Block Hybrid Collocation Method for Higher order Ordinary Differential Equations

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A project report submitted in partial fulfilment of the requirements for the award of MASTER OF MATHEMATICS

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August 2022

DECLARATION

I hereby declare that this project report is based on my original work except for citations and quotations which have been duly acknowledged. I also declare that it has not been previously and concurrently submitted for any other degree or award at UTAR or other institutions.

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APPROVAL FOR SUBMISSION

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ABSTRACT

Five step block hybrid collocation method consist of two off-step points was proposed in this project for direct solution of third order initial value problems of ordinary differential equations. Collocation and interpolation approaches was used to derive the main and additional methods and combine all methods into block form to approximate the solution at the main points and the off-step points simultaneously. The order and stability properties of the method were analysed. Some test problems were used to test the performance of accuracy of the method by compared to existing methods. The method applied to solve the thin film flow problem and compare the result with the existing methods.

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LIST OF SYMBOLS / ABBREVIATIONS

Maxe Maximum error

CHAPTER 1

INTRODUCTION

1.1 General Introduction

Refer to Waeleh and Majid (2016), there are many real-world problems or natural processes can be expressed into the language of mathematics such as differential equation. Differential equation can be classified into various type, for example, partial differential equation (PDE) and ordinary differential equation (ODE). Many problems in science can be solved by ODE which show the important of ODE in the mathematical research. Usually, precise solution may not be available, or the answer may not be given in a convenient form. Hence, many researchers proved the reliability of numerical approximation techniques which used as numerical methods to solve the engineering problems. The problems formulated in the form of higher-order ODE with initial or boundary conditions which are initial value problems (IVPs) andboundary value problems (BVPs) (Modebei, Jator and Ramos, 2020).

1.2 Aims and Objectives

The main purpose of this project is to design a new block hybrid collocation method to solve the third order IVPs directly. The general third order ODEs are defined as:

$$y''' = f(x, y, y', y''),$$
(1)

with the initial conditions

$$y(a) = y_0, y'(a) = y'_0, y''(a) = y''_0, x \in [a, b].$$
 (2)

The objectives of this project are

- Investigate the zero stability and consistency of the newly proposed method to ensure the convergence.
- Analyse the performance of the new method by solving some test problems and compare the results with the methods from literature.
- Apply the new method to solve the thin film flow problem.

1.3 Problem Statement

According to Mohammed and Adeniyi (2014), the mathematical formulation of science and engineering problems were often leads to initial value problems (IVPs). Adeyeye and Omar (2019) also stated that the modelling cases for complex motion such as fluid flow normally result in higher order ordinary differential equations (ODEs). Yap et al. (2014) mentioned that many problems in physical science and engineering can be formulated in third order ODEs such as gravity driven flows, electromagnetic waves and thin film flow. Many numerical and theoretical studies dealing with third order ODEs can be found in literatures. However, the exact solutions may not be found for some third order ODEs with initial conditions. Thus, numerical method is required to solve the third order IVPs to generate the approximate solutions. The conventional approach for direct solution of third order IVPs requires the complicated execution work to obtain starting values. Besides that, it generates only one approximate value at each iteration and lead to longer execution time. Here, we propose the self-starting block method with the features that give as set of numerical approximation simultaneously. The proposed method should be effective in reducing execution time and provide good approximations for third order ODEs.

CHAPTER 2

LITERATURE REVIEW

2.1 Reduction of Order Method

Normally, higher-order ODEs with initial conditions can be solved by reducing it to the system of first order ODEs also known as the reduction of order method (Anake at el. 2012). But the reduction of order method requires longer computational time and heavier computation work to solve the higher order ODEs. Abdelrahim and Omar (2017) stated that the fourth order initial value problems (IVPs) could be solved by reduction of order method which reduced into a system with four equations of first order IVPs and solved by suitable numerical methods. As a consequence, the number of equations needed to be solved also increased.

2.2 Direct Method

The initial values problems (IVPs) were solved directly without reduction of order method in order to overcome the limitation of reduction of order method. Direct method was an alternative approach of the reduction of order method which could be used to solve the IVPs directly with the advantages in speed and accuracy. Refer to Jator (2010), there were various type of direct methods had been proposed such as linear multistep method (LMMs), multistep collocation methods, Runge-Kutta-Nyström methods (RKN), multiderivative methods, exponentially-fitting and trigonometrically fitted methods. Most of the method were implemented in the predictor-corrector mode. In addition, direct method also could be implemented in the block mode.

The linear multistep method was a direct method that usually used to approximate the direct solution of higher order ODEs. Awoyemi (2003) proposed a pstable linear multistep method for solving the general third order IVPs. The method was implemented in predictor-corrector mode and required the starting values from Runge-Kutta methods. The implementation in predictor-corrector mode was complicated. It was also costly in human effort for developing the predictors. Consequently, it leaded in longer computation time and heavy computational work in the part that incorporating subroutines to get the starting values. The implementation in Runge-Kutta methods to supply the starting values increased the computation time because it involved several function evaluations for every step. Adeyeye and Omar (2019) mentioned that the linear multistep method was convergent when it was consistent and zero stable.

2.3 Hybrid Method

According to Dahlquist (1978), the Dahlquist's barrier condition was the basic condition required to implement those direct methods. It highlighted that the zero stable method was better with the order P = k + 1 and P = k + 2 for odd and even number of steps respectively. The hybrid methods were proposed to circumvent the Dahlquist's barrier conditions in order to reduce the number of steps and function evaluations while maintained a high level of accuracy and zero stability at the same time. Anake at el. (2012) stated that the hybrid methods had the advantage that easy change in the step size which similar to the Runge-Kutta methods and evaluated the data at off-step points. The design of the algorithms for hybrid method was tedious because it incorporated the off-step points and the present of non-step function increased the number of predictors required to implement the methods.

Awoyemi and Idowu (2005) recommended the hybrid collocation method for solving general third order ODEs which was quite similar to Awoyemi (2003) but with the additional off-grid point $x_{n+3/2}$. The numerical results showed that the inclusion of off-grid point in hybrid collocation method helped to improve the accuracy.

2.4 Block Hybrid Method

Block hybrid method was the method that group the hybrid formulas into a single block and used for the direct solution of IVPs. The block hybrid method that used to solve higher order IVPs directly computed the numerical approximation at a few points at the same time. Abdelrahim and Omar (2017) remarked that the block hybrid method also could avoid the computational burden and zero stability barrier. Based on Yap et al. (2014), the *m*-point block method was derived by dividing the interval of integration to a series of blocks with *m*-points to generate a block of new solutions simultaneously. Adeyeye and Omar (2019) stated that the zero-stability of block method ensured that the approximation of the solution converge as the step size tends to zero. In other words, the numerical result will converge to the analytical result when step size tends to zero. The block method which is consistent shows that the local truncation error (LTE) tends to zero before the step size tends to zero. Thus, the method is convergent if the method is consistent and zero-stable.

2.4.1 Block Method

The Block method had the features of getting the set of solutions concurrently. Waeleh and Majid (2016) proposed the four-point block method to solve the fifth order IVPs directly by implementing a variable step size strategy. The interval was divided into series of blocks with four equal subintervals and four solutions were computed concurrently. Thus, it was more cost-effective because of the characteristic of the block method that generated a set of solution simultaneously in each iteration. The 4-point block method had better performance in implementing with direct integration approach as less storage required than reduction method with an acceptable accuracy.

2.4.2 Second Order Ordinary Differential Equation

Some authors focused on different block hybrid methods to estimate the solution for the second order IVPs. Jator (2010) proposed a three-step hybrid linear multistep method (HLMM) of order seven to find the approximate solution of the second order IVPs. The proposed method consisted of three non-step points which were $x_{n+1/2}$, $x_{n+3/2}$ and $x_{n+5/2}$. It was a self-starting method that implemented by combining the HLMM as simultaneous integrators for IVPs. The self-starting method solved the issue of conventional method that required a few starting values and predictors to proceed. The approach was derived through interpolation and collocation. The main hybrid method was applied together with these additional methods as a block method to solve IVPs simultaneously.

Anake at el. (2012) developed a continuous one-step implicit block hybrid method to approximate the solution of second order IVPs. The proposed method was improved by included off-grid points to enable the multistep procedure. This ensured the zero stability and the consistency of the method. To derive a zero stable method, the off-step points were carefully chosen as $x_{n+1/3}$, $x_{n+2/3}$. The proposed method was highly accurate with very low error terms.

2.4.3 Third Order Ordinary Differential Equation

Several researchers conducted studies on block hybrid method to approximate the solution for third order ODEs. Mohammed and Adeniyi (2014) derived the hybrid

linear multistep method (HLMM) with the off-step point at $x_{n+8/3}$. The method was convergent and zero stable.

Yap et al. (2014) suggested the three-point block hybrid collocation method with two off-step points ($x_{n+1/2}$ and $x_{n+3/2}$) for solving third order IVPs directly. The method obtained the approximated solution of *y* at three main points and two off-grid points at once for every iteration. As a consequence, this approach required a smaller number of total steps to approximate the solution and provided precise approximation as the step size decreased.

Yap and Ismail (2018) extended the idea to generate a four-point block hybrid collocation method that consist of two off-grid points ($x_{n+1/2}$ and $x_{n+3/2}$) for the solution of third order ODEs with initial condition. It showed the accuracy of the method increased when the step size decreased. The block hybrid collocation method was applicable to solve the physical problem of thin film flow.

2.4.4 Fourth Order Ordinary Differential Equation

Yap and Ismail (2015) recommended a block hybrid collocation method that consists of three off-grid points to approximate the solution of general fourth order initial value problems (IVPs). The proposed method was implemented in the block form of the main and additional methods. The method was self-starting and generated the approximate solution of *y* at the four main points and off-grid points $x_{n+1/2}$, $x_{n+3/2}$, $x_{n+5/2}$ simultaneously.

Abdelrahim and Omar (2017) proposed the four-step block method consist of generalized three off-step points to approximate the solution of fourth order IVPs of ODEs. The strategy used to derive the method was interpolation for the function at selected points and collocation for the function of fourth derivative that cover all points. The proposed approach was implemented in self-starting method which without the predictors that reduced the efficiency of the method. Thus the proposed method worked for chosen off-grid points, it was more robust and flexible. The approaches suggested by Yap and Ismail (2015) and Abderlrahim and Omar (2017) demonstrated that the inclusion of off-step points improved the accuracy when solving higher order ODEs.

Sometimes the block methods were derived with specific points or off-step points to solve the fourth order ODEs. Adeyeye and Omar (2019) suggested a block method to approximate the solution of the fourth order IVPs with five main points with various equally step size. The block method showed faster convergence and better accuracy when the step size decreased.

2.4.5 Boundary Value Problems (BVPs)

According to Ramos and Rufai (2021), the reduction of the order method also could be used to solve the BVPs of ODEs. Similar to initial value problems (IVPs), this method required a lot of human efforts and computational time. Modebei, Jator and Ramos (2020) derived the linear multistep hybrid method consist of four off-step points to approximate the solution of fourth order BVPs in ODEs. The collocation approach was used to obtain the continuous linear multistep formulas and group to an unique block to structure the block hybrid method (BHM). They considered various type of fourth order BVPs with the advantages of flexibility and demonstrated good performance in terms of high accuracy.

Ramos and Rufai (2021) recommended an implicit two step block hybrid method with two off-grid points and two fourth derivatives to solve the linear and nonlinear type of third order BVPs. This proposed method also derived by using collocation and interpolation techniques. They applied to solve the problems of sandwich beam and the Falkner-Skan equations. The numerical approximations of the method converged rapidly.

2.5 Summary on Literature Review

The studies on block hybrid methods to approximate the solution of higher order initial differential equations (IVPs) demonstrated its advantage in accuracy and reduced execution time. Therefore, the block hybrid method is chosen to resolve the IVPs in third order ODEs here. In additional, the reasons to select this approach are self-starting with good accuracy as proven in literature and its simplicity in derivation. More points and closer interval of off-step points are considered in the proposed method with the aim for better performance in accuracy.

CHAPTER 3

METHODOLOGY

3.1 Development of the Five Step Block Hybrid Collocation Method

The five step block hybrid collocation method that results in estimations of y, y' and y'' for the general form of third order ordinary differential equations (ODEs) is defined as below

$$\sum_{j=0}^{k} \alpha_{j} y_{n+j} + \sum_{j=1}^{2} \alpha_{\theta_{j}} y_{n+\theta_{j}} = h^{3} \left(\sum_{j=0}^{k} \beta_{j} f_{n+j} + \sum_{j=1}^{2} \beta_{\theta_{j}} f_{n+\theta_{j}} \right)$$
(3)

On every interval $[x_n, x_n + 5h]$, n = 0, 5, ..., N - 5, we determined the formula by interpolating Y(x) as follows

$$Y(x) = \sum_{j=0}^{r+s-1} \omega_j x^j$$
 (4)

where x in the interval of [a, b], ω_j are the coefficients, r and s are the number of interpolation and collocation points, respectively. The positive integer k denotes the step number of the method. The continuous approximations are obtained by imposing the following conditions

$$Y(x_{n+j}) = y_{n+j}, \ j = 0, 1, 2, \dots, k$$
(5)

$$Y'''(x_{n+\tau}) = f_{n+\tau}, \tau = \{j, \theta_1, \theta_2\}, \ j = 0, 1, 2, \dots, k$$
(6)

where θ_1 and θ_2 are rational numbers. The parameters to develop the method are set as $\theta_1 = 1/3$, $\theta_2 = 2/3$, r = 3, s = 8, k = 5. Via apply the interpolation for three points x_n, x_{n+1}, x_{n+2} with condition (5) and the collocation for eight points $x_n, x_{n+1/3}, x_{n+2/3}, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}$ with condition (6), produce a system of eleven equations. The Cramer's rule is applied to solve the equations and obtain the coefficients ω_j . Next, we substitute the value of ω_j into (4) to get the hybrid collocation methods as below: Main method:

$$y_{n+5} = 6y_n - 15y_{n+1} + 10y_{n+2} + h^3 \left(-\frac{397}{3360} f_n + \frac{40581}{43120} f_{n+\frac{1}{3}} - \frac{194157}{101920} f_{n+\frac{2}{3}} + \frac{1417}{336} f_{n+1} + \frac{697}{160} f_{n+2} + \frac{4741}{2352} f_{n+3} + \frac{17911}{36960} f_{n+4} + \frac{417}{50960} f_{n+5} \right)$$
(7)

Additional methods:

$$\begin{aligned} y_{n+4} &= 3y_n - 8y_{n+1} + 6y_{n+2} \\ &+ h^3 \left(-\frac{191}{3360} f_n + \frac{39609}{86240} f_{n+\frac{1}{3}} - \frac{95013}{101920} f_{n+\frac{2}{3}} + \frac{1171}{560} f_{n+1} \\ &+ \frac{187}{96} f_{n+2} + \frac{11327}{23520} f_{n+3} + \frac{23}{2464} f_{n+4} - \frac{19}{152880} f_{n+5} \right) \\ y_{n+3} &= y_n - 3y_{n+1} + 3y_{n+2} \\ &+ h^3 \left(-\frac{15}{448} f_n + \frac{80433}{344960} f_{n+\frac{1}{3}} - \frac{19197}{40768} f_{n+\frac{2}{3}} + \frac{5473}{6720} f_{n+1} \\ &+ \frac{431}{960} f_{n+2} + \frac{229}{31360} f_{n+3} + \frac{43}{73920} f_{n+4} - \frac{47}{611520} f_{n+5} \right) \\ y_{n+\frac{2}{3}} &= \frac{2}{9} y_n + \frac{8}{9} y_{n+1} - \frac{1}{9} y_{n+2} \\ &+ h^3 \left(-\frac{176529353}{198603044640} f_n + \frac{69247}{6985440} f_{n+\frac{1}{3}} + \frac{41959}{24766560} f_{n+\frac{2}{3}} \right) \\ y_{n+\frac{2}{3}} &= \frac{2}{9} y_n + \frac{8}{9} y_{n+1} - \frac{1}{9} y_{n+2} \\ &+ h^3 \left(-\frac{176529353}{198603044640} f_{n+4} - \frac{38029}{9027411120} f_{n+5} \right) \\ y_{n+\frac{1}{3}} &= \frac{5}{9} y_n + \frac{5}{9} y_{n+1} - \frac{1}{9} y_{n+2} \\ &+ h^3 \left(-\frac{325421}{1382451040} f_{n+4} - \frac{36027}{1027411120} f_{n+5} \right) \\ y_{n+\frac{1}{3}} &= \frac{5}{9} y_n + \frac{5}{9} y_{n+1} - \frac{1}{9} y_{n+2} \\ &+ h^3 \left(-\frac{325421}{1396809280} f_n + \frac{265127}{16765056} f_{n+\frac{1}{3}} + \frac{136187}{16511040} f_{n+\frac{2}{3}} \right) \\ &+ \frac{2862509}{79361856} f_{n+1} + \frac{30503}{11337408} f_{n+2} - \frac{353989}{1111065984} f_{n+3} \\ &+ \frac{215311}{4364902080} f_{n+4} - \frac{154879}{36109644480} f_{n+5} \right) \end{aligned}$$

In order to generate the formulas for the method of first and second derivatives, the values for ω_j are substituted into

$$Y'(x) = \sum_{j=0}^{r+s-1} j\omega_j x^{j-1}$$
$$Y''(x) = \sum_{j=0}^{r+s-1} j(j-1)\omega_j x^{j-2}$$
(9)

Hence, we obtain the formula for the derivatives as follows:

$$\begin{split} hy_n' &= -\frac{3}{2}y_n + 2y_{n+1} - \frac{1}{2}y_{n+2} \\ &+ h^3 \left(\frac{187}{67200}f_n + \frac{201123}{1724800}f_{n+\frac{1}{3}} + \frac{75897}{2038400}f_{n+\frac{2}{3}} + \frac{3349}{20160}f_{n+1} \\ &+ \frac{2369}{201600}f_{n+2} - \frac{127}{94080}f_{n+3} + \frac{457}{2217600}f_{n+4} - \frac{163}{9172800}f_{n+5}\right) \\ hy_{n+\frac{1}{3}}' &= -\frac{7}{6}y_n + \frac{4}{3}y_{n+1} - \frac{1}{6}y_{n+2} \\ &+ h^3 \left(-\frac{2847683}{1322697600}f_n - \frac{31181}{13970880}f_{n+\frac{1}{3}} + \frac{371503}{155110400}f_{n+\frac{2}{3}} \\ &+ \frac{7150051}{132269760}f_{n+1} + \frac{1070663}{264539520}f_{n+2} - \frac{446143}{92588320}f_{n+3} \\ &+ \frac{1088953}{14549673600}f_{n+4} - \frac{392471}{60182740800}f_{n+5}\right) \\ hy_{n+\frac{2}{3}}' &= -\frac{5}{6}y_n + \frac{2}{3}y_{n+1} + \frac{1}{6}y_{n+2} \\ &+ h^3 \left(\frac{1689563}{1322697600}f_n - \frac{3467983}{1322697600}f_{n+2} + \frac{4756861}{165110400}f_{n+\frac{2}{3}} \\ &- \frac{7292953}{1322697600}f_{n+1} - \frac{5327989}{1322697600}f_{n+2} + \frac{176749}{370355328}f_{n+3} \\ &- \frac{1075741}{14549673600}f_{n+4} + \frac{387071}{60182740800}f_{n+5}\right) \\ hy_{n+1}' &= -\frac{1}{2}y_n + \frac{1}{2}y_{n+2} \\ &+ h^3 \left(\frac{869}{201600}f_n - \frac{5751}{156800}f_{n+\frac{1}{3}} + \frac{2511}{81536}f_{n+\frac{2}{3}} - \frac{37}{240}f_{n+1} \\ &- \frac{2453}{201600}f_{n+2} + \frac{407}{282240}f_{n+3} - \frac{1}{4480}f_{n+4} + \frac{89}{4586400}f_{n+5}\right) \end{split}$$

$$\begin{split} hy_{n+2}' &= \frac{1}{2}y_n - 2y_{n+1} + \frac{3}{2}y_{n+2} \\ &+ h^3 \left(-\frac{3953}{201600} f_n + \frac{6399}{49280} f_{n+\frac{1}{3}} - \frac{73791}{291200} f_{n+\frac{2}{3}} + \frac{8171}{20160} f_{n+1} \\ &+ \frac{347}{4480} f_{n+2} - \frac{293}{40320} f_{n+3} + \frac{2423}{2217600} f_{n+4} - \frac{41}{436800} f_{n+5} \right) \\ hy_{n+3}' &= \frac{3}{2}y_n - 4y_{n+1} + \frac{5}{2}y_{n+2} \\ &+ h^3 \left(-\frac{269}{13440} f_n + \frac{161433}{862400} f_{n+\frac{1}{3}} - \frac{794529}{2038400} f_{n+\frac{2}{3}} + \frac{20071}{201600} f_{n+1} \\ &+ \frac{196643}{201600} f_{n+2} + \frac{279}{3136} f_{n+3} - \frac{9479}{2217600} f_{n+4} + \frac{457}{1834560} f_{n+5} \right) \\ hy_{n+4}' &= \frac{5}{2}y_n - 6y_{n+1} + \frac{7}{2}y_{n+2} \\ &+ h^3 \left(-\frac{2581}{28800} f_n + \frac{1024407}{1724800} f_{n+\frac{1}{3}} - \frac{2353131}{2038400} f_{n+\frac{2}{3}} + \frac{639}{320} f_{n+1} \\ &+ \frac{54043}{20800} f_{n+2} + \frac{58315}{56448} f_{n+3} + \frac{58423}{739200} f_{n+4} - \frac{18839}{9172800} f_{n+5} \right) \\ hy_{n+5}' &= \frac{7}{2}y_n - 8y_{n+1} + \frac{9}{2}y_{n+2} \\ &+ h^3 \left(\frac{48661}{201600} f_n - \frac{161757}{156800} f_{n+\frac{1}{3}} + \frac{3635847}{2038400} f_{n+\frac{2}{3}} + \frac{5017}{10080} f_{n+1} \\ &+ \frac{230473}{67200} f_{n+2} + \frac{492949}{282240} f_{n+3} + \frac{222787}{201600} f_{n+\frac{4}{3}} + \frac{23263}{382200} f_{n+5} \right) \\ h^2y_n'' &= y_n - 2y_{n+1} + y_{n+2} \\ &+ h^3 \left(-\frac{3001}{3320} f_n - \frac{170343}{344960} f_{n+\frac{3}{3}} - \frac{729}{31360} f_{n+\frac{2}{3}} - \frac{353}{960} f_{n+1} \\ &- \frac{271}{13440} f_{n+2} + \frac{541}{282240} f_{n+3} - \frac{13}{49280} f_{n+4} + \frac{1}{47040} f_{n+5} \right) \\ h^2y_{n+\frac{3}{3}}'' &= y_n - 2y_{n+1} + y_{n+2} \\ &+ h^3 \left(\frac{104311}{7348320} f_n - \frac{1553}{8960} f_{n+\frac{4}{3}} - \frac{217}{71248} f_{n+\frac{2}{3}} - \frac{9140141}{29393280} f_{n+1} \\ &- \frac{186461}{7348320} f_{n+2} + \frac{36749}{11757312} f_{n+3} - \frac{3613}{7348320} f_{n+4} \\ &+ \frac{16481}{382112640} f_{n+5} \right) \end{split}$$

$$\begin{split} h^2 y_{n+\frac{2}{3}}'' &= y_n - 2y_{n+1} + y_{n+2} \\ &+ h^3 \Big(\frac{211819}{29393280} f_n - \frac{20789}{1034880} f_{n+\frac{1}{3}} + \frac{283}{6272} f_{n+\frac{2}{3}} \\ &- \frac{5066029}{1469640} f_{n+1} - \frac{680261}{29393280} f_{n+2} + \frac{544661}{205752960} f_{n+3} \\ &- \frac{130829}{323326080} f_{n+4} + \frac{719}{20575296} f_{n+5} \Big) \\ h^2 y_{n+1}'' &= y_n - 2y_{n+1} + y_{n+2} \\ &+ h^3 \Big(\frac{37}{3360} f_n - \frac{3159}{62720} f_{n+\frac{1}{3}} + \frac{26001}{101920} f_{n+\frac{2}{3}} - \frac{1553}{8064} f_{n+1} \\ &- \frac{29}{1120} f_{n+2} + \frac{199}{62720} f_{n+3} - \frac{1}{2016} f_{n+4} + \frac{53}{1223040} f_{n+5} \Big) \\ h^2 y_{n+2}'' &= y_n - 2y_{n+1} + y_{n+2} \\ &+ h^3 \Big(- \frac{863}{13440} f_n + \frac{140697}{344960} f_{n+\frac{1}{3}} - \frac{68769}{81536} f_{n+\frac{2}{3}} + \frac{2547}{2240} f_{n+1} \\ &+ \frac{15539}{10320} f_{n+2} - \frac{493}{18816} f_{n+3} + \frac{569}{147840} f_{n+4} - \frac{601}{1834560} f_{n+5} \Big) \\ h^2 y_{n+3}'' &= y_n - 2y_{n+1} + y_{n+2} \\ &+ h^3 \Big(\frac{719}{10080} f_n - \frac{2229149}{689920} f_{n+\frac{1}{3}} + \frac{64881}{101920} f_{n+\frac{2}{3}} + \frac{89}{13440} f_{n+1} \\ &+ \frac{4169}{3360} f_{n+2} + \frac{222239}{564480} f_{n+3} - \frac{223}{12320} f_{n+4} + \frac{1493}{1223040} f_{n+5} \Big) \\ h^2 y_{n+4}'' &= y_n - 2y_{n+1} + y_{n+2} \\ &+ h^3 \Big(- \frac{3263}{13440} f_n + \frac{41067}{31360} f_{n+\frac{1}{3}} - \frac{200961}{81536} f_{n+\frac{2}{3}} + \frac{44299}{20160} f_{n+1} \\ &+ \frac{2299}{4480} f_{n+2} + \frac{42101}{31360} f_{n+3} + \frac{14209}{40320} f_{n+4} - \frac{919}{1223044} f_{n+5} \Big) \\ h^2 y_{n+5}'' &= y_n - 2y_{n+1} + y_{n+2} \\ &+ h^3 \Big(\frac{3877}{3360} f_n - \frac{573723}{98560} f_{n+\frac{1}{3}} + \frac{1701}{160} f_{n+\frac{2}{3}} - \frac{30189}{4480} f_{n+1} \\ &+ \frac{30203}{10080} f_{n+2} - \frac{3133}{26688} f_{n+3} + \frac{59459}{36960} f_{n+4} + \frac{11533}{40320} f_{n+5} \Big) \end{split}$$

We group all the hybrid collocation methods (7), (8) and (10) as block method and then use it to obtain the approximation of y, y' and y'' at five main points ($x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}$) and two off-step points ($x_{n+1/3}, x_{n+2/3}$)

(10)

simultaneously. We name this new method as five step block hybrid collocation method (FVBHCM).

3.2 Order of Method

According to Jator (2010), the difference operator related to (3) is represented as

$$L[y(x);h] = \sum_{j=0}^{k} [\alpha_{j}y(x+jh) - h^{3}\beta_{j}y'''(x+jh)] + \sum_{j=1}^{2} [\alpha_{\theta_{j}}y(x+\theta_{j}h) - h^{3}\beta_{\theta_{j}}y'''(x+\theta_{j}h)]$$
(11)

where y(x) is sufficiently differentiable. The tested functions y(x + jh) and y'''(x + jh) about x are expanding and rearrange to get

$$L[y(x);h] = C_0 y(x) + C_1 h y'(x) + \dots + C_q h^q y^{(q)}(x) + \dots$$
(12)

with the coefficients C_q for $q = 0,1,2 \cdots$ are constants. The matrix of differential equation for the method is defined as

$$\alpha Y_m = h\beta Y'_m + h^2 \gamma Y''_m + h^3 \delta F_m \tag{13}$$

where α , β , γ and δ are the unknown coefficients for the method. The order of the method can be found by using the formula as follow:

$$C_0 = \sum_{j=0}^k \alpha_j + \sum_{j=1}^2 \alpha_{\theta_j}$$

$$C_1 = \sum_{j=1}^k j\alpha_j + \sum_{j=1}^2 \theta_j \alpha_{\theta_j} - \left[\sum_{j=0}^k \beta_j + \sum_{j=1}^2 \beta_{\theta_j}\right]$$

$$C_2 = \frac{1}{2!} \left[\sum_{j=1}^k j^2 \alpha_j + \sum_{j=1}^2 \theta_j^2 \alpha_{\theta_j}\right] - \left[\sum_{j=1}^k j\beta_j + \sum_{j=1}^2 \theta_j \beta_{\theta_j}\right] - \left[\sum_{j=0}^k \gamma_j + \sum_{j=1}^2 \gamma_{\theta_j}\right]$$

$$\vdots$$

$$C_{q} = \frac{1}{q!} \left[\sum_{j=1}^{k} j^{q} \alpha_{j} + \sum_{j=1}^{2} \theta_{j}^{q} \alpha_{\theta_{j}} \right] - \frac{1}{(q-1)!} \left[\sum_{j=1}^{k} j^{q-1} \beta_{j} + \sum_{j=1}^{2} \theta_{j}^{q-1} \beta_{\theta_{j}} \right] - \frac{1}{(q-2)!} \left[\sum_{j=1}^{k} j^{q-2} \gamma_{j} + \sum_{j=1}^{2} \theta_{j}^{q-2} \gamma_{\theta_{j}} \right] - \frac{1}{(q-3)!} \left[\sum_{j=1}^{k} j^{q-3} \delta_{j} + \sum_{j=1}^{2} \theta_{j}^{q-3} \delta_{\theta_{j}} \right]$$
(14)

With reference to Henrici (1962), the linear multistep method is classified as an order P method when

$$C_0 = C_1 = C_2 = \dots = C_{P+2} = 0, \ C_{P+3} \neq 0$$
 (15)

The five step	block hybrid	l collocatio	on meth	od has or	der P	= 8 with	n the error
constants: C	113	151	1877	56401	3	0743	4649
constants, c_{11} a	$\frac{110}{7620480}$,	9525600' 76	204800'	62497461600	' 3333	1979520	1257379200'
46699573	11430989	20	0971	503	6	80107	182981
32998659724800'	8249664931	200' 5029	516800	25660800	502	9516800'	1257379200 '
4908139	121	42719	1832	7097	1	2551	761
5029516800'	38102400', 4	464104400'	249989	846400' 1	05840'	38102400	,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,
2447 2401							

 $\frac{2447}{4233600}$, $-\frac{2491}{680400}$ respectively. According to Fatunla (1991), the method is consistent as it has order P > 1.

3.3 Zero Stability

The hybrid collocation methods (7), (8) and (10) are combined into block form to analyse its zero stability. The matrix finite difference equation to test the zero stability is presented as follows:

$$A^{[0]}Y_{M+1} = A^{[1]}Y_M + h^3 (B^{[0]}F_{M+1} + B^{[1]}F_M) + h^2 C^{[1]}Y_M'' + hD^{[1]}Y_M'$$
(16)

where

$$Y_{M+1} = \begin{bmatrix} y_{n+1/3}, y_{n+2/3}, y_{n+1}, y_{n+2}, y_{n+3}, y_{n+4}, y_{n+5} \end{bmatrix}^{T}$$

$$Y_{M} = \begin{bmatrix} y_{n-4}, y_{n-3}, y_{n-2}, y_{n-1}, y_{n-2/3}, y_{n-1/3}, y_{n} \end{bmatrix}^{T}$$

$$F_{M+1} = \begin{bmatrix} f_{n+1/3}, f_{n+2/3}, f_{n+1}, f_{n+2}, f_{n+3}, f_{n+4}, f_{n+5} \end{bmatrix}^{T}$$

$$F_{M} = \begin{bmatrix} f_{n-4}, f_{n-3}, f_{n-2}, f_{n-1}, f_{n-2/3}, f_{n-1/3}, f_{n} \end{bmatrix}^{T}$$

$$Y''_{M} = \begin{bmatrix} y''_{n-4}, y''_{n-3}, y''_{n-2}, y''_{n-1}, y''_{n-2/3}, y''_{n-1/3}, y''_{n} \end{bmatrix}^{T}$$

$$Y''_{M} = \begin{bmatrix} y'_{n-4}, y'_{n-3}, y'_{n-2}, y''_{n-1}, y'_{n-2/3}, y''_{n-1/3}, y''_{n} \end{bmatrix}^{T}$$

$$Y''_{M} = \begin{bmatrix} y'_{n-4}, y'_{n-3}, y'_{n-2}, y''_{n-1}, y'_{n-2/3}, y''_{n-1/3}, y''_{n} \end{bmatrix}^{T}$$
(17)

The first characteristic polynomial is obtained as follows:

$$p(z) = Det[zA_0 - A_1] = z^6(z - 1)$$
(18)

where

$$A_{0} = [A^{[0]}]^{-1}A^{[0]} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, A_{1} = [A^{[0]}]^{-1}A^{[1]}$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Based on Fatunla (1991), while the roots for first characteristic polynomial are not more than one, the five step block hybrid collocation method is considered as zero stable.

CHAPTER 4

RESULTS AND DICUSSION

4.1 Numerical Examples

To express the performance of the newly proposed method, we consider nine tested problems as follows:

Problem 1. Consider the linear nonhomogeneous problem

$$y''' - 2y'' - 3y' + 10y = 34xe^{-2x} - 16e^{-2x} - 10x^2 + 6x + 34$$
$$y(0) = 3, y'(0) = 0, y''(0) = 0, x \in [0,1].$$

Exact Solution: $y(x) = x^2 e^{-2x} - x^2 + 3$

Source: Majid et al. [16]

Problem 2. Consider the linear homogeneous problem

$$y''' + y' = 0$$

 $y(0) = 0, y'(0) = 1, y''(0) = 2, x \in [0, 20].$

Exact Solution: $y(x) = 2(1 - \cos x) + \sin x$

Source: Majid et al. [16]

Problem 3. Consider the linear nonhomogeneous problem

$$y''' = 3 \sin x$$
$$y(0) = 1, y'(0) = 0, y''(0) = -2, x \in [0,1]$$
ution:
$$y(x) = 3 \cos x + \frac{x^2}{2} - 2$$

Exact Solution: $y(x) = 3\cos x + \frac{x}{2}$

Source: Adesanya et al. [17]

Problem 4. Consider the linear homogeneous problem

$$y''' = -6y'' - 11y' - 6y$$
$$y(0) = 1, y'(0) = 0, y''(0) = 0, x \in [0,10].$$
Exact Solution: $y(x) = 3e^{-x} - 3e^{-2x} + e^{-3x}$

Source: Hochstadt [18]

Problem 5. Consider the linear nonhomogeneous problem

$$y''' = x - 4y'$$

$$y(0) = 0, y'(0) = 0, y''(0) = 1, x \in [0,20].$$

Exact Solution: $y(x) = \frac{3}{16}(1 - \cos 2x) + \frac{1}{8}x^2$
Source: Majid et al. [16]

Problem 6. Consider the nonlinear nonhomogeneous problem

$$y''' = 4(1+x)^{-3} - 2e^{-3y}$$
$$y(0) = 0, y'(0) = 1, y''(0) = -1, x \in [0,4].$$

Exact Solution: $y(x) = x^2 e^{-2x} - x^2 + 3$

Source: Mehrkanoon [20]

Problem 7. Consider the linear nonhomogeneous problem

$$y''' = y^{2} + (\cos x)^{2} - \cos x - 1$$

y(0) = 0, y'(0) = 1, y''(0) = 0, x \in [0,10].

Exact Solution: $y(x) = \sin x$

Source: Mechee et al. [19]

Problem 8. Consider the linear nonhomogeneous problem

$$y''' = y'' - y' + y + e^{x}$$
$$y(0) = 1, y'(0) = 1, y''(0) = 0, x \in [0,2].$$
Exact Solution: $y(x) = \frac{1}{2}xe^{x} + \cos x + \frac{1}{2}\sin x$ Source: Lee et al. [21]

Problem 9. Consider the nonlinear nonhomogeneous system

$$y_1''' = \frac{1}{2}e^{4x}y_3y_2'$$

$$y_2''' = \frac{8}{3}e^{2x}y_1y_3'$$

$$y_3''' = 27y_2y_1'$$

$$y_1(0) = 1, y_1'(0) = -1, y_1''(0) = 1,$$

$$y_2(0) = 1, y_2'(0) = -2, y_2''(0) = 4,$$

$$y_3(0) = 1, y_3'(0) = -3, y_3''(0) = 9, x \in [0,1].$$

Exact Solution: $y_1(x) = e^{-x}, y_2(x) = e^{-2x}, y_3(x) = e^{-3x}$

Source: Fawzi et al. [22]

4.2 Numerical Result and Discussion

We apply our five step block hybrid collocation method (FVBHCM) to solve the nine tested problems in Section 4.1. The results are compared with existing methods in literature. The numerical approximations for the newly proposed method and the existing methods are computed by using Python. The following numerical methods are used to be compared:

- BHCM: Block hybrid collocation method in Yap et al. (2014)
- FBHCM: Four-point block hybrid collocation method in Yap and Ismail (2018)
- HLMM: Hybrid linear multistep method in Mohammed and Adeniyi (2014)
- HCM: Hybrid collocation method in Awoyemi and Idowu (2005)

The Maxe is defined as maximum error between the actual value and numerical result of the methods. Tables 4.1 - 4.11 demonstrate the numerical results of the methods for Problems 1 - 9. Figures 4.1 - 4.11 display the comparison on the performance between our method and the existing methods for Problems 1 - 9.

Step size	Maxe of BHCM	Maxe of FBHCM	Maxe of FVBHCM	Maxe of HLMM	Maxe of HCM
0.00625	7.73E-14	1.13E-15	1.91E-17	2.19E-11	7.98E-07
0.0125	4.83E-12	1.50E-13	4.63E-15	7.24E-10	7.26E-06
0.025	3.34E-10	1.46E-11	1.05E-12	2.78E-08	8.15E-05
0.05	1.91E-08	1.48E-09	2.03E-10	9.87E-07	8.94E-04
0.1	1.58E-06	2.26E-07	2.36E-08	5.78E-05	1.69E-02

Table 4.1: Numerical findings for Problem 1.

Step size	Maxe of BHCM	Maxe of FBHCM	Maxe of FVBHCM	Maxe of HLMM	Maxe of HCM
0.00625	8.49E-16	4.76E-18	3.62E-20	5.87E-13	1.48E-07
0.0125	5.43E-14	6.09E-16	9.28E-18	1.88E-11	1.18E-06
0.025	3.48E-12	7.79E-14	2.38E-15	6.01E-10	9.45E-06
0.05	2.22E-10	9.97E-12	6.08E-13	1.92E-08	7.51E-05
0.1	1.42E-08	1.28E-09	1.56E-10	6.15E-07	5.89E-04

Table 4.2: Numerical findings for Problem 2.

Table 4.3: Numerical findings for Problem 3.

Step size	Maxe of BHCM	Maxe of FBHCM	Maxe of FVBHCM	Maxe of HLMM	Maxe of HCM
0.00625	4.95E-18	1.11E-19	1.94E-22	1.41E-14	3.52E-09
0.0125	3.09E-16	1.48E-17	4.68E-20	4.61E-13	2.88E-08
0.025	2.19E-14	1.52E-15	1.06E-17	1.71E-11	2.65E-07
0.05	1.28E-12	1.67E-13	2.15E-15	5.96E-10	2.27E-06
0.1	1.23E-10	2.84E-11	3.45E-13	3.14E-08	2.96E-05

Table 4.4: Numerical findings for Problem 4.

Step size	Maxe of BHCM	Maxe of FBHCM	Maxe of FVBHCM	Maxe of HLMM	Maxe of HCM
0.00625	1.60E-14	2.83E-16	6.61E-18	3.44E-12	5.02E-08
0.0125	1.03E-12	3.66E-14	1.72E-15	1.09E-10	1.89E-07
0.025	6.67E-11	4.77E-12	4.52E-13	3.43E-09	2.26E-06
0.05	4.36E-09	6.32E-10	1.22E-10	1.05E-07	8.94E-05
0.1	2.89E-07	8.55E-08	3.35E-08	3.11E-06	2.24E-03

Table 4.5: Numerical findings for Problem 5.

Step size	Maxe of BHCM	Maxe of FBHCM	Maxe of FVBHCM	Maxe of HLMM	Maxe of HCM
0.00625	9.13E-15	1.02E-16	1.56E-18	3.15E-12	1.98E-07
0.0125	5.85E-13	1.30E-14	3.99E-16	1.01E-10	1.58E-06
0.025	3.74E-11	1.67E-12	1.02E-13	3.22E-09	1.26E-05
0.05	2.39E-09	2.14E-10	2.61E-11	1.03E-07	9.88E-05
0.1	1.52E-07	2.73E-08	6.67E-09	3.33E-06	7.39E-04

Table 4.6: Numerical findings for Problem 6.

Step size	Maxe of BHCM	Maxe of FBHCM	Maxe of FVBHCM	Maxe of HLMM	Maxe of HCM
0.00625	7.93E-14	3.55E-15	2.48E-16	7.87E-12	1.63E-06
0.0125	4.98E-12	4.91E-13	6.12E-14	2.56E-10	1.37E-05
0.025	3.31E-10	6.42E-11	1.40E-11	9.16E-09	1.22E-04
0.05	1.87E-08	5.01E-09	2.52E-09	3.01E-07	1.14E-03
0.1	1.19E-06	5.18E-07	2.16E-07	1.24E-05	1.29E-02

Step size	Maxe of BHCM	Maxe of FBHCM	Maxe of FVBHCM	Maxe of HLMM	Maxe of HCM
0.00625	7.66E-14	3.84E-16	5.77E-19	4.95E-11	1.24E-05
0.0125	4.87E-12	5.12E-14	8.10E-16	1.60E-09	9.87E-05
0.025	3.17E-10	6.66E-12	2.01E-13	5.39E-08	8.12E-04
0.05	1.97E-08	7.47E-10	4.84E-11	1.79E-06	6.45E-03
0.1	1.34E-06	8.71E-08	1.09E-08	6.98E-05	5.69E-02

Table 4.7: Numerical findings for Problem 7.

Table 4.8: Numerical findings for Problem 8.

Step size	Maxe of BHCM	Maxe of FBHCM	Maxe of FVBHCM	Maxe of HLMM	Maxe of HCM
0.00625	6.70E-16	3.60E-18	3.34E-20	4.92E-13	7.09E-08
0.0125	4.39E-14	4.98E-16	8.28E-18	1.66E-11	6.22E-07
0.025	2.74E-12	6.71E-14	1.99E-15	5.52E-10	5.44E-06
0.05	1.93E-10	6.27E-12	4.46E-13	2.19E-08	5.85E-05
0.1	1.11E-08	6.42E-10	8.70E-11	8.04E-07	5.56E-04

Table 4.9: Numerical findings of y_1 for Problem 9.

Step size	Maxe of BHCM	Maxe of FBHCM	Maxe of FVBHCM	Maxe of HLMM	Maxe of HCM
0.00625	2.01E-15	2.87E-17	3.03E-18	4.82E-13	1.13E-08
0.0125	1.24E-13	5.23E-15	1.71E-16	1.60E-11	9.55E-08
0.025	9.58E-12	8.24E-13	3.67E-14	7.01E-10	1.09E-06
0.05	5.30E-10	3.86E-11	6.25E-12	2.51E-08	1.01E-05
0.1	6.53E-08	9.41E-09	4.13E-10	2.22E-06	2.58E-04

Table 4.10: Numerical findings of y_2 for Problem 9.

Step size	Maxe of BHCM	Maxe of FBHCM	Maxe of FVBHCM	Maxe of HLMM	Maxe of HCM
0.00625	2.15E-14	3.71E-16	4.01E-17	5.56E-12	1.86E-07
0.0125	1.34E-12	5.05E-14	1.67E-15	1.84E-10	1.55E-06
0.025	9.73E-11	7.40E-12	3.84E-13	7.37E-09	1.59E-05
0.05	5.52E-09	4.41E-10	7.00E-11	2.60E-07	1.51E-04
0.1	5.44E-07	8.48E-08	6.14E-09	1.73E-05	2.89E-03

Table 4.11: Numerical findings of y_3 for Problem 9.

Step size	Maxe of BHCM	Maxe of FBHCM	Maxe of FVBHCM	Maxe of HLMM	Maxe of HCM
0.00625	7.11E-14	1.15E-15	3.32E-17	1.71E-11	4.82E-07
0.0125	4.47E-12	1.62E-13	6.47E-15	5.57E-10	4.00E-06
0.025	3.03E-10	2.19E-11	1.53E-12	2.01E-08	3.73E-05
0.05	1.80E-08	1.87E-09	3.22E-10	6.83E-07	3.39E-04
0.1	1.37E-06	2.79E-07	4.54E-08	3.13E-05	4.47E-03



Figure 4.1: Comparison of maximum error for different step sizes in Problem 1.



Figure 4.2: Comparison of maximum error for different step sizes in Problem 2.



Figure 4.3: Comparison of maximum error for different step sizes in Problem 3.



Figure 4.4: Comparison of maximum error for different step sizes in Problem 4.



Figure 4.5: Comparison of maximum error for different step sizes in Problem 5.



Figure 4.6: Comparison of maximum error for different step sizes in Problem 6.



Figure 4.7: Comparison of maximum error for different step sizes in Problem 7.



Figure 4.8: Comparison of maximum error for different step sizes in Problem 8.



Figure 4.9: Maximum error for y_1 versus step size for Problem 9.



Figure 4.10: Maximum error for y_2 versus step size for Problem 9.



Figure 4.11: Maximum error for y_3 versus step size for Problem 9.

Our five step block hybrid collocation method (FVBHCM) is an eighth order method with five main points and two off-step points ($x_{n+1/3}$, $x_{n+2/3}$). The four point block hybrid collocation method (FBHCM) is the seventh order method that consists of four main points and two off-step points ($x_{n+1/2}$, $x_{n+3/2}$). Block hybrid collocation method (BHCM) is the sixth order method that has three main points and two off-step points similar to FBHCM. Hybrid linear multistep method (HLMM) is the fifth order method with three main points and an off-step point ($x_{n+8/3}$). Hybrid collocation method (HCM) is a predictor-corrector method which requires starting values with three main points and an off-step point ($x_{n+3/2}$). All the methods are the self-starting methods except HCM.

Problems 1,3,5,7 and 8 are the linear nonhomogeneous problems and the results are presented in Tables 4.1, 4.3, 4.5, 4.7 and 4.8 respectively. Based on the numerical findings in Table 4.1, FVBHCM outperforms FBHCM, BHCM, HLMM and HCM since it has the smallest maximum error for every step size. For each step size, the maximum error of FVBHCM is smaller than other four methods as shown in Tables 4.3, 4.5, 4.7 and 4.8. The numerical results in Tables 4.1, 4.3, 4.5, 4.7 and 4.8 demonstrated the similar pattern that the FVBHCM has the smallest maximum error and the HCM has the biggest maximum error for each step size. It is clearly shown in the Figures 4.1, 4.3, 4.5, 4.7 and 4.8 that the order of the maximum error increases linearly as the step size decreases.

Next, we discuss the linear homogeneous problems which are Problems 2 and 4. According to the results in Table 4.2, FVBHCM still have the smallest maximum error at each step size. The BHCM demonstrates the better performance than HLMM and HCM for each step size. From the results in Table 4.4, we notice that FVBHCM has the same order of maximum error as FBHCM at step sizes 0.1 and 0.05, but the maximum error of FVBHCM still smaller than FBHCM. The Figure 4.4 clearly shows that the difference of maximum error of FVBHCM and FBHCM becomes bigger as the step size decreases.

Lastly, we discuss the nonlinear nonhomogeneous problem as presented in Problems 6 and 9. Problem 9 is a system of third order ODEs. Based on the findings in Table 4.6, the FVBHCM and FBHCM have same order of maximum error that smaller than other methods at step sizes 0.1, 0.05 and 0.025. But the maximum error of FVBHCM still smaller than FBHCM for each step size. According to the numerical results in Tables 4.9, 4.10 and 4.11, FVBHCM has the best performance in each step size which similar to other problems. FBHCM has the second smaller maximum error and BHCM has the third smaller maximum error at each step size. The maximum error of HLMM and HCM are bigger than others and HCM has the biggest maximum error. This result clearly shown in the Figures 4.9, 4.10 and 4.11 and similar to the other problems.

As conclusion, FVBHCM has shown the better approximation for all the test problems compared to other methods. The numerical results demonstrate the pattern, we can conclude that the smaller the step size, the smaller the maximum error for every method which mean the greater accuracy. FVBHCM always have the best performance in accuracy compared to FBHCM, BHCM, HLMM and HCM.

4.3 Application to Solve the Physical Problem

In fluid dynamics, Tuck and Schwartz (1990) stated that the movement of the fluid on plane surface and viscous forces in the fluid layer without the present of gravity can be represented by third order ODEs. Our five step block hybrid collocation method is applied to approximate the solution of the problem that explains the thin film flow of liquid. The thin film flow equation can be formulated into

$$y''' = f(y),$$
 (19)

where

$$f(y) = -1 + y^{-2},$$

$$f(y) = -1 + (1 + \delta + \delta^{2})y^{-2} - (\delta + \delta^{2})y^{-3},$$

$$f(y) = y^{-2} - y^{-3},$$

$$f(y) = y^{-2}.$$
(20)

Mechee et al. (2013) and Yap et al. (2014) solved the thin film flow problem by considering the special third order IVPs as follows

$$y^{\prime\prime\prime} = y^{-k} \tag{21}$$

where the initial conditions are y(0) = 1, y'(0) = 1, y''(0) = 1 for the cases k = 2and k = 3.

Yap et al. (2014) applied the BHCM to approximate the solution of the third order IVPs (21). For the comparison of the performance between the methods, the numerical solution of the Problem (21) is obtained by our FVBHCM, the existing FBHCM, BHCM, HLMM and HCM.

Tables 4.12 - 4.15 show the numerical results for the cases k = 2 and k = 3 with step lengths of 0.1 and 0.01, respectively. According to the numerical results in Table 4.12, FVBHCM has the similar order of accuracy to FBHCM while it is slightly greater than the BHCM, HLMM and HCM. FVBHCM has similar performance of accuracy to BHCM, FBHCM and HLMM in the case k = 2 with step size of 0.01. The HCM have the worst performance compared to other methods.

There is no analytical solution for the case k = 3. Tables 4.16 and 4.17 show the number of decimal places of numerical results agreed with FVBHCM compared to BHCM, FBHCM, HLMM and HCM in Tables 4.14 and 4.15 respectively. FVBHCM obtained the numerical result that correct to nine decimal places while compared to FBHCM. The similarity of the decimal places between both methods is the highest in Table 4.14. For the case k = 3 with step size 0.01, the numerical results of our method have the same fourteen decimal places as the numerical results for FBHCM. The results in Tables 4.16 and 4.17 are presented graphically in Figures 4.12 and 4.13 respectively for better illustration. Refer to the Figures 4.12 and 4.13, the similarity of decimal places for numerical results between FVBHCM and the existing methods can be arranged in order as FBHCM, BHCM, HLMM and HCM. As a whole, FVBHCM can be applied to solve the thin film flow problem (21) with good performance in accuracy.

Table 4.12: Numerical findings for Problem (21) with step size of 0.1 for the case

X	FVBHCM	BHCM	FBHCM	HLMM	HCM
0.2	1.03E-06	1.03E-06	1.03E-06	1.02E-06	3.65E-07
0.4	1.13E-07	1.12E-07	1.13E-07	8.64E-08	3.78E-06
0.6	6.00E-09	2.07E-09	7.22E-09	5.83E-08	1.07E-05
0.8	2.20E-09	1.02E-08	3.89E-09	1.20E-07	2.18E-05
1	9.48E-07	9.35E-07	9.63E-07	7.59E-07	3.66E-05

k = 2

Table 4.13: Numerical findings for Problem (21) with step size of 0.01 for the case

k	= 2

X	FVBHCM	BHCM	FBHCM	HLMM	HCM
0.2	1.03E-06	1.03E-06	1.03E-06	1.03E-06	1.03E-06
0.4	1.13E-07	1.13E-07	1.13E-07	1.13E-07	1.11E-07
0.6	6.32E-09	6.32E-09	6.32E-09	6.32E-09	2.58E-10
0.8	7.94E-11	7.94E-11	7.94E-11	7.82E-11	1.42E-08
1	9.54E-07	9.54E-07	9.54E-07	9.54E-07	9.29E-07

Table 4.14: Numerical findings for Problem (21) with step size of 0.1 for the case

k = 3

	-	-	
x	Our Method	BHCM	FBHCM
0.2	1.221155142397457721449559	1.221155142680024420006475	1.221155142505802308866799
0.4	1.488105284199660337049335	1.488105287378409574219436	1.488105284769788210975532
0.6	1.804262549558189649204407	1.804262562591302876214688	1.804262545127584554614054
0.8	2.171522808210958078092029	2.171522833301705662990318	2.171522785009719189036218
1	2.59095828555611291591652	2.590958324898400017835074	2.590958229301496199524319

x	HLMM	HCM
0.2	1.221155167028757392000588	1.221157326424750917730007
0.4	1.488105398373026577037829	1.488118147163817701302675
0.6	1.804262830738425911417151	1.804297146002670138794593
0.8	2.171523334089006333453063	2.171591368050456820871119
S1	2.590959139170048807017535	2.59107403602028089935474

Table 4.15: Numerical findings for Problem (21) with step size of 0.01 for the case

1		2
v	_	~
π	_	J

x	Our Method	BHCM	FBHCM
0.2	1.221155142395689116120027	1.221155142395691276223287	1.221155142395688938728668
0.4	1.488105284219075195998742	1.488105284219088803562221	1.488105284219073972950807
0.6	1.804262548146545641918816	1.804262548146582279907453	1.804262548146542281324074
0.8	2.171522798126401874917233	2.171522798126473721294209	2.171522798126395265409646
1	2.59095825911316998116964	2.590958259113289575657368	2.590958259113159012074234

x	HLMM	HCM
0.2	1.221155142395851786851143	1.221155143446293440615906
0.4	1.488105284219995660432451	1.488105291104692416072316
0.6	1.804262548149025334800319	1.804262568303728556967126
0.8	2.17152279813135444975697	2.171522840449143645669975
1	2.590958259121576629077378	2.590958333263425950417269

Table 4.16:Number of decimal places which agreed with numerical result of our

X	BHCM	FBHCM	HLMM	HCM
0.2	9	9	7	5
0.4	8	9	6	4
0.6	7	8	6	4
0.8	7	6	5	4
1	6	7	5	2

method in Table 4.14.

method in Table 4.15.				
x	BHCM	FBHCM	HLMM	HCM
0.2	13	14	12	8
0.4	13	14	12	7
0.6	13	14	11	7
0.8	13	12	10	6
1	12	13	10	6

Table 4.17:Number of decimal places which agreed with numerical result of our



Figure 4.12: Graphical Illustration for the Table 4.16.



Figure 4.13: Graphical Illustration for the Table 4.17.

CHAPTER 5

CONCLUSION

5.1 Conclusion

The five step block hybrid collocation method (FVBHCM) consist of two off-step points ($x_{n+1/3}$, $x_{n+2/3}$) has been proposed for solving third order initial value problem (IVPs) of ordinary differential equations (ODEs) directly in our project. Our newly proposed approach is an eighth order self-starting approach that is consistent and zero stable. The approach is applicable to solve the nonlinear and linear third order ODEs and also the system of the nonlinear third order ODEs. Furthermore, it also can be applied to solve the thin film flow problem. The results show that the smaller the step size, the greater the accuracy of the method. The numerical results for Problems 1 - 9 and the problem of thin film flow (21) draw the same conclusion that our method have the greatest performance in term of accuracy compared to four-point block hybrid collocation method (BHCM) (Yap et al., 2014), hybrid linear multistep method (HLMM) (Mohammed and Adeniyi, 2014) and hybrid collocation method (HCM) (Awoyemi and Idowu, 2005).

5.2 Future Work

Five step block hybrid collocation method (FVBHCM) is proposed to solve the third order ODEs in this project. The idea to include these five main points and two off-step points in derivation can be extended to obtain the numerical methods dealing withs fourth and fifth order IVPs. Besides that, we can develop the method that consist of more main points and considering other off-grid points. Furthermore, we can investigate the effect of the number of main points and off-grid points on the accuracy for the approach.

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