NEW FAMILIES OF UNIVERSAL PORTFOLIOS AND THEIR PERFORMANCES

By

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ABSTRACT

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In the world of investment, various types of investment product are available for the investor to seek for long term investment return. These products include invest in real-estate, invest in foods and beverages, invest in bonds, invest in commodities and many more. Choosing the ’right’ asset to invest is difficult. Losses can occur if the chosen asset behaves in an unfavourable way. In the modern era, investors use portfolio in their investment strategy. A portfolio is a combination or collection of a series of financial instruments like commodities, bonds, stocks, cashes and etc. It is a method that commonly spotted in the market which it can reduce the risk significantly by not allocating all the capital into one basket. Portfolio helps to diversify the investor capital and allocate those wealth onto various options to prevent occurrence of huge loss due to unpleasant event. As an individual investor, we need to determine the optimum allocation for each of the components in the portfolio. A well-diversified portfolio is crucial for any investor to yield a higher return.

Universal portfolio is a strategy of trading on stocks that does not assume any probability model for the stock prices. Universal portfolio is an investment technique where it helps us to generate portfolio vector to produce high wealth return in a long run.

In this research, new possible ways to generate new universal portfolios are studied. Objective functions with different divergence have been tested to obtain the new universal portfolios. The purpose is to obtain the next-day stocks’ allocation of the portfolio which could maximize the wealth return.

Next, the performance of the newly derived universal portfolios is studied by running these universal portfolios on some selected stocks from the Kuala Lumpur Stock Exchange (KLSE). These results will be compared with others and the performances are studied intensively.

Beside comparing the performances, a new family of universal portfolios is developed. Most of the newly derived universal portfolios are linked to the universal portfolio generated by f-divergence and universal portfolio generated by Bregman divergence. These two universal portfolios are the general form of Helmbold universal portfolio.

For numerical experiment of the performance of universal portfolio, it is shown that the newly derived universal portfolios can perform as good as Helmbold universal portfolio. The results of these universal portfolios show that it is possible to increase the wealth of the investor by using these portfolios in investment.

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I, KUANG KEE SENG hereby declare that the thesis is based on my original work except for quotations and citation which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UTAR or other institutions.

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Date: 10/8/2022

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LIST OF ABBREVIATIONS

BCRP Best Constant Rebalanced Portfolio

CRP Constant Rebalanced Portfolio

KLSE Kuala Lumpur Stock Exchange

RPR Reciprocal-Price-Relative

SCRP Successive Constant Rebalanced Portfolio

VBA Visual Basic for Applications

CHAPTER 1

INTRODUCTION

Portfolio is a financial term representing a pool of various types of investments, such as stocks, assets, bonds, commodities, cash and etc. It is always advisable to invest in a portfolio than a single asset as portfolio helps to reduce the risks of unpleasant movement of the market. Famous quote, “Do not put all your eggs in one basket”, is the best advise for the investor who should invest in a portfolio. In this thesis, we focus on the portfolio of stocks only. The performance of the stock market is commonly known as unpredictable. Beside the factor of corporate itself, it’s performance also strongly influenced by both macro and microeconomics factors. Therefore, portfolio selection is always a challenge for those investors who are trying to beat the market and earn tremendous return from the stock market. Furthermore, the fund allocation to each of the stocks in the portfolio provides another challenge to the investors. These investors have to decide the best asset allocation to yield them the best return.

Universal portfolio is a robust investment strategy. The investment decision-making applying universal portfolios adopts the assumption which the stock prices do not follow any probability model. Cover (1991) showed that universal portfolio is capable to achieve a better return than a standard ’buy-and-hold’ strategy. Then Helmbold et al. (1998) introduced a method of generating universal portfolio from the objective function, which maximized the log-optimal growth rate of the portfolio in the long runs. This research mainly focuses on exploring potential methods that can be used to generate new universal portfolios. The performance of these newly generated universal portfolios are then studied by running on some chosen stocks from Bursa

Malaysia.

1.1 Objective

In this research, we extend the work of Helmbold et al. (1998) to generate various universal portfolios by searching potential useful distance functions. Then the empirical result for each newly derived universal portfolios is computed and compared with the benchmark performance. The purpose for this study is to find possible universal portfolio which can outperform the Helmbold universal portfolio (see Helmbold et al. (1998)). A suitable portfolio can help investor to growth his wealth in a long run.

The relationship among the universal portfolios generated by different distance functions is also studied in this research. A possible new parametric family of universal portfolio may be derived by searching for the connections among the universal portfolios.

1.2 Literature Review

A portfolio is an investment strategy which help to reduce the investment risk by diversification of assets. It is shown that portfolio investment outperforms buy-and-hold a single asset strategy in a long run. Markowitz (1952) first introduced the portfolio theory. He proposed a solution for the portfolio assets' selection problem. He suggested to hold a constant variance portfolio while maximizing the mean. He also suggested to hold constant mean portfolio while minimize the variance. This is famously known as the Efficient Frontier Theory. It is a cornerstone of modern portfolio theory. An extension study on Markowitz's work on portfolio analysis had been carried out by Sharpe (1963). A simplified model that study the relationship between securities for practical applications of the Markowitz portfolio analysis technique is introduced by Sharpe.

Kelly (1956) introduced the theory of rebalanced portfolios which the investor's wealth is rebalanced his cumulative wealth based on the knowledge given. The growth rate of wealth can be maximized by the log-optimal investment is shown by Kelly. A constant rebalanced portfolio allocates the same allocations of wealth among the stocks every day. The best constant rebalanced portfolio (BCRP) can achieve a wealth which is expected to grow exponentially at a rate determined by the portfolio's volatility. Robbins (1951) had developed a theory of compound sequential Bayes decision rules. This theory had been further studied by Hanna and Robbins (1955). Then the game-theoretic approachability-excludability theory had introduced by Blackwell (1956a) and Blackwell (1956b). Then, with the theory mentioned above, a best wealth return had achieved by Cover and Gluss (1986). This wealth return had achieved by using the constant-rebalanced-portfolio (CRP) for discrete valued stock markets.

A series of research on universal portfolio had been conducted by Cover (1991), Cover and Ordentlich (1996) and Helmbold et al. (1998). Their purpose was to achieve a higher return in the stock market. Cover and Ordentlich (1996) introduced the class of universal portfolios. However, these portfolios unable to have a better perform than the best constant rebalanced portfolio (BCRP) and successive constant rebalance portfolio (SCRP) introduced by Gaivoronski and Stella (2000).

Cover (1991) introduced the uniform universal portfolio and showed the use of Laplace's method of integration to show that his uniform universal portfolio performs asymptotically as good as the BCRP. He and his research teams tested his algorithm in New York Stock Exchange and showed that it is possible to increase the wealth by a large margin. Cover and Ordentlich (1996) then generalized the uniform universal portfolio into the class of Dirichlet-weighted universal portfolio. Cover and Ordentlich (1996) also introduced the notion of side information and emphasized on the studied of wealth achievable by the uniform and Dirichlet-weighted universal portfolios. They also derived the theoretical performance bounds of the two special Dirichlet-weighted universal portfolios. Helmbold et al. (1998) introduced a universal portfolio which requires lesser memory requirement and computation time. The multiplicative-update universal portfolio is generated using the exponentiated gradient update algorithm that was developed by Kivinen and Warmuth (1997), is introduced by Helmbold et al. (1998). They showed that Helmbold universal portfolio outperforms the uniform universal portfolio based on the same stock data from the New York Stock Exchange in Cover (1991). Then, Tan and Tang (2003) showed that the Helmbold universal portfolio reacts sensitively to the initial wealth allocation for the portfolio. They also showed that the portfolio looks like a constant rebalanced portfolio if the parameter selected is a small positive value.

Helmbold et al. (1998) introduced a method of generating a universal portfolio using the zero-gradient set of objective function containing the Kullback-Leibler divergence of two portfolio vectors. Tan and Lim (2012) extended this method to the zero-gradient set of an objective function containing the Mahalanobis squared-divergence of two portfolio vectors. Tan and Pang (2014) studied the performance of the universal portfolio generated by low order of Brownian Motion. The result they obtained showed that the wealth can be increased by using different parameters. In both Pang et al. (2017) and Pang et al. (2019), the empirical results showed that finite-order universal portfolios generated by stochastic processes can outperform the constant rebalanced portfolio at a certain parameters. Phoon et al. (2020) studied the low order performance of special time series generated universal portfolio. The wealth obtained is close to the wealth achieved by Best Constant Rebalanced Portfolio. In Garivaltis (2021), he generalized Cover's benchmark of the best constant-rebalanced portfolio (1-linear trading strategy) by considering the best bilinear trading strategy determined for the realized sequence of asset prices. They showed that the universal bilinear portfolio asymptotically outperforms the universal 1-linear portfolio.

1.3 Thesis Overview

This thesis consists of a total of six chapters. In chapter 1, an introduction is given by stating the objectives, literature review on this research area and preliminaries definition. In chapter 2, an introduction of Bursa Malaysia is given. Historical daily stock data are collected from Bloomberg. These data have been grouped into few different stock data sets for empirical study. Each stock data set consists of 5 different stock data from local companies.

In chapter 3, we introduce the universal portfolio generated by zero-gradient set of logarithm objective function. We successfully derive four universal portfolios in this chapter. There are Reciprocal-Price-Relative universal portfolios, universal portfolios generated by Rényi divergence and Generalized Kullback-Leibler divergence and reverse Helmbold universal portfolio. We study the performance of these universal portfolios by running them on stock data sets D, E, F, G and H. These universal portfolios are the results obtained during the initial stage of our study.

In chapter 4, we derive a universal portfolio from the well-known Csiszár f-divergence. Then, we obtain the universal portfolio generated by f-divergence. This universal portfolio enables us to obtain more universal portfolios which are presented in chapter 5. We study the performance of this universal portfolio by running it on new stock data sets J, K, L, M and N. We extend the study of this universal portfolio generated by f-divergence and the universal portfolio generated by reverse f-divergence is obtained.

In chapter 5, we study the extension of the universal portfolio given in chapter 4. Few universal portfolios are obtained by applying the suitable convex functions to the universal portfolio derived in chapter 4. The performance of these universal portfolios is studied. We run these universal portfolio on new stock data sets to obtain empirical results. The performance of these universal portfolio is compared with the benchmark universal portfolio.

Chapter 6 gives the conclusion and a comprehensive summary of this research. Some potential directions for the future research are included in this chapter.

1.4 Definitions

This section gives some basic definitions and assumptions used in this thesis.

1.4.1 Some Preliminaries

A market of m numbers of stock is considered in this thesis. The price-relative of a stock in a particular trading day is given by the ratio of the closing price of the stock to its opening price of the stock in that particular trading day. Define $\boldsymbol{x}_n = (x_{ni}) = (x_{n1}, x_{n2}, \cdots, x_{nm})$ as the stock price-relative column vector on the n^{th} trading day, where x_{ni} is the stock price-relative of the i^{th} stock, $n = 1, 2, \cdots$ and $i = 1, 2, \cdots, m$. Denote b_{ni} as the probability or proportion of the current wealth or fund allocated on i^{th} stock on n^{th} trading day. A portfolio vector $\boldsymbol{b}_n = (b_{n1}, b_{n2}, \cdots, b_{nm})$ is a column vector consists of a list of probability where $0 \leq b_{ni} \leq 1$ for $i = 1, 2, \cdots, m$ and $\sum_{i=1}^{m} b_{ni} = 1$ for $n = 1, 2, 3, \cdots$. Since $b_{ni} \geq 0$, we assume that no short-selling is allowed in this strategy. The stock price-relative x_{ni} describes the movement of the i^{th} stock on n^{th} trading day. Hence, the wealth achieved at the end of trading day j is

$$\boldsymbol{b}_j^t \boldsymbol{x}_j = \sum_{i=1}^{m} b_{ji} x_{ji} \tag{1.1}$$

for $i = 1, 2, \cdots, m$.

By assuming the starting wealth, $S_0 = 1$. Then, the wealth achieved by the investor at the end of the investment period, n days is

$$S_n = \prod_{j=1}^{n} \boldsymbol{b}_j^t \boldsymbol{x}_j \tag{1.2}$$

where $\boldsymbol{b}_1, \boldsymbol{b}_2, \cdots, \boldsymbol{b}_n$ is the sequence of proportion of the wealth invested and $\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_n$ is the sequence of the daily price-relative for m numbers of stock.

1.4.2 Objective Function

In Helmbold et al. (1998), they introduced the method to obtain the next-day allocation $\boldsymbol{b}_{n+1}$ by maximizing the function below:

$$F(\boldsymbol{b}_{n+1}) = \xi \log(\boldsymbol{b}_{n+1}^t \boldsymbol{x}_{n+1}) - d(\boldsymbol{b}_{n+1}, \boldsymbol{b}_n) \tag{1.3}$$

where $\xi > 0$ is a parameter and d is a distance measure. This d acts as a penalty term to keep $\boldsymbol{b}_{n+1}$ close to $\boldsymbol{b}_n$. Kivinen and Warmuth (1997) showed that a portfolio can achieve better performance by choosing $\boldsymbol{b}_{n+1}$ that is ”close” to $\boldsymbol{b}_n$.

It is difficult to maximize (1.3) since both terms depend on non-linear on $\boldsymbol{b}_{n+1}$. Fletcher (1987) introduced an approach to use an iterative optimization algorithm to find the maximum vector $\boldsymbol{b}_{n+1}$ that maximizes (1.3) under the constraint $\sum_{i=1}^{m} \boldsymbol{b}_{n+1} = 1$. However, it would be time consuming as it requires solving a different non-linear equation on each trading day. Hence, instead of solving the exact maximizer of (1.3), the first term of (1.3) is replaced by its first-order Taylor series. The Lagrange multiplier, λ to handle the constraint is added to ensure the component of $\boldsymbol{b}_{n+1}$ must sum to one. Then, we will need to maximize the following function:

$$\hat{F}(\boldsymbol{b}_{n+1}) = \xi\left(\log(\boldsymbol{b}_n^t\boldsymbol{x}_n) + \left(\frac{\boldsymbol{b}_{n+1}^t\boldsymbol{x}_n}{\boldsymbol{b}_n^t\boldsymbol{x}_n}\right) - 1\right) - d(\boldsymbol{b}_{n+1}, \boldsymbol{b}_n) + \lambda\left(\sum_{i=1}^{m} b_{n+1,i} - 1\right) \tag{1.4}$$

The objective function (1.4) together with different distance functions, $d(\boldsymbol{b}_{n+1}, \boldsymbol{b}_n)$ are studied. Various universal portfolios are generated in our study and their performance are compared.

CHAPTER 2

DATA COLLECTION

In this chapter, our methods of data collecting and processing will be explained. In our research, we focus on analysing the stock data from Bursa Malaysia, which is explained in Section 2.1. Then, we explained the stocks selection and grouping of the stock data sets in Section 2.2 and Section 2.3.

2.1 Bursa Malaysia

The Bursa Malaysia, or previously known as Kuala Lumpur Stock Exchange (KLSE) dates back to year 1930 when the a formal organisation dealing in securities was set up in Malaysia. It was named as Singapore Stockbrokers' Association. Seven years later, it was re-registered as Malayan Stockbrokers' Association, but public share trading is still not allowed yet. In year 1960, the Malayan Stock Exchange was established and trading of public share were allowed starting on 9 May 1960. In year 1964, the Stock Exchange of Malaysia was officially formed. However, followed by the secession of Singapore from Malaysia, this common stock exchange continued to function under the new name Stock Exchange of Malaysia and Singapore (SEMS). In year 1973, followed by the termination of currency interchangeability between Singapore and Malaysia, the SEMS was forced to separated into the Stock Exchange of Singapore (SES) and the Kuala Lumpur Stock Exchange Bhd (KLSEB).

In year 1976, the Kuala Lumpur Stock Exchange was incorporated as a company limited by guarantee and took over the operation of the Kuala Lumpur Stock Exchange Bhd. In year 1994, the KLSEB then renamed to Kuala Lumpur Stock Exchange (KLSE). On 14 April 2004, Kuala Lumpur Stock Exchange became a demutualised exchange and a new name, Bursa Malaysia was given. The purpose of this demutualised exchange is to respond to global trends in exchange sector and enhance competitive position. On the next year, Bursa Malaysia was listed on the Main Board of Bursa Malaysia Securities Bhd. In year 2008, Bursa Trade Securities, a new trading platform for the securities market was launched. This platform provides better accessibility for investors while enhancing trading efficiency and market transparency. In year 2009, Bursa's benchmark index, known as Kuala Lumpur Composite Index (KLCI), was improved to a new level with the adoption of the Financial Times Stock Exchange (FTSE) international index methodology. Then the enhanced KLCI, now known as the FTSE Bursa Malaysia (FBM) KLCI, is based on globally accepted standards of tradability and investability of the constituents, as well as transparency of the methodology. The corporate history can be found at *Bursa Malaysia Corporate History* (n.d.)

2.2 Stocks Selection

Portfolio is a collection of different stocks. It is always advisable to group stocks from different industries in a portfolio to minimize the risk. In our research, we choose 5 Malaysia company stocks from Bursa Malaysia to form one stock data set. We utilize the Bloomberg terminal, which is one of the facilities offered by Universiti Tunku Abdul Rahman Mary KUOK Pick Hoo Library, to collect the stock data. The daily opening price and daily closing price of each stock is collected from Bloomberg terminal. The reason is we are interested with the daily movement of the stock. As introduce in Section (1.4.1), the stock-price relative is derived by obtaining the ratio of its opening price to its closing price for each stock. These data processing is done via some functions in Microsoft Excel.

2.3 Stock Data Sets

The stock data sets D, E, F, G and H were collected and compiled by previous researchers. These stock data sets consists of stocks of the Malaysian companies that are traded from 1st March 2006 until 2nd August 2012, consisting of a total of 1500 trading days. Table below gives the lists of stocks for data sets D, E, F, G and H.

Table 2.1: List of Malaysian companies in data sets D, E, F, G and H

Data Set	**Portfolio of five Malaysian Companies**
D	IOI Corporation, Carlsberg Brewery Malaysia, British American Tobacco, Nestle, Digi
E	Public Bank, Kulim, KLCC Property Holding, AEON Corporation, Kuala Lumpur Kepong
F	AMMB Holdings, Berjaya Sports TOTO, Air Asia, Gamuda, Genting
G	Hong Leong Bank, DiGi.com, Eco World Development Group, Zecon, United Malacca
H	Digi, Public Bank, KLCC Proporty Holdings, Carlsberg Brewery Malaysia, Kuala Lumpur Kepong

We collect and compile another 5 different stock data sets which the data collected from different stocks of the Malaysian companies that are traded from 3rd January 2005 until 4th September 2015. These data consisting a total of 2500 trading days. Table below provides the lists of stocks for data sets J, K, L, M and

N.

Table 2.2: List of Malaysian companies in data sets J, K, L, M and N

Data Set	**Portfolio of five Malaysian Companies**
J	Public Bank, Nestle Malaysia, Telekom Malaysia, Eco World Development Group, Gamuda
K	AMMB Holding, Air Asia, Encorp, IJM Corp, Genting Plantations
L	Alliance Financial Group, DiGi.com, KSL Holdings, IJM Corp, Kulim Malaysia
M	Hong Leong Bank, DiGi.com, Eco World Development Group, Zecon, United Malacca
N	RHB Capital, Carlsberg Brewery Malaysia, KSL Holdings, Crest Building Holdings, Kulim Malaysia

CHAPTER 3

UNIVERSAL PORTFOLIO GENERATED BY ZERO-GRADIENT SET OF LOGARITHM OBJECTIVE FUNCTION

3.1 Reciprocal-Price-Relative (RPR) Universal Portfolio

In this session, we introduce an additive-update universal portfolio where the updates depend on reciprocal functions of price relatives. Hence, they will be called RPR (Reciprocal-Price-Relative) universal portfolios. We will be using the objective function (1.4), which was introduced by Helmbold et al. (1998). The gradient vector of this objective function $\hat{F}(\boldsymbol{b}_{n+1})$ is defined as: $\nabla F = \left(\frac{\partial \hat{F}}{\partial b_{n+1,i}}\right)$. Then the portfolio components $b_{n+1,i}, \cdots, b_{n+1,m}$ are treated as free variables subject to the constraint $\sum_{i=1}^{m} b_{n+1,i} = 1$, with Lagrange multiplier, λ. $\hat{F}(\boldsymbol{b}_{n+1}, \lambda; \boldsymbol{b}_n, \boldsymbol{x}_n)$ is a function of $\boldsymbol{b}_{n+1}$ and λ given $\boldsymbol{b}_n$ and $\boldsymbol{x}_n$. The zero-gradient set of $\hat{F}(\boldsymbol{b}_{n+1}, \lambda)$ is the set of $\{\boldsymbol{b}_{n+1} : \nabla \hat{F}(\boldsymbol{b}_{n+1}, \lambda) = \boldsymbol{0}\}$.

The pseudo Lagrange multiplier λ is a function of the variable $\boldsymbol{b}_{n+1}$ obtained by some mathematical operation on the zero-gradient equations of the objective function 1.4. Since it is a variable, it is not a valid solution of the zero-gradient equations. The pseudo λ is said to be a *pseudo solution* of the zero-gradient equations.

The quantity $\log(\boldsymbol{b}_{n+1}^t \boldsymbol{x}_{n+1})$ is the rate of growth of wealth on day $n+1$ which can be estimated by $\log(\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n)$ since x_N is unknown on day n. We obtain the Type 1 RPR universal portfolio by approximating $\log(\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n)$ with first-order Taylor series $\left[\log(\boldsymbol{b}_n^t\boldsymbol{x}_n) + \left(\frac{\boldsymbol{b}_{n+1}^t\boldsymbol{x}_n}{\boldsymbol{b}_n^t\boldsymbol{x}_n} - 1\right) - 1\right]$.

3.1.1 Type 1 RPR Universal Portfolio

In this section, the derivation for Type 1 RPR universal portfolio is shown. The empirical results for this universal portfolio are given by running this universal portfolio on the stock data sets.

Proposition 3.1.1: Consider $C = (c_{ij})$ be a non-negative matrix satisfying $\mathbf{1}^t C = \mathbf{1}$ where $\mathbf{1} = (1, 1, \cdots, 1)$ and $\alpha \geq 0$ are given. Given $\xi > 0$, from (1.4), consider the objective function

$$\begin{aligned}\hat{F}(\boldsymbol{b}_{n+1}, \lambda) =& \xi\left[\log(\boldsymbol{b}_n^t\boldsymbol{x}_n) + \left(\frac{\boldsymbol{b}_{n+1}^t\boldsymbol{x}_n}{\boldsymbol{b}_n^t\boldsymbol{x}_n}\right) - 1\right] \\ & - \log\left\{\prod_{i=1}^{m}[\eta_i + (b_{n+1,i} - b_{ni})]\right\} + \lambda\left(\sum_{i=1}^{m} b_{n+1,i} - 1\right)\end{aligned} \tag{3.1}$$

where

$$v_i = [\alpha(\boldsymbol{b}_n^t\boldsymbol{x}_n) + x_{ni}]^{-1} \tag{3.2}$$

for $i = 1, 2, \cdots, m$ and

$$\eta_i = \xi^{-1}(\boldsymbol{b}_n^t\boldsymbol{x}_n)\sum_{j=1}^{m} c_{ij} v_j \tag{3.3}$$

for $i = 1, 2, \cdots, m$. Then we can obtain the pseudo Type 1 RPR universal portfolio generated by the zero-gradient set of $\hat{F}(\boldsymbol{b}_{n+1}, \lambda)$ as followed

$$\boldsymbol{b}_{n+1} = \boldsymbol{b}_n + \xi^{-1}(\boldsymbol{b}_n^t\boldsymbol{x}_n)[v - \boldsymbol{C}v] \tag{3.4}$$

for $i = 1, 2, \cdots, m$, where ξ is any positive scalar satisfying $\boldsymbol{b}_{n+1} \geq 0$.

Proof. We differentiate (3.1) with respect to $b_{n+1,i}$ to obtain

$$\frac{\partial \hat{F}}{\partial b_{n+1,i}} = \xi \frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} - \frac{1}{[\eta_i + (b_{n+1,i} - b_{ni})]} + \lambda \tag{3.5}$$

for $i = 1, 2, \cdots, m$. Then, we let (3.5) equal to 0 and rearrange the equation, we obtain

$$\left[\xi \frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} + \lambda\right] [\eta_i + (b_{n+1,i} - b_{ni})] = 1 \tag{3.6}$$

for $i = 1, 2, \cdots, m$. Then, we multiply (3.5) by b_{ni} and sum over i to get

$$\lambda = \sum_{i=1}^{m} \frac{b_{ni}}{[\eta_i + (b_{n+1,i} - b_{ni})]} - \xi. \tag{3.7}$$

The variable λ that we obtained in (3.7) is known as a pseudo Lagrange multiplier. Next, we substitute this variable λ into (3.6) to obtain

$$\left[\xi \frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} + \sum_{j=1}^{m} \frac{b_{nj}}{\eta_j + (b_{n+1}, j - b_{nj})} - \xi\right] [\eta_i + (b_{n+1,i} - b_{ni})] = 1 \tag{3.8}$$

for $i = 1, 2, \cdots, m$. Let

$$z_i = [\eta_i + (b_{n+1,i} - b_{ni})]^{-1} \tag{3.9}$$

for $i = 1, 2, \cdots, m$ and then (3.8) becomes

$$\sum_{j=1}^{m} b_{nj} z_j - z_i = \xi \left[1 - \frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right] \tag{3.10}$$

Then, we rewrite (3.10) in matrix form, $\boldsymbol{A}\boldsymbol{z} = \boldsymbol{y}$ where

$$y_i = \xi \left[1 - \frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right] \tag{3.11}$$

for $i = 1, 2, \cdots, m$ and $\boldsymbol{A} = (a_{ij})$ is defined as following

$$
\begin{aligned}
a_{ii} &= b_{ni} - 1 \text{ for } i = 1, 2, \cdots, m \\
a_{ij} &= b_{nj} \text{ for } i \neq j.
\end{aligned} \tag{3.12}
$$

Then, the solution to $\boldsymbol{Az} = \boldsymbol{y}$ is $\boldsymbol{z} = \zeta \boldsymbol{1} - \boldsymbol{y}$ where ζ is any real scalar (see (3.1.1)). Hence, we have

$$
z_i = \frac{(\zeta - \xi)(\boldsymbol{b}_n^t \boldsymbol{x}_n) + \xi x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} \tag{3.13}
$$

for $i = 1, 2, \cdots, m$. Then, we reparametrize $\zeta = (\alpha + 1)\xi$ and obtain

$$
z_i^{-1} = \frac{\boldsymbol{b}_n^t \boldsymbol{x}_n}{\xi[\alpha(\boldsymbol{b}_n^t \boldsymbol{x}_n + x_{ni})]} \tag{3.14}
$$

for $i = 1, 2, \cdots, m$. Lastly, we merge the equations (3.2), (3.3) and (3.9) and obtain

$$
\begin{aligned}
b_{n+1,i} =& b_{ni} + z_i^{-1} - \eta_i \\
=& b_{ni} + \xi^{-1}(\boldsymbol{b}_n^t \boldsymbol{x}_n) \left[v_i - \sum_{j=1}^{m} c_{ij} v_j \right]
\end{aligned} \tag{3.15}
$$

where (3.4) is obtained.

Lemma 3.1.1: Let $\boldsymbol{A}$ be the $m \times m$ matrix defined by (3.12).

1. The solution of $\boldsymbol{Az} = \boldsymbol{y}$ where $\boldsymbol{y}$ defined by (3.11) is $\boldsymbol{z} = \zeta \boldsymbol{1} - \boldsymbol{y}$ for any real scalar ζ.

2. The solution to $\boldsymbol{Aq} = \boldsymbol{s}$ where $\boldsymbol{q}$ is defined by

$$
q_i = \phi(\boldsymbol{b}_n, \boldsymbol{x}_n) \left[1 - \frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} \right] \tag{3.16}
$$

for $i = 1, 2, \cdots, m$ and

$$
\phi(\boldsymbol{b}_n, \boldsymbol{x}_n) = \xi \left[1 + \left(\frac{\boldsymbol{b}_n^t \boldsymbol{x}_n - (\boldsymbol{s}^{-1})^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n} \right) \right] \tag{3.17}
$$

is $\boldsymbol{s} = \gamma \boldsymbol{1} - \boldsymbol{q}$ for any real scalar γ where $\boldsymbol{s}^{-1} = (s_i^{-1})$.

Proof.

1. Any $\boldsymbol{z}$ of the form $\boldsymbol{z} = \eta \mathbf{1} - \boldsymbol{y}$ satisfies $\boldsymbol{Az} = \eta(\boldsymbol{A}\mathbf{1}) - \boldsymbol{Ay} = \zeta(\boldsymbol{A}\mathbf{1}) - (-\boldsymbol{y}) = \boldsymbol{y}$. Conversely, if $\boldsymbol{z}$ is a solution to $\boldsymbol{Az} = \boldsymbol{y}$, then $A\boldsymbol{z} = (\boldsymbol{b}_n^t \boldsymbol{z})\mathbf{1} - \boldsymbol{z} = \boldsymbol{y}$, implying that $\boldsymbol{z} = \zeta \mathbf{1} - \boldsymbol{y}$ where $\zeta = \boldsymbol{b}_n^t \boldsymbol{z}_n$.

2. First note that $\boldsymbol{Aq} = (\boldsymbol{b}_n^t \boldsymbol{q})\mathbf{1} - \boldsymbol{q} = -\boldsymbol{q}$ since $\boldsymbol{b}_n^t \boldsymbol{q} = \phi(\boldsymbol{b}_n, \boldsymbol{x}_n) \sum_{i=1}^{m} \left[b_{ni} - \frac{b_{ni} x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} \right] = 0$. Any $\boldsymbol{s}$ of the form $\boldsymbol{s} = \gamma \mathbf{1} - \boldsymbol{q}$ will satisfy $\boldsymbol{As} = \gamma(\boldsymbol{A}\mathbf{1}) - \boldsymbol{Aq} = \boldsymbol{q}$. Conversely, any solution $\boldsymbol{s}$ to $\boldsymbol{As} = \boldsymbol{q}$ must satisfy $A\boldsymbol{s} = (\boldsymbol{b}_n^t \boldsymbol{s})\mathbf{1} - \boldsymbol{s} = \boldsymbol{q}$, implying that $\boldsymbol{s} = \gamma \mathbf{1} - \boldsymbol{q}$ where $\gamma = \boldsymbol{b}^t \boldsymbol{s}$.

By selecting the matrix $\boldsymbol{C}$ as

$$\boldsymbol{C} = \begin{pmatrix} 0.24 & 0.20 & 0.23 & 0.31 & 0.21 \\ 0.17 & 0.20 & 0.33 & 0.22 & 0.11 \\ 0.30 & 0.27 & 0.30 & 0.03 & 0.24 \\ 0.00 & 0.04 & 0.02 & 0.15 & 0.31 \\ 0.29 & 0.29 & 0.13 & 0.29 & 0.13 \end{pmatrix} \tag{3.18}$$

where the sum of the entry in each of the column must be one. The matrix $\boldsymbol{C}$ above is generated randomly which ensure matrix $\boldsymbol{C}$ is a non-negative matrix which satisfy $\mathbf{1}^t C = \mathbf{1}$ where $\mathbf{1} = (1, 1, \cdots, 1)$. A 5×5 matrix is chosen because the stock data sets each consists five stocks.

Then, we run this Type 1 RPR Universal Portfolio on the stock data sets listed in Table 2.1 to obtain empirical results. The period of trading of the stocks selected in Table 2.1 is from 1st March 2006 until 2nd August 2012, consisting of 1500 trading days. There are five company stocks in each data set. We start with taking the initial starting portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$ for all stock data sets. For each stock data sets, the wealth S_{1500} obtained and the final portfolios $\boldsymbol{b}_{1501}$ are calculated for the values of α and best values of ζ are listed in Table 3.1, 3.2, 3.3, 3.4 and 3.5.

Table 3.1: The wealth S_{1500} and the final portfolio $\boldsymbol{b}_{1501}$ achieved by the Type 1 RPR universal portfolio for stock data set D after 1500 trading days, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$

α	ζ	S_{1500}	$\boldsymbol{b}_{1501}$
0	0.00072	2.39	(0.00245, 0.16739, 0.05745, 0.71236, 0.06035)
1	0.00145	2.39	(0.00098, 0.16713, 0.05647, 0.71589, 0.05953)
2	0.00218	2.39	(0.00051, 0.16703, 0.05613, 0.71707, 0.05926)
3	0.00291	2.39	(0.00027, 0.16698, 0.05596, 0.71766, 0.05913)
4	0.00364	2.39	(0.00013, 0.16696, 0.05586, 0.71801, 0.05904)
5	0.00437	2.39	(0.00005, 0.16694, 0.05579, 0.71823, 0.05899)
6	0.00509	2.39	(0.00035, 0.16699, 0.05603, 0.71740, 0.05923)
7	0.00582	2.39	(0.00027, 0.16696, 0.05596, 0.71765, 0.05916)
8	0.00655	2.39	(0.00020, 0.16695, 0.05590, 0.71784, 0.05911)
9	0.00728	2.39	(0.00013, 0.16695, 0.05585, 0.71800, 0.05907)
10	0.00801	2.39	(0.00008, 0.16693, 0.05582, 0.71813, 0.05904)

Table 3.2: The wealth S_{1500} and the final portfolio $\boldsymbol{b}_{1501}$ achieved by the Type 1 RPR universal portfolio for stock data set E after 1500 trading days, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$

α	ζ	S_{1500}	$\boldsymbol{b}_{1501}$
0	0.00072	10.29	(0.00288, 0.16725, 0.05285, 0.71626, 0.06076)
1	0.00145	10.30	(0.00111, 0.16701, 0.05631, 0.71585, 0.05972)
2	0.00218	10.31	(0.00060, 0.16695, 0.05602, 0.71704, 0.05939)
3	0.00291	10.31	(0.00034, 0.16692, 0.05588, 0.71764, 0.05922)
4	0.00364	10.31	(0.00020, 0.16690, 0.05579, 0.71799, 0.05912)
5	0.00437	10.31	(0.00009, 0.16689, 0.05574, 0.71823, 0.05905)
6	0.00509	10.31	(0.00003, 0.16688, 0.05569, 0.71840, 0.05900)
7	0.00582	10.31	(0.00031, 0.16693, 0.05591, 0.71764, 0.05921)
8	0.00655	10.31	(0.00023, 0.16692, 0.05586, 0.71784, 0.05915)
9	0.00728	10.31	(0.00016, 0.16692, 0.05582, 0.71799, 0.05911)
10	0.00801	10.31	(0.00011, 0.16691, 0.05578, 0.71812, 0.05908)

Table 3.3: The wealth S_{1500} and the final portfolio $\boldsymbol{b}_{1501}$ achieved by the Type 1 RPR universal portfolio for stock data set F after 1500 trading days, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$

α	ζ	S_{1500}	$\boldsymbol{b}_{1501}$
0	0	1.27	(0.2, 0.2, 0.2, 0.2, 0.2)
1	0	1.27	(0.2, 0.2, 0.2, 0.2, 0.2)
2	0	1.27	(0.2, 0.2, 0.2, 0.2, 0.2)
3	0	1.27	(0.2, 0.2, 0.2, 0.2, 0.2)
4	0	1.27	(0.2, 0.2, 0.2, 0.2, 0.2)
5	0	1.27	(0.2, 0.2, 0.2, 0.2, 0.2)
6	0	1.27	(0.2, 0.2, 0.2, 0.2, 0.2)
7	0	1.27	(0.2, 0.2, 0.2, 0.2, 0.2)
8	0	1.27	(0.2, 0.2, 0.2, 0.2, 0.2)
9	0	1.27	(0.2, 0.2, 0.2, 0.2, 0.2)
10	0	1.27	(0.2, 0.2, 0.2, 0.2, 0.2)

Table 3.4: The wealth S_{1500} and the final portfolio $\boldsymbol{b}_{1501}$ achieved by the Type 1 RPR universal portfolio for stock data set G after 1500 trading days, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$

α	ζ	S_{1500}	$\boldsymbol{b}_{1501}$
0	0	4.44	(0.2, 0.2, 0.2, 0.2, 0.2)
1	0	4.44	(0.2, 0.2, 0.2, 0.2, 0.2)
2	0	4.44	(0.2, 0.2, 0.2, 0.2, 0.2)
3	0	4.44	(0.2, 0.2, 0.2, 0.2, 0.2)
4	0	4.44	(0.2, 0.2, 0.2, 0.2, 0.2)
5	0	4.44	(0.2, 0.2, 0.2, 0.2, 0.2)
6	0	4.44	(0.2, 0.2, 0.2, 0.2, 0.2)
7	0	4.44	(0.2, 0.2, 0.2, 0.2, 0.2)
8	0	4.44	(0.2, 0.2, 0.2, 0.2, 0.2)
9	0	4.44	(0.2, 0.2, 0.2, 0.2, 0.2)
10	0	4.44	(0.2, 0.2, 0.2, 0.2, 0.2)

Table 3.5: The wealth S_{1500} and the final portfolio $\boldsymbol{b}_{1501}$ achieved by the Type 1 RPR universal portfolio for stock data set H after 1500 trading days, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$

α	ζ	S_{1500}	$\boldsymbol{b}_{1501}$
0	0.00072	5.20	(0.00246, 0.16751, 0.05675, 0.71299, 0.06029)
1	0.00145	5.20	(0.00101, 0.16707, 0.05622, 0.71622, 0.05948)
2	0.00218	5.20	(0.00533, 0.16320, 0.05496, 0.71729, 0.05922)
3	0.00291	5.20	(0.00294, 0.16496, 0.05524, 0.71782, 0.05904)
4	0.00364	5.21	(0.00151, 0.16593, 0.05543, 0.71811, 0.05902)
5	0.00437	5.21	(0.00006, 0.16691, 0.05571, 0.71835, 0.05897)
6	0.00509	5.20	(0.00038, 0.16697, 0.05595, 0.71749, 0.05921)
7	0.00582	5.20	(0.00028, 0.16696, 0.05588, 0.71773, 0.05915)
8	0.00655	5.20	(0.00020, 0.16694, 0.05584, 0.71792, 0.05910)
9	0.00728	5.21	(0.00014, 0.16693, 0.05580, 0.71807, 0.05906)
10	0.00801	5.21	(0.00009, 0.16692, 0.05577, 0.71819, 0.05903)

Table 3.1, 3.2, 3.3, 3.4 and 3.5 give the empirical performance of Type 1 RPR universal portfolio. We can observed that the best wealth of 10.31 units is obtained for data set E corresponding to $\alpha = 6$ and $\zeta = 0.0051$. The lowest wealth of 1.27 units is obtained for data set F corresponding to $\alpha = 0, \cdots, 10$ and $\zeta = 0$. Average wealth of 2.39, 4.44 and 5.20 units are obtained for data set D, G and H respectively. It is also observed that for data sets D, E, and H, a proportion of 70% of the current wealth after 1500 trading days tends to be invested in the fourth company of the portfolio, whereas the proportion invested in the first company tends to zero. This indicates that the fourth and the first stocks are the best and worst stock respectively. For data sets F and G, the portfolios become constant after a long run.

3.1.2 Type 2 RPR Universal Portfolio

In this section, the derivation for Type 2 RPR universal portfolio is shown. The empirical results for this universal portfolio are given by running this universal portfolio on the stock data sets.

Proposition 3.1.2: Consider $C = (c_{ij})$ be a non-negative matrix satisfying $\mathbf{1}^t C = \mathbf{1}$ where a real scalar, α are given. Given $\xi > 0$, from (1.4), consider the objective function

$$\begin{aligned} \hat{F}(\boldsymbol{b}_{n+1}, \lambda) = & \xi \left[\log(\boldsymbol{b}_n^t \boldsymbol{x}_n) + \left(\frac{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n} \right) - \frac{1}{2} \left(\frac{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n} - 1 \right)^2 \right] \\ & - \log \left\{ \prod_{i=1}^{m} [\sigma_i + (b_{n+1,i} - b_{ni})] \right\} + \lambda \left(\sum_{i=1}^{m} b_{n+1,i} - 1 \right) \end{aligned} \tag{3.19}$$

where

$$\sigma_i = \xi^{-1} (\boldsymbol{b}_n^t \boldsymbol{x}_n)^2 \sum_{j=1}^{m} c_{ij} r_j \tag{3.20}$$

for $i = 1, 2, \cdots, m$,

$$r_i = \left[\alpha (\boldsymbol{b}_n^t \boldsymbol{x}_n)^2 + \left(x_{ni} - \beta(\boldsymbol{b}_n, \boldsymbol{x}_n) \right) (\boldsymbol{b}_n^t \boldsymbol{x}_n) + \beta(\boldsymbol{b}_n, \boldsymbol{x}_n) x_{ni} \right]^{-1} > 0 \tag{3.21}$$

for $i = 1, 2, \cdots, m$ and

$$\beta(\boldsymbol{b}_n, \boldsymbol{x}_n) = \xi^{-1} (\boldsymbol{b}_n^t \boldsymbol{x}_n)^2 \boldsymbol{x}_n^t [C\boldsymbol{r} - \boldsymbol{r}]. \tag{3.22}$$

Then, the pseudo Type 2 RPR universal portfolio generated by the zero-gradient set of $\hat{F}(\boldsymbol{b}_{n+1}, \lambda)$ is obtained as follow

$$\boldsymbol{b}_{n+1} = \boldsymbol{b}_n + \xi^{-1} (\boldsymbol{b}_n^t \boldsymbol{x}_n)^2 [\boldsymbol{r} - C\boldsymbol{r}] \tag{3.23}$$

for $n = 1, 2, \cdots$, where ξ is any positive scalar satisfying $\boldsymbol{b}_{n+1} \geq 0$, provided $\boldsymbol{r}$ is a consistent solution of (3.21).

Proof. Then, we differentiate $\hat{F}(\boldsymbol{b}_{n+1}, \lambda)$ in (3.19) with respect to $b_{n+1,i}$ to obtain

$$\frac{\partial \hat{F}}{\partial b_{n+1,i}} = \xi \left[\frac{2x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} - \frac{(\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n) x_{ni}}{(\boldsymbol{b}_n^t \boldsymbol{x}_n)^2} \right] - [\sigma_i (b_{n+1,i} - b_{ni})]^{-1} + \lambda \tag{3.24}$$

for $i = 1, 2, \cdots, m$. Then we let $\frac{\partial \hat{F}}{\partial b_{n+1,i}}$ in (3.24) to be zero and obtain

$$\xi \left[\frac{2x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} - \frac{(\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n) x_{ni}}{(\boldsymbol{b}_n^t \boldsymbol{x}_n)^2} \right] - [\sigma_i + (b_{n+1,i} - b_{ni})]^{-1} + \lambda = 0 \tag{3.25}$$

for $i = 1, 2, \cdots, m$. Next, we multiply (3.25) by b_{ni} and sum over i to obtain the pseudo Lagrange multiplier

$$\lambda = -\xi \left[2 - \frac{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n} \right] + \sum_{j=1}^{m} b_{nj} \big(\sigma_j + (b_{n+1,j} - b_{nj}) \big)^{-1}. \tag{3.26}$$

Then, we substitute the Lagrange multiplier in (3.26) into (3.25) to get

$$\begin{aligned} \xi \left[\frac{2x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} - \frac{(\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n) x_{ni}}{(\boldsymbol{b}_n^t \boldsymbol{x}_n)^2} - 2 + \frac{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n} \right] - [\sigma_i + (b_{n+1,i} - b_{ni})]^{-1}] \\ + \sum_{j=1}^{m} b_{nj} \big(\sigma_j + (b_{n+1,j} - b_{nj}) \big)^{-1} = 0 \end{aligned} \tag{3.27}$$

for $i = 1, 2, \cdots, m$. Then, we let

$$s_i^{-1} = b_{ni} - b_i + \sigma_i \tag{3.28}$$

for $i = 1, 2, \cdots, m$. Then, we substitute (3.28) into (3.27) to obtain

$$\sum_{j=1}^{m} b_{nj} s_j - s_i = \phi(\boldsymbol{b}_n, \boldsymbol{x}_n) \big[1 - \frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} \big] \tag{3.29}$$

for $i = 1, 2, \cdots, m$ where

$$\phi(\boldsymbol{b}_n, \boldsymbol{x}_n) = \xi \big[2 - \frac{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n} \big]. \tag{3.30}$$

We use the vector notation $\boldsymbol{s}^{-1} = (s_i^{-1})$ then (3.28) can become $(\boldsymbol{s}^{-1})^t \boldsymbol{x}_n = \boldsymbol{b}_{n+1}^t \boldsymbol{x}_n - \boldsymbol{b}_n^t \boldsymbol{x}_n + \boldsymbol{\sigma}^t \boldsymbol{x}_n$. This implies

$$\frac{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n} = 1 + \frac{(\boldsymbol{s}^{-1})\boldsymbol{x}_n - \boldsymbol{\sigma}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n}. \tag{3.31}$$

By substituting (3.31) into (3.30), we obtain the equivalent definition of $\phi(\boldsymbol{b}_n, \boldsymbol{x}_n)$ in (3.17). The matrix form of the set of equations on (3.29) is $A\boldsymbol{s} = \boldsymbol{q}$ where A and $\boldsymbol{q}$ are defined by (3.12) and (3.16) respectively. From Lemma (3.1.1), we have shown that the solution to $A\boldsymbol{s} = q$ is $\boldsymbol{s} = \gamma\mathbf{1} - \boldsymbol{q}$ for any real scalar γ. An equivalent definition of $\beta(\boldsymbol{b}_n, \boldsymbol{x}_n)$ in (3.22) is given by

$$\beta(\boldsymbol{b}_n, \boldsymbol{x}_n) = (\boldsymbol{x}_n^t \boldsymbol{\sigma} - \boldsymbol{x}_n^t \boldsymbol{s}^{-1}). \tag{3.32}$$

Hence, we can obtain $s_i = \gamma - q_i = \gamma - \xi[1 + \frac{\beta(\boldsymbol{b}_n, \boldsymbol{x}_n)}{\boldsymbol{b}_n^t \boldsymbol{x}_n}]$ for $i = 1, 2, \cdots, m$ from (3.16), (3.17) and (3.32).

Then we reparametrize the γ as $(\alpha + 1)\xi$, we obtain

$$\begin{aligned} s_i =&(bnx)^{-2}\xi[\alpha(\boldsymbol{b}_n^t \boldsymbol{x}_n)^2 + (x_{ni} - \beta(\boldsymbol{b}_n, \boldsymbol{x}_n))(\boldsymbol{b}_n^t \boldsymbol{x}_n) + \beta(\boldsymbol{b}_n, \boldsymbol{x}_n)x_{ni}] \\ =&(\boldsymbol{b}_n^t \boldsymbol{x}_n)^{-2} r_i^{-1} \end{aligned} \tag{3.33}$$

for $i = 1, 2, \cdots, m$ where r_i is defined by (3.21). From (3.20) and (3.28), we obtain the next-day portfolio

$$\begin{aligned} b_{n+1,i} =&b_{ni} + s_i^{-1} - \sigma_i \\ =&b_{ni} + \xi^{-1}(\boldsymbol{b}_n^t \boldsymbol{x}_n)^2 \left[r_i - \sum_{j=1}^{m} c_{ij} r_j \right] \end{aligned} \tag{3.34}$$

for $i = 1, 2, \cdots, m$ and (3.23) is proved.

We remark that the pseudo Type 2 RPR universal portfolio may be relaxed by assuming that $\beta(\boldsymbol{b}_n, \boldsymbol{x}_n)$ is a constant, which not depending on the $\boldsymbol{b}_n$ and $\boldsymbol{x}_n$. The pseudo relaxed type 2 portfolio has parametric set (C, α, β) where we will choose the β to be $-1 \leq \beta \leq 1$. The scalar α is chosen so that $r_i > 0$ for all $i = 1, 2, \cdots, m$ in (3.21). This is always possible for a large enough α.

Similarly, we run the pseudo relaxed Type 2 RPR universal portfolios with parametric set (C, α, β) on the stock data sets D, E, F, G and H, listed in Table 2.1. We select the matrix C to be an equal-entry matrix with each entry 0.2 and eleven integer values of α from 0 until 10. $\beta = 0.6$ is selected to obtain the empirical results. The best wealth S_{1500} achieved after 1500 trading days on each stock data sets are listed in Table 3.6, 3.7, 3.8, 3.9 and 3.10.

Table 3.6: The wealth S_{1500} and the final portfolio $\boldsymbol{b}_{1501}$ achieved by the pseudo relaxed Type 2 RPR universal portfolio for stock data set D after 1500 trading days, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$

α	ξ	S_{1500}	$\boldsymbol{b}_{1501}$
0	0.1	2.46	(0.24380, 0.21627, 0.21218, 0.17654, 0.15121)
1	0.5	2.53	(0.23456, 0.23665, 0.22449, 0.17672, 0.12758)
2	1.3	2.57	(0.23182, 0.25012, 0.23215, 0.17530, 0.11061)
3	2.2	2.57	(0.22625, 0.25142, 0.23250, 0.17754, 0.11229)
4	3.5	2.58	(0.22423, 0.25482, 0.23428, 0.17777, 0.10890)
5	5.1	2.59	(0.22281, 0.25718, 0.23550, 0.17794, 0.10657)
6	6.9	2.59	(0.22144, 0.25796, 0.23587, 0.17839, 0.10634)
7	9.0	2.59	(0.22048, 0.25876, 0.23626, 0.17865, 0.10585)
8	11.4	2.59	(0.21978, 0.25950, 0.23663, 0.17881, 0.10528)
9	14.0	2.59	(0.21909, 0.25969, 0.23671, 0.17906, 0.10545)
10	16.9	2.59	(0.21857, 0.25998, 0.23684, 0.17923, 0.10538)

Table 3.7: The wealth S_{1500} and the final portfolio $\boldsymbol{b}_{1501}$ achieved by the pseudo relaxed Type 2 RPR universal portfolio for stock data set E after 1500 trading days, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$

α	ξ	S_{1500}	$\boldsymbol{b}_{1501}$
0	0.2	9.17	(0.30314, 0.19905, 0.11999, 0.11239, 0.26543)
1	1.0	9.46	(0.36914, 0.15425, 0.09283, 0.10818, 0.27560)
2	2.1	9.36	(0.37176, 0.14014, 0.09788, 0.12168, 0.26854)
3	3.6	9.32	(0.37245, 0.13381, 0.10079, 0.12810, 0.26485)
4	5.5	9.29	(0.37265, 0.13027, 0.10264, 0.13184, 0.26260)
5	7.9	9.28	(0.37495, 0.12697, 0.10265, 0.13353, 0.26190)
6	10.7	9.28	(0.37604, 0.12487, 0.10298, 0.13492, 0.26119)
7	13.9	9.27	(0.37654, 0.12345, 0.10341, 0.13606, 0.26054)
8	17.5	9.26	(0.37674, 0.12244, 0.10385, 0.13700, 0.25997)
9	21.6	9.26	(0.37761, 0.12128, 0.10381, 0.13753, 0.25977)
10	26.0	9.26	(0.37741, 0.12079, 0.10427, 0.13824, 0.25929)

Table 3.8: The wealth S_{1500} and the final portfolio $\boldsymbol{b}_{1501}$ achieved by the pseudo relaxed Type 2 RPR universal portfolio for stock data set F after 1500 trading days, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$

α	ξ	S_{1500}	$\boldsymbol{b}_{1501}$
0	0.1	1.38	(0.13895, 0.13657, 0.29070, 0.20205, 0.23173)
1	0.6	1.44	(0.11958, 0.12320, 0.33978, 0.16434, 0.25310)
2	1.5	1.46	(0.11436, 0.12027, 0.36283, 0.14144, 0.26110)
3	2.7	1.46	(0.11518, 0.12207, 0.36952, 0.13025, 0.26298)
4	4.2	1.47	(0.11672, 0.12416, 0.37151, 0.12423, 0.26338)
5	6.0	1.46	(0.11817, 0.12595, 0.37187, 0.12068, 0.26333)
6	8.1	1.46	(0.11941, 0.12743, 0.37161, 0.11842, 0.26313)
7	10.5	1.46	(0.12045, 0.12864, 0.37112, 0.11689, 0.26290)
8	13.2	1.46	(0.12132, 0.12965, 0.37055, 0.11582, 0.26266)
9	16.2	1.46	(0.12206, 0.13049, 0.36998, 0.11503, 0.26244)
10	19.6	1.46	(0.12228, 0.13081, 0.37047, 0.11388, 0.26256)

Table 3.9: The wealth S_{1500} and the final portfolio $\boldsymbol{b}_{1501}$ achieved by the pseudo relaxed Type 2 RPR universal portfolio for stock data set G after 1500 trading days, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$

α	ξ	S_{1500}	$\boldsymbol{b}_{1501}$
0	0.1	4.44	(0.2, 0.2, 0.2, 0.2, 0.2)
1	0.5	4.44	(0.2, 0.2, 0.2, 0.2, 0.2)
2	1.1	4.44	(0.2, 0.2, 0.2, 0.2, 0.2)
3	1.9	4.44	(0.2, 0.2, 0.2, 0.2, 0.2)
4	2.9	4.44	(0.2, 0.2, 0.2, 0.2, 0.2)
5	4.1	4.44	(0.2, 0.2, 0.2, 0.2, 0.2)
6	5.6	4.44	(0.2, 0.2, 0.2, 0.2, 0.2)
7	7.3	4.44	(0.2, 0.2, 0.2, 0.2, 0.2)
8	9.2	4.44	(0.2, 0.2, 0.2, 0.2, 0.2)
9	11.3	4.44	(0.2, 0.2, 0.2, 0.2, 0.2)
10	13.7	4.44	(0.2, 0.2, 0.2, 0.2, 0.2)

Table 3.10: The wealth S_{1500} and the final portfolio $\boldsymbol{b}_{1501}$ achieved by the pseudo relaxed Type 2 RPR universal portfolio for stock data set H after 1500 trading days, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$

α	ξ	S_{1500}	$\boldsymbol{b}_{1501}$
0	0.1	4.57	(0.2, 0.2, 0.2, 0.2, 0.2)
1	0.5	4.57	(0.2, 0.2, 0.2, 0.2, 0.2)
2	1.1	4.57	(0.2, 0.2, 0.2, 0.2, 0.2)
3	1.9	4.57	(0.2, 0.2, 0.2, 0.2, 0.2)
4	3.0	4.57	(0.2, 0.2, 0.2, 0.2, 0.2)
5	4.4	4.57	(0.2, 0.2, 0.2, 0.2, 0.2)
6	5.9	4.57	(0.2, 0.2, 0.2, 0.2, 0.2)
7	7.8	4.57	(0.2, 0.2, 0.2, 0.2, 0.2)
8	9.8	4.57	(0.2, 0.2, 0.2, 0.2, 0.2)
9	12.1	4.57	(0.2, 0.2, 0.2, 0.2, 0.2)
10	14.7	4.57	(0.2, 0.2, 0.2, 0.2, 0.2)

Tables 3.6, 3.7, 3.8, 3.9 and 3.10 give the numerical results obtained by the pseudo relaxed Type 2 RPR universal portfolio with parametric set (C, α, β). The matrix C selected is an equal-entry matrix with each entry 0.2. We observed that average wealth of 2.57, 9.27, 1.46, 4.44 and 4.57 units are obtained for data sets D, E, F, G and H respectively. The best portfolio corresponds to data set E exhibiting good performance. In data set E, current-wealth proportions of around 37% and 26% tend to be invested in the first and fifth stock respectively, whereas in set D, proportions of 25% and 23%

tend to be invested in the second and third stocks. For data sets G and H, the portfolios tend to be constant after a long run.

The performance of the pseudo Type 1 and Type 2 RPR portfolios seem to be comparable, with neither exhibiting a superior performance over the other.

3.2 Universal Portfolio Generated by Rényi and Generalized Kullback-Leibler Divergence

Helmbold et al. (1998) introduces a universal portfolio by maximizing an objective function which is a linear sum of an estimated daily growth rate of return and the Kullback-Leibler divergence of two portfolio vectors. In this session, we generalize this portfolio by using a more general order-α Kullback-Leibler divergence and the pseudo Lagrange multiplier. One of the results gives the explicit form of the portfolio corresponds to a reparametrized Helmbold universal portfolio. The Helmbold universal portfolio, $\{\boldsymbol{b}_{n+1}\}$ with parameter η is given in Helmbold et al. (1998) is defined by:

$$b_{n+1,i} = \frac{b_{ni} e^{\frac{\eta x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}}}{\sum_{j=1}^{m} b_{nj} e^{\frac{\eta x_{nj}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}}} \tag{3.35}$$

for $i = 1, 2, \cdots, m$. Another result gives the implicit form of the portfolio, which has a functional relationship with the Rényi universal portfolio. This relationship is studied.

3.2.1 Universal Portfolio Generated by Rényi Divergence

Given two probability distributions $\boldsymbol{p} = (p_i)$ and $\boldsymbol{q} = (q_i)$, the Rényi order-α divergence given in Basu et al. (2011) is defined as:

$$R_\alpha = \frac{1}{\alpha - 1} \log \left[\sum_{i=1}^{m} p_i^{\alpha} q_i^{1-\alpha} \right] \tag{3.36}$$

where $\alpha = 0$ and $\alpha \neq 1$.

Proposition 3.2.1: Consider the objective function $\hat{F}(b_{n+1,i}, \lambda)$ containing the Rényi divergence (3.36):

$$\begin{aligned}\hat{F}(\boldsymbol{b}_{n+1}, \lambda) =& \xi \left(\log(\boldsymbol{b}_n^t \boldsymbol{x}_n) + \left(\frac{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n} \right) - 1 \right) - \frac{1}{\alpha - 1} \log \left[\sum_{i=1}^{m} b_{n+1,i}^{\alpha} b_{ni}^{1-\alpha} \right] \\ & + \lambda \left(\sum_{j=1}^{m} b_{n+1,j} - 1 \right)\end{aligned} \tag{3.37}$$

for $\alpha > 0$ and $\alpha \neq 1$. The pseudo implicit form of the universal portfolio generated by Rényi divergence is

$$b_{n+1,i} = \frac{\psi_i^{\frac{1}{\alpha-1}}}{\sum_{j=1}^{m} \psi_j^{\frac{1}{\alpha-1}} b_{nj}} \tag{3.38}$$

for $i = 1, 2, \cdots, m$ and

$$\psi_i(\boldsymbol{b}_{n+1}, \boldsymbol{b}_n, \boldsymbol{x}_n) = \left\{ \frac{\xi(\alpha - 1)}{\alpha} \left[\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} - \left(\frac{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n} \right) + 1 \right] \right\} \tag{3.39}$$

for $i = 1, 2, \cdots, m$. We will choose the parameter $\xi > 0$ to ensure $\psi_i(\boldsymbol{b}_{n+1}, \boldsymbol{b}_n, \boldsymbol{x}_n) > 0$ for all $i = 1, 2, \cdots, m$. Then the pseudo Rényi (α, ξ) universal portfolio is given by (3.38), in the condition that there is a consistent solution to $\boldsymbol{b}_{n+1}$.

Proof. We differentiate the objective function, $\hat{F}(\boldsymbol{b}_{n+1},\lambda)$ in (3.37) to obtain

$$\frac{\partial \hat{F}}{\partial b_{n+1,i}} = \xi\left[\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right] - \frac{\alpha}{(\alpha-1)} \frac{b_{n+1,i}^{\alpha-1} b_{ni}^{1-\alpha}}{\sum_{j=1}^m b_{n+1,j}^{\alpha} b_{nj}^{1-\alpha}} + \lambda \tag{3.40}$$

for $i = 1, 2, \cdots, m$. Then, we let (3.40) equals to zero and get

$$\xi\left[\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right] - \frac{\alpha}{(\alpha-1)} \frac{b_{n+1,i}^{\alpha-1} b_{ni}^{1-\alpha}}{\sum_{j=1}^m b_{n+1,j}^{\alpha} b_{nj}^{1-\alpha}} + \lambda = 0 \tag{3.41}$$

for $i = 1, 2, \cdots, m$. Then, we multiply $b_{n+1,i}$ into (3.41) and sum over i to get

$$\xi\left(\frac{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right) - \frac{\alpha}{\alpha-1} + \lambda = 0 \tag{3.42}$$

where λ is the pseudo Lagrange multiplier. Then subtract (3.42) from (3.41) and get

$$\xi\left[\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} - \left(\frac{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right)\right] + \frac{\alpha}{\alpha-1} = \frac{\alpha}{\alpha-1} \frac{b_{n+1,i}^{\alpha-1} b_{ni}^{1-\alpha}}{\sum_{j=1}^m b_{n+1,j}^{\alpha} b_{nj}^{1-\alpha}} \tag{3.43}$$

for $i = 1, 2, \cdots, m$. Arrange (3.43) to obtain

$$b_{n+1,i} = \psi_i^{\frac{1}{\alpha-1}} b_{ni} \left[\sum_{j=1}^m b_{n+1,j}^{\alpha} b_{nj}^{1-\alpha}\right]^{\frac{1}{\alpha-1}} \tag{3.44}$$

for $i = 1, 2, \cdots, m$. Summing (3.44) over all i will obtain

$$1 = \left[\sum_{j=1}^m b_{n+1,j}^{\alpha} b_{nj}^{1-\alpha}\right]^{\frac{1}{\alpha-1}} \sum_{i=1}^m \psi_i^{\frac{1}{\alpha-1}} b_{ni} \tag{3.45}$$

Then, substitute (3.45) as normalizing constant to (3.44) and (3.38) is obtained.

3.2.2 Universal Portfolio Generated by Generalized Kullback-Leibler Divergence

Given two probability distributions $\boldsymbol{p} = (p_i)$ and $\boldsymbol{q} = (q_i)$, the more general Kullback-Leibler order-α divergence is defined as:

$$D_\alpha(\boldsymbol{p}||\boldsymbol{q}) = \alpha \sum_{i=1}^{m} p_i \log \frac{p_i}{q_i} \tag{3.46}$$

where $\alpha > 0$ and $\alpha \neq 1$. The case $\alpha = 1$ corresponds to the well-studied Kullback-Leibler divergence.

Proposition 3.2.2: Consider the objective function $\hat{F}(b_{n+1,i}, \lambda)$ from (1.4) which containing the Kullback-Leibler order-α divergence $D_\alpha(\boldsymbol{b}_{n+1}||\boldsymbol{b}_n)$:

$$\begin{aligned}\hat{F}(\boldsymbol{b}_{n+1}, \lambda) =& \xi\left(\log(\boldsymbol{b}_n^t \boldsymbol{x}_n) + \left(\frac{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right) - 1\right) - \alpha \sum_{j=1}^{m} b_{n+1,j} \log \frac{b_{n+1,j}}{b_{nj}} \\ &+ \lambda\left(\sum_{j=1}^{m} b_{n+1,j} - 1\right)\end{aligned} \tag{3.47}$$

for $\alpha > 0$ and $\alpha \neq 1$. We then obtain the pseudo explicit form of the universal portfolio generated by (3.46) is the Helmbold universal portfolio (3.35) with the parameter $\eta = \frac{\xi}{\alpha}$. The pseudo implicit form of the universal portfolio generated by (3.46) is

$$b_{n+1,i} = \frac{\phi_i^{\frac{1}{\alpha-1}} b_{ni}}{\left[\sum_{j=1}^{m} \phi_j^{\frac{1}{\alpha-1}} b_{nj}\right]} \tag{3.48}$$

for $i = 1, 2, \cdots, m$ where

$$\phi_i(\boldsymbol{b}_{n+1}, \boldsymbol{b}_n, \boldsymbol{x}_n) = e^{\psi_i(\boldsymbol{b}_{n+1}, \boldsymbol{b}_n, \boldsymbol{x}_n)} \tag{3.49}$$

for $i = 1, 2, \cdots, m$ and $\psi_i(\boldsymbol{b}_{n+1}, \boldsymbol{b}_n, \boldsymbol{x}_n)$ is defined by (3.39). Then the pseudo implicit Kullback-Leibler (α, ξ) universal portfolio is given by (3.48), in the condition that there is a consistent solution to $\boldsymbol{b}_{n+1}$.

Proof. Differentiate the objective function, $\hat{F}(\boldsymbol{b}_{n+1}, \lambda)$, (3.47) to obtain

$$\frac{\partial \hat{F}}{\partial b_{n+1,i}} = \xi \left[\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} \right] - \alpha \left[1 + \log \frac{b_{n+1,i}}{b_{ni}} \right] + \lambda \tag{3.50}$$

for $i = 1, 2, \cdots, m$. Let (3.50) equal to zero,

$$\xi \left[\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} \right] - \alpha \left[1 + \log \frac{b_{n+1,i}}{b_{ni}} \right] + \lambda = 0 \tag{3.51}$$

for $i = 1, 2, \cdots, m$. Then multiply (3.51) by b_{ni} and sum over i to get

$$\xi - \alpha \left[1 + \sum_{i=1}^{m} b_{nj} \log \frac{b_{n+1,j}}{b_{nj}} \right] + \lambda = 0 \tag{3.52}$$

where λ is the pseudo Lagrange multiplier. Then we take (3.51) minus (3.52) to obtain

$$\xi \left[\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} - 1 \right] + \alpha \left[1 + \sum_{j=1}^{m} b_{nj} \log \frac{b_{n+1,j}}{b_{nj}} - \log \frac{b_{n+1,i}}{b_{ni}} \right] = 0 \tag{3.53}$$

for $i = 1, 2, \cdots, m$ and hence

$$b_{n+1,i} = b_{ni} e^{\frac{\xi}{\alpha} \frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}} e^{-\frac{\xi}{\alpha} + \sum_{j=1}^{m} b_{nj} \log \frac{b_{n+1,j}}{b_{nj}}} \tag{3.54}$$

for $i = 1, 2, \cdots, m$. Sum (3.54) over i and obtain

$$1 = \Big[\sum_{i=1}^{m} b_{ni} e^{\frac{\xi}{\alpha} \frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}} \Big]^{-1} e^{-\frac{\xi}{\alpha} + \sum_{j=1}^{m} b_{nj} \log \frac{b_{n+1,j}}{b_{nj}}}. \tag{3.55}$$

Replacement (3.55) to (3.54) will result a universal portfolio as below:

$$b_{n+1,i} = \frac{b_{ni} e^{\frac{\xi}{\alpha}\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}}}{\sum_{j=1}^m b_{nj} e^{\frac{\xi}{\alpha}\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}}} \tag{3.56}$$

for $i = 1, 2, \cdots, m$. We can see that this $\boldsymbol{b}_{n+1}$ is the Helmbold universal portfolio (3.35) with parameter $\eta = \frac{\xi}{\alpha}$.

Next, if we multiply $b_{n+1,i}$ into (3.51), we obtain the sum is

$$\xi \left[\frac{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n} \right] - \alpha \left[1 + \sum_{j=1}^m b_{n+1,j} \log \frac{b_{n+1,i}}{b_{ni}} \right] + \lambda = 0 \tag{3.57}$$

where λ is a different pseudo Lagrange multiplier from the one defined in (3.52).

Then we take (3.51) minus (3.57) and obtain

$$\xi \left[\frac{x_{ni} - (\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n)}{\boldsymbol{b}_n^t \boldsymbol{x}_n} \right] + \alpha \left[\sum_{j=1}^m b_{b+1,j} \log \frac{b_{n+1,j}}{b_{nj}} - \log \frac{b_{n+1,i}}{b_{ni}} \right] = 0 \tag{3.58}$$

for $i = 1, 2, \cdots, m$. Thus,

$$\begin{aligned} b_{n+1,i} =& b_{ni} e^{\frac{\xi}{\alpha}\left[\frac{x_{ni} - \boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right] + \sum_{j=1}^m b_{n+1,j} \log \frac{b_{n+1,j}}{b_{nj}}} \\ =& b_{ni} e^{\left\{\frac{\xi(\alpha-1)}{\alpha}\left[\frac{x_{ni} - \boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right] + 1\right\}\frac{1}{\alpha-1}} \times e^{\left\{-\frac{1}{\alpha-1} + \sum_{j=1}^m b_{n+1,j} \log \frac{b_{n+1,j}}{b_{nj}}\right\}} \\ =& b_{ni} [e^{\psi_i}]^{\frac{1}{\alpha-1}} e^{\left\{-\frac{1}{\alpha-1} + \sum_{j=1}^m b_{n+1,j} \log \frac{b_{n+1,j}}{b_{nj}}\right\}} \end{aligned} \tag{3.59}$$

for $i = 1, 2, \cdots, m$. We sum (3.59) over all i,

$$1 = \left[\sum_{i=1}^m \left[b_{ni} e^{\psi_i} \right]^{\frac{1}{\alpha-1}} \right] e^{\left\{-\frac{1}{\alpha-1} + \sum_{j=1}^m b_{n+1,j} \log \frac{b_{n+1,j}}{b_{nj}}\right\}} \tag{3.60}$$

Then we substitute (3.60) into (3.59) and get (3.48). We then reparametrize the parameter ξ to a new parameter $\beta > 0$ where $\beta = \frac{1}{\xi} |\frac{\alpha}{\alpha-1}|$ where $\xi > 0$.

Proposition 3.2.3:

1. Let $\alpha > 0$ and $\beta > 0$. If the pseudo implicit Rényi (α, β) universal portfolio is given by

$$b_{n+1,i} = \frac{c_{1i}^{\frac{1}{\alpha-1}} b_{ni}}{\left[\sum_{j=1}^{m} c_{1j}^{\frac{1}{\alpha-1}} b_{nj}\right]} \tag{3.61}$$

for $i = 1, 2, \cdots, m$ where

$$c_{1i}(\boldsymbol{b}_{n+1}, \boldsymbol{b}_n, \boldsymbol{x}_n, \beta) = \beta(\boldsymbol{b}_n^t \boldsymbol{x}_n) - (\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n - x_{ni}) \tag{3.62}$$

for $i = 1, 2, \cdots, m$ and we choose any value for β such that $c_{1i} > 0$ for $i = 1, 2, \cdots, m$, then we have the pseudo implicit Kullback-Leibler (α, β) universal portfolio as below

$$b_{n+1,i} = \frac{\phi_i^{\frac{1}{\alpha-1}} b_{ni}}{\sum_{j=1}^{m} \phi_j^{\frac{1}{\alpha-1}} b_{nj}} \tag{3.63}$$

for $i = 1, 2, \cdots, m$ where

$$\phi_i = e^{\frac{c_{1i}}{\beta(\boldsymbol{b}_n^t \boldsymbol{x}_n)}} \tag{3.64}$$

for $i = 1, 2, \cdots, m$.

2. Let $0 < \alpha < 1$ and $\beta > 0$. If the pseudo implicit Rényi (α, β) universal portfolio is given by

$$b_{n+1,i} = \frac{c_{2i}^{\frac{1}{1-\alpha}} b_{ni}}{\left[\sum_{j=1}^{m} c_{2j}^{\frac{1}{1-\alpha}} b_{nj}\right]} \tag{3.65}$$

for $i = 1, 2, \cdots, m$ where

$$c_{2i}(\boldsymbol{b}_{n+1}, \boldsymbol{b}_n, \boldsymbol{x}_n, \beta) = [\beta(\boldsymbol{b}_n^t \boldsymbol{x}_n) - (\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n - x_{ni})]^{-1} \tag{3.66}$$

for $i = 1, 2, \cdots, m$ and we choose any value for β such that $c_{2i} > 0$ for

$i = 1, 2, \cdots, m$, then we have the pseudo implicit Kullback-Leibler (α, β) universal portfolio as below

$$b_{n+1,i} = \frac{(\phi_i^{-1})^{\frac{1}{1-\alpha}} b_{ni}}{\sum_{j=1}^{m} b_{nj} (\phi_j^{-1})^{\frac{1}{1-\alpha}}} \tag{3.67}$$

for $i = 1, 2, \cdots, m$ where

$$\phi_i^{-1} = e^{\frac{1}{c_{2i}\beta(\boldsymbol{b}_n^t \boldsymbol{x}_n)}} \tag{3.68}$$

for $i = 1, 2, \cdots, m$.

Proof. For $\alpha > 0$, then $\beta = \frac{\alpha}{\xi(\alpha-1)} > 0$. Noting that $\psi_i \beta(\boldsymbol{b}_n) = c_{1i}$, we obtain the results that follow (3.38), (3.48) and (3.49). Then for $0 < \alpha < 1$, $\beta = \frac{\alpha}{\xi(1-\alpha)} > 0$. From the relationship $\psi_i \beta(\boldsymbol{b}_n^t \boldsymbol{x}_n) = c_{2i}^{-1}$, we obtain the results that follow (3.38), (3.48) and (3.49), where $\phi_i^{-1} = e^{-\psi_i}$.

We remark that the explicit and implicit Kullback-Leibler (α, ξ) universal portfolios (3.35) and (3.48) generated by pseudo zero-gradient set of the objective function (3.47) are different from each other. The pseudo implicit Rényi (α, β) and Kullback-Leibler (α, β) universal portfolios (3.61), (3.63), (3.65) and (3.67) in general may not have consistent solution for $\boldsymbol{b}_{n+1}$. Hence, we relax the pseudo Rényi (α, β, γ) and Kullback-Leibler (α, β, γ) universal portfolios by replacing the $\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n$ in (3.62) and (3.66) by

$$\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n = \gamma \min_j \{x_{nj}\} + (1-\gamma) \max_j \{x_{bj}\} \tag{3.69}$$

where γ is a parameter that does not depend on $\boldsymbol{b}_{n+1}$ and $\boldsymbol{x}_n$. The parameter γ is a constant where $0 < \gamma < 1$.

We obtain Helmbold universal portfolio (3.35) from the pseudo Kullback-Leibler (α, β, γ) universal portfolios by replacing $\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n$ in (3.62) and (3.66) by (3.69) with parameters $\eta = [\beta(\alpha-1)]^{-1}$ and $\eta = [\beta(1-\alpha)]^{-1}$ respectively, corresponding to $\alpha > 1$ and $0 < \alpha < 1$.

Tables 3.11, 3.12, 3.13, 3.14 and 3.15 give the empirical results obtained by running the pseudo relaxed Rényi universal portfolio for stock data sets D, E, F, G and H after 1500 trading days. The parameters selected are $\alpha = 10$, $\beta = 6$ and $\gamma = 0.1, 0.2, \cdots, 0.9$. We observed that average wealth of 2.3276, 8.0075, 1.2652, 4.4528 and 4.5760 units are achieved for data sets D, E, F, G and H respectively. The portfolios in decreasing order of performance are data sets E, H, G, D and F. The highest portfolio proportions are in the fifth, third, fourth, third and third stocks for data sets, D, E, F, G and H respectively after 1500 trading days.

Table 3.11: The wealth S_{1500} and the final portfolio $\boldsymbol{b}_{1501}$ achieved by the pseudo relaxed Rényi universal portfolio for stock data set D after 1500 trading days, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$, $\alpha = 10$ and $\beta = 6$.

γ	S_{1500}	$\boldsymbol{b}_{1501}$
0.1	2.3276	(0.1998, 0.1991, 0.1994, 0.2003, 0.2014)
0.2	2.3276	(0.1998, 0.1991, 0.1994, 0.2003, 0.2014)
0.3	2.3276	(0.1998, 0.1991, 0.1994, 0.2003, 0.2014)
0.4	2.3276	(0.1998, 0.1991, 0.1994, 0.2003, 0.2014)
0.5	2.3276	(0.1998, 0.1991, 0.1994, 0.2003, 0.2014)
0.6	2.3276	(0.1998, 0.1991, 0.1994, 0.2003, 0.2014)
0.7	2.3276	(0.1998, 0.1991, 0.1994, 0.2003, 0.2014)
0.8	2.3276	(0.1998, 0.1991, 0.1994, 0.2003, 0.2014)
0.9	2.3276	(0.1998, 0.1991, 0.1994, 0.2003, 0.2014)

Table 3.12: The wealth S_{1500} and the final portfolio $\boldsymbol{b}_{1501}$ achieved by the pseudo relaxed Rényi universal portfolio for stock data set E after 1500 trading days, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$, $\alpha = 10$ and $\beta = 6$.

γ	S_{1500}	$\boldsymbol{b}_{1501}$
0.1	8.0075	(0.198, 0.2009, 0.2010, 0.2008, 0.1995)
0.2	8.0075	(0.198, 0.2009, 0.2010, 0.2008, 0.1995)
0.3	8.0075	(0.198, 0.2009, 0.2010, 0.2008, 0.1995)
0.4	8.0075	(0.198, 0.2009, 0.2010, 0.2008, 0.1995)
0.5	8.0075	(0.198, 0.2009, 0.2010, 0.2008, 0.1995)
0.6	8.0075	(0.198, 0.2009, 0.2010, 0.2008, 0.1995)
0.7	8.0075	(0.198, 0.2009, 0.2010, 0.2008, 0.1995)
0.8	8.0075	(0.198, 0.2009, 0.2010, 0.2008, 0.1995)
0.9	8.0075	(0.198, 0.2009, 0.2010, 0.2008, 0.1995)

Table 3.13: The wealth S_{1500} and the final portfolio $\boldsymbol{b}_{1501}$ achieved by the pseudo relaxed Rényi universal portfolio for stock data set F after 1500 trading days, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$, $\alpha = 10$ and $\beta = 6$.

γ	S_{1500}	$\boldsymbol{b}_{1501}$
0.1	1.2652	(0.2010, 0.2007, 0.1980, 0.2012, 0.1991)
0.2	1.2652	(0.2010, 0.2007, 0.1980, 0.2012, 0.1991)
0.3	1.2652	(0.2010, 0.2007, 0.1980, 0.2012, 0.1991)
0.4	1.2652	(0.2010, 0.2007, 0.1980, 0.2012, 0.1991)
0.5	1.2652	(0.2010, 0.2007, 0.1980, 0.2012, 0.1991)
0.6	1.2652	(0.2010, 0.2007, 0.1980, 0.2012, 0.1991)
0.7	1.2652	(0.2010, 0.2007, 0.1980, 0.2012, 0.1991)
0.8	1.2652	(0.2010, 0.2007, 0.1980, 0.2012, 0.1991)
0.9	1.2652	(0.2010, 0.2007, 0.1980, 0.2012, 0.1991)

Table 3.14: The wealth S_{1500} and the final portfolio $\boldsymbol{b}_{1501}$ achieved by the pseudo relaxed Rényi universal portfolio for stock data set G after 1500 trading days, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$, $\alpha = 10$ and $\beta = 6$.

γ	S_{1500}	$\boldsymbol{b}_{1501}$
0.1	4.4529	(0.2028, 0.1971, 0.2031, 0.1980, 0.1990)
0.2	4.4529	(0.2028, 0.1971, 0.2031, 0.1980, 0.1990)
0.3	4.4528	(0.2028, 0.1971, 0.2031, 0.1980, 0.1990)
0.4	4.4528	(0.2028, 0.1971, 0.2031, 0.1980, 0.1990)
0.5	4.4528	(0.2028, 0.1971, 0.2031, 0.1980, 0.1990)
0.6	4.4528	(0.2028, 0.1971, 0.2031, 0.1980, 0.1990)
0.7	4.4528	(0.2028, 0.1971, 0.2031, 0.1980, 0.1990)
0.8	4.4528	(0.2028, 0.1971, 0.2031, 0.1980, 0.1990)
0.9	4.4528	(0.2028, 0.1971, 0.2031, 0.1980, 0.1990)

Table 3.15: The wealth S_{1500} and the final portfolio $\boldsymbol{b}_{1501}$ achieved by the pseudo relaxed Rényi universal portfolio for stock data set H after 1500 trading days, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$, $\alpha = 10$ and $\beta = 6$.

γ	S_{1500}	$\boldsymbol{b}_{1501}$
0.1	4.5760	(0.1989, 0.2000, 0.2030, 0.1966, 0.2015)
0.2	4.5760	(0.1989, 0.2000, 0.2030, 0.1966, 0.2015)
0.3	4.5760	(0.1989, 0.2000, 0.2030, 0.1966, 0.2015)
0.4	4.5760	(0.1989, 0.2000, 0.2030, 0.1966, 0.2015)
0.5	4.5760	(0.1989, 0.2000, 0.2030, 0.1966, 0.2015)
0.6	4.5760	(0.1989, 0.2000, 0.2030, 0.1966, 0.2015)
0.7	4.5759	(0.1989, 0.2000, 0.2030, 0.1966, 0.2015)
0.8	4.5759	(0.1989, 0.2000, 0.2029, 0.1967, 0.2015)
0.9	4.5759	(0.1989, 0.2000, 0.2029, 0.1967, 0.2015)

Tables 3.16, 3.17, 3.18, 3.19 and 3.20 provide the empirical results achieved by the psudeo relaxed Kullback-Leibler universal portfolio running over data sets D, E, F, G and H for 1500 trading days. The parameters chosen are α and β equal to 0.1 and 2 respectively with nine values of γ from 0.1 to 0.9, with a 0.1 difference. The average wealth achieved are 2.2153, 7.6412, 1.1924, 4.7212 and 4.6942 units for data sets D, E, F, G and H respectively. The best wealth is achieved for data set E and the lowest wealth is achieved for data set F. The highest portfolio proportions are in the fifth, third, fourth, third and third stock for data sets D, E, F, G and H respectively after 1500 trading days. The best performing portfolios are data sets E, G, H, D and F in decreasing order.

We noticed that the pseudo relaxed Rényi universal portfolio performs better than the pseudo relaxed Kullback-Leibler universal portfolio for data sets D, E and F. However, this situation is reversed for data sets G and H. The portfolio performance is data-dependent and a general conclusion cannot be drawn.

Table 3.16: The wealth S_{1500} and the final portfolio $\boldsymbol{b}_{1501}$ achieved by the pseudo relaxed Kullback-Leibler universal portfolio for stock data set D after 1500 trading days, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$, $\alpha = 0.1$ and $\beta = 2$.

γ	S_{1500}	$\boldsymbol{b}_{1501}$
0.1	2.2153	(0.1947, 0.1742, 0.1802, 0.2077, 0.2432)
0.2	2.2153	(0.1947, 0.1742, 0.1802, 0.2077, 0.2432)
0.3	2.2153	(0.1947, 0.1742, 0.1802, 0.2077, 0.2432)
0.4	2.2153	(0.1947, 0.1742, 0.1802, 0.2077, 0.2432)
0.5	2.2153	(0.1947, 0.1742, 0.1802, 0.2077, 0.2432)
0.6	2.2153	(0.1947, 0.1742, 0.1802, 0.2077, 0.2432)
0.7	2.2153	(0.1947, 0.1742, 0.1802, 0.2077, 0.2432)
0.8	2.2153	(0.1947, 0.1742, 0.1802, 0.2077, 0.2432)
0.9	2.2153	(0.1947, 0.1742, 0.1802, 0.2077, 0.2432)

Table 3.17: The wealth S_{1500} and the final portfolio $\boldsymbol{b}_{1501}$ achieved by the pseudo relaxed Kullback-Leibler universal portfolio for stock data set E after 1500 trading days, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$, $\alpha = 0.1$ and $\beta = 2$.

γ	S_{1500}	$\boldsymbol{b}_{1501}$
0.1	7.6412	(0.1465, 0.2234, 0.228, 0.219, 0.1831)
0.2	7.6412	(0.1465, 0.2234, 0.228, 0.219, 0.1831)
0.3	7.6412	(0.1465, 0.2234, 0.228, 0.219, 0.1831)
0.4	7.6412	(0.1465, 0.2234, 0.228, 0.219, 0.1831)
0.5	7.6412	(0.1465, 0.2234, 0.228, 0.219, 0.1831)
0.6	7.6412	(0.1465, 0.2234, 0.228, 0.219, 0.1831)
0.7	7.6412	(0.1465, 0.2234, 0.228, 0.219, 0.1831)
0.8	7.6412	(0.1465, 0.2234, 0.228, 0.219, 0.1831)
0.9	7.6412	(0.1465, 0.2234, 0.228, 0.219, 0.1831)

Table 3.18: The wealth S_{1500} and the final portfolio $\boldsymbol{b}_{1501}$ achieved by the pseudo relaxed Kullback-Leibler universal portfolio for stock data set F after 1500 trading days, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$, $\alpha = 0.1$ and $\beta = 2$.

γ	S_{1500}	$\boldsymbol{b}_{1501}$
0.1	1.1924	(0.2293, 0.218, 0.1502, 0.2296, 0.1729)
0.2	1.1924	(0.2293, 0.218, 0.1502, 0.2296, 0.1729)
0.3	1.1924	(0.2293, 0.218, 0.1502, 0.2296, 0.1729)
0.4	1.1924	(0.2293, 0.218, 0.1502, 0.2296, 0.1729)
0.5	1.1924	(0.2293, 0.218, 0.1502, 0.2296, 0.1729)
0.6	1.1924	(0.2293, 0.218, 0.1502, 0.2296, 0.1729)
0.7	1.1924	(0.2293, 0.218, 0.1502, 0.2296, 0.1729)
0.8	1.1924	(0.2293, 0.218, 0.1502, 0.2296, 0.1729)
0.9	1.1924	(0.2293, 0.218, 0.1502, 0.2296, 0.1729)

Table 3.19: The wealth S_{1500} and the final portfolio $\boldsymbol{b}_{1501}$ achieved by the pseudo relaxed Kullback-Leibler universal portfolio for stock data set G after 1500 trading days, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$, $\alpha = 0.1$ and $\beta = 2$.

γ	S_{1500}	$\boldsymbol{b}_{1501}$
0.1	4.7212	(0.2839, 0.1213, 0.2925, 0.1392, 0.1631)
0.2	4.7212	(0.2839, 0.1213, 0.2925, 0.1392, 0.1631)
0.3	4.7212	(0.2839, 0.1213, 0.2925, 0.1392, 0.1631)
0.4	4.7212	(0.2839, 0.1213, 0.2925, 0.1392, 0.1631)
0.5	4.7212	(0.2839, 0.1213, 0.2925, 0.1392, 0.1631)
0.6	4.7212	(0.2839, 0.1213, 0.2925, 0.1392, 0.1631)
0.7	4.7212	(0.2839, 0.1213, 0.2925, 0.1392, 0.1631)
0.8	4.7212	(0.2839, 0.1213, 0.2925, 0.1392, 0.1631)
0.9	4.7212	(0.2839, 0.1213, 0.2925, 0.1392, 0.1631)

Table 3.20: The wealth S_{1500} and the final portfolio $\boldsymbol{b}_{1501}$ achieved by the pseudo relaxed Kullback-Leibler universal portfolio for stock data set H after 1500 trading days, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$, $\alpha = 0.1$ and $\beta = 2$.

γ	S_{1500}	$\boldsymbol{b}_{1501}$
0.1	4.6942	(0.1623, 0.1897, 0.2943, 0.116, 0.2377)
0.2	4.6942	(0.1623, 0.1897, 0.2943, 0.116, 0.2377)
0.3	4.6942	(0.1623, 0.1897, 0.2943, 0.116, 0.2377)
0.4	4.6942	(0.1623, 0.1897, 0.2943, 0.116, 0.2377)
0.5	4.6942	(0.1623, 0.1897, 0.2943, 0.116, 0.2377)
0.6	4.6942	(0.1623, 0.1897, 0.2943, 0.116, 0.2377)
0.7	4.6942	(0.1623, 0.1897, 0.2943, 0.116, 0.2377)
0.8	4.6942	(0.1623, 0.1897, 0.2943, 0.116, 0.2377)
0.9	4.6942	(0.1623, 0.1897, 0.2943, 0.116, 0.2377)

3.3 Reverse Helmbold Universal Portfolio

In this section, we consider the zero-gradient set of the objective function estimating the next-day portfolio which contains the reverse Kullback-Leibler order-alpha divergence. Then, we obtain the explicit, reverse Helmbold universal portfolio. The performance of this universal portfolio is studied and compared with Helmbold universal portfolio (3.35).

From the Kullback-Leibler order-α divergence (3.46), we obtain Helmbold universal portfolio (3.35) by taking the $\alpha = 1$ in (3.63) and (3.67). Given the reverse Kullback-Leibler order-α divergence of $\boldsymbol{p} = (p_i)$ and $\boldsymbol{q} = (q_i)$ is defined as:

$$D_\alpha(\boldsymbol{q}||\boldsymbol{p}) = \alpha \sum_{j=1}^{m} q_j \log \frac{q_j}{p_j} \tag{3.70}$$

for $\alpha > 0$. The universal portfolio generated by (3.70) is known as the reverse order-α Helmbold portfolio.

Proposition 3.3.1: Let $\boldsymbol{v} = (v_i)$ be a given vector, where

$$v_i = \frac{\xi}{\alpha}\left[\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right] \tag{3.71}$$

for $i = 1, 2, \cdots, m$, $\xi > 0$ and $\alpha > 0$. Then, consider the set of linear equations

$$\boldsymbol{C}\boldsymbol{u} = \boldsymbol{v} \tag{3.72}$$

where $\boldsymbol{C} = (c_{ij})$ is a known matrix given by:

$$c_{ij} = \begin{cases} b_{nj}^2 & \text{for } i \neq j \\ b_{ni}^2 - b_{ni} & \text{for } i = j. \end{cases} \tag{3.73}$$

Then, the solution to the set of equations (3.72) is:

$$u_i = \frac{1}{b_{ni}}(\eta - v_i) \tag{3.74}$$

for $i = 1, 2, \cdots, m$ where η is any real scalar.

Proof. The system (3.72) can be written as:

$$\sum_{j=1}^{m} b_{nj}^2 u_j - b_{ni} u_i = v_i \tag{3.75}$$

for $i = 1, 2, \cdots, m$.

A necessary condition for $\boldsymbol{u} = (u_i)$ be a solution to (3.75) is that $\sum_{j=1}^{m} b_{nj}^2 u_j = b_{ni} u_i + v_i =$ constant independent of i, say η. Then $u_i = \frac{1}{b_{ni}}(\eta - v_i)$ for $i = 1, 2, \cdots, m$.

Conversely, let $\boldsymbol{u} = (u_i)$ be given by (3.74). Then

$$\begin{aligned}
\sum_{j=1}^{m} b_{nj}^2 u_j - b_{ni} u_i =& \sum_{j=1}^{m} b_{nj}(\eta - v_j) - (\eta - v_i) \\
=& \eta - \sum_{j=1}^{m} b_{nj} \frac{\xi}{\alpha} \big[\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} - 1 \big] - \eta + v_i \\
=& \eta - \frac{\xi}{\alpha} + \frac{\xi}{\alpha} - \eta + v_i \\
=& v_i
\end{aligned}$$

which implies that $\boldsymbol{u}$ given by (3.74) for any real scalar η is a solution.

Proposition 3.3.2: Consider the objective function which consists of the reverse Kullback-Leibler order-α divergence (3.70) of $\boldsymbol{b}_{n+1} = (b_{n+1,i})$ and $\boldsymbol{b}_n = (b_{n,i})$,

$$\begin{aligned}
\hat{F}(\boldsymbol{b}_{n+1}, \lambda) =& \xi \left[\log(\boldsymbol{b}_n^t \boldsymbol{x}_n) + \frac{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n} - 1 \right] \\
& - \alpha \sum_{j=1}^{m} b_{nj} \log \frac{b_{nj}}{b_{n+1,j}} + \lambda \big[\sum_{j=1}^{m} b_{n+1,j} - 1 \big]
\end{aligned} \tag{3.76}$$

where $\xi > 0$ and $\alpha > 0$ are given. We obtain the reverse order-α Helmbold universal portfolio generated by the reverse Kullback-Leibler order-α divergence (3.70) as follow:

$$b_{n+1,i} = \frac{b_{ni}[\beta(\boldsymbol{b}_n^t \boldsymbol{x}_n) - x_{ni}]^{-1}}{\sum_{j=1}^{m} b_{nj}[\beta(\boldsymbol{b}_n^t \boldsymbol{x}_n) - x_{nj}]^{-1}} \tag{3.77}$$

for $i = 1, 2, \cdots, m$ where $\beta = \frac{\alpha\eta}{\xi} + 1$ for any real scalar η. The η is chosen such that $\beta(\boldsymbol{b}_n^t \boldsymbol{x}_n) - x_{ni} > 0$ for all $i = 1, 2, \cdots, m$.

Proof. We differentiate the objective function (3.76) with respect to $b_{n+1,i}$ and obtain

$$\frac{\partial \hat{F}}{\partial b_{n+1,i}} = \xi \big[\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} \big] + \alpha \frac{b_{ni}}{b_{n+1,i}} + \lambda \tag{3.78}$$

We then let (3.78) equal to zero and obtain

$$\xi\,[\frac{x_{ni}}{\boldsymbol{b}_n^t\boldsymbol{x}_n}] + \alpha\frac{b_{ni}}{b_{n+1,i}} + \lambda = 0. \tag{3.79}$$

In order to solve for the Lagrange multiplier, λ, we multiply b_{ni} into (3.79) and sum over i to get

$$\xi + \alpha\sum_{j=1}^{m}\frac{b_{nj}^2}{b_{n+1,j}} + \lambda = 0. \tag{3.80}$$

We then replacing λ in (3.79) by the value obtain in (3.80) and obtain

$$\xi\left[\frac{x_{ni}}{\boldsymbol{b}_n^t\boldsymbol{x}_n} - 1\right] + \alpha\left[\frac{b_{ni}}{b_{n+1,i}} - \sum_{j=1}^{m}\frac{b_{nj}^2}{b_{n+1,j}}\right] = 0 \tag{3.81}$$

for $i = 1,2,\cdots,m$. We let $u_i = \frac{1}{b_{n+1,j}}$ for $i = 1,2,\cdots,m$, it leads to the system of equation (3.72). Then, we obtain the solution for this system of equation, is given by (3.74). Then, we obtain

$$\begin{aligned} b_{n+1,i} =& \frac{b_{ni}}{\left\{\eta - \frac{\xi}{\alpha}\left[\frac{x_{ni}}{\boldsymbol{b}_n^t\boldsymbol{x}_n} - 1\right]\right\}} \\ =& \frac{b_{ni}(\boldsymbol{b}_n^t\boldsymbol{x}_n)}{\left\{\left(\eta + \frac{\xi}{\alpha}\right)(\boldsymbol{b}_n^t\boldsymbol{x}_n) - \frac{\xi}{\alpha}x_{ni}\right\}} \\ =& \frac{b_{ni}(\boldsymbol{b}_n^t\boldsymbol{x}_n)\frac{\alpha}{\xi}}{\{\beta(\boldsymbol{b}_n^t\boldsymbol{x}_n) - x_{ni}\}} \end{aligned} \tag{3.82}$$

for $i = 1,2,\cdots,m$ where $\beta = \frac{\alpha\eta}{\xi} + 1$. Summing (3.82) over all i and obtain the normalizing constant

$$(\boldsymbol{b}_n^t\boldsymbol{x}_n)\frac{\alpha}{\xi} = \left\{\sum_{j=1}^{m} b_{nj}[\beta(\boldsymbol{b}_n^t\boldsymbol{x}_n) - x_{nj}]^{-1}\right\}^{-1}. \tag{3.83}$$

Substitute the normalizing constant (3.83) to (3.82) to obtain (3.77).

The reverse order-α Helmbold universal portfolio 3.77 with parameter β is run on the selected data sets D, E,F G and H. The wealth achieved after 1500 trading days are given in Tables 3.21, 3.22, 3.23, 3.24 and 3.25 respectively, together with the final portfolios $\boldsymbol{b}_{1501}$. It is observed that the best wealth achieved for data sets D, E, F, G and H are 2.5245, 8.0192, 1.4358, 5.7101 and 5.0115 units respectively. The good portfolio is E with the best performance. Portfolios G and H exhibit average performance. The portfolios D and F perform poorly. For good portfolio E, the return is almost 8 times the starting wealth after a period of 1500 trading days.

A close examination of Tables 3.21, 3.22, 3.23, 3.24 and 3.25 reveals that the final portfolio for data set E tends to a portfolio with the proportion of wealth almost equally distributed among the five stocks as the parameter β gets large. For data set D and $\beta = 0.3$, the first three stocks have the higher weights in the range $0.23-0.25$, with lower weights for the last two stocks. For set F and the best parameter $\beta = 0.4$, heavier weights are placed on the third and fifth stocks. About half of the wealth is placed on the first stock for data set G and 40% on the third stock corresponding to the best parameter $\beta = 1.3$. The third and fifth stocks in data set H receive higher weights of 48% and 46% for $\beta = 1.3$, with very low weights for the other stocks.

Table 3.21: The wealth S_{1500} and the final portfolio $\boldsymbol{b}_{1501}$ achieved by the reverse Helmbold universal portfolio for stock data set D after 1500 trading days, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$

β	S_{1500}	$\boldsymbol{b}_{1501}$
0.0	2.5066	(0.2182, 0.2432, 0.2266, 0.1778, 0.1342)
0.1	2.5155	(0.2220, 0.2459, 0.2285, 0.1748, 0.1288)
0.2	2.5225	(0.2275, 0.2483, 0.2307, 0.1710, 0.1225)
0.3	2.5245	(0.2358, 0.2496, 0.2332, 0.1661, 0.1153)
0.4	2.5151	(0.2497, 0.2478, 0.2359, 0.1597, 0.1069)
0.5	2.4814	(0.2747, 0.2383, 0.2388, 0.1508, 0.0974)
0.6	2.3967	(0.3255, 0.2099, 0.2406, 0.1374, 0.0866)
0.7	2.2071	(0.4356, 0.1413, 0.2346, 0.1139, 0.0746)
0.8	1.8281	(0.5649, 0.0391, 0.2339, 0.0797, 0.0824)

Table 3.22: The wealth S_{1500} and the final portfolio $\boldsymbol{b}_{1501}$ achieved by the reverse Helmbold universal portfolio for stock data set E after 1500 trading days, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$

β	S_{1500}	$\boldsymbol{b}_{1501}$
10	7.9457	(0.1881, 0.2053, 0.2058, 0.2039, 0.1969)
20	7.9848	(0.1943, 0.2025, 0.2028, 0.2019, 0.1985)
30	7.9969	(0.1963, 0.2016, 0.2018, 0.2012, 0.1991)
40	8.0028	(0.1972, 0.2012, 0.2013, 0.2009, 0.1994)
50	8.0063	(0.1978, 0.2010, 0.2011, 0.2007, 0.1994)
100	8.0132	(0.1989, 0.2005, 0.2005, 0.2004, 0.1997)
300	8.0177	(0.1996, 0.2002, 0.2002, 0.2001, 0.1999)
500	8.0186	(0.1998, 0.2001, 0.2001, 0.2001, 0.1999)
800	8.0191	(0.1999, 0.2001, 0.2000, 0.2000, 0.2000)

Table 3.23: The wealth S_{1500} and the final portfolio $\boldsymbol{b}_{1501}$ achieved by the reverse Helmbold universal portfolio for stock data set F after 1500 trading days, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$

β	S_{1500}	$\boldsymbol{b}_{1501}$
0.0	1.4006	(0.1382, 0.1463, 0.3317, 0.1484, 0.2354)
0.1	1.4106	(0.1315, 0.1396, 0.3462, 0.1449, 0.2378)
0.2	1.4208	(0.1233, 0.1316, 0.3628, 0.1414, 0.2409)
0.3	1.4300	(0.1136, 0.1218, 0.3811, 0.1386, 0.2449)
0.4	1.4358	(0.1022, 0.1100, 0.3986, 0.1379, 0.2513)
0.5	1.4338	(0.0889, 0.0959, 0.4085, 0.1434, 0.2633)
0.6	1.4180	(0.0736, 0.0790, 0.3920, 0.1678, 0.2876)
0.7	1.3813	(0.0537, 0.0560, 0.3071, 0.2560, 0.3272)
0.8	1.2590	(0.0218, 0.0192, 0.1443, 0.5132, 0.3015)

Table 3.24: The wealth S_{1500} and the final portfolio $\boldsymbol{b}_{1501}$ achieved by the reverse Helmbold universal portfolio for stock data set G after 1500 trading days, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$

β	S_{1500}	$\boldsymbol{b}_{1501}$
1.3	5.7101	(0.5143, 0.0035, 0.3854, 0.0807, 0.0161)
1.4	5.5362	(0.5205, 0.0137, 0.3894, 0.0353, 0.0411)
1.5	5.3322	(0.4603, 0.0242, 0.4088, 0.0455, 0.0612)
1.6	5.2014	(0.4200, 0.0353, 0.4084, 0.0574, 0.0789)
1.7	5.1081	(0.3917, 0.0463, 0.3992, 0.0689, 0.0939)
1.8	5.0366	(0.3705, 0.0566, 0.3868, 0.0795, 0.1066)
1.9	4.9792	(0.3539, 0.0662, 0.3737, 0.0891, 0.1171)
2.0	4.9320	(0.3403, 0.0750, 0.3613, 0.0975, 0.1259)

Table 3.25: The wealth S_{1500} and the final portfolio $\boldsymbol{b}_{1501}$ achieved by the reverse Helmbold universal portfolio for stock data set H after 1500 trading days, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$

β	S_{1500}	$\boldsymbol{b}_{1501}$
1.3	5.0115	(0.0237, 0.0409, 0.4762, 0.0041, 0.4551)
1.4	4.9750	(0.0468, 0.0746, 0.5088, 0.0120, 0.3578)
1.5	4.9290	(0.0675, 0.1009, 0.4885, 0.0221, 0.3210)
1.6	4.8893	(0.0849, 0.1210, 0.4594, 0.0330, 0.3017)
1.7	4.8563	(0.0994, 0.1361, 0.4315, 0.0437, 0.2893)
1.8	4.8289	(0.1112, 0.1476, 0.4072, 0.0538, 0.2802)
1.9	4.8060	(0.1209, 0.1564, 0.3866, 0.0630, 0.2731)
2.0	4.7866	(0.1290, 0.1633, 0.3692, 0.0714, 0.2671)

CHAPTER 4

f-DIVERGENCE AND ITS REVERSE

4.1 f-Divergence

Rényi (1961) introduced the f-divergence, which is a function $D_f(\boldsymbol{p}||\boldsymbol{q})$ measuring the difference between two probability distributions $\boldsymbol{p}$ and $\boldsymbol{q}$. This divergence is further studied independently by Csiszár (1963) and this is also known as Csiszár f-divergence. Let $f(t)$ be a convex function on $(0,\infty)$ and is strictly convex at $t=1$ and it satisfies $f(1)=0$. Then, we have the f-divergence between two probability distribution $\boldsymbol{p}=(p_i)$ and $\boldsymbol{q}=(q_i)$ is given by

$$D_f(\boldsymbol{p}||\boldsymbol{q}) = \sum_{i=1}^{n} q_i f\left[\frac{p_i}{q_i}\right]. \tag{4.1}$$

Hence, for two portfolio vectors of $\boldsymbol{b}_{n+1}$ and $\boldsymbol{b}_n$, the f-divergence is given by

$$D_f(\boldsymbol{b}_{n+1}||\boldsymbol{b}_n) = \sum_{i=1}^{m} b_{ni} f\left[\frac{b_{n+1,i}}{b_{ni}}\right]. \tag{4.2}$$

4.1.1 Type k Universal Portfolio Generated by f-Divergence

In this session, we develop a generalization of the universal portfolio generated by f-divergence by considering k-th order Taylor series approximation of $\log\left[\frac{\boldsymbol{b}_{n+1}^t\boldsymbol{x}_n}{\boldsymbol{b}_n^t\boldsymbol{x}_n}\right]$. Then we obtain Type k universal portfolio generated by f-divergence where $k=1,2,\cdots$.

Lemma 4.1.1: The k-th. order approximation of $\log\left[\frac{\mathbf{b}_{n+1}^k \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n}\right]$ is

$$\log\left[\frac{\mathbf{b}_{n+1}^t \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n}\right] \simeq \sum_{r=1}^{k} \frac{(-1)^{r+1}}{r}\left[\frac{\mathbf{b}_{n+1}^t \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n} - 1\right]^r \tag{4.3}$$

and

$$\frac{\partial}{\partial b_{n+1,i}}\left[\log\left(\frac{\mathbf{b}_{n+1}^t \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n}\right)\right] = \frac{x_{ni}}{\mathbf{b}_{n+1}^t \mathbf{x}_n}\left[1 + (-1)^{k+1}\left(\frac{\mathbf{b}_{n+1}^t \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n} - 1\right)^k\right] \tag{4.4}$$

for $0 < \frac{\mathbf{b}_{n+1}^t \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n} < 2$, where $k = 1, 2, \cdots$. Furthermore, (4.4) can be simplified as

$$\frac{\partial}{\partial b_{n+1,i}}\left[\log\left(\frac{\mathbf{b}_{n+1}^t \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n}\right)\right] = \left(\frac{x_{ni}}{\mathbf{b}_n^t \mathbf{x}_n}\right) u_k(\mathbf{b_{n+1}}, \mathbf{b}_n), \tag{4.5}$$

where

$$u_k(\mathbf{b}_{n+1}, \mathbf{b}_n) = \sum_{r=1}^{k} (-1)^{2k+1-r}\binom{k}{r}\left(\frac{\mathbf{b}_{n+1}^t \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n}\right)^r \tag{4.6}$$

for $i = 1, 2, \cdots, m$.

Remarks. The function $u_k(\boldsymbol{b}_{n+1}, \boldsymbol{b}_n)$ in (4.6) is a polynomial in $\left(\frac{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right)$.

1. For $k = 1$, $u_1(\boldsymbol{b}_{n+1}, \boldsymbol{b}_n) = 1$.

2. For $k = 2$, $u_2(\boldsymbol{b}_{n+1}, \boldsymbol{b}_n) = 2 - \left(\frac{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right)$.

3. For $k = 3$, $u_3(\boldsymbol{b}_{n+1}, \boldsymbol{b}_n) = 3 - 3\left(\frac{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right) + \left(\frac{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right)^2$.

Proof. From the Taylor series $\log(1+z) = \sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} z^r$ for $|z| < 1$, it is clear that $\log\, y = \sum_{r=1}^{k} \frac{(-1)^{r+1}}{r}(y-1)^r$ is the k-th approximation of $\log\, y$. Hence (4.3) obtains when $y = \frac{\mathbf{b}_{n+1}^t \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n}$.

Differentiating (4.3) with respect to $b_{n+1,i}$, the derivative is

$$\begin{aligned}
&\frac{x_{ni}}{\mathbf{b}_n^t\mathbf{x}_n}\left[\sum_{r=1}^{k}(-1)^{r+1}\left\{\frac{\mathbf{b}_{n+1}^t\mathbf{x}_n}{\mathbf{b}_n^t\mathbf{x}_n}-1\right\}^{r-1}\right] \\
&\qquad =\frac{x_{ni}}{\mathbf{b}_n^t\mathbf{x}_n}\left[\frac{1+(-1)^{k+1}\left(\frac{\mathbf{b}_{n+1}^t\mathbf{x}_n}{\mathbf{b}_n^t\mathbf{x}_n}-1\right)^k}{1+\left(\frac{\mathbf{b}_{n+1}^t\mathbf{x}_n}{\mathbf{b}_n^t\mathbf{x}_n}-1\right)}\right]
\end{aligned} \tag{4.7}$$

by summing up the geometric series. The derivative simplifies to (4.4). By using the binomial expansion in (4.4), the derivative (4.4) can be written as

$$\begin{aligned}
&\frac{x_{ni}}{\mathbf{b}_n^t\mathbf{x}_n}\left[1+(-1)^{k+1}\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r}\left(\frac{\mathbf{b}_{n+1}^t\mathbf{x}_n}{\mathbf{b}_n^t\mathbf{x}_n}\right)^r\right] \\
&\qquad =\frac{x_{ni}}{\mathbf{b}_n^t\mathbf{x}_n}\left[\sum_{r=1}^{k}(-1)^{2k+1-r}\binom{k}{r}\left(\frac{\mathbf{b}_{n+1}^t\mathbf{x}_n}{\mathbf{b}_n^t\mathbf{x}_n}\right)^r\right]
\end{aligned} \tag{4.8}$$

which leads to (4.5).

Example. (i) For $k=1$,

$$\frac{\partial}{\partial b_{n+1,i}}\left[\log\left(\frac{\mathbf{b}_{n+1}^t\mathbf{x}_n}{\mathbf{b}_n^t\mathbf{x}_n}\right)\right]=\frac{x_{ni}}{\mathbf{b}_n^t\mathbf{x}_n}\ ,\quad i=1,2,\cdots,m. \tag{4.9}$$

(ii) For $k=2$,

$$\begin{aligned}
\frac{\partial}{\partial b_{n+1,i}}\left[\log\left(\frac{\mathbf{b}_{n+1}^t\mathbf{x}_n}{\mathbf{b}_n^t\mathbf{x}_n}\right)\right]&=\left(\frac{x_{ni}}{\mathbf{b}_n^t\mathbf{x}_n}\right)\left[2\left(\frac{\mathbf{b}_{n+1}^t\mathbf{x}_n}{\mathbf{b}_n^t\mathbf{x}_n}\right)-\left(\frac{\mathbf{b}_{n+1}^t\mathbf{x}_n}{\mathbf{b}_n^t\mathbf{x}_n}\right)^2\right] \\
&=\left(\frac{x_{ni}}{\mathbf{b}_n^t\mathbf{x}_n}\right)\left[2-\frac{\mathbf{b}_{n+1}^t\mathbf{x}_n}{\mathbf{b}_n^t\mathbf{x}_n}\right],\quad i=1,2,\cdots,m
\end{aligned} \tag{4.10}$$

Hence, the rate of wealth increase on day $(n+1)$ is $\log(\mathbf{b}_{n+1}^t\mathbf{x}_{n+1})$ which is estimated as $\log(\mathbf{b}_{n+1}^t\mathbf{x}_n)$. From (4.3), the k-th order approximation of $\log(\mathbf{b}_{n+1}^t\mathbf{x}_n)$ is

$$\log(\mathbf{b}_{n+1}^t\mathbf{x}_n)\simeq\log(\mathbf{b}_n^t\mathbf{x}_n)+\sum_{r=1}^{k}\frac{(-1)^{r+1}}{r}\left[\frac{\mathbf{b}_{n+1}^t\mathbf{x}_n}{\mathbf{b}_n^t\mathbf{x}_n}-1\right]^r. \tag{4.11}$$

We then use the result obtained in (4.11) with the objective function

$\hat{F}(\boldsymbol{b}_{n+1};\lambda) = \log(\boldsymbol{b}^t_{n+1}\boldsymbol{x}_n) - D_f(\boldsymbol{b}_{n+1}||\boldsymbol{b}_n)$ to obtain the type k universal portfolio.

Proposition 4.1.1: We define $f(t)$ be a convex function on $(0,\infty)$ satisfying $f(1)=0$ and the objective function

$$\begin{aligned}\hat{F}(\mathbf{b}_{n+1};\lambda) =&\eta\left[\log(\mathbf{b}^t_n\mathbf{x}_n)+\sum_{r=1}^{k}\frac{(-1)^{r+1}}{r}\left(\frac{\mathbf{b}^t_{n+1}\mathbf{x}_n}{\mathbf{b}^t_n\mathbf{x}_n}-1\right)^r\right]\\ &-\sum_{j=1}^{m}b_{nj}f\left[\frac{b_{n+1,j}}{b_{nj}}\right]+\lambda\left(\sum_{j=1}^{m}b_{n+1,j}-1\right),\end{aligned} \tag{4.12}$$

where $\eta > 0$ is a parameter and λ is the Lagrange multiplier. The Type k universal portfolio generated by the f-divergence is given by

$$f'\left[\frac{b_{n+1,i}}{b_{ni}}\right] = \eta u_k(\mathbf{b}_{n+1},\mathbf{b}_n)\left(\frac{x_{ni}}{\mathbf{b}^t_n\mathbf{x}_n}\right)+\xi_n, \tag{4.13}$$

where $u_k(\mathbf{b}_{n+1},\mathbf{b}_n)$ is given by (4.6) and ξ_n is another parameter possible depending on $\mathbf{b}_{n+1}$ and $\mathbf{b}_n$.

Proof. By using (4.3) and (4.5) in the Lemma (4.1.1) and differentiating $\hat{F}(\mathbf{b}_{n+1};\lambda)$ in (4.12). Then we set the derivative equals to zero and obtain the following,

$$\frac{\partial\hat{F}}{\partial b_{n+1,i}} = \eta u_k(\mathbf{b}_{n+1},\mathbf{b}_n)\left(\frac{x_{ni}}{\mathbf{b}^t_n\mathbf{x}_n}\right)-f'\left[\frac{b_{n+1,i}}{b_{ni}}\right]+\lambda=0, \text{ for } \ i=1,2,\cdots,m. \tag{4.14}$$

Then, we multiplying (4.14) by b_{ni} and sum over i to get

$$\lambda=\sum_{j=1}^{m}b_{nj}f'\left[\frac{b_{n+1,j}}{b_{nj}}\right]-\eta u_k(\mathbf{b}_{n+1},\mathbf{b}_n). \tag{4.15}$$

Next, we substitute the value of λ in (4.15) into (4.14) to obtain

$$f'\left[\frac{b_{n+1,i}}{b_{ni}}\right] = \eta u_k(\mathbf{b}_{n+1}, \mathbf{b}_n)\left(\frac{x_{ni}}{\mathbf{b}_n^t \mathbf{x}_n}\right) + \sum_{j=1}^{m} b_{nj} f'\left[\frac{b_{n+1,j}}{b_{nj}}\right] - \eta u_k(\mathbf{b}_{n+1}, \mathbf{b}_n) \tag{4.16}$$

for $i = 1, 2, \cdots, m$.

By reparametrizing,

$$\xi_n = \sum_{j=1}^{m} b_{nj} f'\left[\frac{b_{n+1,j}}{b_{nj}}\right] - \eta u_k(\mathbf{b}_{n+1}, \mathbf{b}_n), \tag{4.17}$$

the form of the universal portfolio (4.13) is derived. For a valid solution to $\nabla \hat{F} = \left(\frac{\partial \hat{F}}{\partial b_{n+1,i}}\right) = \mathbf{0}$, λ in (4.15) should not depend on $\mathbf{b}_{n+1}$. The focus in this paper is on generating a new portfolio instead of finding a valid solution to $\nabla \hat{F} = \mathbf{0}$. If there is no valid solution, (4.13) will be called a *pseudo* solution.

Proposition 4.1.2: We define $f(t)$ be a convex function on $(0, \infty)$ satisfying $f(1) = 0$ and $c > 0$ satisfies $f'(c) < \infty$. The mean-value form of the Type k universal portfolio generated by the f-divergence is given by

$$b_{n+1,i} = b_{ni}\left[c + \frac{1}{f''(s)}\left\{f'\left[\frac{b_{n+1,i}}{b_{ni}}\right] - f'(c)\right\}\right] \text{ for } \quad i = 1, 2, \cdots, m, \tag{4.18}$$

where s is some number between $\frac{b_{n+1,i}}{b_{ni}}$ and c; and $f'\left[\frac{b_{n+1,i}}{b_{ni}}\right]$ is given by (4.13).

Proof. The mean-value theorem states that there exists an s between $\frac{b_{n+1,i}}{b_{ni}}$ and c such that $f'\left[\frac{b_{n+1,i}}{b_{ni}}\right] - f'(c) = \left[\frac{b_{n+1,i}}{b_{ni]}} - c\right] f''(s)$ for $i = 1, 2, \cdots, m$. Rewriting this equation leads to (4.18).

Example. The Helmbold family of universal portfolios is defined by the convex function $f(t) = t \log t$ for $t > 0$. The derivative $f'(t) = \log t + 1$ generates the

Type k Helmbold universal portfolio

$$\log\left(\frac{b_{n+1,i}}{b_{ni}}\right) = \eta u_k(\mathbf{b}_{n+1}, \mathbf{b}_n)\left(\frac{x_{ni}}{\mathbf{b}_n^t \mathbf{x}_n}\right) + \xi_n - 1, \text{ for } \quad i = 1, 2, \cdots, m \quad (4.19)$$

according to (4.13). Exponentiating (4.19), $\frac{b_{n+1,i}}{b_{ni}} = e^{\eta u_k(\mathbf{b}_{n+1},\mathbf{b}_n)\left(\frac{x_{ni}}{\mathbf{b}_n^t\mathbf{x}_n}\right)} e^{\xi_n - 1}$. Evaluating $e^{\xi_n - 1}$, we obtain

$$b_{n+1,i} = \frac{b_{ni} e^{\eta u_k(\mathbf{b}_{n+1},\mathbf{b}_n)\left(\frac{x_{ni}}{\mathbf{b}_n^t\mathbf{x}_n}\right)}}{\sum_{j=1}^{m} b_{nj} e^{\eta u_k(\mathbf{b}_{n+1},\mathbf{b}_n)\left(\frac{x_{nj}}{\mathbf{b}_n^t\mathbf{x}_n}\right)}} \text{ for } \quad i = 1, 2, \cdots, m. \quad (4.20)$$

Recall from (4.6), that $u_k(\mathbf{b}_{n+1}, \mathbf{b}_n)$ is a polynomial in $\frac{\mathbf{b}_{n+1}^t \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n}$. The Type 1 Helmbold universal portfolio for $u_k(\mathbf{b}_{n+1}, \mathbf{b}_n) = 1$ is extensively studied. The Type 2 Helmbold universal portfolio for

$$u_k(\mathbf{b}_{n+1}, \mathbf{b}_n) = 2 - \left(\frac{\mathbf{b}_{n+1}^t \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n}\right) \quad (4.21)$$

is the focus of the empirical study in the next section. The Type 3 Helmbold portfolio is defined for

$$u_k(\mathbf{b}_{n+1}, \mathbf{b}_n) = 3 - 3\left(\frac{\mathbf{b}_{n+1}^t \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n}\right) + \left(\frac{\mathbf{b}_{n+1}^t \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n}\right)^2. \quad (4.22)$$

For the empirical study, $\mathbf{b}_{n+1}^t \mathbf{x}_n$ in (4.21) is replaced by

$$\mathbf{b}_{n+1}^t \mathbf{x}_n = \gamma \min_j \{x_{nj}\} + (1-\gamma) \max_j \{x_{nj}\}. \quad (4.23)$$

where $0 < \gamma < 1$.

Tables 4.1, 4.2, 4.3, 4.4 and 4.5 show the wealth achieved by the Type 1 Helmbold universal portfolio while tables 4.6, 4.2, 4.3, 4.4 and 4.5 provide the empirical result obtained by the Type 2 Helmbold universal portfolio. These universal portfolios are running over the five data sets J, K, L, M and N after

2500 trading days for selected value of the parameters. The results are presented in the tables below.

Table 4.1: The wealth S_{2500} achieved after 2500 trading days by running the Type 1 Helmbold portfolio over the data set J for selected value of η together with final portfolio $\mathbf{b}_{2501}$

η	S_{2500}	$\boldsymbol{b}_{2501}$
0.3	15.4957	(0.126763, 0.127786, 0.114715, 0.513119, 0.117617)
0.4	16.1562	(0.103203, 0.104335, 0.090455, 0.608522, 0.093485)
0.5	16.3094	(0.082917, 0.084071, 0.070421, 0.689242, 0.073349)
0.6	15.9953	(0.066127, 0.067247, 0.054443, 0.755045, 0.057138)
0.7	15.3086	(0.052549, 0.053598, 0.041955, 0.807542, 0.044356)

Table 4.2: The wealth S_{2500} achieved after 2500 trading days by running the Type 1 Helmbold portfolio over the data set K for selected value of η together with final portfolio $\mathbf{b}_{2501}$

η	S_{2500}	$\boldsymbol{b}_{2501}$
-4.2	18.57109	(0.068924, 0.226932, 0.646187, 0.057511, 0.000446)
-4.1	18.67420	(0.080392, 0.263809, 0.587355, 0.067859, 0.000585)
-4.0	18.71009	(0.092274, 0.301252, 0.526939, 0.078782, 0.000753)
-3.9	18.67303	(0.104260, 0.338003, 0.466762, 0.090020, 0.000955)
-3.8	18.55880	(0.116050, 0.372854, 0.408590, 0.101310, 0.001196)

Table 4.3: The wealth S_{2500} achieved after 2500 trading days by running the Type 1 Helmbold portfolio over the data set L for selected value of η together with final portfolio $\mathbf{b}_{2501}$

η	S_{2500}	$\boldsymbol{b}_{2501}$
-2.0	4.42198	(0.261983, 0.009207, 0.504003, 0.198672, 0.026135)
-1.9	4.43670	(0.268802, 0.011158, 0.481837, 0.208073, 0.030130)
-1.8	4.44490	(0.274631, 0.013474, 0.460337, 0.216948, 0.034610)
-1.7	4.44681	(0.279396, 0.016213, 0.439609, 0.225169, 0.039613)
-1.6	4.44287	(0.283043, 0.019440, 0.419736, 0.232612, 0.045169)

Table 4.4: The wealth S_{2500} achieved after 2500 trading days by running the Type 1 Helmbold portfolio over the data set M for selected value of η together with final portfolio $\mathbf{b}_{2501}$

η	S_{2500}	$\boldsymbol{b}_{2501}$
0.3	19.11034	(0.114547, 0.168995, 0.489913, 0.108615, 0.117930)
0.4	19.74197	(0.089628, 0.150149, 0.581495, 0.085567, 0.093161)
0.5	19.97019	(0.068846, 0.130784, 0.661432, 0.066695, 0.072243)
0.6	19.80705	(0.052200, 0.112291, 0.728585, 0.051626, 0.055298)
0.7	19.30544	(0.039230, 0.095428, 0.783594, 0.039797, 0.041951)

Table 4.5: The wealth S_{2500} achieved after 2500 trading days by running the Type 1 Helmbold portfolio over the data set N for selected value of η together with final portfolio $\mathbf{b}_{2501}$

η	S_{2500}	$\boldsymbol{b}_{2501}$
-2.3	4.99236	(0.045748, 0.199536, 0.601143, 0.134370, 0.019203)
-2.2	5.00734	(0.050722, 0.207542, 0.575474, 0.144158, 0.022104)
-2.1	5.01289	(0.056009, 0.214906, 0.549948, 0.153793, 0.025344)
-2.0	5.00873	(0.061599, 0.221550, 0.524765, 0.163140, 0.028946)
-1.9	4.99481	(0.067479, 0.227411, 0.500107, 0.172066, 0.032937)

Table 4.6: The wealth S_{2500} achieved after 2500 trading days by running the Type 2 Helmbold portfolio over the data set J for selected value of η and $\eta = 0.8$ together with final portfolio $\mathbf{b}_{2501}$

η	S_{2500}	$\boldsymbol{b}_{2501}$
-0.1	9.9810	(0.218758, 0.218118, 0.228124, 0.109243, 0.225757)
0.0	11.5238	(0.2, 0.2, 0.2, 0.2, 0.2)
0.1	13.2749	(0.172952, 0.173489, 0.166047, 0.319765, 0.167747)
0.2	14.8741	(0.142037, 0.142943, 0.131064, 0.450227, 0.133729)
0.3	15.9522	(0.112273, 0.113373, 0.099665, 0.572008, 0.102681)
0.4	16.3184	(0.086691, 0.087847, 0.074089, 0.674320, 0.077053)
0.5	15.9952	(0.066127, 0.067246, 0.054443, 0.755044, 0.057140)
0.6	15.1363	(0.050177, 0.051209, 0.039817, 0.816637, 0.042160)
0.7	13.9317	(0.038018, 0.038939, 0.029090, 0.862890, 0.031063)
0.8	12.5510	(0.028815, 0.029618, 0.021267, 0.897403, 0.022897)

Table 4.7: The wealth S_{2500} achieved after 2500 trading days by running the Type 2 Helmbold portfolio over the data set K for selected values of η and $\gamma = 0.4$ together with final portfolio $\mathbf{b}_{2501}$

η	S_{2500}	$\boldsymbol{b}_{2501}$
-2.9	17.4887	(0.029238, 0.095890, 0.851581, 0.023174, 0.000117)
-2.8	17.9828	(0.041217, 0.135798, 0.789504, 0.033285, 0.000196)
-2.7	18.3681	(0.056107, 0.185059, 0.712351, 0.046165, 0.000318)
-2.6	18.6199	(0.073442, 0.241536, 0.622957, 0.061567, 0.000498)
-2.5	18.7100	(0.092273, 0.301251, 0.526939, 0.078781, 0.000756)
-2.4	18.6139	(0.111375, 0.359212, 0.431513, 0.096803, 0.001097)
-2.3	18.3165	(0.129584, 0.410732, 0.343549, 0.114596, 0.001539)
-2.2	17.8147	(0.146095, 0.452562, 0.267874, 0.131375, 0.002094)
-2.1	17.1178	(0.160569, 0.483317, 0.206590, 0.146740, 0.002784)
0	3.9413	(0.2, 0.2, 0.2, 0.2, 0.2)

Table 4.8: The wealth S_{2500} achieved after 2500 trading days by running the Type 2 Helmbold portfolio over the data set L for selected values of η and $\gamma = 0.1$ together with final portfolio $\mathbf{b}_{2501}$

η	S_{2500}	$\boldsymbol{b}_{2501}$
-1.3	4.26914	(0.219783, 0.003563, 0.612543, 0.151316, 0.012795)
-1.2	4.34636	(0.238455, 0.005277, 0.568370, 0.170670, 0.017228)
-1.1	4.40310	(0.255073, 0.007721, 0.524408, 0.189871, 0.022927)
-1.0	4.43670	(0.268802, 0.011157, 0.481837, 0.208072, 0.030132)
-0.9	4.44689	(0.278968, 0.015918, 0.441644, 0.224383, 0.039087)
-0.8	4.43589	(0.285130, 0.022418, 0.404487, 0.237927, 0.050038)
-0.7	4.40787	(0.287079, 0.031159, 0.370636, 0.247949, 0.063177)
-0.6	4.36800	(0.284806, 0.042723, 0.340008, 0.253840, 0.078623)
-0.5	4.32144	(0.278463, 0.057738, 0.312252, 0.255211, 0.096336)
0	4.10782	(0.2, 0.2, 0.2, 0.2, 0.2)

Table 4.9: The wealth S_{2500} achieved after 2500 trading days by running the Type 2 Helmbold portfolio over the data set M for selected values of η and $\gamma = 0.3$ together with final portfolio $\mathbf{b}_{2501}$

η	S_{2500}	$\boldsymbol{b}_{2501}$
-0.1	13.6551	(0.234364, 0.187123, 0.086477, 0.261558, 0.230475)
0	15.6012	(0.2, 0.2, 0.2, 0.2, 0.2)
0.1	17.7752	(0.151584, 0.189215, 0.360377, 0.144705, 0.154116)
0.2	19.4098	(0.104114, 0.161637, 0.527763, 0.098885, 0.107599)
0.3	19.9708	(0.067000, 0.128878, 0.668722, 0.065025, 0.070372)
0.4	19.4294	(0.041559, 0.098650, 0.773498, 0.041932, 0.044358)
0.5	18.0774	(0.025291, 0.073786, 0.846658, 0.026830, 0.027433)
0.6	16.2722	(0.015246, 0.054448, 0.896367, 0.017129, 0.016807)
0.7	14.3040	(0.009147, 0.039835, 0.929830, 0.010934, 0.010251)
0.8	12.3656	(0.005475, 0.028965, 0.952338, 0.006982, 0.006237)

Table 4.10: The wealth S_{2500} achieved after 2500 trading days by running the Type 2 Helmbold portfolio over the data set N for selected values of η and $\gamma = 0.1$ together with final portfolio $\mathbf{b}_{2501}$

η	S_{2500}	$\boldsymbol{b}_{2501}$
-1.6	4.65335	(0.018892, 0.130543, 0.776109, 0.068418, 0.006038)
-1.5	4.76915	(0.024161, 0.148641, 0.735671, 0.083251, 0.008276)
-1.4	4.86902	(0.030519, 0.166947, 0.691488, 0.099839, 0.011207)
-1.3	4.94662	(0.038031, 0.184723, 0.644508, 0.117753, 0.014985)
-1.2	4.99609	(0.046717, 0.201184, 0.596007, 0.136333, 0.019759)
-1.1	5.01290	(0.056554, 0.215604, 0.547411, 0.154742, 0.025689)
-1.0	4.99480	(0.067479, 0.227410, 0.500106, 0.172067, 0.032938)
-0.9	4.94212	(0.079401, 0.236246, 0.455249, 0.187431, 0.041673)
-0.8	4.85759	(0.092203, 0.241972, 0.413638, 0.200127, 0.052060)
0	3.61793	(0.2, 0.2, 0.2, 0.2, 0.2)

The tables 4.6, 4.7, 4.8, 4.9 and 4.10 give the empirical results obtained by running the Type 2 Helmbold universal portfolio on Stock Data Sets J, K, L, M and N, respectively. From the results obtained by Type 2 Helmbold universal portfolio, we can observe that the best wealth is achieved for data set M while the lowest wealth is achieved for data set L. We observed that J, K and M are good portfolios achieving maximum wealth of 16.31847, 18.71009 and 19.97086 units respectively. The empirical result also reveals that L and N are poor portfolios, achieving maximum wealth of 4.44689 and 5.012909 units respectively. The forth stock of set J, third stock of set K and third stock of set

M respectively are performing well. Hence the portfolios assign more weights on them and lead to higher wealth return. Similarly, the fifth stock for both set L and N are performing poorly. Hence, lower weights are assigned on them.

A comparison of the performance between Type 1 Helmbold universal portfolio and Type 2 Helmbold universal is done. The results from tables 4.1, 4.2, 4.3, 4.4 and 4.5 and tables 4.6, 4.2, 4.3, 4.4 and 4.5 are within the same range with small differences after comparing Table 2 and Table 3. Hence, the performance of Type 2 Helmbold universal portfolio is comparable with Type 1 Helmbold universal portfolio with no significant differences.

4.2 Reverse f-divergence

In this session, a reverse f-divergence is used to generate a new universal portfolio. The f-divergence of two probability is defined previous at (4.1). Let the function $f^*(t) = tf(\frac{1}{t})$, which is also convex on $(0,\infty)$ and $f^*(1) = 1$. The function f^* is the reverse f-divergence, which is defined by

$$D_{f^*}(\boldsymbol{p}||\boldsymbol{q}) = D_f(\boldsymbol{q}||\boldsymbol{p}), \tag{4.24}$$

see (Basu et al. (2011)). Hence, we have the reverse f-divergence for the portfolio vectors $\boldsymbol{b}_{n+1}$ and $\boldsymbol{b}_n$ is given by

$$D_f(\boldsymbol{b}_n||\boldsymbol{b}_{n+1}) = \sum_{j=1}^{m} b_{n+1,j} f\left[\frac{b_{nj}}{b_{n+1,j}}\right]. \tag{4.25}$$

Proposition 4.2.1: Let $f(t)$ be a convex function on $(0,\infty)$ satisfying $f(1) = 0$.

Then, let the objective function

$$\begin{aligned}\hat{F}(\boldsymbol{b}_{n+1};\lambda) =& \eta\left\{\log(\boldsymbol{b}_n^t\boldsymbol{x}_n)+\hat{\log}_k\left[\frac{\boldsymbol{b}_{n+1}^t\boldsymbol{x}_n}{\boldsymbol{b}_n^t\boldsymbol{x}_n}\right]\right\} \\ &-\sum_{j=1}^{m} b_{n+1,j} f\left[\frac{b_{nj}}{b_{n+1,j}}\right]+\lambda\left[\sum_{j=1}^{m} b_{n+1,j}-1\right]\end{aligned} \tag{4.26}$$

where $\eta > 0$ and λ is the Lagrange multiplier. The Type k universal portfolio generated by the reverse f divergence $D_{f^*}(\boldsymbol{b}_{n+1}||\boldsymbol{b}_n)$ is

$$f\left(\frac{b_{ni}}{b_{n+1,i}}\right)-\frac{b_{ni}}{b_{n+1,i}}f'\left(\frac{b_{ni}}{b_{n+1,i}}\right)=\eta u_k(\boldsymbol{b}_{n+1},\boldsymbol{b}_n)\left(\frac{x_{ni}}{\boldsymbol{b}_n^t\boldsymbol{x}_n}\right)+\xi_n \quad \text{for } i=1,2,\cdots,m. \tag{4.27}$$

If there is no valid solution to λ, the solutions of Equations (4.27) will be called pseudo solutions.

Proof. The stationary vector of $\hat{F}(\boldsymbol{b}_{n+1};\lambda)$ is $\nabla\hat{F}(\boldsymbol{b}_{n+1};\lambda)=\left(\frac{\partial \hat{F}}{\partial b_{n+1,i}}\right)=\mathbf{0}$.

From the Lemma,

$$\frac{\partial \hat{F}}{\partial b_{n+1,i}}=\eta u_k(\boldsymbol{b}_{n+1},\boldsymbol{b}_n)\left(\frac{x_{ni}}{\boldsymbol{b}_n^t\boldsymbol{x}_n}\right)-\left\{f\left(\frac{b_{ni}}{b_{n+1,i}}\right)-\frac{b_{ni}}{b_{n+1,i}}f'\left(\frac{b_{ni}}{b_{n+1,i}}\right)\right\}+\lambda=0 \tag{4.28}$$

Multiply Equation (4.28) by b_{ni} and sum over i to get

$$\eta u_k(\boldsymbol{b}_{n+1},\boldsymbol{b}_n)-\sum_{j=1}^{m}\left[b_{nj}f\left(\frac{b_{nj}}{b_{n+1,j}}\right)-\frac{b_{nj}^2}{b_{n+1,j}}f'\left(\frac{b_{nj}}{b_{n+1,j}}\right)\right]+\lambda=0 \tag{4.29}$$

Let

$$\xi_n=-\eta u_k(\boldsymbol{b}_{n+1},\boldsymbol{b}_n)+\sum_{j=1}^{m}\left[b_{nj}f\left(\frac{b_{nj}}{b_{n+1,j}}\right)-\frac{b_{nj}^2}{b_{n+1,j}}f'\left(\frac{b_{nj}}{b_{n+1,j}}\right)\right] \tag{4.30}$$

Subtracting Equation (4.29) from Equation (4.28) yields Equation (4.27).

Remarks. The portfolios (4.27) is valid if λ does not depend on $\boldsymbol{b}_{n+1}$.

Proposition 4.2.2: The following gives the Type-1 universal portfolios generated by reverse-f divergence associated with different convex function, $f(t)$.

1. For the associated convex function $f(t) = t\log t - t + 1$, $t > 0$, the reverse Kullback-Leibler universal portfolio is

$$b_{n+1,i} = \frac{b_{ni}[\beta \boldsymbol{b}_n^t \boldsymbol{x}_n - x_{ni}]^{-1}}{\sum_{j=1}^m b_{nj}[\beta \boldsymbol{b}_n^t \boldsymbol{x}_n - x_{nj}]^{-1}},\ i = 1, 2, \cdots, m. \tag{4.31}$$

2. For the associated convex function $f(t) = (t-1)^2$, $t > 0$, the reverse chi-square universal portfolio is

$$b_{n+1,i} = \frac{b_{ni}[\beta \boldsymbol{b}_n^t \boldsymbol{x}_n - x_{ni}]^{-\frac{1}{2}}}{\sum_{j=1}^m b_{nj}[\beta \boldsymbol{b}_n^t \boldsymbol{x}_n - x_{nj}]^{-\frac{1}{2}}},\ i = 1, 2, \cdots, m. \tag{4.32}$$

3. For the associated convex function $f(t) = \frac{t - t^{1-\alpha}}{\alpha(1-\alpha)}$, $t > 0$, the reverse α-divergence universal portfolio is

$$b_{n+1,i} = \frac{b_{ni}[\beta \boldsymbol{b}_n^t \boldsymbol{x}_n - x_{ni}]^{-\frac{1}{1-\alpha}}}{\sum_{j=1}^m b_{nj}[\beta \boldsymbol{b}_n^t \boldsymbol{x}_n - x_{nj}]^{-\frac{1}{1-\alpha}}},\ i = 1, 2, \cdots, m. \tag{4.33}$$

The portfolios are valid provided $[\beta \boldsymbol{b}_n^t \boldsymbol{x}_n - x_{ni}]^s > 0$ for $s = -1, -\frac{1}{2}, -(1-\alpha)^{-1}$.

Proof.

1. For $f(t) = t\log t - t + 1$, $t > 0$, $f'(t) = \log t$ and

$$f\left(\frac{b_{ni}}{b_{n+1,i}}\right) - \frac{b_{ni}}{b_{n+1,i}} f'\left(\frac{b_{ni}}{b_{n+1,i}}\right) = -\frac{b_{ni}}{b_{n+1,i}} + 1 \quad \text{for } i = 1, 2, \cdots, m.$$

From Equation (4.27),

$$\begin{aligned}
-\frac{b_{ni}}{b_{n+1,i}}+1 =& \frac{\eta x_{ni}}{\boldsymbol{b}_n^t\boldsymbol{x}_n}+\xi_n \\
b_{n+1,i} =& b_{ni}\frac{1}{[1-\eta\frac{x_{ni}}{\boldsymbol{b}_n^t\boldsymbol{x}_n}-\xi_n]}, \quad i=1,2,\cdots,m \\
=& \left(\frac{\boldsymbol{b}_n^t\boldsymbol{x}_n}{\eta}\right)\frac{b_{ni}}{\left[\left(\frac{1-\xi_n}{\eta}\right)\boldsymbol{b}_n^t\boldsymbol{x}_n-x_{ni}\right]}
\end{aligned}$$

Evaluating the normalizing constant $\left(\frac{\boldsymbol{b}_n^t\boldsymbol{x}_n}{\eta}\right)$, Equation (4.31) is obtained where $\beta=\frac{(1-\xi_n)}{\eta}$.

2. For $f(t)=(t-1)^2$, $t>0$, $f''(t)=2$.

$$\begin{aligned}
f\left(\frac{b_{ni}}{b_{n+1,i}}\right)-\frac{b_{ni}}{b_{n+1,i}}f'\left(\frac{b_{ni}}{b_{n+1,i}}\right) =& -\left(\frac{b_{ni}}{b_{n+1,i}}\right)^2+1 \quad \text{for } i=1,2,\cdots,m \\
=& \eta\frac{x_{ni}}{\boldsymbol{b}_n^t\boldsymbol{x}_n}+\xi_n
\end{aligned}$$

Thus,

$$\begin{aligned}
b_{n+1,i} =& \frac{b_{ni}}{\left[1-\eta\frac{x_{ni}}{\boldsymbol{b}_n^t\boldsymbol{x}_n}-\xi_n\right]^{\frac{1}{2}}}, \quad i=1,2,\cdots,m \\
=& \left(\frac{\boldsymbol{b}_n^t\boldsymbol{x}_n}{\eta}\right)\frac{b_{ni}}{\left[\left(\frac{1-\xi_n}{\eta}\right)\boldsymbol{b}_n^t\boldsymbol{x}_n-x_{ni}\right]^{\frac{1}{2}}}
\end{aligned}$$

Evaluating the normalizing constant $\left(\frac{\boldsymbol{b}_n^t\boldsymbol{x}_n}{\eta}\right)$, Equation (4.32) is obtained where $\beta=\frac{(1-\xi_n)}{\eta}$.

3. For $f(t)=\frac{t-t^{1-\alpha}}{\alpha(1-\alpha)}$, $t>0$, $f'(t)=\frac{1}{\alpha(1-\alpha)}-\frac{1}{\alpha}t^{-\alpha}$.

Now,

$$\begin{aligned}
f\left(\frac{b_{ni}}{b_{n+1,i}}\right)-\frac{b_{ni}}{b_{n+1,i}}f'\left(\frac{b_{ni}}{b_{n+1,i}}\right) =& \frac{1}{\alpha-1}\left(\frac{b_{ni}}{b_{n+1,i}}\right)^{1-\alpha}, \quad i=1,2,\cdots,m \\
=& \eta\frac{x_{ni}}{\boldsymbol{b}_n^t\boldsymbol{x}_n}+\xi_n,
\end{aligned}$$

leading to

$$
\begin{aligned}
b_{n+1,i} =& b_{ni}\left[\eta(\alpha-1)\frac{x_{ni}}{\boldsymbol{b}_n^t\boldsymbol{x}_n}+\xi_n(\alpha-1)\right]^{-\frac{1}{1-\alpha}} \\
=& \left[\frac{\eta(1-\alpha)}{\boldsymbol{b}_n^t\boldsymbol{x}_n}\right]^{-\frac{1}{1-\alpha}} b_{ni}\left[-x_{ni}-\frac{\xi_n}{\eta}(\boldsymbol{b}_n^t\boldsymbol{x}_n)\right]^{-\frac{1}{1-\alpha}}.
\end{aligned}
$$

Evaluating the normalizing constant results in Equation (4.33) where $\beta = -\frac{\xi_n}{\eta}$.

CHAPTER 5

STRONG FORM AND WEAK FORM OF f-DIVERGENCE

5.1 Bregman Divergence

5.1.1 Universal Portfolio Generated by Bregman Divergence

Given the portfolio $\boldsymbol{b}_n$ and the price-relative vector $\boldsymbol{x}_n$ on the n^{th} trading day, the next-day portfolio $\boldsymbol{b}_{n+1}$ will be determined by maximizing the approximate rate of growth in investment wealth $log[\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n]$ and minimizing the Bregman divergence of $\boldsymbol{b}_{n+1}$ and $\boldsymbol{b}_n$ with respect to a given convex function $f(t)$ for $t > 0$.

Proposition 5.1.1: Let $f(t)$ be a given convex function for $t > 0$ and $B^f(\cdot||\cdot)$ is the Bregman divergence associated with $f(t)$. Consider the objective function $\hat{F}(\boldsymbol{b}_{n+1}, \lambda)$ given by:

$$\begin{aligned}\hat{F}(\boldsymbol{b}_{n+1}, \lambda) =& \xi \left[log(\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n) + \left(\frac{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n} \right) - 1 \right] \\ & - \sum_{j=1}^{m} [f(b_{n+1,j}) - f(b_{nj}) - f'(b_{nj})(b_{n+1,j} - b_{nj})] \\ & + \lambda \left[\sum_{j=1}^{m} b_{n+1,j} - 1 \right]\end{aligned} \tag{5.1}$$

where $\xi > 0$ and λ is the Lagrange multiplier. The explicit form of the Bregman universal portfolio associated with $f(t)$ is

$$f'(b_{n+1,i}) = f'(b_{ni}) + (\eta - \xi) + \xi \left(\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} \right) \text{ for } i = 1, 2, \cdots, m \tag{5.2}$$

where η is a real parameter. The implicit form of the Bregman universal portfolio associated with $f(t)$ is the set of non-linear equations in $(b_{n+1,i})$ given by:

$$f'(b_{n+1,i}) = f'(b_{ni}) + \zeta + \xi\left(\frac{x_{ni} - \boldsymbol{b}^t_{n+1}\boldsymbol{x}_n}{\boldsymbol{b}^t_n\boldsymbol{x}_n}\right) \text{ for } i = 1,2,\cdots,m \tag{5.3}$$

where ζ is a real parameter.

Proof. Differentiating the objective function with respect to $b_{n+1,i}$ and setting the derivatives to zero,

$$\frac{\partial \hat{F}}{\partial b_{n+1,i}} = \xi\left[\frac{x_{ni}}{\boldsymbol{b}^t_n\boldsymbol{x}_n}\right] - \{f'(b_{n+1,i}) - f'(b_{ni})\} + \lambda = 0 \text{ for } i = 1,2,\cdots,m. \tag{5.4}$$

Multiply (5.4) by b_{ni} and summing over i to get

$$\xi - \sum_{j=1}^{m} b_{nj}\{f'(b_{n+1,j}) - f'(b_{nj})\} + \lambda = 0. \tag{5.5}$$

From the difference of (5.4) and (5.5),

$$\xi\left[\frac{x_{ni}}{\boldsymbol{b}^t_n\boldsymbol{x}_n}\right] - \xi - \{f'(b_{n+1,i}) - f'(b_{ni})\} + \sum_{j=1}^{m} b_{nj}\{f'(b_{n+1,j}) - f'(b_{nj})\} = 0 \tag{5.6}$$

for $i = 1,2,\cdots,m$.

Let

$$z_i = b_{ni}[f'(b_{n+1,i}) - f'(b_{ni})] \text{ for } i = 1,2,\cdots,m. \tag{5.7}$$

Then (5.7) can be rewritten as:

$$\sum_{j=1}^{m} z_j = \frac{z_i}{b_{ni}} + \xi\left[1 - \frac{x_{ni}}{\boldsymbol{b}^t_n\boldsymbol{x}_n}\right] \; for \; i = 1,2,\cdots,m. \tag{5.8}$$

From (5.8), it is observed that $\sum_{j=1}^{m} z_j = \eta$ is a constant, not depending on the subscript i. In fact, it can be concluded that the general solution to (5.6) is given by (5.2) where η is any real scalar.

Similarly, multiplying (5.4) by $b_{n+1,i}$ and summing over i,

$$\xi\left[\frac{\boldsymbol{b}_{n+1}^t\boldsymbol{x}_n}{\boldsymbol{b}_n^t\boldsymbol{x}_n}\right]-\sum_{j=1}^{m}b_{n+1,j}[f'(b_{n+1,j})-f'(b_{nj})]+\lambda=0 \text{ for } i=1,2,\cdots,m \tag{5.9}$$

is obtained. The value of λ given by (5.9) is not a valid solution to $\frac{\partial\hat{F}}{\partial b_{n+1,i}}=0$ for $i=1,2,\cdots,m$ because λ depends on the variable $\{b_{n+1,i}\}$. The λ given by (5.9) is known as *pseudo* Lagrange multiplier. However, it can be used to generate a *pseudo* universal portfolio. Substituting the value of λ in (5.9) into (5.5), it is observed that

$$\xi\left[\frac{x_{ni}-\boldsymbol{b}_{n+1}^t\boldsymbol{x}_n}{\boldsymbol{b}_n^t\boldsymbol{x}_n}\right]-\{f'(b_{n+1,i})-f'(b_{ni})\}+\sum_{j=1}^{m}b_{n+1,j}[f'(b_{n+1,j})-f'(b_{nj})]=0 \tag{5.10}$$

for $i=1,2,\cdots,m$.

Let

$$y_i=b_{n+1,i}[f'(b_{n+1,i})-f'(b_{ni})] \text{ for } i=1,2,\cdots,m. \tag{5.11}$$

Again, it is seen that

$$\sum_{j=1}^{m}y_j=\frac{y_i}{b_{n+1,i}}+\xi\left[\frac{\boldsymbol{b}_{n+1}^t\boldsymbol{x}_n-x_{ni}}{\boldsymbol{b}_n^t\boldsymbol{x}_n}\right] \text{ for } i=1,2,\cdots,m \tag{5.12}$$

is a constant, say, ζ, not depending on i. By substituting $\sum_{j=1}^{m}y_j=\zeta$ into (5.10), the implicit form for $\boldsymbol{b}_{n+1}$ in (5.3) is obtained.

Remarks. A pseudo universal portfolio can be derived from (5.3) by replacing $\boldsymbol{b}_{n+1}^t\boldsymbol{x}_n$ on the right-hand side with $\sum_{j=1}^{m}w_jx_{nj}$ for selected weight $\{w_j\}$ such that $0\le w_j\le 1$ and $\sum_{j=1}^{m}w_j=1$. Different ways of replacing $\boldsymbol{b}_{n+1}^t\boldsymbol{x}_n$ with an expression not depending on $\boldsymbol{b}_{n+1}$ are possible.

Proposition 5.1.2: Consider the objective function $\hat{F}(\boldsymbol{b}_{n+1}, \lambda)$ given by (5.1), where $f(t)$ is a convex function for $t > 0$ with associated Bregman divergence. The explicit mean-value form of (5.2) is:

$$b_{n+1,i} = b_{ni} + \frac{1}{f''(r_{ni})}\left[(\eta - \xi) + \xi\left(\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right)\right] \text{ for } i = 1, 2, \cdots, m \qquad (5.13)$$

where η is a real parameter, $b_{ni} \leq r_{ni} \leq b_{n+1}$ or $b_{n+1,i} \leq r_{ni} \leq b_{ni}$. The implicit mean-value form of (5.3) is:

$$b_{n+1,i} = b_{ni} + \frac{1}{f''(r_{ni})}\left[\zeta + \xi\left(\frac{x_{ni} - \boldsymbol{b}_{n+1}^t \boldsymbol{x}_n}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right)\right] \text{ for } i = 1, 2, \cdots, m \qquad (5.14)$$

where ζ is a real parameter.

Proof. The mean-value theorem states that there exists an r_{ni} such that $f'(b_{n+1,i}) - f'(b_{ni}) = (b_{n+1,i} - b_{ni})f''(r_{ni})$ where $b_{ni} \leq r_{ni} \leq b_{n+1,i}$ or vice versa. The rest is obvious.

Remarks. An approximate form of (5.13) may be obtained by approximating r_{ni} as b_{ni}, namely

$$b_{n+1,i} = b_{ni} + \frac{1}{f''(b_{ni})}\left[(\eta - \xi) + \xi\left(\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right)\right] \text{ for } i = 1, 2, \cdots, m. \qquad (5.15)$$

Proposition 5.1.3: Given the convex function $f(t) = \alpha(t+\beta)log[(t+\beta)e^{-1}]$ where the parameters $\alpha > 0$, $\beta > 0$ and $e = 2.718$, consider the Bregman universal portfolio generated by $f(t)$, $t > 0$. The explicit form of the Bregman universal portfolio is given by:

$$b_{n+1,i} = \frac{(1+\beta m)(b_{ni}+\beta)e^{\frac{\gamma x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}}}{\sum_{j=1}^m (b_{nj}+\beta)e^{\frac{\gamma x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}}} - \beta \text{ for } i = 1, 2, \cdots, m \qquad (5.16)$$

where the parameter $\gamma > 0$.

Proof. The derivatives of $f(t)$ are given by: $f'(t) = log(t+\beta)^{\alpha}$, $f''(t) = \frac{\alpha}{t+\beta} > 0$ for $t > 0$, $\alpha > 0$, $\beta > 0$. The explicit Bregman universal portfolio is given by (5.2), namely, $log(b_{n+1,i}+\beta)^{\alpha} = log(b_{ni}+\beta)^{\alpha} + \eta' + \xi(\frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n})$ for $i = 1,2,\cdots,m$ where $\eta' = \eta - \xi$. Hence $(b_{n+1,i}+\beta)^{\alpha} = (b_n+\beta)^{\alpha} e^{\eta'} e^{\frac{\xi x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}}$ and

$$b_{n+1,i} + \beta = (b_{ni}+\beta) e^{\frac{\eta'}{\alpha}} e^{\frac{1}{\alpha}\left(\frac{\xi x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right)} \text{ for } i = 1,2,\cdots,m. \tag{5.17}$$

Summing (5.17) over i to get

$$e^{\frac{\eta'}{\alpha}} = \frac{1+\beta m}{\sum_{j=1}^{m}(b_{nj}+\beta) e^{\left(\frac{\xi}{\alpha}\right)\left(\frac{x_{nj}}{\boldsymbol{b}_n^t \boldsymbol{x}_n}\right)}}. \tag{5.18}$$

Let $\gamma = \frac{\xi}{\alpha}$ and substituting (5.18) into (5.17), the portfolio (5.16) is obtained.

Remarks. (i) When $\gamma < 0$, the portfolio (5.16) is still a valid portfolio. Hence (5.16) is valid for any parameter γ provided $b_{n+1,i} \geq 0$ for $i = 1,2,\cdots,m$.
(ii) The portfolio (5.16) may be regarded as a generalization of the Helmbold portfolio (Helmbold et al. (1998)) given by:

$$b_{n+1,i} = \frac{b_{ni} e^{\frac{\gamma x_{ni}}{(\boldsymbol{b}_n^t \boldsymbol{x}_n)}}}{\sum_{j=1}^{m} b_{nj} e^{\frac{\gamma x_{nj}}{(\boldsymbol{b}_n^t \boldsymbol{x}_n)}}}. \tag{5.19}$$

Observe that when $\beta \to 0$ in (5.16), the portfolio (5.19) is obtained.

The empirical result obtained by running this universal portfolio generated by Bregman divergence on the data sets J, K, L, M and N. The results are shown in Tables 5.1, 5.2, 5.3, 5.4 and 5.5, respectively.

Table 5.1: The wealth S_{2500} and the final portfolio $\boldsymbol{b}_{2501}$ achieved by the Universal Porfolio generated by Bregman Divergence for data set J ,where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$ and $\beta = 0.01$.

γ	S_{2500}	$\boldsymbol{b}_{2501}$
0.43	16.29481	(0.09265, 0.09387, 0.07914, 0.65199, 0.08234)
0.44	16.41119	(0.09049, 0.09172, 0.07700, 0.66061, 0.08019)
0.45	16.32229	(0.08837, 0.08960, 0.07489, 0.66907, 0.07807)
0.46	16.32818	(0.08628, 0.08751, 0.07283, 0.67738, 0.07600)
0.47	16.32890	(0.08423, 0.08546, 0.07081, 0.68552, 0.07397)
0.48	16.32455	(0.08222, 0.08345, 0.06884, 0.69352, 0.07198)
0.49	16.31518	(0.08024, 0.08147, 0.06691, 0.70135, 0.07003)
0.50	16.30090	(0.07830, 0.07953, 0.06502, 0.70903, 0.06813)
0.51	16.28178	(0.07639, 0.07762, 0.06317, 0.71656, 0.06626)
0.52	16.25792	(0.07452, 0.07575, 0.06136, 0.72393, 0.06443)

Table 5.2: The wealth S_{2500} and the final portfolio $\boldsymbol{b}_{2501}$ achieved by the Universal Porfolio generated by Bregman Divergence for data set K ,where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$ and $\beta = 0.001$.

γ	S_{2500}	$\boldsymbol{b}_{2501}$
-3.29	18.07213	(0.16483, 0.48914, 0.19262, 0.15129, 0.00212)
-3.28	17.35192	(0.16565, 0.49053, 0.18946, 0.15220, 0.00217)
-3.27	16.74881	(0.16645, 0.49188, 0.18635, 0.15309, 0.00222)
-3.26	18.25698	(0.16726, 0.49320, 0.18329, 0.15398, 0.00228)
-3.25	18.83454	(0.16805, 0.49446, 0.18029, 0.15486, 0.00233)
-3.24	17.84723	(0.16883, 0.49569, 0.17734, 0.15574, 0.00239)
-3.23	18.25432	(0.16961, 0.49688, 0.17445, 0.15661, 0.00245)
-3.22	17.92804	(0.17038, 0.49802, 0.17161, 0.15748, 0.00250)
-3.21	17.66083	(0.17115, 0.49913, 0.16882, 0.15834, 0.00256)
-3.20	17.78524	(0.17190, 0.50019, 0.16609, 0.15920, 0.00262)

Table 5.3: The wealth S_{2500} and the final portfolio $\boldsymbol{b}_{2501}$ achieved by the Universal Porfolio generated by Bregman Divergence for data set L ,where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$ and $\beta = 0.01$.

γ	S_{2500}	$\boldsymbol{b}_{2501}$
-1.62	4.35246	(0.28376, 0.00936, 0.44250, 0.22904, 0.03534)
-1.61	4.36164	(0.28416, 0.00972, 0.44033, 0.22984, 0.03594)
-1.60	4.31730	(0.28455, 0.01008, 0.43817, 0.23064, 0.03655)
-1.59	4.38673	(0.28492, 0.01045, 0.43602, 0.23143, 0.03717)
-1.58	4.44790	(0.28528, 0.01083, 0.43388, 0.23221, 0.03780)
-1.57	4.44728	(0.28563, 0.01121, 0.43176, 0.23298, 0.03843)
-1.56	4.44660	(0.28597, 0.01160, 0.42964, 0.23374, 0.03907)
-1.55	4.44587	(0.28629, 0.01199, 0.42753, 0.23449, 0.03971)
-1.54	4.44508	(0.28660, 0.01239, 0.42543, 0.23522, 0.04036)
-1.53	4.44423	(0.28689, 0.01279, 0.42334, 0.23595, 0.04102)

Table 5.4: The wealth S_{2500} and the final portfolio $\boldsymbol{b}_{2501}$ achieved by the Universal Porfolio generated by Bregman Divergence for data set M ,where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$ and $\beta = 0.00001$.

γ	S_{2500}	$\boldsymbol{b}_{2501}$
0.47	19.94433	(0.07463, 0.13655, 0.63880, 0.07193, 0.07809)
0.48	19.95694	(0.07266, 0.13462, 0.64648, 0.07014, 0.07610)
0.49	19.96553	(0.07073, 0.13270, 0.65403, 0.06840, 0.07415)
0.50	19.97014	(0.06884, 0.13078, 0.66145, 0.06669, 0.07224)
0.51	19.97080	(0.06699, 0.12888, 0.66874, 0.06502, 0.07037)
0.52	19.96755	(0.06519, 0.12698, 0.67590, 0.06339, 0.06854)
0.53	19.96044	(0.06343, 0.12510, 0.68294, 0.06179, 0.06675)
0.54	19.94952	(0.06170, 0.12323, 0.68984, 0.06023, 0.06500)
0.55	19.93483	(0.06002, 0.12137, 0.69662, 0.05871, 0.06328)
0.56	19.91643	(0.05838, 0.11952, 0.70327, 0.05723, 0.06161)

Table 5.5: The wealth S_{2500} and the final portfolio $\boldsymbol{b}_{2501}$ achieved by the Universal Porfolio generated by Bregman Divergence for data set N ,where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$ and $\beta = 0.001$.

γ	S_{2500}	$\boldsymbol{b}_{2501}$
-2.13	5.01704	(0.05351, 0.21239, 0.56050, 0.15022, 0.02338)
-2.12	5.01741	(0.05405, 0.21312, 0.55793, 0.15119, 0.02371)
-2.11	5.01768	(0.05459, 0.21384, 0.55536, 0.15215, 0.02405)
-2.10	5.01786	(0.05514, 0.21456, 0.55280, 0.15311, 0.02439)
-2.09	5.01793	(0.05569, 0.21527, 0.55024, 0.15407, 0.02474)
-2.08	5.01791	(0.05624, 0.21596, 0.54768, 0.15503, 0.02508)
-2.07	5.01779	(0.05680, 0.21666, 0.54513, 0.15598, 0.02544)
-2.06	5.01757	(0.05735, 0.21734, 0.54258, 0.15693, 0.02579)
-2.05	5.01725	(0.05791, 0.21802, 0.54003, 0.15788, 0.02615)
-2.04	5.01683	(0.05848, 0.21869, 0.53749, 0.15882, 0.02652)

A close examination of 5.1, 5.2, 5.3, 5.4 and 5.5 reveal that J, K, and M are good portfolios achieving maximum wealth of 16.3289, 18.83454 and 19.9708 units respectively. The portfolios L and N perform poorly, achieving maximum wealth of 4.4479 and 5.01793 units respectively. Stocks in the individual portfolios J, K and M contributing to the higher returns are the fourth, second and third respectively. The second and fifth stocks in the portfolio L perform poorly and hence lower weights are assigned on them. Similarly, lower weights are assigned on the first and fifth stocks in the portfolio N due to their poor performance.

We compared the performance of Bregman universal portfolio (5.16) with that of the Helmbold universal portfolio (5.19). The empirical results of Helmbold universal portfolio (5.19) are given by Tables 4.1, 4.2, 4.3, 4.4, 4.5. The comparison indicates that the wealth achieved by the same data set for two different universal portfolios (5.16) and (5.19) are within the same range with small differences. Hence, the performance of universal portfolios (5.16) and (5.19) are comparable with no significant differences. The new Bregman universal portfolio (5.16) introduced here provides an alternative choice to the well-known Helmbold universal portfolio (5.19).

5.1.2 Universal Portfolio Generated by Reverse Bregman Divergence

Let the objective function

$$\begin{aligned}\hat{F}(\boldsymbol{b}_{n+1};\lambda) =&\eta\left\{\log(\boldsymbol{b}_n^t\boldsymbol{x}_n)+\hat{\log}_k\left[\frac{\boldsymbol{b}_{n+1}^t\boldsymbol{x}_n}{\boldsymbol{b}_n^t\boldsymbol{x}_n}\right]\right\}-B_{f^*}(\boldsymbol{b}_{n+1}||\boldsymbol{b}_n)\\&+\lambda\left[\sum_{j=1}^m b_{n+1,j}-1\right]\end{aligned}$$

where $\eta > 0$ and λ is the Lagrange multiplier. The Type k universal portfolio generated by the reverse Bregman divergence $B_{f^*}(\boldsymbol{b}_{n+1}||\boldsymbol{b}_n)$ is

$$f''(b_{n+1,i})(b_{n+1,i}-b_{ni}) = \eta u_k(\boldsymbol{b}_{n+1},\boldsymbol{b}_n)\left(\frac{x_{ni}}{\boldsymbol{b}_n^t\boldsymbol{x}_n}\right)+\alpha_n \quad \text{for } i=1,2,\cdots,m. \tag{5.20}$$

If there is no valid solution to λ, the solutions of Equations (5.20) will be called pseudo solutions.

Note that

$$B_{f^*}(\boldsymbol{b}_{n+1}||\boldsymbol{b}_n)=B_f(\boldsymbol{b}_n||\boldsymbol{b}_{n+1})=\sum_{j=1}^m[f(b_{nj})-f(b_{n+1,j})-f'(b_{n+1,j})(b_{nj}-b_{n+1,j})]. \tag{5.21}$$

From the Lemma,

$$\frac{\partial\hat{F}}{\partial b_{n+1,i}}=\eta\left(\frac{x_{ni}}{\boldsymbol{b}_n^t\boldsymbol{x}_n}\right)+f''(b_{n+1,i})(b_{ni}-b_{n+1,i})+\lambda=0 \quad \text{for } i=1,2,\cdots,m. \tag{5.22}$$

Multiply Equation (5.22) by b_{ni} and sum over i to get

$$\eta+\sum_{j=1}^m f''(b_{n+1,j})b_{nj}(b_{nj}-b_{n+1,j})+\lambda=0 \tag{5.23}$$

Let

$$\alpha_n=-\left[\eta+\sum_{j=1}^m f''(b_{n+1,j})b_{nj}(b_{nj}-b_{n+1,j})\right]. \tag{5.24}$$

The portfolio (5.20) is obtained by subtracting Equation (5.23) from Equation (5.22).

Remarks. The portfolio (5.20) is valid if λ does not depend on $\boldsymbol{b}_{n+1}$.

Proposition 5.1.4: The following gives the Type 1 reverse Bregman universal portfolios associated with different convex functions.

1. For the associated convex function $f(t) = t\log t - t + 1$, $t > 0$, the reverse Bregman universal portfolio is

$$b_{n+1,i} = \frac{b_{ni}[\beta \boldsymbol{b}_n^t \boldsymbol{x}_n - x_{ni}]^{-1}}{\sum_{j=1}^m b_{nj}[\beta \boldsymbol{b}_n^t \boldsymbol{x}_n - x_{nj}]^{-1}}, \quad i = 1, 2, \cdots, m, \tag{5.25}$$

   provided $[\beta \boldsymbol{b}_n^t \boldsymbol{x}_n - x_{ni}]^{-1} > 0$.

2. For the associated convex function $f(t) = (t-1)^2$, $t > 0$, the reverse chi-square Bregman universal portfolio is

$$b_{n+1,i} = b_{ni} + \beta \left[\frac{x_{ni} - \frac{1}{m}\sum_{j=1}^m x_{nj}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} \right], \quad i = 1, 2 \cdots, m, \tag{5.26}$$

   provided $b_{n+1,i} \geq 0$.

Proof.

1. For $f(t) = t\log t - t + 1$, $t > 0$, $f''(t) = \frac{1}{t}$.

   Now

$$\begin{aligned} f''(b_{n+1,i})(b_{n+1,i} - b_{ni}) =& \frac{1}{b_{n+1,i}}(b_{n+1,i} - b_{ni}), \quad i = 1, 2, \cdots, m \\ =& \frac{\eta x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} + \alpha_n \end{aligned}$$

resulting in

$$b_{n+1,i} = \frac{b_{ni}}{\left[1 - \eta \frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} - \alpha_n\right]}, \quad i = 1, 2, \cdots, m.$$

Then, we evaluate the normalizing constant $(\frac{\boldsymbol{b}_n}{\eta})$, (5.25) is obtained.

2. For $f(t) = (t-1)^2$, $t > 0$, $f''(t) = 2$.

   Thus,

$$\begin{aligned} f''(b_{n+1,i})(b_{n+1,i} - b_{ni}) =& 2(b_{n+1,i} - b_{ni}), \quad i = 1, 2, \cdots, m \\ =& \frac{\eta x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} + \alpha_n \end{aligned}$$

   leading to

$$\begin{aligned} b_{n+1,i} =& b_{ni} + \frac{\eta}{2} \frac{x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} + \frac{\alpha_n}{2} \\ =& b_{ni} + \beta \left(\frac{x_{ni} - \frac{1}{m} \sum_{j=1}^m x_{nj}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} \right) \end{aligned}$$

   after evaluating $\frac{\eta}{2}$ where $\beta = \frac{\eta}{2}$.

5.2 f-Disparity Difference

This session gives two universal portfolios generated by f-disparity difference.

5.2.1 Universal Portfolio Generated by f-disparity difference

Let $f(t)$ be a convex function on $(0,\infty)$ and is strictly convex at $t=1$ and satisfies $f(1)=0$. The f-divergence of two probability distribution $\boldsymbol{p}=(p_i)$ and $\boldsymbol{q}=(q_i)$ is defined as

$$D_f(\boldsymbol{p}||\boldsymbol{q}) = \sum_{i=1}^{m} q_i f\left[\frac{p_i}{q_i}\right]. \tag{5.27}$$

The f-divergence (5.27) is known as the *f-disparity difference* between two probability distributions $\boldsymbol{p}=(p_i)$ and $\boldsymbol{q}=(q_i)$ if $f(t)$ is an f-disparity function which is defined as follows. The continuous function $f(t)$ on $(0,\infty)$ is an *f-disparity function* if

1. $f(t)$ is a decreasing function for $0<t<1$,
2. $f(t)$ is a increasing function for $1<t<\infty$,
3. $f(1)=0$,
4. $f(0)$ is determined by the continuous extension of $f(t)$ (see Basu et al. (2011), pg. 29).

The f-disparity function is also known as the phi-disparity function in the statistical inference literature. The convex function $f(t)$ used in the f-divergence is an f-disparity function. The converse is not true. An f-disparity function may not be convex on $(0,\infty)$. Thus, the f-disparity difference (5.27) which may not be a divergence is a weaker form of (5.27).

Example. Let $f(t)=\frac{(1-t)^2}{(1+t^2)}$ for $t\geq 0$. Then

$$f'(t) = -2\left[\frac{1-t^2}{(1+t^2)^2}\right] \tag{5.28}$$

is negative for $0 < t < 1$ and positive for $t > 1$. Noting that

$$f''(t) = \frac{-4t}{(1+t^2)^3}[t^2-3],$$

it is clear that $f''(t) > 0$ for $0 < t < \sqrt{3}$ and $f''(t) < 0$ for $t > \sqrt{3}$. Thus, $f(t)$ is not convex on $(0,\infty)$. However, $f(t)$ is an f-disparity function.

The fact that the f-disparity difference has the same form as the f-divergence implies that the mathematical form of the universal portfolio generated by the f-disparity difference is the same as that generated by the f-divergence. Let $f(t)$ be a given f-disparity difference. Then from (Tan and Kuang (2018)), then the Type-1 universal portfolio $\boldsymbol{b}_{n+1}$ generated by $f(t)$ corresponding to the objective function

$$\begin{aligned}\hat{F}(\boldsymbol{b}_{n+1};\lambda) =& \eta\left[\log(\boldsymbol{b}_n^t\boldsymbol{x}_n) + \left(\frac{\boldsymbol{b}_{n+1}^t\boldsymbol{x}_n}{\boldsymbol{b}_n^t\boldsymbol{x}_n}\right) - 1\right] - D_f(\boldsymbol{b}_{n+1}||\boldsymbol{b}_n) \\ &+ \lambda\left(\sum_{j=1}^{m} b_{n+1,j} - 1\right)\end{aligned} \tag{5.29}$$

is given by

$$f'\left(\frac{b_{n+1,i}}{b_{ni}}\right) = \eta\left(\frac{x_{ni}}{\boldsymbol{b}_n^t\boldsymbol{x}_n}\right) + \xi, \quad i = 1,2,\ldots,m, \tag{5.30}$$

where λ is the Lagrange multiplier and without loss of generality, the parameters η and ξ are assumed to be constants.

Proposition 5.2.1: For the f-disparity function $f(t) = \frac{(1-t)^2}{(1+t^2)}$, $t \geq 0$, a valid version of the universal portfolio $\boldsymbol{b}_{n+1}$ generated by $f(t)$ is given by

$$b_{n+1,i} = \frac{b_{ni}[v_i^{-1} - 1 + v_i^{-1}(1-4v_i)^{\frac{1}{2}}]^{\frac{1}{2}}}{\sum_{j=1}^{m} b_{nj}[v_j^{-1} - 1 + v_j^{-1}(1-4v_j)^{\frac{1}{2}}]^{\frac{1}{2}}}, \quad i = 1,2,\ldots,m \tag{5.31}$$

for $0 < v_i < \frac{1}{4}$, where

$$v_i = \frac{\eta x_{ni}}{\boldsymbol{b}_n^t \boldsymbol{x}_n} + \xi \quad ; \tag{5.32}$$

η and ξ are parameters.

Proof. From (5.28), (5.30) and (5.32),

$$f'\left(\frac{b_{n+1,i}}{b_{ni}}\right) = -2\left[\frac{\left(1 - \frac{b_{n+1,i}^2}{b_{ni}^2}\right)}{\left(1 + \frac{b_{n+1,i}^2}{b_{ni}^2}\right)^2}\right] = v_i, \;\; i = 1,2,\ldots,m. \tag{5.33}$$

Simplifying (5.33),

$$v_i[b_{n+1,i}^4 + 2b_{ni}^2 b_{n+1,i}^2 + b_{ni}^4] + 2b_{ni}^4 - 2b_{ni}^2 b_{n+1,i}^2 = 0$$

or

$$v_i b_{n+1,i}^4 + 2(v_i - 1)b_{ni}^2 b_{n+1,i}^2 + v_i b_{ni}^4 + 2b_{ni}^4 = 0.$$

Solving the quadratic in $b_{n+1,i}^2$, it follows that

$$b_{n+1,i}^2 = \frac{1}{2v_i}\left\{2(1 - v_i)b_{ni}^2 \pm \sqrt{4b_{ni}^4(1 - 4v_i)}\right\}, \;\; i = 1,2,\ldots,m.$$

Choosing the positive root,

$$b_{n+1,i} = b_{ni}[v_i^{-1} - 1 + v_i^{-1}(1 - 4v_i)^{\frac{1}{2}}]^{\frac{1}{2}}, \;\; i = 1,2,\ldots,m \tag{5.34}$$

For a valid portfolio, $0 \le b_{n+1,i} \le 1$. Thus normalizing (5.34) leads to (5.31).

Remarks. (i) For an empirical study, the parameters η and ξ in (5.32) are chosen so that $0 < v_i < \frac{1}{4}$.

(ii) For small v_i, $(1 - 4v_i)^{\frac{1}{2}}$ can be approximated as $(1 - 2v_i)$. The portfolio (5.31) can be replaced by

$$b_{n+1,i} = \frac{b_{ni}[2v_i^{-1} - 3]^{\frac{1}{2}}}{\sum_{j=1}^{m} b_{nj}[2v_j^{-1} - 3]^{\frac{1}{2}}} \tag{5.35}$$

for $i = 1, 2, \cdots, m$.

Tables 5.6, 5.7, 5.8, 5.9 and 5.10 provides the empirical results achieved by running the universal portfolio generated by f-disparity differences on data sets J, K, L, M and N for 2500 trading days. The tables give the accumulated wealth S_{25000} after 2500 trading days for selected value of the parameters η and ξ together with the final portfolio $\boldsymbol{b}_{2501}$.

Table 5.6: The wealth S_{2500} and the final portfolio $\boldsymbol{b}_{2501}$ achieved by the Universal Porfolio generated by f-disparity difference for data set J ,where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$.

η	ξ	S_{2500}	$\boldsymbol{b}_{2501}$
-0.15	0.33	16.0048	(0.0287, 0.0849, 0.0232, 0.8157, 0.0475)
-0.15	0.29	15.9767	(0.0229, 0.0684, 0.0195, 0.8471, 0.0421)
-0.13	0.32	16.1824	(0.0374, 0.1106, 0.0306, 0.7595, 0.0619)
-0.12	0.22	15.9622	(0.0226, 0.0678, 0.0201, 0.8453, 0.0442)
-0.11	0.20	16.0118	(0.0227, 0.0683, 0.0203, 0.8436, 0.0451)
-0.10	0.18	15.9716	(0.0226, 0.0682, 0.0205, 0.8431, 0.0456)
-0.09	0.16	15.8157	(0.0224, 0.0674, 0.0204, 0.8441, 0.0457)
-0.08	0.29	15.7066	(0.0633, 0.1880, 0.0547, 0.5849, 0.1091)
-0.04	0.07	16.2445	(0.0235, 0.0709, 0.0218, 0.8344, 0.0494)
-0.02	0.26	15.7233	(0.0923, 0.2755, 0.0864, 0.3745, 0.1713)

Table 5.7: The wealth S_{2500} and the final portfolio $\boldsymbol{b}_{2501}$ achieved by the Universal Porfolio generated by f-disparity difference for data set K ,where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$.

η	ξ	S_{2500}	$\boldsymbol{b}_{2501}$
0.38	-0.27	7.8305	(0.0643, 0.4770, 0.1591, 0.2789, 0.0207)
0.39	-0.28	7.8439	(0.0629, 0.4823, 0.1567, 0.2785, 0.0196)
0.40	-0.28	7.8766	(0.0679, 0.4814, 0.1445, 0.2858, 0.0204)
0.40	-0.29	7.8421	(0.0614, 0.4879, 0.1538, 0.2782, 0.0187)
0.41	-0.29	7.9185	(0.0666, 0.4852, 0.1435, 0.2852, 0.0195)
0.42	-0.30	7.9446	(0.0653, 0.4893, 0.1420, 0.2848, 0.0186)
0.43	-0.31	7.9554	(0.0641, 0.4937, 0.1400, 0.2846, 0.0176)
0.44	-0.32	7.9511	(0.0629, 0.4984, 0.1376, 0.2844, 0.0167)
0.45	-0.33	7.9318	(0.0617, 0.5033, 0.1348, 0.2843, 0.0159)
0.46	-0.34	7.8964	(0.0605, 0.5085, 0.1314, 0.2843, 0.0153)

Table 5.8: The wealth S_{2500} and the final portfolio $\boldsymbol{b}_{2501}$ achieved by the Universal Porfolio generated by f-disparity difference for data set L ,where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$.

η	ξ	S_{2500}	$\boldsymbol{b}_{2501}$
-0.57	0.69	5.6261	(0.0067, 0.2343, 0.6686, 0.0289, 0.0615)
-0.49	0.57	5.5615	(0.0031, 0.2981, 0.6398, 0.0176, 0.0414)
-0.48	0.58	5.6194	(0.0084, 0.2259, 0.6663, 0.0351, 0.0643)
-0.34	0.41	5.4893	(0.0113, 0.2330, 0.6382, 0.0446, 0.0729)
-0.05	0.06	7.1999	(0.0042, 0.0774, 0.8741, 0.0171, 0.0272)
0.45	-0.28	5.2723	(0.1684, 0.0174, 0.2328, 0.5396, 0.0418)
0.47	-0.31	5.2772	(0.1712, 0.0162, 0.2296, 0.5370, 0.0460)
0.48	-0.32	5.2868	(0.1713, 0.0150, 0.2309, 0.5381, 0.0447)
0.49	-0.33	5.2950	(0.1713, 0.0138, 0.2320, 0.5391, 0.0438)
0.50	-0.34	5.3016	(0.1714, 0.0125, 0.2329, 0.5403, 0.0429)

Table 5.9: The wealth S_{2500} and the final portfolio $\boldsymbol{b}_{2501}$ achieved by the Universal Porfolio generated by f-disparity difference for data set M ,where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$.

η	ξ	S_{2500}	$\boldsymbol{b}_{2501}$
-0.18	0.35	19.8899	(0.0106, 0.0848, 0.8375, 0.0446, 0.0225)
-0.18	0.34	19.6881	(0.0097, 0.0783, 0.8407, 0.0508, 0.0205)
-0.17	0.32	19.9981	(0.0102, 0.0798, 0.8295, 0.0589, 0.0216)
-0.17	0.35	19.4965	(0.0139, 0.1051, 0.8077, 0.0439, 0.0294)
-0.16	0.34	19.5251	(0.0163, 0.1170, 0.7828, 0.0496, 0.0343)
-0.12	0.29	19.8063	(0.0273, 0.1583, 0.6755, 0.0822, 0.0567)
-0.09	0.24	19.4608	(0.0372, 0.1869, 0.5861, 0.1132, 0.0766)
-0.07	0.19	19.7307	(0.0410, 0.1938, 0.5521, 0.1288, 0.0843)
-0.04	0.11	19.5588	(0.0457, 0.2033, 0.5122, 0.1454, 0.0934)
0.09	-0.02	19.9003	(0.0763, 0.0930, 0.0333, 0.6540, 0.1434)

Table 5.10: The wealth S_{2500} and the final portfolio $\boldsymbol{b}_{2501}$ achieved by the Universal Porfolio generated by f-disparity difference for data set N ,where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$.

η	ξ	S_{2500}	$\boldsymbol{b}_{2501}$
0.43	-0.29	5.3227	(0.0361, 0.2926, 0.1889, 0.4295, 0.0529)
0.44	-0.31	5.3005	(0.0357, 0.2727, 0.1933, 0.4382, 0.0601)
0.45	-0.32	5.3329	(0.0352, 0.2713, 0.1950, 0.4374, 0.0611)
0.45	-0.33	5.2841	(0.0359, 0.2583, 0.1981, 0.4289, 0.0788)
0.46	-0.33	5.3663	(0.0347, 0.2700, 0.1966, 0.4361, 0.0626)
0.46	-0.34	5.3036	(0.0354, 0.2561, 0.1985, 0.4257, 0.0843)
0.47	-0.34	5.4044	(0.0343, 0.2694, 0.1986, 0.4329, 0.0648)
0.47	-0.35	5.3193	(0.0348, 0.2534, 0.1984, 0.4212, 0.0922)
0.48	-0.36	5.3283	(0.0341, 0.2495, 0.1973, 0.4144, 0.1047)
0.49	-0.37	5.3221	(0.0330, 0.2429, 0.1939, 0.4028, 0.1274)

From the results above, we can observe that the best wealth is obtained for data set M while the lowest wealth is obtained for data set N. We observed that data sets J and M are good performing portfolios achieving maximum wealth of 16.2445 and 19.9981 respectively. Meanwhile, data sets K and L are poor performing portfolios, achieving maximum wealth of 7.9544 and 7.1999 respectively.

The empirical results reveal that the forth stock of data set J and the third stock of data set M are performing well in the market. Hence, the portfolios assign more weights on these stocks and lead to higher wealth return. Equivalently, the first, second and fifth stock for both data sets J and L are not performing well. Thus, the portfolios assign lower weights on them.

Tables 4.1, 4.2, 4.3, 4.4, 4.5 shows the wealth achieved by the Type 1 Helmbold universal portfolio (5.19). We compared the performance between Type 1 Helmbold universal portfolio and the universal portfolio generated by f-disparity differences. The results shown from both tables imply that universal portfolio generated by f-dispariry differences performs slight better for the data sets L and N while Type 1 Helmbold universal portfolio performs better for data set K. There is no significant differences between the performance of these two universal portfolio for the data sets J and M.

5.2.2 Universal Portfolio Generated by Rational Function

Let $f(t)$ is defined as given f-disparity difference, then the Type-1 universal portfolio $\boldsymbol{b}_{n+1}$ generated by $f(t)$ corresponding to the objective function (5.29) is given by (5.30) where λ is the Lagrange multiplier and the parameters η and ξ are assumed to be constant.

A non-negative, continuously differentiable convex function $g(t)$ on $(0,\infty)$ may not satisfy the condition $g(1) = 0$. The function $g(t)$ is then translated by $g(1)$ to satisfy $f(1) = 0$, e.g. $f(t) = g(t) - g(1)$. Consider function $f(t) = g(t) - g(1)$ on $(0,\infty)$ where function $g(t)$ is a given continuously differentiable, non-negative convex function on $(0,\infty)$. For a translated convex function $g(t)$, a pseudo f-divergence can be defined by

$$f(t) = g(t) - g(1) \tag{5.36}$$

by substituting $f(t)$ in (5.27) by (5.36).

For $t \geq 0$, we consider the convex function

$$g(t) = \frac{d_1}{(\beta+1)}(t+c_1)^{(\beta+1)} + \frac{d_2}{(1-\beta)}\frac{1}{(t+c+2)^{\beta-1}}. \tag{5.37}$$

For constants $c_1 > 0, c_2 > 0, d_1 > 0, d_2 < 0$ and the parameter $\beta > 1$. For $\beta = 1$, $g(t)$ is defined as

$$g(t) = \frac{d_1}{2}(t+c_1)^2 + d_2 \log(t+c_2). \tag{5.38}$$

A pseudo f-divergence is defined by (5.27) for $f(t)$ and $g(t)$ given by (5.36) and (5.37) or (5.38) respectively.

Proposition 5.2.2: Let the convex function $g(t)$ given by (5.37) or (5.38), $f(t)$ given by (5.36) and the objective function (5.29).

For $c_1 = c_2 = c > 0$ and $\beta > 1$, a valid version of the universal portfolio $\boldsymbol{b}_{n+1}$ generated by $f(t)$ is given by

$$b_{n+1,i} = \frac{b_{ni}\left\{\left[\frac{1}{2d_1}(v_i + \sqrt{v_i^2 - 4d_1d_2})\right]^{\frac{1}{\beta}} - c\right\}}{\sum_{j=1}^{m} b_{nj}\left\{\left[\frac{1}{2d_1}(v_j + \sqrt{v_j^2 - 4d_1d_2})\right]^{\frac{1}{\beta}} - c\right\}}. \tag{5.39}$$

for $j = 1, 2, \cdots, m$ where

$$v_i = \frac{\eta x_{ni}}{\boldsymbol{b}_{n+1}^t \boldsymbol{x}_n} + \xi_i \tag{5.40}$$

and $\left[\frac{1}{2d_i}(v_i + \sqrt{v_i^2 - 4d_1d_2})\right]^{\frac{1}{\beta}} > c$ for selected values of v_i and c.

Proof. The universal portfolio $\boldsymbol{b}_{n+1}$ generated by the pseudo f-divergence (5.27) is given by

$$f'(t) = g'(t) = d_1(t+c)^{\beta} + d_2(t+c)^{-\beta} = v_i \tag{5.41}$$

for $i = 1, 2, \cdots, m$ from (5.30), (5.36), (5.37) and (5.40) where $t = \frac{b_{n+1,i}}{b_{ni}}$.

Simplifying the (5.41),

$$d_1(t+c)^{2\beta} - v_i(t+c)^{\beta} + d_2 = 0. \tag{5.42}$$

Solving the quadratic in $(t+c)$ and taking the positive root,

$$t = \left[\frac{1}{2d_1}\left(v_i + \sqrt{v_i^2 - 4d_1d_2}\right)\right]^{\frac{1}{\beta}} - c. \tag{5.43}$$

Then, normalizing (5.43) leads to (5.39).

Proposition 5.2.3: Let the convex function $g(t)$ given by (5.37) or (5.38), $f(t)$ given by (5.36) and the objective function (5.29).

For $\beta = 1$, a valid version of the universal portfolio $\boldsymbol{b}_{n+1}$ generated by $f(t)$ is given by

$$b_{n+1,i} = \frac{b_{ni}\left[\frac{1}{2d_1}[v_i - d_1(c_1+c_2)] + \frac{1}{2d_1}\sqrt{\{v_i - d_1(c_1+c_2)\}^2 - 4d_1[d_1c_1c_2 + d_2 - v_ic_2]}\right]}{\sum_{j=1}^{m} b_{nj}\left[\frac{1}{2d_1}[v_j - d_1(c_1+c_2)] + \frac{1}{2d_1}\sqrt{\{v_j - d_1(c_1+c_2)\}^2 - 4d_1[d_1c_1c_2 + d_2 - v_jc_2]}\right]} \tag{5.44}$$

for $i = 1, 2, \cdots, m$, where v_i is given by (5.40) and the numerator of (5.44) is positive for selected values of v_i, c_1 and c_2.

Proof. The universal portfolio $\boldsymbol{b}_{n+1}$ generated by the pseudo f-divergence (5.27) is given by

$$f'(t) = g'(t) = d_1(t+c) + d_2(t+c_2)^{-1} = v_i \tag{5.45}$$

for $i = 1, 2, \cdots, m$ from (5.30), (5.36), (5.38) and (5.40) where $\dfrac{b_{n+1,i}}{b_{ni}}$.

Simplifying the (5.45),

$$d_1(t+c_1)(t+c_2) + d_2 - v_i(t+c_2) = 0 \tag{5.46}$$

for $i = 1, 2, \cdots, m$.

Solving the quadratic in t and normalizing $b_{n+1,i}$, the universal portfolio (5.44) is obtained.

Tables 5.11, 5.12, 5.13, 5.14 and 5.15 gives the result obtained after universal portfolio (5.39) is run on the data sets J, K, L, M and N, respectively. Tables 5.16, 5.17, 5.18, 5.19 and 5.20 gives the result obtained after universal portfolio (5.44) is run on the same data sets. The results obtained between two universal portfolios (5.39) and (5.44) did not show significant differences among them. Then, we compare the result obtained from universal portfolios (5.39) and (5.44) with the result obtained from Type 1 Helmbold universal portfolio (5.19). The empirical result obtained by Type 1 Helmbold universal portfolio is shown in Tables 4.1, 4.2, 4.3, 4.4 and 4.5.

Table 5.11: The wealth S_{2500} and the final portfolio $\boldsymbol{b}_{2501}$ achieved by the Universal Portfolio generated by rational functions (5.39) for data set J ,where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$ and $c = 0.1$.

η	β	ξ	S_{2500}	$\boldsymbol{b}_{2501}$
1.0	1.1	-2.0	17.2146	(0.0811, 0.0825, 0.0697, 0.6936, 0.0731)
1.0	1.2	-1.8	17.0865	(0.0872, 0.0886, 0.0755, 0.6698, 0.0789)
1.1	1.2	-1.8	16.9870	(0.0782, 0.0795, 0.0666, 0.7058, 0.0699)
1.2	1.3	-1.9	17.0492	(0.0768, 0.0782, 0.0654, 0.7108, 0.0688)
0.9	1.3	-2.0	17.0072	(0.1052, 0.1066, 0.0937, 0.5974, 0.0971)
0.9	1.3	-1.9	16.9640	(0.1044, 0.1058, 0.0928, 0.6007, 0.0963)
1.1	1.4	-2.0	17.2893	(0.0917, 0.0931, 0.0802, 0.6514, 0.0836)
1.2	1.5	-1.8	17.0051	(0.0887, 0.0901, 0.0769, 0.6640, 0.0803)
1.4	1.5	-2.0	16.9972	(0.0746, 0.0759, 0.0633, 0.7197, 0.0665)

Table 5.12: The wealth S_{2500} and the final portfolio $\boldsymbol{b}_{2501}$ achieved by the Universal Portfolio generated by rational functions (5.39) for data set K, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$ and $c = 0.1$.

η	β	ξ	S_{2500}	$\boldsymbol{b}_{2501}$
1.2	1.3	-2.0	19.6095	(0.0634, 0.0569, 0.7123, 0.0614, 0.1060)
1.1	1.2	-1.9	19.4867	(0.0647, 0.0580, 0.7070, 0.0626, 0.1077)
1.1	1.1	-1.9	19.4762	(0.0575, 0.0511, 0.7363, 0.0555, 0.0996)
1.5	1.7	-1.9	19.0312	(0.0655, 0.0584, 0.7033, 0.0633, 0.1095)
1.2	1.8	-2.0	19.0107	(0.0911, 0.0840, 0.6027, 0.0889, 0.1333)
1.6	2.0	-2.0	18.9782	(0.0735, 0.0662, 0.6713, 0.0713, 0.1177)
1.5	1.8	-1.9	18.9756	(0.0704, 0.0632, 0.6836, 0.0682, 0.1146)
1.5	2.1	-2.0	18.9632	(0.0832, 0.0759, 0.6330, 0.0810, 0.1269)
1.4	1.9	-1.9	18.9514	(0.0811, 0.0737, 0.6415, 0.0789, 0.1248)

Table 5.13: The wealth S_{2500} and the final portfolio $\boldsymbol{b}_{2501}$ achieved by the Universal Portfolio generated by rational functions (5.39) for data set L, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$ and $c = 0.1$.

η	β	ξ	S_{2500}	$\boldsymbol{b}_{2501}$
-2.0	1.8	0.2	6.5401	(0.2955, 0.1267, 0.1137, 0.2913, 0.1728)
-2.0	1.7	0.2	6.5400	(0.3014, 0.1227, 0.1088, 0.2968, 0.1703)
-2.0	2.9	0.4	6.5075	(0.2605, 0.1518, 0.1440, 0.2583, 0.1854)
-2.0	2.9	1.0	6.4998	(0.2696, 0.1465, 0.1342, 0.2665, 0.1832)
-2.0	1.2	-0.1	6.4991	(0.3389, 0.0984, 0.0786, 0.3309, 0.1532)
-2.0	1.2	0.2	6.4959	(0.3451, 0.0941, 0.0745, 0.3365, 0.1498)
-1.7	1.1	-0.4	6.4735	(0.3322, 0.1045, 0.0814, 0.3241, 0.1578)
-1.8	1.1	-0.3	6.4734	(0.3390, 0.0995, 0.0770, 0.3305, 0.1540)
-1.6	1.1	-0.6	6.4728	(0.3235, 0.1107, 0.0872, 0.3161, 0.1625)

Table 5.14: The wealth S_{2500} and the final portfolio $\boldsymbol{b}_{2501}$ achieved by the Universal Portfolio generated by rational functions (5.39) for data set M, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$ and $c = 0.1$.

η	β	ξ	S_{2500}	$\boldsymbol{b}_{2501}$
1.1	1.3	-2.0	21.0694	(0.0718, 0.1304, 0.6478, 0.0749, 0.0751)
1.0	1.2	-1.9	20.9321	(0.0740, 0.1328, 0.6397, 0.0762, 0.0773)
1.1	1.4	-1.9	20.9259	(0.0776, 0.1362, 0.6256, 0.0796, 0.0810)
1.2	1.2	-2.0	20.8834	(0.0566, 0.1148, 0.7082, 0.0606, 0.0598)
1.3	1.9	-2.0	20.8725	(0.0875, 0.1447, 0.5877, 0.0892, 0.0909)
1.2	1.8	-2.0	20.8678	(0.0906, 0.1470, 0.5759, 0.0925, 0.0940)
1.3	2.7	-1.9	19.9123	(0.1155, 0.1662, 0.4851, 0.1144, 0.1188)
1.7	2.8	-2.0	19.9114	(0.0977, 0.1557, 0.5508, 0.0946, 0.1012)
1.6	2.6	-1.9	19.8951	(0.0969, 0.1551, 0.5537, 0.0939, 0.1004)

Table 5.15: The wealth S_{2500} and the final portfolio $\boldsymbol{b}_{2501}$ achieved by the Universal Portfolio generated by rational functions (5.39) for data set N, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$ and $c = 0.1$.

η	β	ξ	S_{2500}	$\boldsymbol{b}_{2501}$
-2.0	1.1	1.9	4.6463	(0.1288, 0.2443, 0.3240, 0.2154, 0.0875)
-1.9	1.1	1.8	4.6019	(0.1328, 0.2433, 0.3157, 0.2161, 0.0921)
-2.0	1.3	1.2	4.4774	(0.1413, 0.2342, 0.2910, 0.2282, 0.1053)
-1.9	1.3	1.8	4.4762	(0.1434, 0.2396, 0.2943, 0.2175, 0.1052)
-1.9	1.3	1.9	4.4755	(0.1438, 0.2405, 0.2947, 0.2158, 0.1052)
-2.0	1.5	1.9	4.4153	(0.1484, 0.2374, 0.2848, 0.2177, 0.1117)
-2.0	1.5	1.8	4.4140	(0.1482, 0.2365, 0.2842, 0.2192, 0.1119)
-2.0	1.5	1.7	4.4112	(0.1481, 0.2356, 0.2833, 0.2208, 0.1122)
-2.0	1.5	1.6	4.4070	(0.1481, 0.2346, 0.2823, 0.2223, 0.1127)

Table 5.16: The wealth S_{2500} and the final portfolio $\boldsymbol{b}_{2501}$ achieved by the Universal Portfolio generated by rational functions (5.44) for data set J, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$.

η	ξ	c_1	c_2	S_{2500}	$\boldsymbol{b}_{2501}$
1.2	1.5	-2.0	0.1	17.2924	(0.0889, 0.0904, 0.0774, 0.6623, 0.0810)
1.1	1.4	-1.9	0.1	17.1842	(0.0912, 0.0926, 0.0796, 0.6534, 0.0832)
1.4	1.8	-2.0	0.1	17.1534	(0.0897, 0.0911, 0.0781, 0.6595, 0.0816)
1.4	1.7	-2.0	0.1	17.1461	(0.0850, 0.0864, 0.0734, 0.6784, 0.0768)
1.1	1.5	-1.9	0.1	17.1378	(0.0969, 0.0983, 0.0853, 0.6308, 0.0887)
1.3	1.9	-2.0	0.1	17.1356	(0.1008, 0.1022, 0.0893, 0.6149, 0.0928)
1.2	1.4	-1.9	0.1	17.1351	(0.0830, 0.0844, 0.0714, 0.6864, 0.0748)
1.2	1.6	-1.9	0.1	17.1335	(0.0940, 0.0954, 0.0824, 0.6423, 0.0859)
0.9	1.1	-1.9	0.1	17.1318	(0.0908, 0.0922, 0.0791, 0.6554, 0.0825)

Table 5.17: The wealth S_{2500} and the final portfolio $\boldsymbol{b}_{2501}$ achieved by the Universal Portfolio generated by rational functions (5.44) for data set K, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$.

η	ξ	c_1	c_2	S_{2500}	$\boldsymbol{b}_{2501}$
1.2	1.3	-2.0	0.1	19.6095	(0.0634, 0.0569, 0.7123, 0.0614, 0.1060)
1.3	1.3	-2.0	0.1	19.5290	(0.0558, 0.0496, 0.7431, 0.0538, 0.0977)
1.4	1.5	-2.0	0.1	19.4671	(0.0608, 0.0542, 0.7228, 0.0587, 0.1035)
1.2	1.5	-2.0	0.1	19.4637	(0.0756, 0.0687, 0.6638, 0.0734, 0.1185)
1.4	1.7	-2.0	0.1	19.3886	(0.0714, 0.0644, 0.6803, 0.0692, 0.1147)
1.3	1.4	-1.9	0.1	19.3812	(0.0618, 0.0551, 0.7188, 0.0597, 0.1046)
1.0	1.1	-1.8	0.1	19.3583	(0.0662, 0.0594, 0.7010, 0.0641, 0.1093)
1.0	1.2	-1.9	0.1	19.3573	(0.0742, 0.0672, 0.6692, 0.0720, 0.1174)
1.1	1.2	-1.8	0.1	19.3531	(0.0644, 0.0576, 0.7084, 0.0622, 0.1074)

Table 5.18: The wealth S_{2500} and the final portfolio $\boldsymbol{b}_{2501}$ achieved by the Universal Portfolio generated by rational functions (5.44) for data set L, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$.

η	ξ	c_1	c_2	S_{2500}	$\boldsymbol{b}_{2501}$
-2.0	1.8	0.2	0.1	6.5401	(0.2955, 0.1267, 0.1137, 0.2913, 0.1728)
-2.0	1.9	0.3	0.1	6.5390	(0.2921, 0.1291, 0.1166, 0.2881, 0.1741)
-2.0	1.9	0.2	0.1	6.5389	(0.2902, 0.1304, 0.1182, 0.2864, 0.1748)
-2.0	2.0	0.5	0.1	6.5357	(0.2911, 0.1300, 0.1172, 0.2871, 0.1746)
-2.0	1.6	-0.2	0.1	6.5356	(0.3004, 0.1236, 0.1092, 0.2958, 0.1710)
-2.0	1.4	0.1	0.1	6.5274	(0.3219, 0.1090, 0.0920, 0.3157, 0.1614)
-2.0	1.5	-0.5	0.1	6.5273	(0.3025, 0.1226, 0.1069, 0.2977, 0.1703)
-1.9	1.4	-0.1	0.1	6.5215	(0.3146, 0.1145, 0.0970, 0.3088, 0.1651)
-2.0	2.6	0.4	0.1	6.5185	(0.2678, 0.1464, 0.1375, 0.2652, 0.1831)

Table 5.19: The wealth S_{2500} and the final portfolio $\boldsymbol{b}_{2501}$ achieved by the Universal Portfolio generated by rational functions (5.44) for data set M, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$.

η	ξ	c_1	c_2	S_{2500}	$\boldsymbol{b}_{2501}$
1.1	1.3	-2.0	0.1	21.0694	(0.0718, 0.1304, 0.6478, 0.0749, 0.0751)
1.0	1.1	-1.9	0.1	20.9148	(0.0666, 0.1257, 0.6685, 0.0693, 0.0699)
1.1	1.2	-1.9	0.1	20.9099	(0.0646, 0.1236, 0.6766, 0.0675, 0.0677)
1.2	1.2	-2.0	0.1	20.8834	(0.0566, 0.1148, 0.7082, 0.0606, 0.0598)
1.3	1.9	-2.0	0.1	20.8725	(0.0875, 0.1447, 0.5877, 0.0892, 0.0909)
1.4	2.0	-2.0	0.1	20.8071	(0.0850, 0.1429, 0.5974, 0.0863, 0.0884)
1.0	1.1	-1.8	0.1	20.8060	(0.0661, 0.1255, 0.6707, 0.0683, 0.0694)
1.1	1.3	-1.8	0.1	20.8048	(0.0710, 0.1305, 0.6514, 0.0728, 0.0743)
1.0	1.4	-1.9	0.1	20.8038	(0.0870, 0.1445, 0.5897, 0.0884, 0.0904)

Table 5.20: The wealth S_{2500} and the final portfolio $\boldsymbol{b}_{2501}$ achieved by the Universal Portfolio generated by rational functions (5.44) for data set N, where the initial portfolio $\boldsymbol{b}_1 = (0.2, 0.2, 0.2, 0.2, 0.2)$.

η	ξ	c_1	c_2	S_{2500}	$\boldsymbol{b}_{2501}$
-2.0	1.1	1.9	0.1	4.6463	(0.1288, 0.2443, 0.3240, 0.2153, 0.0876)
-2.0	1.2	1.7	0.1	4.5730	(0.1346, 0.2406, 0.3089, 0.2208, 0.0951)
-1.9	1.2	1.9	0.1	4.5339	(0.1389, 0.2424, 0.3045, 0.2151, 0.0991)
-1.9	1.3	1.9	0.1	4.4755	(0.1438, 0.2405, 0.2947, 0.2158, 0.1052)
-2.0	1.4	1.9	0.1	4.4626	(0.1445, 0.2391, 0.2921, 0.2177, 0.1066)
-2.0	1.4	1.8	0.1	4.4614	(0.1443, 0.2382, 0.2914, 0.2193, 0.1068)
-2.0	1.4	1.3	0.1	4.4337	(0.1451, 0.2335, 0.2850, 0.2266, 0.1098)
-1.9	1.4	1.8	0.1	4.4249	(0.1476, 0.2378, 0.2863, 0.2176, 0.1107)
-2.0	1.4	1.2	0.1	4.4248	(0.1456, 0.2326, 0.2835, 0.2275, 0.1108)

We observe that the result obtained from universal portfolios (5.39) and (5.44) are closed to the result from Type 1 Helmbold universal portfolio. The stock-price data sets J, K and M are good peratforming portfolios while stock-price data sets L and N are poor peratforming portfolios. However, in-term of wealth achieved, both universal portfolio (5.39) and (5.44) peratform slightly better for stock-price data sets J, K, L and M as compare to Type 1 Helmbold universal portfolio. But Type 1 Helmbold universal portfolio peratform better for stock-price data set N. However, the peratformances between Type I Helmbold universal portfolio, universal portfolio (5.39) and (5.44) do not show any significant difference for all stock-price data sets.

CHAPTER 6

CONCLUSION

This chapter concludes the research with a comprehensive summary. Beside, some future works are proposed in this session.

6.1 Summary

The objective of this research is to investigate the new methods to generate new universal portfolio with potential useful distance functions. Then, the performance of the newly generated universal portfolios is studied and compared with the benchmark performance. In this research, a total of 11 new universal portfolios have been derived. There are Type 1 RPR Universal Portfolio, Type 2 RPR Universal Portfolio, Universal Portfolio generated by Rényi Divergence, Universal Portfolio generated by Kullback-Leibler Divergence, Reverse Helmbold Universal Portfolio, Type k universal portfolio generated by f-divergence, universal portfolio generated by reverse f-divergence, universal portfolio generated by Bregman divergence, universal portfolio generated by reverse Bregman divergence, universal portfolio generated by f-disparity difference and universal portfolio generated by rational function.

The core in this new parametric family of universal portfolio is the universal portfolio generated by Csiszar f-divergence. With the right convex functions, we able to derive many other universal portfolios from universal portfolio generated by f-divergence. For instance, the Helmbold universal portfolio can be derived from universal portfolio generated by f-divergence by applying the convex function $f(t) = t\log t - t + 1$, where $t > 0$. We able to derive a few universal portfolios by choosing the right convex functions. These universal portfolios are universal portfolio generated by f-disparity difference and universal portfolio generated by rational function. The performance of these universal portfolio are studied intensively. The performance of these universal portfolios are at the same level as the performance of the benchmark universal portfolio, the Helmbold universal portfolio. The empirical result obtained by running these universal portfolios on the local stock data sets.

There are a few universal portfolios derived from universal portfolios generated by reverse f and Bregman divergences are not studied intensively. There are the reverse Kullback-Leibler universal portfolio, reverse chi-square universal portfolio, the reverse α-divergence universal portfolio, the reverse Bregman universal portfolio and reverse chi-square Bregman universal portfolio. A future study can be done to obtain the empirical results from these universal portfolios. The results will be compared with the result obtained by benchmark universal portfolio.

Figure 6.1 gives the new family tree for universal portfolios derived in this thesis and their relationship.

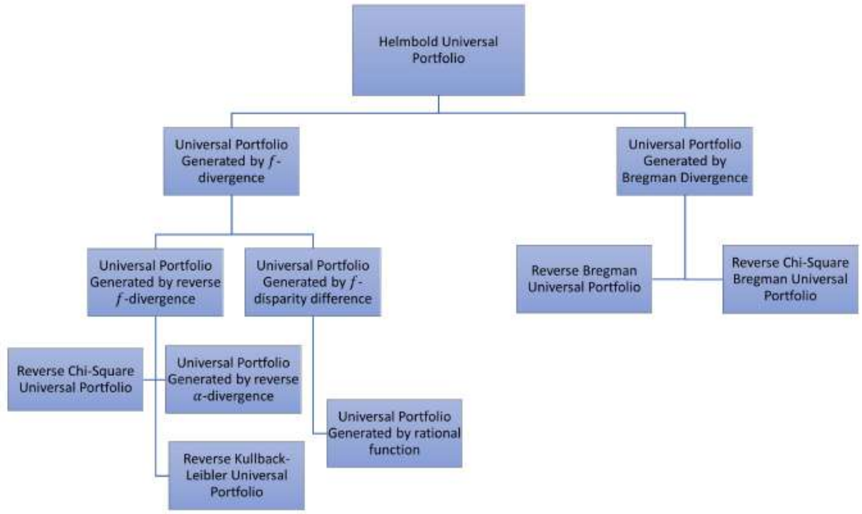

Figure 6.1: New Family of Universal Portfolios

The ability of the universal portfolio to assign proper weights to the constituent stocks to achieve higher investment returns.

6.2 Future Works

We have successfully built a new parametric family of universal portfolio in this research and their performance are on par with well-known Helmbold universal portfolio. However, the newly derived universal portfolios fail to outperform the Helmbold universal portfolio. Therefore, further study to search for new methods to generate universal portfolio is suggested to future work.

Beside searching for new methods generating universal portfolio, a study of practicality of universal portfolio in real stock exchange market is suggested for future work. The current practical limitation of universal portfolio is the daily stock trading that could costs a big amount of commission fee. The universal portfolio optimizes the allocation of the stocks daily, each change in portfolio involves a payment of commission or brokerage fees to the broker. The transaction costs will be costly to the investors. Therefore, further study of universal portfolio including transaction costs is suggested for future work. The number of portfolio changes maybe reduce in order to minimize the effect from transaction cost charged for each portfolio changes. This can be achieved by changing the portfolio when the components have moved to a bigger threshold. By reducing the commission paid to the brokers, the investor's wealth will be increased.

We may consider to include cash-in-hand and gold in the portfolio and observe the performance of the universal portfolios. The cash-in-hand can be spent on the transaction cost while the gold plays an important role in a diversified investment portfolio. The reason is the gold's price tends to increase in value in response to the event that the stocks' value decline. Historically, gold showed that it has always maintain its value over the long term.

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APPENDIX A

This section gives the graphs of the the stock data that were selected from Kuala Lumpur Stock Exchange (KLSE) and Bloomberg. The data were collected for 2500 trading days, for the period from 3rd January 2005 to 4th September 2015. Stock data collected were grouped into 5 different stock data sets. Stock data set J consists of the stocks of Public Bank, Nestle Malaysia, Telekom Malaysia, Eco World Development Group and Gamuda. Stock data set K consists of the stocks of AMMB Holdings, Air Asia, Encorp, IJM Corp and Genting Plantations. Stock data set L consists of stocks of Alliance Financial Group, DiGi.com, KSL Holdings, IJM Corp and Kulim Malaysia. Stock data set M consists of stocks of Hong Leong Bank, DiGi.com, Eco World Development Group, Zecon and United Malacca. Stock data set N consists of stocks of RHB Capital, Carlsberg Brewery Malaysia, KSL Holdings, Crest Building Holdings and Kulim Malaysia.

SET J DATASETS

SET K DATASETS

SET L DATASETS

SET M DATASETS

SET N DATASETS

APPENDIX B

This sessions gives the macros used to study the performance of the universal portfolios derived via Microsoft Excel Visual Basic for Application (VBA).

The empirical results obtained were analyzed further via Microsoft Excel.

Excel VBA Coding for Type 1 RPR universal portfolio

```
Sub void()
Dim alpha, zai As Double
Dim b01, b02, b03, b04, b05 As Double
Dim x11, x12, x13, x14, x15 As Double
Dim x21, x22, x23, x24, x25 As Double
Dim bxn As Double
Dim v1, v2, v3, v4, v5 As Double
Dim c1, c2, c3, c4, c5 As Double
Dim bnxn As Double
Dim sn As Double
Dim stock As Integer
Dim bnxc As Double
Dim count As Integer
Dim sntry As Double
Sheet3.Cells.Clear
stock = 1
Do While stock < 6
count = 1
alpha = 0
Do While alpha < 15
zai = -10
sntry = 1
Do While zai < 5
sn = 1
b01 = 0.2
b02 = 0.2
b03 = 0.2
b04 = 0.2
b05 = 0.2
For i = 1 To 1500
x11 = Sheet2.Cells(i + 1, 1 + 6 * (stock - 1))
x12 = Sheet2.Cells(i + 1, 2 + 6 * (stock - 1))
x13 = Sheet2.Cells(i + 1, 3 + 6 * (stock - 1))
x14 = Sheet2.Cells(i + 1, 4 + 6 * (stock - 1))
x15 = Sheet2.Cells(i + 1, 5 + 6 * (stock - 1))
x21 = Sheet2.Cells(i + 2, 1 + 6 * (stock - 1))
x22 = Sheet2.Cells(i + 2, 2 + 6 * (stock - 1))
x23 = Sheet2.Cells(i + 2, 3 + 6 * (stock - 1))
x24 = Sheet2.Cells(i + 2, 4 + 6 * (stock - 1))
x25 = Sheet2.Cells(i + 2, 5 + 6 * (stock - 1))
bxn = x11 * b01 + x12 * b02 + x13 * b03 +_
x14 * b04 + x15 * b05
v1 = 1 / ( alpha * bxn + x11 )
v2 = 1 / ( alpha * bxn + x12 )
v3 = 1 / ( alpha * bxn + x13 )
v4 = 1 / ( alpha * bxn + x14 )
v5 = 1 / ( alpha * bxn + x15 )
c1 = v1 - ( 0.24 * v1 + 0.20 * v2 + 0.23 * v3 +_
0.31 * v4 + 0.21 * v5)
c2 = v2 - ( 0.17 * v1 + 0.20 * v2 + 0.33 * v3 +_
0.22 * v4 + 0.11 * v5)
```

```
c3 = v3 - ( 0.30 * v1 + 0.27 * v2 + 0.30 * v3 +_
0.03 * v4 + 0.24 * v5)
c4 = v4 - ( 0.00 * v1 + 0.04 * v2 + 0.02 * v3 +_
0.15 * v4 + 0.31 * v5)
c5 = v5 - ( 0.29 * v1 + 0.29 * v2 + 0.13 * v3 +_
0.29 * v4 + 0.13 * v5)
b01 = b01 + bxn * c1 / zai
b02 = b02 + bxn * c2 / zai
b03 = b03 + bxn * c3 / zai
b04 = b04 + bxn * c4 / zai
b05 = b05 + bxn * c5 / zai
If b01 < 0 Or b02 < 0 Or b03 < 0 Or b04 < 0 Or b05 < 0 Then
Exit For
Else
bnxn = b01 * x21 + b02 * x22 + b03 * x23 + b04 * x24 +_
b05 * x25
sn = sn * bnxn
End If
Next i
If b01 > 0 And b02 > 0 And b03 > 0 And_
b04 > 0 And b05 > 0 And sn > sntry Then
sntry = sn
Sheet3.Cells(count, 1 + 10 * (stock - 1)) = alpha
Sheet3.Cells(count, 2 + 10 * (stock - 1)) = zai
Sheet3.Cells(count, 3 + 10 * (stock - 1)) = sn
Sheet3.Cells(count, 5 + 10 * (stock - 1)) = b01
Sheet3.Cells(count, 6 + 10 * (stock - 1)) = b02
Sheet3.Cells(count, 7 + 10 * (stock - 1)) = b03
Sheet3.Cells(count, 8 + 10 * (stock - 1)) = b04
Sheet3.Cells(count, 9 + 10 * (stock - 1)) = b05
End If
zai = zai + 0.01
Loop
count = count + 1
alpha = alpha + 2
Loop
stock = stock + 1
Loop
End Sub
```

Excel VBA Coding for pseudo relaxed Type 2 RPR universal portfolio

```
Sub void()
Dim alpha, zai, beta As Double
Dim b01, b02, b03, b04, b05 As Double
Dim x11, x12, x13, x14, x15 As Double
Dim x21, x22, x23, x24, x25 As Double
Dim bxn As Double
Dim r1, r2, r3, r4, r5 As Double
Dim c1, c2, c3, c4, c5 As Double
Dim bnxn As Double
Dim sn As Double
Dim stock As Integer
Dim bnxc As Double
Dim count As Integer
Dim sntry As Double
Sheet3.Cells.Clear
beta=0.6
stock = 1
Do While stock < 6
count = 1
alpha = 0
Do While alpha < 15
zai = -10
sntry = 1
Do While zai < 30
sn = 1
b01 = 0.2
b02 = 0.2
b03 = 0.2
b04 = 0.2
b05 = 0.2
For i = 1 To 1500
x11 = Sheet2.Cells(i + 1, 1 + 6 * (stock - 1))
x12 = Sheet2.Cells(i + 1, 2 + 6 * (stock - 1))
x13 = Sheet2.Cells(i + 1, 3 + 6 * (stock - 1))
x14 = Sheet2.Cells(i + 1, 4 + 6 * (stock - 1))
x15 = Sheet2.Cells(i + 1, 5 + 6 * (stock - 1))
x21 = Sheet2.Cells(i + 2, 1 + 6 * (stock - 1))
x22 = Sheet2.Cells(i + 2, 2 + 6 * (stock - 1))
x23 = Sheet2.Cells(i + 2, 3 + 6 * (stock - 1))
x24 = Sheet2.Cells(i + 2, 4 + 6 * (stock - 1))
x25 = Sheet2.Cells(i + 2, 5 + 6 * (stock - 1))
bxn = x11 * b01 + x12 * b02 + x13 * b03 +_
x14 * b04 + x15 * b05
r1 = 1 / ( alpha * bxn * bxn + ( x11 - beta ) *_
bxn + beta * x11 )
r2 = 1 / ( alpha * bxn * bxn + ( x12 - beta ) *_
bxn + beta * x12 )
r3 = 1 / ( alpha * bxn * bxn + ( x13 - beta ) *_
bxn + beta * x13 )
r4 = 1 / ( alpha * bxn * bxn + ( x14 - beta ) *_
bxn + beta * x14 )
r5 = 1 / ( alpha * bxn * bxn + ( x15 - beta ) *_
bxn + beta * x15 )
c1 = r1 - 0.2 * (v1 + v2 + v3 + v4 + v5)
c2 = r2 - 0.2 * (v1 + v2 + v3 + v4 + v5)
c3 = r3 - 0.2 * (v1 + v2 + v3 + v4 + v5)
c4 = r4 - 0.2 * (v1 + v2 + v3 + v4 + v5)
c5 = r5 - 0.2 * (v1 + v2 + v3 + v4 + v5)
b01 = b01 + bxn * bxn * c1 / zai
b02 = b02 + bxn * bxn * c2 / zai
b03 = b03 + bxn * bxn * c3 / zai
b04 = b04 + bxn * bxn * c4 / zai
b05 = b05 + bxn * bxn * c5 / zai
If b01 < 0 Or b02 < 0 Or b03 < 0 Or b04 < 0 Or b05 < 0 Then
Exit For
Else
bnxn = b01 * x21 + b02 * x22 + b03 * x23 + b04 * x24 +_
b05 * x25
sn = sn * bnxn
End If
Next i
```

```
If b01 > 0 And b02 > 0 And b03 > 0 And_
b04 > 0 And b05 > 0 And sn > sntry Then
sntry = sn
Sheet3.Cells(count, 1 + 10 * (stock - 1)) = alpha
Sheet3.Cells(count, 2 + 10 * (stock - 1)) = zai
Sheet3.Cells(count, 3 + 10 * (stock - 1)) = sn
Sheet3.Cells(count, 5 + 10 * (stock - 1)) = b01
Sheet3.Cells(count, 6 + 10 * (stock - 1)) = b02
Sheet3.Cells(count, 7 + 10 * (stock - 1)) = b03
Sheet3.Cells(count, 8 + 10 * (stock - 1)) = b04
Sheet3.Cells(count, 9 + 10 * (stock - 1)) = b05
End If
zai = zai + 0.01
Loop
count = count + 1
alpha = alpha + 2
Loop
stock = stock + 1
Loop
End Sub
```

Excel VBA Coding for pseudo relaxed Rényi universal portfolio

```
Sub void()
Dim alpha, beta, gamma As Double
Dim b01, b02, b03, b04, b05 As Double
Dim x11, x12, x13, x14, x15 As Double
Dim x21, x22, x23, x24, x25 As Double
Dim bxn As Double
Dim c1, c2, c3, c4, c5 As Double
Dim m1, m2, m3, m4, m5 As Double
Dim bnxn As Double
Dim sn As Double
Dim stock As Integer
Dim bnxc As Double
Dim count As Integer
Dim sntry As Double
Sheet3.Cells.Clear
stock = 1
alpha = 10
beta = 6
Do While stock < 6
count = 1
gamma = 0
sntry = 1
Do While gamma < 5
sn = 1
b01 = 0.2
b02 = 0.2
b03 = 0.2
b04 = 0.2
b05 = 0.2
For i = 1 To 1500
x11 = Sheet2.Cells(i + 1, 1 + 6 * (stock - 1))
x12 = Sheet2.Cells(i + 1, 2 + 6 * (stock - 1))
x13 = Sheet2.Cells(i + 1, 3 + 6 * (stock - 1))
x14 = Sheet2.Cells(i + 1, 4 + 6 * (stock - 1))
x15 = Sheet2.Cells(i + 1, 5 + 6 * (stock - 1))
x21 = Sheet2.Cells(i + 2, 1 + 6 * (stock - 1))
x22 = Sheet2.Cells(i + 2, 2 + 6 * (stock - 1))
x23 = Sheet2.Cells(i + 2, 3 + 6 * (stock - 1))
x24 = Sheet2.Cells(i + 2, 4 + 6 * (stock - 1))
x25 = Sheet2.Cells(i + 2, 5 + 6 * (stock - 1))
bxn = x11 * b01 + x12 * b02 + x13 * b03 +_
x14 * b04 + x15 * b05
c1 = ( beta * bxn ) - ( gamma - x11 )
c2 = ( beta * bxn ) - ( gamma - x12 )
c3 = ( beta * bxn ) - ( gamma - x13 )
c4 = ( beta * bxn ) - ( gamma - x14 )
c5 = ( beta * bxn ) - ( gamma - x15 )
m1 = ( c1 ^ ( 1 / ( alpha - 1 ) ) ) * b01
m2 = ( c2 ^ ( 1 / ( alpha - 1 ) ) ) * b02
m3 = ( c3 ^ ( 1 / ( alpha - 1 ) ) ) * b03
m4 = ( c4 ^ ( 1 / ( alpha - 1 ) ) ) * b04
m5 = ( c5 ^ ( 1 / ( alpha - 1 ) ) ) * b05
bnxc = m1 + m2 + m3 + m4 + m5
b01 = m1 / ( m1 + m2 + m3 + m4 + m5 )
b02 = m2 / ( m1 + m2 + m3 + m4 + m5 )
b03 = m3 / ( m1 + m2 + m3 + m4 + m5 )
b04 = m4 / ( m1 + m2 + m3 + m4 + m5 )
b05 = m5 / ( m1 + m2 + m3 + m4 + m5 )
If b01 < 0 Or b02 < 0 Or b03 < 0 Or b04 < 0 Or b05 < 0 Then
Exit For
Else
bnxn = b01 * x21 + b02 * x22 + b03 * x23 + b04 * x24 +_
b05 * x25
sn = sn * bnxn
End If
Next i
If b01 > 0 And b02 > 0 And b03 > 0 And_
b04 > 0 And b05 > 0 And sn > sntry Then
sntry = sn
Sheet3.Cells(count, 1 + 10 * (stock - 1)) = alpha
```

```
Sheet3.Cells(count, 2 + 10 * (stock - 1)) = zai
Sheet3.Cells(count, 3 + 10 * (stock - 1)) = sn
Sheet3.Cells(count, 5 + 10 * (stock - 1)) = b01
Sheet3.Cells(count, 6 + 10 * (stock - 1)) = b02
Sheet3.Cells(count, 7 + 10 * (stock - 1)) = b03
Sheet3.Cells(count, 8 + 10 * (stock - 1)) = b04
Sheet3.Cells(count, 9 + 10 * (stock - 1)) = b05
End If
gamma = gamma + 0.01
count = count + 1
Loop
stock = stock + 1
Loop
End Sub
```

Excel VBA Coding for pseudo relaxed Kullback-Leibler universal portfolio

```
Sub void()
Dim alpha, beta, gamma As Double
Dim b01, b02, b03, b04, b05 As Double
Dim x11, x12, x13, x14, x15 As Double
Dim x21, x22, x23, x24, x25 As Double
Dim bxn As Double
Dim c1, c2, c3, c4, c5 As Double
Dim d1, d2, d3, d4, d5 As Double
Dim m1, m2, m3, m4, m5 As Double
Dim bnxn As Double
Dim sn As Double
Dim stock As Integer
Dim bnxc As Double
Dim count As Integer
Dim sntry As Double
Sheet3.Cells.Clear
stock = 1
alpha = 10
beta = 6
Do While stock < 6
count = 1
gamma = 0
sntry = 1
Do While gamma < 5
sn = 1
b01 = 0.2
b02 = 0.2
b03 = 0.2
b04 = 0.2
b05 = 0.2
For i = 1 To 1500
x11 = Sheet2.Cells(i + 1, 1 + 6 * (stock - 1))
x12 = Sheet2.Cells(i + 1, 2 + 6 * (stock - 1))
x13 = Sheet2.Cells(i + 1, 3 + 6 * (stock - 1))
x14 = Sheet2.Cells(i + 1, 4 + 6 * (stock - 1))
x15 = Sheet2.Cells(i + 1, 5 + 6 * (stock - 1))
x21 = Sheet2.Cells(i + 2, 1 + 6 * (stock - 1))
x22 = Sheet2.Cells(i + 2, 2 + 6 * (stock - 1))
x23 = Sheet2.Cells(i + 2, 3 + 6 * (stock - 1))
x24 = Sheet2.Cells(i + 2, 4 + 6 * (stock - 1))
x25 = Sheet2.Cells(i + 2, 5 + 6 * (stock - 1))
bxn = x11 * b01 + x12 * b02 + x13 * b03 +_
x14 * b04 + x15 * b05
c1 = ( beta * bxn ) - ( gamma - x11 )
c2 = ( beta * bxn ) - ( gamma - x12 )
c3 = ( beta * bxn ) - ( gamma - x13 )
c4 = ( beta * bxn ) - ( gamma - x14 )
c5 = ( beta * bxn ) - ( gamma - x15 )
d1 = exp ( c1 / ( beta * bnx ) )
d2 = exp ( c2 / ( beta * bnx ) )
d3 = exp ( c3 / ( beta * bnx ) )
d4 = exp ( c4 / ( beta * bnx ) )
d5 = exp ( c5 / ( beta * bnx ) )
m1 = ( d1 ^ ( 1 / ( alpha - 1 ) ) ) * b01
m2 = ( d2 ^ ( 1 / ( alpha - 1 ) ) ) * b02
m3 = ( d3 ^ ( 1 / ( alpha - 1 ) ) ) * b03
m4 = ( d4 ^ ( 1 / ( alpha - 1 ) ) ) * b04
m5 = ( d5 ^ ( 1 / ( alpha - 1 ) ) ) * b05
b01 = m1 / ( m1 + m2 + m3 + m4 + m5 )
b02 = m2 / ( m1 + m2 + m3 + m4 + m5 )
b03 = m3 / ( m1 + m2 + m3 + m4 + m5 )
b04 = m4 / ( m1 + m2 + m3 + m4 + m5 )
b05 = m5 / ( m1 + m2 + m3 + m4 + m5 )
If b01 < 0 Or b02 < 0 Or b03 < 0 Or b04 < 0 Or b05 < 0 Then
Exit For
Else
bnxn = b01 * x21 + b02 * x22 + b03 * x23 + b04 * x24 +_
b05 * x25
sn = sn * bnxn
```

```
End If
Next i
If b01 > 0 And b02 > 0 And b03 > 0 And_
b04 > 0 And b05 > 0 And sn > sntry Then
sntry = sn
Sheet3.Cells(count, 1 + 10 * (stock - 1)) = alpha
Sheet3.Cells(count, 2 + 10 * (stock - 1)) = zai
Sheet3.Cells(count, 3 + 10 * (stock - 1)) = sn
Sheet3.Cells(count, 5 + 10 * (stock - 1)) = b01
Sheet3.Cells(count, 6 + 10 * (stock - 1)) = b02
Sheet3.Cells(count, 7 + 10 * (stock - 1)) = b03
Sheet3.Cells(count, 8 + 10 * (stock - 1)) = b04
Sheet3.Cells(count, 9 + 10 * (stock - 1)) = b05
End If
gamma = gamma + 0.01
count = count + 1
Loop
stock = stock + 1
Loop
End Sub
```

Excel VBA Coding for Reverse Helmbold universal portfolio

```
Sub void()
Dim beta As Double
Dim b01, b02, b03, b04, b05 As Double
Dim x11, x12, x13, x14, x15 As Double
Dim x21, x22, x23, x24, x25 As Double
Dim bxn As Double
Dim c1, c2, c3, c4, c5 As Double
Dim bnxn As Double
Dim sn As Double
Dim stock As Integer
Dim bnxc As Double
Dim count As Integer
Dim sntry As Double
Sheet3.Cells.Clear
stock = 1
Do While stock < 6
count = 1
gamma = 0
beta = 6
sntry = 1
Do While beta < 1000
sn = 1
b01 = 0.2
b02 = 0.2
b03 = 0.2
b04 = 0.2
b05 = 0.2
For i = 1 To 1500
x11 = Sheet2.Cells(i + 1, 1 + 6 * (stock - 1))
x12 = Sheet2.Cells(i + 1, 2 + 6 * (stock - 1))
x13 = Sheet2.Cells(i + 1, 3 + 6 * (stock - 1))
x14 = Sheet2.Cells(i + 1, 4 + 6 * (stock - 1))
x15 = Sheet2.Cells(i + 1, 5 + 6 * (stock - 1))
x21 = Sheet2.Cells(i + 2, 1 + 6 * (stock - 1))
x22 = Sheet2.Cells(i + 2, 2 + 6 * (stock - 1))
x23 = Sheet2.Cells(i + 2, 3 + 6 * (stock - 1))
x24 = Sheet2.Cells(i + 2, 4 + 6 * (stock - 1))
x25 = Sheet2.Cells(i + 2, 5 + 6 * (stock - 1))
bxn = x11 * b01 + x12 * b02 + x13 * b03 +_
x14 * b04 + x15 * b05
c1 = b01 * 1 / ( ( beta * bnx ) - x11 )
c2 = b02 * 1 / ( ( beta * bnx ) - x12 )
c3 = b03 * 1 / ( ( beta * bnx ) - x13 )
c4 = b04 * 1 / ( ( beta * bnx ) - x14 )
c5 = b05 * 1 / ( ( beta * bnx ) - x15 )
b01 = c1 / ( c1 + c2 + c3 + c4 + c5 )
b02 = c2 / ( c1 + c2 + c3 + c4 + c5 )
b03 = c3 / ( c1 + c2 + c3 + c4 + c5 )
```

```
b04 = c4 / ( c1 + c2 + c3 + c4 + c5 )
b05 = c5 / ( c1 + c2 + c3 + c4 + c5 )
If b01 < 0 Or b02 < 0 Or b03 < 0 Or b04 < 0 Or b05 < 0 Then
Exit For
Else
bnxn = b01 * x21 + b02 * x22 + b03 * x23 + b04 * x24 +_
b05 * x25
sn = sn * bnxn
End If
Next i
If b01 > 0 And b02 > 0 And b03 > 0 And_
b04 > 0 And b05 > 0 And sn > sntry Then
sntry = sn
Sheet3.Cells(count, 1 + 10 * (stock - 1)) = alpha
Sheet3.Cells(count, 2 + 10 * (stock - 1)) = zai
Sheet3.Cells(count, 3 + 10 * (stock - 1)) = sn
Sheet3.Cells(count, 5 + 10 * (stock - 1)) = b01
Sheet3.Cells(count, 6 + 10 * (stock - 1)) = b02
Sheet3.Cells(count, 7 + 10 * (stock - 1)) = b03
Sheet3.Cells(count, 8 + 10 * (stock - 1)) = b04
Sheet3.Cells(count, 9 + 10 * (stock - 1)) = b05
End If
beta = beta + 0.01
count = count + 1
Loop
stock = stock + 1
Loop
End Sub
```

Excel VBA Coding for Type 1 Helmbold universal portfolio

```
Sub void()
Dim eta As Double
Dim b01, b02, b03, b04, b05 As Double
Dim x11, x12, x13, x14, x15 As Double
Dim x21, x22, x23, x24, x25 As Double
Dim bxn As Double
Dim c1, c2, c3, c4, c5 As Double
Dim bnxn As Double
Dim sn As Double
Dim stock As Integer
Dim bnxc As Double
Dim count As Integer
Dim sntry As Double
Sheet3.Cells.Clear
stock = 1
Do While stock < 6
count = 1
gamma = 0
sntry = 1
Do While gamma < 5
sn = 1
b01 = 0.2
b02 = 0.2
b03 = 0.2
b04 = 0.2
b05 = 0.2
For i = 1 To 2500
x11 = Sheet2.Cells(i + 1, 1 + 6 * (stock - 1))
x12 = Sheet2.Cells(i + 1, 2 + 6 * (stock - 1))
x13 = Sheet2.Cells(i + 1, 3 + 6 * (stock - 1))
x14 = Sheet2.Cells(i + 1, 4 + 6 * (stock - 1))
x15 = Sheet2.Cells(i + 1, 5 + 6 * (stock - 1))
x21 = Sheet2.Cells(i + 2, 1 + 6 * (stock - 1))
x22 = Sheet2.Cells(i + 2, 2 + 6 * (stock - 1))
x23 = Sheet2.Cells(i + 2, 3 + 6 * (stock - 1))
x24 = Sheet2.Cells(i + 2, 4 + 6 * (stock - 1))
x25 = Sheet2.Cells(i + 2, 5 + 6 * (stock - 1))
bxn = x11 * b01 + x12 * b02 + x13 * b03 +_
x14 * b04 + x15 * b05
c1 = b01 * exp ( eta * x11 / bnx )
```

```
c2 = b02 * exp ( eta * x12 / bnx )
c3 = b03 * exp ( eta * x13 / bnx )
c4 = b04 * exp ( eta * x14 / bnx )
c5 = b05 * exp ( eta * x15 / bnx )
b01 = c1 / ( c1 + c2 + c3 + c4 + c5 )
b02 = c2 / ( c1 + c2 + c3 + c4 + c5 )
b03 = c3 / ( c1 + c2 + c3 + c4 + c5 )
b04 = c4 / ( c1 + c2 + c3 + c4 + c5 )
b05 = c5 / ( c1 + c2 + c3 + c4 + c5 )
If b01 < 0 Or b02 < 0 Or b03 < 0 Or b04 < 0 Or b05 < 0 Then
Exit For
Else
bnxn = b01 * x21 + b02 * x22 + b03 * x23 + b04 * x24 +_
b05 * x25
sn = sn * bnxn
End If
Next i
If b01 > 0 And b02 > 0 And b03 > 0 And_
b04 > 0 And b05 > 0 And sn > sntry Then
sntry = sn
Sheet3.Cells(count, 1 + 10 * (stock - 1)) = eta
Sheet3.Cells(count, 3 + 10 * (stock - 1)) = sn
Sheet3.Cells(count, 5 + 10 * (stock - 1)) = b01
Sheet3.Cells(count, 6 + 10 * (stock - 1)) = b02
Sheet3.Cells(count, 7 + 10 * (stock - 1)) = b03
Sheet3.Cells(count, 8 + 10 * (stock - 1)) = b04
Sheet3.Cells(count, 9 + 10 * (stock - 1)) = b05
End If
eta = eta + 0.01
count = count + 1
Loop
stock = stock + 1
Loop
End Sub
```

Excel VBA Coding for Type 2 Helmbold universal portfolio

```
Sub void()
Dim eta, gamma As Double
Dim b01, b02, b03, b04, b05 As Double
Dim x11, x12, x13, x14, x15 As Double
Dim x21, x22, x23, x24, x25 As Double
Dim bxn As Double
Dim c1, c2, c3, c4, c5 As Double
Dim bnxn As Double
Dim sn As Double
Dim stock As Integer
Dim bnxc As Double
Dim count As Integer
Dim sntry As Double
Sheet3.Cells.Clear
stock = 1
Do While stock < 6
count = 1
gamma = 0
sntry = 1
Do While gamma < 5
sn = 1
b01 = 0.2
b02 = 0.2
b03 = 0.2
b04 = 0.2
b05 = 0.2
eta = -10
do while eta < 10
For i = 1 To 2500
x11 = Sheet2.Cells(i + 1, 1 + 6 * (stock - 1))
x12 = Sheet2.Cells(i + 1, 2 + 6 * (stock - 1))
x13 = Sheet2.Cells(i + 1, 3 + 6 * (stock - 1))
x14 = Sheet2.Cells(i + 1, 4 + 6 * (stock - 1))
x15 = Sheet2.Cells(i + 1, 5 + 6 * (stock - 1))
```

```
x21 = Sheet2.Cells(i + 2, 1 + 6 * (stock - 1))
x22 = Sheet2.Cells(i + 2, 2 + 6 * (stock - 1))
x23 = Sheet2.Cells(i + 2, 3 + 6 * (stock - 1))
x24 = Sheet2.Cells(i + 2, 4 + 6 * (stock - 1))
x25 = Sheet2.Cells(i + 2, 5 + 6 * (stock - 1))
bxn = x11 * b01 + x12 * b02 + x13 * b03 +_
x14 * b04 + x15 * b05
c1 = b01 * exp ( eta * x11 / bnx )
c2 = b02 * exp ( eta * x12 / bnx )
c3 = b03 * exp ( eta * x13 / bnx )
c4 = b04 * exp ( eta * x14 / bnx )
c5 = b05 * exp ( eta * x15 / bnx )
b01 = c1 / ( c1 + c2 + c3 + c4 + c5 )
b02 = c2 / ( c1 + c2 + c3 + c4 + c5 )
b03 = c3 / ( c1 + c2 + c3 + c4 + c5 )
b04 = c4 / ( c1 + c2 + c3 + c4 + c5 )
b05 = c5 / ( c1 + c2 + c3 + c4 + c5 )
If b01 < 0 Or b02 < 0 Or b03 < 0 Or b04 < 0 Or b05 < 0 Then
Exit For
Else
bnxn = b01 * x21 + b02 * x22 + b03 * x23 + b04 * x24 +_
b05 * x25
sn = sn * bnxn
End If
Next i
If b01 > 0 And b02 > 0 And b03 > 0 And_
b04 > 0 And b05 > 0 And sn > sntry Then
sntry = sn
Sheet3.Cells(count, 1 + 10 * (stock - 1)) = eta
Sheet3.Cells(count, 2 + 10 * (stock - 1)) = gamma
Sheet3.Cells(count, 3 + 10 * (stock - 1)) = sn
Sheet3.Cells(count, 5 + 10 * (stock - 1)) = b01
Sheet3.Cells(count, 6 + 10 * (stock - 1)) = b02
Sheet3.Cells(count, 7 + 10 * (stock - 1)) = b03
Sheet3.Cells(count, 8 + 10 * (stock - 1)) = b04
Sheet3.Cells(count, 9 + 10 * (stock - 1)) = b05
End If
eta = eta + 0.01
count = count + 1
Loop
gamma = gamma + 0.1
Loop
stock = stock + 1
Loop
End Sub
```

Excel VBA Coding for Bregman universal porttfolio

```
Sub void()
Dim beta, gamma As Double
Dim b01, b02, b03, b04, b05 As Double
Dim x11, x12, x13, x14, x15 As Double
Dim x21, x22, x23, x24, x25 As Double
Dim bxn As Double
Dim c1, c2, c3, c4, c5 As Double
Dim bnxn As Double
Dim sn As Double
Dim stock As Integer
Dim bnxc As Double
Dim count As Integer
Dim sntry As Double
Sheet3.Cells.Clear
stock = 1
m = 5
Do While stock < 6
count = 1
eta = -10
sntry = 1
Do while beta < 10
```

```
sn = 1
b01 = 0.2
b02 = 0.2
b03 = 0.2
b04 = 0.2
b05 = 0.2
gamma = 0
Do While gamma < 5
For i = 1 To 2500
x11 = Sheet2.Cells(i + 1, 1 + 6 * (stock - 1))
x12 = Sheet2.Cells(i + 1, 2 + 6 * (stock - 1))
x13 = Sheet2.Cells(i + 1, 3 + 6 * (stock - 1))
x14 = Sheet2.Cells(i + 1, 4 + 6 * (stock - 1))
x15 = Sheet2.Cells(i + 1, 5 + 6 * (stock - 1))
x21 = Sheet2.Cells(i + 2, 1 + 6 * (stock - 1))
x22 = Sheet2.Cells(i + 2, 2 + 6 * (stock - 1))
x23 = Sheet2.Cells(i + 2, 3 + 6 * (stock - 1))
x24 = Sheet2.Cells(i + 2, 4 + 6 * (stock - 1))
x25 = Sheet2.Cells(i + 2, 5 + 6 * (stock - 1))
bxn = x11 * b01 + x12 * b02 + x13 * b03 +_
x14 * b04 + x15 * b05
c1 = ( b01 + beta ) * exp ( gamma * x11 / bnx )
c2 = ( b02 + beta ) * exp ( gamma * x12 / bnx )
c3 = ( b03 + beta ) * exp ( gamma * x13 / bnx )
c4 = ( b04 + beta ) * exp ( gamma * x14 / bnx )
c5 = ( b05 + beta ) * exp ( gamma * x15 / bnx )
b01 = ( 1 + beta * m ) * c1 / ( c1 + c2 + c3 + c4 + c5 )
b02 = ( 1 + beta * m ) * c2 / ( c1 + c2 + c3 + c4 + c5 )
b03 = ( 1 + beta * m ) * c3 / ( c1 + c2 + c3 + c4 + c5 )
b04 = ( 1 + beta * m ) * c4 / ( c1 + c2 + c3 + c4 + c5 )
b05 = ( 1 + beta * m ) * c5 / ( c1 + c2 + c3 + c4 + c5 )
If b01 < 0 Or b02 < 0 Or b03 < 0 Or b04 < 0 Or b05 < 0 Then
Exit For
Else
bnxn = b01 * x21 + b02 * x22 + b03 * x23 + b04 * x24 +_
b05 * x25
sn = sn * bnxn
End If
Next i
If b01 > 0 And b02 > 0 And b03 > 0 And_
b04 > 0 And b05 > 0 And sn > sntry Then
sntry = sn
Sheet3.Cells(count, 1 + 10 * (stock - 1)) = beta
Sheet3.Cells(count, 2 + 10 * (stock - 1)) = gamma
Sheet3.Cells(count, 3 + 10 * (stock - 1)) = sn
Sheet3.Cells(count, 5 + 10 * (stock - 1)) = b01
Sheet3.Cells(count, 6 + 10 * (stock - 1)) = b02
Sheet3.Cells(count, 7 + 10 * (stock - 1)) = b03
Sheet3.Cells(count, 8 + 10 * (stock - 1)) = b04
Sheet3.Cells(count, 9 + 10 * (stock - 1)) = b05
End If
gamma = gamma + 0.01
count = count + 1
Loop
beta = beta + 0.0001
Loop
stock = stock + 1
Loop
End Sub
```

Excel VBA Coding for universal portfolio generated by f-disparity difference

```
Sub void()
Dim eta, xi As Double
Dim b01, b02, b03, b04, b05 As Double
Dim x11, x12, x13, x14, x15 As Double
```

```
Dim x21, x22, x23, x24, x25 As Double
Dim bxn As Double
Dim v1, v2, v3, v4, v5 As Double
Dim c1, c2, c3, c4, c5 As Double
Dim bnxn As Double
Dim sn As Double
Dim stock As Integer
Dim bnxc As Double
Dim count As Integer
Dim sntry As Double
Sheet3.Cells.Clear
stock = 1
m = 5
Do While stock < 6
count = 1
eta = -10
sntry = 1
Do while eta < 10
sn = 1
b01 = 0.2
b02 = 0.2
b03 = 0.2
b04 = 0.2
b05 = 0.2
xi = 0
Do While xi < 5
For i = 1 To 2500
x11 = Sheet2.Cells(i + 1, 1 + 6 * (stock - 1))
x12 = Sheet2.Cells(i + 1, 2 + 6 * (stock - 1))
x13 = Sheet2.Cells(i + 1, 3 + 6 * (stock - 1))
x14 = Sheet2.Cells(i + 1, 4 + 6 * (stock - 1))
x15 = Sheet2.Cells(i + 1, 5 + 6 * (stock - 1))
x21 = Sheet2.Cells(i + 2, 1 + 6 * (stock - 1))
x22 = Sheet2.Cells(i + 2, 2 + 6 * (stock - 1))
x23 = Sheet2.Cells(i + 2, 3 + 6 * (stock - 1))
x24 = Sheet2.Cells(i + 2, 4 + 6 * (stock - 1))
x25 = Sheet2.Cells(i + 2, 5 + 6 * (stock - 1))
bxn = x11 * b01 + x12 * b02 + x13 * b03 +_
x14 * b04 + x15 * b05
v1 = ( eta * x11 )/ ( bxn ) + xi
v2 = ( eta * x12 )/ ( bxn ) + xi
v3 = ( eta * x13 )/ ( bxn ) + xi
v4 = ( eta * x14 )/ ( bxn ) + xi
v5 = ( eta * x15 )/ ( bxn ) + xi
c1 = b01 * ( 2 / v1 - 3 ) ^ ( 0.5 )
c2 = b02 * ( 2 / v2 - 3 ) ^ ( 0.5 )
c3 = b03 * ( 2 / v3 - 3 ) ^ ( 0.5 )
c4 = b04 * ( 2 / v4 - 3 ) ^ ( 0.5 )
c5 = b05 * ( 2 / v5 - 3 ) ^ ( 0.5 )
b01 = c1 / ( c1 + c2 + c3 + c4 + c5 )
b02 = c2 / ( c1 + c2 + c3 + c4 + c5 )
b03 = c3 / ( c1 + c2 + c3 + c4 + c5 )
b04 = c4 / ( c1 + c2 + c3 + c4 + c5 )
b05 = c5 / ( c1 + c2 + c3 + c4 + c5 )
If b01 < 0 Or b02 < 0 Or b03 < 0 Or b04 < 0 Or b05 < 0 Then
Exit For
Else
bnxn = b01 * x21 + b02 * x22 + b03 * x23 + b04 * x24 +_
b05 * x25
sn = sn * bnxn
End If
Next i
If b01 > 0 And b02 > 0 And b03 > 0 And_
b04 > 0 And b05 > 0 And sn > sntry Then
sntry = sn
Sheet3.Cells(count, 1 + 10 * (stock - 1)) = eta
Sheet3.Cells(count, 2 + 10 * (stock - 1)) = xi
Sheet3.Cells(count, 3 + 10 * (stock - 1)) = sn
Sheet3.Cells(count, 5 + 10 * (stock - 1)) = b01
Sheet3.Cells(count, 6 + 10 * (stock - 1)) = b02
Sheet3.Cells(count, 7 + 10 * (stock - 1)) = b03
Sheet3.Cells(count, 8 + 10 * (stock - 1)) = b04
Sheet3.Cells(count, 9 + 10 * (stock - 1)) = b05
End If
gamma = gamma + 0.01
count = count + 1
```

```
Loop
beta = beta + 0.0001
Loop
stock = stock + 1
Loop
End Sub
```

Excel VBA Coding for universal portfolio generated by rational functions (4.28)

```
Sub void()
Dim eta, beta, xi As Double
Dim b01, b02, b03, b04, b05 As Double
Dim x11, x12, x13, x14, x15 As Double
Dim x21, x22, x23, x24, x25 As Double
dim c, d1, d2 as Double
Dim bxn As Double
Dim v1, v2, v3, v4, v5 As Double
Dim c1, c2, c3, c4, c5 As Double
Dim bnxn As Double
Dim sn As Double
Dim stock As Integer
Dim bnxc As Double
Dim count As Integer
Dim sntry As Double
Sheet3.Cells.Clear
stock = 1
Do While stock < 6
eta = -10
Do while eta <10
count = 1
sntry = 1
beta = 1
c1 = -5
Do while c1 < 10
c2 = -5
Do while c2 < 10
sn = 1
b01 = 0.2
b02 = 0.2
b03 = 0.2
b04 = 0.2
b05 = 0.2
xi = -5
Do While xi < 5
For i = 1 To 2500
x11 = Sheet2.Cells(i + 1, 1 + 6 * (stock - 1))
x12 = Sheet2.Cells(i + 1, 2 + 6 * (stock - 1))
x13 = Sheet2.Cells(i + 1, 3 + 6 * (stock - 1))
x14 = Sheet2.Cells(i + 1, 4 + 6 * (stock - 1))
x15 = Sheet2.Cells(i + 1, 5 + 6 * (stock - 1))
x21 = Sheet2.Cells(i + 2, 1 + 6 * (stock - 1))
x22 = Sheet2.Cells(i + 2, 2 + 6 * (stock - 1))
x23 = Sheet2.Cells(i + 2, 3 + 6 * (stock - 1))
x24 = Sheet2.Cells(i + 2, 4 + 6 * (stock - 1))
x25 = Sheet2.Cells(i + 2, 5 + 6 * (stock - 1))
bxn = x11 * b01 + x12 * b02 + x13 * b03 +_
x14 * b04 + x15 * b05
v1 = ( eta * x11 )/ ( bxn ) + xi
v2 = ( eta * x12 )/ ( bxn ) + xi
v3 = ( eta * x13 )/ ( bxn ) + xi
v4 = ( eta * x14 )/ ( bxn ) + xi
v5 = ( eta * x15 )/ ( bxn ) + xi
c1 = b01 * ( ( 1 / ( 2 * d1 ) * ( v1 + sqrt( v1 * v1 -_
4 * d1 * d2 )))^(1/beta)) - c
c2 = b02 * ( ( 1 / ( 2 * d2 ) * ( v2 + sqrt( v2 * v2 -_
4 * d1 * d2 )))^(1/beta)) - c
c3 = b03 * ( ( 1 / ( 2 * d3 ) * ( v3 + sqrt( v3 * v3 -_
4 * d1 * d2 )))^(1/beta)) - c
c4 = b04 * ( ( 1 / ( 2 * d4 ) * ( v4 + sqrt( v4 * v4 -_
```

```
4 * d1 * d2 )))^(1/beta)) - c
c5 = b05 * ( ( 1 / ( 2 * d5 ) * ( v5 + sqrt( v5 * v5 -_
4 * d1 * d2 )))^(1/beta)) - c
c2 = b02 * ( 2 / v2 - 3 ) ^ ( 0.5 )
c3 = b03 * ( 2 / v3 - 3 ) ^ ( 0.5 )
c4 = b04 * ( 2 / v4 - 3 ) ^ ( 0.5 )
c5 = b05 * ( 2 / v5 - 3 ) ^ ( 0.5 )
b01 = c1 / ( c1 + c2 + c3 + c4 + c5 )
b02 = c2 / ( c1 + c2 + c3 + c4 + c5 )
b03 = c3 / ( c1 + c2 + c3 + c4 + c5 )
b04 = c4 / ( c1 + c2 + c3 + c4 + c5 )
b05 = c5 / ( c1 + c2 + c3 + c4 + c5 )
If b01 < 0 Or b02 < 0 Or b03 < 0 Or b04 < 0 Or b05 < 0 Then
Exit For
Else
bnxn = b01 * x21 + b02 * x22 + b03 * x23 + b04 * x24 +_
b05 * x25
sn = sn * bnxn
End If
Next i
If b01 > 0 And b02 > 0 And b03 > 0 And_
b04 > 0 And b05 > 0 And sn > sntry Then
sntry = sn
Sheet3.Cells(count, 1 + 11 * (stock - 1)) = eta
Sheet3.Cells(count, 2 + 11 * (stock - 1)) = c1
Sheet3.Cells(count, 3 + 11 * (stock - 1)) = c1
Sheet3.Cells(count, 4 + 11 * (stock - 1)) = xi
Sheet3.Cells(count, 5 + 11 * (stock - 1)) = sn
Sheet3.Cells(count, 6 + 11 * (stock - 1)) = b01
Sheet3.Cells(count, 7 + 11 * (stock - 1)) = b02
Sheet3.Cells(count, 8 + 11 * (stock - 1)) = b03
Sheet3.Cells(count, 9 + 11 * (stock - 1)) = b04
Sheet3.Cells(count, 10 + 11 * (stock - 1)) = b05
End If
xi = xi + 0.01
count = count + 1
Loop
c2 = c2 + 0.01
Loop
c1 = c1 + 0.01
Loop
eta = eta + 0.01
Loop
stock = stock + 1
Loop
End Sub
```

Excel VBA Coding for universal portfolio generated by rational functions (4.30)

```
Sub void()
Dim eta, beta, xi As Double
Dim b01, b02, b03, b04, b05 As Double
Dim x11, x12, x13, x14, x15 As Double
Dim x21, x22, x23, x24, x25 As Double
dim c1, c2, d1, d2 as Double
Dim bxn As Double
Dim v1, v2, v3, v4, v5 As Double
Dim m1, m2, m3, m4, m5 As Double
Dim bnxn As Double
Dim sn As Double
Dim stock As Integer
Dim bnxc As Double
Dim count As Integer
Dim sntry As Double
Sheet3.Cells.Clear
stock = 1
Do While stock < 6
eta = -10
```

```
Do while eta <10
count = 1
sntry = 1
beta = 0
Do while beta < 10
sn = 1
b01 = 0.2
b02 = 0.2
b03 = 0.2
b04 = 0.2
b05 = 0.2
xi = -5
c=0.1
Do While xi < 5
For i = 1 To 2500
x11 = Sheet2.Cells(i + 1, 1 + 6 * (stock - 1))
x12 = Sheet2.Cells(i + 1, 2 + 6 * (stock - 1))
x13 = Sheet2.Cells(i + 1, 3 + 6 * (stock - 1))
x14 = Sheet2.Cells(i + 1, 4 + 6 * (stock - 1))
x15 = Sheet2.Cells(i + 1, 5 + 6 * (stock - 1))
x21 = Sheet2.Cells(i + 2, 1 + 6 * (stock - 1))
x22 = Sheet2.Cells(i + 2, 2 + 6 * (stock - 1))
x23 = Sheet2.Cells(i + 2, 3 + 6 * (stock - 1))
x24 = Sheet2.Cells(i + 2, 4 + 6 * (stock - 1))
x25 = Sheet2.Cells(i + 2, 5 + 6 * (stock - 1))
bxn = x11 * b01 + x12 * b02 + x13 * b03 +_
x14 * b04 + x15 * b05
v1 = ( eta * x11 )/ ( bxn ) + xi
v2 = ( eta * x12 )/ ( bxn ) + xi
v3 = ( eta * x13 )/ ( bxn ) + xi
v4 = ( eta * x14 )/ ( bxn ) + xi
v5 = ( eta * x15 )/ ( bxn ) + xi
c1 = b01 * ( ( 1 / ( 2 * d1 ) * ( v1 + sqrt( v1 * v1 -_
4 * d1 * d2 )))^(1/beta)) - c
c2 = b02 * ( ( 1 / ( 2 * d2 ) * ( v2 + sqrt( v2 * v2 -_
4 * d1 * d2 )))^(1/beta)) - c
c3 = b03 * ( ( 1 / ( 2 * d3 ) * ( v3 + sqrt( v3 * v3 -_
4 * d1 * d2 )))^(1/beta)) - c
c4 = b04 * ( ( 1 / ( 2 * d4 ) * ( v4 + sqrt( v4 * v4 -_
4 * d1 * d2 )))^(1/beta)) - c
c5 = b05 * ( ( 1 / ( 2 * d5 ) * ( v5 + sqrt( v5 * v5 -_
4 * d1 * d2 )))^(1/beta)) - c
c2 = b02 * ( 2 / v2 - 3 ) ^ ( 0.5 )
c3 = b03 * ( 2 / v3 - 3 ) ^ ( 0.5 )
c4 = b04 * ( 2 / v4 - 3 ) ^ ( 0.5 )
c5 = b05 * ( 2 / v5 - 3 ) ^ ( 0.5 )
b01 = c1 / ( c1 + c2 + c3 + c4 + c5 )
b02 = c2 / ( c1 + c2 + c3 + c4 + c5 )
b03 = c3 / ( c1 + c2 + c3 + c4 + c5 )
b04 = c4 / ( c1 + c2 + c3 + c4 + c5 )
b05 = c5 / ( c1 + c2 + c3 + c4 + c5 )
If b01 < 0 Or b02 < 0 Or b03 < 0 Or b04 < 0 Or b05 < 0 Then
Exit For
Else
bnxn = b01 * x21 + b02 * x22 + b03 * x23 + b04 * x24 +_
b05 * x25
sn = sn * bnxn
End If
Next i
If b01 > 0 And b02 > 0 And b03 > 0 And_
b04 > 0 And b05 > 0 And sn > sntry Then
sntry = sn
Sheet3.Cells(count, 1 + 10 * (stock - 1)) = eta
Sheet3.Cells(count, 2 + 10 * (stock - 1)) = beta
Sheet3.Cells(count, 3 + 10 * (stock - 1)) = xi
Sheet3.Cells(count, 4 + 10 * (stock - 1)) = sn
Sheet3.Cells(count, 5 + 10 * (stock - 1)) = b01
Sheet3.Cells(count, 6 + 10 * (stock - 1)) = b02
Sheet3.Cells(count, 7 + 10 * (stock - 1)) = b03
Sheet3.Cells(count, 8 + 10 * (stock - 1)) = b04
Sheet3.Cells(count, 9 + 10 * (stock - 1)) = b05
End If
xi = xi + 0.01
```

```
count = count + 1
Loop
beta = beta + 0.01
Loop
eta = eta + 0.01
Loop
stock = stock + 1
Loop
End Sub
```