BLOCK HYBRID METHOD FOR DELAY DIFFERENTIAL EQUATIONS IN VARIABLE STEP SIZE

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A project report submitted in partial fulfillment of the requirements for the award of Master of Mathematics

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DECLARATION

I hereby declare that this project report is based on my original work except for citations and quotations which have been duly acknowledged. I also declare that it has not been previously and concurrently submitted for any other degree or award at UTAR or other institutions.

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APPROVAL FOR SUBMISSION

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Date : 15 August 2021

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Date : 16 Aug 2021

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After more than half-year of hard effort, finally, the research is finished. From the initiation of proposing the research undergoes the completion of the algorithm, until the end of the research report, each phase is a new try and challenge for me. This project is also the most prominent research I had accomplished independently for the duration of my study at Universiti Tunku Abdul Rahman.

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ABSTRACT

Delay differential equations have often been used in engineering and science studies. This project proposed the new block-hybrid method to resolve the retarded delay differential equations in variable step size. This technique is constructed on a couple of explicit and implicit equations applied in predictor-corrector mode. The Lagrange interpolation polynomial had been implemented together to come close to the delay solutions. The step size is variated according to the local truncation error. The proposed technique had been analyzed by comparing the numerical results with the existing method when solving the retarded delay differential equations.

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CHAPTER 1

INTRODUCTION

1.1 General Introduction

The differential equation is an equation that consists of the functions and the derivatives that are widely used in many fields of study today. These equations are used to model some problems interconnected with the rate of change. We need the results of the equations to solve real-life problems. However, not all the problems can be written as a simple differential equation. The ordinary differential equations (ODEs) are more realistic as they consider the calculation of the current state.

ODEs have a general form as follows:

$$y'(t) = g(t, y(t)) \quad \text{for} \quad \alpha \leq t \leq \beta,$$
$$y(t_0) = y_0. \tag{1.1.1}$$

ODEs are very limited when they come to some dynamic problems because ODEs depend on the current time only. Therefore, Delay Differential Equations (DDEs) are another type of differential equation that are more precise on this kind of problem. DDEs involve the past event in the calculation of the current state. The limitation of ODEs can be overcome by modeling the problem using DDEs. In the actual situation, the delay is always there, and exploration around DDEs becomes essential.

We know that not all differential equations can be solved by analytical methods. So, the numerical solution of DDEs is studied in this project. DDEs are the unique type of differential equations whose derivatives at a particular time depend on the preceding time. DDEs can be further divided into two types, which are Retarded Delay Differential Equations (RDDEs) and Neutral Delay Differential Equations (NDDEs). Both RDDEs and NDDEs involve the function and the delay solution, but the only difference is NDDEs have one more term, the first derivative of the delay term.

The general form of RDDEs and NDDEs are defined as follows:

RDDEs:

$$\begin{aligned} y'(t) &= g(t, y(t), y(t-\tau)) \quad , \quad \alpha \leq t \leq \beta, \\ y(t) &= \omega(t) \quad , \quad t < \alpha; \end{aligned} \tag{1.1.2}$$

NDDEs:

$$\begin{aligned} y'(t) &= g(t, y(t), y(t-\tau), y'(t-\tau)) \quad , \quad \alpha \leq t \leq \beta, \\ y(t) &= \omega(t) \quad , \quad t < \alpha; \end{aligned} \tag{1.1.3}$$

where

τ	is the delay,
$t-\tau$	is the previous time,
$y(t-\tau)$	is the delay solution, and
$\omega(t)$	is an initial function.

There are three types of delay terms in DDEs as follows:

- Constant Delay

$$y'(t) = g(t, y(t), y(t-\tau)) \quad , \quad y(t) \in \mathbb{R}, \tag{1.1.4}$$

where $\tau > 0$ and is a constant.

- Time Dependent Delay

$$y'(t) = g(t, y(t)y(t-\tau(t))) \quad , \quad y(t) \in \mathbb{R}, \tag{1.1.5}$$

where $\tau(t) > 0$ and is a function that depends on t.

- State Dependent Delay

$$y'(t) = g(t, y(t), y(t-\tau(t-y(t)))) \quad , \quad y(t) \in \mathbb{R}, \tag{1.1.6}$$

where $\tau(t-y(t)) > 0$ and is a function that depends on t and $y(t)$.

1.2 Multistep Method

The multistep method is one of the techniques used to solve DDEs. It uses previous points as a reference to approximate the value y at the targeted point, y_{n+1}. The block method uses the main points as reference points. The step size between all the main points is h. Figure 1.1 shows the relationship between reference points and the targeted point by the step size.

The general form of Block Method is defined as follows:

$$y_{n+1} = g(y_n, y_{n-1}, y_{n-2}, ...) \tag{1.2.1}$$

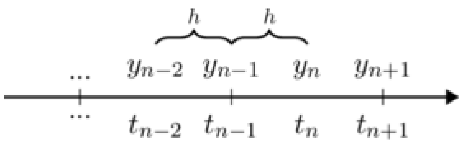

Figure 1.1: Block Method

The block-hybrid method is different from the block method as the main and off-step points will be used together as reference points. The off-step points are the points half step, $\frac{h}{2}$ from the main points. Both the value of y at the main and off step targeted points can be approximate together by this method. Figure 1.2 shows the relationship between reference points and the targeted point by half the step size.

The general form of Block Hybrid Method is defined as follows:

$$\begin{aligned} y_{n+\frac{1}{2}} &= g\left(y_n, y_{n-\frac{1}{2}}, y_{n-1}, y_{n-\frac{3}{2}}, y_{n-2}, \cdots\right) \\ y_{n+1} &= g\left(y_n, y_{n-\frac{1}{2}}, y_{n-1}, y_{n-\frac{3}{2}}, y_{n-2}, \cdots\right) \end{aligned} \tag{1.2.2}$$

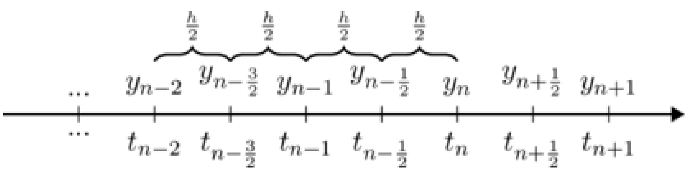

Figure 1.2: Block Hybrid Method

1.3 Importance of Study

DDEs are commonly found in engineering and science studies. Salpeter and Salpeter (1998) estimated the reproductive number and the infection using DDEs Model for epidemiology data on tuberculosis. Makroglou et al. (2006) used DDEs to model the glucose-insulin regulatory system to treat diabetes. It provided a possible mechanism in the secretion of pancreatic insulin. Kajiwara et al. (2012) also used DDEs to construct Lyapunov functional in virology and epidemiology. Gopalsamy (2013) investigated the application of DDEs on population dynamics by looking at stability and oscillations.

1.4 Problem Statement

Various mathematical methods have been derived to solve DDEs today. However, not all types of equations can be solved by the analytical method. As such, the numerical method comes in to solve the unsolvable parts. The block method that finds the approximation at a few main points concurrently is commonly used to solve ODEs. However, these methods might not get a good approximation when facing the delay term in DDEs. In this study, the block-hybrid method will use information from the main and off-step points to estimate the delay solution to obtain a more refined approximation of the targeted point. Implementation in constant step size requires many iterations when considering a smaller step size to get a more minor accumulated error. The higher number of iterations causes the computation time to become longer. Variable step size is implemented to minimize the total steps or have a more efficient way to solve the problems with minor errors.

1.5 Aim and Objectives

The objectives of this research are to:

(i) Derive a new block-hybrid method based on the divided difference formula to solve RDDEs. New predictor and corrector equations will be derived as the explicit method and implicit method, respectively.

(ii) Ensure the stability of the new method.

(iii) Implement the new method in variable step size. The step size of the targeted point depends on the local truncation error of the reference points so that the step size can be varied depending on the curve of the equations to minimize the total steps.

(iv) Analyze the accuracy and effectiveness of the new method with a set of specific tolerances. The analysis is carried out in a few aspects like the number of successful steps/total steps (TS), number of failure steps (FS), number of function evaluations (FCN), the maximum error of absolute value between the computed solution and exact solution, and the time for the algorithm.

1.6 Scope of Study

The primary focus of this study will be on Retarded Delay Differential Equations:

$$
\begin{aligned}
y'(t) &= g(t, y(t), y(t-\tau)) \quad , \quad \alpha \leq t \leq \beta, \\
y(t) &= \omega(t) \quad , \quad t < \alpha; \qquad (1.6.1)
\end{aligned}
$$

where

τ	is the delay,
$t-\tau$	is the previous time,
$y(t-\tau)$	is the delay solution, and
$\omega(t)$	is an initial function.

A new block-hybrid method is used to approximate the numerical solutions of RDDEs. This method is used in predictor-corrector (PECE) mode, with the explicit equation serving as the predictor and the implicit equation serving as the corrector. The step size is varied in every iteration depends on the local truncation error. Three types of delay terms will be used to show how the new method compares to the existing method in terms of accuracy and efficiency.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

There are several numerical methods implemented to solve the DDEs. Runge-Kutta method is a popular method used to solve ODEs and can be extended to DDEs. However, it has its limitation when confronting the delay term. So, some research had been carried out by modifying the existing method or implementing a new method to fight this delay term. Xie (1992) said the stability and uniqueness of slow oscillate solution would influence most in solving DDEs.

Continuous Runge-Kutta Methods was proposed in Enright and Hayashi (1997) to solve RDDEs and NDDEs. An iterative system by extrapolation is used as a new idea to handle the vanish delays. This idea is further evaluated in Xu et al. (2010) by using Exponential Runge-Kutta Methods. In Zhang and Chen (2010), it is proved that the Block Boundary Value Methods is convergent of an order under the classical Lipschitz condition.

El-Morshedy and Ruiz-Herrera (2017) proposed a new method using a scalar function to lead the nonlinear terms in the system to obtain the delay-dependent results to cover almost all the delay-independent conditions. Hu and Xiao (2018) study the delay-dependent for a class of nonlinear NDDEs and derived it by generalized Halanay's inequality. Jamilla et al. (2020) used the Lambert W function to solve NDDEs which function is defined as $W(a)e^{W(a)} - a = 0$. These are some of the parts that the effort being studied to handle DDEs.

2.2 Block Method

Look closer to the research related to the block method, Hue et al. (2011) proposed the Variable Order Coupled Block Method as the numerical results for DDEs. It used the 2 point 2 step block method of order 5 and 3 point 2 step block method of order 6 to solve DDEs. Majid et al. (2013) derived a 5 point 1 step block method constructed by the divided difference of Newton backward to solve DDEs. Aziz and Majid (2013) modified the 2 Points Block Method by computing the numerical solution of 2 points simultaneously to produce 2 new equal spaced solutions within the block. This method is based on the pair of implicit and explicit Adams formulas and implemented in predictor-corrector mode. Newton divided difference is derived for interpolation of the delay solutions. In the following year, Aziz et al. (2014) modified its explicit and implicit methods by recalculating the predictor and corrector formula depends on the step size changed. Yashkun and Aziz (2020) used the new two point Adams predictor-corrector block method derived by the divided difference of Newton to solve NDDEs. The neutral delay is approximated with the technique of central divided difference.

2.3 Block Hybrid Method

For the block hybrid method, Yap and Ismail (2015) proposed a method of order 3 in predictor-corrector mode to solve DDEs. The method is improved by using the order 6 block hybrid method in Yap et al. (2020). Ismail et al. (2020) derived the block hybrid method based on the Taylor series to solve NDDEs.

2.4 Variable Step Size

Variable step size strategy had been widely used in many research to solve the DDEs. Hue et al. (2011) implemented the strategy in its variable coupled block method. It varies the step size by choosing the maximum step size on the next block. Aziz and Majid (2013) also used this strategy in its modified 2 point block method. It extended its method by using the Runge-Kutta Fehlberg step size as the strategy to improve the results.

CHAPTER 3

RESEARCH METHODOLOGY

To derive a new block-hybrid method, both off step and main points are required to proceed with calculation. To implement it in predictor-corrector mode, at least a pair of explicit and implicit methods is needed.

3.1 Divided Difference

The block-hybrid method is constructed based on the divided differences, which are the recursive division process.

Table 3.1: Divided Differences

t	$f(t)$	First Divided Differences	Second Divided Differences
t_0	$f\left[t_0\right]$		
		$f\left[t_0, t_{\frac{1}{2}}\right] = \frac{f\left[t_{\frac{1}{2}}\right]-f[t_0]}{t_{\frac{1}{2}}-t_0}$	
$t_{\frac{1}{2}}$	$f\left[t_{\frac{1}{2}}\right]$		$f\left[t_0, t_{\frac{1}{2}}, t_1\right] = \frac{f\left[t_{\frac{1}{2}},t_1\right]-f\left[t_0,t_{\frac{1}{2}}\right]}{t_1-t_0}$
		$f\left[t_{\frac{1}{2}}, t_1\right] = \frac{f[t_1]-f\left[t_{\frac{1}{2}}\right]}{t_1-t_{\frac{1}{2}}}$	
t_1	$f\left[t_1\right]$		$f\left[t_{\frac{1}{2}}, t_1, t_{\frac{3}{2}}\right] = \frac{f\left[t_1,t_{\frac{3}{2}}\right]-f\left[t_{\frac{1}{2}},t_1\right]}{t_{\frac{3}{2}}-t_{\frac{1}{2}}}$
		$f\left[t_1, t_{\frac{3}{2}}\right] = \frac{f\left[t_{\frac{3}{2}}\right]-f[t_1]}{t_{\frac{3}{2}}-t_1}$	
$t_{\frac{3}{2}}$	$f\left[t_{\frac{3}{2}}\right]$		$f\left[t_1, t_{\frac{3}{2}}, t_2\right] = \frac{f\left[t_{\frac{3}{2}},t_2\right]-f\left[t_1,t_{\frac{3}{2}}\right]}{t_2-t_1}$
		$f\left[t_{\frac{3}{2}}, t_2\right] = \frac{f[t_2]-f\left[t_{\frac{3}{2}}\right]}{t_2-t_{\frac{3}{2}}}$	
t_2	$f\left[t_2\right]$		$f\left[t_{\frac{3}{2}}, t_2, t_{\frac{5}{2}}\right] = \frac{f\left[t_2,t_{\frac{5}{2}}\right]-f\left[t_{\frac{3}{2}},t_2\right]}{t_{\frac{5}{2}}-t_{\frac{3}{2}}}$
		$f\left[t_2, t_{\frac{5}{2}}\right] = \frac{f\left[t_{\frac{5}{2}}\right]-f[t_2]}{t_{\frac{5}{2}}-t_2}$	
$t_{\frac{5}{2}}$	$f\left[t_{\frac{5}{2}}\right]$		

The n^{th} divided differences have a general form as follows:

$$f\left[t_0, t_{\frac{1}{2}}, t_1, ..., t_{n-1}, t_{n-\frac{1}{2}}, t_n\right] = \frac{\begin{matrix} f\left[t_{\frac{1}{2}}, t_1, ..., t_{n-1}, t_{n-\frac{1}{2}}, t_n\right] \\ -f\left[t_0, t_{\frac{1}{2}}, t_1, ..., t_{n-1}, t_{n-\frac{1}{2}}\right] \end{matrix}}{t_n - t_0} \tag{3.1.1}$$

Then a list of divided differences is calculated and will be used later.

$$f\left[t_k\right] = f_k$$

$$\begin{aligned} f\left[t_{k-\frac{1}{2}}, t_k\right] &= \frac{f\left[t_k\right] - f\left[t_{k-\frac{1}{2}}\right]}{t_k - t_{k-\frac{1}{2}}} \\ &= \frac{f_k - f_{k-\frac{1}{2}}}{\frac{h}{2}} \\ &= \frac{2}{h}\left(f_k - f_{k-\frac{1}{2}}\right) \end{aligned}$$

$$\begin{aligned} f\left[t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] &= \frac{f\left[t_{k-\frac{1}{2}}, t_k\right] - f\left[t_{k-1}, t_{k-\frac{1}{2}}\right]}{t_k - t_{k-1}} \\ &= \frac{\frac{2}{h}\left(f_k - f_{k-\frac{1}{2}}\right) - \frac{2}{h}\left(f_{k-\frac{1}{2}} - f_{k-1}\right)}{h} \\ &= \frac{2}{h^2}\left(f_k - 2f_{k-\frac{1}{2}} + f_{k-1}\right) \end{aligned}$$

$$\begin{aligned} f\left[t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] &= \frac{f\left[t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] - f\left[t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}\right]}{t_k - t_{k-\frac{3}{2}}} \\ &= \frac{\begin{matrix} \frac{2}{h^2}\left(f_k - 2f_{k-\frac{1}{2}} + f_{k-1}\right) \\ -\frac{2}{h^2}\left(f_{k-\frac{1}{2}} - 2f_{k-1} + f_{k-\frac{3}{2}}\right) \end{matrix}}{\frac{3}{2}h} \\ &= \frac{4}{3h^3}\left(f_k - 3f_{k-\frac{1}{2}} + 3f_{k-1} - f_{k-\frac{3}{2}}\right) \end{aligned}$$

$$\begin{aligned} f\left[t_{k-2}, t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] &= \frac{\begin{matrix} f\left[t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] \\ -f\left[t_{k-2}, t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}\right] \end{matrix}}{t_k - t_{k-2}} \\ &= \frac{\begin{matrix} \frac{4}{3h^3}\left(f_k - 3f_{k-\frac{1}{2}} + 3f_{k-1} - f_{k-\frac{3}{2}}\right) \\ -\frac{4}{3h^3}\left(f_{k-\frac{1}{2}} - 3f_{k-1} + 3f_{k-\frac{3}{2}} - f_{k-2}\right) \end{matrix}}{2h} \\ &= \frac{2}{3h^4}\left(f_k - 4f_{k-\frac{1}{2}} + 6f_{k-1} - 4f_{k-\frac{3}{2}} + f_{k-2}\right) \end{aligned}$$

$$f\left[t_{k-\frac{5}{2}}, t_{k-2}, t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right]$$
$$= \frac{f\left[t_{k-2}, t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] - f\left[t_{k-\frac{5}{2}}, t_{k-2}, t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}\right]}{t_k - t_{k-\frac{5}{2}}}$$
$$= \frac{\frac{2}{3h^4}\left(f_k - 4f_{k-\frac{1}{2}} + 6f_{k-1} - 4f_{k-\frac{3}{2}} + f_{k-2}\right) - \frac{2}{3h^4}\left(f_{k-\frac{1}{2}} - 4f_{k-1} + 6f_{k-\frac{3}{2}} - 4f_{k-2} + f_{k-\frac{5}{2}}\right)}{\frac{5}{2}h}$$
$$= \frac{4}{15h^5}\left(f_k - 5f_{k-\frac{1}{2}} + 10f_{k-1} - 10f_{k-\frac{3}{2}} + 5f_{k-2} - f_{k-\frac{5}{2}}\right)$$

$$f\left[t_{k-3}, t_{k-\frac{5}{2}}, t_{k-2}, t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right]$$
$$= \frac{f\left[t_{k-\frac{5}{2}}, t_{k-2}, t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] - f\left[t_{k-3}, t_{k-\frac{5}{2}}, t_{k-2}, t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}\right]}{t_k - t_{k-\frac{5}{2}}}$$
$$= \frac{\frac{4}{15h^5}\left(f_k - 5f_{k-\frac{1}{2}} + 10f_{k-1} - 10f_{k-\frac{3}{2}} + 5f_{k-2} - f_{k-\frac{5}{2}}\right) - \frac{4}{15h^5}\left(f_{k-\frac{1}{2}} - 5f_{k-1} + 10f_{k-\frac{3}{2}} - 10f_{k-2} + 5f_{k-\frac{5}{2}} - f_{k-3}\right)}{3h}$$
$$= \frac{4}{45h^6}\left(f_k - 6f_{k-\frac{1}{2}} + 15f_{k-1} - 20f_{k-\frac{3}{2}} + 15f_{k-2} - 6f_{k-\frac{5}{2}} + f_{k-3}\right)$$

3.2 Lagrange Interpolation Polynomial

The k^{th} Lagrange interpolation polynomial, $P_k(x)$, is derived based on the divided differences, where

$$\begin{aligned} P_k(t) =& f[t_k] + f\left[t_{k-\frac{1}{2}}, t_k\right](t - t_k) + f\left[t_{k-1}, t_{k-\frac{1}{2}}, t_k\right](t - t_k)\left(t - t_{k-\frac{1}{2}}\right) \\ &+ f\left[t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right](t - t_k)\left(t - t_{k-\frac{1}{2}}\right)(t - t_{k-1}) + ... \\ &+ f\left[t_0, t_{\frac{1}{2}}, t_1, ..., t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right](t - t_k)...(t - t_1)\left(t - t_{\frac{1}{2}}\right) \end{aligned} \tag{3.2.1}$$

3.2.1 The Predictor

Set a variable $s = \dfrac{t - t_k}{h}$ to measure t in the unit of h, where it starts at $t = t_k$. Then, $t = t_k + sh$.

$$\begin{aligned} P_k(t) =& P_k(t_k + sh) \\ =& f[t_k] + shf\left[t_{k-\frac{1}{2}}, t_k\right] + s\left(s + \frac{1}{2}\right)h^2 f\left[t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] \\ &+ s\left(s + \frac{1}{2}\right)(s + 1)h^3 f\left[t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] + ... \end{aligned} \tag{3.2.2}$$

3.2.2 The Corrector

Set a variable $s = \dfrac{t - t_{k+\frac{1}{2}}}{h}$, where the starting point is $t = t_{k+\frac{1}{2}}$. Then, $t = t_{k+\frac{1}{2}} + sh$.

$$\begin{aligned} P_k(t) =& P_k\left(t_{k+\frac{1}{2}} + sh\right) \\ =& f\left[t_{k+\frac{1}{2}}\right] + shf\left[t_k, t_{k+\frac{1}{2}}\right] + s\left(s + \frac{1}{2}\right)h^2 f\left[t_{k-\frac{1}{2}}, t_k, t_{k+\frac{1}{2}}\right] \\ &+ s\left(s+\frac{1}{2}\right)(s+1)h^3 f\left[t_{k-1}, t_{k-\frac{1}{2}}, t_k, t_{k+\frac{1}{2}}\right] + \ldots \end{aligned} \tag{3.2.3}$$

Set a variable $s = \dfrac{t - t_{k+1}}{h}$, where the starting point is $t = t_{k+1}$. Then, $t = t_{k+1} + sh$.

$$\begin{aligned} P_k(t) =& P_k\left(t_{k+1} + sh\right) \\ =& f\left[t_{k+1}\right] + shf\left[t_{k+\frac{1}{2}}, t_{k+1}\right] + s\left(s + \frac{1}{2}\right)h^2 f\left[t_k, t_{k+\frac{1}{2}}, t_{k+1}\right] \\ &+ s\left(s+\frac{1}{2}\right)(s+1)h^3 f\left[t_{k-\frac{1}{2}}, t_k, t_{k+\frac{1}{2}}, t_{k+1}\right] + \ldots \end{aligned} \tag{3.2.4}$$

Set a variable $s = \dfrac{t - t_{k+\frac{3}{2}}}{h}$, where the starting point is $t = t_{k+\frac{3}{2}}$. Then, $t = t_{k+\frac{3}{2}} + sh$.

$$\begin{aligned} P_k(t) =& P_k\left(t_{k+\frac{3}{2}} + sh\right) \\ =& f\left[t_{k+\frac{3}{2}}\right] + shf\left[t_{k+1}, t_{k+\frac{3}{2}}\right] + s\left(s + \frac{1}{2}\right)h^2 f\left[t_{k+\frac{1}{2}}, t_{k+1}, t_{k+\frac{3}{2}}\right] \\ &+ s\left(s+\frac{1}{2}\right)(s+1)h^3 f\left[t_k, t_{k+\frac{1}{2}}, t_{k+1}, t_{k+\frac{3}{2}}\right] + \ldots \end{aligned} \tag{3.2.5}$$

Set a variable $s = \dfrac{t - t_{k+2}}{h}$, where the starting point is $t = t_{k+2}$. Then, $t = t_{k+2} + sh$.

$$\begin{aligned} P_k(t) =& P_k\left(t_{k+2} + sh\right) \\ =& f\left[t_{k+2}\right] + shf\left[t_{k+\frac{3}{2}}, t_{k+2}\right] + s\left(s + \frac{1}{2}\right)h^2 f\left[t_{k+1}, t_{k+\frac{3}{2}}, t_{k+2}\right] \\ &+ s\left(s+\frac{1}{2}\right)(s+1)h^3 f\left[t_{k+\frac{1}{2}}, t_{k+1}, t_{k+\frac{3}{2}}, t_{k+2}\right] + \ldots \end{aligned} \tag{3.2.6}$$

3.3 Block Hybrid Method

The explicit and implicit methods are derived by integrating the Lagrange interpolation polynomial in Section 3.2. The equation of the explicit method will calculate the predicted y, y^p as an initial estimation for the targeted value by using few previous points as the reference points. Then, the implicit method equation will recalculate the y^p to corrected y, y^c as the final estimation for the targeted value by using both the y^p and few previous points as the reference points. Here, the derivation of two new methods will be discussed, which are 2 step block-hybrid method and 1 step block-hybrid method.

3.3.1 Explicit 2 Step Block Hybrid Method

First Off Step Point

Set the off step point $t_{k+\frac{1}{2}} = t_k + \dfrac{1}{2}h$. Integrate both sides of equation (1.1.2) from t_k to $t_{k+\frac{1}{2}}$

$$\int_{t_k}^{t_{k+\frac{1}{2}}} y'(t)\,dt = \int_{t_k}^{t_{k+\frac{1}{2}}} g(t, y(t), y(t-\tau))\,dt$$

The function $g(t, y(t), y(t-\tau))$ is interpolated by the Lagrange interpolation polynomial equation, (3.2.2).

$$y_{k+\frac{1}{2}} - y_k = \int_{t_k}^{t_{k+\frac{1}{2}}} P_k(t)\,dt$$

Use $s = \dfrac{t-t_k}{h}$, then $t = t_k + sh$, so $dt = hds$. To adjust the limits of integration, change the limits when $t = t_k$, $s = 0$ and when $t = t_{k+\frac{1}{2}}$, $s = \frac{1}{2}$.

$$y_{k+\frac{1}{2}} - y_k = \int_0^{\frac{1}{2}} P_k\left(t_k + sh\right) h\,ds$$

$$\begin{aligned} y_{k+\frac{1}{2}} =& y_k + \int_0^{\frac{1}{2}} f\left[t_k\right] + shf\left[t_{k-\frac{1}{2}}, t_k\right] + s\left(s+\frac{1}{2}\right)h^2 f\left[t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] \\ &+ s\left(s+\frac{1}{2}\right)(s+1)\,h^3 f\left[t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] + \dots\,ds \end{aligned} \tag{3.3.1}$$

where

$$
\begin{aligned}
f\left[t_k\right] &= f_k \\
f\left[t_{k-\frac{1}{2}}, t_k\right] &= \frac{2}{h}\left(f_k - f_{k-\frac{1}{2}}\right) \\
f\left[t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] &= \frac{2}{h^2}\left(f_k - 2f_{k-\frac{1}{2}} + f_{k-1}\right) \\
f\left[t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] &= \frac{4}{3h^3}\left(f_k - 3f_{k-\frac{1}{2}} + 3f_{k-1} - f_{k-\frac{3}{2}}\right) \\
f\left[t_{k-2}, t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] &= \\
&\frac{2}{3h^4}\left(f_k - 4f_{k-\frac{1}{2}} + 6f_{k-1} - 4f_{k-\frac{3}{2}} + f_{k-2}\right) \\
f\left[t_{k-\frac{5}{2}}, t_{k-2}, t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] &= \\
\frac{4}{15h^5}&\left(f_k - 5f_{k-\frac{1}{2}} + 10f_{k-1} - 10f_{k-\frac{3}{2}} + 5f_{k-2} - f_{k-\frac{5}{2}}\right)
\end{aligned}
$$

Then, the explicit formula for the first off-step point is derived as follows:

$$
\begin{aligned}
y_{k+\frac{1}{2}} = y_k + h\Bigg(&\frac{4277}{2880}f_k - \frac{2641}{960}f_{k-\frac{1}{2}} + \frac{4991}{1440}f_{k-1} \\
&-\frac{3649}{1440}f_{k-\frac{3}{2}} + \frac{959}{960}f_{k-2} - \frac{95}{576}f_{k-\frac{5}{2}}\Bigg)
\end{aligned}
\tag{3.3.2}
$$

<u>First Main Point</u>

Set the main point $t_{k+1} = t_k + h$. Integrate both sides of equation (1.1.2) from t_k to t_{k+1}

$$\int_{t_k}^{t_{k+1}} y'(t)\,dt = \int_{t_k}^{t_{k+1}} g(t, y(t), y(t-\tau))\,dt$$

The function $g(t, y(t), y(t-\tau))$ is interpolated by the Lagrange interpolation polynomial equation, (3.2.2).

$$y_{k+1} - y_k = \int_{t_k}^{t_{k+1}} P_k(t)\,dt$$

Use $s = \dfrac{t - t_k}{h}$, then $t = t_k + sh$, so $dt = hds$. To adjust the limits of integration, change the limits when $t = t_k$, $s = 0$ and when $t = t_{k+1}$, $s = 1$.

$$
\begin{aligned}
y_{k+1} - y_k = \int_0^1 P_k\left(t_k + sh\right) h\, ds \\
y_{k+1} = & y_k + \int_0^1 f\left[t_k\right] + shf\left[t_{k-\frac{1}{2}}, t_k\right] + s\left(s + \frac{1}{2}\right) h^2 f\left[t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] \\
& + s\left(s + \frac{1}{2}\right)(s+1)\, h^3 f\left[t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] + \ldots ds
\end{aligned}
\tag{3.3.3}
$$

where

$$
\begin{aligned}
f\left[t_k\right] &= f_k \\
f\left[t_{k-\frac{1}{2}}, t_k\right] &= \frac{2}{h}\left(f_k - f_{k-\frac{1}{2}}\right) \\
f\left[t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] &= \frac{2}{h^2}\left(f_k - 2f_{k-\frac{1}{2}} + f_{k-1}\right) \\
f\left[t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] &= \frac{4}{3h^3}\left(f_k - 3f_{k-\frac{1}{2}} + 3f_{k-1} - f_{k-\frac{3}{2}}\right) \\
f\left[t_{k-2}, t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] &= \\
& \frac{2}{3h^4}\left(f_k - 4f_{k-\frac{1}{2}} + 6f_{k-1} - 4f_{k-\frac{3}{2}} + f_{k-2}\right) \\
f\left[t_{k-\frac{5}{2}}, t_{k-2}, t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] &= \\
\frac{4}{15h^5} & \left(f_k - 5f_{k-\frac{1}{2}} + 10f_{k-1} - 10f_{k-\frac{3}{2}} + 5f_{k-2} - f_{k-\frac{5}{2}}\right)
\end{aligned}
$$

Then, the explicit formula for the first main point is derived as follows:

$$
\begin{aligned}
y_{k+1} = y_k + h\Bigg(& \frac{344}{45} f_k - \frac{3881}{180} f_{k-\frac{1}{2}} + \frac{919}{30} f_{k-1} \\
& - \frac{2143}{90} f_{k-\frac{3}{2}} + \frac{877}{90} f_{k-2} - \frac{33}{20} f_{k-\frac{5}{2}} \Bigg)
\end{aligned}
\tag{3.3.4}
$$

Second Off Step Point

Set the off step point $t_{k+\frac{3}{2}} = t_k + \dfrac{3}{2}h$. Integrate both sides of equation (1.1.2) from t_k to $t_{k+\frac{3}{2}}$

$$
\int_{t_k}^{t_{k+\frac{3}{2}}} y'(t)\, dt = \int_{t_k}^{t_{k+\frac{3}{2}}} g(t, y(t), y(t-\tau))\, dt
$$

The function $g(t, y(t), y(t-\tau))$ is interpolated by the Lagrange interpolation polynomial equation, (3.2.2).

$$y_{k+\frac{3}{2}} - y_k = \int_{t_k}^{t_{k+\frac{3}{2}}} P_k(t)\,dt$$

Use $s = \dfrac{t - t_k}{h}$, then $t = t_k + sh$, so $dt = hds$. To adjust the limits of integration, change the limits when $t = t_k$, $s = 0$ and when $t = t_{k+\frac{3}{2}}$, $s = \frac{3}{2}$.

$$y_{k+\frac{3}{2}} - y_k = \int_0^{\frac{3}{2}} P_k\left(t_k + sh\right) h\,ds$$

$$\begin{aligned} y_{k+\frac{3}{2}} =& y_k + \int_0^{\frac{3}{2}} f\left[t_k\right] + shf\left[t_{k-\frac{1}{2}}, t_k\right] + s\left(s + \frac{1}{2}\right) h^2 f\left[t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] \\ &+ s\left(s + \frac{1}{2}\right)\left(s + 1\right) h^3 f\left[t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] + \ldots\,ds \end{aligned} \tag{3.3.5}$$

where

$$\begin{aligned} f\left[t_k\right] &= f_k \\ f\left[t_{k-\frac{1}{2}}, t_k\right] &= \frac{2}{h}\left(f_k - f_{k-\frac{1}{2}}\right) \\ f\left[t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] &= \frac{2}{h^2}\left(f_k - 2f_{k-\frac{1}{2}} + f_{k-1}\right) \\ f\left[t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] &= \frac{4}{3h^3}\left(f_k - 3f_{k-\frac{1}{2}} + 3f_{k-1} - f_{k-\frac{3}{2}}\right) \\ f\left[t_{k-2}, t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] &= \\ &\frac{2}{3h^4}\left(f_k - 4f_{k-\frac{1}{2}} + 6f_{k-1} - 4f_{k-\frac{3}{2}} + f_{k-2}\right) \\ f\left[t_{k-\frac{5}{2}}, t_{k-2}, t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] &= \\ &\frac{4}{15h^5}\left(f_k - 5f_{k-\frac{1}{2}} + 10f_{k-1} - 10f_{k-\frac{3}{2}} + 5f_{k-2} - f_{k-\frac{5}{2}}\right) \end{aligned}$$

Then, the explicit formula for the second off-step point is derived as follows:

$$\begin{aligned} y_{k+\frac{3}{2}} = y_k + h\Bigg(&\frac{8253}{320} f_k - \frac{27771}{320} f_{k-\frac{1}{2}} + \frac{21207}{160} f_{k-1} \\ &- \frac{17193}{160} f_{k-\frac{3}{2}} + \frac{14469}{320} f_{k-2} - \frac{2499}{320} f_{k-\frac{5}{2}}\Bigg) \end{aligned} \tag{3.3.6}$$

Second Main Point

Set the main point $t_{k+2} = t_k + 2h$. Integrate both sides of equation (1.1.2) from

t_k to t_{k+2}

$$\int_{t_k}^{t_{k+2}} y'(t)\,dt = \int_{t_k}^{t_{k+2}} g(t, y(t), y(t-\tau))\,dt$$

The function $g(t, y(t), y(t-\tau))$ is interpolated by the Lagrange interpolation polynomial equation, (3.2.2).

$$y_{k+2} - y_k = \int_{t_k}^{t_{k+2}} P_k(t)\,dt$$

Use $s = \dfrac{t - t_k}{h}$, then $t = t_k + sh$, so $dt = hds$. To adjust the limits of integration, change the limits when $t = t_k$, $s = 0$ and when $t = t_{k+2}$, $s = 2$.

$$\begin{aligned} y_{k+2} - y_k &= \int_0^2 P_k\left(t_k + sh\right) h\,ds \\ y_{k+2} =& y_k + \int_0^2 f\left[t_k\right] + shf\left[t_{k-\frac{1}{2}}, t_k\right] + s\left(s + \frac{1}{2}\right) h^2 f\left[t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] \\ &+ s\left(s + \frac{1}{2}\right)(s+1)\, h^3 f\left[t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] + \ldots\, ds \end{aligned} \tag{3.3.7}$$

where

$$\begin{aligned} f\left[t_k\right] &= f_k \\ f\left[t_{k-\frac{1}{2}}, t_k\right] &= \frac{2}{h}\left(f_k - f_{k-\frac{1}{2}}\right) \\ f\left[t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] &= \frac{2}{h^2}\left(f_k - 2f_{k-\frac{1}{2}} + f_{k-1}\right) \\ f\left[t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] &= \frac{4}{3h^3}\left(f_k - 3f_{k-\frac{1}{2}} + 3f_{k-1} - f_{k-\frac{3}{2}}\right) \\ f\left[t_{k-2}, t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] &= \\ &\frac{2}{3h^4}\left(f_k - 4f_{k-\frac{1}{2}} + 6f_{k-1} - 4f_{k-\frac{3}{2}} + f_{k-2}\right) \\ f\left[t_{k-\frac{5}{2}}, t_{k-2}, t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] &= \\ &\frac{4}{15h^5}\left(f_k - 5f_{k-\frac{1}{2}} + 10f_{k-1} - 10f_{k-\frac{3}{2}} + 5f_{k-2} - f_{k-\frac{5}{2}}\right) \end{aligned}$$

Then, the explicit formula for the second main point is derived as follows:

$$\begin{aligned} y_{k+2} = y_k + h\bigg(&\frac{625}{9} f_k - \frac{3856}{15} f_{k-\frac{1}{2}} + \frac{18532}{45} f_{k-1} \\ &- \frac{15488}{45} f_{k-\frac{3}{2}} + \frac{2219}{15} f_{k-2} - \frac{1168}{45} f_{k-\frac{5}{2}}\bigg) \end{aligned} \tag{3.3.8}$$

3.3.2 Explicit 1 Step Block Hybrid Method

Evaluate equations (3.3.1) and (3.3.7) up to the term

$$f\left[t_{k-3}, t_{k-\frac{5}{2}}, t_{k-2}, t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] = \frac{4}{45h^6}\left(f_k - 6f_{k-\frac{1}{2}} + 15f_{k-1} - 20f_{k-\frac{3}{2}} + 15f_{k-2} - 6f_{k-\frac{5}{2}} + f_{k-3}\right).$$

Off Step Point

The explicit formula for off-step point is derived as follows:

$$y_{k+\frac{1}{2}} = y_k + h\left(\frac{198721}{120960}f_k - \frac{18637}{5040}f_{k-\frac{1}{2}} + \frac{235183}{40320}f_{k-1} - \frac{5377}{945}f_{k-\frac{3}{2}} + \frac{135713}{40320}f_{k-2} - \frac{5603}{5040}f_{k-\frac{5}{2}} + \frac{19087}{120960}f_{k-3}\right) \tag{3.3.9}$$

Main Point

The explicit formula for the main point is derived as follows:

$$y_{k+1} = y_k + h\left(\frac{14281}{1512}f_k - \frac{2039}{63}f_{k-\frac{1}{2}} + \frac{145261}{2520}f_{k-1} - \frac{56534}{945}f_{k-\frac{3}{2}} + \frac{92621}{2520}f_{k-2} - \frac{3923}{315}f_{k-\frac{5}{2}} + \frac{13613}{7560}f_{k-3}\right) \tag{3.3.10}$$

3.3.3 Implicit 2 Step Block Hybrid Method

First Off Step Point

Set the off step point $t_{k+\frac{1}{2}} = t_k + \dfrac{1}{2}h$. Integrate both sides of equation (1.1.2) from t_k to $t_{k+\frac{1}{2}}$

$$\int_{t_k}^{t_{k+\frac{1}{2}}} y'(t)\, dt = \int_{t_k}^{t_{k+\frac{1}{2}}} f(t, y(t), y(t-\tau))\, dt$$

The function $f(t, y(t), y(t-\tau))$ is interpolated by the Lagrange interpolation polynomial equation, (3.2.3).

$$y_{k+\frac{1}{2}} - y_k = \int_{t_k}^{t_{k+\frac{1}{2}}} P_k(t)\, dt$$

Use $s = \dfrac{t - t_{k+\frac{1}{2}}}{h}$, then $t = t_{k+\frac{1}{2}} + sh$, so $dt = hds$. To adjust the limits of integration, change the limits when $t = t_k$, $s = -\frac{1}{2}$ and when $t = t_{k+\frac{1}{2}}$, $s = 0$.

$$y_{k+\frac{1}{2}} - y_k = \int_{-\frac{1}{2}}^{0} P_k\left(t_{k+\frac{1}{2}} + sh\right) h\, ds$$

$$\begin{aligned} y_{k+\frac{1}{2}} =& y_k + \int_{-\frac{1}{2}}^{0} f\left[t_{k+\frac{1}{2}}\right] + shf\left[t_k, t_{k+\frac{1}{2}}\right] \\ &+ s\left(s + \frac{1}{2}\right) h^2 f\left[t_{k-\frac{1}{2}}, t_k, t_{k+\frac{1}{2}}\right] \\ &+ s\left(s + \frac{1}{2}\right)(s+1)\, h^3 f\left[t_{k-2}, t_{k-\frac{1}{2}}, t_k, t_{k+\frac{1}{2}}\right] + ...\, ds \end{aligned} \tag{3.3.11}$$

where

$$
\begin{aligned}
f\left[t_{k+\frac{1}{2}}\right] &= f_{k+\frac{1}{2}} \\
f\left[t_k, t_{k+\frac{1}{2}}\right] &= \frac{2}{h}\left(f_{k+\frac{1}{2}} - f_k\right) \\
f\left[t_{k-\frac{1}{2}}, t_k, t_{k+\frac{1}{2}}\right] &= \frac{2}{h^2}\left(f_{k+\frac{1}{2}} - 2f_k + f_{k-\frac{1}{2}}\right) \\
f\left[t_{k-1}, t_{k-\frac{1}{2}}, t_k, t_{k+\frac{1}{2}}\right] &= \frac{4}{3h^3}\left(f_{k+\frac{1}{2}} - 3f_k + 3f_{k-\frac{1}{2}} - f_{k-1}\right) \\
f\left[t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k, t_{k+\frac{1}{2}}\right] &= \\
&\frac{2}{3h^4}\left(f_{k+\frac{1}{2}} - 4f_k + 6f_{k-\frac{1}{2}} - 4f_{k-1} + f_{k-\frac{3}{2}}\right) \\
f\left[t_{k-2}, t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k, t_{k+\frac{1}{2}}\right] &= \\
\frac{4}{15h^5}&\left(f_{k+\frac{1}{2}} - 5f_k + 10f_{k-\frac{1}{2}} - 10f_{k-1} + 5f_{k-\frac{3}{2}} - f_{k-2}\right)
\end{aligned}
$$

Then, the implicit formula for the first off-step point is derived as follows:

$$
\begin{aligned}
y_{k+\frac{1}{2}} = y_k + h\bigg(&\frac{95}{576}f_{k+\frac{1}{2}} + \frac{1427}{2880}f_k - \frac{133}{480}f_{k-\frac{1}{2}} \\
&+\frac{241}{1440}f_{k-1} - \frac{173}{2880}f_{k-\frac{3}{2}} + \frac{3}{320}f_{k-2}\bigg)
\end{aligned}
\tag{3.3.12}
$$

First Main Point

Set the main point $t_{k+1} = t_k + h$. Integrate both sides of equation (1.1.2) from t_k to t_{k+1}

$$\int_{t_k}^{t_{k+1}} y'(t)\,dt = \int_{t_k}^{t_{k+1}} f(t, y(t), y(t-\tau))\,dt$$

The function $f(t, y(t), y(t-\tau))$ is interpolated by the Lagrange interpolation polynomial equation, (3.2.4).

$$y_{k+1} - y_k = \int_{t_k}^{t_{k+1}} P_k(t)\,dt$$

Use $s = \dfrac{t - t_{k+1}}{h}$, then $t = t_{k+1} + sh$, so $dt = hds$. To adjust the limits of integration, change the limits when $t = t_k$, $s = -1$ and when $t = t_{k+1}$, $s = 0$.

$$
\begin{aligned}
y_{k+1} - y_k &= \int_{-1}^{0} P_k \left(t_k + sh\right) h \, ds \\
y_{k+1} = &y_k + \int_{-1}^{0} f\left[t_{k+1}\right] + shf\left[t_{k+\frac{1}{2}}, t_{k+1}\right] \\
&+ s\left(s + \frac{1}{2}\right) h^2 f\left[t_k, t_{k+\frac{1}{2}}, t_{k+1}\right] \\
&+ s\left(s + \frac{1}{2}\right)(s+1) h^3 f\left[t_{k-\frac{1}{2}}, t_k, t_{k+\frac{1}{2}}, t_{k+1}\right] + ... \, ds
\end{aligned}
\tag{3.3.13}
$$

where

$$
\begin{aligned}
f\left[t_{k+1}\right] &= f_k \\
f\left[t_{k+\frac{1}{2}}, t_{k+1}\right] &= \frac{2}{h}\left(f_{k+1} - f_{k+\frac{1}{2}}\right) \\
f\left[t_k, t_{k+\frac{1}{2}}, t_{k+1}\right] &= \frac{2}{h^2}\left(f_{k+1} - 2f_{k+\frac{1}{2}} + f_k\right) \\
f\left[t_{k-\frac{1}{2}}, t_k, t_{k+\frac{1}{2}}, t_{k+1}\right] &= \frac{4}{3h^3}\left(f_{k+1} - 3f_{k+\frac{1}{2}} + 3f_k - f_{k-\frac{1}{2}}\right) \\
f\left[t_{k-1}, t_{k-\frac{1}{2}}, t_k, t_{k+\frac{1}{2}}, t_{k+1}\right] &= \\
&\frac{2}{3h^4}\left(f_{k+1} - 4f_{k+\frac{1}{2}} + 6f_k - 4f_{k-\frac{1}{2}} + f_{k-1}\right) \\
f\left[t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k, t_{k+\frac{1}{2}}, t_{k+1}\right] &= \\
\frac{4}{15h^5}&\left(f_{k+1} - 5f_{k+\frac{1}{2}} + 10f_k - 10f_{k-\frac{1}{2}} + 5f_{k-1} - f_{k-\frac{3}{2}}\right)
\end{aligned}
$$

Then, the implicit formula for the first main point is derived as follows:

$$
\begin{aligned}
y_{k+1} = y_k + h\Bigg(&\frac{7}{45} f_{k+1} + \frac{43}{60} f_{k+\frac{1}{2}} + \frac{7}{90} f_k \\
&+ \frac{7}{90} f_{k-\frac{1}{2}} - \frac{1}{30} f_{k-1} + \frac{1}{180} f_{k-\frac{3}{2}}\Bigg)
\end{aligned}
\tag{3.3.14}
$$

<u>Second Off Step Point</u>

Set the off step point $t_{k+\frac{3}{2}} = t_{k+1} + \frac{1}{2}h$. Integrate both sides of equation (1.1.2) from t_{k+1} to $t_{k+\frac{3}{2}}$

$$
\int_{t_{k+1}}^{t_{k+\frac{3}{2}}} y'(t) \, dt = \int_{t_{k+1}}^{t_{k+\frac{3}{2}}} f(t, y(t), y(t-\tau)) \, dt
$$

The function $f(t, y(t), y(t-\tau))$ is interpolated by the Lagrange interpolation polynomial equation, (3.2.5).

$$y_{k+\frac{3}{2}} - y_{k+1} = \int_{t_{k+1}}^{t_{k+\frac{3}{2}}} P_k(t)\, dt$$

Use $s = \dfrac{t - t_{k+\frac{3}{2}}}{h}$, then $t = t_{k+\frac{3}{2}} + sh$, so $dt = hds$. To adjust the limits of integration, change the limits when $t = t_{k+1}$, $s = -\frac{1}{2}$ and when $t = t_{k+\frac{3}{2}}$, $s = 0$.

$$\begin{aligned} y_{k+\frac{3}{2}} - y_{k+1} =& \int_{-\frac{1}{2}}^{0} P_k\left(t_{k+\frac{3}{2}} + sh\right) h\, ds \\ y_{k+\frac{3}{2}} =& y_{k+1} + \int_{0}^{\frac{3}{2}} f\left[t_k\right] + shf\left[t_{k-\frac{1}{2}}, t_k\right] \\ &+ s\left(s+\frac{1}{2}\right) h^2 f\left[t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] \\ &+ s\left(s+\frac{1}{2}\right)(s+1)\, h^3 f\left[t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] + \ldots ds \end{aligned} \tag{3.3.15}$$

where

$$\begin{aligned} f\left[t_{k+\frac{3}{2}}\right] &= f_{k+\frac{3}{2}} \\ f\left[t_{k+1}, t_{k+\frac{3}{2}}\right] &= \frac{2}{h}\left(f_{k+\frac{3}{2}} - f_{k+1}\right) \\ f\left[t_{k+\frac{1}{2}}, t_{k+1}, t_{k+\frac{3}{2}}\right] &= \frac{2}{h^2}\left(f_{k+\frac{3}{2}} - 2f_{k+1} + f_{k+\frac{1}{2}}\right) \\ f\left[t_k, t_{k+\frac{1}{2}}, t_{k+1}, t_{k+\frac{3}{2}}\right] &= \frac{4}{3h^3}\left(f_{k+\frac{3}{2}} - 3f_{k+1} + 3f_{k+\frac{1}{2}} - f_k\right) \\ f\left[t_{k-\frac{1}{2}}, t_k, t_{k+\frac{1}{2}}, t_{k+1}, t_{k+\frac{3}{2}}\right] &= \\ &\frac{2}{3h^4}\left(f_{k+\frac{3}{2}} - 4f_{k+1} + 6f_{k+\frac{1}{2}} - 4f_k + f_{k-\frac{1}{2}}\right) \\ f\left[t_{k-1}, t_{k-\frac{1}{2}}, t_k, t_{k+\frac{1}{2}}, t_{k+1}, t_{k+\frac{3}{2}}\right] &= \\ \frac{4}{15h^5}&\left(f_{k+\frac{3}{2}} - 5f_{k+1} + 10f_{k+\frac{1}{2}} - 10f_k + 5f_{k-\frac{1}{2}} - f_{k-1}\right) \end{aligned}$$

Then, the implicit formula for the second off-step point is derived as follows:

$$\begin{aligned} y_{k+\frac{3}{2}} = y_{k+1} + h\bigg(&\frac{8253}{320} f_{k+\frac{3}{2}} - \frac{27771}{320} f_{k+1} + \frac{21207}{160} f_{k+\frac{1}{2}} \\ &- \frac{17193}{160} f_k + \frac{14469}{320} f_{k-\frac{1}{2}} - \frac{2499}{320} f_{k-1}\bigg) \end{aligned} \tag{3.3.16}$$

Second Main Point

Set the main point $t_{k+2} = t_{k+1} + h$. Integrate both sides of equation (1.1.2) from t_{k+1} to t_{k+2}

$$\int_{t_{k+1}}^{t_{k+2}} y'(t)\,dt = \int_{t_{k+1}}^{t_{k+2}} f(t, y(t), y(t-\tau))\,dt$$

The function $f(t, y(t), y(t-\tau))$ is interpolated by the Lagrange interpolation polynomial equation, (3.2.6).

$$y_{k+2} - y_{k+1} = \int_{t_{k+1}}^{t_{k+2}} P_k(t)\,dt \tag{3.3.17}$$

Use $s = \dfrac{t - t_{k+2}}{h}$, then $t = t_{k+2} + sh$, so $dt = hds$. To adjust the limits of integration, change the limits when $t = t_{k+1}$, $s = -1$ and when $t = t_{k+2}$, $s = 0$.

$$\begin{aligned} y_{k+2} - y_{k+1} &= \int_{-1}^{0} P_k\left(t_{k+2} + sh\right) h\,ds \\ y_{k+2} =& y_{k+1} + \int_{-1}^{0} f\left[t_{k+2}\right] + shf\left[t_{k+\frac{3}{2}}, t_{k+2}\right] \\ &+ s\left(s + \frac{1}{2}\right) h^2 f\left[t_{k+1}, t_{k+\frac{3}{2}}, t_{k+2}\right] \\ &+ s\left(s + \frac{1}{2}\right)(s+1)\, h^3 f\left[t_{k+\frac{1}{2}}, t_{k+1}, t_{k+\frac{3}{2}}, t_{k+2}\right] + ...\, ds \end{aligned} \tag{3.3.18}$$

where

$$\begin{aligned} f\left[t_{k+2}\right] &= f_{k+2} \\ f\left[t_{k+\frac{3}{2}}, t_{k+2}\right] &= \frac{2}{h}\left(f_{k+2} - f_{k+\frac{3}{2}}\right) \\ f\left[t_{k+1}, t_{k+\frac{3}{2}}, t_{k+2}\right] &= \frac{2}{h^2}\left(f_{k+2} - 2f_{k+\frac{3}{2}} + f_{k+1}\right) \\ f\left[t_{k+\frac{1}{2}}, t_{k+1}, t_{k+\frac{3}{2}}, t_{k+2}\right] &= \frac{4}{3h^3}\left(f_{k+2} - 3f_{k+\frac{3}{2}} + 3f_{k+1} - f_{k+\frac{1}{2}}\right) \\ f\left[t_k, t_{k+\frac{1}{2}}, t_{k+1}, t_{k+\frac{3}{2}}, t_{k+2}\right] &= \\ \frac{2}{3h^4}&\left(f_{k+2} - 4f_{k+\frac{3}{2}} + 6f_{k+1} - 4f_{k+\frac{1}{2}} + f_k\right) \\ f\left[t_{k-\frac{1}{2}}, t_k, t_{k+\frac{1}{2}}, t_{k+1}, t_{k+\frac{3}{2}}, t_{k+2}\right] &= \\ \frac{4}{15h^5}&\left(f_{k+2} - 5f_{k+\frac{3}{2}} + 10f_{k+1} - 10f_{k+\frac{1}{2}} + 5f_k - f_{k-\frac{1}{2}}\right) \end{aligned}$$

Then, the implicit formula for the second main point is derived as follows:

$$\begin{aligned} y_{k+2} = y_{k+1} + h \bigg(& \frac{7}{45} f_{k+2} + \frac{43}{60} f_{k+\frac{3}{2}} + \frac{7}{90} f_{k+1} \\ & + \frac{7}{90} f_{k+\frac{1}{2}} - \frac{1}{30} f_k + \frac{1}{180} f_{k-\frac{1}{2}} \bigg) \end{aligned} \tag{3.3.19}$$

3.3.4 Implicit 1 Step Block Hybrid Method

Off Step Point

Evaluate equation (3.3.11) up to the term

$$f\left[t_{k-\frac{5}{2}}, t_{k-2}, t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k, t_{k+\frac{1}{2}}\right] = \frac{4}{45h^6}\left(f_{k+\frac{1}{2}} - 6f_k + 15f_{k-\frac{1}{2}} - 20f_{k-1} + 15f_{k-\frac{3}{2}} - 6f_{k-2} + f_{k-\frac{5}{2}}\right).$$

Then, the implicit formula for off-step point is derived as follows:

$$\begin{aligned} y_{k+\frac{1}{2}} = y_k + h \bigg(& \frac{19087}{120960} f_{k+\frac{1}{2}} + \frac{2713}{5040} f_k - \frac{15487}{40320} f_{k-\frac{1}{2}} + \frac{293}{945} f_{k-1} \\ & - \frac{6737}{40320} f_{k-\frac{3}{2}} + \frac{263}{5040} f_{k-2} - \frac{863}{120960} f_{k-\frac{5}{2}} \bigg) \end{aligned} \tag{3.3.20}$$

Main Point

Evaluate equation (3.3.13) up to the term

$$f\left[t_{k-3}, t_{k-\frac{5}{2}}, t_{k-2}, t_{k-\frac{3}{2}}, t_{k-1}, t_{k-\frac{1}{2}}, t_k\right] = \frac{4}{45h^6}\left(f_k - 6f_{k-\frac{1}{2}} + 15f_{k-1} - 20f_{k-\frac{3}{2}} + 15f_{k-2} - 6f_{k-\frac{5}{2}} + f_{k-3}\right).$$

Then, the implicit formula for main point is derived as follows:

$$\begin{aligned} y_{k+1} = y_k + h \bigg(& \frac{1139}{7560} f_{k+1} + \frac{47}{63} f_{k+\frac{1}{2}} + \frac{11}{2520} f_k + \frac{166}{945} f_{k-\frac{1}{2}} \\ & - \frac{269}{2520} f_{k-1} + \frac{11}{315} f_{k-\frac{3}{2}} - \frac{37}{7560} f_{k-2} \bigg) \end{aligned} \tag{3.3.21}$$

3.4 Order of The Method

The accuracy for the approximation of the system mainly depends on the order of the explicit and implicit methods. According to Henrici (1962), the order of the linear multistep system can be checked. If the linear multistep method stated is order r then $C_0 = C_1 = C_2 = ... = C_r = \mathbf{0}$ and $C_{r+1} \neq \mathbf{0}$.

The difference operator (C_r) has a general form as follows:

$$C_r = \frac{1}{r!}\left[\sum_{i=1}^{k} i^r \alpha_i + \sum_{i=1}^{p} {v_i}^r \alpha_{v_i} - r\left(\sum_{i=1}^{k} i^{r-1}\beta_i + \sum_{i=1}^{p} {v_i}^{r-1}\beta_{v_i}\right)\right] \quad (3.4.1)$$

where

k is the index of the main point of α or β,

p is the index of v as the off-step point of α or β.

3.4.1 Explicit 2 Step Block Hybrid Method

Equations (3.3.2), (3.3.4), (3.3.6), and (3.3.8) are expressed in matrix form with coefficients α and β.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} Y_2 + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix} Y_0 =$$

$$h\begin{pmatrix} -\frac{3649}{1440} & \frac{4991}{1440} & -\frac{2641}{960} & \frac{4277}{2880} \\ -\frac{2143}{90} & \frac{919}{30} & -\frac{3881}{180} & \frac{344}{45} \\ -\frac{17193}{160} & \frac{21207}{160} & -\frac{27771}{320} & \frac{8253}{320} \\ -\frac{15488}{45} & \frac{18532}{45} & -\frac{3856}{15} & \frac{625}{9} \end{pmatrix}\begin{bmatrix} f_{k-\frac{3}{2}} \\ f_{k-1} \\ f_{k-\frac{1}{2}} \\ f_k \end{bmatrix} + h\begin{pmatrix} 0 & 0 & -\frac{95}{576} & \frac{959}{960} \\ 0 & 0 & -\frac{33}{20} & \frac{877}{90} \\ 0 & 0 & -\frac{2499}{320} & \frac{14469}{320} \\ 0 & 0 & -\frac{1168}{45} & \frac{2219}{15} \end{pmatrix}\begin{bmatrix} f_{k-\frac{7}{2}} \\ f_{k-3} \\ f_{k-\frac{5}{2}} \\ f_{k-2} \end{bmatrix} \quad (3.4.2)$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} Y_{\frac{11}{2}} + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix} Y_{\frac{7}{2}} =$$

$$h \begin{pmatrix} -\frac{3649}{1440} & \frac{4991}{1440} & -\frac{2641}{960} & \frac{4277}{2880} \\ -\frac{2143}{90} & \frac{919}{30} & -\frac{3881}{180} & \frac{344}{45} \\ -\frac{17193}{160} & \frac{21207}{160} & -\frac{27771}{320} & \frac{8253}{320} \\ -\frac{15488}{45} & \frac{18532}{45} & -\frac{3856}{15} & \frac{625}{9} \end{pmatrix} \begin{bmatrix} f_{k+2} \\ f_{k+\frac{5}{2}} \\ f_{k+3} \\ f_{k+\frac{7}{2}} \end{bmatrix} + h \begin{pmatrix} 0 & 0 & -\frac{95}{576} & \frac{959}{960} \\ 0 & 0 & -\frac{33}{20} & \frac{877}{90} \\ 0 & 0 & -\frac{2499}{320} & \frac{14469}{320} \\ 0 & 0 & -\frac{1168}{45} & \frac{2219}{15} \end{pmatrix} \begin{bmatrix} f_k \\ f_{k+\frac{1}{2}} \\ f_{k+1} \\ f_{k+\frac{3}{2}} \end{bmatrix}$$

where

$$\alpha_0 = \alpha_{\frac{1}{2}} = \alpha_1 = \alpha_{\frac{3}{2}} = \alpha_2 = \alpha_{\frac{5}{2}} = \alpha_3 = \mathbf{0},$$

$$\alpha_{\frac{7}{2}} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \alpha_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \alpha_{\frac{9}{2}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \alpha_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \alpha_{\frac{11}{2}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\beta_0 = \beta_{\frac{1}{2}} = \mathbf{0}, \beta_1 = \begin{bmatrix} -\frac{95}{576} \\ -\frac{33}{20} \\ -\frac{-499}{320} \\ -\frac{1168}{45} \end{bmatrix}, \beta_{\frac{3}{2}} = \begin{bmatrix} \frac{959}{960} \\ \frac{877}{90} \\ \frac{14469}{320} \\ \frac{2219}{15} \end{bmatrix}, \beta_2 = \begin{bmatrix} -\frac{3649}{1440} \\ -\frac{2143}{90} \\ -\frac{17193}{160} \\ -\frac{15488}{45} \end{bmatrix}, \beta_{\frac{5}{2}} = \begin{bmatrix} \frac{4991}{1440} \\ \frac{919}{30} \\ \frac{21207}{160} \\ \frac{18532}{45} \end{bmatrix},$$

$$\beta_3 = \begin{bmatrix} -\frac{2641}{960} \\ -\frac{3881}{180} \\ -\frac{27771}{320} \\ -\frac{3856}{15} \end{bmatrix}, \beta_{\frac{7}{2}} = \begin{bmatrix} \frac{4277}{2880} \\ \frac{344}{45} \\ \frac{8253}{320} \\ \frac{625}{9} \end{bmatrix}, \beta_4 = \beta_{\frac{9}{2}} = \beta_5 = \beta_{\frac{11}{2}} = \mathbf{0}$$

and

$$Y_0 = \begin{bmatrix} y_{k-\frac{3}{2}} \\ y_{k-1} \\ y_{k-\frac{1}{2}} \\ y_k \end{bmatrix}, Y_2 = \begin{bmatrix} y_{k+\frac{1}{2}} \\ y_{k+1} \\ y_{k+\frac{3}{2}} \\ y_{k+2} \end{bmatrix}, Y_{\frac{7}{2}} = \begin{bmatrix} y_{k+2} \\ y_{k+\frac{5}{2}} \\ y_{k+3} \\ y_{k+\frac{7}{2}} \end{bmatrix}, Y_{\frac{11}{2}} = \begin{bmatrix} y_{k+4} \\ y_{k+\frac{9}{2}} \\ y_{k+5} \\ y_{k+\frac{11}{2}} \end{bmatrix}$$

Calculate equation (3.4.1) by $k = 5$ and $p = 6$ with α and β values listed above, and let $v_1 = \frac{1}{2}, v_2 = \frac{3}{2}, v_3 = \frac{5}{2}, v_4 = \frac{7}{2}, v_5 = \frac{9}{2}, v_6 = \frac{11}{2}$. The corresponding difference operators are $C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = \mathbf{0}$ and $C_7 = \begin{bmatrix} \frac{19087}{7741440} & \frac{13613}{483840} & \frac{43021}{286720} & \frac{8399}{15120} \end{bmatrix}^T$. Therefore, the explicit 2 step block-hybrid method (3.3.2), (3.3.4), (3.3.6), and (3.3.8) is verified to be an order six system.

3.4.2 Implicit 2 Step Block Hybrid Method

Equations (3.3.12), (3.3.14), (3.3.16), and (3.3.19) are expressed in matrix form with coefficients α and β.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} Y_2 + \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} Y_1 =$$

$$h \begin{pmatrix} \frac{95}{576} & 0 & 0 & 0 \\ \frac{43}{60} & \frac{7}{45} & 0 & 0 \\ -\frac{133}{480} & \frac{1427}{2880} & \frac{95}{576} & 0 \\ \frac{7}{90} & \frac{7}{90} & \frac{43}{60} & \frac{7}{45} \end{pmatrix} \begin{bmatrix} f_{k+\frac{1}{2}} \\ f_{k+1} \\ f_{k+\frac{3}{2}} \\ f_{k+2} \end{bmatrix} + h \begin{pmatrix} -\frac{173}{2880} & \frac{241}{1440} & -\frac{133}{480} & \frac{1427}{2880} \\ \frac{1}{180} & -\frac{1}{30} & \frac{7}{90} & \frac{7}{90} \\ 0 & \frac{3}{320} & -\frac{173}{2880} & \frac{241}{1440} \\ 0 & 0 & \frac{1}{180} & -\frac{1}{30} \end{pmatrix} \begin{bmatrix} f_{k-\frac{3}{2}} \\ f_{k-1} \\ f_{k-\frac{1}{2}} \\ f_k \end{bmatrix}$$

$$+ h \begin{pmatrix} 0 & 0 & 0 & \frac{3}{320} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} f_{k-\frac{7}{2}} \\ f_{k-3} \\ f_{k-\frac{5}{2}} \\ f_{k-2} \end{bmatrix} \tag{3.4.3}$$

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&-1&1&0\\0&-1&0&1\end{pmatrix}Y_{\frac{11}{2}}+\begin{pmatrix}0&0&0&-1\\0&0&0&-1\\0&0&0&0\\0&0&0&0\end{pmatrix}Y_{\frac{7}{2}}=$$

$$h\begin{pmatrix}\frac{95}{576}&0&0&0\\\frac{43}{60}&\frac{7}{45}&0&0\\-\frac{133}{480}&\frac{1427}{2880}&\frac{95}{576}&0\\\frac{7}{90}&\frac{7}{90}&\frac{43}{60}&\frac{7}{45}\end{pmatrix}\begin{bmatrix}f_{k+4}\\f_{k+\frac{9}{2}}\\f_{k+5}\\f_{k+\frac{11}{2}}\end{bmatrix}+h\begin{pmatrix}-\frac{173}{2880}&\frac{241}{1440}&-\frac{133}{480}&\frac{1427}{2880}\\\frac{1}{180}&-\frac{1}{30}&\frac{7}{90}&\frac{7}{90}\\0&\frac{3}{320}&-\frac{173}{2880}&\frac{241}{1440}\\0&0&\frac{1}{180}&-\frac{1}{30}\end{pmatrix}\begin{bmatrix}f_{k+2}\\f_{k+\frac{5}{2}}\\f_{k+3}\\f_{k+\frac{7}{2}}\end{bmatrix}$$

$$+h\begin{pmatrix}0&0&0&\frac{3}{320}\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{bmatrix}f_{k}\\f_{k+\frac{1}{2}}\\f_{k+1}\\f_{k+\frac{3}{2}}\end{bmatrix}$$

where

$$\alpha_0=\alpha_{\frac{1}{2}}=\alpha_1=\alpha_{\frac{3}{2}}=\alpha_2=\alpha_{\frac{5}{2}}=\alpha_3=\mathbf{0},$$

$$\alpha_{\frac{7}{2}}=\begin{bmatrix}-1\\-1\\0\\0\end{bmatrix},\alpha_4=\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\alpha_{\frac{9}{2}}=\begin{bmatrix}0\\1\\-1\\-1\end{bmatrix},\alpha_5=\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\alpha_{\frac{11}{2}}=\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$\beta_0=\beta_{\frac{1}{2}}=\beta_1=\mathbf{0},\beta_{\frac{3}{2}}=\begin{bmatrix}\frac{3}{320}\\0\\0\\0\end{bmatrix},\beta_2=\begin{bmatrix}-\frac{173}{2880}\\\frac{1}{180}\\0\\0\end{bmatrix},\beta_{\frac{5}{2}}=\begin{bmatrix}\frac{241}{1440}\\-\frac{1}{30}\\\frac{3}{320}\\0\end{bmatrix},\beta_3=\begin{bmatrix}-\frac{133}{480}\\\frac{7}{90}\\-\frac{173}{2880}\\\frac{1}{180}\end{bmatrix},$$

$$\beta_{\frac{7}{2}}=\begin{bmatrix}\frac{1427}{2880}\\\frac{7}{90}\\\frac{241}{1440}\\-\frac{1}{30}\end{bmatrix},\beta_4=\begin{bmatrix}\frac{95}{576}\\\frac{43}{60}\\-\frac{133}{480}\\\frac{7}{90}\end{bmatrix},\beta_{\frac{9}{2}}=\begin{bmatrix}0\\\frac{7}{45}\\\frac{1427}{2880}\\\frac{7}{90}\end{bmatrix},\beta_5=\begin{bmatrix}0\\0\\\frac{95}{576}\\\frac{43}{60}\end{bmatrix},\beta_{\frac{11}{2}}=\begin{bmatrix}0\\0\\0\\\frac{7}{45}\end{bmatrix}$$

and

$$Y_0 = \begin{bmatrix} y_{k-\frac{3}{2}} \\ y_{k-1} \\ y_{k-\frac{1}{2}} \\ y_k \end{bmatrix}, Y_2 = \begin{bmatrix} y_{k+\frac{1}{2}} \\ y_{k+1} \\ y_{k+\frac{3}{2}} \\ y_{k+2} \end{bmatrix}, Y_{\frac{7}{2}} = \begin{bmatrix} y_{k+2} \\ y_{k+\frac{5}{2}} \\ y_{k+3} \\ y_{k+\frac{7}{2}} \end{bmatrix}, Y_{\frac{11}{2}} = \begin{bmatrix} y_{k+4} \\ y_{k+\frac{9}{2}} \\ y_{k+5} \\ y_{k+\frac{11}{2}} \end{bmatrix}$$

Calculate equation (3.4.1) by $k = 5$ and $p = 6$ with α and β values listed above, and let $v_1 = \frac{1}{2}, v_2 = \frac{3}{2}, v_3 = \frac{5}{2}, v_4 = \frac{7}{2}, v_5 = \frac{9}{2}, v_6 = \frac{11}{2}$. The corresponding difference operators are $C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = \mathbf{0}$ and $C_7 = \begin{bmatrix} -\frac{863}{7741440} & -\frac{37}{483840} & -\frac{863}{7741440} & -\frac{37}{483840} \end{bmatrix}^T$. Therefore, the explicit 2 step block-hybrid method (3.3.12), (3.3.14), (3.3.16), and (3.3.19) is verified to be an order six system.

3.4.3 Explicit 1 Step Block Hybrid Method

Equations (3.3.9) and (3.3.10) are expressed in matrix form with coefficients α and β.

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Y_1 + \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix} Y_0 = \\
&h \begin{pmatrix} -\frac{18637}{5040} & \frac{198721}{120960} \\ -\frac{2039}{63} & \frac{14281}{1512} \end{pmatrix} \begin{bmatrix} f_{k-\frac{1}{2}} \\ f_k \end{bmatrix} + h \begin{pmatrix} -\frac{5377}{945} & \frac{235183}{40320} \\ -\frac{56534}{945} & \frac{145261}{2520} \end{pmatrix} \begin{bmatrix} f_{k-\frac{3}{2}} \\ f_{k-1} \end{bmatrix} \\
&+h \begin{pmatrix} -\frac{5603}{5040} & \frac{135713}{40320} \\ -\frac{3923}{315} & \frac{92621}{2520} \end{pmatrix} \begin{bmatrix} f_{k-\frac{5}{2}} \\ f_{k-2} \end{bmatrix} + h \begin{pmatrix} 0 & \frac{19087}{120960} \\ 0 & \frac{13613}{7560} \end{pmatrix} \begin{bmatrix} f_{k-\frac{7}{2}} \\ f_{k-3} \end{bmatrix}
\end{aligned} \quad (3.4.4)$$

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Y_{\frac{9}{2}} + \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix} Y_{\frac{7}{2}} = \\
&h \begin{pmatrix} -\frac{18637}{5040} & \frac{198721}{120960} \\ -\frac{2039}{63} & \frac{14281}{1512} \end{pmatrix} \begin{bmatrix} f_{k+3} \\ f_{k+\frac{7}{2}} \end{bmatrix} + h \begin{pmatrix} -\frac{5377}{945} & \frac{235183}{40320} \\ -\frac{56534}{945} & \frac{145261}{2520} \end{pmatrix} \begin{bmatrix} f_{k+2} \\ f_{k+\frac{5}{2}} \end{bmatrix} \\
&+h \begin{pmatrix} -\frac{5603}{5040} & \frac{135713}{40320} \\ -\frac{3923}{315} & \frac{92621}{2520} \end{pmatrix} \begin{bmatrix} f_{k+1} \\ f_{k+\frac{3}{2}} \end{bmatrix} + h \begin{pmatrix} 0 & \frac{19087}{120960} \\ 0 & \frac{13613}{7560} \end{pmatrix} \begin{bmatrix} f_k \\ f_{k+\frac{1}{2}} \end{bmatrix}
\end{aligned}$$

where

$$\alpha_0 = \alpha_{\frac{1}{2}} = \alpha_1 = \alpha_{\frac{3}{2}} = \alpha_2 = \alpha_{\frac{5}{2}} = \alpha_3 = \mathbf{0}, \alpha_{\frac{7}{2}} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \alpha_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \alpha_{\frac{9}{2}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\beta_0 = \mathbf{0}, \beta_{\frac{1}{2}} = \begin{bmatrix} \frac{19087}{120960} \\ \frac{13613}{7560} \end{bmatrix}, \beta_1 = \begin{bmatrix} -\frac{5603}{5040} \\ -\frac{3923}{315} \end{bmatrix}, \beta_{\frac{3}{2}} = \begin{bmatrix} \frac{135713}{40320} \\ \frac{92621}{2520} \end{bmatrix}, \beta_2 = \begin{bmatrix} -\frac{5377}{945} \\ -\frac{56534}{945} \end{bmatrix},$$

$$\beta_{\frac{5}{2}} = \begin{bmatrix} \frac{235183}{40320} \\ \frac{145261}{2520} \end{bmatrix}, \beta_3 = \begin{bmatrix} -\frac{18637}{5040} \\ -\frac{2039}{63} \end{bmatrix}, \beta_{\frac{7}{2}} = \begin{bmatrix} \frac{198721}{120960} \\ \frac{14281}{1512} \end{bmatrix}, \beta_4 = \beta_{\frac{9}{2}} = \mathbf{0}$$

and

$$Y_0 = \begin{bmatrix} y_{k-\frac{1}{2}} \\ y_k \end{bmatrix}, Y_1 = \begin{bmatrix} y_{k+\frac{1}{2}} \\ y_{k+1} \end{bmatrix}, Y_{\frac{7}{2}} = \begin{bmatrix} y_{k+3} \\ y_{k+\frac{7}{2}} \end{bmatrix}, Y_{\frac{9}{2}} = \begin{bmatrix} y_{k+4} \\ y_{k+\frac{9}{2}} \end{bmatrix}$$

Calculate equation (3.4.1) by $k = 4$ and $p = 5$ with α and β values listed above, and let $v_1 = \frac{1}{2}, v_2 = \frac{3}{2}, v_3 = \frac{5}{2}, v_4 = \frac{7}{2}, v_5 = \frac{9}{2}$. The corresponding difference operators are $C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = C_7 = \mathbf{0}$ and $C_8 = \begin{bmatrix} \frac{5257}{4423680} & \frac{23}{1512} \end{bmatrix}^T$. Therefore, the explicit block-hybrid method (3.3.9) and (3.3.10) is verified to be an order seven system.

3.4.4 Implicit Block Hybrid Method

Equations (3.3.20) and (3.3.21) are expressed in matrix form with coefficients α and β.

$$\begin{aligned} &\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Y_1 + \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix} Y_0 = \\ &+h \begin{pmatrix} \frac{19087}{120960} & 0 \\ \frac{47}{63} & \frac{1139}{7560} \end{pmatrix} \begin{bmatrix} f_{k+\frac{1}{2}} \\ f_{k+1} \end{bmatrix} + h \begin{pmatrix} -\frac{15487}{40320} & \frac{2713}{5040} \\ \frac{166}{945} & \frac{11}{2520} \end{pmatrix} \begin{bmatrix} f_{k-\frac{1}{2}} \\ f_k \end{bmatrix} \\ &+h \begin{pmatrix} -\frac{6737}{40320} & \frac{293}{945} \\ \frac{11}{315} & -\frac{269}{2520} \end{pmatrix} \begin{bmatrix} f_{k-\frac{3}{2}} \\ f_{k-1} \end{bmatrix} + h \begin{pmatrix} -\frac{863}{120960} & \frac{263}{5040} \\ 0 & -\frac{37}{7560} \end{pmatrix} \begin{bmatrix} f_{k-\frac{5}{2}} \\ f_{k-2} \end{bmatrix} \end{aligned} \tag{3.4.5}$$

$$\begin{aligned} &\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Y_{\frac{9}{2}} + \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix} Y_{\frac{7}{2}} = \\ &+h \begin{pmatrix} \frac{19087}{120960} & 0 \\ \frac{47}{63} & \frac{1139}{7560} \end{pmatrix} \begin{bmatrix} f_{k+3} \\ f_{k+\frac{7}{2}} \end{bmatrix} + h \begin{pmatrix} -\frac{15487}{40320} & \frac{2713}{5040} \\ \frac{166}{945} & \frac{11}{2520} \end{pmatrix} \begin{bmatrix} f_{k+2} \\ f_{k+\frac{5}{2}} \end{bmatrix} \\ &+h \begin{pmatrix} -\frac{6737}{40320} & \frac{293}{945} \\ \frac{11}{315} & -\frac{269}{2520} \end{pmatrix} \begin{bmatrix} f_{k+1} \\ f_{k+\frac{3}{2}} \end{bmatrix} + h \begin{pmatrix} -\frac{863}{120960} & \frac{263}{5040} \\ 0 & -\frac{37}{7560} \end{pmatrix} \begin{bmatrix} f_k \\ f_{k+\frac{1}{2}} \end{bmatrix} \end{aligned}$$

where

$$\alpha_0 = \alpha_{\frac{1}{2}} = \alpha_1 = \alpha_{\frac{3}{2}} = \alpha_2 = \mathbf{0}, \alpha_{\frac{5}{2}} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \alpha_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \alpha_{\frac{7}{2}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\beta_0 = \begin{bmatrix} -\frac{863}{120960} \\ 0 \end{bmatrix}, \beta_{\frac{1}{2}} = \begin{bmatrix} \frac{263}{5040} \\ -\frac{37}{7560} \end{bmatrix}, \beta_1 = \begin{bmatrix} -\frac{6737}{40320} \\ \frac{11}{315} \end{bmatrix}, \beta_{\frac{3}{2}} = \begin{bmatrix} \frac{293}{945} \\ -\frac{269}{2520} \end{bmatrix},$$
$$\beta_2 = \begin{bmatrix} -\frac{15487}{40320} \\ \frac{166}{945} \end{bmatrix}, \beta_{\frac{5}{2}} = \begin{bmatrix} \frac{2713}{5040} \\ \frac{11}{2520} \end{bmatrix}, \beta_3 = \begin{bmatrix} \frac{19087}{120960} = \\ \frac{47}{63} \end{bmatrix}, \beta_{\frac{7}{2}} = \begin{bmatrix} 0 \\ \frac{1139}{7560} \end{bmatrix}$$

and

$$Y_0 = \begin{bmatrix} y_{k-\frac{1}{2}} \\ y_k \end{bmatrix}, Y_1 = \begin{bmatrix} y_{k+\frac{1}{2}} \\ y_{k+1} \end{bmatrix}, Y_{\frac{7}{2}} = \begin{bmatrix} y_{k+3} \\ y_{k+\frac{7}{2}} \end{bmatrix}, Y_{\frac{9}{2}} = \begin{bmatrix} y_{k+4} \\ y_{k+\frac{9}{2}} \end{bmatrix}$$

Calculate equation (3.4.1) by $k = 3$ and $p = 4$ with α and β values listed above, and let $v_1 = \frac{1}{2}, v_2 = \frac{3}{2}, v_3 = \frac{5}{2}, v_4 = \frac{7}{2}$. The corresponding difference operators are $C_0 = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = C_7 = \mathbf{0}$ and $C_8 = \begin{bmatrix} -\frac{275}{6193152} & -\frac{1}{30240} \end{bmatrix}^T$. Therefore, the implicit block-hybrid method (3.3.20) and (3.3.21) is verified to be an order seven system.

3.5 The Method's Stability

This section discusses the stability analysis of the proposed block-hybrid method by finding the method's Q-stability applying the following test equation:

$$y'(t) = \mu y(t-\tau). \qquad (3.5.1)$$

The test equation is substituted into the method and transform into matrix finite difference form (MFDF) to find the Q-stability polynomial. This polynomial is solved for H_2, where $H_2 = h\mu$. Then, the Q-stability region is located by applying the boundary locus technique.

3.5.1 Explicit 2 Step Block Hybrid Method of Order Six

The equations of explicit 2 step block-hybrid method (3.3.2), (3.3.4),(3.3.6), and (3.3.8) are transformed into MFDF (3.4.2).

$$P_2 Y_{K+2} + P_1 Y_{K+1} = h(R_1 F_{K+1} + R_0 F_K) \qquad (3.5.2)$$

where

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, P_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

$$R_1 = \begin{pmatrix} -\frac{3649}{1440} & \frac{4991}{1440} & -\frac{2641}{960} & \frac{4277}{2880} \\ -\frac{2143}{90} & \frac{919}{30} & -\frac{3881}{180} & \frac{344}{45} \\ -\frac{17193}{160} & \frac{21207}{160} & -\frac{27771}{320} & \frac{8253}{320} \\ -\frac{15488}{45} & \frac{18532}{45} & -\frac{3856}{15} & \frac{625}{9} \end{pmatrix}, R_0 = \begin{pmatrix} 0 & 0 & -\frac{95}{576} & \frac{959}{960} \\ 0 & 0 & -\frac{33}{20} & \frac{877}{90} \\ 0 & 0 & -\frac{2499}{320} & \frac{14469}{320} \\ 0 & 0 & -\frac{1168}{45} & \frac{2219}{15} \end{pmatrix},$$

$$Y_{K+2} = \begin{bmatrix} y_{k+\frac{1}{2}} \\ y_{k+1} \\ y_{k+\frac{3}{2}} \\ y_{k+2} \end{bmatrix}, Y_{K+1} = \begin{bmatrix} y_{k-\frac{3}{2}} \\ y_{k-1} \\ y_{k-\frac{1}{2}} \\ y_k \end{bmatrix}, F_{K+1} = \begin{bmatrix} f_{k-\frac{3}{2}} \\ f_{k-1} \\ f_{k-\frac{1}{2}} \\ f_k \end{bmatrix}, F_K = \begin{bmatrix} f_{k-\frac{7}{2}} \\ f_{k-3} \\ f_{k-\frac{5}{2}} \\ f_{k-2} \end{bmatrix}$$

The test equation (3.5.1) is applied on (3.5.2).

$$P_2 Y_{K+2} + P_1 Y_{K+1} - H_2(R_1 Y_{K+1-d} + R_0 Y_{K-d}) = 0$$

Get the Q-stability polynomial by finding the determinant.

$$\left| P_2 \zeta^{d+2} + P_1 \zeta^{d+1} - H_2(R_1 \zeta^1 + R_0) \right| = 0$$

With $d = 1$, the polynomial is derived as follows.

$$\begin{aligned}
&\zeta^{12} - \zeta^{11} - \frac{3443\zeta^{10} H_2}{320} - \frac{13517\zeta^9 H_2}{1440} + \zeta^8 \left(\frac{52261 H_2}{2880} + \frac{795397 {H_2}^2}{17280} \right) \\
&- \frac{408841\zeta^7 {H_2}^2}{8640} + \zeta^6 \left(\frac{917581 {H_2}^2}{17280} + \frac{217393 {H_2}^3}{17280} \right) + \frac{106699\zeta^5 {H_2}^3}{1440} \\
&+ \zeta^4 \left(\frac{120611 {H_2}^3}{17280} + \frac{1690097 {H_2}^4}{28800} \right) + \frac{44773\zeta^3 {H_2}^4}{14400} + \frac{197\zeta^2 {H_2}^4}{28800} = 0
\end{aligned} \tag{3.5.3}$$

The Q-stability polynomial (3.5.3) is solved with respect to H_2. Figure 3.1 shows the Q-stability regions of the explicit 2 step block-hybrid method after applying the boundary locus technique and sketched on the complex H_2-plane. Any points that lie inside the boundary represent the Q-stability region.

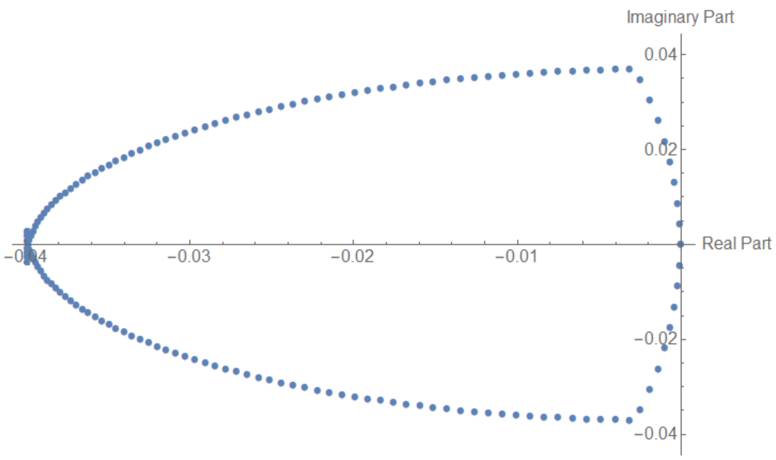

Figure 3.1: Q-Stability area of explicit 2 step block-hybrid method (3.3.2), (3.3.4), (3.3.6), and (3.3.8) of order six.

3.5.2 Implicit 2 Step Block Hybrid Method of Order Six

The equations of implicit 2 step block-hybrid method (3.3.12), (3.3.14),(3.3.16), and (3.3.19) are transformed into MFDF (3.4.3).

$$P_2 Y_{K+2} + P_1 Y_{K+1} = h(R_2 F_{K+2} + R_1 F_{K+1} + R_0 F_K) \qquad (3.5.4)$$

where

$$P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, P_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} \frac{95}{576} & 0 & 0 & 0 \\ \frac{43}{60} & \frac{7}{45} & 0 & 0 \\ -\frac{133}{480} & \frac{1427}{2880} & \frac{95}{576} & 0 \\ \frac{7}{90} & \frac{7}{90} & \frac{43}{60} & \frac{7}{45} \end{pmatrix},$$

$$R_1 = \begin{pmatrix} -\frac{173}{2880} & \frac{241}{1440} & -\frac{133}{480} & \frac{1427}{2880} \\ \frac{1}{180} & -\frac{1}{30} & \frac{7}{90} & \frac{7}{90} \\ 0 & \frac{3}{320} & -\frac{173}{2880} & \frac{241}{1440} \\ 0 & 0 & \frac{1}{180} & -\frac{1}{30} \end{pmatrix}, R_0 = \begin{pmatrix} 0 & 0 & 0 & \frac{3}{320} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, Y_{K+2} = \begin{bmatrix} y_{k+\frac{1}{2}} \\ y_{k+1} \\ y_{k+\frac{3}{2}} \\ y_{k+2} \end{bmatrix},$$

$$Y_{K+1} = \begin{bmatrix} y_{k-\frac{3}{2}} \\ y_{k-1} \\ y_{k-\frac{1}{2}} \\ y_k \end{bmatrix}, F_{K+2} = \begin{bmatrix} f_{k+\frac{1}{2}} \\ f_{k+1} \\ f_{k+\frac{3}{2}} \\ f_{k+2} \end{bmatrix}, F_{K+1} = \begin{bmatrix} f_{k-\frac{3}{2}} \\ f_{k-1} \\ f_{k-\frac{1}{2}} \\ f_k \end{bmatrix}, F_K = \begin{bmatrix} f_{k-\frac{7}{2}} \\ f_{k-3} \\ f_{k-\frac{5}{2}} \\ f_{k-2} \end{bmatrix}$$

The test equation (3.5.1) is applied on (3.5.4).

$$P_2 Y_{K+2} + P_1 Y_{K+1} - H_2(R_2 Y_{K+2-d} + R_1 Y_{K+1-d} + R_0 Y_{K-d}) = 0$$

Get the Q-stability polynomial by finding the determinant.

$$\left| P_2 \zeta^{d+2} + P_1 \zeta^{d+1} - H_2(R_2 \zeta^2 + R_1 \zeta^1 + R_0) \right| = 0$$

With $d = 1$, the polynomial is derived as follows.

$$\zeta^{12} + \zeta^{11}\left(-1 - \frac{923H_2}{1440}\right) + \zeta^{10}\left(-\frac{221H_2}{180} + \frac{425843{H_2}^2}{2764800}\right)$$
$$+ \zeta^{9}\left(-\frac{21H_2}{160} - \frac{617717{H_2}^2}{552960} - \frac{122759{H_2}^3}{7464960}\right)$$
$$+ \zeta^{8}\left(-\frac{36103{H_2}^2}{2764800} - \frac{119545249{H_2}^3}{746496000} + \frac{17689{H_2}^4}{26873856}\right)$$
$$+ \zeta^{7}\left(-\frac{271{H_2}^2}{61440} + \frac{39391{H_2}^3}{1024000} - \frac{42598034929{H_2}^4}{268738560000}\right)$$
$$+ \zeta^{6}\left(-\frac{639209{H_2}^3}{248832000} + \frac{541460293{H_2}^4}{22394880000}\right) + \zeta^{5}\left(-\frac{7{H_2}^3}{1024000} - \frac{66546527{H_2}^4}{134369280000}\right)$$
$$- \frac{117521\zeta^4{H_2}^4}{67184640000} - \frac{\zeta^3{H_2}^4}{368640000} = 0 \tag{3.5.5}$$

The Q-stability polynomial (3.5.5) is solved with respect to H_2. Figure 3.2 shows the Q-stability regions of the implicit 2 step block-hybrid method after applying the boundary locus technique and sketched on the complex H_2-plane. Any points that lie inside the boundary represent the Q-stability region.

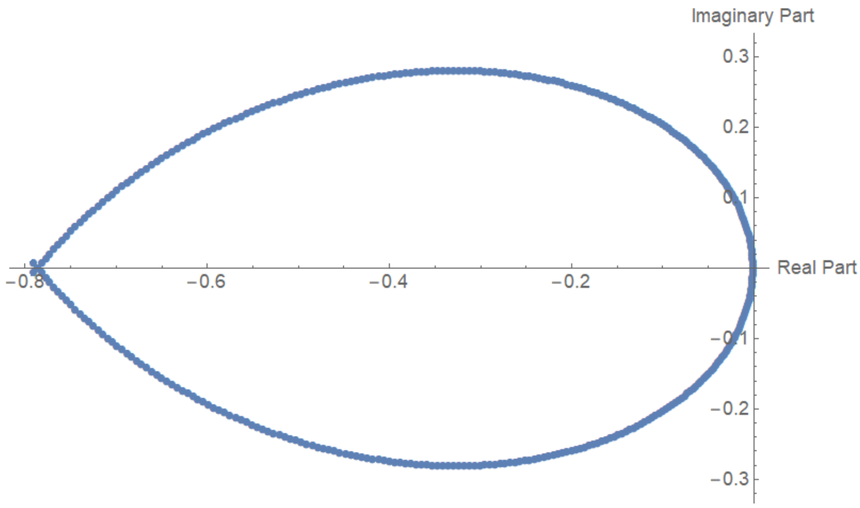

Figure 3.2: Q-Stability area of implicit 2 step block-hybrid method (3.3.12), (3.3.14), (3.3.16), and (3.3.19) of order six.

3.5.3 Explicit 1 Step Seven Block Hybrid Method of Order Seven

The equations of explicit 1 step block-hybrid method (3.3.9) and (3.3.10) are transformed into MFDF (3.4.4).

$$P_4Y_{K+4} + P_3Y_{K+3} = h(R_3F_{K+3} + R_2F_{K+2} + R_1F_{K+1} + R_0F_K) \qquad (3.5.6)$$

where

$$P_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, P_3 = \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}, R_3 = \begin{pmatrix} -\frac{18637}{5040} & \frac{198721}{120960} \\ -\frac{2039}{63} & \frac{14281}{1512} \end{pmatrix},$$
$$R_2 = \begin{pmatrix} -\frac{5377}{945} & \frac{235183}{40320} \\ -\frac{56534}{945} & \frac{145261}{2520} \end{pmatrix}, R_1 = \begin{pmatrix} -\frac{5603}{5040} & \frac{135713}{40320} \\ -\frac{3923}{315} & \frac{92621}{2520} \end{pmatrix}, R_0 = \begin{pmatrix} 0 & \frac{19087}{120960} \\ 0 & \frac{13613}{7560} \end{pmatrix}$$
$$Y_{K+4} = \begin{bmatrix} y_{k+\frac{1}{2}} \\ y_{k+1} \end{bmatrix}, Y_{K+3} = \begin{bmatrix} y_{k-\frac{1}{2}} \\ y_k \end{bmatrix}, F_{K+3} = \begin{bmatrix} f_{k-\frac{1}{2}} \\ f_k \end{bmatrix},$$
$$F_{K+2} = \begin{bmatrix} f_{k-\frac{3}{2}} \\ f_{k-1} \end{bmatrix}, F_{K+1} = \begin{bmatrix} f_{k-\frac{5}{2}} \\ f_{k-2} \end{bmatrix}, F_K = \begin{bmatrix} f_{k-\frac{7}{2}} \\ f_{k-3} \end{bmatrix}$$

The test equation (3.5.1) is applied on (3.5.6).

$$P_4Y_{K+4} + P_3Y_{K+3} - H_2(R_3Y_{K+3-d} + R_2Y_{K+2-d} + R_1Y_{K+1-d} + R_0Y_{K-d}) = 0$$

Get the Q-stability polynomial by finding the determinant.

$$\left| P_4\zeta^{d+4} + P_3\zeta^{d+3} - H_2(R_3\zeta^3 + R_2\zeta^2 + R_1\zeta^1 + R_0) \right| = 0$$

With $d = 1$, the polynomial is derived as follows.

$$\zeta^{10} - \zeta^9 - \frac{86899\zeta^8 H_2}{15120} - \frac{70417\zeta^7 H_2}{3024} + \zeta^6\left(\frac{55919H_2}{3024} + \frac{9931223{H_2}^2}{544320}\right)$$
$$+ \zeta^5\left(\frac{144269H_2}{15120} + \frac{54893593{H_2}^2}{2721600}\right) + \frac{5374679\zeta^4{H_2}^2}{1360800}$$
$$- \frac{1032431\zeta^3{H_2}^2}{1360800} + \frac{688823\zeta^2{H_2}^2}{2721600} - \frac{99667\zeta{H_2}^2}{2721600} \qquad (3.5.7)$$

The Q-stability polynomial (3.5.7) is solved with respect to H_2. Figure 3.3 shows the Q-stability regions of the explicit block-hybrid method after applying the boundary locus technique and sketched on the complex H_2-plane. Any points that lie inside the boundary represent the Q-stability region.

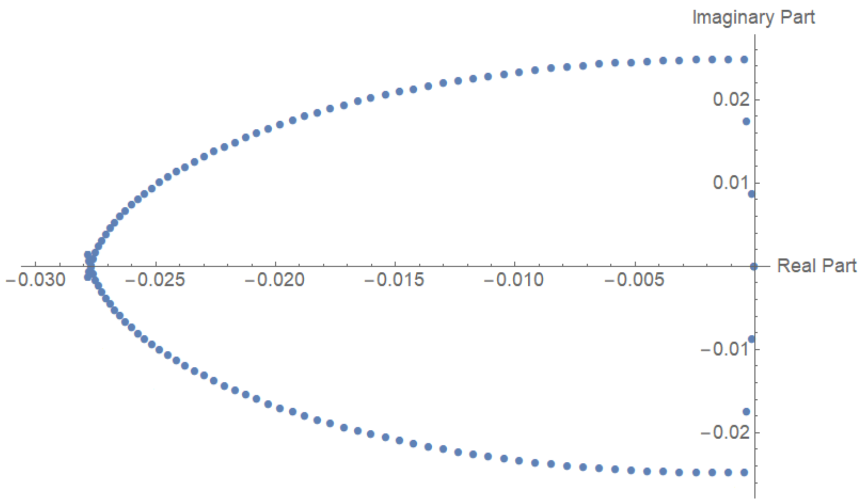

Figure 3.3: Q-Stability area of explicit 1 step block-hybrid method (3.3.9) and (3.3.10) of order seven.

3.5.4 Implicit 1 Step Block Hybrid Method

The equations of implicit 1 step block-hybrid method (3.3.20) and (3.3.21) are transformed into MFDF (3.4.5).

$$P_3Y_{K+3} + P_2Y_{K+2} = h(R_3F_{K+3} + R_2F_{K+2} + R_1F_{K+1} + R_0F_K) \tag{3.5.8}$$

where

$$P_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, P_2 = \begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}, R_3 = \begin{pmatrix} \frac{19087}{120960} & 0 \\ \frac{47}{63} & \frac{1139}{7560} \end{pmatrix}$$
$$R_2 = \begin{pmatrix} -\frac{15487}{40320} & \frac{2713}{5040} \\ \frac{166}{945} & \frac{11}{2520} \end{pmatrix}, R_1 = \begin{pmatrix} -\frac{6737}{40320} & \frac{293}{945} \\ \frac{11}{315} & -\frac{269}{2520} \end{pmatrix}, R_0 = \begin{pmatrix} -\frac{863}{120960} & \frac{263}{5040} \\ 0 & -\frac{37}{7560} \end{pmatrix}$$
$$Y_{K+3} = \begin{bmatrix} y_{k+\frac{1}{2}} \\ y_{k+1} \end{bmatrix}, Y_{K+2} = \begin{bmatrix} y_{k-\frac{1}{2}} \\ y_k \end{bmatrix}, F_{K+3} = \begin{bmatrix} f_{k+\frac{1}{2}} \\ f_{k+1} \end{bmatrix}, F_{K+2} = \begin{bmatrix} f_{k-\frac{1}{2}} \\ f_k \end{bmatrix},$$
$$F_{K+1} = \begin{bmatrix} f_{k-\frac{3}{2}} \\ f_{k-1} \end{bmatrix}, F_K = \begin{bmatrix} f_{k-\frac{5}{2}} \\ f_{k-2} \end{bmatrix}$$

The test equation (3.5.1) is applied on (3.5.6).

$$P_3Y_{K+3} + P_2Y_{K+2} - H_2(R_3Y_{K+3-d} + R_2Y_{K+2-d} + R_1Y_{K+1-d} + R_0Y_{K-d}) = 0$$

Get the Q-stability polynomial by finding the determinant.

$$\left| P_3\zeta^{d+3} + P_2\zeta^{d+2} - H_2(R_3\zeta^3 + R_2\zeta^2 + R_1\zeta^1 + R_0) \right| = 0$$

With $d = 1$, the polynomial is derived as follows.

$$\begin{aligned} &\zeta^8 + \zeta^7\left(-1 - \frac{12437H_2}{40320}\right) + \zeta^6\left(-\frac{1261H_2}{6048} + \frac{21740093{H_2}^2}{914457600}\right) \\ &+ \zeta^5\left(-\frac{17293H_2}{60480} - \frac{5826679{H_2}^2}{12700800}\right) + \zeta^4\left(-\frac{383H_2}{2016} - \frac{37549783{H_2}^2}{101606400}\right) \\ &+ \zeta^3\left(-\frac{863H_2}{120960} - \frac{2108021{H_2}^2}{28576800}\right) - \frac{31393\zeta^2{H_2}^2}{101606400} - \frac{617\zeta{H_2}^2}{2540160} + \frac{31931{H_2}^2}{914457600} \end{aligned} \tag{3.5.9}$$

The Q-stability polynomial (3.5.9) is solved with respect to H_2. Figure 3.4 shows the Q-stability regions of the implicit 1 step block-hybrid method after applying the boundary locus technique and sketched on the complex H_2-plane. Any points that lie inside the boundary represent the Q-stability region.

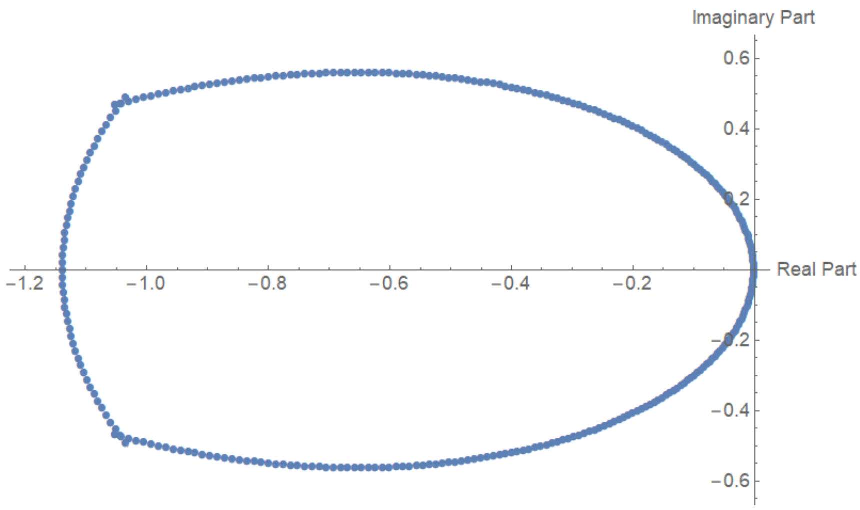

Figure 3.4: Q-Stability area of implicit 1 step block-hybrid method (3.3.20) and (3.3.21) of order seven.

3.5.5 Q-Stability Analysis

For the explicit method, the region of 2 step block-hybrid method is greater than 1 step block-hybrid method. For the implicit method, the region of 2 step block-hybrid method is smaller than the 1 step block-hybrid method. Furthermore, the regions of the implicit method are always greater than the corresponding explicit method.

3.6 Local Truncation Error

Local truncation error (LTE) is the error produced by the multistep method during a single iteration. LTE is used to determine the variable step size. LTE can be calculated by taking the difference between the method of r order and $r-1$ order. The general equation is as follows:

$$LTE = \left| y^{(r)} - y^{(r-1)} \right|, \tag{3.6.1}$$

where r is the order of the method.

3.6.1 LTE for 2 Step Block Hybrid Method

By considering the equation (3.3.18) up to the term

$$f\left[t_k, t_{k+\frac{1}{2}}, t_{k+1}, t_{k+\frac{3}{2}}, t_{k+2}\right].$$

Then, the implicit formula for main point of order 5 is derived as follows:

$$y_{k+2} = y_{k+1} + h\left(\frac{29}{180}f_{k+2} + \frac{31}{45}f_{k+\frac{3}{2}} + \frac{2}{15}f_{k+1} + \frac{1}{45}f_{k+\frac{1}{2}} - \frac{1}{180}f_k\right) \tag{3.6.2}$$

Therefore, by taking the difference between equations (3.3.19) and (3.6.2), the LTE for 2 step block-hybrid method is obtained.

$$\begin{aligned}
LTE &= \left|{y_{k+2}}^{c(6)} - {y_{k+2}}^{c(5)}\right| \\
&= \left|\left[y_{k+1} + h\left(\frac{7}{45}f_{k+2} + \frac{43}{60}f_{k+\frac{3}{2}} + \frac{7}{90}f_{k+1} + \frac{7}{90}f_{k+\frac{1}{2}} - \frac{1}{30}f_k + \frac{1}{180}f_{k-\frac{1}{2}}\right)\right]\right. \\
&\quad \left.- \left[y_{k+1} + h\left(\frac{29}{180}f_{k+2} + \frac{31}{45}f_{k+\frac{3}{2}} + \frac{2}{15}f_{k+1} + \frac{1}{45}f_{k+\frac{1}{2}} - \frac{1}{180}f_k\right)\right]\right| \\
&= \left|h\left(-\tfrac{1}{180}f_{k+2} + \tfrac{1}{36}f_{k+\frac{3}{2}} - \tfrac{1}{18}f_{k+1} + \tfrac{1}{18}f_{k+\frac{1}{2}} - \tfrac{1}{36}f_k + \tfrac{1}{180}f_{k-\frac{1}{2}}\right)\right|
\end{aligned} \tag{3.6.3}$$

3.6.2 LTE for 1 Step Block Hybrid Method

Similarly, by taking the difference between equations (3.3.21) and (3.3.14), the LTE for 1 step block-hybrid method is obtained.

$$\begin{aligned}
LTE &= \left|{y_{k+1}}^{c(7)} - {y_{k+1}}^{c(6)}\right| \\
&= \left|\left[y_k + h\left(\frac{1139}{7560}f_{k+1} + \frac{47}{63}f_{k+\frac{1}{2}} + \frac{11}{2520}f_k + \frac{166}{945}f_{k-\frac{1}{2}}\right.\right.\right. \\
&\quad \left.\left. -\frac{269}{2520}f_{k-1} + \frac{11}{315}f_{k-\frac{3}{2}} - \frac{37}{7560}f_{k-2}\right)\right] \\
&\quad \left.\left[y_k + h\left(\frac{7}{45}f_{k+1} + \frac{43}{60}f_{k+\frac{1}{2}} + \frac{7}{90}f_k + \frac{7}{90}f_{k-\frac{1}{2}} - \frac{1}{30}f_{k-1} + \frac{1}{180}f_{k-\frac{3}{2}}\right)\right]\right| \\
&= \left|37h\left(-\frac{1}{7560}f_{k+1} + \frac{1}{1260}f_{k+\frac{1}{2}} - \frac{1}{504}f_k + \frac{1}{378}f_{k-\frac{1}{2}}\right.\right. \\
&\quad \left.\left. -\frac{1}{504}f_{k-1} + \frac{1}{1260}f_{k-\frac{3}{2}} - \frac{1}{7560}f_{k-2}\right)\right|
\end{aligned} \tag{3.6.4}$$

3.7 Algorithm

A programming code is designed using python to perform the numerical method in solving the RDDEs. The following algorithms explain the process when implementing the new methods in constant and variable step sizes. The maximum error is calculated by taking the maximum absolute value between the approximated values and the exact values.

3.7.1 2 Step Block Hybrid Method

In 2 step block-hybrid method of order six, 6 starting points are needed to be initialized, (t_0, y_0) to ($t_{\frac{5}{2}}$, $y_{\frac{5}{2}}$). The four explicit methods will be evaluated at the same time. After evaluating the predicted function value at initial points, the value y at the first off-step and main points are calculated. It is followed by the estimation of y at the second off-step and main points. The second implicit method utilizes the values from the first implicit method to do the evaluation.

Constant Step Size Algorithm

S1:	Get the boundary $[\alpha, \beta]$ and starting position $t_0 = \alpha$ from the problem.
S2:	Let a step size, h.
S3:	Initialize the starting values $t_i = t_0 + ih$ and y_i, where $i = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$.
S4:	Evaluate the delay argument, D_i and delay solution, yD_i.
S5:	Evaluate the function value, f_i.
S6:	Compute the t_j and predictor value, $y_j{}^p$, where $j = 3, \frac{7}{2}, 4, \frac{9}{2}$ using the explicit block-hybrid method (3.3.2), (3.3.4), (3.3.6), and (3.3.8).
S7:	Evaluate the delay argument, D_j.
S8:	**if** $(D_j < \alpha)$
S9:	$yD_j = \omega(D_j)$
S10:	**else** $(D_j \geq \alpha)$
S11:	Locate the position of D_j, then evaluate yD_j by Lagrange interpolation polynomial with four nearest points.
S12:	Evaluate the predicted function value, $f_j{}^p$.
S13:	Compute the corrected value, $y_k{}^c$, where $k = 3, \frac{7}{2}$ using the implicit block-hybrid method (3.3.12) and (3.3.14).
S14:	Repeat **S7-S11** by changing j to k.
S15:	Evaluate the corrected function value, $f_k{}^c$.
S16:	Repeat **S13-S15** by changing $k = 4, \frac{9}{2}$ and using the implicit block-hybrid method (3.3.16) and (3.3.19).
S17:	Repeat **S6-S16** by changing $j = j + 2$ and $k = k + 2$ until $t_j \geq b$.
S18:	Compute the maximum error.

Variable Step Size Algorithm

S1:	Get the boundary $[\alpha, b]$ and starting position $t_0 = \alpha$ from the problem.
S2:	Set the tolerance (TOL) and step size, $h = TOL$.
S3:	Initialize the starting values $t_i = t_0 + ih$ and y_i, where $i = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$.
S4:	Evaluate the delay argument, D_i and delay solution, yD_i.
S5:	Evaluate the function value, f_i.
S6:	Compute the $t_j = t_{j-\frac{1}{2}} + \frac{h}{2}$, where $j = 3, \frac{7}{2}, 4, \frac{9}{2}$.
S7:	Find the previous 6 points by using the first $j = 3$, $t_{j-(3-p)} = t_{j=3} - (3-p)h$, where $p = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}$.
S8:	**if** (value $t_{j-(3-p)}$ exists previously)
S9:	Use corresponding $f_{j-(3-p)}$.
S10:	**else** (value $t_{j-(3-p)}$ does not exists previously)
S11:	**if** $(t_{j-(3-p)} < \alpha)$
S12:	$y_{j-(3-p)} = \omega(t_{j-(3-p)})$
S13:	**else** $(t_{j-(3-p)} \geq \alpha)$
S14:	Locate the position of $t_{j-(3-p)}$, then evaluate $y_{j-(3-p)}$ by Lagrange interpolation polynomial with four nearest points.
S15:	Evaluate the delay argument, $D_{j-(3-p)}$.
S16:	**if** $(D_{j-(3-p)} < \alpha)$
S17:	$yD_{j-(3-p)} = \omega(D_{j-(3-p)})$.
S18:	**else** $(D_{j-(3-p)} \geq \alpha)$
S19:	Locate the position of $D_{j-(3-p)}$, then evaluate $yD_{j-(3-p)}$ by Lagrange interpolation polynomial with four nearest points.
S20:	Evaluate the function value, $f_{j-(3-p)}$.
S21:	Compute the predictor value, ${y_j}^p$ using the explicit block-hybrid method (3.3.2), (3.3.4), (3.3.6), and (3.3.8).
S22:	Evaluate the delay argument, D_j.
S23:	**if** $(D_j < \alpha)$
S24:	$yD_j = \omega(D_j)$
S25:	**else** $(D_j \geq \alpha)$
S26:	Locate the position of D_j, then evaluate yD_j by Lagrange interpolation polynomial with four nearest points.
S27:	Evaluate the predicted function value, ${f_j}^p$.
S28:	Compute the corrected value, ${y_k}^c$, where $k = 3, \frac{7}{2}$ using the implicit block-hybrid method (3.3.12) and (3.3.14).
S29:	Repeat **S22-S26** by changing j to k.
S30:	Evaluate the corrected function value, ${f_k}^c$.
S31:	Repeat **S28-S30** by changing $k = 4, \frac{9}{2}$ and using the implicit block-hybrid method (3.3.16) and (3.3.19).
S32:	**[Convergence Test] if** (for any j, $\left\|({y_j}^c)^{[t]} - ({y_j}^c)^{[t-1]}\right\| \leq 0.1 \times TOL$)
S33:	Repeat from **S28** by changing ${f_j}^p = {f_j}^c$.

S34:	Calculate the local truncation error (LTE).
S35:	**[Variable Step] if** ($LTE \leq TOL$) *Successful step*
S36:	$h_{new} = C \times \left(\frac{TOL}{LTE}\right)^{\frac{1}{r}} \times h$, where $C = 0.5$ is the safety factor and $r = 6$ is the order of the method.
S37:	**if** ($h_{new} > 4 \times h$) $h_{new} = 4 \times h$
S38:	**if** ($t_j + 2 \times h_{new} > b$) $h_{new} = \frac{b-t_j}{2}$, where j is the last j.
S39:	**else** ($LTE > TOL$) *Failure step*
S40:	$h_{new} = 0.5 \times h$
S41:	Repeat **S6-S40** by changing $j = j + 2$, $k = k + 2$, and $h = h_{new}$ until $t_j \geq b$.
S42:	Compute the maximum error.

3.7.2 1 Step Block Hybrid Method

In 1 step block-hybrid method of order seven, 7 starting points are needed to be initialized, (t_0, y_0) to (t_3, y_3). The two explicit methods will be evaluated at the same time. The two implicit methods also will be evaluated simultaneously after the function evaluation of the predictor.

Constant Step Size Algorithm

S1:	Get the boundary $[\alpha, \beta]$ and starting position $t_0 = \alpha$ from the problem.
S2:	Let a step size, h.
S3:	Initialize the starting values $t_i = t_0 + ih$ and y_i, where $i = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$.
S4:	Evaluate the delay argument, D_i and delay solution, yD_i.
S5:	Evaluate the function value, f_i.
S6:	Compute the t_j and predictor value, $y_j{}^p$, where $j = \frac{7}{2}, 4$ using the explicit block-hybrid method (3.3.9) and (3.3.10).
S7:	Evaluate the delay arguments, D_j.
S8:	**if** ($D_j < \alpha$)
S9:	$yD_j = \omega(D_j)$
S10:	**else** ($D_j \geq \alpha$)
S11:	Locate the position of D_j, then evaluate yD_j by Lagrange interpolation polynomial with four nearest points.
S12:	Evaluate the predicted function value, $f_j{}^p$.
S13:	Compute the corrected value, $y_j{}^c$ using the implicit block-hybrid method (3.3.20) and (3.3.21).
S14:	Repeat **S7-S11**.
S15:	Evaluate the corrected function value, $f_j{}^c$.
S16:	Repeat **S6-S15** by changing $j = j + 1$ and $k = k + 1$ until $t_j \geq b$.
S17:	Compute the maximum error.

Variable Step Size Algorithm

S1:	Get the boundary $[\alpha, \beta]$ and starting position $t_0 = \alpha$ from the problem.
S2:	Set the tolerance (TOL) and step size, $h = TOL$.
S3:	Initialize the starting values $t_i = t_0 + ih$ and y_i, where $i = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$.
S4:	Evaluate the delay argument, D_i and delay solution, yD_i.
S5:	Evaluate the function value, f_i.
S6:	Compute the $t_j = t_{j-\frac{1}{2}} + \frac{h}{2}$, where $j = \frac{7}{2}, 4$.
S7:	Find the previous 7 points by using the first $j = \frac{7}{2}$, $t_{j-(3-p)} = t_{j=3} - (3-p)h$, where $p = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$.
S8:	**if** (value $t_{j-(3-p)}$ exists previously)
S9:	Use corresponding $f_{j-(3-p)}$.
S10:	**else** (value $t_{j-(3-p)}$ does not exists previously)
S11:	**if** ($t_{j-(3-p)} < \alpha$)
S12:	$y_{j-(3-p)} = \omega(t_{j-(3-p)})$
S13:	**else** ($t_{j-(3-p)} \geq \alpha$)
S14:	Locate the position of $t_{j-(3-p)}$, then evaluate $y_{j-(3-p)}$ by Lagrange interpolation polynomial with four nearest points.
S15:	Evaluate the delay argument, $D_{j-(3-p)}$.
S16:	**if** ($D_{j-(3-p)} < \alpha$)
S17:	$yD_{j-(3-p)} = \omega(D_{j-(3-p)})$
S18:	**else** ($D_{j-(3-p)} \geq \alpha$)
S19:	Locate the position of $D_{j-(3-p)}$, then evaluate $yD_{j-(3-p)}$ by Lagrange interpolation polynomial with four nearest points.
S20:	Evaluate the function value, $f_{j-(3-p)}$.
S21:	Compute the predictor value ${y_j}^p$ using the explicit block-hybrid method (3.3.9) and (3.3.10).
S22:	Evaluate the delay argument, D_j.
S23:	**if** ($D_j < \alpha$)
S24:	$yD_j = \omega(D_j)$
S25:	**else** ($D_j \geq \alpha$)
S26:	Locate the position of D_j, then evaluate yD_j by Lagrange interpolation polynomial with four nearest points.
S27:	Evaluate the predicted function value, ${f_j}^p$.

S28:	Compute the corrected value, ${y_j}^c$ using the implicit block-hybrid method (3.3.20) and (3.3.21).
S29:	Repeat **S22-S26**.
S30:	Evaluate the corrected function value, ${f_j}^c$.
S31:	**[Convergence Test] if** (for any j, $\left\|({y_j}^c)^{[t]} - ({y_j}^c)^{[t-1]}\right\| \leq 0.1 \times TOL$)
S32:	Repeat from **S28** by changing ${f_j}^p = {f_j}^c$.
S33:	Calculate the local truncation error (LTE).
S34:	**[Variable Step] if** ($LTE \leq TOL$) *Successful step*
S35:	$h_{new} = C \times \left(\frac{TOL}{LTE}\right)^{\frac{1}{r}} \times h$, where $C = 0.5$ is the safety factor and $r = 7$ is the order of the method.
S36:	**if** ($h_{new} > 2 \times h$) $h_{new} = 2 \times h$
S37:	**if** ($t_j + h_{new} > b$) $h_{new} = b - t_j$, where j is the last j.
S38:	**else** ($LTE > TOL$) *Failure step*
S39:	$h_{new} = 0.5 \times h$
S40:	Repeat **S6-S39** by changing $j = j + 1$ and $h = h_{new}$ until $t_j \geq b$.
S41:	Compute the maximum error.

CHAPTER 4

RESULTS AND DISCUSSION

4.1 Introduction

In this chapter, five test problems are solved using the new block-hybrid methods which are 1 step and 2 step block-hybrid method of order seven and order six, respectively. The performance of the two strategies, constant step size strategy and variable step size strategy, will be studied in both methods. Adams-Bashforth and Adams-Moulton method (ABAM) is the standard to compare with new methods. Here, three different types of delays differential equations will be considered. They consist of constant delay (1.1.4), time dependent delay (1.1.5), and state dependent delay (1.1.6). Below are the problems used to illustrate the performance of the methods.

Problem 1: Constant Delay

$y'(t) = -2y(t) - \frac{\pi}{2}e^{-2}y(t-\tau) \quad \text{for} \quad 0 \leq t \leq 5,$

$y(t) = e^{-2t}sin\left(\frac{\pi}{2}t\right) \quad \text{for} \quad t \leq 0,$

$\tau(t) = 1.$

Actual Solution Equation:

$y(t) = e^{-2t}sin\left(\frac{\pi}{2}t\right)$

Problem 2: Time Dependent Delay

$y'(t) = 1 - y(t-\tau) \quad \text{for} \quad 1 \leq t \leq 10,$

$y(t) = ln(t) \quad \text{for} \quad 0 < t \leq 1,$

$\tau(t) = t - e^{1-\frac{1}{t}}.$

Actual Solution Equation:

$y(t) = ln(t)$

Problem 3: Time Dependent Delay

$y'(t) = -y(t-\tau) + sin(t-\tau) + cos(t) \quad \text{for} \quad 0 \leq t \leq 10,$

$y(t) = sin(t) \quad \text{for} \quad t \leq 0,$

$\tau(t) = 1 - \frac{1}{e^t}.$

Actual Solution Equation:

$y(t) = sin(t)$

Problem 4: Time Dependent Delay

$y'(t) = \dfrac{t^4 - 3}{t^5 + t} \dfrac{1}{ln(t - \tau + (t - \tau)^{-3})} y(t - \tau)$ for $2 \leq t \leq 10$,

$y(t) = ln\left(t + \frac{1}{t^3}\right)$ for $1.5 \leq t \leq 2$,

$\tau(t) = \frac{1}{t^3}$.

Actual Solution Equation:

$y(t) = ln\left(t + \frac{1}{t^3}\right)$

Problem 5: State Dependent Delay

$y'(t) = y(t - \tau)cos(t)$ for $2 \leq t \leq 10$,

$y(t) = 1$ for $1.5 \leq t \leq 2$,

$\tau(t) = t - y(t) + 2$.

Actual Solution Equation:

$y(t) = sin(t) + 1$

4.2 Implementation in Constant Step Size

This section discusses the proposed methods' performance which is all implemented in constant step size strategy follows the algorithm in Sections 3.7.1 and 3.7.2. The numerical results of the newly proposed block-hybrid methods in this project and the ABAM are shown in Tables 4.1 to 4.5. The following are the definitions of the shortened form used in the table:

h	Step size
FCN	Number of function evaluation
MAXE	Maximum error
Time	Time taken in second(s)
2BHM6	2 step block-hybrid method of order 6 proposed by this research
BHM7	1 step block-hybrid method of order 7 proposed by this research
ABAM	Adams-Bashforth and Adams-Moulton method

Figures 4.1 to 4.5 show the accuracy of all the methods over the different step sizes. The decimal logarithm of maximum error, $log_{10}(MAXE)$, is used as the responding variable to clarify the results. Figures 4.6 to 4.10 show the efficiency of all the methods over the time taken.

Table 4.1: Numerical results for Problem 1.

h	Method	FCN	MAXE	Time
0.1	2BHM6	198	8.579×10^{-4}	0.026
	BHM7	199	2.871×10^{-8}	0.023
	ABAM	102	2.121×10^{-3}	0.015
0.05	2BHM6	398	2.444×10^{-8}	0.082
	BHM7	399	8.668×10^{-11}	0.099
	ABAM	202	7.244×10^{-4}	0.038
0.01	2BHM6	1998	2.388×10^{-13}	1.063
	BHM7	1999	3.331×10^{-16}	0.742
	ABAM	1002	3.698×10^{-5}	0.259
0.005	2BHM6	3998	2.165×10^{-15}	2.842
	BHM7	3999	1.270×10^{-15}	3.211
	ABAM	2002	9.528×10^{-6}	0.737
0.001	2BHM6	19998	6.405×10^{-15}	59.828
	BHM7	19995	6.536×10^{-15}	54.122
	ABAM	10002	3.904×10^{-7}	13.15

Table 4.2: Numerical results for Problem 2.

h	Method	FCN	MAXE	Time
0.1	2BHM6	358	2.964×10^{-7}	0.097
	BHM7	359	4.412×10^{-7}	0.051
	ABAM	182	4.876×10^{-4}	0.026
0.05	2BHM6	718	3.377×10^{-8}	0.212
	BHM7	715	2.381×10^{-8}	0.15
	ABAM	362	1.310×10^{-4}	0.063
0.01	2BHM6	3598	4.556×10^{-11}	2.325
	BHM7	3599	4.709×10^{-11}	2.803
	ABAM	1802	6.011×10^{-6}	0.726
0.005	2BHM6	7198	2.643×10^{-12}	8.699
	BHM7	7195	2.823×10^{-12}	9.826
	ABAM	3602	1.532×10^{-6}	2.259
0.001	2BHM6	35998	4.001×10^{-13}	192.664
	BHM7	35999	3.992×10^{-13}	200.838
	ABAM	18002	6.225×10^{-8}	59.243

Table 4.3: Numerical results for Problem 3.

h	Method	FCN	MAXE	Time
0.1	2BHM6	398	1.197×10^{-7}	0.063
	BHM7	399	1.255×10^{-7}	0.076
	ABAM	182	4.876×10^{-4}	0.021
0.05	2BHM6	798	9.677×10^{-9}	0.176
	BHM7	795	9.656×10^{-9}	0.237
	ABAM	362	1.310×10^{-4}	0.058
0.01	2BHM6	3998	1.764×10^{-11}	2.462
	BHM7	3999	1.767×10^{-11}	2.716
	ABAM	1802	6.011×10^{-6}	0.571
0.005	2BHM6	7998	9.938×10^{-13}	10.716
	BHM7	7995	9.943×10^{-13}	15.02
	ABAM	3602	1.532×10^{-6}	1.85
0.001	2BHM6	39998	5.455×10^{-13}	228.635
	BHM7	39999	5.468×10^{-13}	237.272
	ABAM	18002	6.225×10^{-8}	44.634

Table 4.4: Numerical results for Problem 4.

h	Method	FCN	MAXE	Time
0.1	2BHM6	318	7.867×10^{-9}	0.043
	BHM7	319	2.338×10^{-8}	0.044
	ABAM	162	3.406×10^{-5}	0.018
0.05	2BHM6	638	2.746×10^{-9}	0.125
	BHM7	635	2.475×10^{-9}	0.132
	ABAM	322	1.178×10^{-6}	0.096
0.01	2BHM6	3198	4.686×10^{-12}	1.567
	BHM7	3199	5.170×10^{-12}	1.601
	ABAM	1602	2.708×10^{-7}	0.549
0.005	2BHM6	6398	3.291×10^{-13}	5.633
	BHM7	6395	2.980×10^{-13}	5.976
	ABAM	3202	7.939×10^{-8}	1.736
0.001	2BHM6	31998	1.390×10^{-13}	138.597
	BHM7	31995	1.483×10^{-13}	151.662
	ABAM	16002	3.561×10^{-9}	45.424

Table 4.5: Numerical results for Problem 5.

h	Method	FCN	MAXE	Time
0.1	2BHM6	398	9.197×10^{-11}	0.002
	BHM7	399	6.567×10^{-12}	0.003
	ABAM	202	7.831×10^{-5}	0.005
0.05	2BHM6	798	1.306×10^{-12}	0.009
	BHM7	795	6.894×10^{-14}	0.012
	ABAM	402	9.771×10^{-6}	0.012
0.01	2BHM6	3998	3.965×10^{-14}	0.037
	BHM7	3999	4.296×10^{-14}	0.073
	ABAM	2002	7.813×10^{-8}	0.041
0.005	2BHM6	7998	2.439×10^{-13}	0.068
	BHM7	7995	2.384×10^{-13}	0.103
	ABAM	4002	9.766×10^{-9}	0.073
0.001	2BHM6	39998	1.613×10^{-12}	0.306
	BHM7	39999	1.605×10^{-12}	0.479
	ABAM	20002	7.812×10^{-11}	0.366

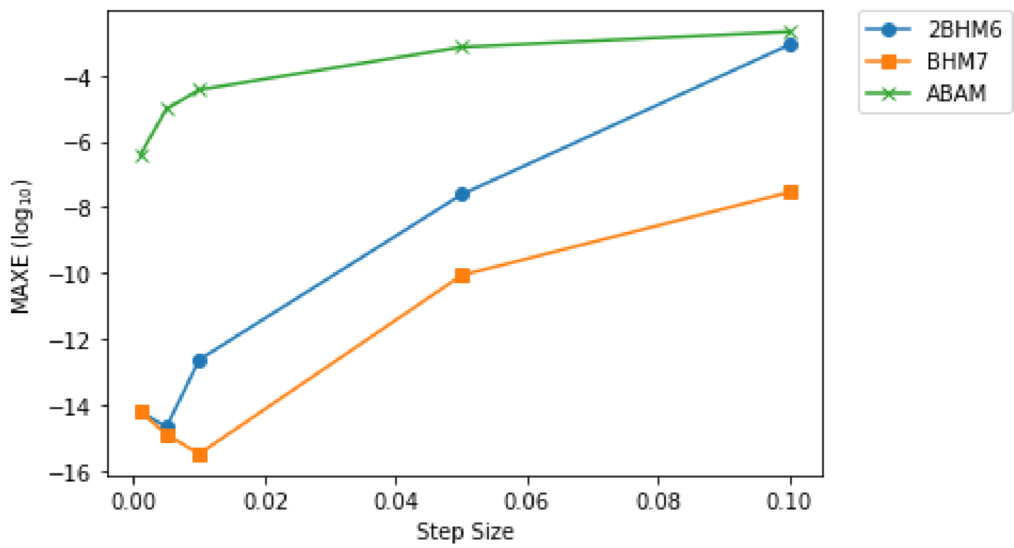

Figure 4.1: Maximum error (log_{10}) versus step size for Problem 1.

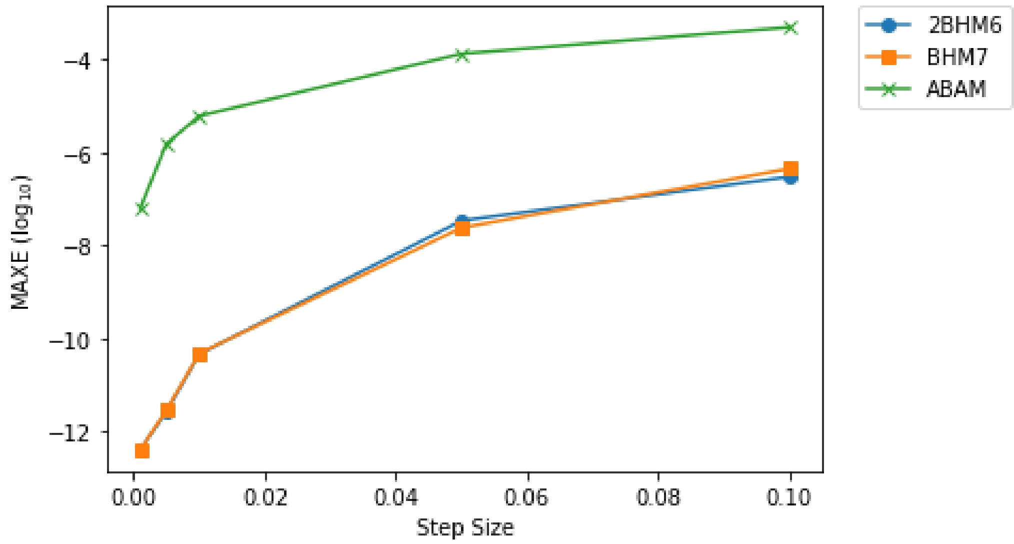

Figure 4.2: Maximum error (log_{10}) versus step size for Problem 2

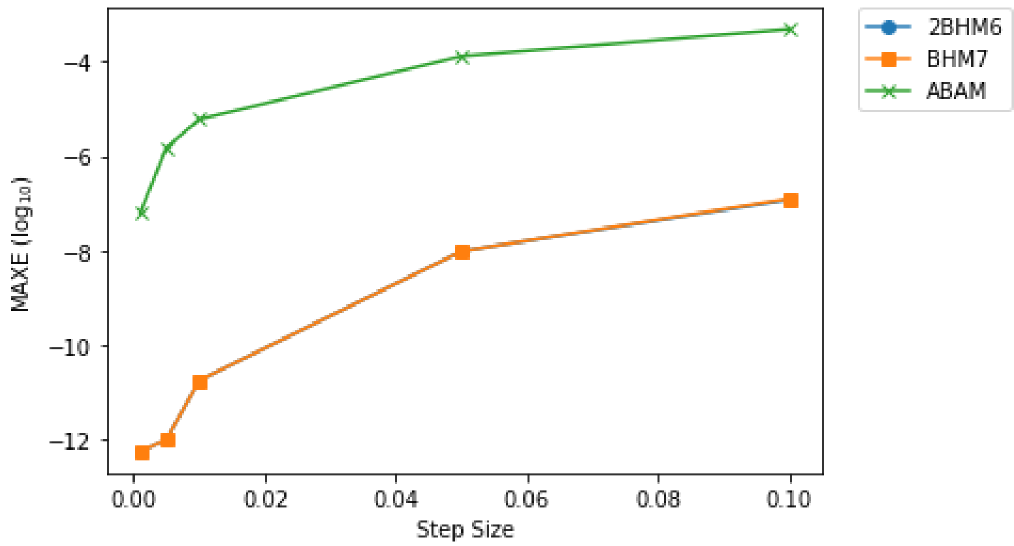

Figure 4.3: Maximum error (log_{10}) versus step size for Problem 3.

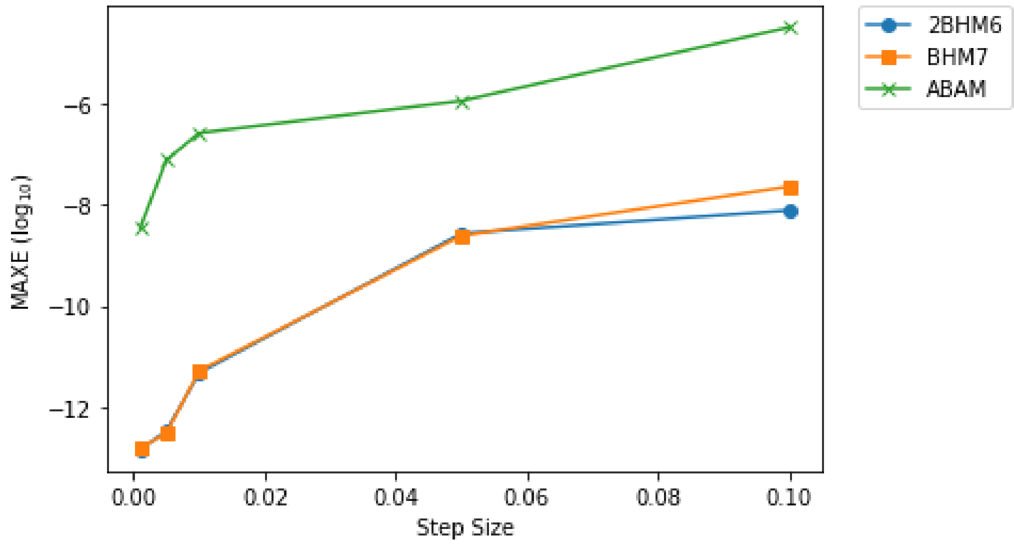

Figure 4.4: Maximum error (log_{10}) versus step size for Problem 4.

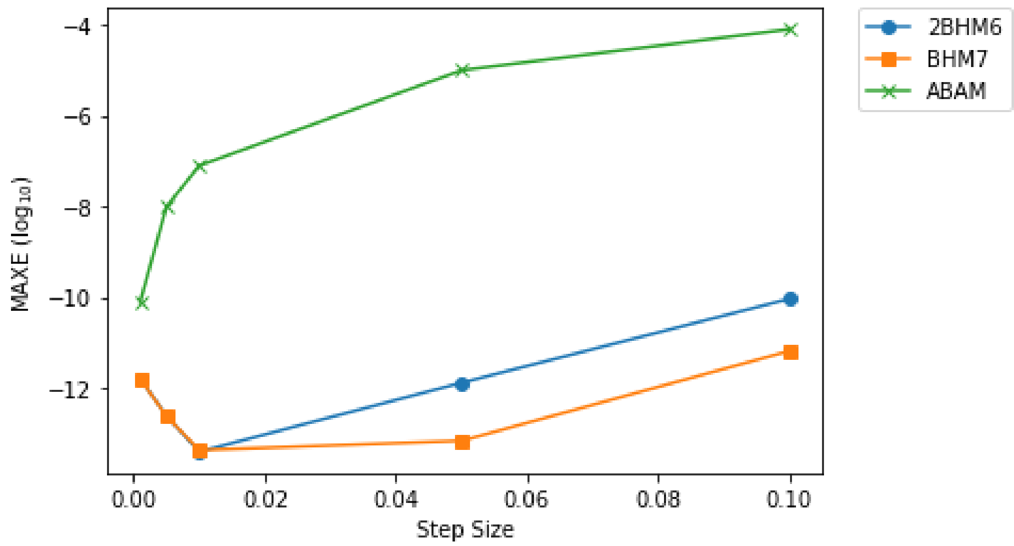

Figure 4.5: Maximum error (log_{10}) versus step size for Problem 5.

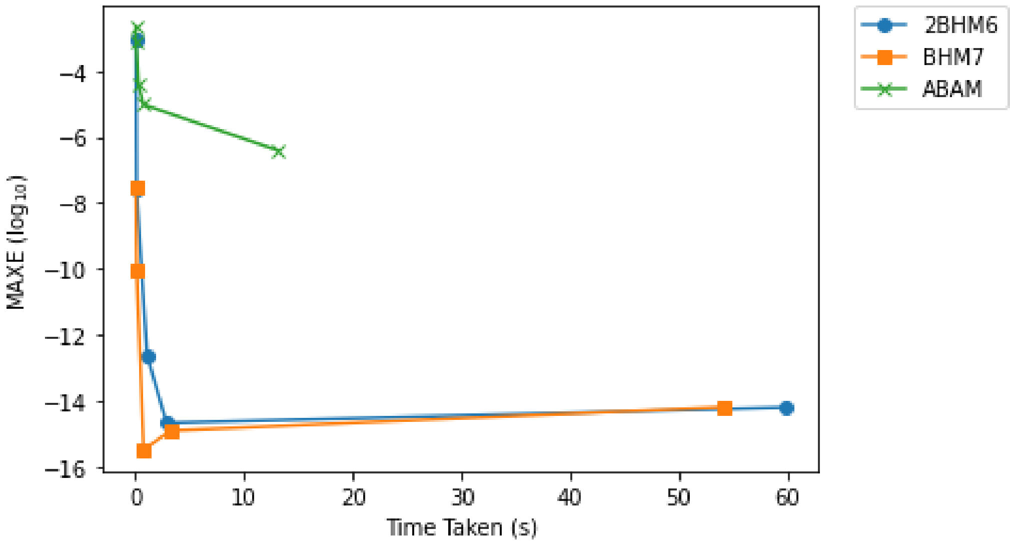

Figure 4.6: Maximum error (log_{10}) versus time taken for Problem 1.

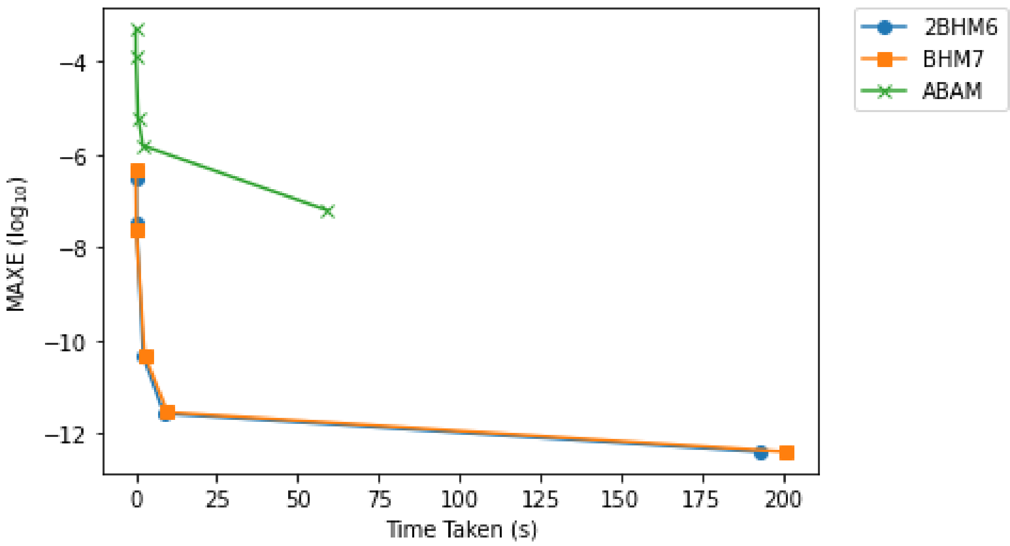

Figure 4.7: Maximum error (log_{10}) versus time taken for Problem 2

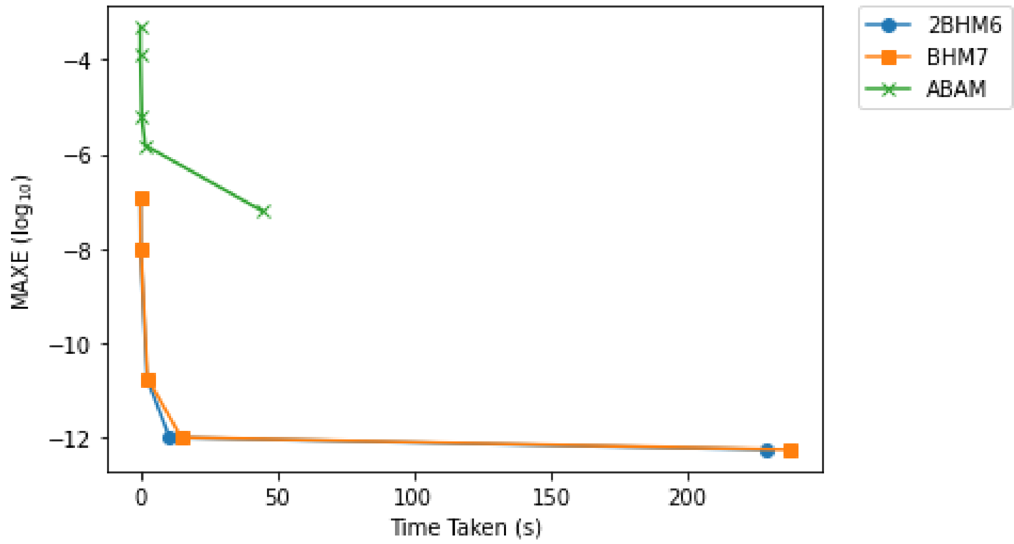

Figure 4.8: Maximum error (log_{10}) versus time taken for Problem 3.

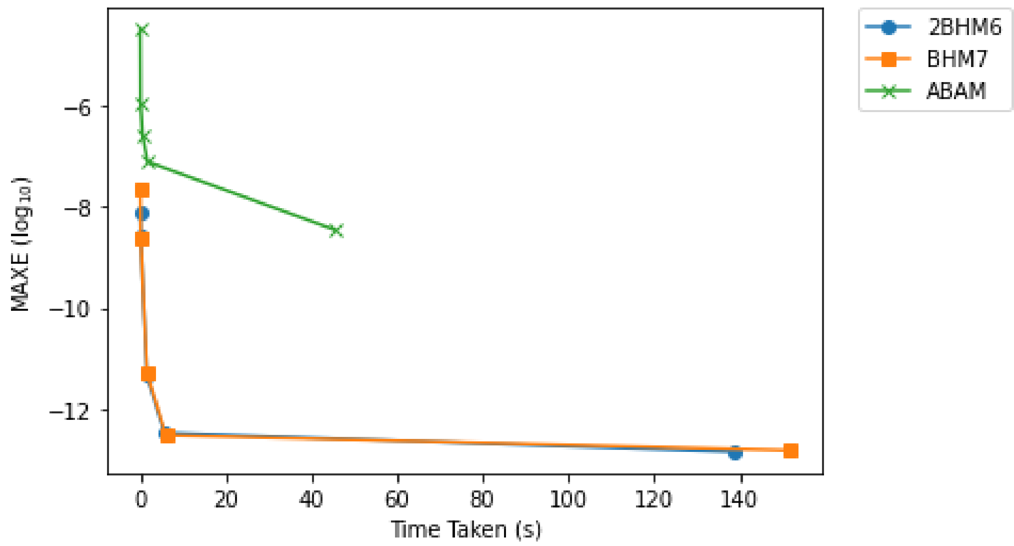

Figure 4.9: Maximum error (log_{10}) versus time taken for Problem 4.

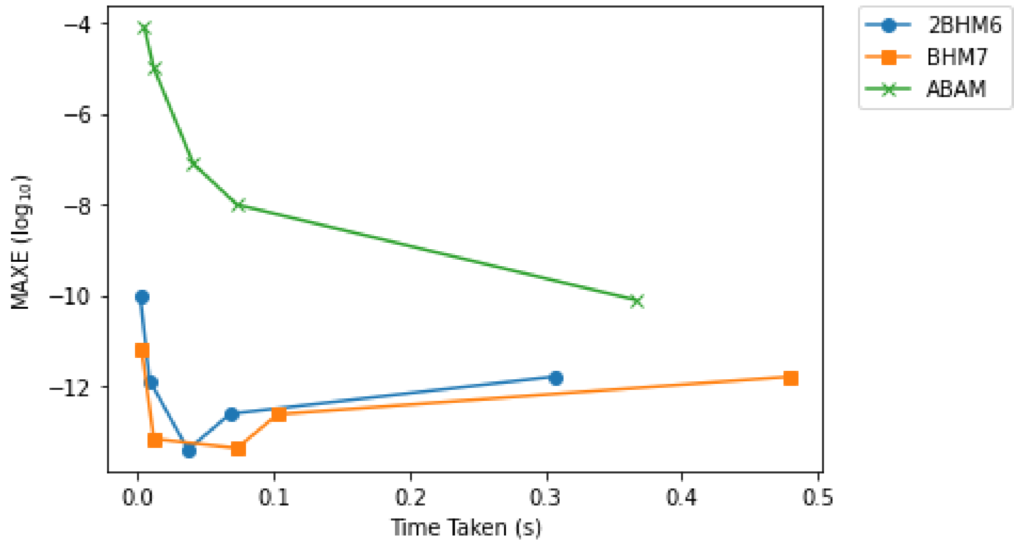

Figure 4.10: Maximum error (log_{10}) versus time taken for Problem 5.

4.2.1 Discussion I

Due to the constant step size strategy, the total steps of every method will be nearly similar. The number of function evaluations is also not much different. So, more focus will be put on the accuracy of the method. With the constant step size strategy, the performance of BHM7 is slightly better than 2BHM6. The two methods proposed are significantly performed better than the ABAM. In a higher order method, more reference points are used to interpolate the next step, consequently the better the approximation of the next step. Therefore, BHM7 has a minor maximum error compared to other methods.

According to Figures 4.1 to 4.5, both the line of BHM7 and 2BHM6 are significantly lower than the line of ABAM. It shows the accuracy of the newly proposed method is more accurate than ABAM. Overall, BHM7 has a minor maximum error than 2BHM6 except for Problems 2 and 4 at the step size $h = 0.1$. When the step size is smaller than or equal to 0.005, BHM7 and 2BHM6 having almost the same accuracy.

Figures 4.6 to 4.10 show that the line of BHM7 and 2BHM6 are significantly lower than the line of ABAM. It means 2BHM6 and BHM7 can get a minor maximum error in a shorter time. This shows the involvement of off step points in the block-hybrid method had played a significant role in the interpolation of delay solution, which leads the solution of DDEs toward smaller maximum error.

4.3 Implementation in Variable Step Size

This section discusses the performance of the methods, which are all implemented in variable step size strategy follows the algorithm in Sections 3.7.1 and 3.7.2. The numerical results of the newly proposed block-hybrid methods in this project and the ABAM using the variable step size strategy are shown in Tables 4.6 to 4.10. The following are the definitions of the shortened form used in the table:

TOL	Tolerance
TS	Total steps / Number of successful steps
FS	Number of failure steps
FCN	Number of function evaluation
MAXE	Maximum error
Time	Time taken in second(s)
2BHM6	2 step block-hybrid method of order 6 proposed by this research
BHM7	1 step block-hybrid method of order 7 proposed by this research
BHM6	Block hybrid method of order 6 proposed by Yap et al. (2020)
ABAM	Adams-Bashforth and Adams-Moulton method

Figures 4.11 to 4.15 show the efficiency of all the methods over the total steps. The decimal logarithm of maximum error, $log_{10}(MAXE)$, is used as the responding variable to clarify the results.

Table 4.6: Numerical results for Problem 1.

TOL	Method	TS	FS	FCN	MAXE	Time
10^{-2}	2BHM6	7	0	183	2.102×10^{-4}	0.013
	BHM7	13	0	181	2.351×10^{-4}	0.02
	BHM6	13	0	179	2.841×10^{-4}	0.016
	ABAM	17	0	122	1.067×10^{-3}	0.011
10^{-4}	2BHM6	13	0	292	2.801×10^{-6}	0.032
	BHM7	24	0	309	1.044×10^{-5}	0.041
	BHM6	25	0	317	9.082×10^{-5}	0.055
	ABAM	37	0	242	1.072×10^{-4}	0.036
10^{-6}	2BHM6	23	0	507	2.583×10^{-7}	0.054
	BHM7	46	1	602	5.787×10^{-6}	0.084
	BHM6	46	0	536	4.683×10^{-7}	0.1
	ABAM	82	0	507	9.534×10^{-7}	0.068
10^{-8}	2BHM6	41	0	859	1.898×10^{-8}	0.107
	BHM7	93	0	1216	2.641×10^{-8}	0.193
	BHM6	84	0	978	2.888×10^{-8}	0.177
	ABAM	184	0	1103	1.462×10^{-8}	0.185
10^{-10}	2BHM6	78	0	1627	2.334×10^{-9}	0.276
	BHM7	205	1	2633	4.950×10^{-10}	0.866
	BHM6	174	2	2104	7.707×10^{-10}	0.501
	ABAM	424	3	2577	2.488×10^{-9}	0.58

Table 4.7: Numerical results for Problem 2.

TOL	Method	TS	FS	FCN	MAXE	Time
10^{-2}	2BHM6	7	0	147	7.461×10^{-3}	0.016
	BHM7	14	0	159	1.539×10^{-4}	0.017
	BHM6	13	0	135	7.915×10^{-4}	0.014
	ABAM	18	0	105	2.413×10^{-3}	0.012
10^{-4}	2BHM6	13	0	209	7.424×10^{-6}	0.041
	BHM7	25	0	278	2.811×10^{-5}	0.05
	BHM6	25	0	249	7.108×10^{-6}	0.05
	ABAM	37	0	211	8.128×10^{-5}	0.044
10^{-6}	2BHM6	22	0	358	6.152×10^{-7}	0.055
	BHM7	46	0	518	1.733×10^{-6}	0.096
	BHM6	44	0	456	9.022×10^{-7}	0.091
	ABAM	78	0	451	2.605×10^{-6}	0.131
10^{-8}	2BHM6	39	0	641	3.167×10^{-8}	0.116
	BHM7	82	0	936	1.948×10^{-7}	0.183
	BHM6	77	0	796	1.705×10^{-8}	0.183
	ABAM	171	0	998	7.273×10^{-8}	0.248
10^{-10}	2BHM6	72	1	1230	1.933×10^{-9}	0.255
	BHM7	158	0	1858	9.893×10^{-9}	0.457
	BHM6	150	0	1608	1.795×10^{-9}	0.421
	ABAM	396	0	2338	1.955×10^{-9}	0.881

Table 4.8: Numerical results for Problem 3.

TOL	Method	TS	FS	FCN	MAXE	Time
10^{-2}	2BHM6	8	0	150	9.791×10^{-4}	0.02
	BHM7	16	0	196	6.092×10^{-3}	0.024
	BHM6	15	0	166	6.484×10^{-3}	0.024
	ABAM	21	1	141	4.047×10^{-2}	0.018
10^{-4}	2BHM6	17	0	280	1.302×10^{-4}	0.056
	BHM7	33	0	383	9.920×10^{-5}	0.078
	BHM6	34	1	377	8.490×10^{-4}	0.131
	ABAM	55	1	341	3.699×10^{-4}	0.066
10^{-6}	2BHM6	33	0	555	1.116×10^{-5}	0.123
	BHM7	79	0	935	1.873×10^{-5}	0.213
	BHM6	64	0	703	3.671×10^{-6}	0.211
	ABAM	134	3	819	1.418×10^{-5}	0.159
10^{-8}	2BHM6	81	0	1371	2.449×10^{-7}	0.437
	BHM7	191	1	2344	3.741×10^{-7}	0.674
	BHM6	134	0	1487	1.114×10^{-7}	0.42
	ABAM	332	0	1980	3.684×10^{-8}	0.502
10^{-10}	2BHM6	164	1	2832	1.093×10^{-8}	1.024
	BHM7	479	0	5818	3.718×10^{-8}	2.802
	BHM6	363	1	4107	1.021×10^{-8}	1.856
	ABAM	789	0	4738	1.538×10^{-9}	1.874

Table 4.9: Numerical results for Problem 4.

TOL	Method	TS	FS	FCN	MAXE	Time
10^{-2}	2BHM6	9	0	132	2.803×10^{-4}	0.012
	BHM7	13	0	152	3.044×10^{-4}	0.018
	BHM6	13	0	135	4.169×10^{-4}	0.015
	ABAM	15	0	83	1.060×10^{-3}	0.007
10^{-4}	2BHM6	17	0	242	4.788×10^{-6}	0.035
	BHM7	23	0	262	2.279×10^{-5}	0.056
	BHM6	26	0	263	7.776×10^{-7}	0.044
	ABAM	29	1	173	2.025×10^{-4}	0.027
10^{-6}	2BHM6	28	0	402	8.329×10^{-8}	0.056
	BHM7	37	0	399	6.369×10^{-8}	0.096
	BHM6	45	0	450	3.068×10^{-8}	0.07
	ABAM	61	0	347	1.403×10^{-6}	0.06
10^{-8}	2BHM6	44	0	644	4.622×10^{-9}	0.108
	BHM7	61	0	674	1.248×10^{-8}	0.17
	BHM6	81	0	752	1.800×10^{-9}	0.162
	ABAM	130	0	749	4.370×10^{-8}	0.142
10^{-10}	2BHM6	74	0	1118	2.920×10^{-10}	0.231
	BHM7	117	0	1354	1.027×10^{-9}	0.381
	BHM6	163	0	1615	7.855×10^{-11}	0.48
	ABAM	296	0	1736	1.358×10^{-9}	0.428

Table 4.10: Numerical results for Problem 5.

TOL	Method	TS	FS	FCN	MAXE	Time
10^{-2}	2BHM6	8	0	132	3.808×10^{-2}	0.005
	BHM7	15	0	177	2.714×10^{-4}	0.01
	BHM6	15	0	162	4.490×10^{-3}	0.009
	ABAM	13	2	95	$1.038 \times 10^{+0}$	0.004
10^{-4}	2BHM6	16	0	262	3.105×10^{-6}	0.012
	BHM7	29	0	333	3.036×10^{-5}	0.027
	BHM6	31	0	330	1.343×10^{-4}	0.029
	ABAM	53	1	328	4.794×10^{-4}	0.029
10^{-6}	2BHM6	30	0	497	2.119×10^{-7}	0.038
	BHM7	51	0	576	6.020×10^{-8}	0.06
	BHM6	59	0	634	6.680×10^{-7}	0.051
	ABAM	130	1	793	1.161×10^{-5}	0.064
10^{-8}	2BHM6	56	0	946	1.889×10^{-9}	0.067
	BHM7	89	0	1025	9.454×10^{-10}	0.09
	BHM6	113	0	1205	4.578×10^{-9}	0.095
	ABAM	318	0	1902	3.597×10^{-8}	0.176
10^{-10}	2BHM6	112	0	1898	1.799×10^{-11}	0.154
	BHM7	156	0	1753	7.584×10^{-12}	0.218
	BHM6	224	0	2347	1.051×10^{-11}	0.264
	ABAM	774	0	4654	7.993×10^{-10}	0.876

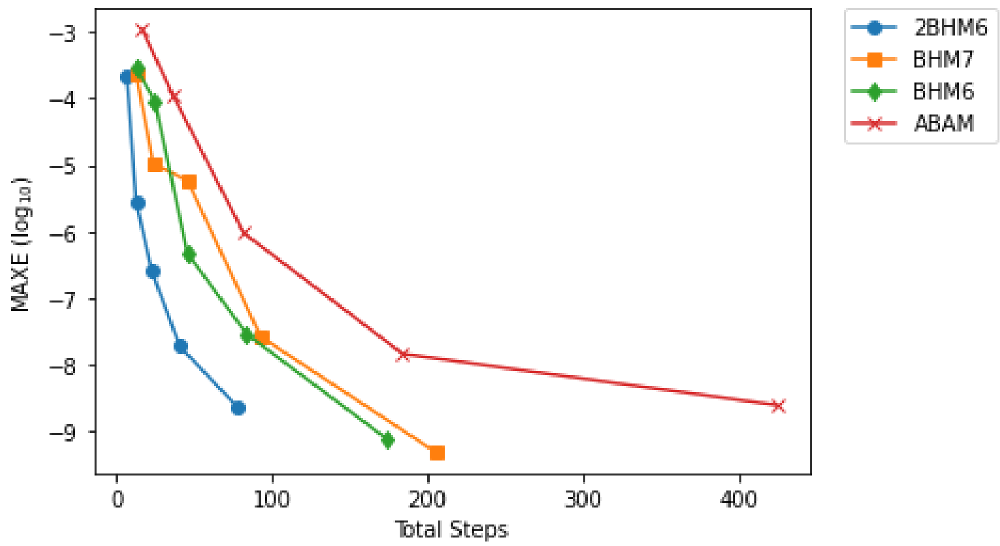

Figure 4.11: Maximum error (log_{10}) versus total steps for Problem 1.

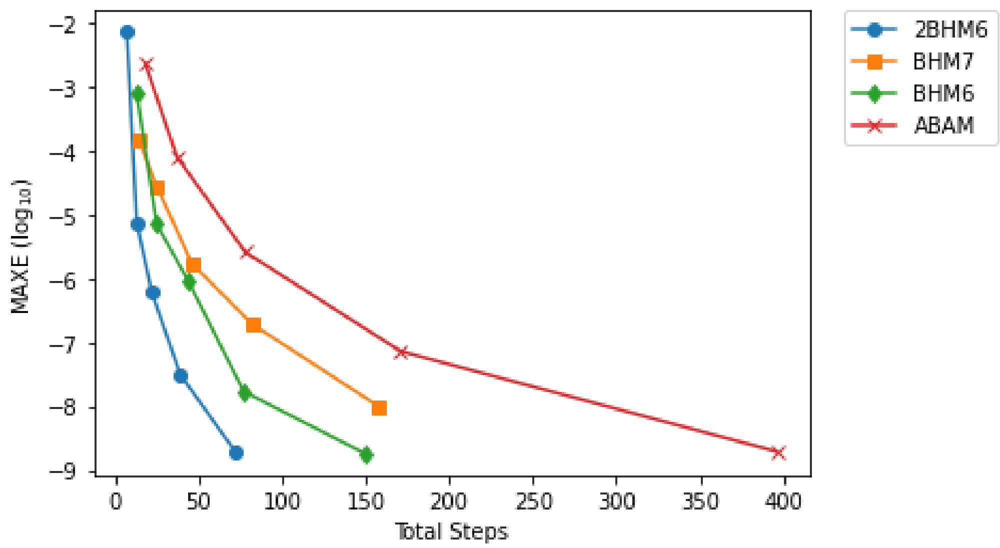

Figure 4.12: Maximum error (log_{10}) versus total steps for Problem 2

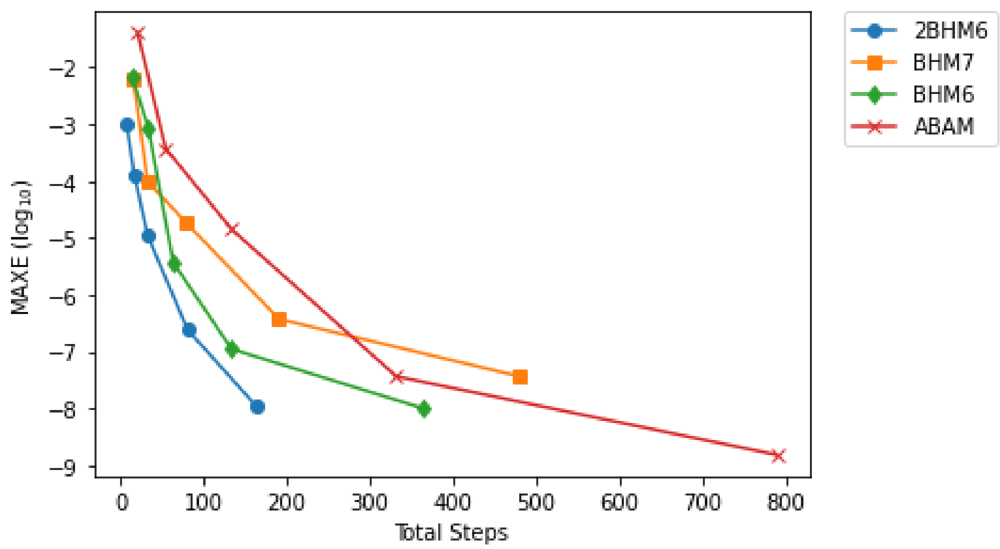

Figure 4.13: Maximum error (log_{10}) versus total steps for Problem 3.

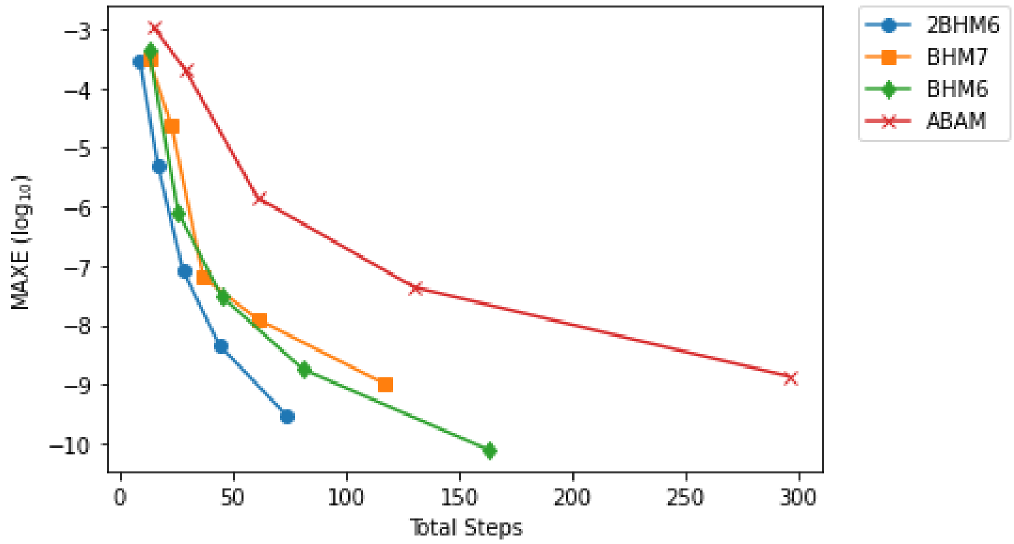

Figure 4.14: Maximum error (log_{10}) versus total steps for Problem 4.

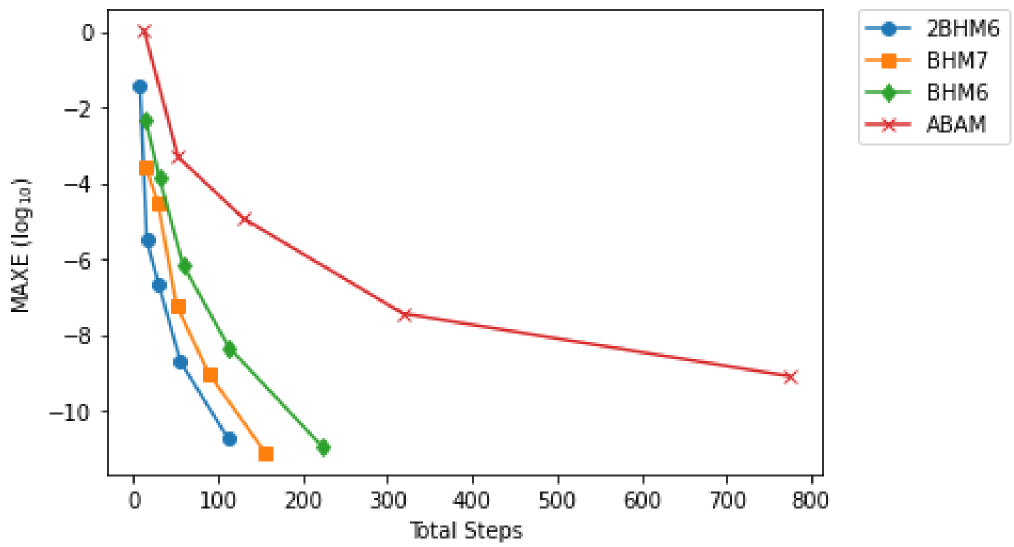

Figure 4.15: Maximum error (log_{10}) versus total steps for Problem 5.

4.3.1 Discussion II

The maximum error in the variable step size strategy does not give meaningful analysis if compared based on the tolerances, as the variable step size strategy makes the error nearer to the given tolerance. So, as long as the maximum error is around or smaller than the given tolerances, the method can be used to solve DDEs. More focus will be put on the efficiency of the method. With the variable step size strategy, the performance of 2BHM6 is significantly better than the other methods. With an extra step involved in a single iteration, the total steps were reduced significantly. Therefore, 2BHM6 requires fewer total steps compared to other methods.

Figures 4.11 to 4.15 show that the line of 2BHM6 is significantly lower than the other lines. This means 2BHM6 has minor total steps compared to other methods. Besides that, the time taken for 2BHM6 is shorter than other methods at all tolerances. This also can be expected through the number of function evaluations because it affects the execution time of an algorithm.

4.4 Compare and Contrast

There are some existing results for Problems 2, 3, and 5 found in the literature. This section will compare our findings with these existing results for the implementation in variable step size. The numerical results of the new block-hybrid methods in this project and the existing results using the variable step size strategy are displayed in Tables 4.11 to 4.13. The following are the definitions of the shortened form used in the table:

TOL	Tolerance
TS	Total steps / Number of successful steps
FS	Number of failure steps
FCN	Number of function evaluation
MAXE	Maximum error
2BHM6	2 step block-hybrid method of order 6 proposed by this research
BHM7	1 step block-hybrid method of order 7 proposed by this research
BHM6	Block hybrid method of order 6 proposed by Yap et al. (2020)
ABAM	Adams-Bashforth and Adams-Moulton method
CB(5,6)	Coupled block method proposed by Hue et al. (2011)
M2BM	Modified 2-point block method proposed by Aziz and Majid (2013)
S2PBTI	Variable order 2 point block method proposed by Ishak et al. (2011)

Figures 4.16 to 4.18 show the efficiency of all the methods over the total steps. The decimal logarithm of maximum error, $log_{10}(MAXE)$, is used as the responding variable to clarify the results.

Table 4.11: Numerical results for Problem 2.

TOL	Method	TS	FS	FCN	MAXE	Time
10^{-2}	2BHM6	7	0	147	7.461×10^{-3}	0.016
	BHM7	14	0	159	1.539×10^{-4}	0.017
	BHM6	13	0	135	7.915×10^{-4}	0.014
	ABAM	18	0	105	2.413×10^{-3}	0.012
	S2PBTI	25	0		3.029×10^{-3}	
10^{-4}	2BHM6	13	0	209	7.424×10^{-6}	0.041
	BHM7	25	0	278	2.811×10^{-5}	0.05
	BHM6	25	0	249	7.108×10^{-6}	0.05
	ABAM	37	0	211	8.128×10^{-5}	0.044
	S2PBTI	37	0		6.518×10^{-6}	
10^{-6}	2BHM6	22	0	358	6.152×10^{-7}	0.055
	BHM7	46	0	518	1.733×10^{-6}	0.096
	BHM6	44	0	456	9.022×10^{-7}	0.091
	ABAM	78	0	451	2.605×10^{-6}	0.131
	S2PBTI	53	0		9.514×10^{-7}	
10^{-8}	2BHM6	39	0	641	3.167×10^{-8}	0.116
	BHM7	82	0	936	1.948×10^{-7}	0.183
	BHM6	77	0	796	1.705×10^{-8}	0.183
	ABAM	171	0	998	7.273×10^{-8}	0.248
	S2PBTI	72	0		1.834×10^{-8}	
10^{-10}	2BHM6	72	1	1230	1.933×10^{-9}	0.255
	BHM7	158	0	1858	9.893×10^{-9}	0.457
	BHM6	150	0	1608	1.795×10^{-9}	0.421
	ABAM	396	0	2338	1.955×10^{-9}	0.881
	S2PBTI	97	0		8.921×10^{-11}	

Table 4.12: Numerical results for Problem 3.

TOL	Method	TS	FS	FCN	MAXE	Time
10^{-2}	2BHM6	8	0	150	9.791×10^{-4}	0.02
	BHM7	16	0	196	6.092×10^{-3}	0.024
	BHM6	15	0	166	6.484×10^{-3}	0.024
	ABAM	21	1	141	4.047×10^{-2}	0.018
	S2PBTI	18	0	-	3.331×10^{-2}	
	CB(5,6)	20	0	-	3.224×10^{-3}	
	M2BM	20	0		4.205×10^{-4}	
10^{-4}	2BHM6	17	0	280	1.302×10^{-4}	0.056
	BHM7	33	0	383	9.920×10^{-5}	0.078
	BHM6	34	1	377	8.490×10^{-4}	0.131
	ABAM	55	1	341	3.699×10^{-4}	0.066
	S2PBTI	25	0	-	1.133×10^{-4}	
	CB(5,6)	29	0	-	1.690×10^{-4}	
	M2BM	37	0		3.218×10^{-6}	
10^{-6}	2BHM6	33	0	555	1.116×10^{-5}	0.123
	BHM7	79	0	935	1.873×10^{-5}	0.213
	BHM6	64	0	703	3.671×10^{-6}	0.211
	ABAM	134	3	819	1.418×10^{-5}	0.159
	S2PBTI	38	0	-	2.555×10^{-7}	
	CB(5,6)	42	0	-	4.095×10^{-7}	
	M2BM	76	2		5.771×10^{-7}	
10^{-8}	2BHM6	81	0	1371	2.449×10^{-7}	0.437
	BHM7	191	1	2344	3.741×10^{-7}	0.674
	BHM6	134	0	1487	1.114×10^{-7}	0.42
	ABAM	332	0	1980	3.684×10^{-8}	0.502
	S2PBTI	55	1	-	2.890×10^{-9}	
	CB(5,6)	80	0	-	1.544×10^{-9}	
	M2BM	164	3		5.470×10^{-9}	
10^{-10}	2BHM6	164	1	2832	1.093×10^{-8}	1.024
	BHM7	479	0	5818	3.718×10^{-8}	2.802
	BHM6	363	1	4107	1.021×10^{-8}	1.856
	ABAM	789	0	4738	1.538×10^{-9}	1.874
	S2PBTI	68	0	-	1.073×10^{-10}	
	CB(5,6)	151	0	-	6.302×10^{-12}	

Table 4.13: Numerical results for Problem 5.

TOL	Method	TS	FS	FCN	MAXE	Time
10^{-2}	2BHM6	8	0	132	3.808×10^{-2}	0.005
	BHM7	15	0	177	2.714×10^{-4}	0.01
	BHM6	15	0	162	4.490×10^{-3}	0.009
	ABAM	13	2	95	$1.038 \times 10^{+0}$	0.004
	CB(5,6)	19	0		2.492×10^{-4}	
	M2BM	20	0		3.705×10^{-5}	
10^{-4}	2BHM6	16	0	262	3.105×10^{-6}	0.012
	BHM7	29	0	333	3.036×10^{-5}	0.027
	BHM6	31	0	330	1.343×10^{-4}	0.029
	ABAM	53	1	328	4.794×10^{-4}	0.029
	CB(5,6)	29	0		1.909×10^{-5}	
	M2BM	37	0		3.370×10^{-6}	
10^{-6}	2BHM6	30	0	497	2.119×10^{-7}	0.038
	BHM7	51	0	576	6.020×10^{-8}	0.06
	BHM6	59	0	634	6.680×10^{-7}	0.051
	ABAM	130	1	793	1.161×10^{-5}	0.064
	CB(5,6)	42	0		4.325×10^{-7}	
	M2BM	74	1		3.318×10^{-7}	
10^{-8}	2BHM6	56	0	946	1.889×10^{-9}	0.067
	BHM7	89	0	1025	9.454×10^{-10}	0.09
	BHM6	113	0	1205	4.578×10^{-9}	0.095
	ABAM	318	0	1902	3.597×10^{-8}	0.176
	CB(5,6)	80	0		2.254×10^{-9}	
	M2BM	155	3		5.491×10^{-9}	
10^{-10}	2BHM6	112	0	1898	1.799×10^{-11}	0.154
	BHM7	156	0	1753	7.584×10^{-12}	0.218
	BHM6	224	0	2347	1.051×10^{-11}	0.264
	ABAM	774	0	4654	7.993×10^{-10}	0.876
	CB(5,6)	151	0		1.171×10^{-11}	

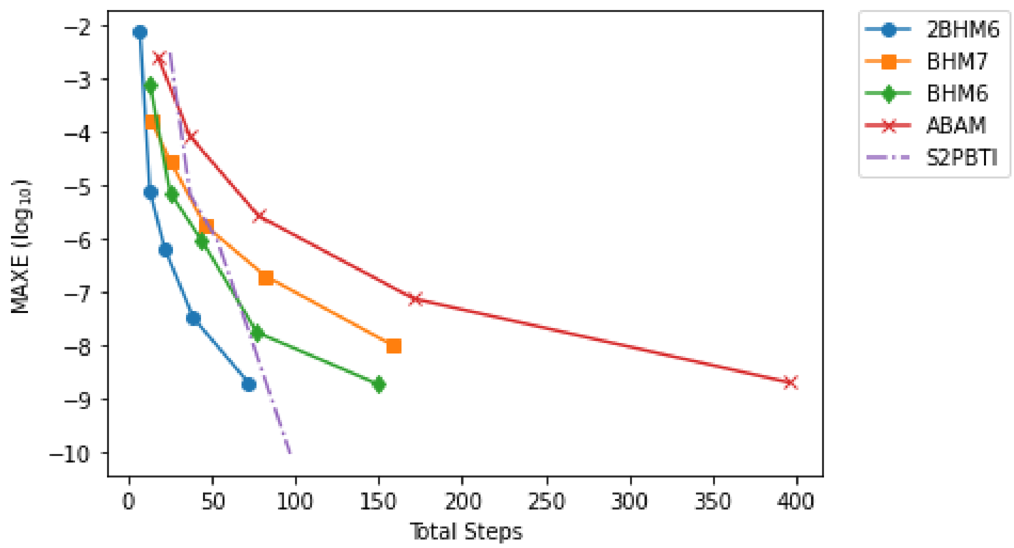

Figure 4.16: Maximum error (log_{10}) versus total steps for Problem 2.

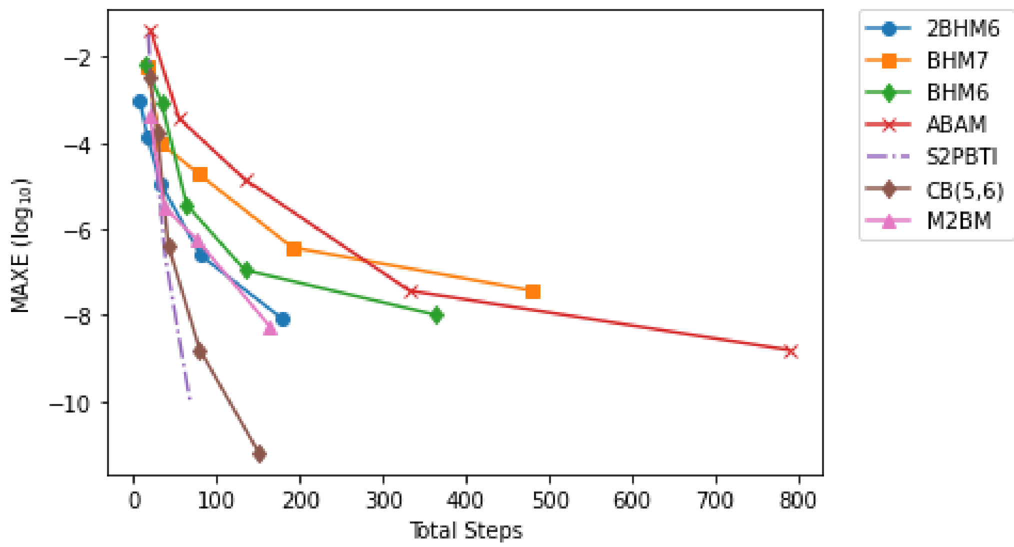

Figure 4.17: Maximum error (log_{10}) versus total steps for Problem 3.

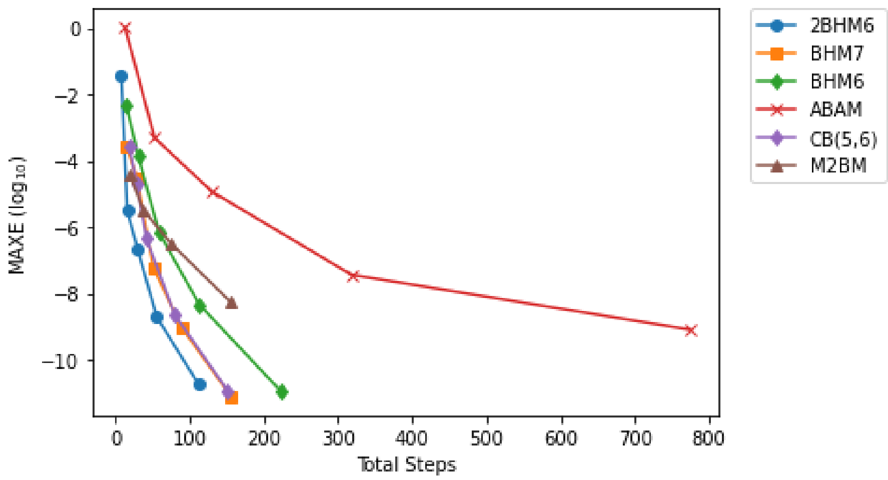

Figure 4.18: Maximum error (log_{10}) versus total steps for Problem 5.

4.4.1 Discussion III

The solutions of S2PBTI for solving Problem 2 are listed at Table 4.11. The solutions by S2PBTI, CB(5,6), and M2BM for solving Problem 3 are listed at Table 4.12. The solutions by CB(5,6) and M2BM to solve Problem 5 are listed at Table 4.13. The solutions of M2BM are empty at $TOL = 10^{-10}$ due to a lack of information.

In Problems 2 and 5, Figures 4.16 and 4.18 show that the line of 2BHM6 is more to the left than the line of other methods. That means 2BHM6 uses a less total step to reach the given tolerances. 2BHM6 is more efficient in solving Problems 2 and 5 among these methods. In Problem 2, S2PBTI achieves excellent accuracy and total steps when the tolerance decrease to 10^{-10}. 2BHM6 is less accurate compared to S2PBTI at tolerance 10^{-10}.

In Problem 3, Table 4.17 shows different outcomes compared to Problems 2 and 5. When the tolerance is less than or equal to 10^{-6}, S2PBTI and BC(5,6) have minor maximum error than the 2BHM6, and the maximum error of M2BM is almost the same as the 2BHM6. However, for the tolerance greater than 10^{-6}, 2BHM6 uses fewer total steps to reach a minor maximum error.

4.5 Conclusion

The 2 step block-hybrid method of order 6 and 1 step block-hybrid method of order 7 based on the predictor-corrector that had been derived can be used to solve RDDEs and achieve the first objective of this research. All the explicit and implicit methods derived have their stability regions. Implementating the variable step size strategy showed its efficiency in reducing the total steps or time taken to get the solution with errors within the desired tolerance compared to the constant step size strategy. The higher the order of the method, the accurate the results are calculated due to more reference points included in approximating the next point. The strategy of 2 step block-hybrid method has shown its ability to significantly decreasing the total steps for shortening the execution time compared to 1 step block-hybrid method. As a whole, the 2 step block-hybrid method of order six implemented in variable step size demonstrates significant improvement when solving RDDEs compared to BHM7 and the existing methods.

4.6 Future Research

This research can be further investigated in many directions. The strategy of involving more than 2 step in the block-hybrid method can be studied to investigate the maximum step to obtain a better result in accuracy and effectiveness. The off step point used in this research only considers the half step size from the main point. More off step points with different step sizes can be considered to get more accurate results when interpolating the delay values. Besides that, the method proposed in this research can be modified to solve the problem of NDDEs by inserting the step of function evaluation for the delay derivative.

In the variable step size strategy, the safety factor, C can be changed depends on the result or the number of failure steps. The LTE used in this research only considered the last main point, more LTE of the method can be included together to ensure good step size change. The ratio of $\frac{TOL}{LTE}$ needs not always be 1:1 and can be changed to 1:2, 1:3, or others.

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