(I_n) = 0_m$, then $\mathrm{rk}(\Upsilon(A)) \leqslant m-2$ for all $A \in \mathbb{H}_n(\mathbb{F})$. It is impossible as Υ is onto. Next, according to **Lemma 3.4.7**, Υ is bijective as Υ is one-to-one. Besides that, we also know that for any $A, B \in \mathbb{H}_n(\mathbb{F}), \mathrm{rk}(A-B) = n$ if and only if $\mathrm{rk}(\Upsilon(A) - \Upsilon(B)) = m$. Now we intend to separate our proof into the following two cases.

Case 1: $\mathbb{F}^- = GF(2)$. It follows from **Lemma 3.4.3** and **Lemma 3.4.10** that Υ is a rank-one non-increasing additive mapping. By referring to the **Theorem 3.3.2** and **Theorem 3.3.3**, we see that Υ has the form $\Upsilon(A) = \xi Z A^{\phi} Z^H$ for all $A \in \mathbb{H}_n(\mathbb{F})$ where ξ is a non-zero element in $\mathbb{F}^-, Z$ is a non-singular matrix in $\mathbb{M}_n(\mathbb{F})$ and ϕ is a non-zero field monomorphism of $\mathbb{F}$ with $\phi(\overline{\rho}) = \overline{\phi(\rho)}$ for all $\rho \in \mathbb{F}$. Since Υ is onto, then for all $\alpha \in \mathbb{F}$, there exists some $X \in \mathbb{H}_n(\mathbb{F})$ such that $\Upsilon(X) = \xi Z(\alpha E_{12} + \overline{\alpha} E_{21}) Z^H$. This leads to $X^{\phi} = \alpha E_{12} + \overline{\alpha} E_{21}$. This means for all $\alpha \in \mathbb{F}$, there exists some $X_{12} \in \mathbb{F}$ such that $\phi(X_{12}) = \alpha$. Consequently, ϕ is a field automorphism $\mathbb{F}$ as ϕ is onto. Also the desired results $C_{n-1}(Z) = \omega Z$ for some non-zero element $\omega \in \mathbb{F}$ with $\xi^{n-2}\omega\overline{\omega} = 1$ follows immediately from **Lemma 4.2.4**.

Case 2: $||\mathbb{F}^-|| > 3$ and $\mathrm{char}(\mathbb{F}) \neq 2$ when $-$ is identity. It follows from **Theorem 3.2.2** that Υ preserves the adjacency and $m = n$. Besides that, by **Theorem 3.2.3**, Υ has the form $\Upsilon(A) = \xi Z A^{\phi} Z^H + \mathcal{R}_0$ for all $A \in \mathbb{H}_n(\mathbb{F})$ where ξ is a non-zero element in $\mathbb{F}^-, Z$ is a non-singular matrix in $\mathbb{M}_n(\mathbb{F}), \phi$ is a field automorphism of $\mathbb{F}$ with $\phi(\overline{\rho}) = \overline{\phi(\rho)}$ and $\mathcal{R}_0 \in \mathbb{H}_n(\mathbb{F})$. As a reason that $\Upsilon(0_n) = 0_m$, so we attain $\mathcal{R}_0 = 0_m$. It follows that $\Upsilon(A) = \xi Z A^{\phi} Z^H$ for all $A \in \mathbb{H}_n(\mathbb{F})$. By referring to **Lemma 4.2.4**, we obtain $C_{n-1}(Z) = \omega Z$ for some non-zero element $\omega \in \mathbb{F}$ with $\xi^{n-2}\omega\overline{\omega} = 1$.

Conversely, by the fact given, clearly $\Upsilon(I_n) \neq 0_m$. Further, for any $A, B \in \mathbb{H}_n(\mathbb{F})$,

$$
\begin{aligned}
\Upsilon(C_{n-1}(A-B)) &= \xi Z C_{n-1}(A-B)^{\phi} Z^{H} \\
&= \xi(\omega^{-1}C_{n-1}(Z))C_{n-1}(A-B)^{\phi}(\overline{\omega}^{-1}C_{n-1}(Z^{H})) \\
&= \xi\overline{\omega}^{-1}\omega^{-1}C_{n-1}(Z)C_{n-1}(A-B)^{\phi}C_{n-1}(Z^{H}) \\
&= \xi\xi^{n-2}C_{n-1}(Z(A-B)^{\phi}Z^{H}) \\
&= \xi^{n-1}C_{n-1}(Z(A-B)^{\phi}Z^{H}) \\
&= C_{n-1}(\xi Z(A-B)^{\phi}Z^{H}) \\
&= C_{n-1}(\xi Z(A-1B)^{\phi}Z^{H}) \\
&= C_{n-1}(\xi Z A^{\phi} Z^{H} - \phi(1)\xi Z B^{\phi} Z^{H}) \\
&= C_{n-1}(\xi Z A^{\phi} Z^{H} - \xi Z B^{\phi} Z^{H}) \\
&= C_{n-1}(\Upsilon(A) - \Upsilon(B)) \\
&= C_{m-1}(\Upsilon(A) - \Upsilon(B)).
\end{aligned}
$$

□

4.3 Characterisation of Compound-Commuting Mappings on Symmetric Matrices

We recall again, $\mathbb{H}_n(\mathbb{F}) = \mathbb{S}_n(\mathbb{F})$ when $-$ is identity.

Theorem 4.3.1. Let $\mathbb{F}$ be a field with either $\mathbb{F} = GF(2)$ or $||\mathbb{F}|| > n+1$. Let $m, n \in \mathbb{N}$ with $m, n > 2$. If $\Upsilon\colon \mathbb{S}_n(\mathbb{F}) \to \mathbb{S}_m(\mathbb{F})$ is satisfying $\Upsilon(C_{n-1}(A+\alpha B)) = C_{m-1}(\Upsilon(A) + \alpha\Upsilon(B))$ for any $A, B \in \mathbb{S}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}$ if and only if

$$\Upsilon(A) = 0_m \text{ for any } A \in \mathbb{S}_n(\mathbb{F}) \text{ with } \mathrm{rk}(A) \leqslant 1,$$

$$\Upsilon(C_{n-1}(A+\alpha B)) = 0_m \text{ for any } A, B \in \mathbb{S}_n(\mathbb{F}) \text{ and } \alpha \in \mathbb{F}$$

and

$$\mathrm{rk}(\Upsilon(A) + \alpha\Upsilon(B)) \leqslant m-2 \text{ for any } A, B \in \mathbb{S}_n(\mathbb{F}) \text{ and } \alpha \in \mathbb{F},$$

or $m = n$ and there exist some non-zero element $\xi \in \mathbb{F}$, non-singular matrix $Z \in \mathbb{M}_n(\mathbb{F})$ with $C_{n-1}(Z) = \omega Z$ for some non-zero element $\omega \in \mathbb{F}$ for which $\xi^{n-2}\omega^2 = 1$ such that

$$\Upsilon(A) = \xi Z A Z^{T} \text{ for all } A \in \mathbb{S}_n(\mathbb{F}).$$

Proof.

We begin our proof with the necessity part. In case $\Upsilon(I_n) = 0_m$, the results follow immediately from **Lemma 4.2.1**. Afterward, if $\Upsilon(I_n) \neq 0_m$, by **Lemma 4.2.3**, we have $m = n$. Furthermore, Υ is a rank-one non-increasing additive mapping which follows from **Lemma 3.4.3**, **Lemma 3.4.9** and **Lemma 3.4.10**. As mentioned by **Theorem 3.3.4** , we recognize that Υ has the following form,

$$\Upsilon(A) = \xi Z A^{\phi} Z^T \text{ for all } A \in \mathbb{S}_n(\mathbb{F}) \tag{4.1}$$

or

$$\Upsilon(A) = \mathcal{Q}\Upsilon'(A)\mathcal{Q}^T \text{ for all } A \in \mathbb{S}_3(GF(2)) \tag{4.2}$$

where ξ is a non-zero element in $\mathbb{F}$, Z is a non-singular matrix in $\mathbb{M}_n(\mathbb{F})$ and ϕ is a non-zero field monomorphism of $\mathbb{F}$, $\mathcal{Q}$ is a non-singular matrix in $\mathbb{M}_3(GF(2))$ and Υ' is an additive mapping from $\mathbb{S}_3(GF(2))$ to $\mathbb{S}_3(GF(2))$.

If Υ has the form (4.1), by referring to **Lemma 4.2.4**, we have $C_{n-1}(Z) = \omega Z$ for some non-zero element $\omega \in \mathbb{F}$ with $\xi^{n-2}\omega^2 = 1$. Additionally, according to **Lemma 4.2.5**, we obtain ϕ is identity. So we have

$$\Upsilon(A) = \xi Z A Z^T \text{ for all } A \in \mathbb{S}_n(\mathbb{F}).$$

Now we consider that Υ has the form (4.2). Due to the fact that $\Upsilon(C_2(A)) = C_2(\Upsilon(A))$ for all $A \in \mathbb{S}_3(GF(2))$, so we know that

$$\begin{aligned}
&\mathcal{Q}\Upsilon'(C_2(A))\mathcal{Q}^T = C_2(\mathcal{Q}\Upsilon'(A)\mathcal{Q}^T) \\
\Rightarrow\ &\mathcal{Q}\Upsilon'(C_2(A))\mathcal{Q}^T = C_2(\mathcal{Q})C_2(\Upsilon'(A))C_2(\mathcal{Q}^T) \\
\Rightarrow\ &\Upsilon'(C_2(A)) = \mathcal{Q}^{-1}C_2(\mathcal{Q})C_2(\Upsilon'(A))C_2(\mathcal{Q}^T)(\mathcal{Q}^T)^{-1} \\
\Rightarrow\ &\Upsilon'(C_2(A)) = \mathcal{Q}^{-1}C_2(\mathcal{Q})C_2(\Upsilon'(A))(\mathcal{Q}^{-1}C_2(\mathcal{Q}))^T \\
\Rightarrow\ &\Upsilon'(C_2(A)) = XC_2(\Upsilon'(A))X^T
\end{aligned} \tag{4.3}$$

where $X = \mathcal{Q}^{-1}C_2(\mathcal{Q}) \in \mathbb{M}_3(GF(2))$ with $\mathrm{rk}(X) = 3$. Obviously, $\mathrm{rk}(\Upsilon'(A)) = \mathrm{rk}(\mathcal{Q}\Upsilon'(A)\mathcal{Q}^T) = \mathrm{rk}(\Upsilon(A))$ for all $A \in \mathbb{S}_3(GF(2))$ and $\Upsilon'(A) \in \mathbb{S}_3(GF(2))$ for all $A \in \mathbb{S}_3(GF(2))$. Since $\Upsilon(I_3) \neq 0_3$, then $\mathrm{rk}(\Upsilon'(I_3)) = \mathrm{rk}(\Upsilon(I_3)) = 3$. Therefore we see that

$$
\begin{aligned}
3 &= \mathrm{rk}(\Upsilon'(I_3)) \\
&= \mathrm{rk}(\Upsilon'(E_{11}+E_{22}+E_{33})) \\
&= \mathrm{rk}(\Upsilon'(E_{11})+\Upsilon'(E_{22})+\Upsilon'(E_{33})) \\
&\leqslant \mathrm{rk}(\Upsilon'(E_{11}))+\mathrm{rk}(\Upsilon'(E_{22}))+\mathrm{rk}(\Upsilon'(E_{33})).
\end{aligned}
$$

Thus $\mathrm{rk}(\Upsilon'(E_{11}))+\mathrm{rk}(\Upsilon'(E_{22}))+\mathrm{rk}(\Upsilon'(E_{33})) \geqslant 3$. We conclude that $\mathrm{rk}(\Upsilon'(E_{11})) = \mathrm{rk}(\Upsilon'(E_{22})) = \mathrm{rk}(\Upsilon'(E_{33})) = 1$. By using the same method as above, we also conclude that $\mathrm{rk}(\Upsilon'(E_{ii}+E_{jj})) = 2$ for all $1 \leqslant i < j \leqslant 3$.

Now we find a matrix $G \in \mathbb{M}_3(GF(2))$ with $\mathrm{rk}(G) = 3$ such that $\Upsilon'(E_{ii}) = GE_{ii}G^T$ for all $i \in \{1,2,3\}$. By **Corollary 2.1.8.4**, we know that there exists some $G_1 \in \mathbb{M}_3(GF(2))$ with $\mathrm{rk}(G_1) = 3$ such that $\Upsilon'(E_{11}) = G_1E_{11}G_1^T$.

We let

$$
\Upsilon'(E_{22}) = G_1 \begin{bmatrix} \chi_1 & \mathcal{J}_1 \\ \mathcal{J}_1^T & \mathcal{K}_1 \end{bmatrix} G_1^T
$$

where $\chi_1 \in GF(2), \mathcal{J}_1 \in \mathbb{M}_{1\times 2}(GF(2))$ and $\mathcal{K}_1 \in \mathbb{S}_2(GF(2))$. Suppose that $\mathcal{K}_1 = 0_2$. Inevitably, $\mathcal{J}_1 = 0_{1\times 2}$ as $\mathrm{rk}(\Upsilon'(E_{22})) = 1$. Consequently, $\Upsilon'(E_{11}) + \Upsilon'(E_{22}) = (1+\chi_1)G_1E_{11}G_1^T$. This contradicts to the fact that $\mathrm{rk}(\Upsilon'(E_{11}) + \Upsilon'(E_{22})) = 2$. So $\mathcal{K}_1 \neq 0_2$. If $\mathrm{rk}(\mathcal{K}_1) = 2$, then $\mathrm{rk}\left(\begin{bmatrix} \mathcal{J}_1^T & \mathcal{K}_1 \end{bmatrix}\right) = 2$. This implies $\mathrm{rk}(\Upsilon'(E_{22})) \geqslant 2$. It is impossible. As a result, $\mathrm{rk}(\mathcal{K}_1) = 1$. By applying **Corollary 2.1.8.4** again, there exists some $G_2 \in \mathbb{M}_2(GF(2))$ with $\mathrm{rk}(G_2) = 2$ such that $\mathcal{K}_1 = G_2E_{11}G_2^T$. Since $G_2 \in \mathbb{M}_2(GF(2))$ with $\mathrm{rk}(G_2) = 2$, then there exists a $P = \begin{bmatrix} p_0 & p_1 \end{bmatrix} \in \mathbb{M}_{1\times 2}(GF(2))$ such that $\mathcal{J}_1 = PG_2^T$. Thereupon we have

$$
\begin{aligned}
\Upsilon'(E_{22}) &= G_1 \begin{bmatrix} \chi_1 & \mathcal{J}_1 \\ \mathcal{J}_1^T & G_2E_{11}G_2^T \end{bmatrix} G_1^T \\
&= G_1 \begin{bmatrix} \chi_1 & PG_2^T \\ G_2P^T & G_2E_{11}G_2^T \end{bmatrix} G_1^T \\
&= G_1 \begin{bmatrix} 1 & 0_{1\times 2} \\ 0_{2\times 1} & G_2 \end{bmatrix} \begin{bmatrix} \chi_1 & P \\ P^T & E_{11} \end{bmatrix} \begin{bmatrix} 1 & 0_{1\times 2} \\ 0_{2\times 1} & G_2^T \end{bmatrix} G_1^T \\
&= G_1 \begin{bmatrix} 1 & 0_{1\times 2} \\ 0_{2\times 1} & G_2 \end{bmatrix} \begin{bmatrix} \chi_1 & p_0 & p_1 \\ p_0 & 1 & 0 \\ p_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0_{1\times 2} \\ 0_{2\times 1} & G_2^T \end{bmatrix} G_1^T.
\end{aligned}
$$

By the fact that $\mathrm{rk}\left(\begin{bmatrix} \chi_1 & p_0 & p_1 \\ p_0 & 1 & 0 \\ p_1 & 0 & 0 \end{bmatrix}\right) = 1$, we know that there exist some non-trivial $c_0, c_1 \in GF(2)$ (i.e, $c_0 \neq 0$ or $c_1 \neq 0$) such that

$$\begin{aligned} c_0 \begin{bmatrix} p_0 & 1 & 0 \end{bmatrix} &= c_1 \begin{bmatrix} p_1 & 0 & 0 \end{bmatrix} \\ \Rightarrow c_0 &= 0, c_1 = 1 \\ \Rightarrow p_1 &= 0. \end{aligned}$$

Moreover, there exist some non-trivial $c_2, c_3 \in GF(2)$ (i.e, $c_2 \neq 0$ or $c_3 \neq 0$) such that

$$\begin{aligned} c_2 \begin{bmatrix} \chi_1 & p_0 & 0 \end{bmatrix} &= c_3 \begin{bmatrix} p_0 & 1 & 0 \end{bmatrix} \\ \Rightarrow c_2 &= 1 \quad (\because c_2 = 0 \text{ implies } c_3 = 0) \\ \Rightarrow \chi_1 &= c_3 p_0, p_0 = c_3 \\ \Rightarrow \chi_1 &= p_0^2. \end{aligned}$$

This leads to

$$\begin{aligned} \Upsilon'(E_{22}) &= G_1 \begin{bmatrix} 1 & 0_{1\times 2} \\ 0_{2\times 1} & G_2 \end{bmatrix} \begin{bmatrix} p_0^2 & p_0 & 0 \\ p_0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0_{1\times 2} \\ 0_{2\times 1} & G_2^T \end{bmatrix} G_1^T \\ &= G_1 \begin{bmatrix} 1 & 0_{1\times 2} \\ 0_{2\times 1} & G_2 \end{bmatrix} \begin{bmatrix} 1 & p_0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ p_0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0_{1\times 2} \\ 0_{2\times 1} & G_2^T \end{bmatrix} G_1^T. \end{aligned}$$

We let $G_3 = G_1 \begin{bmatrix} 1 & 0_{1\times 2} \\ 0_{2\times 1} & G_2 \end{bmatrix} \begin{bmatrix} 1 & p_0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. It is clear that $G_3 \in \mathbb{M}_3(GF(2))$ with $\mathrm{rk}(G_3) = 3$. Hence we have $\Upsilon'(E_{22}) = G_3 E_{22} G_3^T$. Since

$$\begin{aligned} G_3 E_{11} G_3^T &= G_1 \begin{bmatrix} 1 & 0_{1\times 2} \\ 0_{2\times 1} & G_2 \end{bmatrix} \begin{bmatrix} 1 & p_0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ p_0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0_{1\times 2} \\ 0_{2\times 1} & G_2^T \end{bmatrix} G_1^T \\ &= G_1 E_{11} G_1^T, \end{aligned}$$

then $G_3 E_{11} G_3^T = \Upsilon'(E_{11})$. Consequently, $\Upsilon'(E_{ii}) = G_3 E_{ii} G_3^T$ for all $i \in \{1, 2\}$.

Next, we let

$$\Upsilon'(E_{33}) = G_3 \begin{bmatrix} \mathcal{K}_2 & \mathcal{J}_2 \\ \mathcal{J}_2^T & \chi_2 \end{bmatrix} G_3^T$$

where $\chi_2 \in GF(2)$, $\mathcal{J}_2 = \begin{bmatrix} \mathcal{J}_{21} \\ \mathcal{J}_{22} \end{bmatrix} \in \mathbb{M}_{2\times 1}(GF(2))$ and $\mathcal{K}_2 = \begin{bmatrix} t_1 & t_3 \\ t_3 & t_2 \end{bmatrix} \in \mathbb{S}_2(GF(2))$. We suppose that $\chi_2 = 0$. Together with the fact that $\mathrm{rk}(\Upsilon'(E_{33})) = 1$ and $\mathrm{rk}(\Upsilon'(I_3)) = 3$, we know that

$$\begin{aligned}
&\left| G_3 \begin{bmatrix} \mathcal{K}_2 & \mathcal{J}_2 \\ \mathcal{J}_2^T & \chi_2 \end{bmatrix} G_3^T \right| = 0 \quad \text{and} \quad \left| G_3 \left(E_{11} + E_{22} + \begin{bmatrix} \mathcal{K}_2 & \mathcal{J}_2 \\ \mathcal{J}_2^T & \chi_2 \end{bmatrix} \right) G_3^T \right| = 1 \\
\Rightarrow &\begin{vmatrix} t_1 & t_3 & \mathcal{J}_{21} \\ t_3 & t_2 & \mathcal{J}_{22} \\ \mathcal{J}_{21} & \mathcal{J}_{22} & 0 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} t_1+1 & t_3 & \mathcal{J}_{21} \\ t_3 & t_2+1 & \mathcal{J}_{22} \\ \mathcal{J}_{21} & \mathcal{J}_{22} & 0 \end{vmatrix} = 1 \\
\Rightarrow &\, t_2\mathcal{J}_{21}^2 + t_1\mathcal{J}_{22}^2 = 0 \quad \text{and} \quad (t_2+1)\mathcal{J}_{21}^2 + (t_1+1)\mathcal{J}_{22}^2 = 1 \\
\Rightarrow &\, \mathcal{J}_{21}^2 + \mathcal{J}_{22}^2 = 1.
\end{aligned}$$

We see that either $\mathcal{J}_{21} = 1, \mathcal{J}_{22} = 0, t_2 = 0, t_1, t_3 \in GF(2)$ or $\mathcal{J}_{21} = 0, \mathcal{J}_{22} = 1, t_1 = 0, t_2, t_3 \in GF(2)$. Both the solutions give us

$$\mathrm{rk}\left(\begin{bmatrix} \mathcal{K}_2 & \mathcal{J}_2 \\ \mathcal{J}_2^T & \chi_2 \end{bmatrix} \right) = \mathrm{rk}\left(\begin{bmatrix} t_1 & t_3 & \mathcal{J}_{21} \\ t_3 & t_2 & \mathcal{J}_{22} \\ \mathcal{J}_{21} & \mathcal{J}_{22} & 0 \end{bmatrix} \right) = 2.$$

This contradicts to the fact that $\mathrm{rk}(\Upsilon'(E_{33})) = 1$. So χ_2 must be equal to 1. Since $\mathrm{rk}\left(\begin{bmatrix} t_1 & t_3 & \mathcal{J}_{21} \\ t_3 & t_2 & \mathcal{J}_{22} \\ \mathcal{J}_{21} & \mathcal{J}_{22} & 1 \end{bmatrix} \right) = 1$, therefore there exist some non-trivial $c_4, c_5 \in GF(2)$ (i.e, $c_4 \neq 0$ or $c_5 \neq 0$) and non-trivial $c_6, c_7 \in GF(2)$ (i.e, $c_6 \neq 0$ or $c_7 \neq 0$) such that

$$\begin{aligned}
&c_4 \begin{bmatrix} t_1 & t_3 & \mathcal{J}_{21} \end{bmatrix} = c_5 \begin{bmatrix} \mathcal{J}_{21} & \mathcal{J}_{22} & 1 \end{bmatrix} \quad \text{and} \quad c_6 \begin{bmatrix} t_3 & t_2 & \mathcal{J}_{22} \end{bmatrix} = c_7 \begin{bmatrix} \mathcal{J}_{21} & \mathcal{J}_{22} & 1 \end{bmatrix} \\
\Rightarrow &\, c_4 = 1 \quad (\because c_4 = 0 \text{ implies } c_5 = 0) \quad \text{and} \quad c_6 = 1 \quad (\because c_6 = 0 \text{ implies } c_7 = 0) \\
\Rightarrow &\, t_1 = c_5\mathcal{J}_{21}, t_3 = c_5\mathcal{J}_{22}, \mathcal{J}_{21} = c_5 \quad \text{and} \quad t_3 = c_7\mathcal{J}_{21}, t_2 = c_7\mathcal{J}_{22}, \mathcal{J}_{22} = c_7 \\
\Rightarrow &\, t_1 = \mathcal{J}_{21}^2, t_3 = \mathcal{J}_{21}\mathcal{J}_{22}, t_2 = \mathcal{J}_{22}^2.
\end{aligned}$$

Hence we acquire

$$\begin{bmatrix} t_1 & t_3 & \mathcal{J}_{21} \\ t_3 & t_2 & \mathcal{J}_{22} \\ \mathcal{J}_{21} & \mathcal{J}_{22} & 1 \end{bmatrix} = \begin{bmatrix} \mathcal{J}_{21}^2 & \mathcal{J}_{21}\mathcal{J}_{22} & \mathcal{J}_{21} \\ \mathcal{J}_{21}\mathcal{J}_{22} & \mathcal{J}_{22}^2 & \mathcal{J}_{22} \\ \mathcal{J}_{21} & \mathcal{J}_{22} & 1 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} \mathcal{K}_2 & \mathcal{J}_2 \\ \mathcal{J}_2^T & 1 \end{bmatrix} = \begin{bmatrix} \mathcal{J}_2\mathcal{J}_2^T & \mathcal{J}_2 \\ \mathcal{J}_2^T & 1 \end{bmatrix}.$$

So we see that

$$\begin{aligned} \Upsilon'(E_{33}) &= G_3 \begin{bmatrix} \mathcal{K}_2 & \mathcal{J}_2 \\ \mathcal{J}_2^T & 1 \end{bmatrix} G_3^T \\ &= G_3 \begin{bmatrix} \mathcal{J}_2\mathcal{J}_2^T & \mathcal{J}_2 \\ \mathcal{J}_2^T & 1 \end{bmatrix} G_3^T \\ &= G_3 \begin{bmatrix} I_2 & \mathcal{J}_2 \\ 0_{1\times 2} & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I_2 & 0_{2\times 1} \\ \mathcal{J}_2^T & 1 \end{bmatrix} G_3^T \\ &= GE_{33}G^T \end{aligned}$$

where $G = G_3 \begin{bmatrix} I_2 & \mathcal{J}_2 \\ 0_{1\times 2} & 1 \end{bmatrix} \in \mathbb{M}_3(GF(2))$ with $\mathrm{rk}(G) = 3$. Because

$$GE_{11}G^T = G_3 \begin{bmatrix} I_2 & \mathcal{J}_2 \\ 0_{1\times 2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_2 & 0_{2\times 1} \\ \mathcal{J}_2^T & 1 \end{bmatrix} G_3^T = G_3E_{11}G_3^T = \Upsilon'(E_{11})$$

and

$$GE_{22}G^T = G_3 \begin{bmatrix} I_2 & \mathcal{J}_2 \\ 0_{1\times 2} & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_2 & 0_{2\times 1} \\ \mathcal{J}_2^T & 1 \end{bmatrix} G_3^T = G_3E_{22}G_3^T = \Upsilon'(E_{22}),$$

thus $\Upsilon(E_{ii}) = GE_{ii}G^T$ for all $i \in \{1, 2, 3\}$.

Next we prove that $\Upsilon(E_{ij} + E_{ji}) = G(E_{ij} + E_{ji})G^T$ for all $1 \leqslant i < j \leqslant 3$. Now we write down all the three possibilities of $E_{ij} + E_{ji}$ which are

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

From the above matrices, it is not difficult to see that $C_2(E_{ij}+E_{ji}) = E_{4-k,4-k}$ where $1 \leqslant k \leqslant 3$ with $k \neq i, j$. When we set $A = E_{ij}+E_{ji}$ in (4.3), we know that

$$\begin{aligned}
&\Upsilon'(C_2(E_{ij}+E_{ji})) = XC_2(\Upsilon'(E_{ij}+E_{ji}))X^T \\
\Rightarrow\ &\Upsilon'(E_{4-k,4-k}) = XC_2(\Upsilon'(E_{ij}+E_{ji}))X^T \\
\Rightarrow\ &GE_{4-k,4-k}G^T = XC_2(\Upsilon'(E_{ij}+E_{ji}))X^T \\
\Rightarrow\ &C_2(\Upsilon'(E_{ij}+E_{ji})) = (X^{-1}G)E_{4-k,4-k}(X^{-1}G)^T \\
\Rightarrow\ &\mathrm{rk}(C_2(\Upsilon'(E_{ij}+E_{ji}))) = 1 \\
\Rightarrow\ &\mathrm{rk}(\Upsilon'(E_{ij}+E_{ji})) = 2.
\end{aligned}$$

Let

$$\Upsilon'(E_{ij}+E_{ji}) = GYG^T$$

where $Y \in \mathbb{S}_3(GF(2))$ with $\mathrm{rk}(Y) = 2$. We hope to get $Y = E_{ij}+E_{ji}$. Due to $\mathrm{rk}(E_{ii}+E_{jj}+E_{ij}+E_{ji}) = 1$, then $\mathrm{rk}(\Upsilon'(E_{ii}+E_{jj}+E_{ij}+E_{ji})) = \mathrm{rk}(\Upsilon(E_{ii}+E_{jj}+E_{ij}+E_{ji})) \leqslant 1$. We assume that $\mathrm{rk}(\Upsilon'(E_{ii}+E_{jj}+E_{ij}+E_{ji})) = 0$. This implies

$$\begin{aligned}
&\Upsilon'(E_{ii}+E_{jj}+E_{ij}+E_{ji}) = 0_3 \\
\Rightarrow\ &\Upsilon'(E_{ii})+\Upsilon'(E_{jj})+\Upsilon'(E_{ij}+E_{ji}) = 0_3 \\
\Rightarrow\ &\Upsilon'(E_{ij}+E_{ji}) = \Upsilon'(E_{ii})+\Upsilon'(E_{jj}) \\
\Rightarrow\ &\Upsilon'(E_{ij}+E_{ji})+\Upsilon'(E_{jj})+\Upsilon'(E_{kk}) = \Upsilon'(E_{ii})+\Upsilon'(E_{jj})+\Upsilon'(E_{jj})+\Upsilon'(E_{kk}) \\
\Rightarrow\ &\Upsilon'(E_{ij}+E_{ji})+\Upsilon'(E_{jj})+\Upsilon'(E_{kk}) = \Upsilon'(E_{ii})+\Upsilon'(E_{kk}) \\
\Rightarrow\ &\Upsilon'(E_{ij}+E_{ji}+E_{jj}+E_{kk}) = G(E_{ii}+E_{kk})G^T \\
\Rightarrow\ &\mathrm{rk}(\Upsilon'(E_{ij}+E_{ji}+E_{jj}+E_{kk})) = 2.
\end{aligned}$$

This contradicts to the fact that $\mathrm{rk}(\Upsilon'(E_{ij}+E_{ji}+E_{jj}+E_{kk})) = \mathrm{rk}(\Upsilon(E_{ij}+E_{ji}+E_{jj}+E_{kk})) = 3$ as $\mathrm{rk}(E_{ij}+E_{ji}+E_{jj}+E_{kk}) = 3$. This contradiction shows that our supposition $\mathrm{rk}(\Upsilon'(E_{ii}+E_{jj}+E_{ij}+E_{ji})) = 0$ is false. Consequently, $\mathrm{rk}(\Upsilon'(E_{ii}+E_{jj}+E_{ij}+E_{ji})) = \mathrm{rk}(G(Y+E_{ii}+E_{jj})G^T) = \mathrm{rk}(Y+E_{ii}+E_{jj}) = 1$. Now we presume that $Y_{kk} = 1$. So all the three possibilities of $Y+E_{ii}+E_{jj}$ are

$$\begin{bmatrix} 1 & Y_{ki} & Y_{kj} \\ Y_{ik} & Y_{ii}+1 & Y_{ij} \\ Y_{jk} & Y_{ji} & Y_{jj}+1 \end{bmatrix}, \begin{bmatrix} Y_{ii}+1 & Y_{ik} & Y_{ij} \\ Y_{ki} & 1 & Y_{kj} \\ Y_{ji} & Y_{jk} & Y_{jj}+1 \end{bmatrix} \text{ and } \begin{bmatrix} Y_{ii}+1 & Y_{ij} & Y_{ik} \\ Y_{ji} & Y_{jj}+1 & Y_{jk} \\ Y_{ki} & Y_{kj} & 1 \end{bmatrix}.$$

This implies

$$\begin{bmatrix} 1 & Y_{ki} & Y_{kj} \\ Y_{ki} & Y_{ii}+1 & Y_{ij} \\ Y_{kj} & Y_{ij} & Y_{jj}+1 \end{bmatrix}, \begin{bmatrix} Y_{ii}+1 & Y_{ki} & Y_{ij} \\ Y_{ki} & 1 & Y_{kj} \\ Y_{ij} & Y_{kj} & Y_{jj}+1 \end{bmatrix} \text{ and } \begin{bmatrix} Y_{ii}+1 & Y_{ij} & Y_{ki} \\ Y_{ij} & Y_{jj}+1 & Y_{kj} \\ Y_{ki} & Y_{kj} & 1 \end{bmatrix}$$

as $Y + E_{ii} + E_{jj} \in \mathbb{S}_3(GF(2))$. As a reason that $\mathrm{rk}(Y + E_{ii} + E_{jj}) = 1$, thus there exist some non-trivial $c_8, c_9 \in GF(2)$ (i.e, $c_8 \neq 0$ or $c_9 \neq 0$) and non-trivial $c_{10}, c_{11} \in GF(2)$ (i.e, $c_{10} \neq 0$ or $c_{11} \neq 0$) such that

$$c_8 \begin{bmatrix} 1 & Y_{ki} & Y_{kj} \end{bmatrix} = c_9 \begin{bmatrix} Y_{ki} & Y_{ii}+1 & Y_{ij} \end{bmatrix} \quad \text{and} \quad c_{10} \begin{bmatrix} 1 & Y_{ki} & Y_{kj} \end{bmatrix} = c_{11} \begin{bmatrix} Y_{kj} & Y_{ij} & Y_{jj}+1 \end{bmatrix}$$

$$\Rightarrow c_9 = 1 \quad (\because c_9 = 0 \text{ implies } c_8 = 0) \quad \text{and} \quad c_{11} = 1 \quad (\because c_{11} = 0 \text{ implies } c_{10} = 0)$$

$$\Rightarrow c_8 = Y_{ki}, c_8 Y_{ki} = (Y_{ii}+1), c_8 Y_{kj} = Y_{ij}$$

$$\text{and } c_{10} = Y_{kj}, c_{10} Y_{ki} = Y_{ij}, c_{10} Y_{kj} = (Y_{jj}+1)$$

$$\Rightarrow Y_{ki}^2 = Y_{ii}+1, Y_{ki}Y_{kj} = Y_{ij}, Y_{kj}^2 = Y_{jj}+1.$$

By using the same method as above, we obtain the same results for another two matrices. So all the three possibilities of Y are

$$\begin{bmatrix} 1 & Y_{ki} & Y_{kj} \\ Y_{ki} & Y_{ki}^2+1 & Y_{ki}Y_{kj} \\ Y_{kj} & Y_{ki}Y_{kj} & Y_{kj}^2+1 \end{bmatrix}, \begin{bmatrix} Y_{ki}^2+1 & Y_{ki} & Y_{ki}Y_{kj} \\ Y_{ki} & 1 & Y_{kj} \\ Y_{ki}Y_{kj} & Y_{kj} & Y_{kj}^2+1 \end{bmatrix} \text{ and } \begin{bmatrix} Y_{ki}^2+1 & Y_{ki}Y_{kj} & Y_{ki} \\ Y_{ki}Y_{kj} & Y_{kj}^2+1 & Y_{kj} \\ Y_{ki} & Y_{kj} & 1 \end{bmatrix}.$$

Next, we let $\lambda_1, \lambda_2, \lambda_3 \in GF(2)$. Since $\lambda_1 \begin{bmatrix} 1 & Y_{ki} & Y_{kj} \end{bmatrix} + \lambda_2 \begin{bmatrix} Y_{ki} & Y_{ki}^2+1 & Y_{ki}Y_{kj} \end{bmatrix} + \lambda_3 \begin{bmatrix} Y_{kj} & Y_{ki}Y_{kj} & Y_{kj}^2+1 \end{bmatrix} = 0_{1\times 3}$ has only trivial solutions (i.e, $\lambda_1 = \lambda_2 = \lambda_3 = 0$). Thereupon $\mathrm{rk}(Y) = 3$. We obtain the same results for the other two matrices. This contradicts to the fact that $\mathrm{rk}(Y) = 2$. This contradiction shows that our supposition $Y_{kk} = 1$ is false. Then $Y_{kk} = 0$.

Now the three possibilities of Y are

$$\begin{bmatrix} 0 & Y_{ki} & Y_{kj} \\ Y_{ki} & Y_{ii} & Y_{ij} \\ Y_{kj} & Y_{ij} & Y_{jj} \end{bmatrix}, \begin{bmatrix} Y_{ii} & Y_{ki} & Y_{ij} \\ Y_{ki} & 0 & Y_{kj} \\ Y_{ij} & Y_{kj} & Y_{jj} \end{bmatrix} \text{ and } \begin{bmatrix} Y_{ii} & Y_{ij} & Y_{ki} \\ Y_{ij} & Y_{jj} & Y_{kj} \\ Y_{ki} & Y_{kj} & 0 \end{bmatrix}.$$

Evidently three possibilities of $Y + E_{ii} + E_{jj}$ are

$$\begin{bmatrix} 0 & Y_{ki} & Y_{kj} \\ Y_{ki} & Y_{ii}+1 & Y_{ij} \\ Y_{kj} & Y_{ij} & Y_{jj}+1 \end{bmatrix}, \begin{bmatrix} Y_{ii}+1 & Y_{ki} & Y_{ij} \\ Y_{ki} & 0 & Y_{kj} \\ Y_{ij} & Y_{kj} & Y_{jj}+1 \end{bmatrix} \text{ and } \begin{bmatrix} Y_{ii}+1 & Y_{ij} & Y_{ki} \\ Y_{ij} & Y_{jj}+1 & Y_{kj} \\ Y_{ki} & Y_{kj} & 0 \end{bmatrix}.$$

By the fact that $\mathrm{rk}(Y) = 2$ and $\mathrm{rk}(Y + E_{ii} + E_{jj}) = 1$, hence

$$\begin{aligned} & |Y| = 0 \quad \text{and} \quad |Y + E_{ii} + E_{jj}| = 0 \\ \Rightarrow\ & Y_{ki}^2 Y_{jj} + Y_{kj}^2 Y_{ii} = 0 \quad \text{and} \quad Y_{ki}^2(Y_{jj}+1) + Y_{kj}^2(Y_{ii}+1) = 0 \\ \Rightarrow\ & Y_{ki}^2 + Y_{kj}^2 = 0. \end{aligned}$$

Consider $Y_{ki} = Y_{kj} = 1$. Then three possibilities of $Y + E_{ii} + E_{jj}$ are

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & Y_{ii}+1 & Y_{ij} \\ 1 & Y_{ij} & Y_{jj}+1 \end{bmatrix}, \begin{bmatrix} Y_{ii}+1 & 1 & Y_{ij} \\ 1 & 0 & 1 \\ Y_{ij} & 1 & Y_{jj}+1 \end{bmatrix} \text{ and } \begin{bmatrix} Y_{ii}+1 & Y_{ij} & 1 \\ Y_{ij} & Y_{jj}+1 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

This shows us that $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$ & $\begin{bmatrix} 1 & Y_{ii}+1 & Y_{ij} \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$ & $\begin{bmatrix} Y_{ii}+1 & 1 & Y_{ij} \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$ & $\begin{bmatrix} Y_{ii}+1 & Y_{ij} & 1 & 1 \end{bmatrix}$ are linearly independent. It is impossible as $\mathrm{rk}(Y + E_{ii} + E_{jj}) = 1$. So we are forced to conclude that $Y_{ki} = Y_{kj} = 0$. So we have

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & Y_{ii}+1 & Y_{ij} \\ 0 & Y_{ij} & Y_{jj}+1 \end{bmatrix}, \begin{bmatrix} Y_{ii}+1 & 0 & Y_{ij} \\ 0 & 0 & 0 \\ Y_{ij} & 0 & Y_{jj}+1 \end{bmatrix} \text{ and } \begin{bmatrix} Y_{ii}+1 & Y_{ij} & 0 \\ Y_{ij} & Y_{jj}+1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Because $\mathrm{rk}(Y + E_{ii} + E_{jj}) = 1$, thereupon $\begin{bmatrix} Y_{ii}+1 & Y_{ij} \end{bmatrix}$ and $\begin{bmatrix} Y_{ij} & Y_{jj}+1 \end{bmatrix}$ are linearly dependent. This leads to

$$\begin{vmatrix} Y_{ii}+1 & Y_{ij} \\ Y_{ij} & Y_{jj}+1 \end{vmatrix} = 0 \Rightarrow Y_{ij}^2 = (Y_{ii}+1)(Y_{jj}+1).$$

In the case that $Y_{ij} = 0$, then Y_{ii} and Y_{jj} must be equal to 1. This implies $Y + E_{ii} + E_{jj} = 0_3$. It is impossible as $\mathrm{rk}(Y + E_{ii} + E_{jj}) = 1$. Thereby we obtain $Y_{ij} = 1$ and $Y_{ii} = Y_{jj} = 0$. Consequently, $Y = E_{ij} + E_{ji}$. It follows that $\Upsilon'(E_{ij} + E_{ji}) = G(E_{ij} + E_{ji})G^T$ for all $1 \leqslant i < j \leqslant 3$.

Together with the fact that Υ' is additive, $\Upsilon'(E_{ii}) = GE_{ii}G^T$ for all $i \in \{1,2,3\}$ and $\Upsilon'(E_{ij} + E_{ji}) = G(E_{ij} + E_{ji})G^T$ for all $i, j \in \mathbb{N}$ with $1 \leqslant i < j \leqslant 3$ where $G \in \mathbb{M}_3(GF(2))$ with $\mathrm{rk}(G) = 3$. Accordingly $\Upsilon'(A) = GAG^T$ for all $A \in \mathbb{H}_3(GF(2))$. Thus we attain

$$\Upsilon(A) = ZAZ^T \text{ for all } A \in \mathbb{H}_3(GF(2))$$

where $Z = QG \in \mathbb{M}_3(GF(2))$ with $\mathrm{rk}(QG) = 3$.

Furthermore, by applying **Lemma 4.2.1** and **Lemma 4.2.6**, the sufficiency part follows immediately. Finally, we accomplished our proof.

□

As an immediate consequence of **Theorem 4.2.8**, when $-$ is identity, we have the following theorem.

Theorem 4.3.2. Let $\mathbb{F}$ be a field with either $\mathbb{F} = GF(2)$, or $||\mathbb{F}|| > 3$ and $\mathrm{char}(\mathbb{F}) \neq 2$. Let $m, n \in \mathbb{N}$ with $m, n > 2$. If $\Upsilon\colon \mathbb{S}_n(\mathbb{F}) \to \mathbb{S}_m(\mathbb{F})$ is satisfying $\Upsilon(C_{n-1}(A - B)) = C_{m-1}(\Upsilon(A) - \Upsilon(B))$ for any $A, B \in \mathbb{H}_n(\mathbb{F})$ with Υ is onto if and only if $m = n$ and there exists some non-zero element $\xi \in \mathbb{F}$, non-singular matrix $Z \in \mathbb{M}_n(\mathbb{F})$ with $C_{n-1}(Z) = \omega Z$ for some non-zero element $\omega \in \mathbb{F}$ for which $\xi^{n-2}\omega^2 = 1$ and non-zero field automorphism of $\mathbb{F}$, ϕ such that

$$\Upsilon(A) = \xi Z A^{\phi} Z^T \text{ for all } A \in \mathbb{S}_n(\mathbb{F}).$$

CHAPTER 5

CONCLUSION

In order to characterise a general form of a mapping $\Upsilon\colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ satisfying [P1] or [P2], we need to impose some assumptions on Υ. If Υ satisfies [P1] with $\Upsilon(I_n) \neq 0_m$, then Υ is satisfied $\mathrm{rk}(A - B) = n$ if and only if $\mathrm{rk}(\Upsilon(A) - \Upsilon(B)) = m$ for any $A, B \in \mathbb{H}_n(\mathbb{F})$ (see **Lemma 3.4.7**). Thus we apply the theorem in **Section 3.2** by imposing onto condition into Υ. Also, if Υ satisfies [P2] with $\Upsilon(I_n) \neq 0_m$, then Υ is a rank-one non-increasing additive mapping (see **Lemma 3.4.3**, **Lemma 3.4.9** and **Lemma 3.4.10**). Hence we use the theorem in **Section 3.3**. In the case of Υ satisfies [P2] with $\Upsilon(I_n) = 0_m$, we have $\Upsilon(A) = 0_m$ for any $A \in \mathbb{H}_n(\mathbb{F})$ with $\mathrm{rk}(A) \leqslant 1, \Upsilon(C_{n-1}(A)) = 0_m$ for any $A \in \mathbb{H}_n(\mathbb{F})$ and $\mathrm{rk}(\Upsilon(A)) \leqslant m-2$ for any $A \in \mathbb{H}_n(\mathbb{F})$ (see **Corollary 3.4.6**). In this project, we are unable to find the general form of such mappings. But, we use the properties in **Corollary 3.4.6** to construct some examples of non-zero mapping $\Upsilon\colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ satisfying [P2] with $\Upsilon(I_n) = 0_m$ in **Section 4.1**.

In conclusion, this project's main goal is to study the compound-commuting mappings on Hermitian matrices and symmetric matrices. In this project, we characterise the $\Upsilon\colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ that satisfies one of the following conditions:

[P1] $\mathbb{F}$ with either $\mathbb{F}^- = GF(2)$ or $||\mathbb{F}^-|| > 3$ and Υ satisfies $\Upsilon(C_{n-1}(A - B)) = C_{m-1}(\Upsilon(A) - \Upsilon(B))$ for any $A, B \in \mathbb{H}_n(\mathbb{F})$ with Υ is onto;

[P2] $\mathbb{F}$ with either $\mathbb{F}^- = GF(2)$ or $||\mathbb{F}^-|| > n+1$ and Υ satisfies $\Upsilon(C_{n-1}(A+\alpha B)) = C_{m-1}(\Upsilon(A) + \alpha\Upsilon(B))$ for any $A, B \in \mathbb{H}_n(\mathbb{F})$ and $\alpha \in \mathbb{F}^-$

in **Theorem 4.2.7**, **Theorem 4.2.8** and **Theorem 4.3.1**. Also, we construct some examples of non-zero mapping $\Upsilon\colon \mathbb{H}_n(\mathbb{F}) \to \mathbb{H}_m(\mathbb{F})$ satisfying

$$\Upsilon(C_{n-1}(A + \alpha B)) = C_{m-1}(\Upsilon(A) + \alpha\Upsilon(B)) \text{ for any } A, B \in \mathbb{H}_n(\mathbb{F}) \text{ and } \alpha \in \mathbb{F}^-$$

with $\Upsilon(I_n) = 0_m$ in **Example 4.1.1**. Thus our objectives are met.

In this project, we have also found several unresolved concerns for future study. In the proofs of **Theorem 4.2.7** and **Theorem 4.3.1**, we have used **Lemma 3.4.9**. Besides that, in the proofs of **Theorem 4.2.8**, we have applied **Theorem 3.2.2**. Because

Lemma 3.4.9 and **Theorem 3.2.2** do not include the situation for $3 \leqslant |\mathbb{F}^-| \leqslant n+1$ and $|\mathbb{F}^-| = 3$, respectively, the theorems are not proven for the situation $3 \leqslant |\mathbb{F}^-| \leqslant n+1$ and $|\mathbb{F}^-| = 3$, respectively. Besides that, **Theorem 4.2.7** and **Theorem 4.3.1** does not contain the general form of Υ with $\Upsilon(I_n) = 0_m$. If these concerns are able to be solved, the theorems are more complete. Moreover, by using the idea of Chooi and Ng (2010), we can continue our study by considering other matrix spaces like skew-Hermitian matrices, skew-symmetric matrices, triangular matrices, alternate matrices, etc.

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