ON SYMMETRICAL AND PANDIAGONAL MAGIC SQUARES

By

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A project report submitted in partial fulfilment of the
requirements for the award of Master of Mathematics

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DECLARATION OF ORIGINALITY

I hereby declare that this project report entitled "**ON SYMMETRICAL AND PANDIAGONAL MAGIC SQUARES**" is my own work except for citations and quotations which have been duly acknowledged. I also declare that it has not been previously and concurrently submitted for any other degree or award at UTAR or other institutions.

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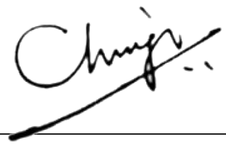

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NG POH TECK

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ABSTRACT

My project entitled "On Symmetrical and Pandiagonal Magic Squares". Magic square is one of the branches of mathematics under the field of combinatorial and recreational. Magic square already existed for a very long time ago. It is starting from a scroll called *Lo-Shu* according to one of the versions of the magic square legend. A magic square of order n is an $n \times n$ array of natural numbers from $1, 2, ..., n^2$ that are arranged in the form of square, such that the sums of each row, column and diagonal are the same constant number. There are still many properties about the magic squares that are yet to be discovered. Besides, there are many interesting types of magic squares that attract the attention of mathematicians. In this project, we have studied on the symmetrical and pandiagonal magic squares. A symmetrical magic square is a magic square of order n where every pair of numbers that are symmetrically opposite with respect to the center sum to $n^2 + 1$. A pandiagonal magic square is a magic square with the additional property that each broken diagonal sums to the magic constant. We also studied the methods of construction for magic square with different types and order. After that, we explore new methods of construction for these types of magic squares. Lastly, We will try to explore the possibility applying magic squares on cryptography.

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CHAPTER 1: INTRODUCTION

1-1 Introduction

Magic square is a fascinating topic in Mathematics under combinatorial and recreational mathematics. It has been attracting attention of many mathematicians all around the world due to its very unique properties. Even though many researchers found out many new properties, but it still has more properties haven't been discovered yet.

A *magic square of order* n is an $n \times n$ array of natural numbers from $1, 2, ..., n^2$ that are arranged in the form of square. Sum of entries in every diagonal, row, and column are the same constant number, which is called as the *magic constant* or *magic sum*.

4	9	2
3	5	7
8	1	6

Figure 1.1: Magic square of order 3

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

Figure 1.2: Magic square of order 5

There are many types of the magic squares. In this project, we will be studying on symmetrical and pandiagonal magic squares.

Symmetrical Magic Squares

A *symmetrical* magic square is also known as an *associative* magic squares. It is a magic square of order n where every pair of numbers that are symmetrically opposite with respect to the center sum to $n^2 + 1$.

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

Figure 1.3: Symmetrical magic square of order 5

The example above showed that the number in the cell with the same colour sum to $5^2 + 1 = 26$. All those numbers are opposite of each other from the center cell, which is 13.

Pandiagonal Magic Squares

A *pandiagonal* magic square or *panmagic* square in short, is a magic square with the additional property such that each broken diagonal sums to the magic constant.

7	12	1	14
2	13	8	11
16	3	10	5
9	6	15	4

Figure 1.4: Pandiagonal magic square of order 4

20	22	4	6	13
9	11	18	25	2
23	5	7	14	16
12	19	21	3	10
1	8	15	17	24

Figure 1.5: Pandiagonal magic square of order 5

The two examples above showed that all the numbers in cell with the same colour, which is along the broken diagonal in Figure 1.4 and Figure 1.5 sum to 34 and 65 respectively.

1-2 Background on Magic Squares

There are a few versions of the magic square legend. One of the versions says that, the earliest magic square was discovered in China, during the Xia dynasty, which is called *Lo-Shu* (Figure 1.6) or *scroll of river Lo*. During the reign of Emperor Yü around 2200 B.C., it is said that there was a huge flood that destroyed the crops and affected the civilians. The civilians offered sacrifices to the god of river Lo to calm the wrath of the river god but the situation remain unchanged in the end. Every time they make offering, there was a turtle emerged. Emperor Yü tried to find a way to overcome this problem, until he noticed the turtle one day. The turtle shell had a unique pattern, like a square with a 3-by-3 grid with dots on it. Emperor Yü managed to figure out a way from the pattern of the turtle shell. They need to sacrifice 15 people, which is the magic sum, to calm the river god down, and the flood did not happen again afterwards. The magic square of order 3 (Figure 1.7) was constructed by counting the amount of dots on every small subsequent pattern from Lo-Shu.

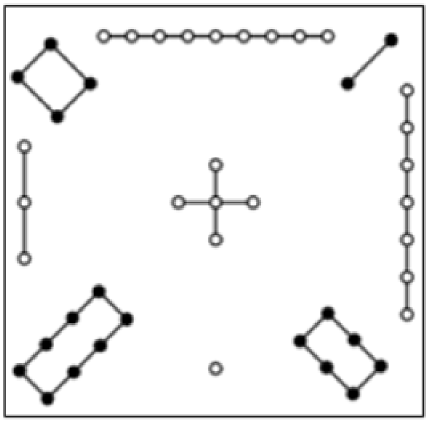

Figure 1.6: Lo-Shu

4	9	2
3	5	7
8	1	6

Figure 1.7: Magic square of order 3

There are odd numbers and even numbers in the entries of magic square. According to Tchi (2018), it is similar to the *Yin* and *Yang* concept from *Feng-Shui*. For example, the *Yang* quality is represented by the odd number entries while the *Yin* energy is represented by the even number entries in the magic square.

1-3 Objectives

The aim of this project is to investigate the methods of constructions for symmetrical and pandiagonal magic square as well as their properties. The method of constructions for symmetrical magic squares and pandiagonal magic squares are different. After the construction is completed, the next objective is to investigate the discovered properties of both of symmetrical and pandiagonal magic square. After that, we will try to explore new methods of construction for these types of magic squares.

1-4 Problem Statement

(i) To explore new method of construction for symmetrical magic square or pandiagonal magic square by using or modifying the existing methods.

(ii) To investigate whether there are new properties of these classes of magic squares.

(iii) To explore the possibility of applying magic squares on cryptography.

1-5 Work Schedule

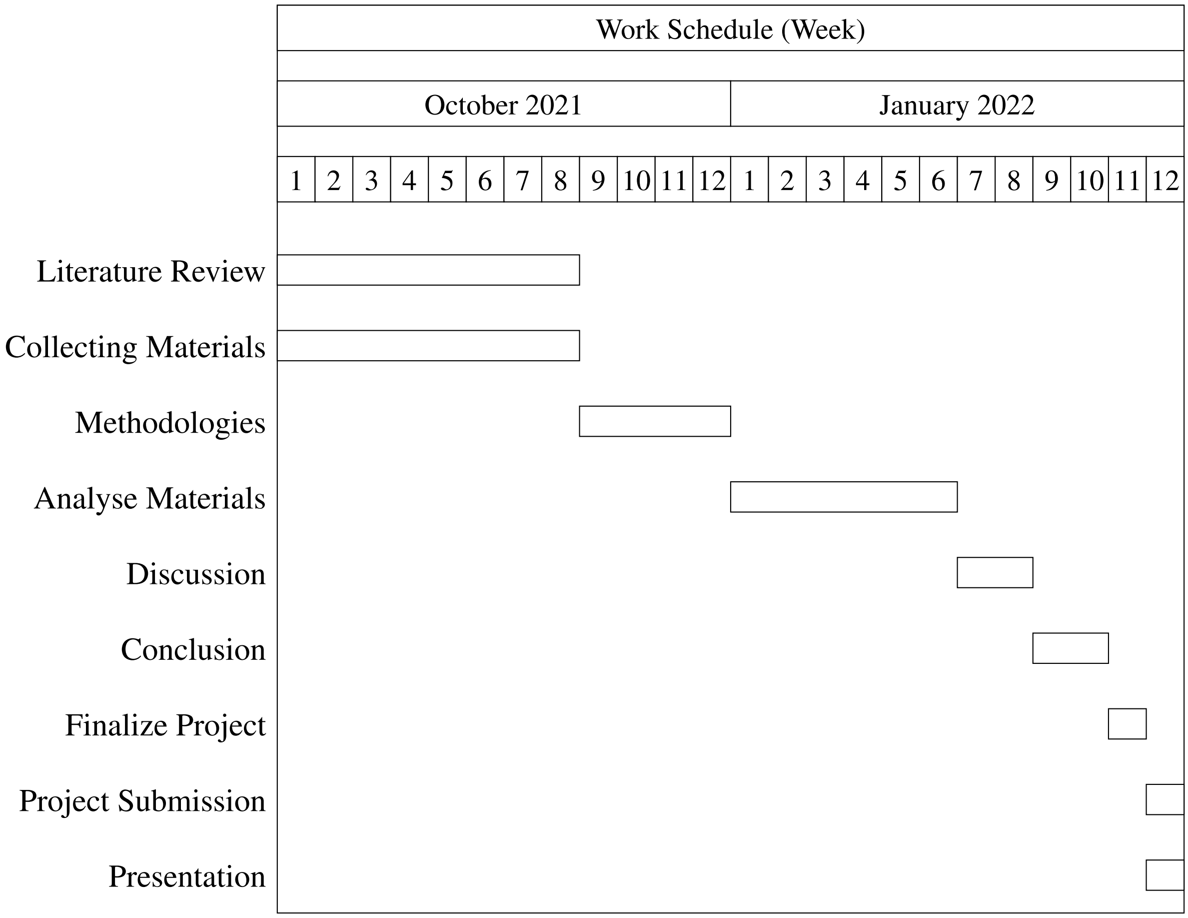

Figure 1.8: Work schedule for this project

Chapter 2: Literature Review

Nowadays, there are many mathematicians still conducting research on magic squares to discover more interesting properties or new branches. The results throughout the years have never failed to attract the experts to continue finding out the mystery behind magic squares.

One of the earliest magic square was created from Lo-Shu as mentioned earlier. From the magic square of order 3 (Figure 1.7) and Lo-Shu (Figure 1.6), the sum of each diagonal, row and column is 15. According to Sorici (2010), the number 15 corresponds to the number of days in every 24 cycles of the Chinese solar year.

According to Leite, Jacquemin and Boillot (2016), there was a German painter named Albrecht Dúrer who introduced an interesting magic square of order 4 with some additional properties in one of his paintings called *Melencolia I*.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Figure 2.1: Magic square in *Melencolia I*

The magic sum in this magic square is 34. The interesting part of this magic square is the magic sum not only appearing on each row, column and diagonal, but also appearing in many other sums.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Figure 2.2: Four quadrants of Magic square in *Melencolia I*

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Figure 2.3: Center of Magic square in *Melencolia I*

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Figure 2.4: Four corners of magic square in *Melencolia I*

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Figure 2.5: Four sides of magic square in *Melencolia I*

From the four figure above, we can notice that the four quadrants (Figure 2.2), center (Figure 2.3), four corners (Figure 2.4) and four sides (Figure 2.5) are sum to 34. Besides, the properties shown in Figure 2.3 and Figure 2.4 are true for any magic square of order 4. This can be proved.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Figure 2.6: Symmetrical property in the magic square in *Melencolia I*

Furthermore, the magic square also shows symmetrical property. The eight pairs of number above are symmetrically opposite of each other from the center and sum to 17. In an article written by Benjamin and Yasuda (1999), they proved a theorem such that every magic squares of order 3 and every symmetrical magic squares are square-palindromic.

Magic squares can be of odd or even order. An odd order magic square, is a magic square of order $1, 3, 5, ...$ While for even order, there are two categories, singly-even order magic square and doubly-even order magic square. Singly-even order magic square are those order with $n \equiv 2 \pmod 4$, such as $6, 10, 14, ...$ For doubly-even order magic square are those order with $n \equiv 0 \pmod 4$, or the multiple of 4, such as $4, 8, 12, ...$ Besides, Chia (2018) presented a new method on constructing doubly-even order magic square.

According to Weisstein (n.d.), the Lo-Shu is a symmetrical magic square but not a pandiagonal magic square. For magic squares of order 4, it can be symmetrical, pandiagonal or neither but not both. The smallest symmetrical and pandiagonal magic square is order 5. By referring to Chee (1981), the result that there exists no pandiagonal magic square of singly-even order is due to Planck.

Furthermore, Chen, Li and Zhang (2016) proved that, symmetrical pandiagonal magic squares exist. They obtained a method to construct magic squares which are symmetrical and pandiagonal. They also showed that a magic square of order n which is pandiagonal and symmetrical exists if and only if $n \geq 5$ and $n \not\equiv 2 \ (mod\ 4)$.

Moreover, magic square can be used in cryptography. According to Meenu and Ojha (2012), they applied magic squares' concept into their technique of encryption or decryption. Adachi and Sugita (2017) describe the algorithm in cryptosystem based on magic square. The algorithm is for magic square of order $4, 8$ and 16. In additional, Lok and Chin (2018) used magic square as a cipher in cryptography to encrypt and decrypt information.

CHAPTER 3: METHODOLOGY

As mentioned earlier, sums of each row, column and diagonal of the magic square are the same constant number, which is magic sum. The magic sum can be calculated by using the following formula:

$$S_n = \frac{1}{2}n(n^2 + 1)$$

Example

Magic square of order $n = 3$:

$$\begin{aligned} S_3 &= \frac{1}{2}(3)(3^2 + 1) \\ &= 15 \end{aligned}$$

Magic square of order $n = 4$:

$$\begin{aligned} S_4 &= \frac{1}{2}(4)(4^2 + 1) \\ &= 34 \end{aligned}$$

Therefore, magic sum for magic square of order 3 and 4 are 15 and 34 respectively.

The size of magic square can be unlimited. Therefore, required different method to construct. Since there are singly-even order magic squares, doubly-even order magic squares and odd order magic squares. So, the order of magic squares need to be identified first, then only decide which method of construction to be used.

3-1 Odd Order Magic Squares

By referring to Chee (1981), the method of construction for odd ordered magic squares is called the De la Loubére method. The order of this magic square can be represented by

$$n = 2b + 1$$

where b is a positive integer.

We explain the method by using a magic square of order 5 below as an illustration.

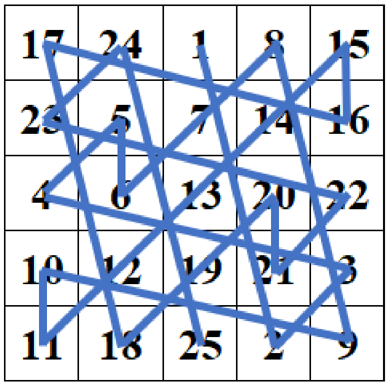

Figure 3.1: Magic square of order 5

Steps:

1. Place "1" in the center cell at the top row. Then, continue to fill the numbers $45°$ diagonally to the upper right-hand side.

2. When reached the top row, the next number will be filled to the last row with the column next to the previous entry.

3. When reached the last column, the next number will be filled to the first column with the row above the previous entry.

4. When the upper right cell is filled, the next number will be filled right below of the previous entry.

Sum of each row, column and diagonal of this magic square is 15 which tallies with the formula S_n given earlier.

3-2 Singly-Even Order Magic Squares

Singly-even magic squares are constructed by using the Ralph Strachy method according to Chee (1981). The order of this magic square can be defined by

$$n = 2(2b + 1) = 4b + 2$$

where b is a positive integer.

We explain the method by taking $n = 10$, where $b = 2$.

Steps:

1. Divide the square into four subsquares W, X, Y, and Z of order $2b+1$ by referring to the figure below.

W	Y
Z	X

Figure 3.2: Four subsquares

2. Construct the magic square by using the De la Loubére method on each subsquare, the entries for each subsquare are:

   - W contains numbers from 1 until $\frac{n^2}{4}$.
   - X contains numbers from $\frac{n^2}{4} + 1$ until $\frac{n^2}{2}$.
   - Y contains numbers from $\frac{n^2}{2} + 1$ until $\frac{3n^2}{4}$.
   - Z contains numbers from $\frac{3n^2}{4} + 1$ until n^2.

This brings the result as below:

17	24	1	8	15	67	74	51	58	65
23	5	7	14	16	73	55	57	64	66
4	6	13	20	22	54	56	63	70	72
10	12	19	21	3	60	62	69	71	53
11	18	25	2	9	61	68	75	52	59
92	99	76	83	90	42	49	26	33	40
98	80	82	89	91	48	30	32	39	41
79	81	88	95	97	29	31	38	45	47
85	87	94	96	78	35	37	44	46	28
86	93	100	77	84	36	43	50	27	34

Figure 3.3: De la Loubére method on four subsquares

3. Take the $b-1$ columns from the rightmost in the subsquare X to exchange vertically with the same column of subsquare Y, which is shown as green colour in Figure 3.3.

4. In the middle row of the subsquare W, take the b cells starting from the second cell to exchange with the corresponding cells in the subsquare Z, which is shown in blue colour in Figure 3.3.

5. In the subsquare W, take the leftmost $b \times b$ subsquare above (respectively below) W is to be exchanged with the corresponding leftmost subsquare of Z which is shown in yellow colour in Figure 3.3.

Then, a magic square of order 10 is resulted as below:

92	99	1	8	15	67	74	51	58	40
98	80	7	14	16	73	55	57	64	41
4	81	88	20	22	54	56	63	70	47
85	87	19	21	3	60	62	69	71	28
86	93	25	2	9	61	68	75	52	34
17	24	76	83	90	42	49	26	33	65
23	5	82	89	91	48	30	32	39	66
79	6	13	95	97	29	31	38	45	72
10	12	94	96	78	35	37	44	46	53
11	18	100	77	84	36	43	50	27	59

Figure 3.4: Magic square of order 10

The magic constant for the magic square above is calculated as below:

$$n = 10 \quad \text{implies} \quad S_{10} = \frac{1}{2}(10)(10^2 + 1)$$
$$= 505$$

However, the method mentioned above is only applicable when $b \geq 2$. When $b = 1$, which results to magic square of order 6, Step (3) will be ignored while the other steps remain unchanged.

8	1	6	26	19	24
3	5	7	21	23	25
4	9	2	22	27	20
35	28	33	17	10	15
30	32	34	12	14	16
31	36	29	13	18	11

→

35	1	6	26	19	24
3	32	7	21	23	25
31	9	2	22	27	20
8	28	33	17	10	15
30	5	34	12	14	16
4	26	29	13	18	11

Figure 3.5: Magic square of order 6

The magic constant for the magic square of order 6 above is calculated as below:

$$n = 6 \quad \text{implies} \quad S_6 = \frac{1}{2}(6)(6^2 + 1)$$
$$= 111$$

3-3 Doubly-Even Order Magic Squares

The Generalized Doubly-Even Method as described in Kurdle and Menard (2007) can be used to construct doubly-even order magic squares. The order of these magic squares can be represented by

$$n = 2(2b) = 4b$$

where b is a positive integer.

We explain the method by taking $n = 8$, where $b = 2$.

Step:

1. Arrange the numbers from 1 until n^2 in a natural order as shown in Figure 3.6.
2. Divide the square into b^2 subsquares of order 4.
3. Draw a line on the main diagonal and off diagonal on each subsquare.

4. Interchange those numbers in the cell that cut through by the line in reverse ordering about the center of the square, which is the blue colour dot in Figure 3.6.

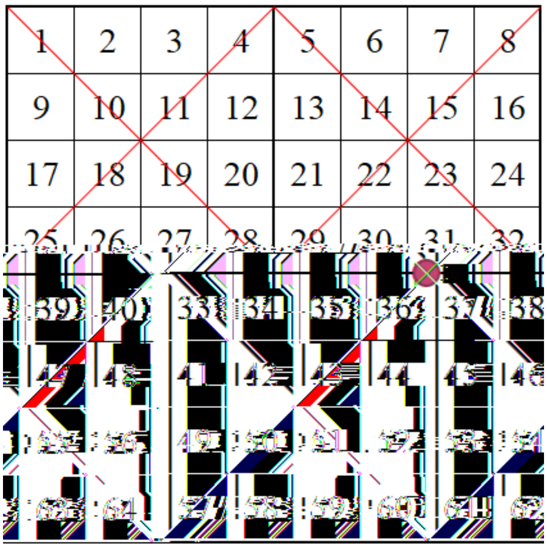

Figure 3.6: Four subsquares of order 4

This leads to the magic square of order 8 as below.

64	2	3	61	60	6	7	57
9	55	54	12	13	51	50	16
17	47	46	20	21	43	42	24
40	26	27	37	36	30	31	33
32	34	35	29	28	38	39	25
41	23	22	44	45	19	18	48
49	15	14	52	53	11	10	56
8	58	59	5	4	62	63	1

Figure 3.7: Magic square of order 8

The magic constant for the magic square of order 8 above is calculated as below:

$$n = 8 \quad \text{implies} \quad S_8 = \frac{1}{2}(8)(8^2+1)$$
$$= 260$$

Chapter 4: Results and Discussion

After the research had been done, some results had been generated, and some interesting properties are found.

4-1 Self-Complementary Magic Squares

Let S be a magic square of order n. When each entry x of S is replaced by n^2+1-x, the resulting square is called the *complement* of S denoted by $\bar{S}$, which is also a magic square.

After that, two transformations can be applied on $\bar{S}$, which are letting $\bar{S}$ goes through a $180°$ clockwise rotation with respect to the center of the square or goes through a vertical or horizontal reflection with respect to the central axis of the square.

When a magic square goes through a $180°$ clockwise rotation about the center of the square, it becomes its complement, we call it *ro-symmetrical magic square*. While when a magic square goes through a vertical or horizontal reflection with respect to the central axis, it becomes its complement, we call it *ref-symmetrical magic square*.

A magic square S is said to be *self-complementary* if S is equivalent to its complement. The following is the mapping of *self-complementary magic squares*.

$$S \xrightarrow{n^2+1-x} \bar{S} \xrightarrow{Transformation} S$$

4-1-1 ro-symmetrical Magic Squares

Here is the example of ro-symmetrical magic squares. Let A be a magic square of order 5 and its complement which are showed in Figure 4.1 and Figure 4.2 respectively as below:

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

Figure 4.1: Magic square A

9	2	25	18	11
3	21	19	12	10
22	20	13	6	4
16	14	7	5	23
15	8	1	24	17

Figure 4.2: Complement of A

We can see that, when the complement of A undergoes a $180°$ will return back to its original form in Figure 4.1. Furthermore, both A and its complement showed the same and unique pattern as in the Figure 4.3 and Figure 4.4 below:

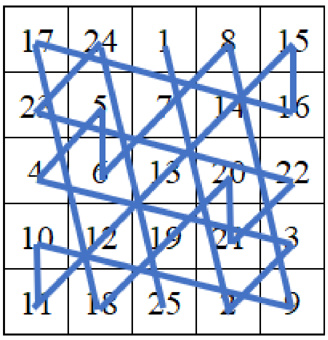

Figure 4.3: Pattern on magic square A

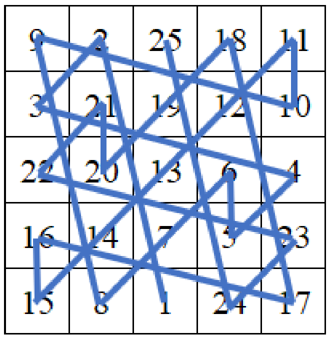

Figure 4.4: Pattern on complement of A

Therefore, A is a ro-symmetrical magic square and so it is a self-complementary magic square too.

4-1-2 ref-symmetrical Magic Squares

Next, here is the example of ref-symmetrical magic squares. Let B be a magic square of order 8 and its complement which are showed in Figure 4.5 and Figure 4.6 respectively as below:

64	55	46	37	28	19	10	1
2	9	20	27	38	45	56	63
3	12	17	26	39	48	53	62
61	54	47	40	25	18	11	4
60	51	42	33	32	23	14	5
6	13	24	31	34	41	52	59
7	16	21	30	35	44	49	58
57	50	43	36	29	22	15	8

Figure 4.5: Magic square B

1	10	19	28	37	46	55	64
63	56	45	38	27	20	9	2
62	53	48	39	26	17	12	3
4	11	18	25	40	47	54	61
5	14	23	32	33	42	51	60
59	52	41	34	31	24	13	6
58	49	44	35	30	21	16	7
8	15	22	29	36	43	50	57

Figure 4.6: Complement of B

We can see that, when complement of B undergoes a vertical reflection with respect to the middle axis, it returns to its original form in Figure 4.5. Moreover, there are some interesting properties in this magic square. It shows the properties of semi-pandiagonal magic squares.

64	55	46	37	28	19	10	1
2	9	20	27	38	45	56	63
3	12	17	26	39	48	53	62
61	54	47	40	25	18	11	4
60	51	42	33	32	23	14	5
6	13	24	31	34	41	52	59
7	16	21	30	35	44	49	58
57	50	43	36	29	22	15	8

Figure 4.7: Main broken diagonals start with odd column

64	55	46	37	28	19	10	1
2	9	20	27	38	45	56	63
3	12	17	26	39	48	53	62
61	54	47	40	25	18	11	4
60	51	42	33	32	23	14	5
6	13	24	31	34	41	52	59
7	16	21	30	35	44	49	58
57	50	43	36	29	22	15	8

Figure 4.8: Off broken diagonals start with even column

In the first row, the main broken diagonals start with odd column, and the off broken diagonals start with even column sum to magic constant, 260 which are highlighted with the same colour in both Figure 4.7 and Figure 4.8 shown above. Therefore, B is a ref-symmetrical magic square and so it is a self-complementary magic square as well.

According to Chia and Lee (2014), ro-symmetrical magic squares are also called as symmetrical or associative magic squares. ro-symmetrical magic squares have a long history while ref-symmetrical magic squares were introduced only recently by Chia and Lee (2014). For ro-symmetrical magic squares, it can be constructed by using the well-known De la Loubére method, but for ref-symmetrical magic squares do not have a way to construct it, until Chia and Lee (2014) presented a way to construct ref-symmetrical magic square of order n where $n \geq 4$ is even. Besides, Chia (2018) also presented another new way to construct it.

After learning the construction method for ref-symmetrical, more new results can be generated, like a ref-symmetrical magic square of order 16 below:

256	239	222	205	192	175	158	141	116	99	82	65	52	35	18	1
2	17	36	51	66	81	100	115	142	157	176	191	206	221	240	255
3	20	33	50	67	84	97	114	143	160	173	190	207	224	237	254
253	238	223	208	189	174	159	144	113	98	83	68	49	34	19	4
252	235	218	201	188	171	154	137	120	103	86	69	56	39	22	5
6	21	40	55	70	85	104	119	138	153	172	187	202	217	236	251
7	24	37	54	71	88	101	118	139	156	169	186	203	220	233	250
249	234	219	204	185	170	155	140	117	102	87	72	53	38	23	8
248	231	214	197	184	167	150	133	124	107	90	73	60	43	26	9
10	25	44	59	74	89	108	123	134	149	168	183	198	213	232	247
11	28	41	58	75	92	105	122	135	152	165	182	199	216	229	246
245	230	215	200	181	166	151	136	121	106	91	76	57	42	27	12
244	227	210	193	180	163	146	129	128	111	94	77	64	47	30	13
14	29	48	63	78	93	112	127	130	145	164	179	194	209	228	243
15	32	45	62	79	96	109	126	131	148	161	178	195	212	225	242
241	226	211	196	177	162	147	132	125	110	95	80	61	46	31	16

Figure 4.9: ref-symmetrical magic square of order 16

4-2 Compositions of Magic Squares

When the order of magic square increases, more time is needed to construct it. Even though the method is easy to be applied. To overcome this situation, we can apply composition method on it. It uses two or more magic squares of lower orders to generate a higher order magic square.

4-2-1 Compositions of Magic Squares

We will use a magic square of order 3 to illustrate the composition by using the following diagram.

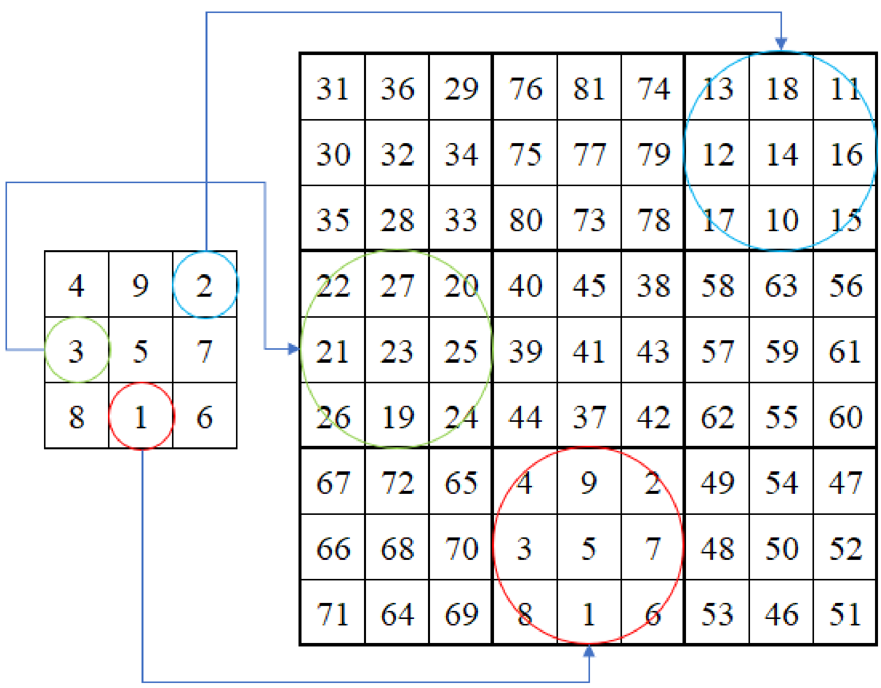

Figure 4.10: Composition method

The square is divided into nine blocks, then the nine blocks will be divided into another nine sub-blocks. After that, fill in the numbers from 1 to 9 according to the style of the magic square until all the cells are filled out.

Let P and Q be two magic squares of order 5 as shown in Figure 4.11 and Figure 4.12 respectively.

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

Figure 4.11: Magic square P

1	8	15	17	24
12	19	21	3	10
23	5	7	14	16
9	11	18	25	2
20	22	4	6	13

Figure 4.12: Magic square Q

We can generate magic squares of order 25 with different composition order as it able to generate different results. We performed the composition method to find the composite of P and Q, we denote it as $P \circ Q$ in short. Besides, we also find the composition of $Q \circ P$, $P \circ P$ and $Q \circ Q$. For $P \circ P$ and $Q \circ Q$, it means that P and Q composite with itself respectively. The results can be found on the following pages.

Result for $P \circ Q$:

401	408	415	417	424	576	583	590	592	599	1	8	15	17	24	176	183	190	192	199	351	358	365	367	374
412	419	421	403	410	587	594	596	578	585	12	19	21	3	10	187	194	196	178	185	362	369	371	353	360
423	405	407	414	416	598	580	582	589	591	23	5	7	14	16	198	180	182	189	191	373	355	357	364	366
409	411	418	425	402	584	586	593	600	577	9	11	18	25	2	184	186	193	200	177	359	361	368	375	352
420	422	404	406	413	595	597	579	581	588	20	22	4	6	13	195	197	179	181	188	370	372	354	356	363
551	558	565	567	574	101	108	115	117	124	151	158	165	167	174	326	333	340	342	349	376	383	390	392	399
562	569	571	553	560	112	119	121	103	110	162	169	171	153	160	337	344	346	328	335	387	394	396	378	385
573	555	557	564	566	123	105	107	114	116	173	155	157	164	166	348	330	332	339	341	398	380	382	389	391
559	561	568	575	552	109	111	118	125	102	159	161	168	175	152	334	336	343	350	327	384	386	393	400	377
570	572	554	556	563	120	122	104	106	113	170	172	154	156	163	345	347	329	331	338	395	397	379	381	388
76	83	90	92	99	126	133	140	142	149	301	308	315	317	324	476	483	490	492	499	526	533	540	542	549
87	94	96	78	85	137	144	146	128	135	312	319	321	303	310	487	494	496	478	485	537	544	546	528	535
98	80	82	89	91	148	130	132	139	141	323	305	307	314	316	498	480	482	489	491	548	530	532	539	541
84	86	93	100	77	134	136	143	150	127	309	311	318	325	302	484	486	493	500	477	534	536	543	550	527
95	97	79	81	88	145	147	129	131	138	320	322	304	306	313	495	497	479	481	488	545	547	529	531	538
226	233	240	242	249	276	283	290	292	299	451	458	465	467	474	501	508	515	517	524	51	58	65	67	74
237	244	246	228	235	287	294	296	278	285	462	469	471	453	460	512	519	521	503	510	62	69	71	53	60
248	230	232	239	241	298	280	282	289	291	473	455	457	464	466	523	505	507	514	516	73	55	57	64	66
234	236	243	250	227	284	286	293	300	277	459	461	468	475	452	509	511	518	525	502	59	61	68	75	52
245	247	229	231	238	295	297	279	281	288	470	472	454	456	463	520	522	504	506	513	70	72	54	56	63
251	258	265	267	274	426	433	440	442	449	601	608	615	617	624	26	33	40	42	49	201	208	215	217	224
262	269	271	253	260	437	444	446	428	435	612	619	621	603	610	37	44	46	28	35	212	219	221	203	210
273	255	257	264	266	448	430	432	439	441	623	605	607	614	616	48	30	32	39	41	223	205	207	214	216
259	261	268	275	252	434	436	443	450	427	609	611	618	625	602	34	36	43	50	27	209	211	218	225	202
270	272	254	256	263	445	447	429	431	438	620	622	604	606	613	45	47	29	31	38	220	222	204	206	213

Figure 4.13: Composite of P with Q

Result for $Q \circ P$:

17	24	1	8	15	192	199	176	183	190	367	374	351	358	365	417	424	401	408	415	592	599	576	583	590
23	5	7	14	16	198	180	182	189	191	373	355	357	364	366	423	405	407	414	416	598	580	582	589	591
4	6	13	20	22	179	181	188	195	197	354	356	363	370	372	404	406	413	420	422	579	581	588	595	597
10	12	19	21	3	185	187	194	196	178	360	362	369	371	353	410	412	419	421	403	585	587	594	596	578
11	18	25	2	9	186	193	200	177	184	361	368	375	352	359	411	418	425	402	409	586	593	600	577	584
292	299	276	283	290	467	474	451	458	465	517	524	501	508	515	67	74	51	58	65	242	249	226	233	240
298	280	282	289	291	473	455	457	464	466	523	505	507	514	516	73	55	57	64	66	248	230	232	239	241
279	281	288	295	297	454	456	463	470	472	504	506	513	520	522	54	56	63	70	72	229	231	238	245	247
285	287	294	296	278	460	462	469	471	453	510	512	519	521	503	60	62	69	71	53	235	237	244	246	228
286	293	300	277	284	461	468	475	452	459	511	518	525	502	509	61	68	75	52	59	236	243	250	227	234
567	574	551	558	565	117	124	101	108	115	167	174	151	158	165	342	349	326	333	340	392	399	376	383	390
573	555	557	564	566	123	105	107	114	116	173	155	157	164	166	348	330	332	339	341	398	380	382	389	391
554	556	563	570	572	104	106	113	120	122	154	156	163	170	172	329	331	338	345	347	379	381	388	395	397
560	562	569	571	553	110	112	119	121	103	160	162	169	171	153	335	337	344	346	328	385	387	394	396	378
561	568	575	552	559	111	118	125	102	109	161	168	175	152	159	336	343	350	327	334	386	393	400	377	384
217	224	201	208	215	267	274	251	258	265	442	449	426	433	440	617	624	601	608	615	42	49	26	33	40
223	205	207	214	216	273	255	257	264	266	448	430	432	439	441	623	605	607	614	616	48	30	32	39	41
204	206	213	220	222	254	256	263	270	272	429	431	438	445	447	604	606	613	620	622	29	31	38	45	47
210	212	219	221	203	260	262	269	271	253	435	437	444	446	428	610	612	619	621	603	35	37	44	46	28
211	218	225	202	209	261	268	275	252	259	436	443	450	427	434	611	618	625	602	609	36	43	50	27	34
492	499	476	483	490	542	549	526	533	540	92	99	76	83	90	142	149	126	133	140	317	324	301	308	315
498	480	482	489	491	548	530	532	539	541	98	80	82	89	91	148	130	132	139	141	323	305	307	314	316
479	481	488	495	497	529	531	538	545	547	79	81	88	95	97	129	131	138	145	147	304	306	313	320	322
485	487	494	496	478	535	537	544	546	528	85	87	94	96	78	135	137	144	146	128	310	312	319	321	303
486	493	500	477	484	536	543	550	527	534	86	93	100	77	84	136	143	150	127	134	311	318	325	302	309

Figure 4.14: Composite of Q with P

Result for $P \circ P$:

417	424	401	408	415	592	599	576	583	590	17	24	1	8	15	192	199	176	183	190	367	374	351	358	365
423	405	407	414	416	598	580	582	589	591	23	5	7	14	16	198	180	182	189	191	373	355	357	364	366
404	406	413	420	422	579	581	588	595	597	4	6	13	20	22	179	181	188	195	197	354	356	363	370	372
410	412	419	421	403	585	587	594	596	578	10	12	19	21	3	185	187	194	196	178	360	362	369	371	353
411	418	425	402	409	586	593	600	577	584	11	18	25	2	9	186	193	200	177	184	361	368	375	352	359
567	574	551	558	565	117	124	101	108	115	167	174	151	158	165	342	349	326	333	340	392	399	376	383	390
573	555	557	564	566	123	105	107	114	116	173	155	157	164	166	348	330	332	339	341	398	380	382	389	391
554	556	563	570	572	104	106	113	120	122	154	156	163	170	172	329	331	338	345	347	379	381	388	395	397
560	562	569	571	553	110	112	119	121	103	160	162	169	171	153	335	337	344	346	328	385	387	394	396	378
561	568	575	552	559	111	118	125	102	109	161	168	175	152	159	336	343	350	327	334	386	393	400	377	384
92	99	76	83	90	142	149	126	133	140	317	324	301	308	315	492	499	476	483	490	542	549	526	533	540
98	80	82	89	91	148	130	132	139	141	323	305	307	314	316	498	480	482	489	491	548	530	532	539	541
79	81	88	95	97	129	131	138	145	147	304	306	313	320	322	479	481	488	495	497	529	531	538	545	547
85	87	94	96	78	135	137	144	146	128	310	312	319	321	303	485	487	494	496	478	535	537	544	546	528
86	93	100	77	84	136	143	150	127	134	311	318	325	302	309	486	493	500	477	484	536	543	550	527	534
242	249	226	233	240	292	299	276	283	290	467	474	451	458	465	517	524	501	508	515	67	74	51	58	65
248	230	232	239	241	298	280	282	289	291	473	455	457	464	466	523	505	507	514	516	73	55	57	64	66
229	231	238	245	247	279	281	288	295	297	454	456	463	470	472	504	506	513	520	522	54	56	63	70	72
235	237	244	246	228	285	287	294	296	278	460	462	469	471	453	510	512	519	521	503	60	62	69	71	53
236	243	250	227	234	286	293	300	277	284	461	468	475	452	459	511	518	525	502	509	61	68	75	52	59
267	274	251	258	265	442	449	426	433	440	617	624	601	608	615	42	49	26	33	40	217	224	201	208	215
273	255	257	264	266	448	430	432	439	441	623	605	607	614	616	48	30	32	39	41	223	205	207	214	216
254	256	263	270	272	429	431	438	445	447	604	606	613	620	622	29	31	38	45	47	204	206	213	220	222
260	262	269	271	253	435	437	444	446	428	610	612	619	621	603	35	37	44	46	28	210	212	219	221	203
261	268	275	252	259	436	443	450	427	434	611	618	625	602	609	36	43	50	27	34	211	218	225	202	209

Figure 4.15: Composite of P with P

Result for $Q \circ Q$:

1	8	15	17	24	176	183	190	192	199	351	358	365	367	374	401	408	415	417	424	576	583	590	592	599
12	19	21	3	10	187	194	196	178	185	362	369	371	353	360	412	419	421	403	410	587	594	596	578	585
23	5	7	14	16	198	180	182	189	191	373	355	357	364	366	423	405	407	414	416	598	580	582	589	591
9	11	18	25	2	184	186	193	200	177	359	361	368	375	352	409	411	418	425	402	584	586	593	600	577
20	22	4	6	13	195	197	179	181	188	370	372	354	356	363	420	422	404	406	413	595	597	579	581	588
276	283	290	292	299	451	458	465	467	474	501	508	515	517	524	51	58	65	67	74	226	233	240	242	249
287	294	296	278	285	462	469	471	453	460	512	519	521	503	510	62	69	71	53	60	237	244	246	228	235
298	280	282	289	291	473	455	457	464	466	523	505	507	514	516	73	55	57	64	66	248	230	232	239	241
284	286	293	300	277	459	461	468	475	452	509	511	518	525	502	59	61	68	75	52	234	236	243	250	227
295	297	279	281	288	470	472	454	456	463	520	522	504	506	513	70	72	54	56	63	245	247	229	231	238
551	558	565	567	574	101	108	115	117	124	151	158	165	167	174	326	333	340	342	349	376	383	390	392	399
562	569	571	553	560	112	119	121	103	110	162	169	171	153	160	337	344	346	328	335	387	394	396	378	385
573	555	557	564	566	123	105	107	114	116	173	155	157	164	166	348	330	332	339	341	398	380	382	389	391
559	561	568	575	552	109	111	118	125	102	159	161	168	175	152	334	336	343	350	327	384	386	393	400	377
570	572	554	556	563	120	122	104	106	113	170	172	154	156	163	345	347	329	331	338	395	397	379	381	388
201	208	215	217	224	251	258	265	267	274	426	433	440	442	449	601	608	615	617	624	26	33	40	42	49
212	219	221	203	210	262	269	271	253	260	437	444	446	428	435	612	619	621	603	610	37	44	46	28	35
223	205	207	214	216	273	255	257	264	266	448	430	432	439	441	623	605	607	614	616	48	30	32	39	41
209	211	218	225	202	259	261	268	275	252	434	436	443	450	427	609	611	618	625	602	34	36	43	50	27
220	222	204	206	213	270	272	254	256	263	445	447	429	431	438	620	622	604	606	613	45	47	29	31	38
476	483	490	492	499	526	533	540	542	549	76	83	90	92	99	126	133	140	142	149	301	308	315	317	324
487	494	496	478	485	537	544	546	528	535	87	94	96	78	85	137	144	146	128	135	312	319	321	303	310
498	480	482	489	491	548	530	532	539	541	98	80	82	89	91	148	130	132	139	141	323	305	307	314	316
484	486	493	500	477	534	536	543	550	527	84	86	93	100	77	134	136	143	150	127	309	311	318	325	302
495	497	479	481	488	545	547	529	531	538	95	97	79	81	88	145	147	129	131	138	320	322	304	306	313

Figure 4.16: Composite of Q with Q

Note that, P is a symmetrical magic square while Q is pandiagonal magic square. According to Chia (1983), when the composition method applied on two pandiagonal magic squares, the results will be also a pandiagonal magic square. From Figure 4.16, which is $Q \circ Q$, we already verified that is it a pandiagonal magic square. Same thing happens to symmetrical magic squares as well. From Figure 4.15, which is $P \circ P$, we already verified that it is a symmetrical magic square as well. While for $P \circ Q$ and $Q \circ P$ in Figure 4.13 and Figure 4.14 respectively, they are neither symmetrical magic square nor pandiagonal magic square, but they are just a normal magic square of order 25.

4-2-2 *Yang-Hui* Composition

The *Yang-Hui* composition is similar to the previous method. We will use a magic square of order 3 to illustrate the *Yang-Hui* composition method by using the following diagram.

4	9	2
3	5	7
8	1	6

31	76	13	36	81	18	29	74	11
22	40	58	27	45	63	20	38	56
67	4	49	72	9	54	65	2	47
30	75	12	32	77	14	34	79	16
21	39	57	23	41	59	25	43	61
66	3	48	68	5	50	70	7	52
35	80	17	28	73	10	33	78	15
26	44	62	19	37	55	24	42	60
71	8	53	64	1	46	69	6	51

Figure 4.17: *Yang-Hui* composition method

The magic square of order 9 generated above is also known as the *Giant Lo-Shu*. Similar to the previous method, the square is divided into nine blocks, then the nine will be divided into another nine sub-blocks. After that, fill in the numbers from 1 to 9 according to the style of the magic square, which is different from the previous one, until all the cells are filled out.

We will use back the magic square P and Q from the previous section to generate different results by using *Yang-Hui* composition method. We will perform the same composite, which are $P * Q$, $Q * P$, $P * P$ and $Q * Q$. The results can be found on the following pages.

Result for $P * Q$:

17	192	367	417	592	24	199	374	424	599	1	176	351	401	576	8	183	358	408	583	15	190	365	415	590
292	467	517	67	242	299	474	524	74	249	276	451	501	51	226	283	458	508	58	233	290	465	515	65	240
567	117	167	342	392	574	124	174	349	399	551	101	151	326	376	558	108	158	333	383	565	115	165	340	390
217	267	442	617	42	224	274	449	624	49	201	251	426	601	26	208	258	433	608	33	215	265	440	615	40
492	542	92	142	317	499	549	99	149	324	476	526	76	126	301	483	533	83	133	308	490	540	90	140	315
23	198	373	423	598	5	180	355	405	580	7	182	357	407	582	14	189	364	414	589	16	191	366	416	591
298	473	523	73	248	280	455	505	55	230	282	457	507	57	232	289	464	514	64	239	291	466	516	66	241
573	123	173	348	398	555	105	155	330	380	557	107	157	332	382	564	114	164	339	389	566	116	166	341	391
223	273	448	623	48	205	255	430	605	30	207	257	432	607	32	214	264	439	614	39	216	266	441	616	41
498	548	98	148	323	480	530	80	130	305	482	532	82	132	307	489	539	89	139	314	491	541	91	141	316
4	179	354	404	579	6	181	356	406	581	13	188	363	413	588	20	195	370	420	595	22	197	372	422	597
279	454	504	54	229	281	456	506	56	231	288	463	513	63	238	295	470	520	70	245	297	472	522	72	247
554	104	154	329	379	556	106	156	331	381	563	113	163	338	388	570	120	170	345	395	572	122	172	347	397
204	254	429	604	29	206	256	431	606	31	213	263	438	613	38	220	270	445	620	45	222	272	447	622	47
479	529	79	129	304	481	531	81	131	306	488	538	88	138	313	495	545	95	145	320	497	547	97	147	322
10	185	360	410	585	12	187	362	412	587	19	194	369	419	594	21	196	371	421	596	3	178	353	403	578
285	460	510	60	235	287	462	512	62	237	294	469	519	69	244	296	471	521	71	246	278	453	503	53	228
560	110	160	335	385	562	112	162	337	387	569	119	169	344	394	571	121	171	346	396	553	103	153	328	378
210	260	435	610	35	212	262	437	612	37	219	269	444	619	44	221	271	446	621	46	203	253	428	603	28
485	535	85	135	310	487	537	87	137	312	494	544	94	144	319	496	546	96	146	321	478	528	78	128	303
11	186	361	411	586	18	193	368	418	593	25	200	375	425	600	2	177	352	402	577	9	184	359	409	584
286	461	511	61	236	293	468	518	68	243	300	475	525	75	250	277	452	502	52	227	284	459	509	59	234
561	111	161	336	386	568	118	168	343	393	575	125	175	350	400	552	102	152	327	377	559	109	159	334	384
211	261	436	611	36	218	268	443	618	43	225	275	450	625	50	202	252	427	602	27	209	259	434	609	34
486	536	86	136	311	493	543	93	143	318	500	550	100	150	325	477	527	77	127	302	484	534	84	134	309

Figure 4.18: *Yang-Hui* composition of P with Q

Result for $Q * P$:

401	576	1	176	351	408	583	8	183	358	415	590	15	190	365	417	592	17	192	367	424	599	24	199	374
551	101	151	326	376	558	108	158	333	383	565	115	165	340	390	567	117	167	342	392	574	124	174	349	399
76	126	301	451	526	83	133	308	458	533	90	140	315	465	540	92	142	317	467	542	99	149	324	474	549
226	276	451	501	51	233	283	458	508	58	240	290	465	515	65	242	292	467	517	67	249	299	474	524	74
251	426	601	26	201	258	433	608	33	208	265	440	615	40	215	267	442	617	42	217	274	449	624	49	224
412	587	12	187	362	419	594	19	194	369	421	596	21	196	371	403	578	3	178	353	410	585	10	185	360
562	112	162	337	387	569	119	169	344	394	571	121	171	346	396	553	103	153	328	378	560	110	160	335	385
87	137	312	462	537	94	144	319	469	544	96	146	321	471	546	78	128	303	453	528	85	135	310	460	535
237	287	462	512	62	244	294	469	519	69	246	296	471	521	71	228	278	453	503	53	235	285	460	510	60
262	437	612	37	212	269	444	619	44	219	271	446	621	46	221	253	428	603	28	203	260	435	610	35	210
423	598	23	198	373	405	580	5	180	355	407	582	7	182	357	414	589	14	189	364	416	591	16	191	366
573	123	173	348	398	555	105	155	330	380	557	107	157	332	382	564	114	164	339	389	566	116	166	341	391
98	148	323	473	548	80	130	305	455	530	82	132	307	457	532	89	139	314	464	539	91	141	316	466	541
248	298	473	523	73	230	280	455	505	55	232	282	457	507	57	239	289	464	514	64	241	291	466	516	66
273	448	623	48	223	255	430	605	30	205	257	432	607	32	207	264	439	614	39	214	266	441	616	41	216
409	584	9	184	359	411	586	11	186	361	418	593	18	193	368	425	600	25	200	375	402	577	2	177	352
559	109	159	334	384	561	111	161	336	386	568	118	168	343	393	575	125	175	350	400	552	102	152	327	377
84	134	309	459	534	86	136	311	461	536	93	143	318	468	543	100	150	325	475	550	77	127	302	452	527
234	284	459	509	59	236	286	461	511	61	243	293	468	518	68	250	300	475	525	75	227	277	452	502	52
259	434	609	34	209	261	436	611	36	211	268	443	618	43	218	275	450	625	50	225	252	427	602	27	202
420	595	20	195	370	422	597	22	197	372	404	579	4	179	354	406	581	6	181	356	413	588	13	188	363
570	120	170	345	395	572	122	172	347	397	554	104	154	329	379	556	106	156	331	381	563	113	163	338	388
95	145	320	470	545	97	147	322	472	547	79	129	304	454	529	81	131	306	456	531	88	138	313	463	538
245	295	470	520	70	247	297	472	522	72	229	279	454	504	54	231	281	456	506	56	238	288	463	513	63
270	445	620	45	220	272	447	622	47	222	254	429	604	29	204	256	431	606	31	206	263	438	613	38	213

Figure 4.19: *Yang-Hui* composition of Q with P

Result for $P * P$:

417	592	17	192	367	424	599	24	199	374	401	576	1	176	351	408	583	8	183	358	415	590	15	190	365
567	117	167	342	392	574	124	174	349	399	551	101	151	326	376	558	108	158	333	383	565	115	165	340	390
92	142	317	492	542	99	149	324	499	549	76	126	301	476	526	83	133	308	483	533	90	140	315	490	540
242	292	467	517	67	249	299	474	524	74	226	276	451	501	51	233	283	458	508	58	240	290	465	515	65
267	442	617	42	217	274	449	624	49	224	251	426	601	26	201	258	433	608	33	208	265	440	615	40	215
423	598	23	198	373	405	580	5	180	355	407	582	7	182	357	414	589	14	189	364	416	591	16	191	366
573	123	173	348	398	555	105	155	330	380	557	107	157	332	382	564	114	164	339	389	566	116	166	341	391
98	148	323	498	548	80	130	305	480	530	82	132	307	482	532	89	139	314	489	539	91	141	316	491	541
248	298	473	523	73	230	280	455	505	55	232	282	457	507	57	239	289	464	514	64	241	291	466	516	66
273	448	623	48	223	255	430	605	30	205	257	432	607	32	207	264	439	614	39	214	266	441	616	41	216
404	579	4	179	354	406	581	6	181	356	413	588	13	188	363	420	595	20	195	370	422	597	22	197	372
554	104	154	329	379	556	106	156	331	381	563	113	163	338	388	570	120	170	345	395	572	122	172	347	397
79	129	304	479	529	81	131	306	481	531	88	138	313	488	538	95	145	320	495	545	97	147	322	497	547
229	279	454	504	54	231	281	456	506	56	238	288	463	513	63	245	295	470	520	70	247	297	472	522	72
254	429	604	29	204	256	431	606	31	206	263	438	613	38	213	270	445	620	45	220	272	447	622	47	222
410	585	10	185	360	412	587	12	187	362	419	594	19	194	369	421	596	21	196	371	403	578	3	178	353
560	110	160	335	385	562	112	162	337	387	569	119	169	344	394	571	121	171	346	396	553	103	153	328	378
85	135	310	485	535	87	137	312	487	537	94	144	319	494	544	96	146	321	496	546	78	128	303	478	528
235	285	460	510	60	237	287	462	512	62	244	294	469	519	69	246	296	471	521	71	228	278	453	503	53
260	435	610	35	210	262	437	612	37	212	269	444	619	44	219	271	446	621	46	221	253	428	603	28	203
411	586	11	186	361	418	593	18	193	368	425	600	25	200	375	402	577	2	177	352	409	584	9	184	359
561	111	161	336	386	568	118	168	343	393	575	125	175	350	400	552	102	152	327	377	559	109	159	334	384
86	136	311	486	536	93	143	318	493	543	100	150	325	500	550	77	127	302	477	527	84	134	309	484	534
236	286	461	511	61	243	293	468	518	68	250	300	475	525	75	227	277	452	502	52	234	284	459	509	59
261	436	611	36	211	268	443	618	43	218	275	450	625	50	225	252	427	602	27	202	259	434	609	34	209

Figure 4.20: *Yang-Hui* composition of P with P

Result for $Q * Q$:

1	176	351	401	576	8	183	358	408	583	15	190	365	415	590	17	192	367	417	592	24	199	374	424	599
276	451	501	51	226	283	458	508	58	233	290	465	515	65	240	292	467	517	67	242	299	474	524	74	249
551	101	151	326	376	558	108	158	333	383	565	115	165	340	390	567	117	167	342	392	574	124	174	349	399
201	251	426	601	26	208	258	433	608	33	215	265	440	615	40	217	267	442	617	42	224	274	449	624	49
476	526	76	126	301	483	533	83	133	308	490	540	90	140	315	492	542	92	142	317	499	549	99	149	324
12	187	362	412	587	19	194	369	419	594	21	196	371	421	596	3	178	353	403	578	10	185	360	410	585
287	462	512	62	237	294	469	519	69	244	296	471	521	71	246	278	453	503	53	228	285	460	510	60	235
562	112	162	337	387	569	119	169	344	394	571	121	171	346	396	553	103	153	328	378	560	110	160	335	385
212	262	437	612	37	219	269	444	619	44	221	271	446	621	46	203	253	428	603	28	210	260	435	610	35
487	537	87	137	312	494	544	94	144	319	496	546	96	146	321	478	528	78	128	303	485	535	85	135	310
23	198	373	423	598	5	180	355	405	580	7	182	357	407	582	14	189	364	414	589	16	191	366	416	591
298	473	523	73	248	280	455	505	55	230	282	457	507	57	232	289	464	514	64	239	291	466	516	66	241
573	123	173	348	398	555	105	155	330	380	557	107	157	332	382	564	114	164	339	389	566	116	166	341	391
223	273	448	623	48	205	255	430	605	30	207	257	432	607	32	214	264	439	614	39	216	266	441	616	41
498	548	98	148	323	480	530	80	130	305	482	532	82	132	307	489	539	89	139	314	491	541	91	141	316
9	184	359	409	584	11	186	361	411	586	18	193	368	418	593	25	200	375	425	600	2	177	352	402	577
284	459	509	59	234	286	461	511	61	236	293	468	518	68	243	300	475	525	75	250	277	452	502	52	227
559	109	159	334	384	561	111	161	336	386	568	118	168	343	393	575	125	175	350	400	552	102	152	327	377
209	259	434	609	34	211	261	436	611	36	218	268	443	618	43	225	275	450	625	50	202	252	427	602	27
484	534	84	134	309	486	536	86	136	311	493	543	93	143	318	500	550	100	150	325	477	527	77	127	302
20	195	370	420	595	22	197	372	422	597	4	179	354	404	579	6	181	356	406	581	13	188	363	413	588
295	470	520	70	245	297	472	522	72	247	279	454	504	54	229	281	456	506	56	231	288	463	513	63	238
570	120	170	345	395	572	122	172	347	397	554	104	154	329	379	556	106	156	331	381	563	113	163	338	388
220	270	445	620	45	222	272	447	622	47	204	254	429	604	29	206	256	431	606	31	213	263	438	613	38
495	545	95	145	320	497	547	97	147	322	479	529	79	129	304	481	531	81	131	306	488	538	88	138	313

Figure 4.21: *Yang-Hui* composition of Q with Q

The results are similar to the previous section. We already verified that $P * P$ in Figure 4.20 is a symmetrical magic square, and $Q * Q$ in Figure 4.21 is pandiagonal magic square. While for $P * Q$ and $Q * P$ in Figure 4.18 and Figure 4.19 respectively, they are neither symmetrical magic square nor pandiagonal magic square, but a normal magic square of order 25.

Note that, if magic square M is ro-symmetrical or ref-symmetrical or pandiagonal, then so are the composite magic square $M \circ M$ and $M * M$.

4-3 Ultramagic Squares

Based on the research by Al-Ashhab (2011), symmetrical and pandiagonal magic squares are called *ultramagic squares*. As mentioned earlier in literature review, Chen, Li and Zhang (2016) proved that there exist symmetrical and pandiagonal magic squares. However, due to the complexity of the construction method, it might need more advance knowledge to understand the proving and the construction method.

After some researched, we found out that there is a website which shows various ways to construct different types of magic squares with different order. The author of the website is Arie (n.d.). He illustrated the construction methods by using diagrams and he provided Excel file to download for each example. Among the methods, the construction methods for ultramagic squares are included as well. Besides, there are different ways to construct ultramagic squares with different order as well. One of it is the construction of ultramagic squares of order n such that n is a prime number and $n \geq 5$, which is the easiest to understand by comparing with other methods. Furthermore, this method does not appear in any research journal or related paper that we have gone through. So, we would like to learn and share out his idea to construct. Since the method mentioned in the website is not clear enough, so we summarize it to make it clearer and easier to be understood.

We will only focus on ultramagic squares of order n such that n is a prime number and $n \geq 5$. We explain the method by taking $n = 5$. The steps are shown at the following pages.

Steps:

1. Fill in the first row of the square U with order $0, n-1, 1, 2, ..., n-2$ from left to right.

0	4	1	2	3

Figure 4.22: First row filled up in square U

2. Copy the first row to fill into the next row but shift 2 columns ring-wise.

0	4	1	2	3
2	3	0	4	1

Figure 4.23: Second row filled up in square U

3. Repeat the step by copying current row into the next row by shifting 2 columns ring-wise until the squares is completely filled up.

0	4	1	2	3
2	3	0	4	1
4	1	2	3	0
3	0	4	1	2
1	2	3	0	4

Figure 4.24: Square U

4. Transpose the square.

0	2	4	3	1
4	3	1	0	2
1	0	2	4	3
2	4	3	1	0
3	1	0	2	4

Figure 4.25: U^T

5. Perform the following operation, $U + n \times U^T + 1$.

Then, an ultramagic square of order 5 is resulted as below:

1	15	22	18	9
23	19	6	5	12
10	2	13	24	16
14	21	20	7	3
17	8	4	11	25

Figure 4.26: Ultramagic square of order 5

The magic constant is 65. It is symmetrical and pandiagonal.

After that, we apply the composition method and *Yang-Hui* composition method on the ultramagic square that we constructed in Figure 4.26 to generate an ultramagic square of order 25 and check its properties.

Composition of ultramagic squares:

1	15	22	18	9	351	365	372	368	359	526	540	547	543	534	426	440	447	443	434	201	215	222	218	209
23	19	6	5	12	373	369	356	355	362	548	544	531	530	537	448	444	431	430	437	223	219	206	205	212
10	2	13	24	16	360	352	363	374	366	535	527	538	549	541	435	427	438	449	441	210	202	213	224	216
14	21	20	7	3	364	371	370	357	353	539	546	545	532	528	439	446	445	432	428	214	221	220	207	203
17	8	4	11	25	367	358	354	361	375	542	533	529	536	550	442	433	429	436	450	217	208	204	211	225
551	565	572	568	559	451	465	472	468	459	126	140	147	143	134	101	115	122	118	109	276	290	297	293	284
573	569	556	555	562	473	469	456	455	462	148	144	131	130	137	123	119	106	105	112	298	294	281	280	287
560	552	563	574	566	460	452	463	474	466	135	127	138	149	141	110	102	113	124	116	285	277	288	299	291
564	571	570	557	553	464	471	470	457	453	139	146	145	132	128	114	121	120	107	103	289	296	295	282	278
567	558	554	561	575	467	458	454	461	475	142	133	129	136	150	117	108	104	111	125	292	283	279	286	300
226	240	247	243	234	26	40	47	43	34	301	315	322	318	309	576	590	597	593	584	376	390	397	393	384
248	244	231	230	237	48	44	31	30	37	323	319	306	305	312	598	594	581	580	587	398	394	381	380	387
235	227	238	249	241	35	27	38	49	41	310	302	313	324	316	585	577	588	599	591	385	377	388	399	391
239	246	245	232	228	39	46	45	32	28	314	321	320	307	303	589	596	595	582	578	389	396	395	382	378
242	233	229	236	250	42	33	29	36	50	317	308	304	311	325	592	583	579	586	600	392	383	379	386	400
326	340	347	343	334	501	515	522	518	509	476	490	497	493	484	151	165	172	168	159	51	65	72	68	59
348	344	331	330	337	523	519	506	505	512	498	494	481	480	487	173	169	156	155	162	73	69	56	55	62
335	327	338	349	341	510	502	513	524	516	485	477	488	499	491	160	152	163	174	166	60	52	63	74	66
339	346	345	332	328	514	521	520	507	503	489	496	495	482	478	164	171	170	157	153	64	71	70	57	53
342	333	329	336	350	517	508	504	511	525	492	483	479	486	500	167	158	154	161	175	67	58	54	61	75
401	415	422	418	409	176	190	197	193	184	76	90	97	93	84	251	265	272	268	259	601	615	622	618	609
423	419	406	405	412	198	194	181	180	187	98	94	81	80	87	273	269	256	255	262	623	619	606	605	612
410	402	413	424	416	185	177	188	199	191	85	77	88	99	91	260	252	263	274	266	610	602	613	624	616
414	421	420	407	403	189	196	195	182	178	89	96	95	82	78	264	271	270	257	253	614	621	620	607	603
417	408	404	411	425	192	183	179	186	200	92	83	79	86	100	267	258	254	261	275	617	608	604	611	625

Figure 4.27: Composite of ultramagic square with itself

Yang-Hui composition of ultramagic squares:

1	351	526	426	201	15	365	540	440	215	22	372	547	447	222	18	368	543	443	218	9	359	534	434	209
551	451	126	101	276	565	465	140	115	290	572	472	147	122	297	568	468	143	118	293	559	459	134	109	284
226	26	301	576	376	240	40	315	590	390	247	47	322	597	397	243	43	318	593	393	234	34	309	584	384
326	501	476	151	51	340	515	490	165	65	347	522	497	172	72	343	518	493	168	68	334	509	484	159	59
401	176	76	251	601	415	190	90	265	615	422	197	97	272	622	418	193	93	268	618	409	184	84	259	609
23	373	548	448	223	19	369	544	444	219	6	356	531	431	206	5	355	530	430	205	12	362	537	437	212
573	473	148	123	298	569	469	144	119	294	556	456	131	106	281	555	455	130	105	280	562	462	137	112	287
248	48	323	598	398	244	44	319	594	394	231	31	306	581	381	230	30	305	580	380	237	37	312	587	387
348	523	498	173	73	344	519	494	169	69	331	506	481	156	56	330	505	480	155	55	337	512	487	162	62
423	198	98	273	623	419	194	94	269	619	406	181	81	256	606	405	180	80	255	605	412	187	87	262	612
10	360	535	435	210	2	352	527	427	202	13	363	538	438	213	24	374	549	449	224	16	366	541	441	216
560	460	135	110	285	552	452	127	102	277	563	463	138	113	288	574	474	149	124	299	566	466	141	116	291
235	35	310	585	385	227	27	302	577	377	238	38	313	588	388	249	49	324	599	399	241	41	316	591	391
335	510	485	160	60	327	502	477	152	52	338	513	488	163	63	349	524	499	174	74	341	516	491	166	66
410	185	85	260	610	402	177	77	252	602	413	188	88	263	613	424	199	99	274	624	416	191	91	266	616
14	364	539	439	214	21	371	546	446	221	20	370	545	445	220	7	357	532	432	207	3	353	528	428	203
564	464	139	114	289	571	471	146	121	296	570	470	145	120	295	557	457	132	107	282	553	453	128	103	278
239	39	314	589	389	246	46	321	596	396	245	45	320	595	395	232	32	307	582	382	228	28	303	578	378
339	514	489	164	64	346	521	496	171	71	345	520	495	170	70	332	507	482	157	57	328	503	478	153	53
414	189	89	264	614	421	196	96	271	621	420	195	95	270	620	407	182	82	257	607	403	178	78	253	603
17	367	542	442	217	8	358	533	433	208	4	354	529	429	204	11	361	536	436	211	25	375	550	450	225
567	467	142	117	292	558	458	133	108	283	554	454	129	104	279	561	461	136	111	286	575	475	150	125	300
242	42	317	592	392	233	33	308	583	383	229	29	304	579	379	236	36	311	586	386	250	50	325	600	400
342	517	492	167	67	333	508	483	158	58	329	504	479	154	54	336	511	486	161	61	350	525	500	175	75
417	192	92	267	617	408	183	83	258	608	404	179	79	254	604	411	186	86	261	611	425	200	100	275	625

Figure 4.28: *Yang-Hui* composition of ultramagic square with itself

From the results in Figure 4.27 and Figure 4.28, they are symmetrical and pandiagonal. So, both of the them are ultramagic squares of order 25.

4-4 Modification in Ralph Strachy Method

In this section, we make some modifications to the Ralph Strachy method that are used for constructing singly-even order magic squares; an account of the method is given in Chee (1981). We would like to investigate whether the magic square still can be generated after some modifications.

We replace the De la Loubére method by the construction method for ultramagic squares mentioned in the previous section. The Ralph Strachy method can produce magic square with order $n = 2(2b+1)$. Since the construction method for ultramagic square only applicable for magic squares of order n such that n is a prime number and $n \geq 5$, so $(2b+1) \geq 5$ and $b \geq 2$.

Here is the step to construct magic squares of singly-even order after applying the modification.

Step:

1. Divide the square into four subsquares W, X, Y and Z of order $2b+1$ by referring to the figure below.

W	Y
Z	X

Figure 4.29: Four subsquares

2. Construct the magic square by using ultramagic square method on each subsquare, the entries for each subsquare are:

   - W contains numbers from 1 until $\frac{n^2}{4}$.
   - X contains numbers from $\frac{n^2}{4} + 1$ until $\frac{n^2}{2}$.
   - Y contains numbers from $\frac{n^2}{2} + 1$ until $\frac{3n^2}{4}$.
   - Z contains numbers from $\frac{3n^2}{4} + 1$ until n^2.

This brings the result as below:

1	15	22	18	9	51	65	72	68	59
23	19	6	5	12	73	69	56	55	62
10	2	13	24	16	60	52	63	74	66
14	21	20	7	3	64	71	70	57	53
17	8	4	11	25	67	58	54	61	75
76	90	97	93	84	26	40	47	43	34
98	94	81	80	87	48	44	31	30	37
85	77	88	99	91	35	27	38	49	41
89	96	95	82	78	39	46	45	32	28
92	83	79	86	100	42	33	29	36	50

Figure 4.30: Ultramagic square method on four subsquares

3. Take the $b-1$ columns from the rightmost in the subsquare X to exchange vertically with the same column of subsquare Y, which is shown as green colour in Figure 4.30.

4. In the middle row of the subsquare W, take the b cells starting from the second cell to exchange with the corresponding cells in the subsquare Z, which is shown in blue colour in Figure 4.30.

5. In the subsquare W, take the leftmost $b \times b$ subsquare above (respectively below) A is to be exchanged with the corresponding leftmost subsquare of Z which is shown in yellow colour in Figure 4.30.

Then, a magic square of order 10 is resulted as below:

76	90	22	18	9	51	65	72	68	34
98	94	6	5	12	73	69	56	55	37
10	77	88	24	16	60	52	63	74	41
89	96	20	7	3	64	71	70	57	28
92	83	4	11	25	67	58	54	61	50
1	15	97	93	84	26	40	47	43	59
23	19	81	80	87	48	44	31	30	62
85	2	13	99	91	35	27	38	49	66
14	21	95	82	78	39	46	45	32	53
17	8	79	86	100	42	33	29	36	75

Figure 4.31: Magic square of order 10

The result we get in Figure 4.31 is just a normal magic square of order 10 with magic constant equals to 505.

Here is another example, we generate another ultramagic square of order 7 in Figure 4.32 below.

1	35	16	45	39	26	13
47	41	22	14	2	31	18
10	4	33	20	43	42	23
21	44	38	25	12	6	29
27	8	7	30	17	46	40
32	19	48	36	28	9	3
37	24	11	5	34	15	49

Figure 4.32: Ultramagic square of order 7

After that, we use it to construct magic square of order $n = 14$ where $(2b+1) = 7$ and $b = 3$ in Figure 4.33 below by using the modified method.

148	182	163	45	39	26	13	99	133	114	143	137	75	62
194	188	169	14	2	31	18	145	139	120	112	100	80	67
157	151	180	20	43	42	23	108	102	131	118	141	91	72
21	191	185	172	12	6	29	119	142	136	123	110	55	78
174	155	154	30	17	46	40	125	106	105	128	115	95	89
179	166	195	36	28	9	3	130	117	146	134	126	58	52
184	171	158	5	34	15	49	135	122	109	103	132	64	98
1	35	16	192	186	173	160	50	84	65	94	88	124	111
47	41	22	161	149	178	165	96	90	71	63	51	129	116
10	4	33	167	190	189	170	59	53	82	69	92	140	121
168	44	38	25	159	153	176	70	93	87	74	61	104	127
27	8	7	177	164	193	187	76	57	56	79	66	144	138
32	19	48	183	175	156	150	81	68	97	85	77	107	101
37	24	11	152	181	162	196	86	73	60	54	83	113	147

Figure 4.33: Magic square of order 14

We get a singly-even magic square of order 14 with magic constant equals to 1379.

We believe that the above construction will produce a magic square of order $2(2b+1)$ for any ultramagic square of prime order. However, we are yet to produce a general proof.

4-5 Cryptography on Magic Squares

Due to the uniqueness of magic squares, some researchers have been applying magic square into cryptography. Meenu and Ojha (2012) used magic squares of order 8 generate key to encrypt data by using the technique mentioned in their paper. Besides, Adachi and Sugita (2017) proposed a way to encrypt data by using different cipher texts which is generated by using magic squares with different orders. Moreover, Lok and Chin (2018) used magic square in an algorithm as a cipher in cryptography as well.

However, these are the only published papers related to magic squares and cryptography. Due to lack of materials and lack of knowledge related to cryptography, we are not able to find out a way to implement the concept of magic squares into cryptography. But, we believe that magic squares can be applied into cryptography and so it can be extended in the future which will required more knowledge about both magic squares and cryptography.

CHAPTER 5: CONCLUSION

In conclusion, there are various methods to construct singly-even order magic squares, doubly-even order magic squares and odd order magic squares. Each of the method are fairly easy to be understood.

After we have learned the construction methods, we use them to generate magic squares with different orders, and we discovered some interesting and unique properties in self-complementary magic squares. In ro-symmetrical magic squares, complement of the magic squares has the same pattern with the original magic squares. While for ref-symmetrical magic squares, in Figure 4.8 shows that it has the properties of semi-pandiagonal magic squares, but not sure if this property is true in general.

Moreover, composition method of magic squares and *Yang-Hui* composition method save times to generate a higher order magic squares by using two or more magic squares of lower orders.

Furthermore, we summarized the method of construction for ultramagic squares of prime ordered to make it more clear and easy to understand, as well as applying it into the Ralph Strachy method to do some modifications to create singly-even order magic squares. Even though the method is modified, but we still able to generate magic squares with the modified method as the results shown.

However, we are not able to conclude any results related to magic squares and cryptography, due to lack of materials and knowledge in the field of cryptography. But, we believe that, it is possible to apply magic squares on cryptography, and to be extended in the future.

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