POWER DOMINATING NUMBERS IN GRAPHS

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A project report submitted in partial fulfilment of the requirements for the award of Bachelor of Science (Hons.) Applied Mathematics with Computing

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DECLARATION

I hereby declare that this project report is based on my original work except for citations and quotations which have been duly acknowledged. I also declare that it has not been previously and concurrently submitted for any other degree or award at UTAR or other institutions.

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APPROVAL FOR SUBMISSION

I certify that this project report entitled "**POWER DOMINATING NUMBERS IN GRAPHS**" was prepared by **CHAN KAR MAN** has met the required standard for submission in partial fulfilment of the requirements for the award of Bachelor of Science (Hons.) Applied Mathematics with Computing at Universiti Tunku Abdul Rahman.

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ABSTRACT

A Phase Measurement Unit (PMU) is a device to monitor the electrical activity and every electrical company uses it. Since PMU comes at a high cost and the company wants to use the least amount of PMU while monitoring all the electrical network stations to make sure they could respond to any emergency situation. When we convert this problem into Graph Theory, we have the power dominating problem which is to find the minimum cardinality of the smallest power dominating set (PDS) of a graph (i.e. the power dominating number).

In this project, we will be investigating the power dominating number of a specific graph called twisted torus, which is a variation of torus graph. To find the power dominating number, we have to understand the observation rules and apply it properly. Then, we have to study and analyze the power dominating problem for various graphs. For example, the torus and the cylinder graph have the closest resemblance of a twisted torus.

Once we have gone through that, we will begin the first phase of the proof. That is, find the zero forcing number for the twisted torus such that we could apply it to a known theorem in order to find the lower bound of the power dominating number. To find the zero forcing number, we have broken down the problem into different parts in order to get a good grasp on it.

If the zero forcing number is found, we may enter the second and the last phase of the proof. That is, find the lower bound and the upper bound of power dominating number of the twisted torus. In this phase, we will show the construction of the PDS and the bounded region of the power dominating number of the twisted torus.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF APPENDICES

CHAPTER 1

INTRODUCTION

1.1 Background

Every electrical company has to monitor their electrical networks consistently to prevent any deficiency in their networks. To do that, they installed some devices called Phase Measurement Units (PMUs) at various locations in the network system. What PMU does is it measures the currents and phase differences (or phase angles) of all transmission lines connected to the PMU's location. Since PMU comes with a high cost, it will be a financial burden for the electrical company to buy a bulk of PMUs. To prevent any financial issue, they have to reduce the number of PMU that is needed for the electrical company as much as possible and place the devices at proper locations such that they will be able to monitor the entire network. Thus, this turn into PMU placement problem.

PMU placement problem is also known as the *Power Dominating Set Problem*, which is a variation of dominating set problem that uses the idea of Graph Theory together with the properties of Kirchhoff's Current Law and Ohm's Law. Barrera (2009) states that this problem has been proven to be NP-Complete. In other words, we can't find a minimal power dominating set for a graph efficiently.

1.2 Objectives

The main goal of this project is to find the upper bound of the twisted torus graph, a variation of torus graph. We will begin the search by understanding the observation rules and apply them on a solved problem to get a better understanding. Then, we will begin the first phase of the proof. That is, find the zero forcing number of the twisted torus. Once that is found, we may apply it on a theorem and proceed to the next and the last phase of the proof. That is, find the lower and upper bound of twisted torus. It is possible to be done by a specific way of constructing the power dominating set, but analytic reasoning is required to explain the how and the why.

1.3 Project Timeline

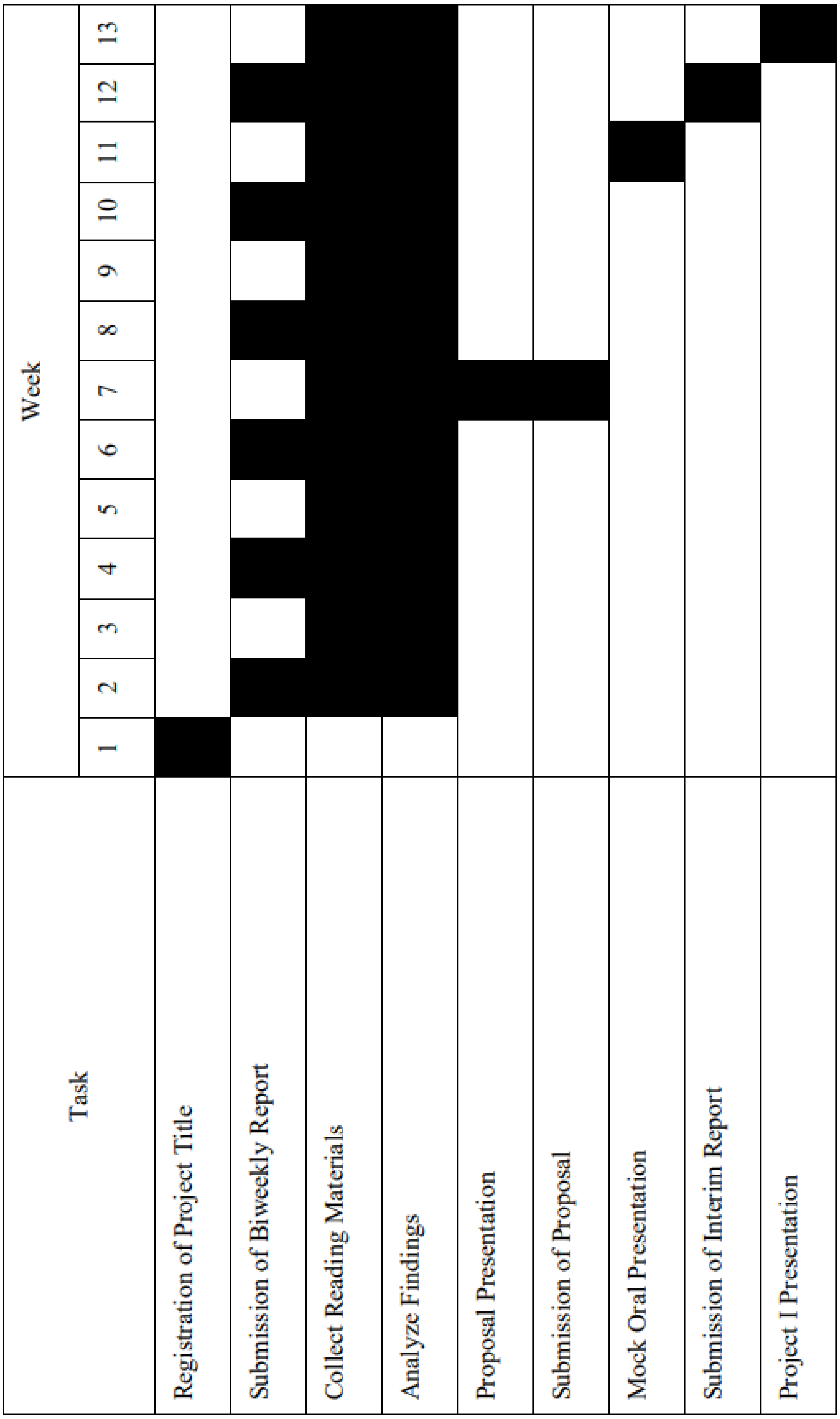

Figure 1.1: Project I Timeline (May 2020)

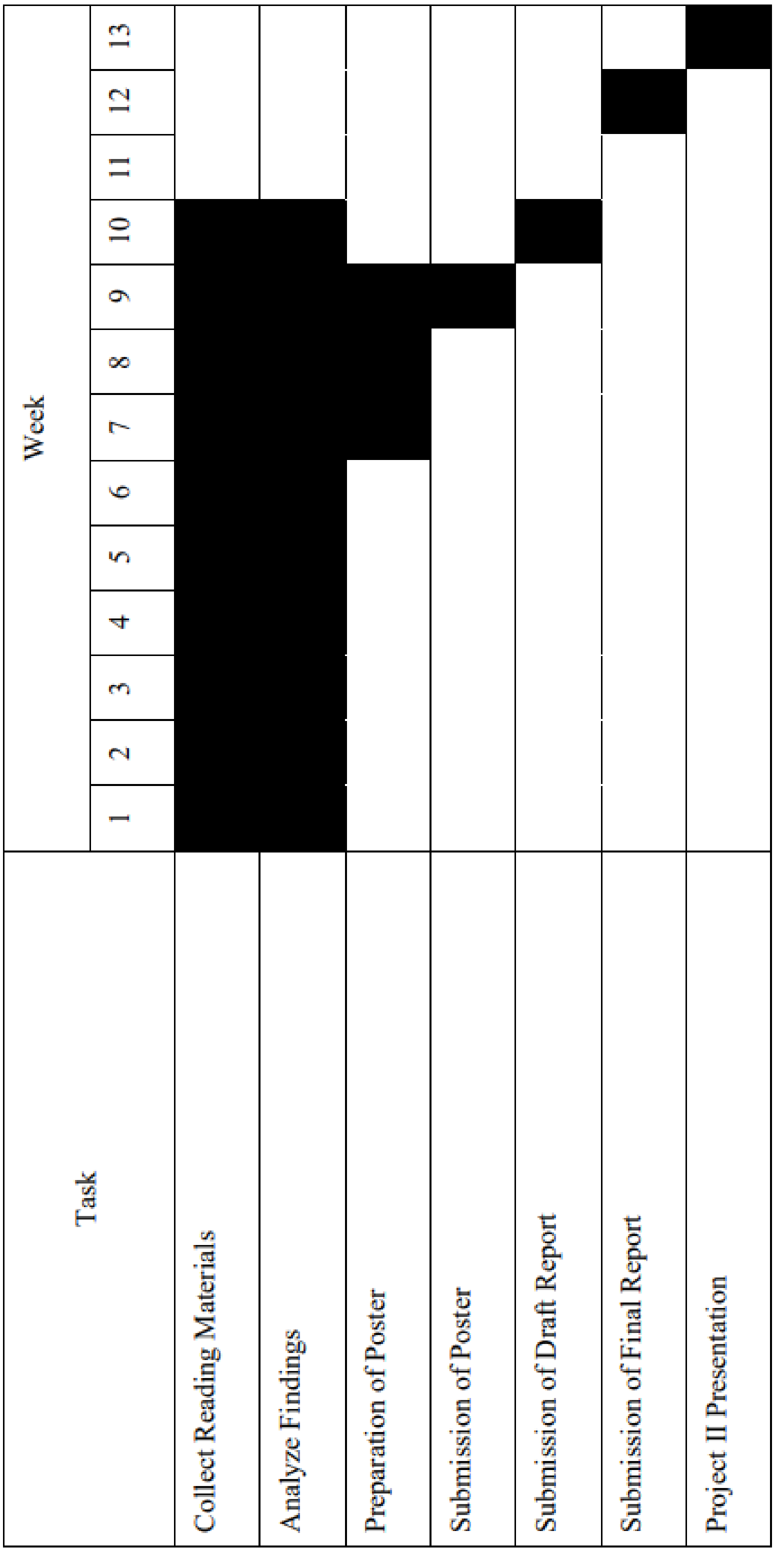

Figure 1.2: Project II Timeline (Jan 2021)

CHAPTER 2

DEFINITIONS AND NOTATIONS

2.1 Graph

Let graph $G = (V, E)$ denote a graph where V and E are the vertex set and edge set of G respectively. Given any vertex $v \in V$, the *neighborhood* of v, denoted by $N(v)$, is the set of all vertices adjacent to v, and its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. For any set $S \subseteq V$, its neighborhood of S is defined to be $N(S) = \bigcup_{v \in S} N(V)$ and its closed neighborhood of S is defined to be $N[S] = N(S) \cup S$.

A *path graph* P_n is a connected graph with n vertices where the start-vertex and the end-vertex has degree of 1 and the rest of the vertices have degree of 2.

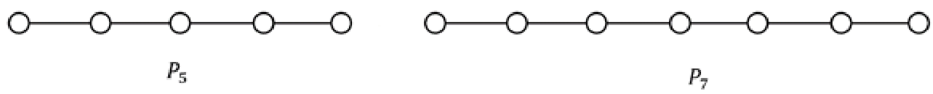

Figure 2.1: Example of path graph

A *cycle graph* C_n is a connected graph with n vertices where all the vertices only has degree of 2. If a vertex is removed from the cycle graph (so are the edges incident to the vertex), it will become a path graph.

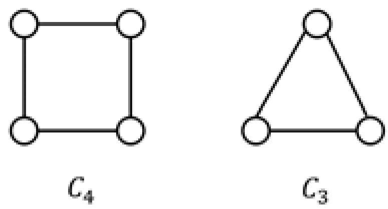

Figure 2.2: Example of cycle graph

The *Cartesian product of two graphs* G_1, G_2, denoted $G = G_1 \square G_2$ is the graph with vertex set of $V(G) = V(G_1) \times V(G_2) = \{(x_1, x_2) \mid x_i \in V(G_i), i = 1, 2\}$. Given any two vertices such as (u_1, u_2) and (v_1, v_2) from G, we say they are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$, or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$. There are some notable Cartesian product of graphs such as grid $P_n \square P_m$, cylinder $P_n \square C_m$, and torus $C_n \square C_m$.

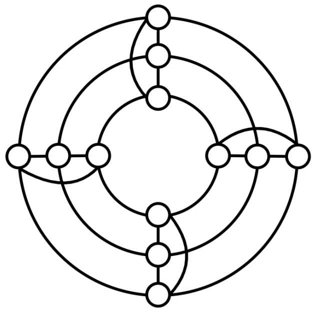

Figure 2.3: Example of torus graph $C_3\Box C_4$

Let $C_m\Box C_n$ denote the *torus graph* which consists of m copies of the n-cycle

$$x_{i,1}x_{i,2}\cdots x_{i,n}x_{i,1},$$

$i=1,2,\ldots,m$ together with n copies of the m-cycle

$$x_{1,j}x_{2,j}\cdots x_{m,j}x_{1,j},$$

$j=1,2,\ldots,n$. Here $m,n\geq 3$. Let $C_m\Box C_n^t$ denote the *twisted torus* which is the graph obtained from the torus $C_m\Box C_n$ by first deleting the set of m edges

$$x_{1,n}x_{1,1},\ x_{2,n}x_{2,1},\ldots,x_{m,n}x_{m,1}$$

and then join the resulting graph with a set of m new edges

$$x_{1,n}x_{2,1},\ x_{2,n}x_{3,1},\ldots,x_{m-1,n}x_{m,1},\ x_{m,n}x_{1,1}.$$

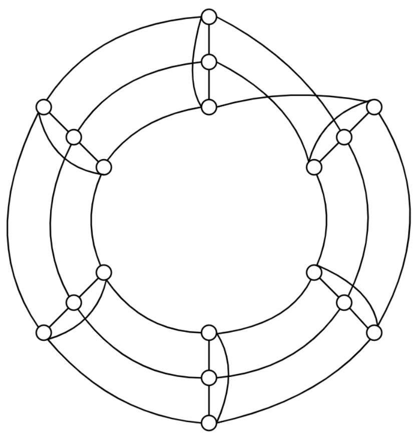

Figure 2.4: An example of twisted torus graph $C_6\Box C_3^t$

2.2 Dominating Set

Given that S is a subset of vertices of graph G, we say S is a *dominating set* if any vertex from G that is not in S is adjacent to one or more vertex in S. The *domination number* of a graph G (denoted by $\gamma(G)$) is the cardinality of the smallest dominating set.

We may also apply some conditions or rules on the vertices of the dominating set to create a different set of problem. For example, *power dominating set* (denoted by S_P) is a problem that applies observation rules, and *zero forcing set* (denoted by S_Z) is a problem that applies color-change rule. The details of these rules can be found in Chapter 4. We use γ_P and γ_Z to represent the *power dominating number* and the *zero forcing number*.

Another notable problem is the Eight Queens Problem. In the original statement, we want to find eight distinct positions to place the queens with the condition that none of the queens could attack each other on a normal 8 by 8 chessboard. When the positions are identified, we noticed that all the squares on the chessboard are in the line of attack by at least one queen. Then, we ask the following question. Using the condition from above, what is the minimum number of queens (i.e. the dominating number) needed such that all the squares are attackable by at least one queen? The answer is $\gamma(G) = 5$ (Gibbons, 1985).

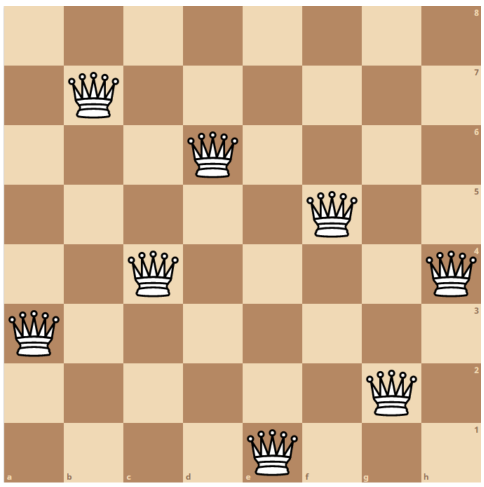

Figure 2.5: Solution for Eight Queens Problem

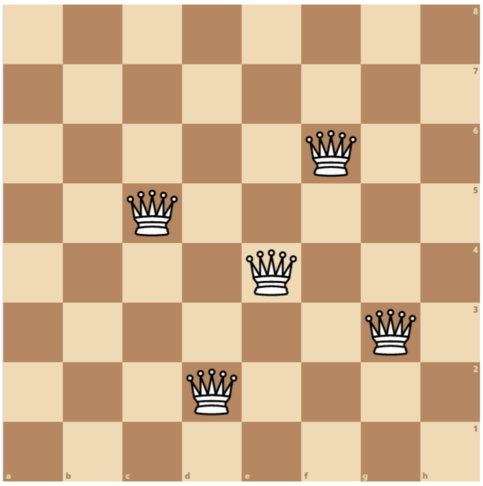

Figure 2.6: Solution for the dominating number of the Eight Queens Problem

2.3 Matrix

A *square matrix* $\mathcal{M}_n$ is an $n \times n$ array of numbers arranged into rows and columns. Each entry in the matrix is denoted by a_{ij}, and it should be read as: the entry at at i-th row and j-th column.

A *symmetric matrix* $\mathcal{S}_n$ is a matrix where $a_{ij} = a_{ji}$ for all i and j. A *permutation matrix* $\mathcal{P}_n$ is a matrix where every row and column has exactly one entry 1. A *block matrix* is a matrix that has been broken into sections called *blocks*. For example, the matrix

$$\mathcal{M} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

can be partitioned into four 2×2 blocks

$$\mathcal{M}_{11} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \mathcal{M}_{12} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}, \quad \mathcal{M}_{21} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}, \quad \mathcal{M}_{22} = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}.$$

Then, the matrix $\mathcal{M}$ can be rewritten as

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{bmatrix}$$

Let G be a simple undirected graph. The *adjacency matrix* of G, denoted by $\mathcal{A}(G)$, is a $(0,1)$-matrix such that $a_{ij} = 1$ if and only if the vertices v_i and v_j are adjacent in G.

The *Kronecker product*, denoted by $\otimes$, is an operation on two matrices that results in a block matrix. This operation is useful for the Cartesian product when we want to construct an adjacency matrix for Cartesian product of graphs.

The *eigenvalue* of $\mathcal{M}$, denoted by λ, is a number such that $\mathcal{M}\mathbf{x} = \lambda\mathbf{x}$ for some nonzero vector $\mathbf{x}$. The *eigenvector* of $\mathcal{M}$, denoted by $\mathbf{x}$, is a nonzero vector such that $\mathcal{M}\mathbf{x} = \lambda\mathbf{x}$ for some number λ.

2.4 Kirchhoff's Current Law and Ohm's Law

Kirchhoff's current law states that given any point P in a circuit, the sum of the currents entering P is same as the sum of currents exiting P. We can write the statement as an equation such as $\sum I_{in} = \sum I_{out}$ where I is the currents in the circuit.

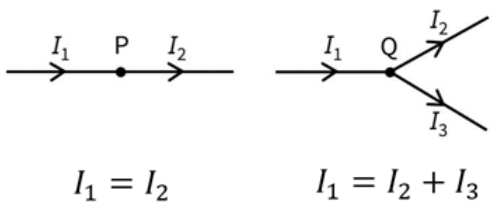

Figure 2.7: Examples of Kirchhoff's Current Law (Sang et al., 2014)

Ohm's law states that the potential difference V in a circuit is directly proportional to the currents I. In mathematical terms, we have $V \propto I$, and $V = kI$ when we convert it to an equation where k is a constant. Since the only constant in a circuit is resistance R, this indicate $k = R$. Therefore, the formula of Ohm's law is $V = IR$.

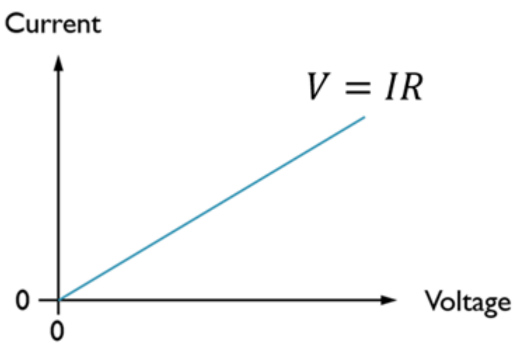

Figure 2.8: Illustration of Ohm's Law

CHAPTER 3

LITERATURE REVIEW

In Graph Theory, there are many different types of graphs. Some graphs are named according to their appearance such as cycle graph, tree graph, and so on. Some graphs are named after the person who discovered them such as the Petersen graph. It is obvious that different type of graph comes with different properties. Because of this, the researchers are facing different challenges with different scales of difficulties when they are working on a certain problem.

When it comes to power dominating set (PDS) problem, Haynes et al. (2002) has proved that this problem is NP-Complete even when we are restricted to bipartite graphs or chordal graphs. There is no general formula that can help us identify the exact power domination number or an algorithm that can help us to find the proper PMU placement for all kind of graphs.

The structure of grid graph $P_n \square P_m$ may look simple, that is, a rectangle or a square filled with $(n-1)(m-1)$ number of one by one squares. In fact, finding the dominating number of grid graph is challenging for the researches. Based on the statement from Dorfling and Henning (2006), the researchers have yet to determine the exact dominating number for the $n \times m$ grid graph when $n \geq 7$ and m can be any positive integer. When it comes to power dominating number, we can determine it by using the following formula.

$$\gamma_P(P_n \square P_m) = \begin{cases} \left\lceil \frac{n+1}{4} \right\rceil & \text{if } n \equiv 4 (\text{mod } 8) \\ \left\lceil \frac{n}{4} \right\rceil & \text{otherwise} \end{cases}$$

Barrera and Ferrero (2011) has constructed two formulas to calculate the upper bounds of power dominating number for cylinder and torus graph. A few year later, Koh and Soh (2016, 2019) briefly survey the upper bounds through the work of Barrera and Ferrero at first, and they managed to find the exact power dominating number with a solid proof. For a cylinder graph $P_n \square C_m$ where $m \geq 3$,

$$\gamma_P(P_n \square C_m) = \begin{cases} \left\lceil \frac{n+1}{2} \right\rceil & \text{if } n \equiv 2 (\text{mod } 4) \text{ and } m \geq 2n \\ \left\lceil \frac{m+1}{4} \right\rceil & \text{if } m \equiv 4 (\text{mod } 8) \text{ and } m \leq 2n \\ \min\left\{ \left\lceil \frac{n}{2} \right\rceil, \left\lceil \frac{m}{4} \right\rceil \right\} & \text{otherwise} \end{cases}$$

and for torus graph $C_n \Box C_m$ where $m \geq n \geq 3$,

$$\gamma_P(C_n \Box C_m) = \begin{cases} \left\lceil \frac{n+1}{2} \right\rceil & \text{if } n \equiv 2 (\text{mod } 4) \\ \left\lceil \frac{n}{2} \right\rceil & \text{otherwise} \end{cases}$$

Zhao et al. (2020) investigate the problem posed by Xu and Kang (2011) which is to find the exact power dominating number for the generalized Petersen graph $P(4k, k)$ or $P(ck, k)$ where $c \geq 4$. They managed to prove $\gamma_P(P(ck, k)) \leq \left\lceil \frac{2k+2}{3} \right\rceil$ for integer $k \geq 2$ and $c \geq 4$, and it is considered as sharp upper bound, meaning the equality holds for at least one value of k. Not only that, they showed the exact values for $P(4k, k)$ to be

$$\gamma_P(P(4k, k)) = \begin{cases} 2 & \text{for } k = 1, \\ 3 & \text{for } k = 3, \\ \left\lceil \frac{2k}{3} \right\rceil & \text{for } k = 2 \text{ or } k \geq 4 \end{cases}$$

At the end, the authors made a conjecture as follow. For $k \geq 2$ and $c \geq 5$, $\gamma_P(P(ck, k)) = \left\lceil \frac{2k}{3} \right\rceil$ if $2k \equiv 1(\text{mod } 3$ and $\gamma_P(P(ck, k)) \in \left\{ \left\lceil \frac{2k}{3} \right\rceil, \left\lceil \frac{2k}{3} \right\rceil + 1 \right\}$ if $2k \equiv 0$ or $2k \equiv 1(\text{mod } 3$. Maybe we can expect to see Kang to find the power domination number for $P(5k, k)$ in near future since he has found the exact power domination number for $P(3k, k)$ and gave a sharp upper bound for $P(n, k)$ with Xu back in 2011.

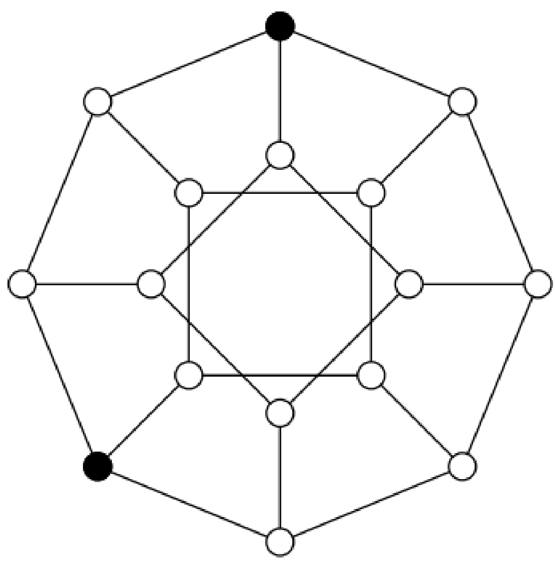

Figure 3.1: Generalized Petersen graph $P(8, 2)$ with two black vertices (PMU placement) (Zhao et al., 2020)

Lastly, the results that we have shown so far are purely for the ideal situation. Meaning the formulas will be more practical or useful if the electrical company's network structure happened to have the same structure with the graphs that have been investigated by the researchers.

CHAPTER 4

METHODOLOGY

Given a graph G, we can find the power dominating set $S_P(G)$ by using the observation rules. There are three rules in total, and we have two approaches for these rules. The first approach uses the idea of Physics whereas the second approach uses the idea of Set Theory. As for the zero forcing set $S_Z(G)$, we will be using the color-change rule.

4.1 The Observation Rules (Physics)

The power dominating problem of an electric network can be illustrated by using a graph. The vertex represents the power stations, and the edge represents the connectivity between the stations. Whenever a PMU is placed on a vertex, say v, the PMU will be able to detect all the power stations that are connected to v. In other words, the vertices adjacent to v and the edges incident to v are observed by the PMU. The observability of any other vertices and edges will be determined by the following observation rules based on Kirchhoff's Current Law and Ohm's Law.

1. Ohm's Law, $V = IR$: If a vertex is incident to an observed edge, then the vertex is observed.

2. Ohm's Law, $I = V/R$: If an edge is incident to two observed vertices, then the edge is observed.

3. Kirchhoff's Current Law: For $k \geq 2$, if a vertex is incident with k edges and $k-1$ of these edges are observed, then all k of them are observed.

To illustrate the usage of these rules, we will use a grid graph $P_5 \Box P_7$ as an example.

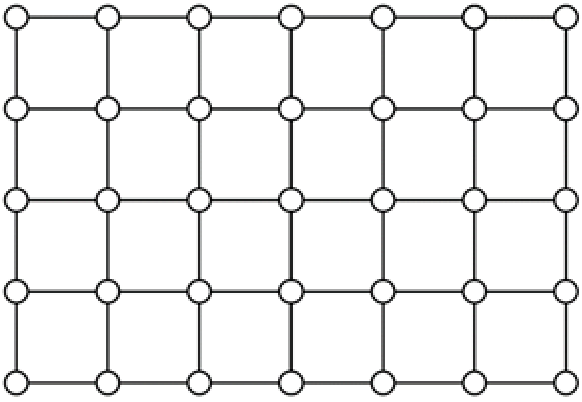

Figure 4.1: Grid Graph $P_5 \square P_7$

Taking a closer look on Figure 4.1, the grid graph has the resemblance of a matrix. Using this idea, we can pinpoint a specific vertex by writing a_{ij} where i and j represent the row number and the column number respectively. Assuming that we know the number of PMU needed and its placement, we get the following graph.

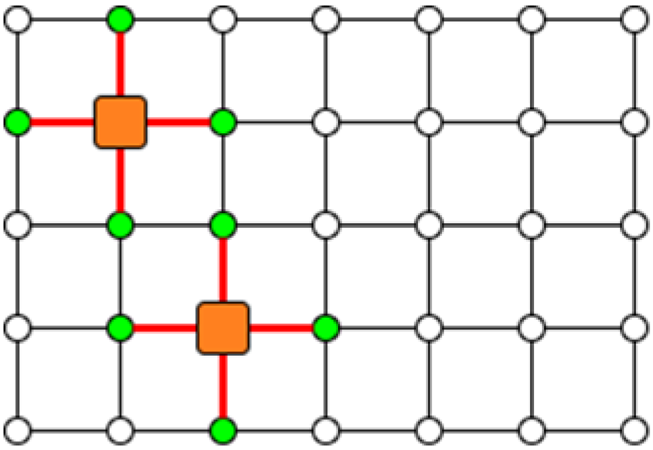

Figure 4.2: Placement of PMUs

In Figure 4.2, we use orange rounded square to represent the PMUs, the observed edges and the observed vertices are colored in red and green, respectively. Furthermore, we may apply the second observation rule since there are three different edges incident to three different pairs of observed vertices such as (a_{32}, a_{42}), (a_{32}, a_{33}) and (a_{23}, a_{33}).

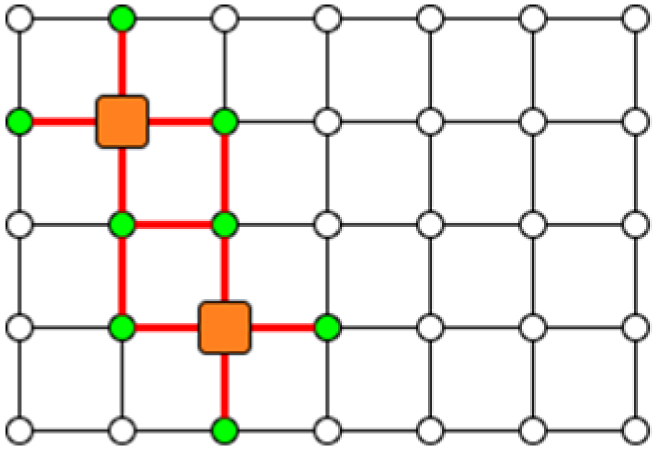

Figure 4.3: Result from applying the second observation rule

In Figure 4.3, we can see that a_{32} and a_{33} have three out of four edges colored. By the third observation rule, the remaining uncolored edge will be colored, which mean the edge will be observed.

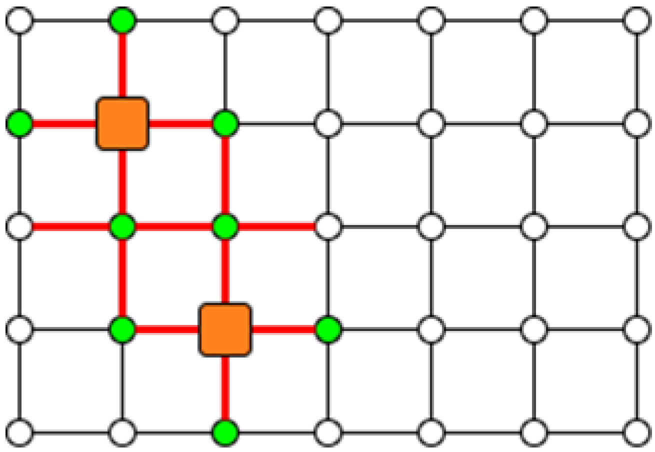

Figure 4.4: Result from applying the third observation rule

In Figure 4.4, we can see that a_{31} and a_{34} are incident with an observed edge. By the first observation rule, a_{31} and a_{34} are observed.

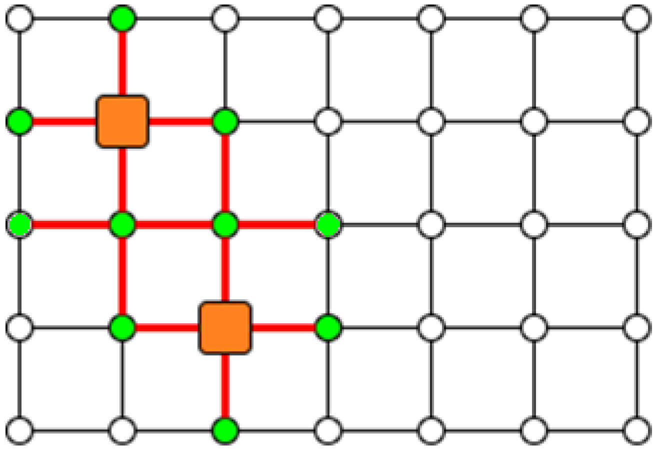

Figure 4.5: Result from applying the first observation rule

From here on out, we will be applying the observation rules repeatedly until: a) all of the edges and vertices are observed, b) the condition of stopping criteria is met, or c) none of the observation rules are applicable for any unobserved edge and vertex. In our example, we may stop using the observation rules once we have the following observed graph.

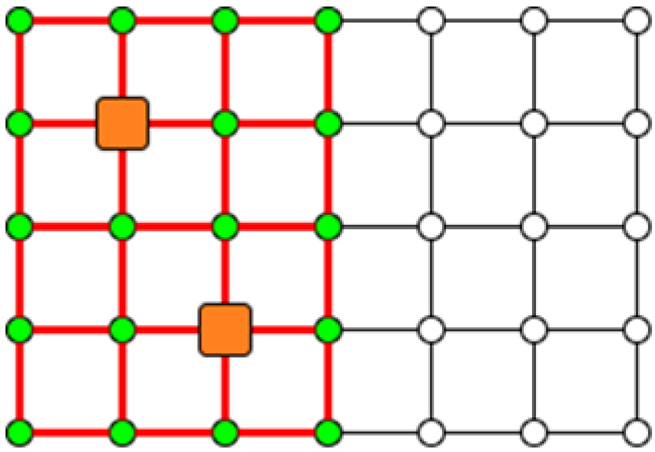

Figure 4.6: Partially observed graph

For a grid graph, whenever we achieved a result like Figure 4.6, or at least two rows or columns of vertices are observed, we can conclude that the rest of the vertices and edges will be observed as well. Using Figure 4.6, every vertex on the fourth column are eligible to apply the third observation rule, follow by the first rule, then the second rule, and we apply the third rule once again on the fifth column's vertices. In other words, this is a repetitive process and we can stop as soon as we achieve a result that is similar to the pattern similar to Figure 4.6.

4.2 The Observation Rules (Set Theory)

Given a graph G, we have a $P \subseteq V(G)$ where P contains the vertices of PMUs' location. The observed set of vertices C and edges F can be determined by using the following algorithm.

(i) Let $C = P$ and $F = \{e \in E(G) \mid e$ is incident to a vertex in $P\}$.

(ii) Add any vertex to C if it is incident to an edge in F and it is not in C.

(iii) Add any edge to F if it is not in F and one of the following conditions is satisfied:

  (a) both of its end-vertices are in C, or

  (b) it is incident to a vertex in C with a degree greater than one and all the other edges are already in F.

(iv) If we can't find any new vertices or edges to add into C and F respectively, we will stop the search. Otherwise, we will continue the search by repeating steps 2 and 3.

The power domination problem is considered solved if $C = V(G)$, $F = E(G)$ and $|P|$ is at its lowest possible. Applying this algorithm to the example in Chapter 4.1 will give us the same result. The connections between Chapter 4.1 and Chapter 4.2 are as follow:

- Step 2 is equivalent to the first observation rule.
- Step 3(a) is equivalent to the second observation rule.
- Step 3(b) is equivalent to the third observation rule.

4.3 Simplified Observation Rule (SOR)

SOR is built on the foundation of the observation rules in Chapter 4.1. Given an observed vertex v that is adjacent to k vertices. If $k - 1$ of them are observed, then the remaining vertex will be observed as well.

The flow of figures from Figure 4.2 to Figure 4.5 show that the vertices of a_{32} and a_{33} are trying to color the remaining unobserved vertices of a_{31} (adjacent to a_{32}) and a_{34} (adjacent to a_{33}) while they are adjacent to three out of four observed vertices.

SOR is fairly similar to the third observation rule in Chapter 4.1 except that we replace the keyword 'edge' to 'vertex'. Overall, the SOR mainly focuses on the observability of the vertices of graph, and the edges serve as a connection between the vertices.

4.4 Color-Change Rule

Consider a colored black vertex u. If u has exactly one neighbor v (i.e. u is adjacent to v) that is colored white, then the color of v will be changed to black. This rule is similar with SOR, and their difference is illustrated in the following example.

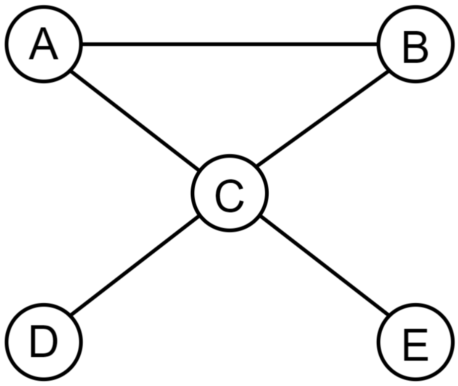

Figure 4.7: A simple graph G

Using Figure 4.7 as our graph G. By using the observation rules, the power dominating set $S_P(G)$ is $\{C\}$ since C is adjacent to all other vertices. In other words, placing a PMU at C will observe all other vertices. Therefore, $\gamma_p(G) = 1$.

Similar to power dominating number, a zero forcing number $\gamma_Z(G)$ is the minimum cardinality of the smallest zero forcing set $S_Z(G)$. Assuming only one vertex is needed in $S_Z(G)$, it is easily to see that we need more than one vertex by using the color-change rule. A proof by contradiction can be used to show that $S_Z(G)$ contains at least two elements.

Suppose there exist one vertex that will allow us to color the entire graph to black. If we color vertex A, B or C to black, they do not have exactly one neighbor who is colored white. If we color vertex D or E to black, vertex C will be colored black, but now it has three neighbors who are colored white. This contradict with our assumption, and thus, $\gamma_Z(G) \geq 2$.

There are several solutions for this example such as $S_Z(G) = \{A, D\}$. When vertex A and D are colored black, vertex C becomes black since it is vertex D's only colored white neighbor. Then, vertex B becomes black as well because it is vertex A's only colored white neighbor. Finally, vertex E becomes black because it is vertex C's only colored white neighbor. All of vertices are colored black, thus, $\gamma_Z(G) = 2$. The sets $\{B, E\}$, $\{A, E\}$, and $\{B, D\}$ are zero forcing set as well.

CHAPTER 5

PRELIMINARY RESULT

In this chapter, we consider the twisted torus with $m \geq n$. This is because we want to simplify the process of applying both the color change rule and the SOR. Also, we will reconstruct the twisted torus graph in the resemblance of a cylinder graph.

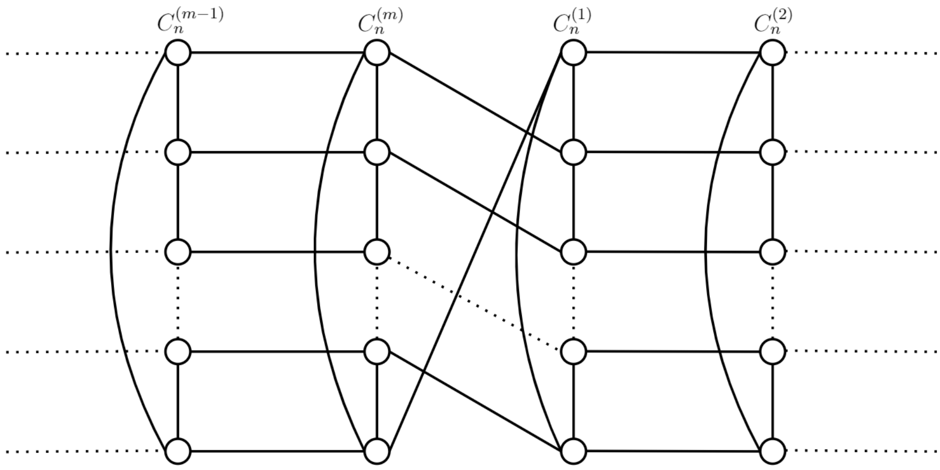

Figure 5.1: Reconstructed twisted torus graph

Let $C_n^{(i)}$ denotes the sequence of cycle C_n where $i = 1, \ldots, m$, denote the vertices of cycle $C_n^{(i)}$ by $v_j^{(i)}, j = 1, \ldots, n$.

Separate the graph into two regions, the left-hand side (LHS) and the right-hand side (RHS). The LHS will contain $C_n^{(m)}, C_n^{(m-1)}, \ldots, C_n^{(\lceil m/2 \rceil)}$, and the RHS will contain $C_n^{(1)}, C_n^{(2)}, \ldots, C_n^{(\lfloor m/2 \rfloor)}$. Note that the LHS has one extra C_n more than RHS. When m is even, remove $C_n^{(\lceil m/2 \rceil)}$ on the LHS and rewrite $C_n^{(\lfloor m/2 \rfloor)}$ as $C_n^{(m/2)}$. Then, $C_n^{(m/2)}$ is adjacent to $C_n^{(\lceil m/2 \rceil + 1)}$.

5.1 Zero Forcing Number of $C_m \square C_n^t$

Lemma 5.1. *Any two successive cycles of C_n such as $C_n^{(i)}$ and $C_n^{(i+1)}$ in graph $C_m \square C_n^t$ will form a zero forcing set. Then, $\gamma_Z(C_m \square C_n^t) \leq 2n$.*

Proof. Since the zero forcing number of $C_m \square C_n$ has been thoroughly investigated by Benson et al. (2018), we will solely investigate the twisted sector.

Based on the definition of twisted torus, the two successive cycles of C_n of the twisted sector are $C_n^{(m)}$ and $C_n^{(1)}$, and the zero forcing set is $\{v_1^{(m)}, \ldots,$

$v_n^{(m)}, v_1^{(1)}, \ldots, v_n^{(1)}\}$. Each of the vertices from $C_n^{(m)}$ are adjacent to 4 vertices and 3 of them are colored black (2 vertices from $C_n^{(m)}$ and 1 vertex from $C_n^{(1)}$). The remaining white vertex is from $C_n^{(m-1)}$. By the color change rule, we can *force* the vertices of $C_n^{(m-1)}$ to black.

Similarly, we may repeat the same process on $C_n^{(1)}$ and force all the vertices in $C_n^{(2)}$ to black. Then, repeat this process on any applicable $C_n^{(i)}$. Eventually, all vertices in $C_m \Box C_n^t$ will be forced to black. Thus, $\gamma_Z(C_m \Box C_n^t) \leq 2n$. □

For a torus graph, any two successive cycles of C_m or C_n would produce a zero forcing set. As for the twisted torus graph, all of the C_m are gone when the graph is twisted. Which is why we highlight C_n only in Lemma 5.1. But there is C_{m+1} if we involve the edges from either $C_n^{(m)}$ or $C_n^{(1)}$.

Proposition 5.2. *Let $S_V \subseteq V(C_m \Box C_n^t)$ and $S_V = \{v_j^{(i)}, v_{j+1}^{(i)} \ : \ i = 1, \ldots, m\}$ for some $j = 1, \ldots, n$. If all the elements in S_V are colored black, then the entire graph will be colored black.*

Proof. Similar proof with Lemma 5.1 except that the forcing or coloring direction is vertical. □

Note that S_V is the set of vertices of two successive cycles of C_m (before twist), and S_V can never be a zero forcing set except when $m = n$.

Consider that there are I iterations of applying the color change rule, and the iteration stops when there exists a $C_n^{(i)}$ that can no longer apply the rule for each side.

Assume all the black vertices in $C_n^{(i)}$ are continuous and the black vertices are filling from bottom to top. Denote the number of black vertices in $C_n^{(i)}$ by $b^{(i)}$ where $i = 1, \ldots, m$.

Observation 5.3. Given that $b^{(m)} > b^{(1)}$. After the first iteration, LHS would lose at least 1 black vertex (depending on the amount of black vertices that are adjacent to another one or two black vertices) whereas RHS loses 2 black vertices. Afterward, both sides would lose 2 vertices for the rest of iteration. If $b^{(m)} = b^{(1)}$ and $b^{(m)} \neq n$, then both sides would lose 2 vertices for every iteration.

Observation 5.4. As the number of black vertices decreases for each iteration, they "move" slowly towards the middle of all black vertices in each $C_n^{(i)}$.

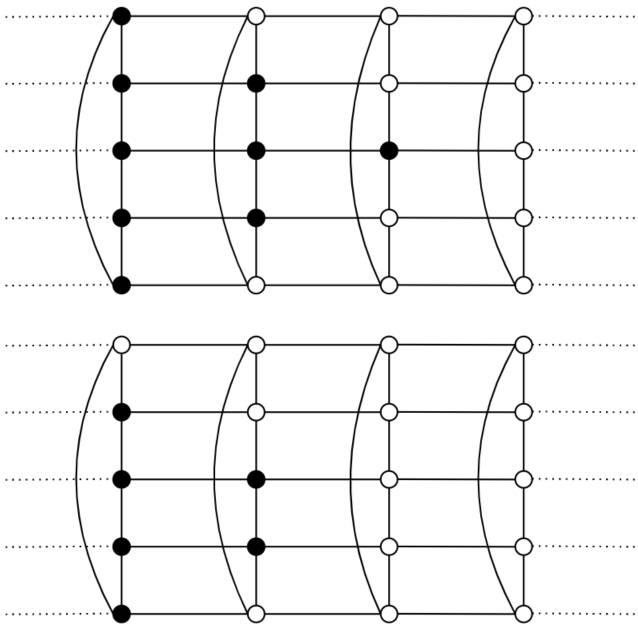

Figure 5.2: Example of the "movement" of black vertices

Define β to be the *iteration factor* and $\beta = \min(b^{(m)}, b^{(1)})$. The iteration factor is important in this section as it would give us the exact i for $C_n^{(i)}$ in both LHS and RHS when the iteration stops and the exact position(s) of the vertex or vertices in $C_n^{(i)}$. Additionally, let I_L and I_R be the number of iteration of LHS and RHS, respectively.

Proposition 5.5. *Given that $b^{(m)} > b^{(1)}$, then*

$$I_R = \begin{cases} \lfloor \frac{\beta}{2} \rfloor & \text{if } \beta \text{ is odd,} \\ \frac{\beta}{2} - 1 & \text{if } \beta \text{ is even} \end{cases}$$

and

$$I_L = I_R + 1$$

Proof. If $b^{(m)} > b^{(1)}$, then by Observation 5.3, the RHS is losing 2 black vertices consistently. If β is even, the iteration stops when there are only 2 black vertices. If β is odd, the iteration stops when there is only 1 black vertex. Convert these situations into equations and with a some algebraic manipulation, the RHS has $\lfloor \frac{\beta}{2} \rfloor$ iterations when β is odd and $\frac{\beta}{2} - 1$ iterations when β is even.

As for the LHS, notice that $b^{(m-1)} = b^{(1)}$ after its first iteration, which mean LHS has 1 iteration more than the RHS. Thus, LHS has $\left\lfloor \frac{\beta}{2} \right\rfloor + 1 = \left\lceil \frac{\beta}{2} \right\rceil$ iterations when β is odd and $\frac{\beta}{2}$ iterations when β is even. □

Remark. If $b^{(m)} = b^{(1)}$ and $b^{(m)} \neq n$, then $I_L = I_R$.

Observation 5.6. Given that $b^{(m)} > b^{(1)}$, then the placement of black vertices in $C_n^{(m-1)}, C_n^{(m-2)}, \ldots, C_n^{(\lceil m/2 \rceil)}$ is mirrored to the placement of black vertices in $C_n^{(1)}, C_n^{(2)}, \ldots, C_n^{(\lfloor m/2 \rfloor)}$, respectively, and shift the placement upward by one level. If $b^{(m)} = b^{(1)}$ and $b^{(m)} \neq n$, then the shifting can be omitted.

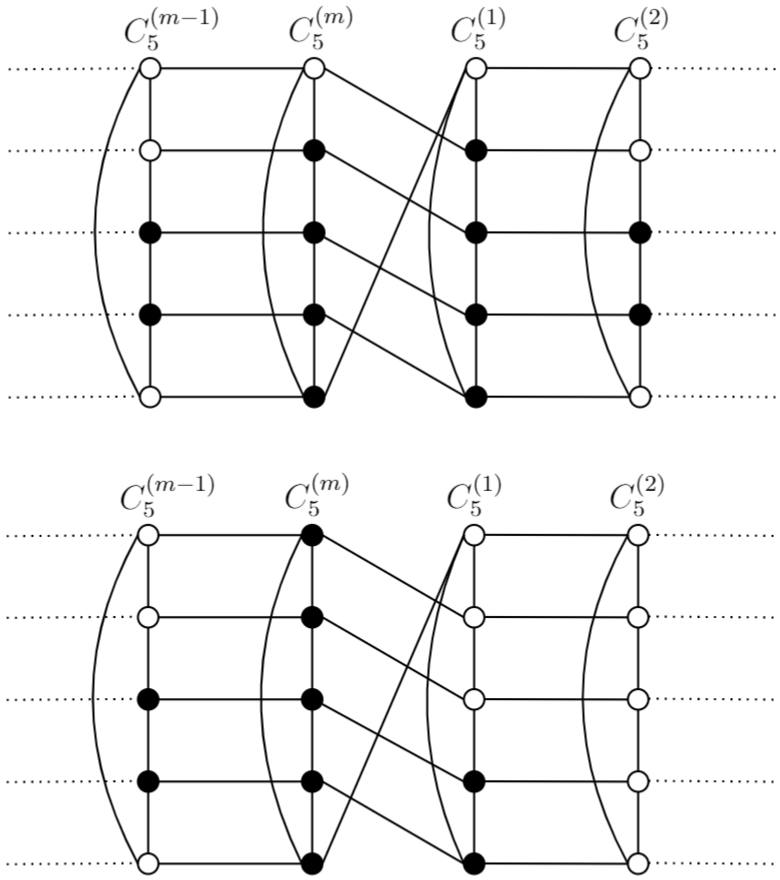

Figure 5.3: Examples of Observation 5.6

Lemma 5.7. *Let S_1 and S_2 be the set of black vertices in both $C_n^{(m-I_L)}$ and $C_n^{(1+I_R)}$. Assign S_1 with odd β and S_2 with even β. Then,*

$$S_1 = \left\{ v_{(n-I_L)}^{(m-I_L)},\ v_{(n-I_R)}^{(1+I_R)} \right\}$$

and

$$S_2 = \left\{ v_{(n-I_L-1)}^{(m-I_L)},\ v_{(n-I_L)}^{(m-I_L)},\ v_{(n-I_R-1)}^{(1+I_R)},\ v_{(n-I_R)}^{(1+I_R)} \right\}.$$

Proof. Begin with $b^{(m)} > b^{(1)}$ and $b^{(1)} \neq n$. When β is odd, the iteration stops at $C_n^{(m-\lceil \beta/2 \rceil)}$ on the LHS and $C_n^{(1+\lfloor \beta/2 \rfloor)}$ on the RHS. Since $1 + \lfloor \frac{\beta}{2} \rfloor = \lceil \frac{\beta}{2} \rceil$ and hence, $C_n^{(1+\lfloor \beta/2 \rfloor)} = C_n^{(\lceil \beta/2 \rceil)}$. Similarly, when β is even, the iteration stops at $C_n^{(m-\beta/2)}$ on the LHS and $C_n^{(\beta/2)}$ on the RHS.

With the knowledge of Observation 5.6, it is possible to find all the positions of black vertices on the LHS by solely applying the color change rule on the RHS. Based on Observation 5.3 and Observation 5.4, the RHS is losing 1 black vertex on the top and another one on the bottom consistently until there is only 1 or 2 black vertices remain (depending on β's value). This can also be thought as the bottom black vertex has shifted upward once for each iteration (similar to the top black vertex).

By using Proposition 5.5, if the β is odd, then there is only 1 black vertex remained on the RHS after the last iteration, and the black vertex is $v_{(n-\lfloor \beta/2 \rfloor)}^{(\lceil \beta/2 \rceil)}$. As stated in Observation 5.6, the position of black vertex in $C_n^{(m-\lceil \beta/2 \rceil)}$ is mirrored to $C_n^{(\lceil \beta/2 \rceil)}$ and shifted upward by one level, which produce $v_{(n-\lceil \beta/2 \rceil)}^{(m-\lceil \beta/2 \rceil)}$. Thus, these two vertices are matched with the black vertices from S_1.

Similarly, if the β is even, then there are 2 black vertices remained on the RHS after the last iteration, and the black vertices are $v_{(n-\beta/2)}^{(\beta/2)}$ and $v_{(n-\beta/2+1)}^{(\beta/2)}$. Refer to Observation 5.6, the black vertices on the LHS after its last iteration are $v_{(n-\beta/2-1)}^{(m-\beta/2)}$ and $v_{(n-\beta/2)}^{(m-\beta/2)}$. Thus, these four vertices are matched with the black vertices from S_2.

If $b^{(m)} = b^{(1)}$ and $b^{(1)} \neq n$, then $I_L = I_R$. By using Proposition 5.5, if the β is odd, then there is only 1 black vertex remained on the RHS after the last iteration, and the black vertex is $v_{(n-\lfloor \beta/2 \rfloor)}^{(\lceil \beta/2 \rceil)}$. As stated in Observation 5.6, the position of black vertex in $C_n^{(m-\lceil \beta/2 \rceil)}$ is mirrored to $C_n^{(\lceil \beta/2 \rceil)}$ and no shifting is needed, which produce $v_{(n-\lfloor \beta/2 \rfloor)}^{(m-\lceil \beta/2 \rceil)}$. Thus, these two vertices are matched with the black vertices from S_1.

Similarly, if the β is even, then there are 2 black vertices remained on the RHS after the last iteration, and the black vertices are $v_{(n-\beta/2)}^{(\beta/2)}$ and $v_{(n-\beta/2+1)}^{(\beta/2)}$. Refer to Observation 5.6, the black vertices on the LHS after its last iteration are $v_{(n-\beta/2)}^{(m-\beta/2)}$ and $v_{(n-\beta/2+1)}^{(m-\beta/2)}$. Thus, these four vertices are matched with the black vertices from S_2. $\square$

Lemma 5.8. *Let I_L and I_R be the number of iteration of LHS and RHS, respectively. Given that $b^{(m)} > b^{(1)}$ and $\beta = b^{(1)}$. If*

$$(m - I_L) - \left\lceil \frac{m}{2} \right\rceil \geq 1,$$

and

$$\left\lfloor \frac{m}{2} \right\rfloor - (1 + I_R) \geq 1.$$

then there exists at least one $C_n^{(i)}$ contains 0 black vertices in between $C_n^{(m-I_L)}$ and $C_n^{(1+I_R)}$ where $1 + I_R < i < m - I_L$.

Proof. (by Contrapositive) Suppose that there are no $C_n^{(i)}$ containing 0 black vertices in between $C_n^{(m-I_L)}$ and $C_n^{(1+I_R)}$, this indicate that $C_n^{(m-I_L)} = C_n^{(\lceil m/2 \rceil)}$ and $C_n^{(1+I_R)} = C_n^{(\lfloor m/2 \rfloor)}$. Thus,

$$(m - I_L) - \left\lceil \frac{m}{2} \right\rceil < 1$$

and

$$\left\lfloor \frac{m}{2} \right\rfloor - (1 + I_R) < 1,$$

□

Note that the values of I_L and I_R is depending on β. We can think the $C_n^{(i)}$ that contains 0 black vertices as a *barrier* because it is forbidding the black vertices from the last iteration on both LHS and RHS to be each other's neighbor.

Algorithm 5.9. Determine $\gamma_Z(C_m \square C_n^t)$ with brute force method.

1. Let U be the upper bound of $\gamma_Z(C_m \square C_n^t)$.
2. Assume exactly U vertices are needed to form a zero forcing set.
3. Suppose that there exist a zero forcing set with $U - 1$ vertices.
4. Determine the value of β.
5. Determine the value of I_L, I_R by using Proposition 5.5.
6. Determine the existence of barrier using Lemma 5.8.
   (a) If there is no barrier, proceed to Step 6.
   (b) Otherwise, $\gamma_Z(C_m \square C_n^t) = U$.
7. Substitute the value of m, n, I_L, I_R into either S_1 or S_2 (depending on β) in Lemma 5.7.

8. Determine the adjacency of the vertices in S_1 or S_2.
   (a) If there are at least 2 vertices are adjacent to each other, determine the existence of S_V from Proposition 5.2.
      i. If true, then $U - 1$ becomes the new U. Repeat from Step 2.
      ii. Else, determine the applicability of color change rule.
         A. If true, repeat from Step 8(a).
         B. Otherwise, $\gamma_Z(C_m \Box C_n^t) = U$.
   (b) Otherwise, $\gamma_Z(C_m \Box C_n^t) = U$.

Note that if we can't force the graph to black with 1 missing vertex, then it is impossible with more than 1 missing vertices. Thus, $\gamma_Z(C_m \Box C_n^t) = U$.

Theorem 5.10. *For* $n \geq m \geq 3$

$$\gamma_Z(C_m \Box C_n^t) = \begin{cases} 2n - 1 & \textit{if } m = n, \\ 2n & \textit{otherwise} \end{cases}$$

Proof. To prove this theorem, we will separate it into two cases.

Case 1: $m \neq n$.

By Lemma 5.1, we know that $\gamma_Z(C_m \Box C_n^t) \leq 2n$. Apply Algorithm 5.9, assume that $\gamma_Z(C_m \Box C_n^t) = 2n$, and suppose that there exist a zero forcing set that contain $2n - 1$ vertices from the two successive cycles $C_n^{(m)}$ and $C_n^{(1)}$. Because $2n - 1$ is an odd number, one of the two cycles will have $n - 1$ black vertices. Assume that $b^{(m)} = n$ and $b^{(1)} = n - 1 = \beta$. Then, we add the following vertices into the zero forcing set S_Z.

$$S_Z = \left\{ v_1^{(m)}, \ldots, v_n^{(m)}, v_2^{(1)}, \ldots, v_n^{(1)} \right\}$$

It is obvious that when m is much larger than n, S_Z becomes invalid because of Lemma 5.8. In other words, there are many barriers in between $C_n^{(m-I_L)}$ and $C_n^{(1+I_R)}$ which lead to the failure of forcing the next C_n to black.

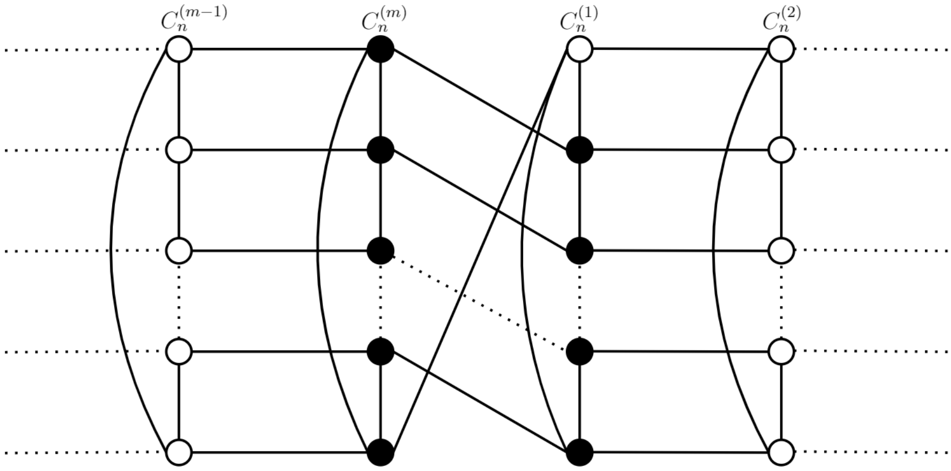

Figure 5.4: Illustration of S_Z

Assume that the difference between m and n is as small as possible. By Proposition 5.5, the iteration of applying color change rule is depending on whether β is even or odd. Thus, we have four sub-cases.

Sub-case 1: If m, n are even, then $m = n + 2$ and $\beta = n - 1$ is odd. By Proposition 5.5, $I_L = \lceil \frac{\beta}{2} \rceil$ and $I_R = \lfloor \frac{\beta}{2} \rfloor$. By using the equations in Lemma 5.8, there is 1 barrier in between $C_n^{(m-\lceil \beta/2 \rceil)}$ and $C_n^{(\lceil \beta/2 \rceil)}$. Thus, it is impossible to force the entire graph to black and $\gamma_Z(C_m \Box C_n^t) = 2n$.

Sub-case 2: If m, n are odd, then $m = n + 2$ and $\beta = n - 1$ is even. By Proposition 5.5, $I_L = \frac{\beta}{2}$ and $I_R = \frac{\beta}{2} - 1$. By using the equations in Lemma 5.8, there is 1 barrier in between $C_n^{(m-\beta/2)}$ and $C_n^{(\beta/2)}$. Thus, it is impossible to force the entire graph to black and $\gamma_Z(C_m \Box C_n^t) = 2n$.

Sub-case 3: If m is odd and n is even, then $m = n + 1$ and $\beta = n - 1$ is odd. By Proposition 5.5, $I_L = \lceil \frac{\beta}{2} \rceil$ and $I_R = \lfloor \frac{\beta}{2} \rfloor$. By using the equations in Lemma 5.8, there are 0 barriers in between $C_n^{(m-\lceil \beta/2 \rceil)}$ and $C_n^{(\lceil \beta/2 \rceil)}$. This indicate that $C_n^{(m-\lceil \beta/2 \rceil)} = C_n^{(\lceil m/2 \rceil)}$ and $C_n^{(\lceil \beta/2 \rceil)} = C_n^{(\lfloor m/2 \rfloor)}$, and the vertices in these cycles are adjacent to each other. Then, substitute the value of m, n, I_L, I_R into the black vertices from S_1 in Lemma 5.7, we have

$$S_1 = \left\{ v_{(n-\lceil \beta/2 \rceil)}^{(\lceil m/2 \rceil)},\ v_{(n-\lfloor \beta/2 \rfloor)}^{(\lfloor m/2 \rfloor)} \right\}.$$

The difference between the subscripts' value of both black vertices is 1. This imply that one of the black vertices is positioned 1 level higher than another one and they are not on the same cycle since $\lceil \frac{m}{2} \rceil \neq \lfloor \frac{m}{2} \rfloor$. Thus, these two black vertices aren't adjacent to each other and the color change rule is not applicable. Hence, it is impossible to force the entire graph to black and $\gamma_Z(C_m \square C_n^t) = 2n$.

Sub-case 4: If m is even and n is odd, then $m = n + 1$ and $\beta = n - 1$ is even. By Proposition 5.5, $I_L = \frac{\beta}{2}$ and $I_R = \frac{\beta}{2} - 1$. By using the equations in Lemma 5.8, there is 1 barrier in between $C_n^{(m-\beta/2)}$ and $C_n^{(\beta/2)}$. Thus, it is impossible to force the entire graph to black and $\gamma_Z(C_m \square C_n^t) = 2n$.

Therefore, by Algorithm 5.9, $\gamma_Z(C_m \square C_n^t) = 2n$ for $m \neq n$.

Case 2: $m = n$

Sub-case 1: n is even. By Algorithm 5.9, assume that $\gamma_Z(C_m \square C_n^t) = 2n$, and suppose that there exist a zero forcing set that contain $2n - 1$ vertices from two successive cycles, $C_n^{(m)}$ and $C_n^{(1)}$. Using the same setup as *Case 1*, that is, $b^{(m)} > b^{(1)}$ and $b^{(1)} = n - 1 = \beta$.

Due to Lemma 5.8, there are no barriers in between $C_n^{(m-\lceil\beta/2\rceil)}$ and $C_n^{(\lceil\beta/2\rceil)}$. Thus, $C_n^{(m-\lceil\beta/2\rceil)} = C_n^{(\lceil m/2\rceil)}$ and $C_n^{(\lceil\beta/2\rceil)} = C_n^{(\lfloor m/2\rfloor)}$ (similar to *Case 1: Sub-case 3*). Since m is even, $\left\lceil \frac{m}{2} \right\rceil = \left\lfloor \frac{m}{2} \right\rfloor = \frac{m}{2}$. Hence, the iteration of both sides stop at $C_n^{(m/2)}$.

By using Proposition 5.5, $I_L = \left\lceil \frac{\beta}{2} \right\rceil, I_R = \left\lfloor \frac{\beta}{2} \right\rfloor$. Substitute the value of m, n, I_L, I_R into the black vertices from S_1 in Lemma 5.7, we have

$$S_1 = \left\{ v_{(n-\lceil\beta/2\rceil)}^{(m/2)},\ v_{(n-\lfloor\beta/2\rfloor)}^{(m/2)} \right\}.$$

The difference between the subscripts' value of both black vertices is 1. This indicate that one of the black vertices is positioned 1 level higher than another one and they are on the same cycle (refer Figure 5.6). Based on the "movement" of the black vertices on both sides (refer to Observation 5.4), there exists a set of black vertices same as S_V from Proposition 5.2. Thus, a zero forcing set can be formed with $2n - 1$ vertices. By Algorithm 5.9, $\gamma_Z(C_m \square C_n^t) \leq 2n - 1$ and follow by a new assumption.

Assume that $\gamma_Z(C_m \square C_n^t) = 2n - 1$ and suppose that there exist a zero forcing set with $2n - 2$ vertices. Since $2n - 2$ is even, the black vertices can be distributed evenly such as $b^{(m)} = n - 1 = b^{(1)}$ or let $b^{(m)} = n$ and $b^{(1)} = n - 2$.

If the black vertices are distributed evenly, then $\beta = n - 1$. Let S_Z to be the zero forcing set such as

$$S_Z = \left\{ v_2^{(m)}, \ldots, v_n^{(m)}, v_2^{(1)}, \ldots, v_n^{(1)} \right\}.$$

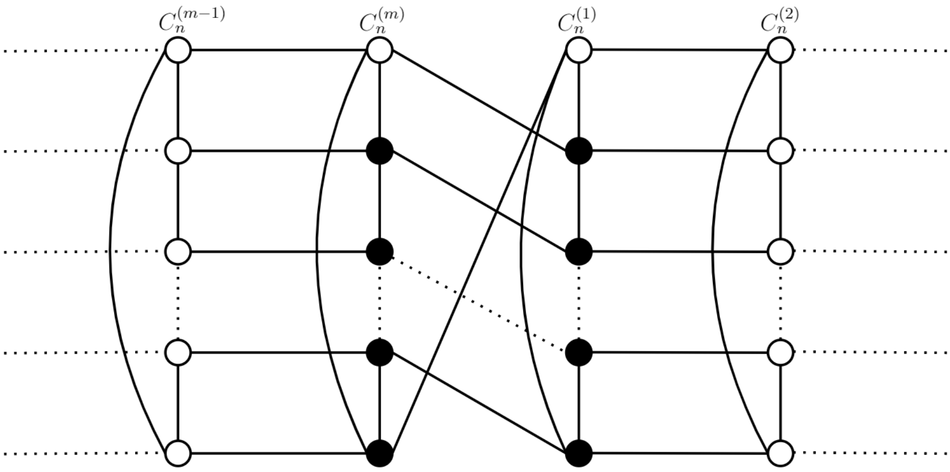

Figure 5.5: Illustration of S_Z

By the Proposition 5.5, $I_L = I_R = \left\lfloor \frac{\beta}{2} \right\rfloor$. By using the equations in Lemma 5.8, there is 1 barrier on the LHS and 0 barriers on the RHS. This is due to $m - \left\lfloor \frac{\beta}{2} \right\rfloor = \left\lceil \frac{m}{2} \right\rceil + 1$ instead of $\left\lceil \frac{m}{2} \right\rceil$. Furthermore, $C_n^{(\lceil m/2 \rceil)}$ is removed from the LHS since it is equivalent to $C_n^{(\lfloor m/2 \rfloor)}$ on the RHS. Other than that, Lemma 5.8 is only valid when both equations produce a value more than or equal to 1 (or less than 1). Thus, there is no barrier in between $C_n^{(m-\lfloor \beta/2 \rfloor)}$ and $C_n^{(\lceil \beta/2 \rceil)}$. This also indicate that the vertices in these cycles are adjacent to each other, and the superscript's expression of the cycles can be rewritten as $C_n^{(\lceil m/2 \rceil + 1)}$ and $C_n^{(m/2)}$, respectively.

Substitute the value of m, n, I_L, I_R into the black vertices from S_1 in Lemma 5.7, we have

$$S_1 = \left\{ v_{(n-\lfloor \beta/2 \rfloor)}^{(\lceil m/2 \rceil + 1)},\ v_{(n-\lfloor \beta/2 \rfloor)}^{(m/2)} \right\}.$$

The subscript's value of both black vertices are indicating that they are on the same level (supported by Observation 5.6). By comparing the superscript's value, we have $\left\lceil \frac{m}{2} \right\rceil + 1 \neq \frac{m}{2}$. This indicate that these 2 black vertices are adjacent to each other on the same level, but different cycle. Since these black vertices are adjacent to another 2 black vertices, the color change rule is not applicable. Hence, $\gamma_Z(C_m \square C_n^t) = 2n - 1$ when n is even.

Note that if the black vertices are distributed unevenly such as $b^{(m)} = n$ and $b^{(1)} = n - 2 = \beta$. Then, $I_L = \frac{\beta}{2}$ and $I_R = \frac{\beta}{2} - 1$ by Proposition 5.5. Substitute the value of m, I_L, I_R into the equations in Lemma 5.8, there is 1 barrier in between $C_n^{(m-\beta/2)}$ and $C_n^{(\beta/2)}$. Therefore, our conclusion holds.

Sub-case 2: m is odd

By Algorithm 5.9, assume that $\gamma_Z(C_m \Box C_n^t) = 2n$, and suppose that there exist a zero forcing set that contain $2n - 1$ vertices from two successive cycles, $C_n^{(m)}$ and $C_m^{(1)}$. Using the same setup as *Case 1*, that is, $b^{(m)} > b^{(1)}$ and $b^{(1)} = n-1 = \beta$. Then, $I_L = \frac{\beta}{2}, I_R = \frac{\beta}{2} - 1$ by Proposition 5.5. Due to Lemma 5.8, there are no barriers in between $C_n^{(m-\beta/2)}$ and $C_n^{(\beta/2)}$. This indicate that the vertices in between these cycles are adjacent to each other, and the superscript's expression can be rewritten as $C_n^{(\lceil m/2 \rceil)}$ and $C_n^{(\lfloor m/2 \rfloor)}$, respectively.

Substitute the value m, n, I_L, I_R into the black vertices from S_2 in Lemma 5.7, we have

$$S_2 = \left\{ v_{(n-\beta/2-1)}^{(\lceil m/2 \rceil)}, v_{(n-\beta/2)}^{(\lceil m/2 \rceil)}, v_{(n-\beta/2)}^{(\lfloor m/2 \rfloor)}, v_{(n-\beta/2+1)}^{(\lfloor m/2 \rfloor)} \right\}.$$

Based on the value of the subscript of second and third black vertex in S_2, these two black vertices are on the same level and they are adjacent to each other. More importantly, these two black vertices are adjacent to 3 black vertices (refer to Figure 5.7). Thus, they can force $v_{(n-\beta/2+1)}^{(\lceil m/2 \rceil)}$ and $v_{(n-\beta/2+1)}^{(\lfloor m/2 \rfloor)}$ to black.

Based on the "movement" of the black vertices on both sides (refer to Observation 5.4), there exists a set of black vertices same as S_V from Proposition 5.2. Thus, a zero forcing set can be formed with $2n - 1$ vertices. By Algorithm 5.9, $\gamma_Z(C_m \Box C_n^t) \leq 2n - 1$ and follow by a new assumption.

Assume that $\gamma_Z(C_m \Box C_n^t) = 2n - 1$ and suppose that there exist a zero forcing set with $2n - 2$ vertices. If the black vertices are distributed evenly to $C_n^{(m)}$ and $C_n^{(1)}$ such as $b^{(m)} = b^{(1)} = n - 1 = \beta$, then we have $I_L = I_R = \frac{\beta}{2} - 1$ by Proposition 5.5. By Lemma 5.8, there exist a barrier in between $C_n^{(m-\beta/2+1)}$ and $C_n^{(\beta/2)}$. Thus, it is impossible to force the entire graph to black.

If the black vertices are distributed unevenly such as $b^{(m)} = n$ and $b^{(1)} = n - 2 = \beta$. Then, $I_L = \left\lceil \frac{\beta}{2} \right\rceil, I_R = \left\lfloor \frac{\beta}{2} \right\rfloor$ by Proposition 5.5. By Lemma 5.8, there are no barriers in between $C_n^{(m-\lceil \beta/2 \rceil)}$ and $C_n^{(\lceil \beta/2 \rceil)}$. This indicate that the vertices in between these cycles are adjacent to each other, and the superscript's expression can be rewritten as $C_n^{(\lceil m/2 \rceil)}$ and $C_n^{(\lfloor m/2 \rfloor)}$, respectively. Substitute m, n, I_L, I_R into the black vertices from S_1 in Lemma 5.7, we have

$$S_1 = \left\{ v_{(n-\lceil \beta/2 \rceil)}^{(\lceil m/2 \rceil)},\ v_{(n-\lfloor \beta/2 \rfloor)}^{(\lfloor m/2 \rfloor)} \right\}.$$

The difference between the subscripts' value of both black vertices is 1. This indicate that one of the black vertices is 1 level above of another one, but they are not on the same cycle since $\lceil \frac{m}{2} \rceil \neq \lfloor \frac{m}{2} \rfloor$. Thus, the color change rule is not applicable and it is impossible to force the entire graph to black.

Since both of the distribution methods couldn't force the entire graph to black. Thus, $\gamma_Z(C_m \square C_n^t) = 2n - 1$ when n is odd.

In conclusion, $\gamma_Z(C_m \square C_n^t) = 2n - 1$ when $m = n$. $\square$

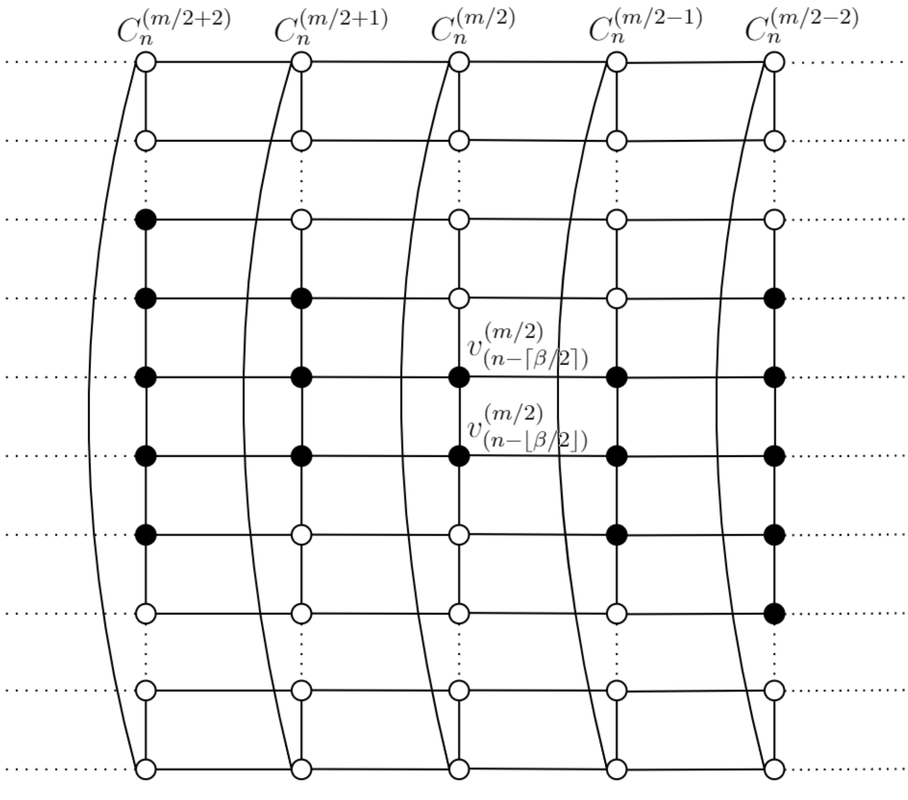

Figure 5.6: The opposite of twisted sector after the last iteration on both sides (*Case 2: Sub-case 1*, $\beta = n - 1$)

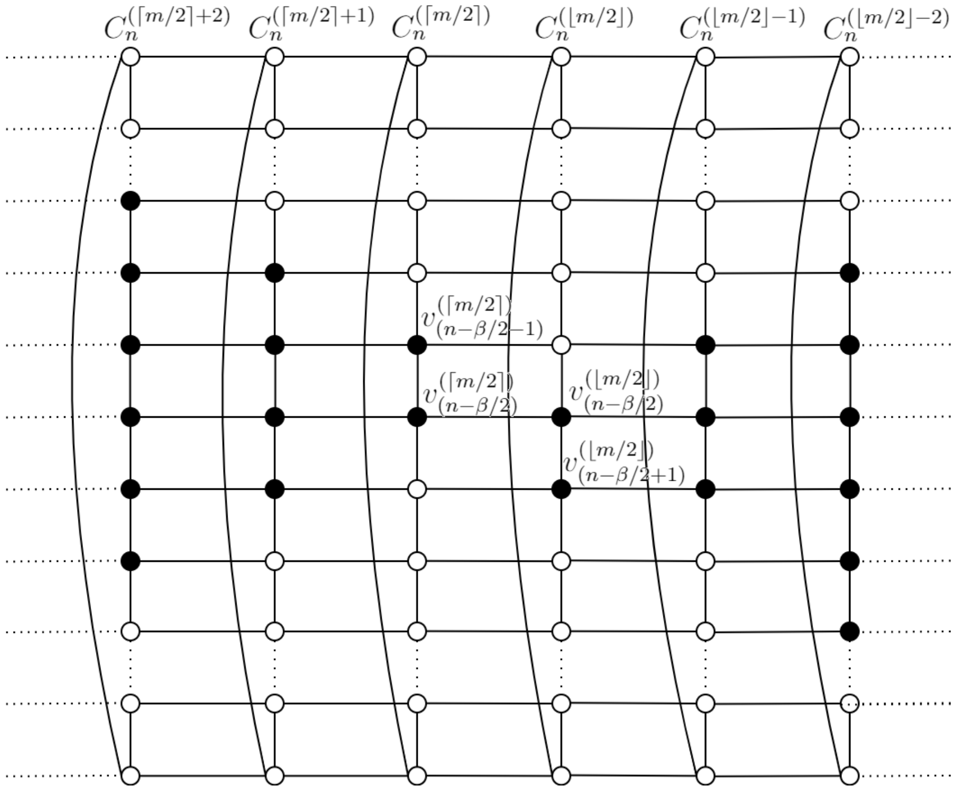

Figure 5.7: The opposite of twisted sector after the last iteration on both sides (*Case 2: Sub-case 2*, $\beta = n - 1$)

5.2 Power Dominating Number of $C_m \square C_n^t$

The relationship between the zero forcing set S_Z and power dominating set (PDS) S_P can be thought as if all the vertices from S_P has observed all the vertices in S_Z, then the graph is observed. This is due to the similarity of the color change rule and the simplified observation rule (SOR). Additionally, this relationship is also implying that $S_P \subseteq S_Z$. Thus, we need to find a closed neighborhood of S_P such that $N[S_P] \cap S_Z = S_Z$.

Based on the zero forcing number of twisted torus, we need to observed at least $2n - 1$ vertices when $m = n$ and $2n$ vertices when $m \neq n$. Similar to previous section, we put our focus on $C_n^{(m)}$ and $C_n^{(1)}$ since non-twisted sector (or torus graph) is thoroughly investigated by Koh and Soh (2019). The following two algorithms are the modified construction of PDS by Barrera and Ferrero (2011).

Algorithm 5.11. Construction of PDS that observe $2n$ vertices for $m \geq n \geq 3$.

1. Let S_{P_1} be the PDS.

2. Add the second vertex of every four vertices from $C_n^{(m)}$ into S_{P_1}.
3. Add the first vertex of every four vertices from $C_n^{(1)}$ into S_{P_1}.

Algorithm 5.12. Construction of PDS that observe at least $2n-1$ vertices for $m = n \geq 3$.

1. Let S_{P_2} be the PDS.
2. Add the fourth vertex of every four vertices from $C_n^{(m)}$ into S_{P_2}.
3. Add the vertex $v_n^{(m)}$ from $C_n^{(m)}$ into S_{P_2}
4. Add the third vertex of every four vertices from $C_n^{(1)}$ into S_{P_2}.

Note that we are considering the observed vertices from $C_n^{(m)}$ and $C_n^{(1)}$ only. Additionally, Step 2 and Step 3 in Algorithm 5.12 can be omitted if $n = 3$ and $n \equiv 0 (\bmod 4)$, respectively. Convert the Algorithm 5.11 and Algorithm 5.12 into a mathematical expression for analysis. For $m \geq n \geq 3$,

$$S_{P_1} = \left\{ v_k^{(m)} \ : \ k \equiv 2 (\bmod 4) \right\} \cup \left\{ v_l^{(1)} \ : \ l \equiv 1 (\bmod 4) \right\},$$

and for $m = n \geq 3$,

$$S_{P_2} = \left\{ v_k^{(m)} \ : \ k \equiv 0 (\bmod 4) \right\} \cup \left\{ v_l^{(1)} \ : \ l \equiv 3 (\bmod 4) \right\} \cup v_n^{(m)}$$

where $1 \leq k, l \leq n$. The following theorem will help us to obtain the lower bound of $\gamma_P(C_m \square C_n^t)$.

Theorem 5.13. *(Benson et al., 2018) If G is a nontrivial graph, then* $\left\lceil \frac{\gamma_Z(G)}{\Delta(G)} \right\rceil \leq \gamma_P(G)$*, and this bound is tight.*

$\Delta(G)$ denotes the greatest vertex degree of graph G. Since all the vertices in $C_m \square C_n^t$ has the same vertex degree, thus, $\Delta(C_m \square C_n^t) = 4$. Hence, we have the following theorem.

Theorem 5.14. *For $m \geq n \geq 3$, the lower bound of power dominating number of twisted torus graph is* $\gamma_P(C_m \square C_n^t) \geq \left\lceil \frac{n}{2} \right\rceil$.

Proof. The lower bound can be shown by substituting the zero forcing number from Theorem 5.10 and $\Delta(C_m \square C_n^t)$ into Theorem 5.13. Begin with substituting $\gamma_Z(C_m \square C_n^t) = 2n - 1$ into Theorem 5.13 when $m = n$, then

$$\gamma_P(C_m \square C_n^t) \geq \left\lceil \frac{2n-1}{4} \right\rceil \ \Rightarrow \ \gamma_P(C_m \square C_n^t) \geq \left\lceil \frac{n}{2} \right\rceil$$

regardless of whether the value of n is odd or even. Substitute $\gamma_Z(C_m\square C_n^t) = 2n$ into Theorem 5.13 when $m \neq n$, then

$$\gamma_P(C_m\square C_n^t) \geq \left\lceil \frac{2n}{4} \right\rceil \Rightarrow \gamma_P(C_m\square C_n^t) \geq \left\lceil \frac{n}{2} \right\rceil.$$

Thus, $\gamma_P(C_m\square C_n^t) \geq \lceil \frac{n}{2} \rceil$. $\square$

Theorem 5.15. *For $m \geq n \geq 3$, $\gamma_P(C_m\square C_3^t) = \gamma_P(C_m\square C_4^t) = 2$*

Proof. When $n = 3$, $\gamma_P(C_m\square C_3^t) \geq 2$ by Theorem 5.14. Let S_P be the power dominating set of $\gamma_P(C_m\square C_3^t)$. If $m \neq n$, by Algorithm 5.11, we have

$$S_P = \left\{v_2^{(m)}, v_1^{(1)}\right\}$$

and these vertices observed all the vertices in $C_n^{(m)}$ and $C_n^{(1)}$. Thus, $C_m\square C_3^t$ is observed. If $m = n$, by Algorithm 5.12, we have

$$S_P = \left\{v_3^{(m)}, v_3^{(1)}\right\}.$$

and these vertices observed all the vertices in $C_n^{(m)}$ and $C_n^{(1)}$. Thus, $C_m\square C_3^t$ is observed. Hence, $\gamma_P(C_m\square C_3^t) = 2$ (similar proof for $n = 4$). $\square$

The upper bound of power dominating number of twisted torus can be determined by analyzing the mathematical expression of Algorithm 5.11 and Algorithm 5.12.

Theorem 5.16. *For $m \geq n \geq 3$, the upper bound of power dominating number of $\gamma_P(C_m\square C_n)$ is*

$$\gamma_P(C_m\square C_n^t) \leq \begin{cases} \lceil \frac{n+1}{2} \rceil & \textit{if } n \equiv 2 (\bmod 4) \textit{ and } m \neq n \\ \lceil \frac{n}{2} \rceil & \textit{otherwise} \end{cases}$$

Proof. Begin with S_{P_1}. Convert the congruence into equation, $|S_{P_1}|$ increases by 1 whenever $n = 4k+1$ and $n = 4l+2$ for all $k, l = 1, 2, \ldots, n$. Since $|S_{P_1}| = 2$ when $3 \leq n \leq 4$, if $k, l = 1$, then $|S_{P_1}| = 3$ when $n = 5$ and $|S_{P_1}| = 4$ when $n = 6$. Similarly, if $k, l = 2$, then $|S_{P_1}| = 5$ when $n = 9$ and $|S_{P_1}| = 6$ when $n = 10$. From the series of $|S_{P_1}|$, we can deduce that when $m \neq n$, $\gamma_P(C_m\square C_n^t) = \lceil \frac{n+1}{2} \rceil$ when $n \equiv 2 (\bmod 4)$. Otherwise, $\gamma_P(C_m\square C_n^t) = \lceil \frac{n}{2} \rceil$.

Similar to S_{P_1}, begin with converting the congruence in S_{P_2} into equation. $|S_{P_2}|$ increases by 1 consistently whenever $n = 4k$ and $n = 4l + 3$ for all

$k, l = 1, 2, \ldots, n$. As for the vertex $v_n^{(m)}$, notice that it never stays at the same spot as n increases. Thus, if $4k < n < 4l + 3$, then $|S_{P_2}|$ for n is $x + 1$ where x is the $|S_{P_2}|$ for $4k$.

For example, $|S_{P_2}| = 2$ when $n = 4$ by Theorem 5.15. If $k, l = 1$, then $|S_{P_2}| = 3$ when $4 < n < 7$. Continuing this series, $|S_{P_2}| = 4$ when $n = 7$ and 8, $|S_{P_2}| = 5$ when $8 < n < 11$. From the series of $|S_{P_2}|$, we can deduce that when $m = n$, $\gamma_P(C_m \square C_n^t) = \left\lceil \frac{n}{2} \right\rceil$.

Since the value of $\gamma_P(C_m \square C_n)$ is produced from the construction of PDS (Algorithm 5.11 and Algorithm 5.12) and it was never claim to be the most optimal construction. Thus, the PDS guarantees the highest possible number of vertices needed only. Hence, $\gamma_P(C_m \square C_n^t) \leq \left\lceil \frac{n+1}{2} \right\rceil$ when $n \equiv 2 (\bmod 4)$ and $m \neq n$. Otherwise, $\gamma_P(C_m \square C_n^t) \leq \left\lceil \frac{n}{2} \right\rceil$. $\square$

The following two figures shows the observed twisted torus with the construction of PDS with Algorithm 5.11 and Algorithm 5.12. In the figure, the black vertex represents the vertex from PDS, the green vertex represents the vertex observed directly by the black vertex, and the orange vertex represents the vertex observed by the black vertex via SOR.

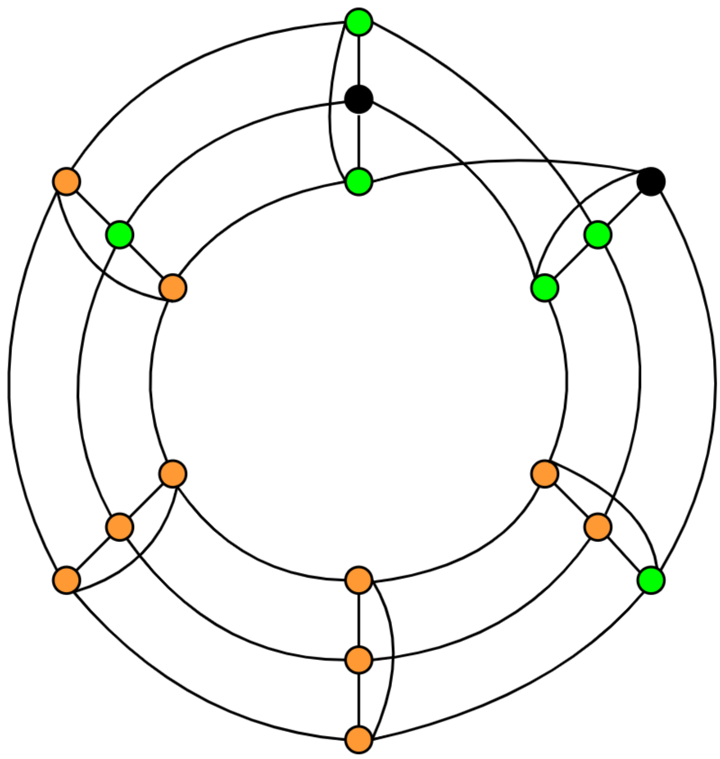

Figure 5.8: $C_6 \square C_3^t$ with Algorithm 5.11

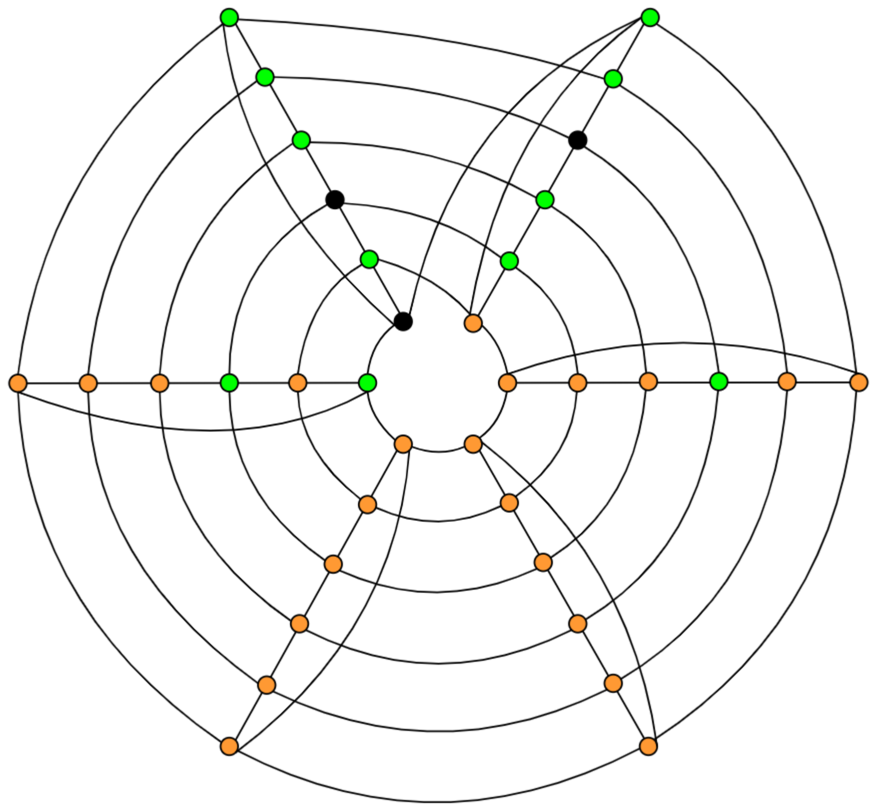

Figure 5.9: $C_6 \square C_6^t$ with Algorithm 5.12

CHAPTER 6

CONCLUSION AND RECOMMENDATIONS

6.1 Conclusion

Throughout this project, it is highly suspected the result of zero forcing number and power dominating number of the twisted torus would be the same as the torus. Surprisingly, a twisted torus requires one less condition in zero forcing number, but one more condition in power dominating number compare to a torus.

More importantly, the highlight of this project is determining the upper bound of the twisted torus. When we realized that the zero forcing number requires lesser condition, we have hope that perhaps it is possible to compute the power dominating number without any conditions. But the construction of PDS in Algorithm 5.11 and Algorithm 5.12 seems to be giving us the best PDS possible.

6.2 Recommendations For Future Work (1) - Zero Forcing Number

The process of finding the zero forcing number of twisted torus in this project is tedious and lengthy. But once we recognize a pattern among this problem, it shouldn't take long to identify whether the given number of vertices is enough to form a zero forcing set.

Another way to approach the zero forcing number is by converting the twisted torus graph into an adjacency matrix. For example, a torus graph can be expressed as a Kronecker product $\mathcal{K}$ as follow.

$$\mathcal{K}(C_m \Box C_n) = \mathcal{M}(C_m) \otimes I_n + I_m \otimes \mathcal{M}(C_n)$$

where $\mathcal{M}(C_m), \mathcal{M}(C_n)$ is the adjacency matrix of C_m, C_n respectively.

There isn't a way to express twisted torus as a Kronecker product, but we can expand the Kronecker product of torus as a normal matrix and change a few specific entry according to the appearance of the twisted torus. Expanding Kronecker product as a normal matrix is only feasible when the values of m, n are low. We can try to expand Kronecker product as $mn \times mn$ a block matrix $\mathcal{B}$ as follow.

$$\mathcal{B} = \begin{bmatrix} \mathcal{M}(C_n) & \mathcal{I}_m & \mathcal{O} & \cdots & \mathcal{I}_m \\ \mathcal{I}_m & \mathcal{M}(C_n) & \mathcal{I}_m & \cdots & \mathcal{O} \\ \mathcal{O} & \mathcal{I}_m & \mathcal{M}(C_n) & \cdots & \mathcal{O} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{I}_m & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{M}(C_n) \end{bmatrix}$$

where $\mathcal{O}$ is a zero block matrix and $\mathcal{I}_m$ is a identity block matrix. Note that each row and column must have exactly two $\mathcal{I}_m$ and one $\mathcal{M}(C_n)$.

In this project, we often focus on the m-th and the 1st cycle of C_n. But as long as that we are twisting two successive cycles of C_n, we can redefine the twisted torus definition. For example, we twist the first and the second cycle. This corresponds to $\mathcal{B}_{12}$ and $\mathcal{B}_{21}$. By changing the values inside the matrices, we have the new block matrix $\mathcal{B}$

$$\mathcal{B} = \begin{bmatrix} \mathcal{M}(C_n) & \mathcal{P}_m & \mathcal{O} & \cdots & \mathcal{I}_m \\ \mathcal{P}_m^\top & \mathcal{M}(C_n) & \mathcal{I}_m & \cdots & \mathcal{O} \\ \mathcal{O} & \mathcal{I}_m & \mathcal{M}(C_n) & \cdots & \mathcal{O} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{I}_m & \mathcal{O} & \mathcal{O} & \cdots & \mathcal{M}(C_n) \end{bmatrix}$$

where $\mathcal{P}_m$ is a permutation matrix.

If m, n are of larger values, then it is impossible to do it by hand. We may rely on a computer to help us to compute the eigenvalues and eigenvectors. Only then, we may have a chance find the zero forcing number through the linear algebra approach. We have wrote a simple Python programme to create such matrix which can be found in Appendix A.

6.3 Recommendations For Future Work (2) - Power Dominating Number

We have the upper bound of $\gamma_P(C_m \Box C_n^t)$ and it is possible to prove its equality. Koh and Soh (2019) has proven the equality of $\gamma_P(C_m \Box C_n)$ made by Barrera and Ferrero (2011) with the of *closure*, *star closure* and the bipartite graph of a Cartesian product of graphs. We believe the same approach can be used in twisted torus.

Other than that, what if we could twist the torus even further? For example, we redefine twisted torus as follow.

Let $C_m \square C_n$ denote the torus graph which consists of m copies of the n-cycle

$$x_{i,1}x_{i,2}\cdots x_{i,n}x_{i,1},$$

$i = 1, 2, \ldots, m$ together with n copies of the m-cycle

$$x_{1,j}x_{2,j}\cdots x_{m,j}x_{1,j},$$

$j = 1, 2, \ldots, n$. Here $m, n \geq 3$. Let $C_m \square C_n^{t=\tau}$ denote the twisted torus which is the graph obtained from the torus $C_m \square C_n$ by first deleting the set of m edges

$$x_{1,n}x_{1,1},\ x_{2,n}x_{2,1}, \ldots, x_{m,n}x_{m,1}$$

and then join the resulting graph with a set of m new edges

$$x_{1,n}x_{1+\tau,1},\ x_{2,n}x_{2+\tau,1}, \ldots, x_{m-1,n}x_{m-1+\tau,1},\ x_{m,n}x_{1+\tau,1}.$$

where τ is the *twisting degree* ranging from 1 to $\left\lfloor \frac{n}{2} \right\rfloor$. Thus, we have been working on $\gamma_P(C_m \square C_n^{t=1})$ in this project. Will all the twisted torus with $t \geq 1$ has the same zero forcing number and power dominating number?

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APPENDICES

APPENDIX A: A Simple Python Code

```
import numpy as np

def cycle_graph(n):
    A = np.zeros((n,n))

    for i in range(n):
        a = (i+1)%n
        b = (n+i-1)%n
        A[i][a] = 1
        A[i][b] = 1

    return A

def cycle_digraph(m):
    A = np.zeros((m,m))

    for i in range(m):
        a = (i+1)%m
        A[i][a] = 1

    return A

# assign t to be any other value if we want to investigate the Chapter 6.3 problem
def twisted_torus(n, m, t=1):
    # adjacency matrix of cycle digraph = permutation matrix
    perm_matrix = np.linalg.matrix_power(cycle_digraph(m), t)
    transpose_RM = perm_matrix.T

    M = np.kron(cycle_graph(n), np.eye(m)) +
            np.kron(np.eye(n), cycle_graph(m))

    for i in range(m):
        for j in range(m):
            M[i][m+j] = perm_matrix[i][j]
```

```
            M[m+i][j] = transpose_RM[i][j]

    return M
```