SOME PROPERTIES OF g(x)-NIL CLEAN GRAPHS OF RINGS

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A project report submitted in partial fulfilment of the requirements for the award of Bachelor of Science (Honours) Applied Mathematics with Computing

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DECLARATION

I hereby declare that this project report is based on my original work except for citations and quotations which have been duly acknowledged. I also declare that it has not been previously and concurrently submitted for any other degree or award at UTAR or other institutions.

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APPROVAL FOR SUBMISSION

I certify that this project report entitled **"SOME PROPERTIES OF g(x)-NIL CLEAN GRAPHS OF RINGS"** was prepared by **VIVEK MENON NARAYANAN** has met the required standard for submission in partial fulfilment of the requirements for the award of Bachelor of Science (Honours) Applied Mathematics with Computing at Universiti Tunku Abdul Rahman.

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VIVEK MENON NARAYANAN

ABSTRACT

This project aims to study the relationship between ring theory and graph theory, specifically the study of nil-clean graph rings. Rings are those that satisfy the property of being an abelian group under addition and being closed under multiplication. Graphs are structures that consist of a vertex set and an edge set which together, will form various types of connections. Using these concepts, a nil-clean ring R is defined such that for each element $x \in R$, it is the sum of a nilpotent n and an idempotent e. Together with a brief review of previous literature, the properties of nil clean graphs of rings are discussed in the main reference article by Basnet an Bhattacharyya (2017), where two vertices x and y are considered adjacent if and only if $x + y$ is a nil clean element in R. This concept is then extended into $g(x)$-nil clean rings which are defined such that for each element $r \in R$, it is the sum of a nilpotent an a root of g(x). For the purpose of this project, $g(x) = x^2 - 3x$ is chosen, which yields a $x(x-3)$-nil clean ring. Subsequently, the graphs of $x(x-3)$-nil clean rings are generated and visualized with respect to $\mathbb{Z}_n$ and the properties of these graphs are generalised according to connectedness and presence of Hamiltonian cycles an paths.

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CHAPTER 1

INTRODUCTION

1.1 Background

In order to study the properties of g(x)-nil clean graphs of rings effectively, the foundation to this research involves the understanding of two main components, namely rings and graphs.

1.1.1 Rings

Rings are algebraic structures that generalize fields, such that any ring is a set R equipped with two binary operations, which are $+$ and $\cdot$. It follows a few basic properties listed below:

I. R forms an abelian group under addition, whereby the addition is always commutative, and that there is also an additive identity, which we will usually denote by 0.

II. R is closed under multiplication. A ring displaying multiplicative identity is called a ring with unity, whereas a ring that obeys multiplicative commutativity is known as a commutative ring.

We can use exponents to denote compounded multiplication and associativity can assure that the usual exponential rules are still applicable. It is important to note that for the purpose of this research, only finite commutative rings that are commutative with identity are studied.

1.1.2 Graphs

A graph G is defined to be a pair $(V(G), E(G))$, where $V(G)$ is a non-empty finite set of elements called vertices, and $E(G)$ a finite set of unordered pairs of elements of $V(G)$ called edges. $V(G)$ is known as the vertex-set of G and $E(G)$ is the edge-set of G. An edge $\{v, w\}$ is said to join the vertices v and w, and is usually abbreviated to vw. The number of vertices in G is often called the order of G, while the number of edges is referred to as its size. Generally, these graphs can be classified as directed or undirected, weighted or unweighted as well as cyclic or acyclic.

For this project, the concepts of graphs and rings are to be consolidated when analyzing the main properties and graphical components of commutative nil clean graphs as discussed in the main article of reference by Basnet and Bhattacharyya (2017).

1.2 Aim and Objectives

For the duration of Project I, the objective of this project is to understand the basic properties of graphs and rings and link them together in the study of nil clean graphs of rings for the analysis of mathematical proofs based on the publication by Basnet and Bhattacharyya (2017).

As for the duration of Project II, the objective of this project is to apply and extend the concepts of general nil-clean graphs of rings to $x(x-3)$-nil clean graphs of rings, with the intended aim of using programming to visualize such graphs.

1.3 Scope of the Study

The first stage of the project focuses on analysis of finite commutative rings with respect to certain properties in the field of Graph Theory, namely dominating sets, girth and diameter, with an emphasis on simple undirected graphs.

The second stage of the project would apply similar graph properties to a more specific set of graphs of rings. Once the first stage of the project has been completed, it would provide a suitable setting to specify the scope of the next phase.

1.4 Project Timeline

The schedule for working on the project is shown in the Gantt Chart below:

TASKS	WEEK												
	01	02	03	04	05	06	07	08	09	10	11	12	13
Title registration	■												
Preliminary discussion with supervisor		■											
Sourcing study and research materials		■	■	■									
Biweekly report submissions			■		■		■		■		■		■
Compilation of materials for literature review				■	■	■							
Project proposal submission							■						
Continuous project analysis							■	■					
Proposal mock presentation								■					
Report writing and final revision									■	■	■		
Interim report submission												■	
Project oral presentation													■

Table 1(a): Schedule for Project I during the January 2021 Trimester

TASKS	WEEK												
	01	02	03	04	05	06	07	08	09	10	11	12	13
Project Discussion with Supervisor	■	■	■	■	■	■	■	■	■	■	■	■	
Continue research and material compilation	■	■	■	■	■	■	■	■					
Submission of Mid-Semester Monitoring Form							■						
Project Poster Preparation										■	■		
Final Report Preparation										■	■	■	
Project Poster submission												■	
Final Report submission												■	
Project oral presentation													■

Table 1(b): Schedule for Project II during the May 2021 Trimester

This timeline above had been strictly adhered to, which allowed for sufficient time to complete the necessary tasks to accomplish the goals of this project.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

This section showcases some relevant literature referenced in the research paper by Basnet and Bhattacharyya (2017). The preliminary work done by previous mathematicians serve as an introduction to the comprehensive concepts presented in the primary article of research.

2.2 Review of Literature in Primary Article

Let R be a commutative ring. Consider R as a simple graph whose vertices are the elements of R, such that two different elements x and y are adjacent if and only if $xy = 0$. Let a be an element of a ring R. It has been established that a is:

1. nilpotent if $a^k = 0$ for some $k \in \mathbb{N}$; in any ring, 0 is a nilpotent element and is generally known as a trivial nilpotent.
2. idempotent if $a^2 = a$.
3. a unit if a has a multiplicative inverse, i.e., if an element b exists in R such that $ab = 1 = ba$;
4. a zero divisor if there exists a non-zero element $b \in R$ such that $ab = ba = 0$ where $a \neq 0$.

Using the concept of chromatic number of a graph, the idea of finitely coloured commutative rings was first highlighted by Beck (1988). Soon after, the concept of unit rings was highlighted in studies conducted by Ashrafi (2010). For any ring R, let $U(R)$ be the set of unit elements of R. The unit graph of R, denoted $G(R)$, is the graph obtained by setting all the elements of R to be distinct vertices such that x and y are adjacent if and only if $x + y \in U(R)$. If the word “distinct” was omitted in the definition, a closed unit graph denoted $G(R)$ is obtained, whereby the graph may contain loops – which is out of the scope of study.

To explain the idempotency and nilpotency of a ring, let the sets of idempotents and nilpotents of R to be denoted by $Id(R)$ and $Nil(R)$

respectively. Nicholson (1977) has defined that an element x in R is said to be clean if there exist an $e \in Id(R)$ such that $x - e$ is a unit of R. In a more generalized manner, presented by Han & Nicholson (2001), an element x in R is called clean if it can be expressed as the sum of an idempotent and a unit in R. The ring R is called a clean ring if every element of R is clean.

A more specified outlook was introduced by Diesl (2013), which are nil clean rings and strongly nil clean rings. A ring R is called nil clean ring if for each $x \in R$ such that $x = n + e$, for some $n \in Nil(R)$ and $e \in Id(R)$. A study by Danchev and McGovern (2015) further branched into weakly nil clean ring. An element $x \in$ R is called weakly nil clean if $x = n - e$ or $x = n + e$ for some $n \in Nil(R)$ and $e \in Id(R)$. The above result was also visible in a study from Kosan & Zhou (2016). However, Danchev and McGovern (2015) also mentions that an element in a ring is unipotent if it can be written as $1 + b$ for some nilpotent $b \in R$. It is imperative to examine that any given ring is weakly nil clean if and only if every element can be written as either the sum of a nilpotent and an idempotent, or of a unipotent and an idempotent.

2.3 Further Concepts

If considering the graph of a nil clean ring R, denoted by $G_N(R)$, it is defined by applying R as a vertex set. Two distinct vertices x and y are adjacent if $x + y$ is a nil clean element in R, not taking into consideration loops at a particular vertex.

Firstly, there are some important information to consider for the duration of this project. For instance, the number of edges joined to a vertex on a graph G, where $x \in V(G)$ can be identified as the degree of the vertex, $deg(x)$. In addition, the neighbourhood of a vertex x is defined as $N_G(x) = \{y \in V(G) : x \text{ and } y \text{ are adjacent}\}$, where $N_G[x] = N_G(x) \cup \{x\}$. Also, it is important to understand the concept of a complete graph, whereby every pair of distinct vertices within the graph are connected by an edge that is unique.

As described in the scope of this project, some Graph Theory properties that will be considered are chromatic index, diameter and girth. For instance, girth would refer to the length of the shortest cycle in the graph. Using the $GF(25)$ graph, a finite field of 25 elements, illustrated by Basnet and Bhattacharyya (2017) as an example to show the application of girth:

$$\begin{aligned} GF(25) &\cong \mathbb{Z}_5[x]/\langle x^2 + x + 1\rangle \\ &= \{ax + b + \langle x^2 + x + 1\rangle : a, b \in \mathbb{Z}_5\}. \end{aligned}$$

Let us define $\alpha := x + \langle x^2 + x + 1\rangle$. Then we have

$$\begin{aligned} GF(25) = \{&\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \alpha, 2\alpha, 3\alpha, 4\alpha, 1+\alpha, 1+2\alpha, 1+3\alpha, \\ &1+4\alpha, 2+\alpha, 2+2\alpha, 2+3\alpha, 2+4\alpha, 3+\alpha, 3+2\alpha, \\ &3+3\alpha, 3+4\alpha, 4+\alpha, 4+2\alpha, 4+3\alpha, 4+4\alpha\}. \end{aligned}$$

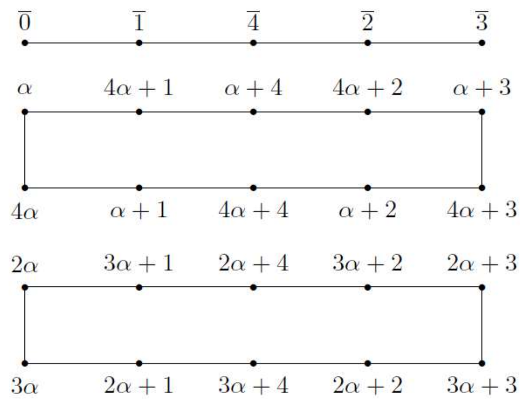

Figure 1: The nil clean graph of $GF(25)$

By looking at the above nil-clean graph figure, it can be seen that the length of the shortest cycle is 10, hence indicative of the girth of $GF(25)$ which is 10.

Another important concept to be studied in relation to the project is the chromatic index. It refers to the minimum number of colours needed to make a proper colouring for a graph G such that for any two adjacent edges $x, y \in E(G)$, the colour of x and y will not be the same (Chartrand et. al., 2009) . In the case of nil clean graphs of rings, we assume Δ to be the maximum vertex degree of $G_N(R)$. According to Vizing's Theorem, $\Delta \leq \chi'\big(G_N(R)\big) \leq \Delta + 1$ and it further classifies graphs into class 1 and 2 for graphs that satisfy the conditions $\chi'\big(G_N(R)\big) = \Delta$ and $\chi'\big(G_N(R)\big) = \Delta+1$ respectively. To illustrate the colouring concept further, we can examine the diagram below:

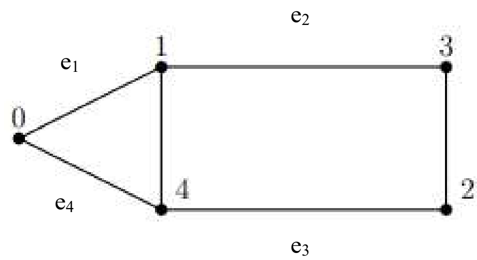

Figure 2: Part of the nil clean graph of $GF(25)$

Based on the diagram above, it can be seen that vertex 1 is connected to three other vertices, namely 0, 3 and 4 via edges $e_1, e_2$and e_3 respectively. From the definition, these three edges cannot be coloured with the same colour. However, since e_2 and e_4 are not adjacent, these edges may have the same colouring. In addition, the maximum vertex degree belongs to vertices 1 and 4, with a degree of 3. Hence, according to Vizing's theorem, Δ = 3, satisfying the condition 3 $\leq \chi'\left(G_N(R)\right) \leq 4$.

Another important property that will be studied in this project is the diameter of a graph $G_N(R)$. Firstly, it is important to note that the distance between two vertices x and y, $d(x, y)$ is referred to as the number of edges that lie on the shortest path between the two vertices. If there is no established path between the vertices, then $d(x, y) = \infty$. The maximum distance of all distinct vertex pairs within a graph is referred to as the diameter, $diam(G_N(R)$ (Chen & Garfinkel, 1982).

CHAPTER 3

PRELIMINARY RESULTS

3.1 Methodology

The majority of this project requires working on the chosen research article by Basnet and Bhattacharyya (2017) with a critical and analytical outlook as well as a systematic collection of relevant information from other similar material and articles.

Considering the nature of this project to be more theoretical, the project will require extensive research and acquisition of materials pertaining ring theory, graph theory as well as linear algebra fundamentals. With this basic understanding, a proper analysis of the main research article can be conducted thoroughly. This would include breaking down the proofs presented in each section to firmly establish the properties of nil-clean ring graphs.

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3.2 Basic Properties of the Nil Clean Graphs

This section examines the theories that develop the fundamental properties of the nil-clean graph of a finite commutative ring.

Theorem 3.1. *The nil clean graph $G_N(R)$ is considered complete if and only if R is a nil clean ring.*

Proof: (⇒): For a ring R, let $G_N(R)$ be a complete nil clean graph. For all $r \in R$, if r and 0 are adjacent to each other, there exists a path connecting r and 0 such that $r + 0 = r$. This implies that the nil clean property is satisfied, hence, R is considered to be nil clean.

(⇐): The converse is true by the nil clean graph definition. Suppose R is a nil clean ring. For any elements $x, y \in R$, they are connected to each other if and only if $x + y$ is nil clean. It implies that distinct element pairings of the ring are to form a unique edge set. Hence $G_N(R)$ is said to be a complete nil clean graph.

Lemma 3.2. *Let R be a ring and idempotents lift modulo $Nil(R)$. If elements $x + Nil(R)$ and $y + Nil(R)$ are adjacent in $G_N(R/Nil(R))$, then the adjacency of the elements $x + Nil(R)$ and $y + Nil(R)$ holds true in the nil clean graph of $G_N(R)$.*

Proof: If the idempotents lift modulo $Nil(R)$, it implies that for any ring element $x^2 = x \in R/Nil(R)$, there exists an idempotent $e^2 = e \in R$ such that $e - x \in R/Nil(R)$. Subsequently, we have

$$x + Nil(R) + y + Nil(R) = n + e + Nil(R)$$

$$x + Nil(R) + y + Nil(R) = e + Nil(R), \text{ as } n \in Nil(R)$$

$$(x + y) + Nil(R) = e + Nil(R)$$

Thus, we have

$$x + y + n_1 = f + n_2, \text{ where } n_1, n_2 \in Nil(R) \text{ and } f \in R.$$

Hence,

$$x + y = f - n_1 + n_2 = f + n', \text{ where } n' = -n_1 + n_2$$

The above result implies that $x + y \in R$, generalized from the idea that $x + y \in R/Nil(R)$.

Lemma 3.3. *Let $G_N(R)$ be the nil clean graph of a ring R. For $x \in R$ we have the following:*

(i) If $2x$ is nil clean, then $deg(x) = |NC(R)| - 1$.

(ii) If $2x$ is not nil clean, then $deg(x) = |NC(R)|$.

Proof: Suppose $x \in R$ and it is evident that $x + R = R$. Hence there exists a unique element $x_y \in R$ for each $y \in NC(R)$ such that $x + x_y = y$.

So, $deg(x) \leq |NC(R)|$.

To show (i): Suppose $\{x_1, x_2, x_3, \ldots, x\} \subseteq R$ and $\{x + x_1, x + x_2, x + x_3, \ldots, 2x\} \subseteq NC(R)$. An illustration of the graph is shown below, by taking into account that the graph has no loops and multiple edges.

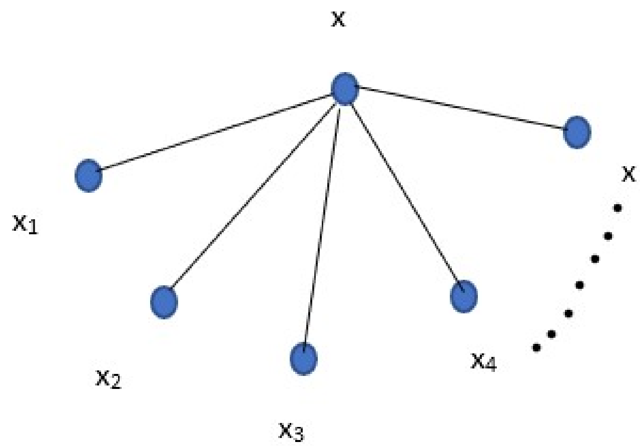


Figure 3: Nil clean graph of x elements – including x

Since $N_{G_N(R)}(x)$ comprises of a set of elements $y \subseteq V(G_N(R))$ such that x and y are adjacent to each other and that y = $\{x_1, x_2, x_3, \dots, x\}$, we can note a clear one-to-one correspondence. Therefore,

$$deg(x) = |N_{G_N(R)}(x)| = |N_{G_N(R)}[x]| - 1 = |NC(R)| - 1.$$

To show (ii): Suppose $\{x_1, x_2, x_3, \dots, x_y, x\} \subseteq R$ and $\{x + x_1, x + x_2, x + x_3, \dots, x + x_y\} \subseteq NC(R)$ but $2x$ is no longer nil clean. Leaning into the same prerequisites as Figure 3, we can show the following illustration:

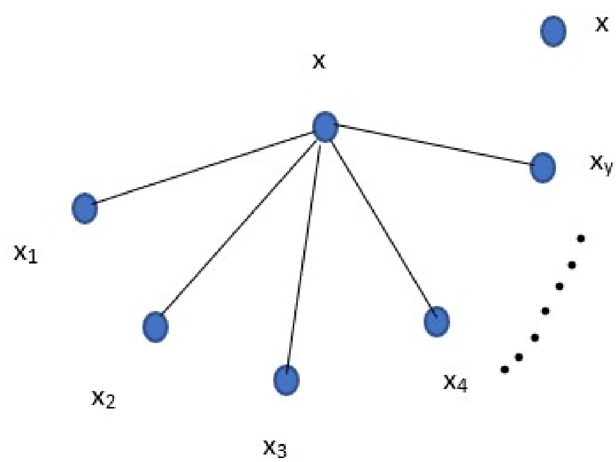


Figure 4: Nil clean graph of x elements – not including x

Similar to case (i), $N_{G_N(R)}(x)$ comprises of a set of elements $y \subseteq V(G_N(R))$ such that x and y are adjacent to each other and that y = $\{x_1, x_2, x_3, \dots, x_y\}$. Therefore,

$$deg(x) = |N_{G_N(R)}(x)| = |N_{G_N(R)}[x] = |NC(R)|.$$

Lemma 3.4. *A reduced ring R is finite commutative with no non-trivial idempotents if and only if R is a finite field.*

Proof: For any reduced ring R that is finite commutative, it implies that R only has zero nilpotent elements. Hence, we assume the nilpotent to be $\overline{0}$ and the idempotents to be $\overline{0}$ and $\overline{1}$.

(⇒): Suppose $\overline{0} \neq x \in R$. We have a finite set $\{x^k\}$, where k is a natural number. There exists a value $m \geq l$, such that $x^l = x^m$. Using a specific example, we study the field of $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ using $x = 5$ as an example.

If $l = 1$ and $m = 2$, the following result is obtained:

$$5^1 = 5 \, mod \, 4 = 1 \text{ and } 5^2 = 25 \, mod \, 4 = 1.$$

Hence, $5^1 = 5^2$ ($m > l$, proven).

In more general terms:

$x^l = x^m = x^{m-l+l} = x^{m-l} \cdot x^l = x^{m-l} \cdot x^m = x^{(2m-l)+l-l} = x^{2(m-l)+l}$

This can continue to generalise into $x^{k(m-l)+l}$, where k is a natural number.

We now have $\left(x^{l(m-l)}\right)^2$

$$\begin{aligned}
&= x^{l(m-l)} \cdot x^{l(m-l)} \\
&= x^{l(m-l)+l(m-l)+l-l} \\
&= x^{l(m-l)+l} \cdot x^{l(m-l)-l} \\
&= x^l \cdot x^{l(m-l)-l} \text{ (since } x^l = x^{k(m-l)+l}\text{)} \\
&= x^{l+l(m-l)-l} \\
&= x^{l(m-l)}
\end{aligned}$$

The above result indicates that $x^{l(m-l)}$ is an idempotent of ring R. Hence, $x^{l(m-l)} = \overline{1}$, implying that x is a unit. Also, since $x^{l(m-l)} = x^l \cdot x^{l(m-l)-l} = \overline{1}$, x will have an inverse, which is $x^{l(m-l)-l}$. Since an inverse exists, R is therefore, a finite field.

(⇐): Let R be a finite field. It implies that an inverse will exist for all $x \in R$. Assume x is a non-zero nilpotent element in R. So, $x^n = 0$ for some n ∈ ℕ. The presence of an inverse would indicate that $x^n\ (x^{-1})^{n-1} = 0(x^{-1})^{n-1}$, implying that $x = 0$. This is a contradiction since x is assumed to be non-zero. Hence, R cannot have any non-zero nilpotents.

3.3 Invariants of Nil Clean Graphs

This section examines the proofs of nil clean graph properties in relation to graph theory invariants.

3.3.1 Girth

Theorem 3.5. *The statements below are true for the nil clean graph $G_N(R)$:*

(i) If R is not a field, then the girth of $G_N(R)$ = 3

(ii) If R is a field, then

(a) the girth of $G_N(R)$= 2p if R ≅ GF(p^k) (the field of order p^k), where p is an odd prime number and k > 1; and

(b) otherwise, the girth of $G_N(R)$ is infinite, making $G_N(R)$ a path.

Proof: To show (i): Based on Lemma 3.4., if a ring R is not a field, then it cannot be considered commutative with trivial idempotents. Hence, it implies that R must contain a minimum of one non-trivial nilpotent or idempotent. Consider a non-trivial idempotent $e \in R$, there exists a cycle of length 3 in $G_N(R)$ as shown below:

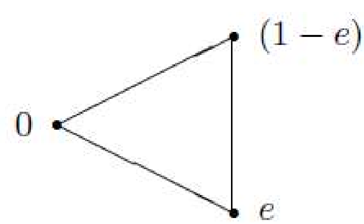


Figure 5: A cycle of length 3 for a non-trivial idempotent

The diagram above implies the girth of $G_N(R)$ to be 3. Alternatively, we may consider a non-trivial nilpotent $n \in R$, there exists a cycle of length 3 in $G_N(R)$ as show below:

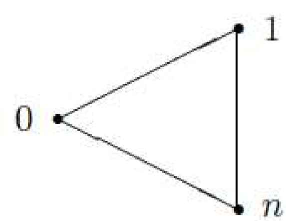


Figure 6: A cycle of length 3 for a non-trivial nilpotent

To show (ii): For a finite field, the nil clean set of elements is defined as $\{\bar{0}, \bar{1}\}$. Hence the nil clean graph for $\mathbb{Z}_p$, for any prime number p is illustrated below:

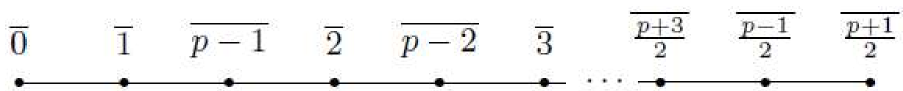


Figure 7: The nil clean graph of $\mathbb{Z}_p$

It is evident from the above figure that $G_N(\mathbb{Z}_p)$ possesses an infinite girth, thus rendering statement (ii)(b) to be true. It can be implied that the nil clean graph of $GF(p^k)$, where $p > 2$, is disconnected based on how finite field are categorized. Moreover, the graph of $GF(p^k)$ is made up of a path of length p containing $\frac{p^{k-1}-1}{2}$ $2p$-cycles. Consider $GF(p^k)$ =$\mathbb{Z}_p[X]/\langle f(x)\rangle$, $f(x)$ being an irreducible polynomial of degree k over $\mathbb{Z}_p$. Suppose $B \subseteq GF(p^k)$, where B is made up of linear combinations of $x, x^2, \dots, x^{k-1}$ consisting of coefficients from $\mathbb{Z}_p$. Taking these facts into consideration, if $g(x) + \langle f(x)\rangle \in B$, then $-\ g(x) + \langle f(x)\rangle \notin B$. Hence, B can be described as follows:

$$B = \{\overline{g_\iota(x)} = \ g_i(x) + \langle f(x)\rangle : 1 \le i \le \frac{p^{k-1}-1}{2}\}$$

This produces a nil clean graph of $GF(p^k)$ as follows:

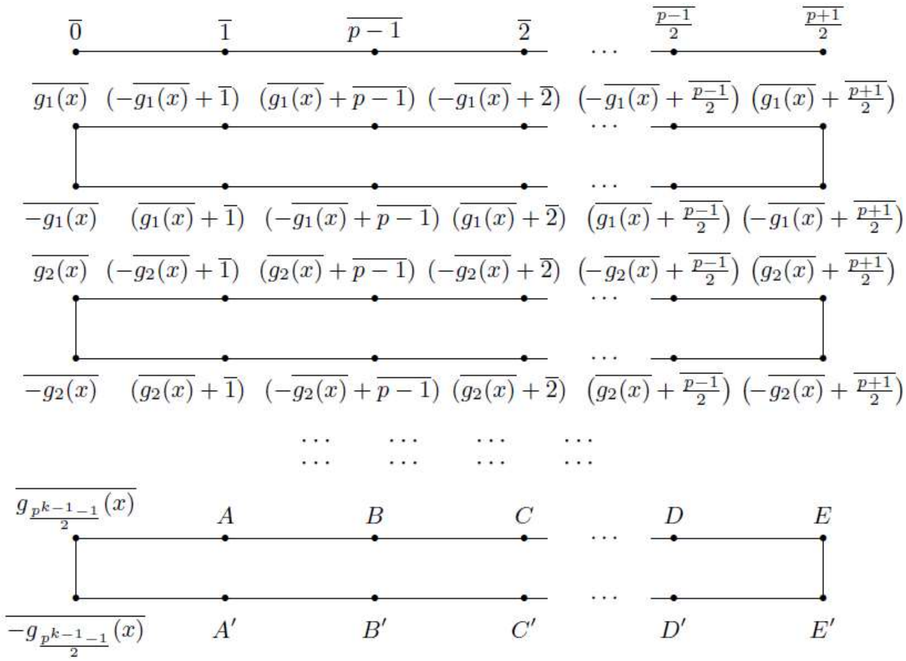


Figure 8: The nil clean graph of $GF(p^k)$

Based on the graph, we have the following:

$$
\begin{aligned}
&A = -\overline{g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{1}, \quad B = -\overline{g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{p-1},\\
&C = \overline{-g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{2}, \quad D = \overline{-g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{\frac{p-1}{2}},\\
&E = -\overline{g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{\frac{p+1}{2}},\\
&A' = \overline{g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{1}, \quad B' = -\overline{g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{p-1},\\
&C' = \overline{g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{2}, \quad D' = \overline{g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{\frac{p-1}{2}},\\
&E' = -\overline{g_{\frac{p^{k-1}-1}{2}}(x)} + \overline{\frac{p+1}{2}}.
\end{aligned}
$$

3.3.2 Chromatic Index

Theorem 3.6. *If R is a finite commutative ring, then nil clean graph of R is of class 1.*

Proof: An edge ab is coloured by the colour $a + b$. By this technique, it can be said that any two distinct edges ab and ac would be coloured differently. Suppose a set of colours $C = \{a + b : ab \text{ is an edge in } G_N(R)\}$.

To illustrate this concept, let $\{x, x_1, x_2, \dots, x_n\} \subseteq R$ where $C = \{x + x_1, x + x_2, \dots, x + x_n\} \subset NC(R)$. The graph below is obtained:

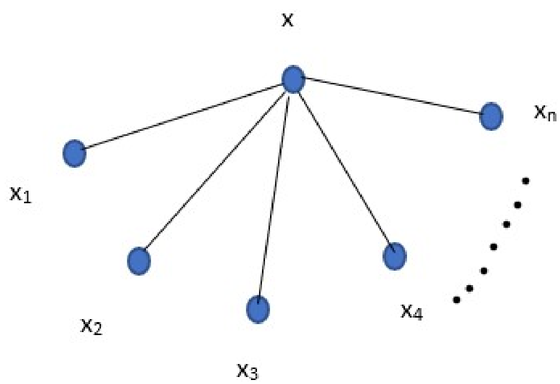


Figure 9: The graph of $G_N(R)$ containing $(n + 1)$ elements

The above graph would contain a $|C|$ -edge colouring, implying that $\chi'(G_N(R)) \leq C$. As established, $C \subseteq NC(R)$ and so, $|C| \leq |NC(R)|$. Hence, $\chi'(G_N(R)) \leq |NC(R)|$. From Lemma 3.3, it is clear that $deg(x) \leq |NC(R)|$ and $deg \leq |NC(R)| - 1$. It would then imply that $deg(x) \leq deg(x) + 1 \leq |NC(R)|$. So, for $deg(x) = \Delta = |C|$ and $\Delta \leq |NC(R)|$. By Vizing's Theorem, $\chi'(G_N(R)) \geq \Delta = |NC(R)|$. Hence, $\chi'(G_N(R)) = |NC(R)| = 1$. In other words, $G_N(R)$ is of class 1.

3.3.3 Diameter

Lemma 3.7. *A ring R is a nil clean ring if and only if* $diam(G_N(R)) = 1$.

Proof: (⇒): Suppose R is a nil clean ring. It implies that all elements within the ring must be nil clean elements. It is a fact that the nil clean elements are the sum of a nilpotent and an idempotent, resulting in the diameter of $G_N(R)$ to be 1, as illustrated below:

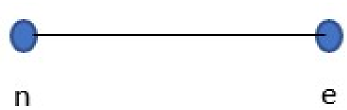


Figure 10: The nil clean graph for the nilpotent and idempotent

(⇐): Suppose $diam(G_N(R))$ = 1, where the maximum distance of each pair of distinct vertices in $G_N(R)$ is 1. For any nil clean elements $x, y \in R$, $x + y$ must be nil clean as well. Following this, it implies all elements in *R* must be nil-clean, which indicates that R must be a nil clean ring.

Theorem 3.8. *If R is a weakly nil clean ring with no non-trivial idempotents, but not nil-clean, then* $diam(G_N(R))$ *= 2.*

Proof: Assume *R* as weakly nil clean with no non-trivial idempotents, taking $x = n_1 - 1, n_1, n_1 + 1$ and $y = n_2 - 1, n_2, n_2 + 1$, where $x, y \in R$ for some n_1, $n_2 \in Nil(R)$. We will have nine possible combination of distances $d(x, y)$ as follows:

$G_N(R)$	$d(x, y)$
$[n_1]$ —— $[n_2]$	1
$[n_1]$ —— $[n_2+1]$	1
$[n_1]$ —— [1] —— $[n_2 - 1]$	2
$[n_1+1]$ —— $[n_2]$	1
$[n_1+1]$ —— [-1] —— $[n_2 + 1]$	2
$[n_1+1]$ —— $[n_2 - 1]$	1
$[n_1 - 1]$ —— [1] —— $[n_2]$	2
$[n_1 - 1]$ —— $[n_2 + 1]$	1
$[n_1 - 1]$ —— [1] —— $[n_2 - 1]$	2

Table 2: The possible values of $d(x, y)$ for each $G_N(R)$

The table clearly shows that $diam(G_N(R)) \leq 2$. But since R is weakly nil clean and not nil clean, there is at least one $x \in R$ which is not considered nil clean. In other words, $x = n - 1$ but $x \neq n + 1$. This would indicate $d(0, x) = 2$ as [0] —— [1] —— [n – 1], thus making $diam(G_N(R)) \geq 2$. The combination of the two results shows that $diam(G_N(R))$ = 2.

Theorem 3.9. *Let $R = A \times B$ such that A is nil clean and B weakly nil clean with no non-trivial idempotents. Then $diam(G_N(R)) = 2$.*

Proof: From the above, we can say that $Id(R) = \{(e, 0_B), (e, 1_B) : e \in Id(A)\}$ since A is nil clean with no non-trivial idempotents. Suppose (a_1, b_1), (a_2, b_2) $\in R$. If $(a_1, b_1) + (a_2, b_2)$ is nil clean, then $d((a_1, b_1), (a_2, b_2)) = 1$ in $G_N(R)$. However, if $(a_1, b_1) + (a_2, b_2)$ is not nil clean, it can be said that $b_1 + b_2$ is not nil clean.

To assess further, let $b_1 = n_1 - 1, n_1, n_1 + 1$ and $b_2 = n_2 - 1, n_2, n_2 + 1$. We can study the occurrences in four separate cases as follows:

<table>
<tr><th>b_1</th><th>b_2</th><th>Path formed</th><th>$d((a1, b1), (a2, b2))$</th></tr>
<tr><td>n_1</td><td>$n_2 - 1$</td><td rowspan="3">[a1, b1] — [0,1] — [a2, b2]</td><td rowspan="4">≤ 2</td></tr>
<tr><td>$n_1 - 1$</td><td>n_2</td></tr>
<tr><td>$n_1 - 1$</td><td>$n_2 - 1$</td></tr>
<tr><td>$n_1 + 1$</td><td>$n_2 + 1$</td><td>[a1, b1] — [0,-1] — [a2, b2]</td></tr>
</table>

Table 3: The distance for each case in the idempotent set where $b_1 + b_2$ is not nil clean.

The table clearly shows that $diam(G_N(R)) \leq 2$. However, since R is not nil clean, $diam(G_N(R)) \geq 2$. As such, $diam(G_N(R)) = 2$.

3.4 Contradiction in Proof of Theorem

This section discusses an inconsistency in the proof of a theorem published by Basnet and Bhattacharyya (2017).

Theorem: *Let R be a weakly nil clean ring such that R has no non-trivial idempotents. Then {1, 2} is a dominating set for* $G_N(R)$.

The authors assume R to be a weakly nil clean ring with only trivial idempotents, where $Id(R) = \{0,1\}$. Since a weakly nil clean ring can be defined as $n - e$ or $n + e$, it implies that for any $a \in R$, it can be defined as:

(i) $a = n - 0 = n + 0 = n$

(ii) $a = n + 1$

(iii) $a = n - 1$.

We can prove that $\{1\}$ is a dominating set element of $G_N(R)$.

Case (i): If $a = n$, then $a + 1 = n + 1 \in NC(R)$

Case (ii): If $a = n + 1$, then $a - 1 = (n + 1) - 1 = n \in NC(R)$

Case (iii): If $a = n - 1$, then $a + 1 = (n - 1) + 1 = n \in NC(R)$

The above three cases imply that a is connected to 1.

However $\{2\}$ being a dominating set element of $G_N(R)$ is not necessarily true as indicated by the article. For an initial assumption of $2 = n_1+1$ for some nilpotent $n_1 \in R$, it implies that $n_1 = 1$. Based on the definition of a nilpotent, the above implication is false since $1^n \neq 0$ for any value of n. Hence, it disproves $\{2\}$ from being a dominating set element of $G_N(R)$.

CHAPTER 4

$g(x)$-NIL CLEAN GRAPHS

4.1 Introduction

For any ring R, let $C(R)$ br the center of the ring with $g(x)$ as a fixed polynomial in $C(R)[x]$. An element $r \in R$ is said to be $g(x)$-nil clean if every element in R is the sum of a nilpotent and a root of $g(x)$. In other words, $r = n + s$, where $n \in Nil(R)$ and $s \in R$ such that $g(s) = 0$ (Handam & Khashan, 2017). If every element in R is $g(x)$-nil clean, then the ring R is classified as $g(x)$-nil clean.

Let $G_{N*}(R)$ denote the $g(x)$-nil clean graph of ring R, where the set of $g(x)$-nil clean elements of ring R denote by $N^*(R)$. For any two distinct vertices x and y which are the elements from the $g(x)$-nil clean ring R, it is said that x and y are adjacent if and only if $x + y \in N^*(R)$.

4.2 The $x(x-3)$-nil clean graphs

Throughout the rest of the study, a specific $g(x)$ is chosen to be studied, where $g(x) = x^2 - 3x \in C(R)[x]$. It implies that the roots of $g(x)$ are 0 and 3. Hence, some properties and examples of such $g(x)$-nil clean graphs of rings will be analyzed in the subsequent sections.

4.2.1 The graph of GF(25) as a $x(x-3)$-nil clean graph

In section 2.3, it has been previously established that $GF(25)$ is a finite field containing 25 elements and is nil-clean. However, GF (25) is also $x(x-3)$-nil clean as shown below.

$$\begin{aligned} GF(25) &\cong \mathbb{Z}_5[x]/\langle x^2+x+1\rangle \\ &= \{ax+b+\langle x^2+x+1\rangle : a, b \in \mathbb{Z}_5\}. \end{aligned}$$

Let us define $\alpha := x + \langle x^2+x+1\rangle$. Then we have

$$\begin{aligned} GF(25) = \{&\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \alpha, 2\alpha, 3\alpha, 4\alpha, 1+\alpha, 1+2\alpha, 1+3\alpha, \\ &1+4\alpha, 2+\alpha, 2+2\alpha, 2+3\alpha, 2+4\alpha, 3+\alpha, 3+2\alpha, \\ &3+3\alpha, 3+4\alpha, 4+\alpha, 4+2\alpha, 4+3\alpha, 4+4\alpha\}. \end{aligned}$$

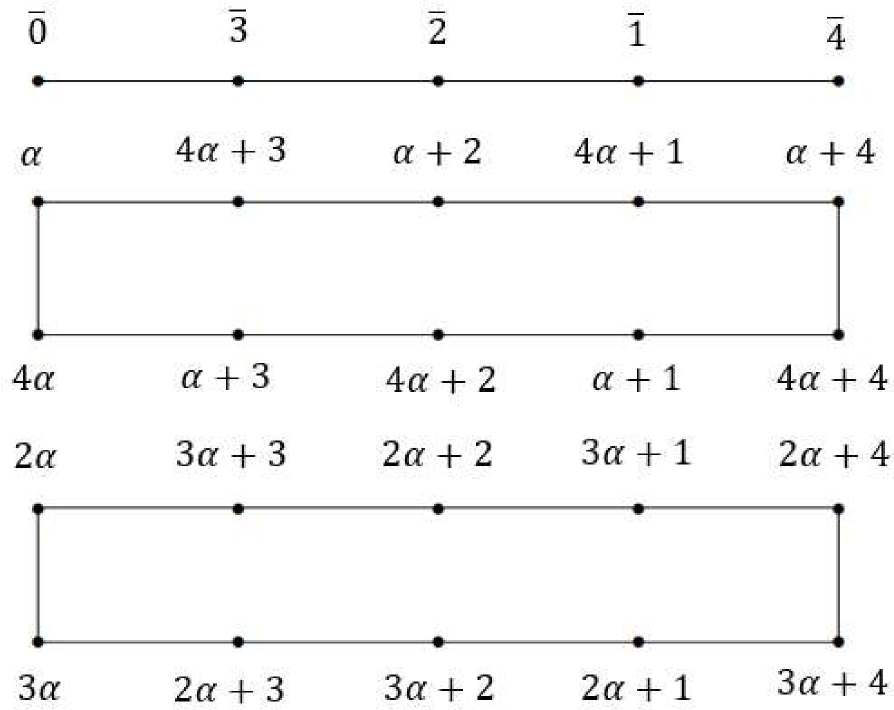


Figure 11: $g(x)$-nil clean graph of $GF(25)$

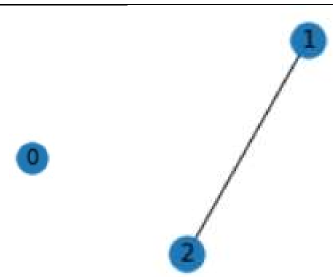
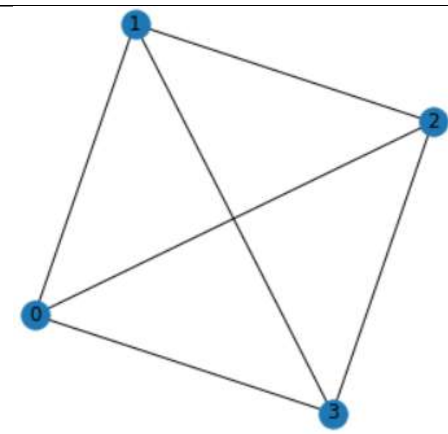
4.2.2 $\mathbb{Z}_n$ graphs as $x(x-3)$-nil clean graphs

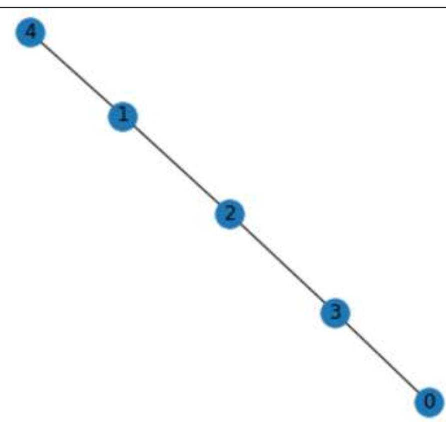
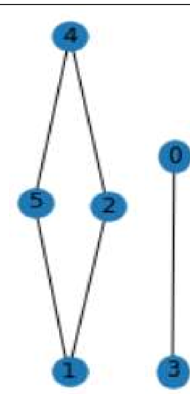
In section 3.3.1, $\mathbb{Z}_n$ graphs have been established as nil-clean. However, the graphs of $\mathbb{Z}_n$ are also $x(x-3)$-nil clean. The following set of diagrams illustrate all graphs of $\mathbb{Z}_n$ where the vertex set of $\mathbb{Z}_n$, denoted by $V(\mathbb{Z}_n) = \{0, 1, 2, \ldots, n-1\}$ for $3 \leq n \leq 40$.

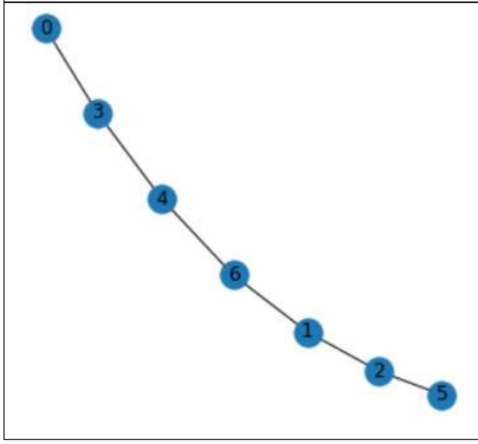
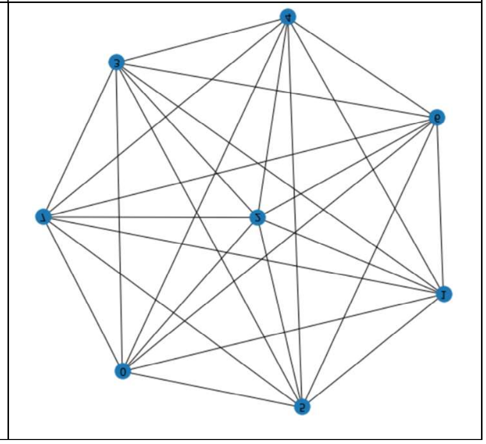
A sample of the code used is shown in the screenshot below. The codes in the figure below are specific to $\mathbb{Z}_{27}$. However, all values of n were changed manually to print out the graphs within the range studied.

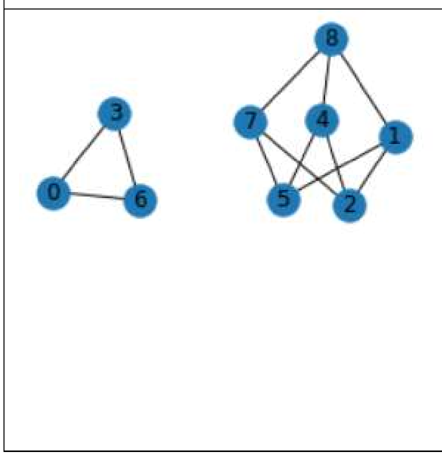
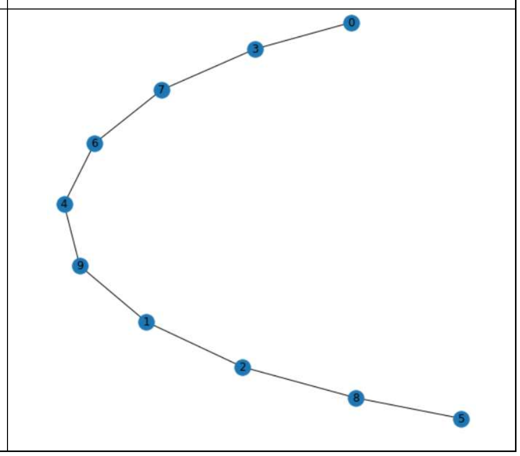
```
G = nx.Graph()
for x in range (27):
    for y in range (27):
        nil = x + y
        for n in range(10):
            if ((nil)**n % 27 == 0):
                G.add_edge(x,y)
            if ((nil - 3)**n % 27 == 0):
                G.add_edge(x,y)
            if (nil) % 27 == 0 or (nil) % 27 == 3:
                G.add_edge(x,y)

plt.figure(figsize=(9,9))
nx.draw_networkx(G)
```

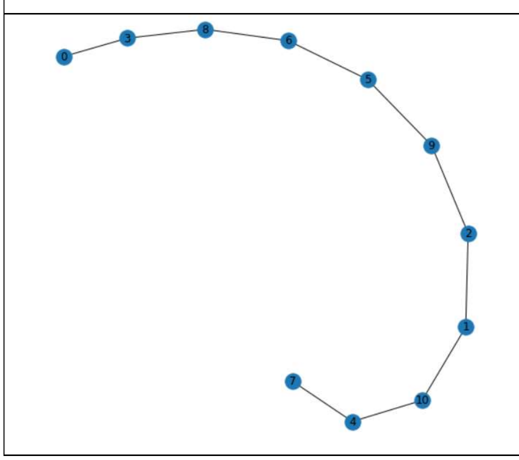
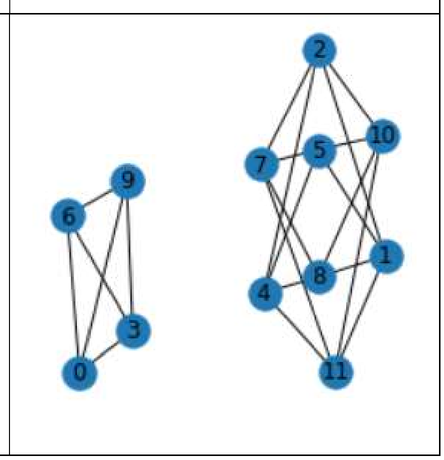
Figure 12: Coding screenshot for the generation of $G_{N*}(\mathbb{Z}_{27})$

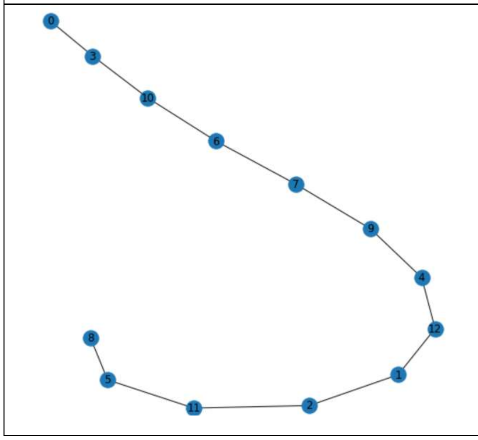
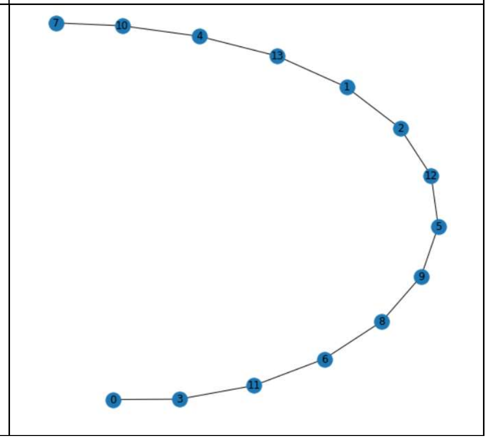
Graph of $\mathbb{Z}_3$	Graph of $\mathbb{Z}_4$
	

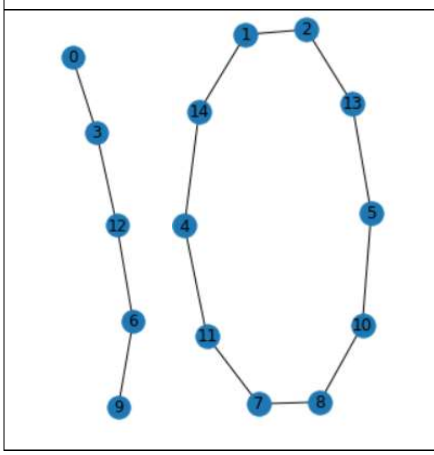
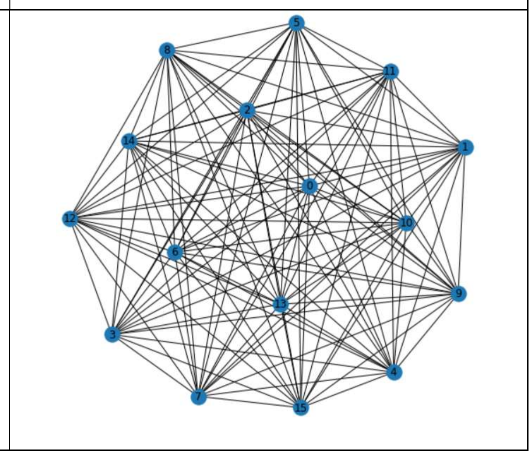
Graph of $\mathbb{Z}_5$	Graph of $\mathbb{Z}_6$
	

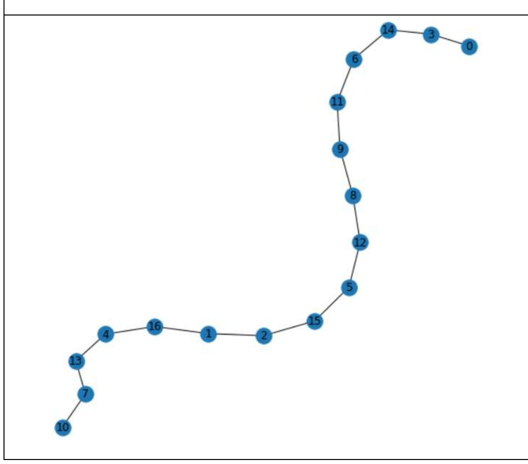
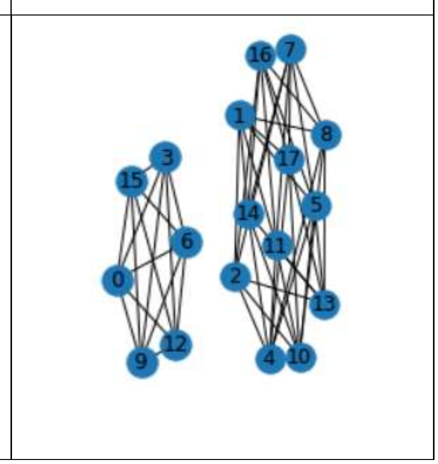
Graph of $\mathbb{Z}_7$	Graph of $\mathbb{Z}_8$
	

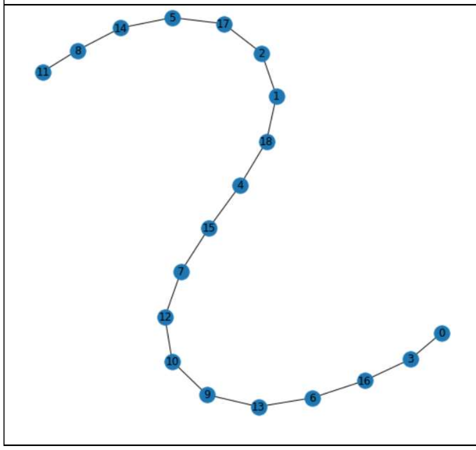
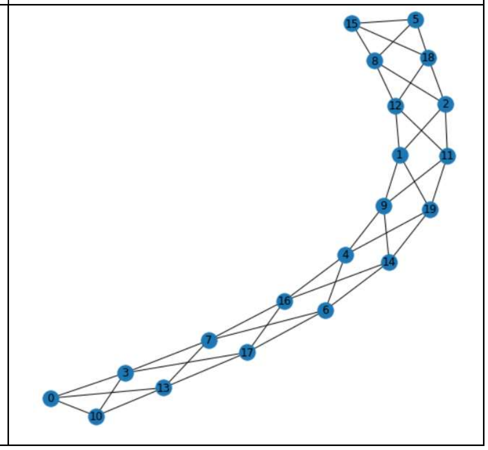
Graph of $\mathbb{Z}_9$	Graph of $\mathbb{Z}_{10}$
	

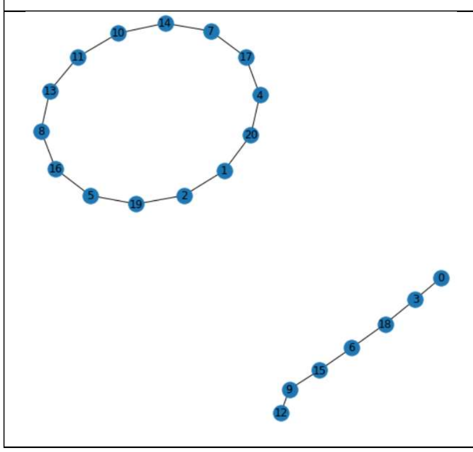
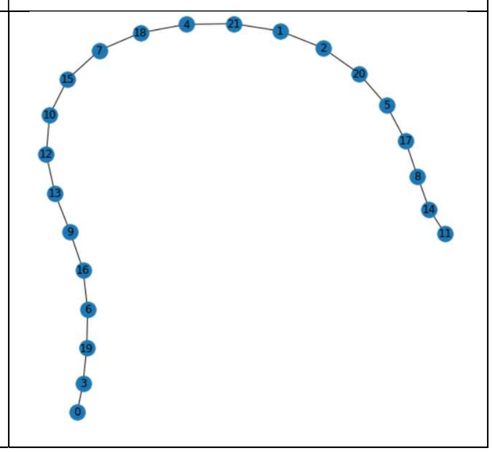
Graph of $\mathbb{Z}_{11}$	Graph of $\mathbb{Z}_{12}$
	

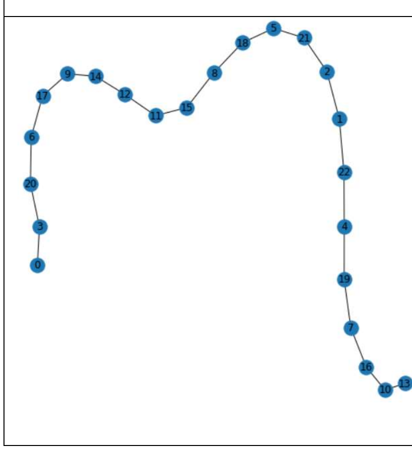
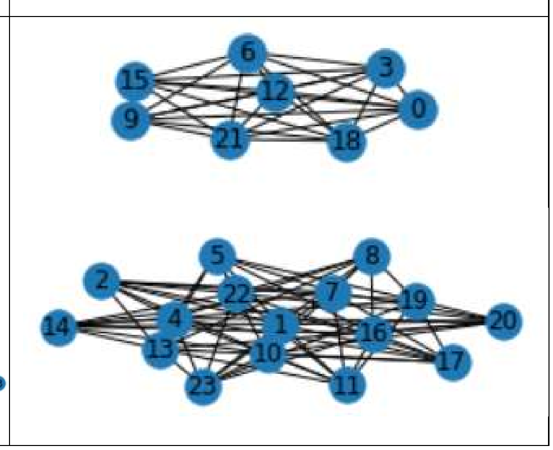
Graph of $\mathbb{Z}_{13}$	Graph of $\mathbb{Z}_{14}$
	

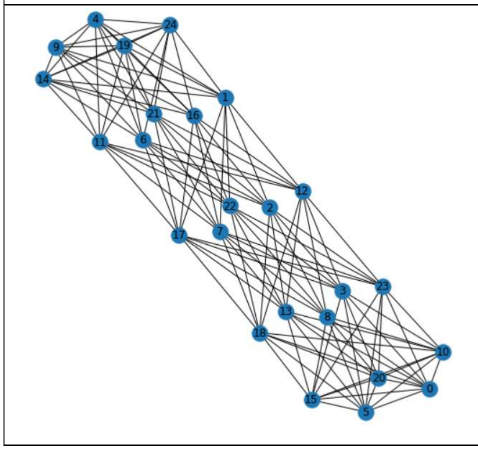
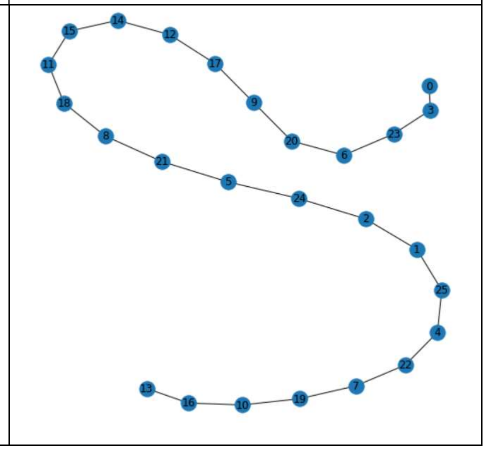
Graph of $\mathbb{Z}_{15}$	Graph of $\mathbb{Z}_{16}$
	

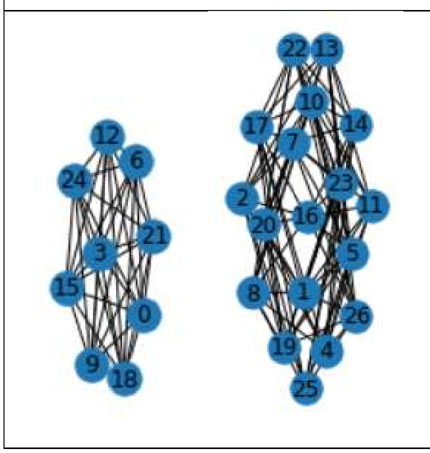
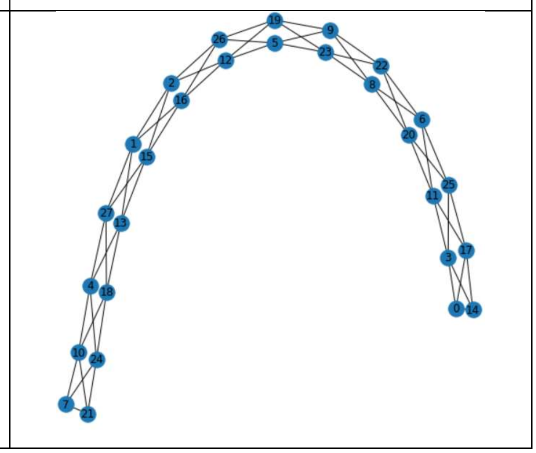
Graph of $\mathbb{Z}_{17}$	Graph of $\mathbb{Z}_{18}$
	

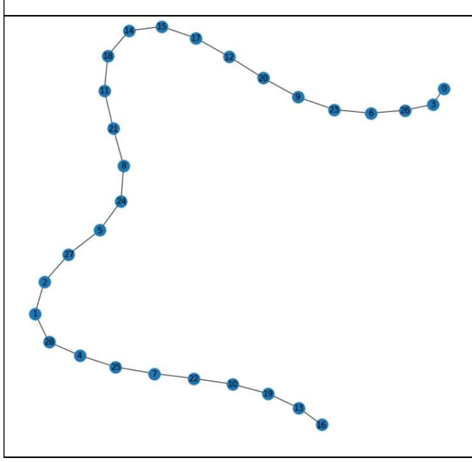
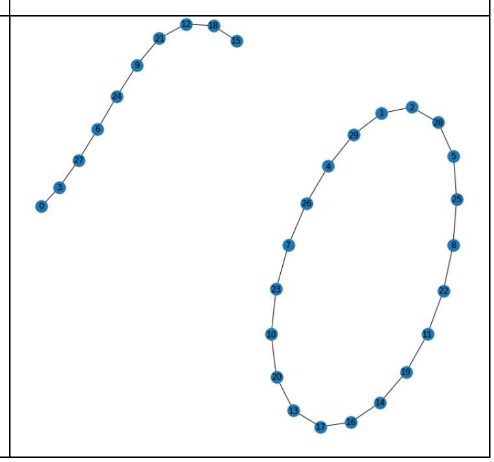
Graph of $\mathbb{Z}_{19}$	Graph of $\mathbb{Z}_{20}$
	

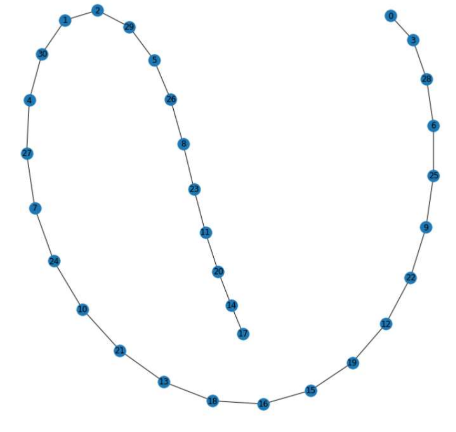
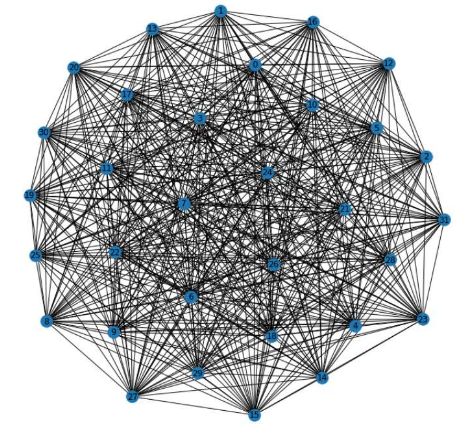
Graph of $\mathbb{Z}_{21}$	Graph of $\mathbb{Z}_{22}$
	

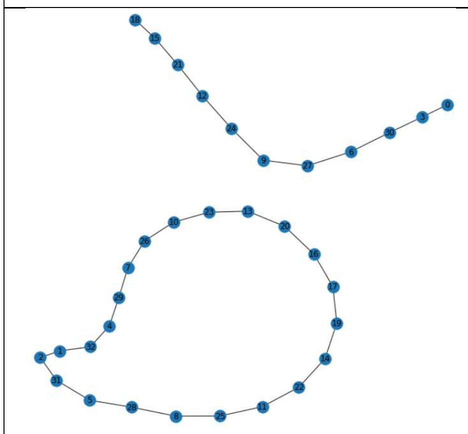
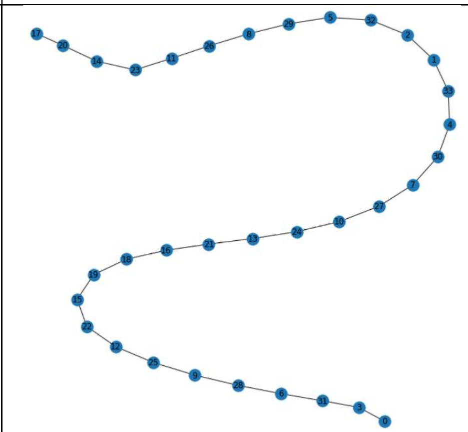
Graph of $\mathbb{Z}_{23}$	Graph of $\mathbb{Z}_{24}$
	

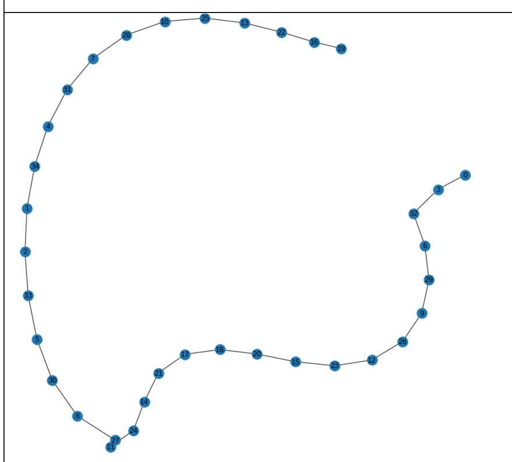
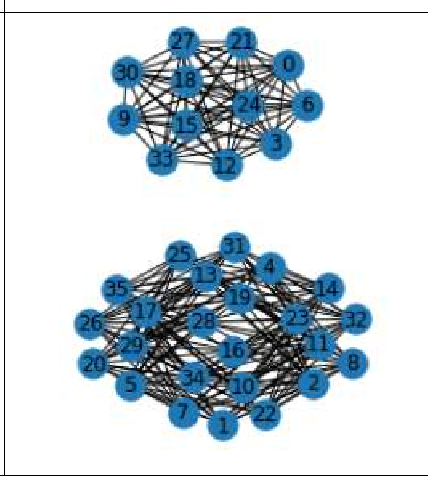
Graph of $\mathbb{Z}_{25}$	Graph of $\mathbb{Z}_{26}$
	

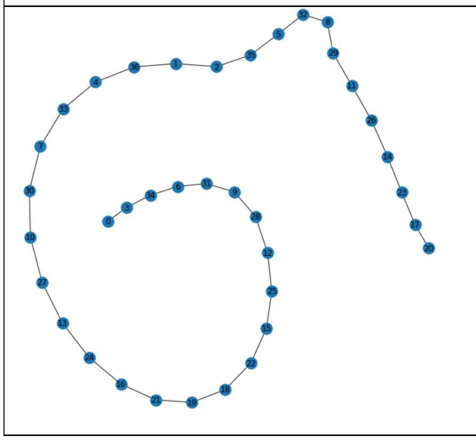
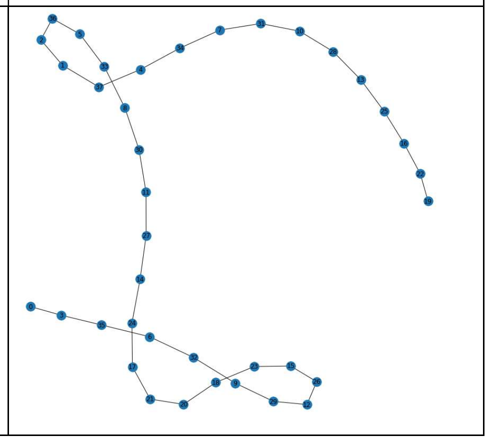
Graph of $\mathbb{Z}_{27}$	Graph of $\mathbb{Z}_{28}$
	

Graph of $\mathbb{Z}_{29}$	Graph of $\mathbb{Z}_{30}$
	

Graph of $\mathbb{Z}_{31}$	Graph of $\mathbb{Z}_{32}$
	

Graph of $\mathbb{Z}_{33}$	Graph of $\mathbb{Z}_{34}$
	

Graph of $\mathbb{Z}_{35}$	Graph of $\mathbb{Z}_{36}$
	

Graph of $\mathbb{Z}_{37}$	Graph of $\mathbb{Z}_{38}$
	

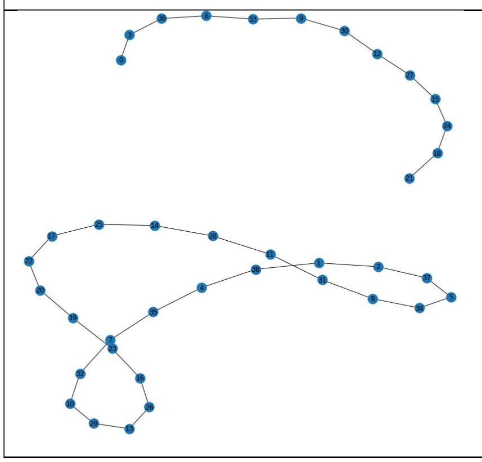
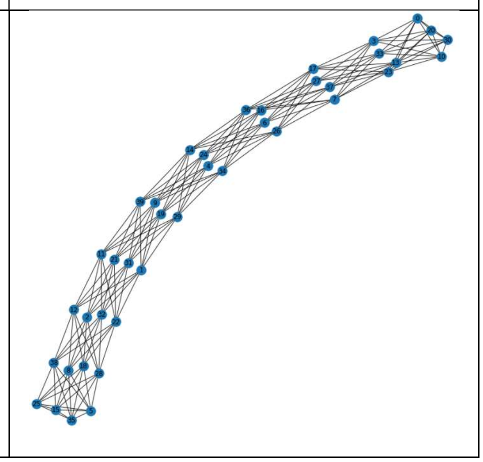
Graph of $\mathbb{Z}_{39}$	Graph of $\mathbb{Z}_{40}$
	

Figure 13: $g(x)$-nil clean graphs of $\mathbb{Z}_{\mathrm{n}}$ where $3 \leq n \leq 40$

It is important to note that for each of these graphs, all the possible nilpotents besides $\overline{0}$ were calculated. In other words, a general code was created to fulfil the conditions of a nilpotent element by considering vertices x and y such that $(x+y-0)^k$ or $(x+y-3)^k$ for some $k \in \mathbb{N}$, seeing as the roots of $g(x)$ are 0 and 3. The nilpotent outputs from the Python coding are shown below:

```
p = 1
while p < 41:
    result = []
    for x in range (p):
        for y in range (p):
            nil = (x + y)
            for n in range(10):
                if ((nil)**n % p == 0 and 0 < nil < p):
                    result.append(nil)
                if ((nil - 3)**n % p == 0 and 0 < (nil-3) < p):
                    result.append(nil-3)

    result = list(dict.fromkeys(result))
    print("other nilpotents in Z" + str(p) +" are:")
    print(result)

    p = p + 1
```

```
other nilpotents in Z1 are:
[]
other nilpotents in Z2 are:
[]
other nilpotents in Z3 are:
[]
other nilpotents in Z4 are:
[2]
other nilpotents in Z5 are:
[]
other nilpotents in Z6 are:
[]
other nilpotents in Z7 are:
[]
other nilpotents in Z8 are:
[2, 4, 6]
other nilpotents in Z9 are:
[3, 6]
other nilpotents in Z10 are:
[]
```

```
other nilpotents in Z11 are:
[]
other nilpotents in Z12 are:
[6]
other nilpotents in Z13 are:
[]
other nilpotents in Z14 are:
[]
other nilpotents in Z15 are:
[]
other nilpotents in Z16 are:
[2, 4, 6, 8, 10, 12, 14]
other nilpotents in Z17 are:
[]
other nilpotents in Z18 are:
[6, 12]
other nilpotents in Z19 are:
[]
other nilpotents in Z20 are:
[10]
```

```
other nilpotents in Z21 are:
[]
other nilpotents in Z22 are:
[]
other nilpotents in Z23 are:
[]
other nilpotents in Z24 are:
[6, 12, 18]
other nilpotents in Z25 are:
[5, 10, 15, 20]
other nilpotents in Z26 are:
[]
other nilpotents in Z27 are:
[3, 6, 9, 12, 15, 18, 21, 24]
other nilpotents in Z28 are:
[14]
other nilpotents in Z29 are:
[]
other nilpotents in Z30 are:
[]
```

```
other nilpotents in Z31 are:
[]
other nilpotents in Z32 are:
[2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30]
other nilpotents in Z33 are:
[]
other nilpotents in Z34 are:
[]
other nilpotents in Z35 are:
[]
other nilpotents in Z36 are:
[6, 12, 18, 24, 30]
other nilpotents in Z37 are:
[]
other nilpotents in Z38 are:
[]
other nilpotents in Z39 are:
[]
other nilpotents in Z40 are:
[10, 20, 30]
```

Figure 14: Coding and output of set of nilpotent elements of $G_{N*}(\mathbb{Z}_n)$

An interesting pattern is observed upon finding the set of nilpotents for each $\mathbb{Z}_n$. If n is expressed as a product of their prime factors, the set of nilpotent elements are such that they are multiples of the prime factor product without including the power of the prime factor.

In other words, if $n = {p_1}^{b_1}{p_2}^{b_2} \dots {p_i}^{b_i}$ such that $p_1, p_2, \dots, p_i$ are prime factors for $i = 1,2,3, \dots$, then the set of nilpotent elements are the multiples of $p_1 \times p_2 \times \dots \times p_i$ less than n.

4.3 Some properties of $G_{N*}(\mathbb{Z}_n)$

This section examines some of the generalised properties of the $g(x)$-nil clean graphs of $\mathbb{Z}_n$ in relation to graph theory invariants.

4.3.1 Connectedness of $G_{N*}(\mathbb{Z}_n)$

In general, the connecteness of the $g(x)$-nil clean graphs of $\mathbb{Z}_n$ can be studied based on two major categories – where n is either a multiple of 3 or not a multiple of 3. Each of these categories can be further subdivided based on their behaviour of n.

Theorem 4.1 *The following properties hold for $G_{N*}(\mathbb{Z}_n)$, where $n > 3$:*

(I) *If $n = 3m$ for all $m \in \mathbb{N}$, then $G_{N*}(\mathbb{Z}_n)$ is a disconnected graph. If n is expressed as a product of prime factors where $n = 3^a {p_1}^{b_1}{p_2}^{b_2} \dots {p_i}^{b_i}$ such that $p_1, p_2, \dots, p_i$ are prime factors other than 3:*

(A) If at least one of $a, b_1, b_2, \dots, b_i > 1$, then the graph is disconnected in two parts – one complete graph made up of vertices that are multiples of 3 and one complete bipartite graph made up of vertices that are not multiples of 3.

(B) If $a = b_1 = b_2 = \cdots = b_i = 1$, then the graph is disconnected in two parts – one linear graph made up of vertices that are multiples of 3 and one cycle made up of vertices that are not multiples of 3.

(II) *If $n \neq 3m$ for all $m \in \mathbb{N}$, then $G_{N*}(\mathbb{Z}_n)$ is a single connected graph.*

(A) If $n = 2^m$ for all $m \in \mathbb{N}$ where $m \geq 2$, then the graph is complete, K_n.

(B) If n is an odd prime, then the graph is linear.

(C) If n is a non-prime where $n = {p_1}^{b_1}{p_2}^{b_2} \ldots {p_i}^{b_i}$ such that p_i is a prime number other than 3:

(i) If every $b_i = 1$, then the graph is linear.

(ii) If at least one $b_i > 1$, then the graph is connected such that $x + y < 2n$ and $x + y$ are all possible nilpotents of $G_{N*}(\mathbb{Z}_n)$.

Proof:

<u>Case (I):</u> Consider n as being a multiple of 3. In other words, $n = 3m$, where $m = 1,2,3 \ldots$ for each $\mathrm{G_{N*}}(\mathbb{Z}_\mathrm{n})$.

(A): Let K_m be a complete graph and $K_{m,m}$ be a complete bipartite graph. For each $\mathrm{G_{N*}}(\mathbb{Z}_\mathrm{n})$ that falls in this category, it is displayed as one K_m and one $K_{m,m}$ in the following pattern:

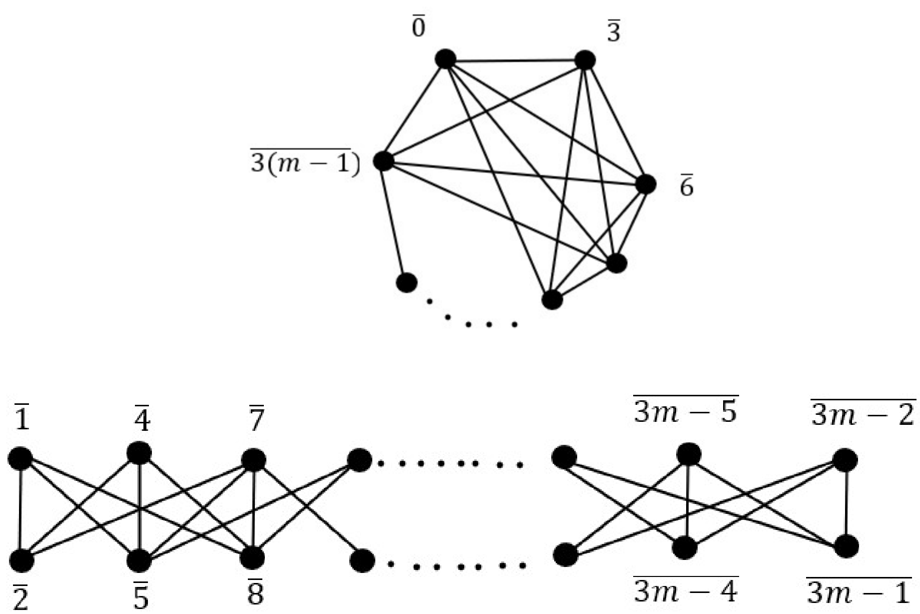


Figure 15: The graphs of $\mathrm{G_{N*}}(\mathbb{Z}_\mathrm{n})$ for case (I)(A)

Graphs of $\mathrm{G}_{\mathrm{N}*}(\mathbb{Z}_\mathrm{n})$ that fall within this that fall in this category are such that $n = 3^a {p_1}^{b_1} {p_2}^{b_2} \dots {p_i}^{b_i}$ where $p_1, p_2, \dots, p_i$ are prime factors other than 3 and at least one $a, b_1, b_2, \dots, b_i > 1$. Such examples are indicated below:

$\mathbb{Z}_9$ where $n = 3^2$

$\mathbb{Z}_{12}$ where $n = 3 \times 2^2$

$\mathbb{Z}_{18}$ where $n = 3^2 \times 2$

$\mathbb{Z}_{24}$ where $n = 3 \times 2^3$

$\mathbb{Z}_{27}$ where $n = 3^3$

$\mathbb{Z}_{36}$ where $n = 3^2 \times 2^2$

In each case, there is always at least one power of the prime factor that satisfies the condition of "greater than one".

(B): Let C_{2m} be a cycle with $2m$ vertices. For each $\mathrm{G}_{\mathrm{N}*}(\mathbb{Z}_\mathrm{n})$ that falls in this category, it is displayed as one C_{2m} and one linear graph with m vertices in the following pattern:

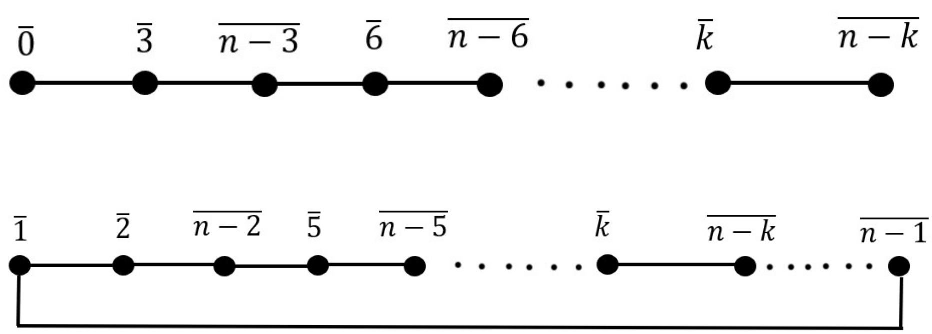


Figure 16: The graphs of $\mathrm{G}_{\mathrm{N}*}(\mathbb{Z}_\mathrm{n})$ for case (I)(B)

Graphs of $\mathrm{G}_{\mathrm{N}*}(\mathbb{Z}_\mathrm{n})$ that fall within this that fall in this category are such that $n = 3^a {p_1}^{b_1} {p_2}^{b_2} \dots {p_i}^{b_i}$ where $p_1, p_2, \dots, p_i$ are prime factors other than 3 and $a = b_1 = b_2 = \cdots = b_i = 1$. Such examples are indicated below:

$\mathbb{Z}_6$ where $n = 3 \times 2$

$\mathbb{Z}_{15}$ where $n = 3 \times 5$

$\mathbb{Z}_{21}$ where $n = 3 \times 7$

$\mathbb{Z}_{30}$ where $n = 3 \times 2 \times 5$

$\mathbb{Z}_{33}$ where $n = 3 \times 11$

$\mathbb{Z}_{39}$ where $n = 3 \times 13$

For the linear sequence, it is important to note that for even values of n, the vertex sequence ends when $k = \frac{n}{2}$. Such examples include graphs of $\mathbb{Z}_6$ and $\mathbb{Z}_{30}$. However, in the case of odd values of n, the vertex sequence ends when $n - k$ of the previous pair of vertices is equal to k of the next pair of vertices. The said vertex only appears once as all graphs of $\mathrm{G}_{\mathrm{N}*}(\mathbb{Z}_\mathrm{n})$ are assumed to contain no loops. Such examples include the graphs of $\mathbb{Z}_{15}$ and $\mathbb{Z}_{21}$. As for the cycle, the vertex sequence ultimately reaches a point where $k = n - 1$. Hence, $n - k = 1$. This implies a connection of the vertex k to vertex 1, resulting in the formation of a cycle. The graph of $\mathbb{Z}_{21}$ is used to illustrate the above phenomenon.

Case (II): Now, we consider n as not being a multiple of 3 for each $\mathrm{G}_{\mathrm{N}*}(\mathbb{Z}_\mathrm{n})$.
(A): If $n = 2^m$ for all $m \in \mathbb{N}$ where $m \geq 2$, then the graph is complete, K_n. Since the set of nilpotents for $\mathrm{G}_{\mathrm{N}*}(\mathbb{Z}_{2^m})$ are multiples of 2, all distinct vertex pairs $x + y$ subject to the roots 0 and 3 will yield zero when raised to any power of n.

(B) If n is an odd prime, then the graph is linear in the following sequence:

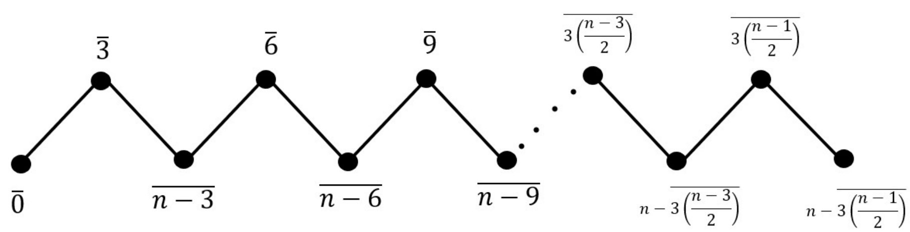


Figure 17: The graphs of $\mathrm{G}_{\mathrm{N}*}(\mathbb{Z}_\mathrm{n})$ for case (II)(B)

(C) If n is a non-prime where $n = {p_1}^{b_1}{p_2}^{b_2} \ldots {p_i}^{b_i}$ such that p_i is a prime number other than 3, one of two possible outcomes will be obtained:
(i): A linear graph is generated if every $b_i = 1$ according to the pattern below:

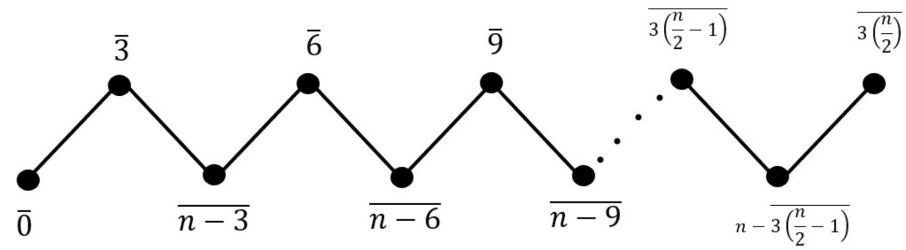


Figure 18: The graphs of $\mathrm{G_{N*}}(\mathbb{Z}_\mathrm{n})$ for case (II)(C)(i)

Such examples of $\mathrm{G_{N*}}(\mathbb{Z}_\mathrm{n})$ are indicated below:

$\mathbb{Z}_{10}$ where $n = 2 \times 5$

$\mathbb{Z}_{14}$ where $n = 2 \times 7$

$\mathbb{Z}_{22}$ where $n = 2 \times 11$

$\mathbb{Z}_{26}$ where $n = 2 \times 13$

$\mathbb{Z}_{34}$ where $n = 2 \times 17$

$\mathbb{Z}_{38}$ where $n = 2 \times 19$

(ii) If at least one $b_i > 1$, then the graph is connected such that $x + y < 2n$ and $x + y$ are all possible nilpotents of $G_{N*}(\mathbb{Z}_n)$. Such examples of $\mathrm{G_{N*}}(\mathbb{Z}_\mathrm{n})$ are indicated below:

$\mathbb{Z}_{20}$ where $n = 5 \times 2^2$

$\mathbb{Z}_{25}$ where $n = 5^2$

$\mathbb{Z}_{28}$ where $n = 7 \times 2^2$

$\mathbb{Z}_{40}$ where $n = 5 \times 2^3$

Upon analysis, there are two noteworthy occurences for the above set of graphs. Firstly, there is at least one additional nilpotent element besides 0 for each graph. Secondly, these graphs use smaller $\mathbb{Z}_\mathrm{n}$ graphs as building blocks in achieving their pattern and structure.

Consider the multiplication of the prime factors of the above examples of n without the attached power. The information on the trend is displayed in the table below:

$\mathbb{Z}_\mathrm{n}$	Prime factor product (without powers)	Set of nilpotent elements	Linear graph with similar pattern
$\mathbb{Z}_{20}$	$5 \times 2 = 10$	{0, 10}	$\mathbb{Z}_{10}$
$\mathbb{Z}_{25}$	5	{0, 5, 10, 15, 20}	$\mathbb{Z}_{5}$
$\mathbb{Z}_{28}$	$7 \times 2 = 14$	{0, 14}	$\mathbb{Z}_{14}$
$\mathbb{Z}_{40}$	$5 \times 2 = 10$	{0, 10, 20, 30}	$\mathbb{Z}_{10}$

Table 4: Identification of smaller linear graph with similar patterns and its relationship to the set of nilpotent elements.

Observe the following illustration for the sequence of vertex connections of $G_{N*}(\mathbb{Z}_{10})$ and $G_{N*}(\mathbb{Z}_{20})$ below.

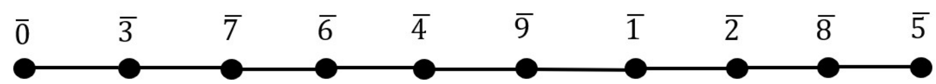


Figure 19: The graph of $G_{N*}(\mathbb{Z}_{10})$

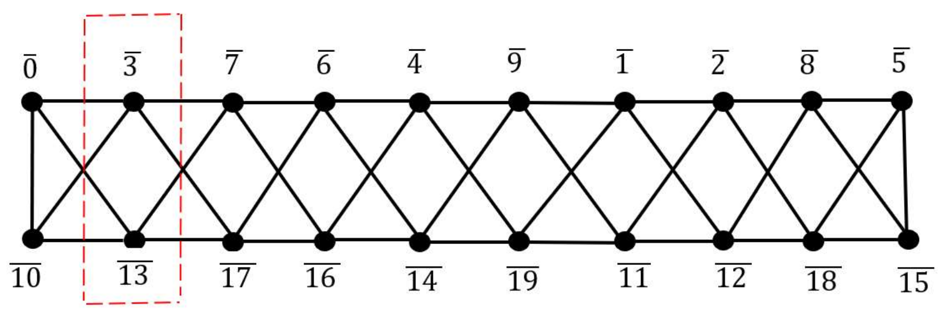


Figure 20: The graph of $G_{N*}(\mathbb{Z}_{20})$

We can see that the bottom row of elements are generated through the sum of each vertex with the extra nilpotent 10. Additionally, only the end vertices in each row will connect with each other as they fulfil the condition that $(x + y - 0)^k$ or $(x + y - 3)^k$ for some $k \in \mathbb{N}$. This concept can be extended to $\mathbb{Z}_{25}$ having 5 nilpotent elements and hence 5 similar rows of elements, while $\mathbb{Z}_{40}$ having 4 nilpotent elements and hence 4 similar rows of elements.

In general, for any $\mathbb{Z}_n$ in this category having a similar pattern structure as $\mathbb{Z}_p$, where p is the product of prime factors of n (excluding the power) and having q nilpotent elements, the graph will be made up of q rows of vertices, each consisting p vertices. The general structure of the graph can be illustrated as follows:

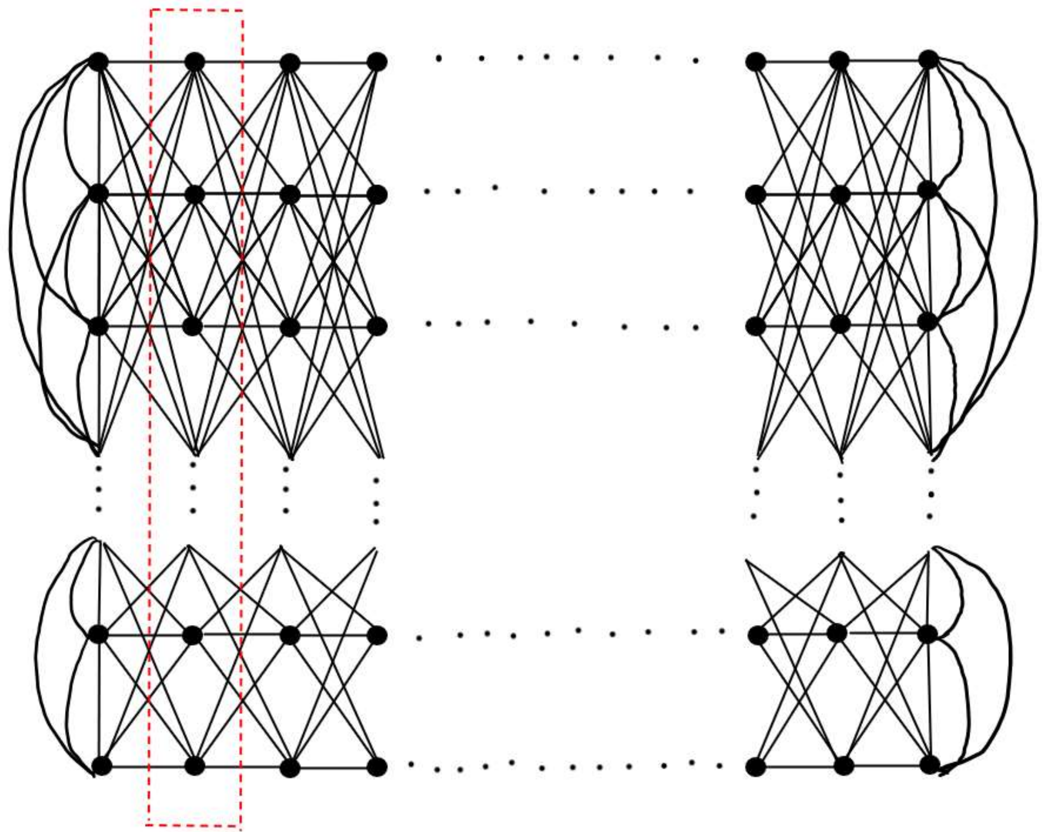


Figure 21: General pattern of $\mathrm{G}_{\mathrm{N}*}(\mathbb{Z}_\mathrm{n})$ as per case (II)(C)(ii)

Note that the dotted red box are vertices of equivalent value with respect to the nilpotents. This can be achieved since the pattern in each row is identical.

4.3.2 Hamiltonian paths and cycles of $G_{N*}(\mathbb{Z}_n)$

A graph is said to contain a Hamiltonian path if it visits each vertex in the graph only once. Furthermore, if the said path is able to return to its starting vertex without passing through another vertex more than once, it forms a cycle. Hence it will be called a Hamiltonian cycle. In this section, we investigate the presence of Hamiltonian paths and cycles with reference to the classification from Theorem 4.1.

Theorem 4.2 *The following properties hold for $G_{N*}(\mathbb{Z}_n)$, where $n > 3$:*

(I) *If $n = 3m$ for all $m \in \mathbb{N}$, then $G_{N*}(\mathbb{Z}_n)$ will not contain a Hamiltonian path since it contains two distinct connected components. However if the structures are studied separately and n is expressed as a product of prime factors where $n = 3^a {p_1}^{b_1} {p_2}^{b_2} \dots {p_i}^{b_i}$ such that $p_1, p_2, \dots, p_i$ are prime factors other than 3:*

- *(A) If at least one of $a, b_1, b_2, \dots, b_i > 1$, then both distinct structures form Hamiltonian cycles.*
- *(B) If $a = b_1 = b_2 = \cdots = b_i = 1$, then one of the structures is a Hamiltonian cycle and the other is a Hamiltonian path.*

(II) *If $n \neq 3m$ $for\ all$ $m \in \mathbb{N}$, then $G_{N*}(\mathbb{Z}_n)$ is a single connected graph.*

- *(A) If $n = 2^m$ for all $m \in \mathbb{N}$ where $m \geq 2$, then the graph will have a Hamiltonian cycle.*
- *(B) If n is an odd prime, then the graph will have a Hamiltonian path.*
- *(C) If n is a non-prime where $n = {p_1}^{b_1} {p_2}^{b_2} \dots {p_i}^{b_i}$ such that p_i is a prime number other than 3:*
  - *(i) If every $b_i = 1$, then the graph will have a Hamiltonian path.*
  - *(ii) If at least one $b_i > 1$, then the graph will have a Hamiltonian cycle*

.

Proof:

(I): By Theorem 4.1, the graphs are disconnected, hence will not have a Hamiltonian path if looked at as a whole structure. But if studied separately:

The structures in case (A) will be considered Hamiltonian cycles since both complete graphs and complete bipartite graphs are considered cyclic.

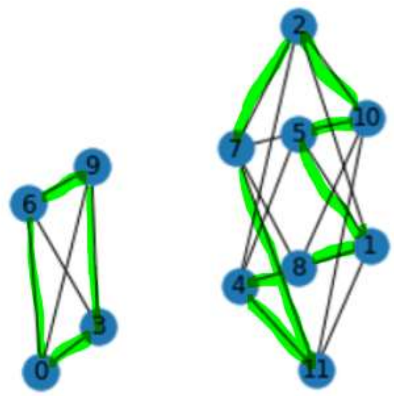


Figure 22: The Hamiltonian cycles of $\mathbb{Z}_{12}$

The structures in case (B) which consists of a linear graph and a cycle will have a Hamiltonian path and cycle respectively due to its obvious nature.

(II): By theorem 4.1, the graphs are connected, hence should contain at least a Hamiltonian path. To specify, in case (A) it has been established that all graphs in this category are complete and hence will contain a Hamiltonian cycle. In case (B) and C(i), the graphs have already been defined as linear and hence will have a Hamiltonian path. In case (C)(ii), the graphs may be a derivation of an odd prime linear graph or a non-prime linear graph. When combined as several rows, the first and last columns of vertices resemble a complete graph with each vertex connecting to every other vertex in the column whereas the second column until the penultimate column resemble connections that of a complete bipartite graph. Hence, there will always exist at least one clear path that passes through each vertex once and being able to return to its original start vertex.

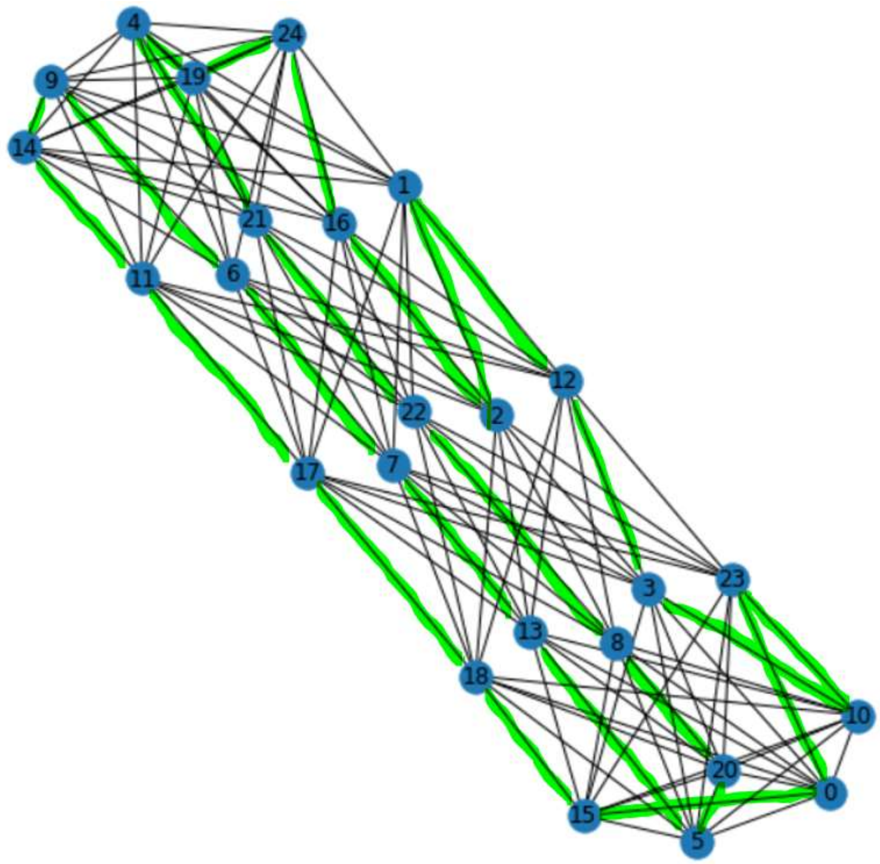


Figure 23: The Hamiltonian cycle of $\mathbb{Z}_{25}$

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

This project encompassed a comprehensive theoretical requirement that involved sourcing out the necessary information to make sense of the information pertaining nil clean graphs of rings in the main research paper by Basnet and Bhattacharyya (2017). The proofs discussed in Chapter 3 were essential to proceed as some of these properties have been applied to the generation of the $g(x)$-nil clean ring graphs in the second part of this project discussed in Chapter 4.

Using programming, we have been able to visualise and categorise $x(x-3)$-nil clean graphs of rings based on common and generalizable properties. Most importantly, we have used some of the concepts from the reference article to establish a few original theorems and findings about the connectedness of the graphs of rings as well the presence of Hamiltonian cycles and paths.

5.2 Recommendations for future work

There are many more properties to be studied within this specified $x(x-3)$-nil clean graphs of rings. Properties such as diameter, chromatic number and adjacency matrices are some of the areas that could possibly be looked into to see if generalizable properties can be theorized. Besides, there are many different polynomial functions for $g(x)$ can could be considered by future researchers to compare and contrast the properties.

Theoretical analysis and proving techniques as per the attempt in this project should be looked into further to widen the horizon and understanding of the concept of $g(x)$-nil clean graphs of rings. There are still not many sources of literature on the subject, therefore it could be a potential field to explore.

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