ANTI-MAGIC LABELING ON A CLASS OF SPARSE GRAPHS

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By

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ABSTRACT

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Tai Yu Bin

In 1990, Hartsfield and Ringel first introduced the anti-magic labeling and conjectured that every graph other than the complete graph with 2 vertices has an anti-magic labeling. This conjecture has been verified for regular graphs and some classes of trees. In this dissertation we shall prove the anti-magicness of a class of sparse graphs.

The thesis begins with a survey on some graph labelings, including anti-magic labeling. The thesis continues by introducing graph decompositions and some applications of graph labelings. In the next chapter, we proved that multi-bridge graphs are anti-magic.

The thesis is concluded with a discussion on the anti-magicness of families of sparse graphs obtained by overlapping the multi-bridge graph with itself or with some extended friendship graph. The proof of the anti-magicness of these families of sparse graphs is left as an open problem for future research.

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Date: **_13/2/2023_______**

SUBMISSION OF DISSERTATION

It is hereby certified that **TAI YU BIN** (ID No: **20UEM00891**) has completed this dissertation entitled "ANTI-MAGIC LABELING ON A CLASS OF SPARSE GRAPHS" under the supervision of PROF. DR. CHIA GEK LING (Supervisor) from the Department of MATHEMATICAL AND ACTUARIAL SCIENCES, LEE KONG CHIAN FACULTY OF ENGINEERING AND SCIENCE, and ASSIST. PROF. DR. ONG POH HWA (Co-Supervisor) from the Department of MATHEMATICAL AND ACTUARIAL SCIENCES, LEE KONG CHIAN FACULTY OF ENGINEERING AND SCIENCE.

I understand that University will upload softcopy of my dissertation in pdf format into UTAR Institutional Repository, which may be made accessible to UTAR community and public.

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(*Tai Yu Bin*)

APPROVAL SHEET

This dissertation/thesis entitled "**ANTI-MAGIC LABELING ON A CLASS OF SPARSE GRAPHS**" was prepared by TAI YU BIN and submitted as partial fulfillment of the requirements for the degree of Master of Science at Universiti Tunku Abdul Rahman.

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DECLARATION

I, Tai Yu Bin hereby declare that the dissertation is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UTAR or other institutions.

(TAI YU BIN)

Date : 13/2/2023

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

1.1 Introduction

The study of graph theory originated from Euler's work on the Königsberg Bridge Problem in 1735. Nowadays graph theory is a topic of mathematics which is quite popular. Graphs are capable of representing models of relations. Therefore, graph theory has a huge range of applications in other topics of mathematics such as geometry, number theory, linear algebra and topology.

1.2 Overview on this Dissertation

As a subtopic in graph theory, graph labeling is capable of modeling numerous kinds of relations in real-life situations. Anti-magic labeling is among the most famous graph labelings. Since 1990, much efforts have been made to find the anti-magic labeling of graphs. We focus on finding anti-magic labelings of a class of sparse graphs in this dissertation.

This dissertation consists of five chapters. In Chapter 1, we give the general introduction, some basic definitions and notations that will be used in the subsequent chapters.

A survey on graph labelings and the 1-2-3 Conjecture are provided in Chapter 2. In Chapter 3, a brief introduction on graph decomposition is presented together with some applications of graph labelings.

In Chapter 4, we prove that all multi-bridge graphs admit anti-magic labeling. In Chapter 5, we investigate the anti-magicness of another class of sparse graph, denoted $G(r, s)$ which is obtained from the multi-bridge graph and the extended friendship graph. We conjecture that $G(r, s)$ is anti-magic for all natural numbers r and s.

1.3 Preliminaries and Definitions

Some definitions and notations which will be used frequently all over this dissertation are presented in this section. We shall refer to West (2001) for all notations and terminologies not explained in this dissertation.

A graph G is made up of a finite collection of vertex set $V(G)$ and a finite edge set $E(G)$, where $V(G) \neq \emptyset$. Note that the elements in $V(G)$ and $E(G)$ are *vertices* and *edges* respectively. The *order* of G, denoted $|V(G)|$, is the number of vertices of G. The *size* of G, denoted $|E(G)|$, is the number of edges of G.

Two vertices s and t of a graph $G(V, E)$ are said to be *adjacent* if there is an edge e connecting them, and the vertices s and t are then said to be incident to e. An edge joining a vertex to itself is called a *loop*. Meanwhile, *multiple edges* are edges connected to the same pair of endpoints. A graph that contains neither loops nor multiple edges is called a *simple* graph.

For any vertex t in a graph $G(V, E)$, the *neighborhood* of t, denoted by $N(t)$, is the set of all vertices of $G(V, E)$ adjacent to t. The *degree* of a vertex t in a graph G, written as $d_G(t)$ or $deg(t)$, is the number of edges incident to t. The maximum of all $d_G(t)$ in a graph G, denoted $\Delta(G)$, is the *maximum degree* of G.

A graph $G(V, E)$ is *edge-labeled* if every edge in $E(G)$ is labeled with one positive real number. For any edge-labeled graph $G(V, E)$, the vertex sum of a vertex $t \in V$, denoted $w(t)$, is the sum of labels of all edges incident to t.

A graph G is *r-regular* if $d_G(t) = r$ for any vertex t in G. Note that a 3-regular graph is also known as a *cubic* graph.

In a graph G, a *walk* of length m is a list of k edges of G arranged in the form $u_0u_1, u_1u_2, u_2u_3, ..., u_{m-1}\,u_m$. The walk is known as a *trail* if all the edges of a walk are different. The walk is called the path with m vertices, denoted P_m, if all the vertices and edges are different. An *m-cycle*, denoted C_m, is a path with m vertices and $u_1 = u_m$.

An *acyclic* graph is a graph which does not contain any cycles. A graph H where every pair of vertices are linked by a path in H is known as a *connected* graph. Otherwise, we say H is *disconnected*.

A *tree* on m vertices is a connected acyclic graph. A tree with only one vertex u having $d_G(u) \geq 2$ is called a *spider*.

A *complete* graph with m vertices, denoted K_m is a simple graph where any pair of vertices is exactly connected by one edge. A *bipartite* graph G is a graph where V(G) is decomposable into two disjoint subsets C and D in such a way that each edge of the graph connects a vertex in C to a vertex in D. That is,

$$(i) \quad V(G) = C \bigcup D, \qquad C \bigcap D = \emptyset$$

$$(ii) \quad \text{for all } cd \in E(G), \;\; c \in C, \;\; d \in D \;\; \text{or} \;\; c \in D, \;\; d \in C$$

A bipartite graph with bipartition C and D is denoted by $G(C, D)$. A bipartite graph in which each vertex in C is joined to each vertex in D by exactly one edge is called a *complete bipartite* graph. If $|C| = m$ and $|D| = n$, the complete bipartite graph is denoted by $K_{m,n}$. A *star* S_n is the graph $K_{1,n}$. The graph shown in Figure 1.1 is the *Petersen graph*. It was originated from a paper written by J. Petersen in 1898.

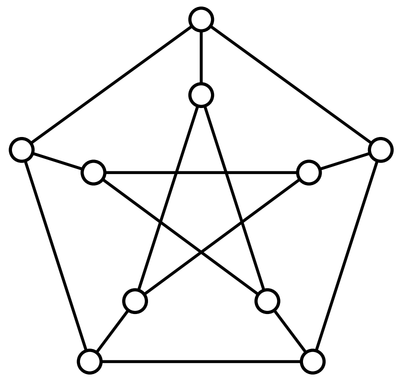


Figure 1.1: The Petersen graph

Let n and m be two integers such that $1 \leq m \leq n-1$. The *generalized Petersen* graph, denoted by $P(n, m)$ is a graph having vertex set $\{s_i, t_i : i = 0, 1, \ldots, n-1\}$ and edge-set $\{s_i s_{i+1}, s_i t_i, t_i t_{i+n} : i = 0, 1, \ldots, n-1$ with subscripts reduced modulo $n\}$. $P(5, 2)$ is the classical Petersen graph. Another example of the generalized Petersen graph $P(8, 3)$ is shown in Figure 1.2.

Suppose $G(C, D)$ is a bipartite graph. A *matching* from C to D is a set M

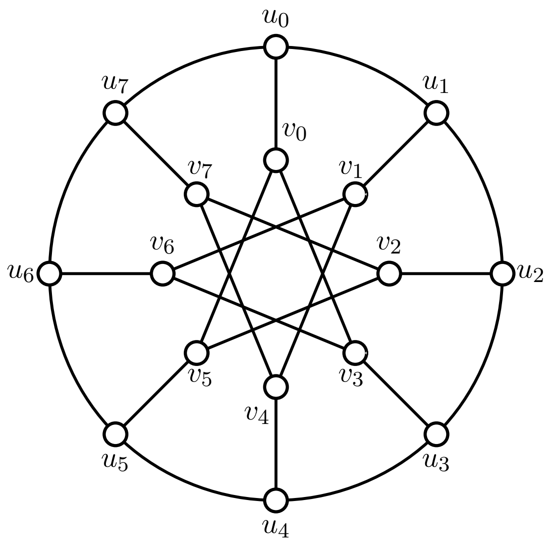


Figure 1.2: $P(8,3)$

of independent edges. We call this particular matching M a *complete matching* from C to D if every vertex in C is incident with an edge in M.

Suppose H is a graph. A graph $J(V', E')$ is known as a *subgraph* of H if $V' \subseteq V$ and $E' \subseteq E$. We say that J is an *induced subgraph* of H if for two vertices $s, t \in V'$, $(s,t) \in E'$ if and only if $(s,t) \in E$. Two graphs H and J are *isomorphic*, written as $H \cong J$, if there occurs a one-to-one correspondence between $V(H)$ and $V(J)$ which preserves adjacency.

The *join* of two graphs H_1 and H_2, denoted by $H_1 + H_2$, is the graph obtained from the disjoint union of H_1 and H_2 by connecting all vertices of H_1 to all vertices of H_2.

Let $G \,\square\, J$ denote the *Cartesian product* of the graphs G and J which is the graph having the vertex set $V(G \,\square\, J) = V(G) \times V(J)$, and the edge set

$$E(G \,\square\, J) = \{(x_p, y_q)(x_r, y_s) \mid x_p = x_r \text{ and } y_q y_s \in E(G) \text{ or } y_q = y_s \text{ and } x_p x_r \in E(J)\}.$$

The Cartesian product of P_3 with C_4 is shown in Figure 1.3.

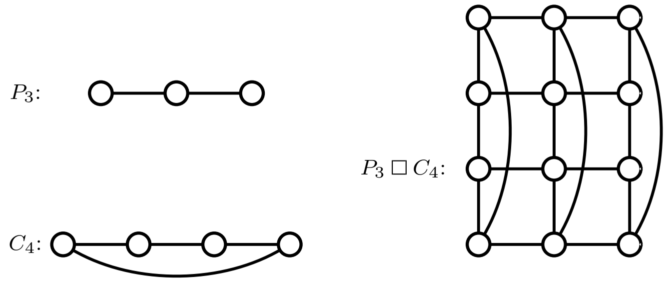


Figure 1.3: $P_3 \,\square\, C_4$

A graph J is the *subdivision* of a graph G if J is constructed from G by performing a series of subdivisions on the edges of G.

The *wheel* with m vertices W_m is obtained by connecting all vertices of C_{m-1} where $m \geq 4$ to a single vertex K_1. That particular vertex K_1 is known as the *hub* of the wheel. The edges that are incident to the hub are known as the *spokes* of the wheel, while the remaining edges are the *rims* of the wheel. A *fan* graph $K_1 + P_{m-1}$ can be composed from a wheel W_n by deleting a rim edge.

CHAPTER 2

GRAPH LABELINGS

2.1 Introduction

In this chapter, we briefly discuss about graph labelings and the 1-2-3 Conjecture. Unless otherwise stated, we suppose that the graphs mentioned throughout the dissertation are connected and simple.

For graph labelings, we outline some important graph labelings such as graceful labeling, magic labeling and anti-magic labeling. We shall also present a brief survey on the 1-2-3 Conjecture, a conjecture which is popular and having a relatively short history.

Suppose there is a graph $G(V, E)$. A *graph labeling* is a process of assigning numbers, most likely integers to $V(G)$ or $E(G)$, or both with certain constraints. We may trace back the origin of most of the graph labelings to Rosa's paper (1967), in which he introduced four valuations or labelings. The *ρ-labeling, σ-labeling, β-labeling and α-labeling.*

First of all, the definition of ρ-labeling is given as follows.

Definition 2.1. *Suppose $|E(L)| = m$. Let l be a one-to-one function that maps $V(L)$ to $\{0, 1, 2, \ldots, 2m\}$. Suppose the edge labeling function l' induced from l satisfies the following conditions.*

$$l'(st) = \begin{cases} |l(s) - l(t)|, & \textit{if} \quad |l(s) - l(t)| \leq m \\ 2m + 1 - |l(s) - l(t)|, & \textit{if} \quad |l(s) - l(t)| > m \end{cases}$$

where st is an edge of L. Then l is called a ρ-labeling of L.

An example of K_5 which admits a ρ-labeling is shown in Figure 2.1.

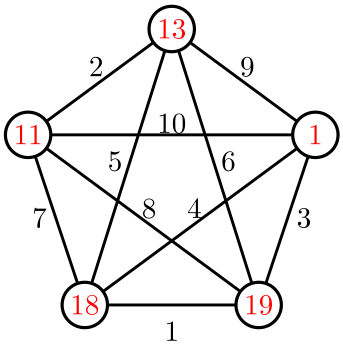


Figure 2.1: A ρ-labeling on K_5

Next, we define σ-labeling which is a stronger version of ρ-labeling.

Definition 2.2. *Suppose $|E(L)| = m$. Let l be a one-to-one function that maps $V(L)$ to $\{0, 1, 2, \ldots, 2m\}$. Suppose the edge labeling function l' induced from l satisfies the following conditions.*

$$l'(st) = |l(s) - l(t)|$$

where st is an edge of L and the edge values range over $\{1, 2, 3, \ldots, m\}$. Then l is called a σ-labeling of L.

A σ-labeling on the complete graph K_4 is shown in Figure 2.2.

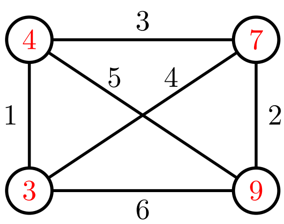


Figure 2.2: A σ-labeling on K_4

Next, we introduce β-labeling, a stronger version of σ-labeling.

Definition 2.3. *Suppose* $|E(L)| = m$. *A* σ*-labeling* l *of* L *is said to be a* β*-labeling if the codomain of* l *changes into* $\{0, 1, 2, \ldots, m\}$.

Figure 2.3 depicts an example of a tree with a β-labeling.

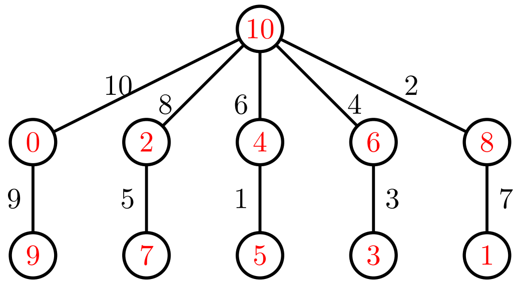


Figure 2.3: A β-labeling of a tree with 11 vertices

Finally, we have α-labeling, a stronger version of β-labeling.

Definition 2.4. *Suppose* $|E(L)| = m$. *A beta-labeling* l *of* G *is also known as an* α*-labeling of* L*, if there occurs a number* $\nu \in \mathbb{N}$ *so that either* $l(s) \leq \nu < l(t)$ *or* $l(s) > \nu \geq l(t)$ *for every edge* $st \in E(L)$.

Figure 2.4 depicts an example of a tree with an α-labeling.

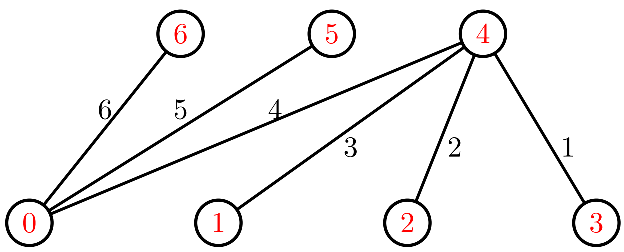


Figure 2.4: An α-labeling of a tree with 7 vertices

2.2 Graceful Labeling

Note that Golomb (1972) used the term *graceful labeling* to represent β-labeling. The study of graceful labeling originated from the Graceful Tree Conjecture, which is a conjecture proposed by Rosa in 1967.

Conjecture 2.1. (Rosa, 1967) *All trees are graceful.*

For example, Figure 2.5 illustrates a gracefully labeled complete graph with 4 vertices K_4.

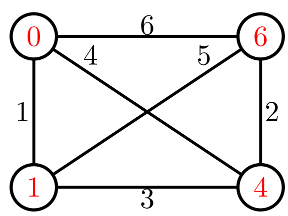


Figure 2.5: An example of a gracefully labeled K_4

Many mathematicians are interested in finding the graceful labeling of other graphs. Golomb (1972) and Simmons (1974) proved the following theorem which is related to the gracefulness of complete graphs.

Theorem 2.1. *The complete graph K_m admits a graceful labeling if and only if $m \leq 4$.*

For the gracefulness of complete bipartite graphs, both Rosa (1967) and Golomb (1972) proved the theorem below.

Theorem 2.2. *The complete bipartite graph $K_{n,m}$ admits a graceful labeling.*

For example, Figure 2.6 depicts a gracefully labeled complete bipartite graph $K_{3,3}$.

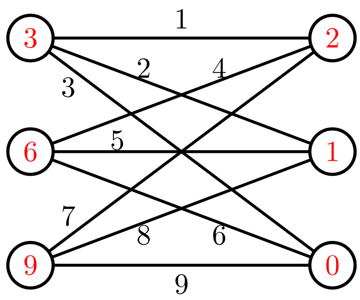


Figure 2.6: An example of a gracefully labeled $K_{3,3}$

In the same paper, Rosa (1967) determined a necessary and sufficient condition for the cycle to have a graceful labeling.

Theorem 2.3. (Rosa, 1967) *The i-cycle admits a graceful labeling if and only if $i \equiv 0$ or 3 (mod 4).*

Figure 2.7 illustrates a gracefully labeled cycle with 8 vertices.

Recall that wheels are one of the cycle-related graphs. Frucht (1979), Hoede and Kuiper (1987) have studied the gracefulness of wheels.

Theorem 2.4. *All wheels are graceful.*

Figure 2.8 depicts a gracefully labeled wheel with 6 vertices.

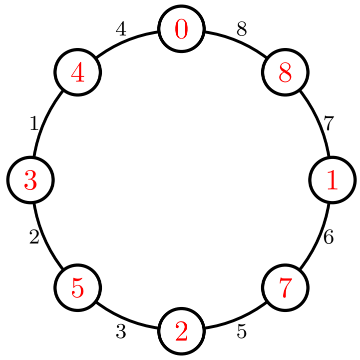


Figure 2.7: An example of a gracefully labeled C_8

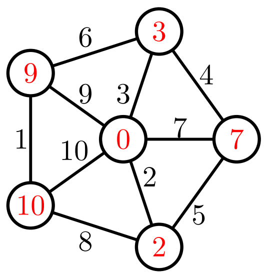


Figure 2.8: An example of a gracefully labeled W_6

In the same paper, Frucht (1979) proved the following theorem.

Theorem 2.5. (Frucht, 1979) *The Petersen graph is graceful.*

Note that Conjecture 2.1 is also known as the Rosa-Kotzig-Ringel Conjecture. In the last 50 years, many researchers in graph theory have put a lot of effort in proving Conjecture 2.1 and some special classes of trees have been proved to have graceful labeling. A *leaf* is a vertex u in a tree with $d_G(u) = 1$. A *caterpillar* is a subclass of tree where the deletion of all leaves yields a path. Note that a path is also a subclass of caterpillars. Rosa (1967) proved the theorem below.

Theorem 2.6. (Rosa, 1967) *Every caterpillar is graceful.*

Figure 2.9 illustrates a gracefully labeled path with 7 vertices while Figure

2.10 shows a gracefully labeled caterpillar with 12 vertices.

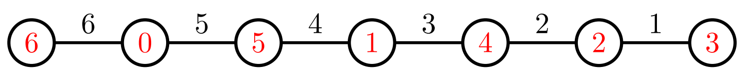


Figure 2.9: An example of a gracefully labeled P_7

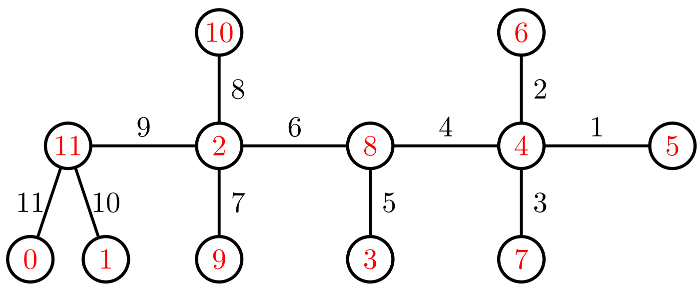


Figure 2.10: An example of a gracefully labeled caterpillar with 12 vertices

Recall that a spider is a connected tree which contains only one vertex u satisfying $d_G(u) \geq 2$. We shall denote that particular vertex having degree exceeding 2 by u^*. Bahls et al. (2010) proved that for any spider S, if the difference in lengths of any path from u^* to a leaf is not more than two for S, then S admits a graceful labeling.

A *banana tree* is a tree constructed by connecting a new vertex u to one leaf of each star from a collection of stars. Note that u is not in any of the stars. Sethuraman and Jesintha (2009) proved the result below on banana trees.

Theorem 2.7. (Sethuraman and Jesintha, 2009) *All banana trees admit graceful labelings.*

Figure 2.11 illustrates an example of a gracefully labeled banana trees.

Recently, Gnang posted two manuscripts with a proof of Conjecture 2.1 (Gnang, 2018, 2022). However, we are uncertain of the correctness of the proofs provided in these two manuscripts.

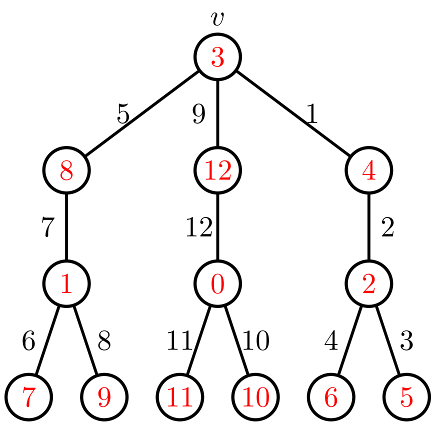


Figure 2.11: An example of a gracefully labeled banana tree with 13 vertices

For more references on graceful graphs, we refer the reader to Gallian (2021). For some recent progresses on graceful graphs, we refer the reader to Kotul'ová and Haviar (2020).

2.3 Magic Labeling

A *magic square* is an array of different positive integers arranged in the form of a square grid in such a way that the sum of entries in each row, each column and each diagonal equals to a constant. Sedláček (1963) introduced magic labeling based on the concept of magic squares in number theory.

Now we have the definition of magic labeling.

Definition 2.5. *Suppose $G(V, E)$ is a graph. G is a magic graph if $E(G)$ can be labeled using different positive integers from $\mathbb{N}$ such that for any vertex $u \in V(G)$, the vertex sum $w(u)$ is the same.*

Figure 2.12 illustrates an example on how to construct the magic labeling of $K_{3,3}$ based on a 3×3 magic square.

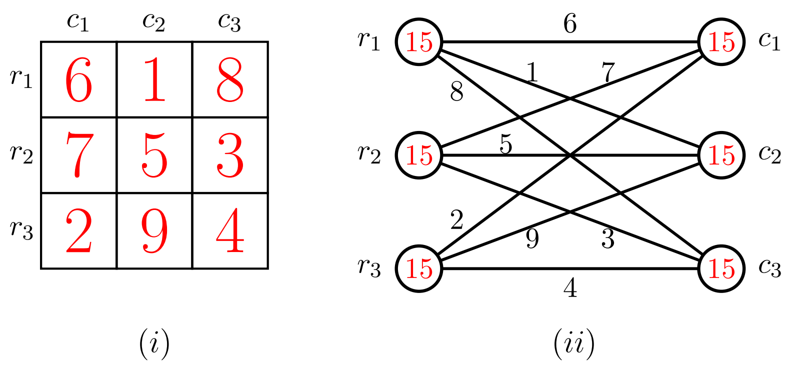


Figure 2.12: A 3×3 magic square and a magic labeling of $K_{3,3}$

Stewart (1966) have proved the magicness of complete graphs and complete bipartite graphs in the following theorems.

Theorem 2.8. *K_n is magic when $n = 2$ or $n \geq 5$.*

Theorem 2.9. *$K_{n,n}$ is magic when $n \geq 3$.*

For example, Figure 2.13 illustrates a magic labeling of K_5.

In the same paper, Stewart (1966) also proved the theorems below.

Theorem 2.10. *A fan graph $P_{j-1} + K_1$ admits a magic labeling if and only if $j \geq 3$ and j is odd.*

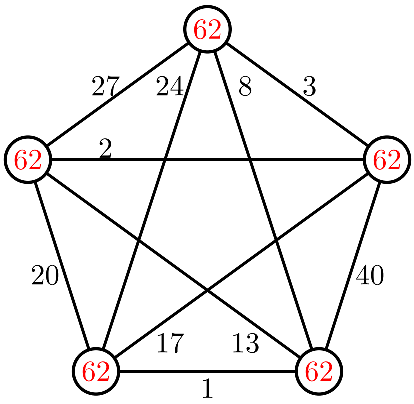


Figure 2.13: An example of a magic labeling of K_5

Theorem 2.11. *W_n is magic when $n \geq 4$.*

Figure 2.14 shows a magic labeling of a wheel which has 6 vertices W_6 and Figure 2.15 depicts an example of magic labeling of a fan which has 5 vertices $P_4 + K_1$.

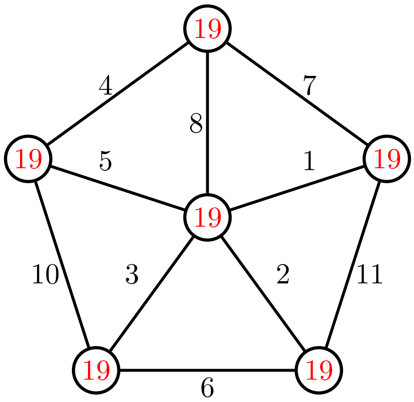


Figure 2.14: An example of a magic labeling of W_6

Meanwhile, Doob (1978) discovered a condition for regular graphs of large degree to have a magic labeling in the theorem below.

Theorem 2.12. (Doob, 1978) *Let H be a regular graph with degree $c \geq 5$ and m vertices. Then H admits a magic labeling if $c > m/2$.*

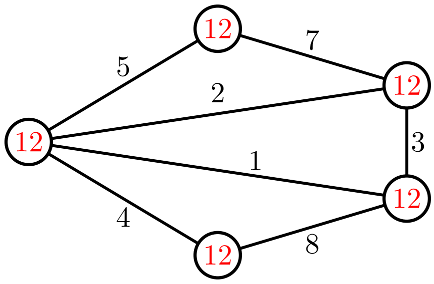


Figure 2.15: An example of a magic labeling of $P_4 + K_1$

Let H be a connected graph which has c vertices and d edges excluding P_2. Trenklér (2000) proved that the necessary and sufficient condition for H to admit a magic labeling is $\frac{5c}{4} < d \leq \frac{c(c-1)}{2}$. For more references on magic graphs, we refer the reader to Gallian (2021).

2.4 Anti-Magic Labeling

The definition of anti-magic labeling is given as follows.

Definition 2.6. *Suppose $G = (V, E)$ is a graph with p edges. G admits an anti-magic labeling if $E(G)$ can be labeled with different integers from $\{1, 2, 3, \ldots, p\}$ so that the vertex sums of all vertices are different.*

An anti-magic labeling on $K_{3,3}$ is shown in Figure 2.16.

The concept of anti-magic graphs is introduced in the book written by Hartsfield and Ringel (1994). In the same book, they proposed the following conjectures.

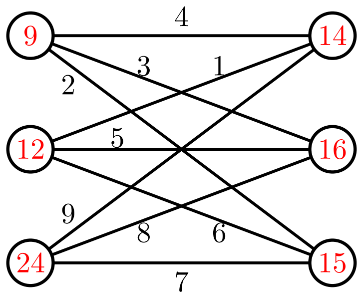


Figure 2.16: An anti-magic labeling on $K_{3,3}$

Conjecture 2.2. (Hartsfield and Ringel 1994)

Every connected graphs excluding K_2 is anti-magic.

Conjecture 2.3. (Hartsfield and Ringel 1994)

Every tree excluding K_2 admits an anti-magic labeling.

Subsequently, many researchers in graph theory focus on solving the problem of deciding which graphs are anti-magic. However, the conjectures remain unsettled.

A connected graph H is *dense* if $|E(H)| = \Theta(n^2)$. The following result of Alon et al. (2004) is the most important progress of Conjecture 2.2.

Theorem 2.13. *All dense graphs are anti-magic.*

Precisely, they proved that there exists an absolute constant d such that all graphs on m vertices with minimum degree at least d log m are anti-magic. Besides that, they also proved the theorem below.

Theorem 2.14. *Complete partite graphs but K_2 admit anti-magic labelings.*

Hartsfield and Ringel (1994) proved the anti-magicness of two general classes of graphs in the following theorems.

Theorem 2.15. (Hartsfield and Ringel, 1994) *All paths are anti-magic.*

Theorem 2.16. (Hartsfield and Ringel, 1994) *All wheels are anti-magic.*

For example, Figure 2.17 shows an anti-magic labeling of P_7 and Figure 2.18 shows an anti-magic labeling of W_6.

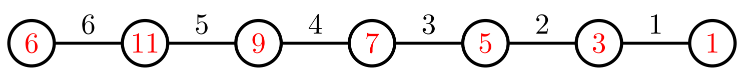


Figure 2.17: An example of P_7 which admits an anti-magic labeling

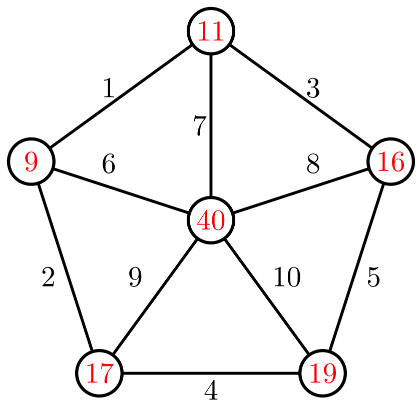


Figure 2.18: An example of an anti-magic labeling on W_6

By confining the attention on regular graphs, the situation turns out to be a lot more delightful. Recall that cycles are 2-regular. Hartsfield and Ringel (1994) proved the following theorem.

Theorem 2.17. *Cycles are anti-magic.*

For example, Figure 2.19 shows an anti-magic labeling of C_6.

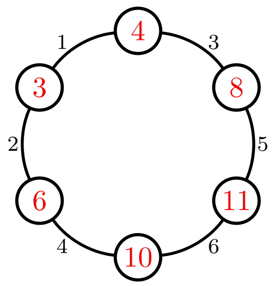


Figure 2.19: An example of a cycle which has 6 vertices C_6 which admits an anti-magic labeling

A graph J is *regular bipartite* if J is both regular and bipartite.

Lemma 2.1. (Kőnig-Hall Theorem)

Suppose $G(S,T)$ is a bipartite graph, and for each subset C of S, let $N(C)$ be the set of vertices of T that are adjacent to at least one vertex of C. A complete matching from S to T exists if and only if $|C| \leq |N(C)|$ for subset C of S.

Using Lemma 2.1, it is not difficult to derive that every r-regular bipartite graph $G(S,T)$ can be decomposed into r complete matchings from S to T. By altering the way to combine these complete matchings in $G(S,T)$, Cranston (2009) proved the theorem below.

Theorem 2.18. (Cranston, 2009) *Every regular bipartite graph where its degree ≥ 2 admits an anti-magic labeling.*

For k-regular graphs, Liang and Zhu (2014) proved the following theorem for the case $k = 3$.

Theorem 2.19. (Liang and Zhu, 2014) *All cubic graphs are anti-magic.*

Figure 2.20 depicts an example of an anti-magic labeling of a 3-regular graph which has 6 vertices.

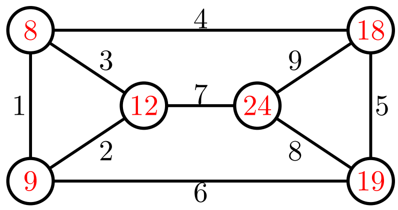


Figure 2.20: An anti-magic labeling of 3-regular graph which has 6 vertices

Cranston et al. (2015) proved the theorem below.

Theorem 2.20. (Cranston et al., 2015) *All regular graphs which have odd degree admit anti-magic labelings.*

Chang et al. (2016) extended the result using the same general idea to verify the anti-magicness of regular graphs with even degree. Meanwhile, Bérczi et al. (2015) also proved the following theorem by changing the argument used in Cranston et al. (2015) .

Theorem 2.21. *All regular graphs with even degree are anti-magic.*

Figure 2.21 depicts an example of an anti-magic labeling of a 4-regular graph which has 6 vertices.

For more details on the definition of rooted tree, we refer the reader to Anick (2016). In an attempt to solve Conjecture 2.3, Kaplan et al. (2009) introduced

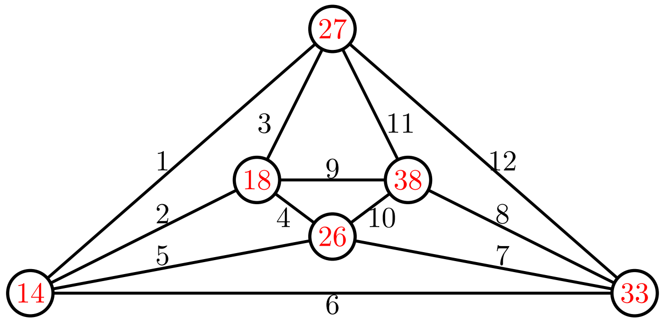


Figure 2.21: An anti-magic labeling of a 4-regular graph which has 6 vertices

a subclass of trees.

Definition 2.7. *In any tree, if a vertex b instantly precedes vertex c on the path from the root to c, then b is a parent of c and c is a child of b.*

Definition 2.8. *A vertex b is called a descendant of another vertex c (and c is called an ancestor of b), if c is on the unique path from the root to b.*

Recall that a leaf is a vertex u in a tree with $d_G(u) = 1$. In a rooted tree, a leaf is any vertex without any children. Figure 2.22 shows an example of a rooted tree with 8 vertices.

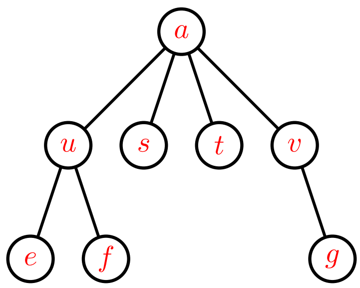


Figure 2.22: An example of a rooted tree with 8 vertices

In Figure 2.22, if the vertex a is the root of the tree, then the vertices u, s, t and v are the children of a. Besides that, the vertices e, f and g are the descendants of a. Note that s, t, e, f and g are the leaves of the tree.

Definition 2.9. *A 2-tree T is a rooted tree, where every vertex $u \in V(T)$ which is not a leaf is connected to at least two leaves in T.*

After defining 2-tree, Kaplan et al. (2009) proved the theorem below.

Theorem 2.22. *Every 2-tree $T(V, E)$ which satisfies $|V| = n$ and $n \geq 2$ is anti-magic.*

Liang et al. (2014) corrected an error in the proof of the above result. Figure 2.23 illustrates an example of a 2-tree which admits an anti-magic labeling.

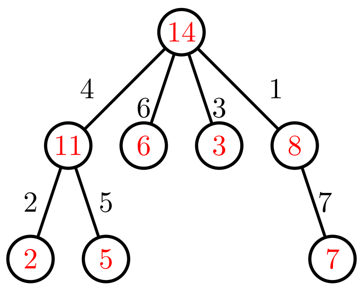


Figure 2.23: An example of a 2-tree with 8 vertices which admits an anti-magic labeling

Define $V_i(T)$ as the set of vertices of T with $d_T(v) = i$ for any tree T. In the same paper, Liang et al. (2014) introduced another subclass of trees T^* and determined a condition for T^* to be anti-magic.

Definition 2.10. *For any tree T, a tree T^* is constructed from T by subdividing every edge of T only once.*

Theorem 2.23. *Suppose T is a tree with $V_2(T) = \emptyset$ and T admits an anti-magic labeling. Then T^* also admits an anti-magic labeling.*

For example, Figure 2.24 depicts two anti-magic labelings of T^* and its corresponding T.

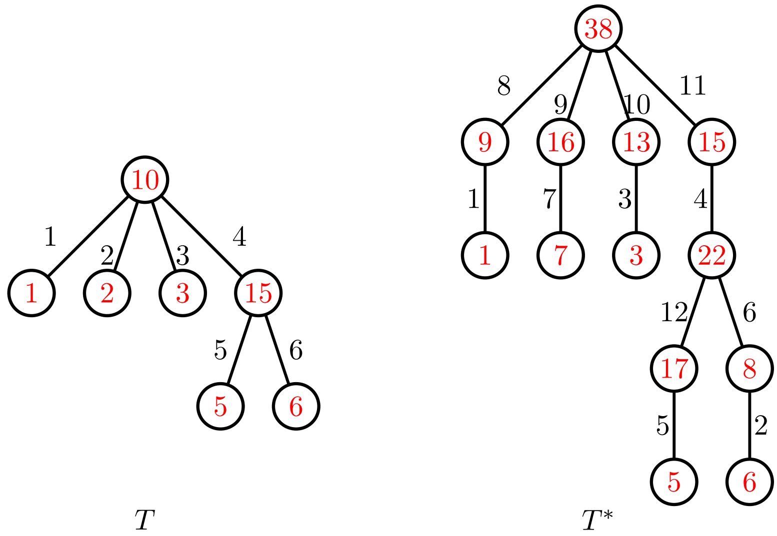


Figure 2.24: Anti-magic labelings on T^* and its corresponding T

Besides that, Liang et al. (2014) also proved the theorem below.

Theorem 2.24. *Let T be a tree. If $V_2(T)$ induces a path and the vertex degrees of all other vertices of T are odd. Then T admits an anti-magic labeling.*

Recently, Lozano et al. (2022) extended Theorem 2.24 in Liang et al. (2014) by showing that trees whose $V_{2i}(T)$ induce a path are anti-magic.

Recall that a spider is a subclass of tree. Shang (2015) proved the following theorem.

Theorem 2.25. *Every spider is anti-magic.*

Recall that a star is the graph $K_{1,n}$. A *star forest*, denoted by $\cup_{i=1}^{r} K_{1,k_i}$ is a graph containing r number of disjoint stars. Note that k_i represents the number of vertices of each star. Shang et al. (2015) investigated the anti-magicness of star forests.

Theorem 2.26. *A star forest* $\cup_{i=1}^{r} K_{1,k_i}$ *with* $k_1 \geq 2$ *and* $k_i \geq 3$ *for* $i = 2, 3, \ldots, r$ *where* $r \geq 2$ *is anti-magic.*

A *double spider* is a tree which has exactly two vertices s and t, where $deg(s) \geq 2$ and $deg(t) \geq 2$. Chang et al. (2020) proved the theorem below.

Theorem 2.27. *Every double spider is anti-magic.*

Recall that a caterpillar is a subclass of tree where the deletion of all leaves yields a path. Deng and Li (2019) proved the theorem below.

Theorem 2.28. (Deng and Li, 2019) *All caterpillars with maximum degree 3 are anti-magic.*

Lozano et al. (2019) determined sufficient conditions for a caterpillar to admit an anti-magic labeling. Recently, Lozano et al. (2021) extended the previous results on the anti-magicness of caterpillars.

Theorem 2.29. *Caterpillars are anti-magic.*

Recall that $H \square J$ represents the Cartesian product of two graphs H and J. Some research has been carried out on the anti-magicness of Cartesian products. The Cartesian products of two cycles is known as a *toroidal* graph, written as $C_m \square C_n$. Wang (2005) proved the following theorem related to torodial graphs.

Theorem 2.30. (Wang, 2005) *Toroidal graphs admit anti-magic labelings.*

The Cartesian products of two paths are called *lattice grids* and the Cartesian products of a cycle and a path are called the *prisms*. Cheng (2007) studied the anti-magicness of lattice grids and prisms and proved the following theorems.

Theorem 2.31. (Cheng, 2007) *Lattice grids $P_{m+1} \square P_{n+1}$ admit anti-magic labelings for all integers $m, n \geq 1$.*

Theorem 2.32. (Cheng, 2007) *Prism graphs $C_m \square P_{n+1}$ admit anti-magic labelings for all integers $m \geq 3$ and $n \geq 1$.*

Figure 2.25 illustrates an example of an anti-magic labeling on $P_3 \square P_3$.

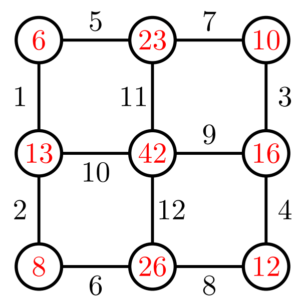


Figure 2.25: An example of an anti-magic labeling admitted on $P_3 \square P_3$

Wang and Hsiao (2008) defined a *generalized prism grid* graph as the Cartesian product of a k-regular graph and a path and a *generalized toroidal grid* graph as the Cartesian product of a cycle and a k-regular graph. They proved the theorem below.

Theorem 2.33. (Wang and Hsiao, 2008) *Generalized prism grid graphs and generalized toroidal grid graphs admit anti-magic labelings.*

Based on Theorem 2.30, Cheng (2008) proved that the Cartesian products of two or more regular graphs admit anti-magic labelings.

Theorem 2.34. *All Cartesian products of two or more regular graphs admit anti-magic labelings.*

By combining the results in Wang and Hsiao (2008) and Cheng (2008), Zhang and Sun (2009) have proved the anti-magicness of the Cartesian products of a connected graph and an anti-magic k-regular graph.

Theorem 2.35. *Suppose H is an anti-magic k-regular graph and G is a connected graph. The Cartesian product of $H \square G$ is anti-magic.*

Moreover, Liang and Zhu (2013) extended the above result to the following.

Theorem 2.36. *Let J be a k-regular graph and G be a connected graph. The Cartesian product of $J \square G$ admits an anti-magic labeling.*

Wang and Zhang (2012) investigated the anti-magicness of the generalized Petersen graphs.

Theorem 2.37. *All generalized Petersen graphs are anti-magic.*

For a bipartite graph $G(S,T)$, G is said to be (k,k')-*biregular*, if each vertex in S has the degree k, while each vertex in T has the degree k'. Deng and Li (2020) have proved the anti-magicness of some biregular bipartite graphs.

Theorem 2.38. *Each $(k, k^2 + y)$-biregular bipartite graph is anti-magic for all integers $k \geq 3$ and $y \geq 1$.*

For more references on anti-magic graphs, we refer the reader to Gallian (2021). For some recent progresses on anti-magic graphs, we refer the reader to Simanjuntak et al. (2021).

2.5 The 1-2-3 Conjecture

In 2004, Karoński, Łuczak and Thomason proposed a well-known conjecture which is called the 1-2-3 Conjecture (Karoński et al., 2004).

Conjecture 2.4. (1-2-3 Conjecture)

For any connected graph G which is not isomorphic to K_2, there exists a way to label the edges of G using the numbers from $\{1, 2, 3\}$ in such a way that for any two adjacent vertices s and t, $w(s) \neq w(t)$.

The figure below illustrates the 1-2-3 Conjecture for the cycle C_6.

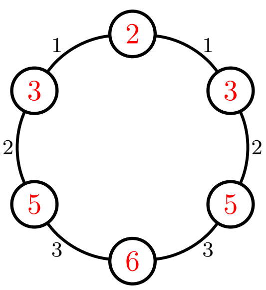


Figure 2.26: The 1-2-3 Conjecture is true for C_6

The following are some definitions for rewriting the 1-2-3 Conjecture in a more concise way.

Definition 2.11. *Suppose $G(V, E)$ is a simple graph. A l-edge-weighting of G is a mapping $h : E(G) \to 1, 2, \ldots, l$.*

Definition 2.12. *An edge-weighting h of a graph H induces a vertex coloring $f_w : V(H) \to \mathbb{N}$ defined by $f_w(v) = \sum_{v \in e} w(e)$. If $f_w(s) \neq f_w(t)$ for any edge st, then this coloring is a proper vertex-coloring.*

Denote by $\mu(H)$ the minimum value of l so that a graph H obtains a proper vertex-coloring l-edge-weighting. For any graph H, if there is no connected component isomorphic to K_2 in H, then H is *nice*. Thus, the 1-2-3 Conjecture can be rewritten as follows.

Conjecture 2.4. (1-2-3 Conjecture)

$\mu(H) \leq 3$, where H is any nice graph.

$\mu(H) \leq 3$ is the best possible in general. For example, if H is a cycle having length not divisble by 4, then $\mu(H) \neq 2$. Therefore, researchers tend to improve the general upper bounds on $\mu(H)$. Addario-Berry et al. (2007) obtained the first general upper bound in the theorem below.

Theorem 2.39. $\mu(H) \leq 30$.

Next, the result was improved to $\mu(H) \leq 16$ by Addario-Berry, Dalal and Reed (2008) and then to $\mu(H) \leq 13$ by Wang and Yu (2008). A significant improvement is made by Kalkowski, Karoński and Pfender (2010) which greatly decreases $\mu(H)$.

Theorem 2.40. $\mu(H) \leq 5$.

Recently, Keusch (2022) improved the general upper bound again in the theorem below.

Theorem 2.41. (Keusch, 2022) $\mu(H) \leq 4$.

Chang et al. (2011) proved that $\mu(H) \leq 2$ if H is a bipartite r-regular graph for $r \geq 3$. Lu et al. (2011) proved that for any nice graph H which is bipartite and 3-connected, $\mu(H) \leq 2$. Then, Davoodi and Omooni (2015) proved that if Conjecture 2.4 holds for two graphs H and G, then it also holds for $H \,\square\, G$. Khatirinejad et al. (2012) proved that $\mu(H) \leq 2$ for any graph H containing only cycles of length divisible by 4.

Przybyło (2021) studied on the $\mu(H)$ of r-regular graphs and proved the following theorems.

Theorem 2.42. (Przybyło, 2021) *For every k-regular graphs H, $\mu(H) \leq 4$.*

Theorem 2.43. *For every k-regular graphs H, $\mu(H) \leq 3$ if $k \geq 10^8$.*

Meanwhile, Bensmail et al. (2017) related Conjecture 2.4 to the anti-magic labeling of graphs. To get further details on Conjecture 2.4 and its related problems, we suggest the reader to refer the paper written by Seamone (2012).

CHAPTER 3

APPLICATIONS OF GRAPH LABELINGS

3.1 Introduction

The study of graph decompositions can be traced back from Euler's work on *Latin squares* more than two hundred years ago. A *Latin square* of order m is an $m \times m$ array of m unique symbols such that each symbol appears once in every row and column. A *transversal* of an $m \times m$ Latin square is a set of m distinct entries no two of which occur in same row or column. Euler initiated the study of transversals in Latin squares. In fact, the study of transversals equals to a graph decomposition problem. For a graph $H = (V, E)$, can we partition $E(H)$ into disjoint copies of another graph J?

3.2 Graph decomposition

A *decomposition* of a graph $L = (V, E)$ is a set of subgraphs $\{G_1, G_2, \ldots, G_s\}$ whose edge sets $(E_1, E_2, \ldots, E_s)$ is a partition of $E(L)$. For example, the figure below shows a decomposition of K_6 into three different subgraphs.

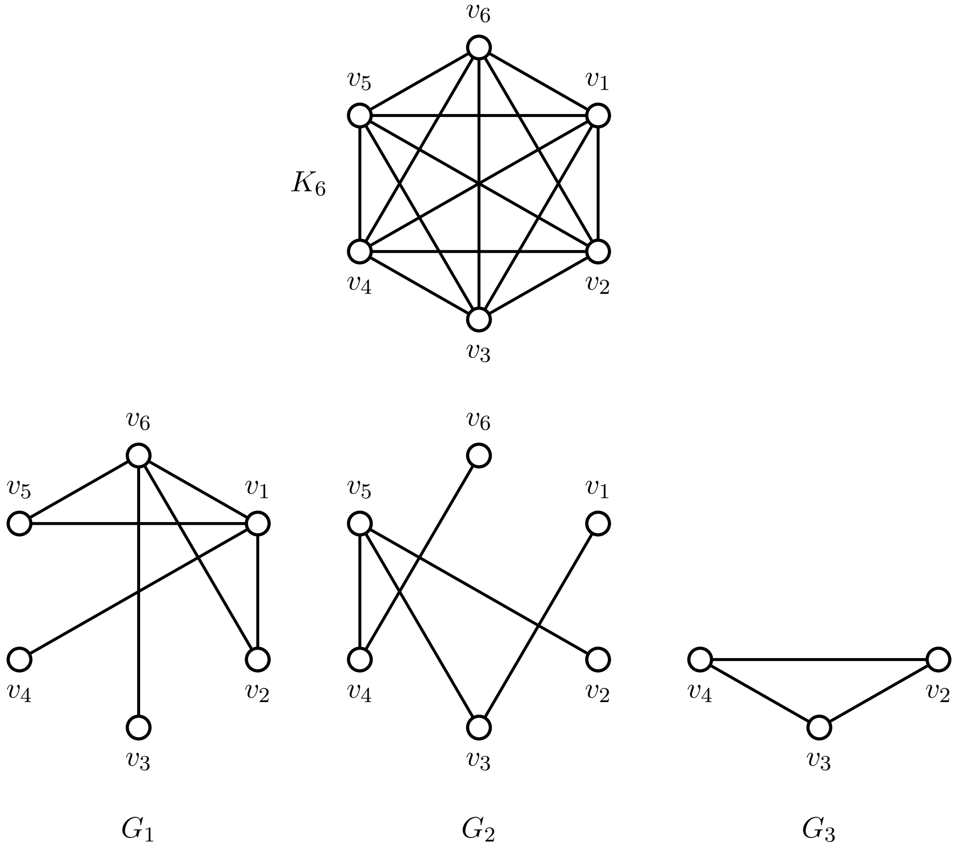


Figure 3.1: Decomposition of K_6 into G_1, G_2 and G_3

We say that a graph L contains a *G-decomposition* if L has a decomposition $\{G_1, G_2, \ldots, G_s\}$ and each graph G_j is isomorphic to G, for any $1 \leq j \leq s$. The G-decomposition of L is called a *cycle-decomposition* of L if G is a cycle. For example, the figure below shows a cycle-decomposition of K_5.

The G-decomposition of L is called a *tree-decomposition* (respectively *path-decomposition*) of L if G is a tree (respectively path). In 1847, Krikman investigated the decompositions of the complete graphs K_s and proved that K_s contains a G-decomposition $\{G_1, G_2, \ldots, G_{2s-1}\}$ where each subgraph is isomorphic to a triangle if and only if $s \equiv 1$ or $3 \pmod 6$. Since then, the decomposition of K_s gathered the main interest of mathematicians. Tarsi (1983) has completely solved the path-decomposition of K_s. For cycle-decomposition of K_s, Alspach and Gavlas (2001) has solved it for the odd values of s and Sajna

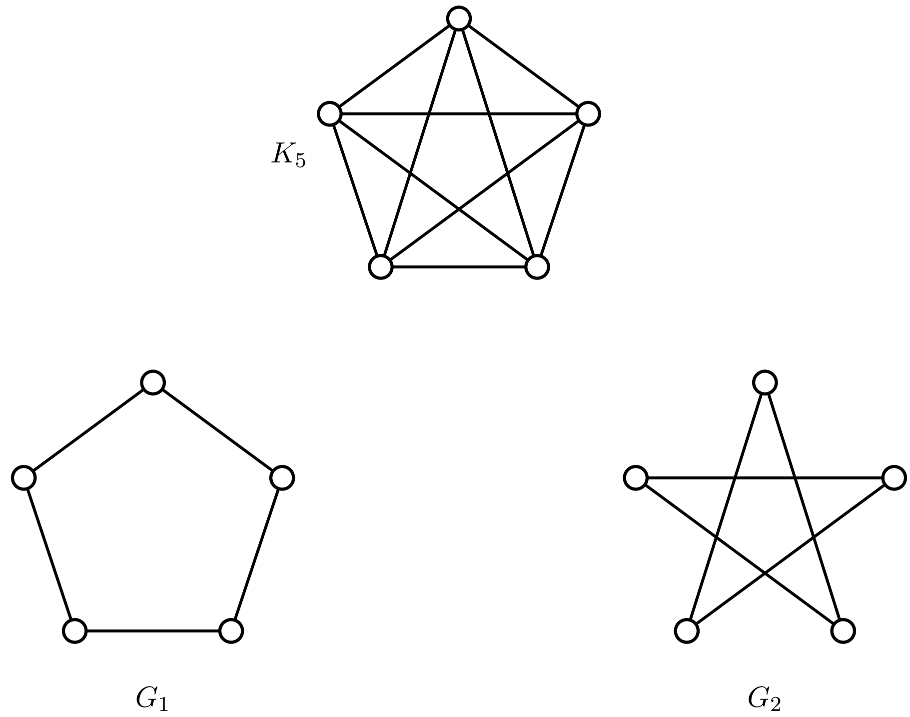


Figure 3.2: Cycle-decomposition of K_5

(2002) has solved it for the even values of s. Meanwhile for star-decomposition of K_s, Yamamoto et al. (1975) and Tarsi (1979) have completely solved the problem independently. However, the tree-decomposition of K_s is still open.

Concerning the tree-decomposition of K_s, Ringel (1963) proposed the following famous conjecture in the Smolenice symposium.

Conjecture 3.1. (Ringel's Conjecture 1963)

The complete graph K_{2s-1} is decomposable into $2s-1$ copies of trees T which has s vertices.

The figure below shows an example of Ringel's Conjecture with $s = 4$.

Given the complete graph K_s, we label $V(K_s)$ using the non-negative in-

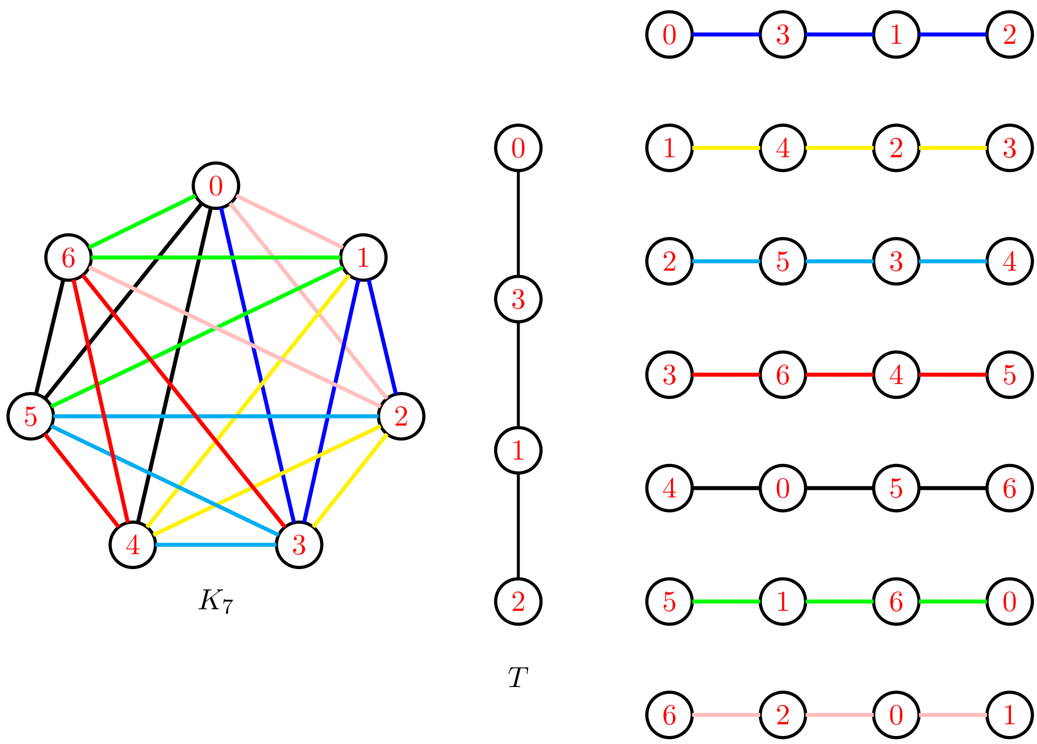


Figure 3.3: T-decomposition of K_7

tegers $\{0, 1, 2, \ldots, s-1\}$. Let $ij \in E(K_s)$. A *turning of the edge ij* happens when both labels of ij increase by one, i.e. the edge $(i+1)(j+1)$, where the addition is taken modulo s. A *turning of a subgraph* H of K_s is the simultaneous turning of all the edges in H. We say that a decomposition of K_s is *cyclic* when the following condition satisfies.

If the decomposition consists of a graph G, then it also consists of the graph G' constructed by turning G.

Based on the concept of cyclic decomposition, Kotzig (1965) proposed another conjecture which is considered a stronger version of Ringel's conjecture.

Conjecture 3.2. (Kotzig's Conjecture 1965)

If S is any tree which has s vertices, then the complete graph K_{2s-1} is cyclically decomposable into $2s-1$ copies of S.

Clearly Conjecture 3.2 is the stronger version of Conjecture 3.1. Recall that Rosa (1967) introduced four types of valuations and proposed the Graceful Tree Conjecture. In fact, the notion of graceful labeling is an approach used to tackle both Conjecture 3.2 and Conjecture 3.1. In the same paper (Rosa, 1967), Rosa proved the important theorem below.

Theorem 3.1. (Rosa 1967)

If a tree S admits a graceful labeling, then K_{2s-1} is cyclically decomposed into $2s-1$ copies of S.

The significance of Theorem 3.1 is that we may perform the tree-decomposition for complete graphs if we can find the graceful labelings of all trees. In other words, Ringel's Conjecture, Kotzig's Conjecture and the Graceful Tree Conjecture are highly related. Montgomery et al. (2021) proved Ringel's conjecture when the size of the graph is sufficiently large. Meanwhile, Barrientos and Minion (2016) found a way to decrease the number of trees that needed for the investigation on proving Kotzig's Conjecture. Barrientos and Minion (2019) found the method to perform cyclic decomposition for a few subfamilies of trees.

Using an example, we wish to elaborate the relationship between the graceful labeling of a tree and the cyclic decomposition of complete graphs. Suppose T is a tree which contains 4 vertices with a graceful labeling as shown in Figure 3.4.

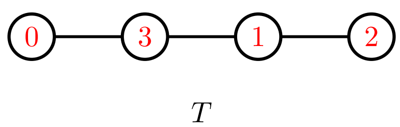


Figure 3.4: T with graceful labeling

We wish to pack T in Figure 3.4 into a complete graph K_7 cyclically. Take 7 vertices 0, 1, 2, ..., 6 and place them in the form of a cyele. Then place T as shown in Figure 3.4 according to the labels on its vertices to get Figure 3.5 (a). By rotating T cyclically once, we obtain the second copy of T as illustrated in Figure 3.5 (b). Continue the cyclic rotation in this way until all vertices have been covered by rotation and we obtain the tree decomposition of T depicted in Figure 3.3.

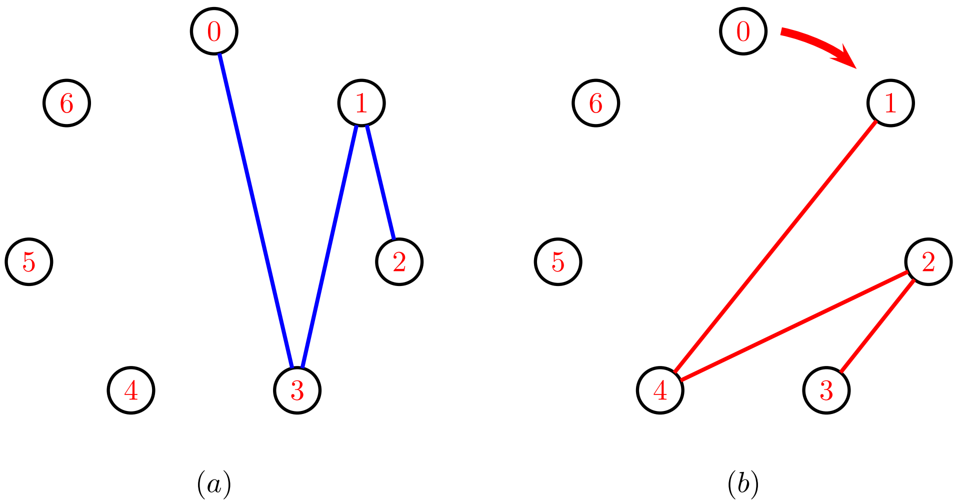


Figure 3.5: Cyclic T-decomposition of K_7

3.3 Application in graph decomposition

In this section, we show that graph labeling can also be applied in (H, J)- decomposition, when H is a tree. The definition of (H, J)- decomposition is given below followed by an example of a (P_4, C_3)-decomposition.

Definition 3.1. *We say that a graph L contains a (H, J)-decomposition if L has a decomposition $\{H_1, H_2, \ldots, H_s\}$ and for each k, $1 \le k \le s$, H_k is isomorphic* *to either a graph H or a graph J in such a way that there exist at least one i and at least one j in such a way that $1 \leq i < j \leq s$ with H_i isomorphic to H and J_j isomorphic to J.*

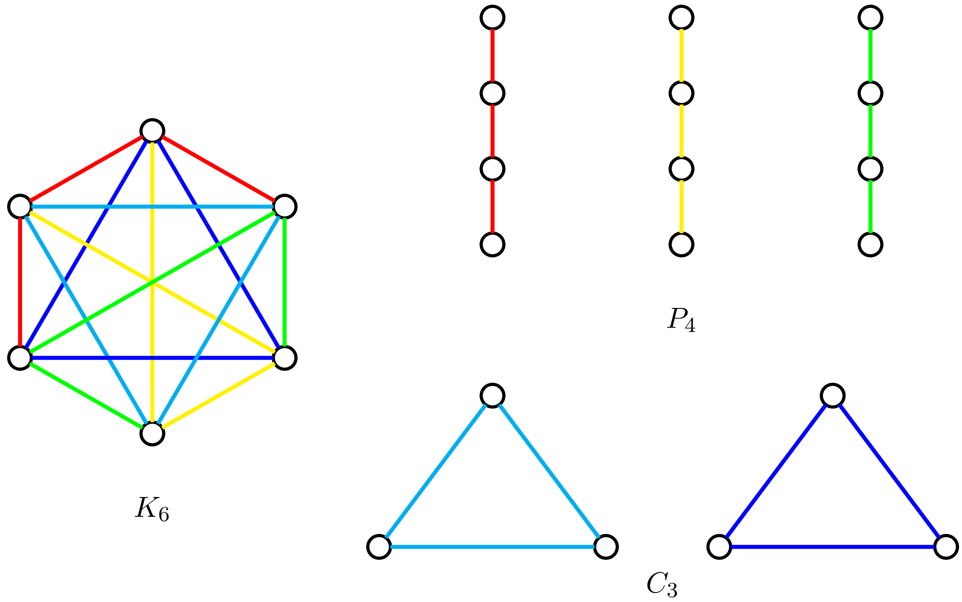


Figure 3.6: (P_4, C_3)-decomposition of K_6

Sethuraman and Murugan (2021) proved the following theorem.

Theorem 3.2. (Sethuraman and Murugan 2021)

Suppose L is a path or a star with n vertices. The complete graph K_{4n-3} is decomposable into $4n-3$ copies of a random tree which has n vertices and $4n-3$ copies of graph L.

In order to achieve the result, they introduced δ-labeling.

Definition 3.2. *Suppose $|E(L)| = m$. Let l be a one-to-one function that maps $V(L)$ to $\{0, 1, 2, \ldots, 4m\}$. Suppose the edge labeling function l' induced from l satisfies the following condition.*

$$l'(st) = min\{|l(s) - l(t)|, 4m + 1 - |l(s) - l(t)|\},$$

where st is an edge of L and the edge values range over $\{1, 2, 3, \ldots, 2m\}$. *Then l is known as a* δ*-labeling of L.*

Figure 3.7 depicts a δ-labeling of a cycle with 6 vertices C_6 and Figure 3.8 depicts an example of a δ-labeling of a tree with 7 vertices.

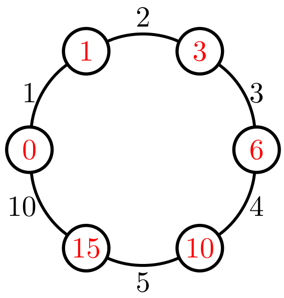


Figure 3.7: A δ-labeling of C_6

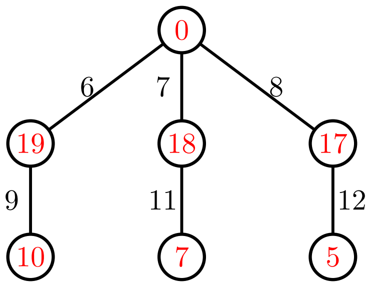


Figure 3.8: A δ-labeling of a tree with 7 vertices

Recall that in a ρ-labeling of a graph H, the numbers used to label the vertex set $V(H)$ range over $\{0, 1, 2, \ldots, 2m\}$ and the edge values range over $\{1, 2, 3, \ldots, m\}$. Therefore, δ-labeling is also a weaker version of ρ-labeling. Based on Definition 3.2, Sethuraman and Murugan (2021) introduced *two graph* ρ^{-}*-labeling pair.* For more information on this concept, we refer the reader to Sethuraman and Murugan (2021).

Suppose H and J are graphs where both of them admit δ-labeling. Using the concept of two graph ρ^--labeling pair, Sethuraman and Murugan has shown a way to join H and J into a graph $H \cup J$ which admits ρ-labeling. Let H be C_6 and J be a tree with 7 vertices. Figure 3.9 shows an example of $H \cup J$.

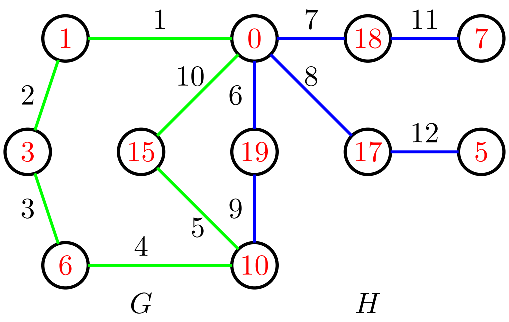


Figure 3.9: A ρ-labeling of $H \cup J$

Since $H \cup J$ admits ρ-labeling, by Theorem 3.1, Sethuraman and Murugan found that the complete graph K_{4m+1} is cyclically decomposable into $4m + 1$ copies of $H \cup J$. As $H \cup J$ can be naturally decomposed into the graphs H and J, they proved the theorem below.

Theorem 3.3. (Sethuraman and Murugan 2021)

Suppose H and J are two graphs which contain n edges respectively and both of them admit two graph ρ^--labeling pair. The complete graph K_{4n+1} is decomposable into $4n + 1$ copies of H and $4n + 1$ copies of J.

Based on Theorem 3.3, they proved another theorem.

Theorem 3.4. (Sethuraman and Murugan 2021)

Let H be a graph which has n edges and H admits a δ-labeling. Suppose J is either the star S_n or the path P_n. The complete graph K_{4n+1} is decomposable into $4n + 1$ copies of H and $4n + 1$ copies of J.

Theorem 3.4 is a special case of Theorem 3.3 by restricting the graph J to be a path or star. For example, Figure 3.10 shows a δ-labeling of a path with 7 vertices P_7. Let H be C_6 and J be P_7. Figure 3.11 shows an example of $H \cup J$.

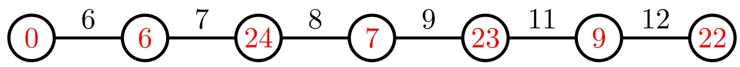


Figure 3.10: A δ-labeling of P_7

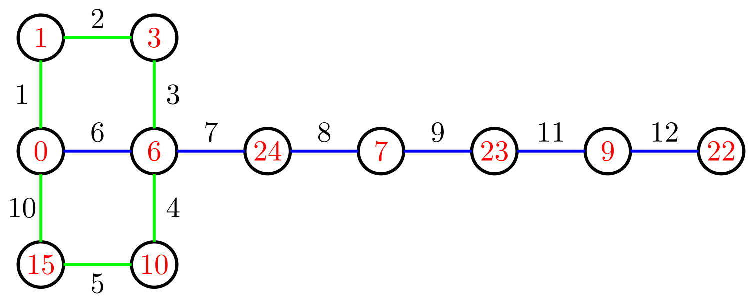


Figure 3.11: A ρ-labeling of $C_6 \cup P_7$

In order to apply Theorem 3.3 by restricting the graph H to be a tree, they proved the theorem below.

Theorem 3.5. (Sethuraman and Murugan 2021)

Every tree admits a δ-labeling.

For example, the figure below shows a δ-labeling of a tree graph with 14 vertices.

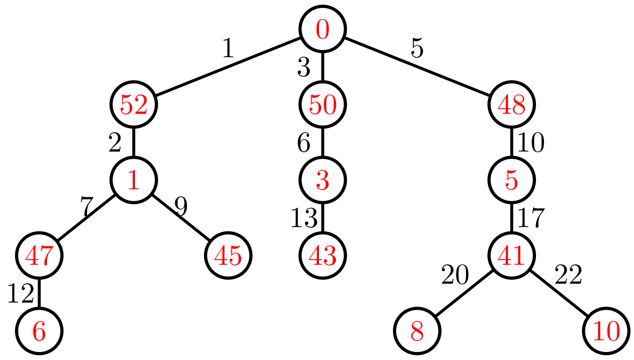


Figure 3.12: δ-labeling of a graph with 14 vertices

Based on Theorems 3.3, 3.4 and 3.5, they have successfully proved Theorem 3.2.

CHAPTER 4

ANTI-MAGICNESS OF MULTI-BRIDGE GRAPHS

4.1 Introduction

Recall that a regular graph of degree r is a graph G, where every vertex in G has the same vertex degree r. In this chapter, we focus on studying the anti-magicness of multi-bridge graphs, a class of graphs which are close to being regular. The definition of multi-bridge graph is given as follows.

Definition 4.1. *Consider a graph with only two vertices and having r multiple edges joining them, $r \geq 3$. Subdivide the edges of this graph arbitrarily so that at most one edge is not subdivided. Call the result graph an r-bridge graph and denote it by $\theta(m_1, m_2, \ldots, m_r)$ if the lengths of the paths are $m_1, m_2, \ldots, m_r$ respectively.*

The main purpose of this chapter is to prove the following result.

Theorem 4.1. *Every r-bridge graph is anti-magic.*

4.2 The proof of Theorem 4.1

Throughout this section, we shall assume that in the graph $\theta(m_1, m_2, \ldots, m_r)$, the path lengths satisfy the condition $m_1 \geq m_2 \geq \cdots \geq m_r$. Also, we shall call the paths in $\theta(m_1, m_2, \ldots, m_r)$ the m_i-path, $i = 1, 2, \ldots, r$.

Let x and y denote the two vertices of degree r in $\theta(m_1, m_2, \ldots, m_r)$ and let $w(x), w(y)$ denote the vertex sums of x, y respectively.

The proof is divided into three cases.

Case I: $r = 3k$.

Suppose $k = 1$.

The labelings depicted in Figure 4.1 show that if $m_1 \leq 2$, the 3-bridge graph is anti-magic. Hence we assume that $m_1 \geq 3$.

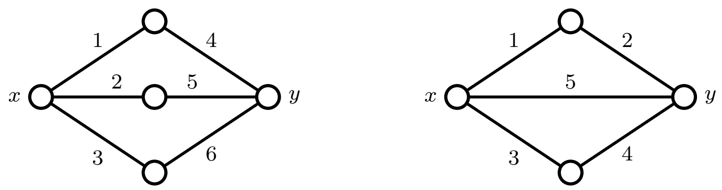


Figure 4.1: Anti-magic labelings where $m_1 = 2$.

Subcase I.1: $m_1 + m_2 + m_3$ is odd.

Let φ_0 denote the following edge labeling on the 3-bridge graph.

(i) Label the edges of the m_1-path with $1, 2, \ldots, m_1$ successively starting from the vertex x.

(ii) Label the edges of the m_3-path with $m_1 + 1, m_1 + 2, \ldots, m_1 + m_3$ successively starting from the vertex y.

(iii) Label the edges of the m_2-path with $m_1+m_3+1, m_1+m_3+2, \ldots, m_1+m_3 + m_2$ successively starting from the vertex x.

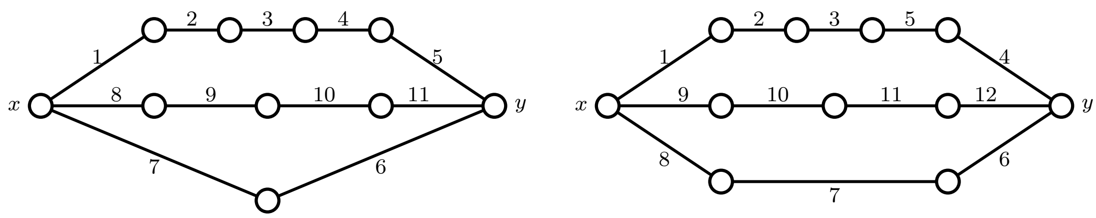
Figure 4.2(i) illustrates the case $(m_1, m_2, m_3) = (5, 4, 2)$.

Note that the vertex sums of the degree-2 vertices include distinct odd natural numbers and that the vertex sums of x and y are both even and are given by $w(x) = 2(m_1 + m_3 + 1)$ and $w(y) = 2m_1 + m_1 + m_2 + m_3 + 1$ respectively.

This shows that φ_0 is an anti-magic labeling of the 3-bridge graph.

Subcase I.2: $m_1 + m_2 + m_3$ is even.

In this case, an anti-magic labeling is acquired by swapping the labels m_1-



(i) (ii)

Figure 4.2: Two anti-magic labelings on 3-bridges.

$1, m_1$ (on the last two edges of the m_1-path) from the anti-magic labeling φ_0 given in Case I. Note that there are only three vertices whose vertex-sums are even, namely x, y and the second last vertex on the m_1-path. Since the vertex-sums are $2(m_1+m_3+1), 2m_1+m_1+m_2+m_3$ and $2m_1-2$ respectively, they are distinct natural numbers.

The vertex-sums of the rest of the vertices are distinct odd natural numbers.

Figure 4.2(ii) illustrates the case $(m_1, m_2, m_3) = (5, 4, 3)$.

Now suppose $k \geq 2$.

For each $i = 1, 2, \ldots, k$, let H_i denote the 3-bridge subgraph induced by the m_{3i-2}-path, m_{3i-1}-path and the m_{3i}-path.

Define $p_0 = 0$ and $p_i = p_{i-1} + m_{3i-2} + m_{3i-1} + m_{3i}$ for $i \geq 1$.

For each $i = 1, 2, \ldots, k$, label the edges of H_i so that

(i) the edges of the m_{3i-2}-path receive the labels $p_{i-1}+1, p_{i-1}+2, \ldots, p_{i-1}+m_{3i-2}$ successively staring from the vertex x,

(ii) and then label the edges of the m_{3i}-path with $p_{i-1} + m_{3i-2} + 1, p_{i-1} + m_{3i-2} + 2, \ldots, p_{i-1} + m_{3i-2} + m_{3i}$ successively starting from the vertex y.

(iii) Finally, label the edges of the m_{3i-1}-path with $p_{i-1} + m_{3i-2} + m_{3i} + 1, p_{i-1} + m_{3i-2} + m_{3i} + 2, \ldots, p_{i-1} + m_{3i-2} + m_{3i} + m_{3i-1}$ starting from the vertex x.

Figure 4.3 illustrates the cases $(m_1, m_2, \ldots, m_6) = (6, 6, 5, 4, 3, 2)$ and $(m_1, m_2, \ldots, m_6) = (2, 2, \ldots, 2)$.

It is routine to check that the vertex sums of x and y are given by

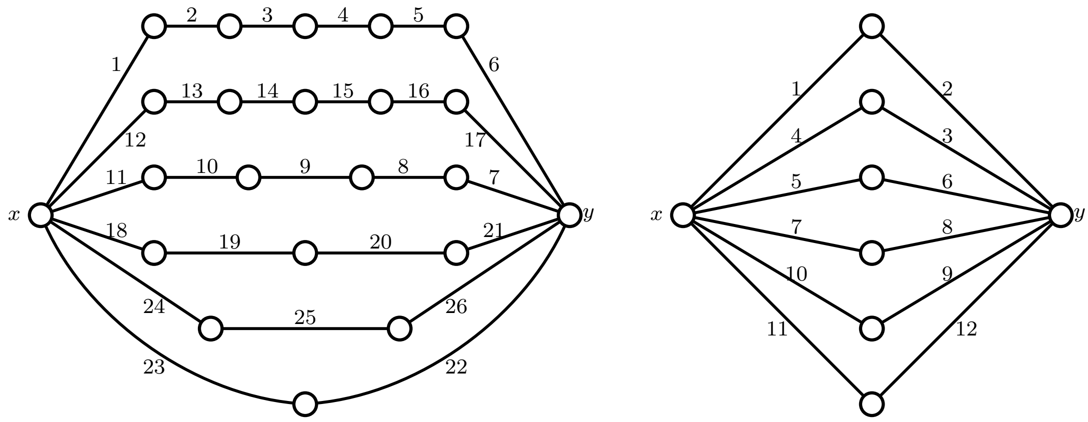


Figure 4.3: Two anti-magic labelings on 6-bridges.

$$w(x) = 2k + 2p_k - 2\sum_{i=1}^{k} m_{3i-1} + 3\sum_{i=1}^{k-1} p_i$$

and

$$w(y) = k + p_k + 2\sum_{i=1}^{k} m_{3i-2} + 3\sum_{i=1}^{k-1} p_i.$$

respectively.

Also, note that the vertex sums of degree-2 vertices consist of odd distinct natural numbers and are less than either of $w(x)$ and $w(y)$.

This completes the proof for Case I.

Case II: $r = 3k + 1$.

Suppose $k = 1$.

Subcase II.1: Not all paths have the same length.

Let φ_1 denote the following edge labeling on the 4-bridge graph.

(i) Label the edges of the m_1-path with $1, 2, \ldots, m_1$ successively starting from the vertex x.

(ii) Label the edges of the m_2-path with $m_1 + 1, m_1 + 2, \ldots, m_1 + m_2$ successively starting from the vertex x.

(iii) Label the edges of the m_3-path with $m_1+m_2+1, m_1+m_2+2, \ldots, m_1+m_2 + m_3$ successively starting from the vertex y.

(iv) Label the edges of the m_4-path with $m_1 + m_2 + m_3 + 1, m_1 + m_2 + m_3 + 2, \ldots, m_1 + m_2 + m_3 + m_4$ successively starting from the vertex y.

Figure 4.4(i) illustrates the case $(m_1, m_2, m_3, m_4) = (5, 4, 3, 2)$.

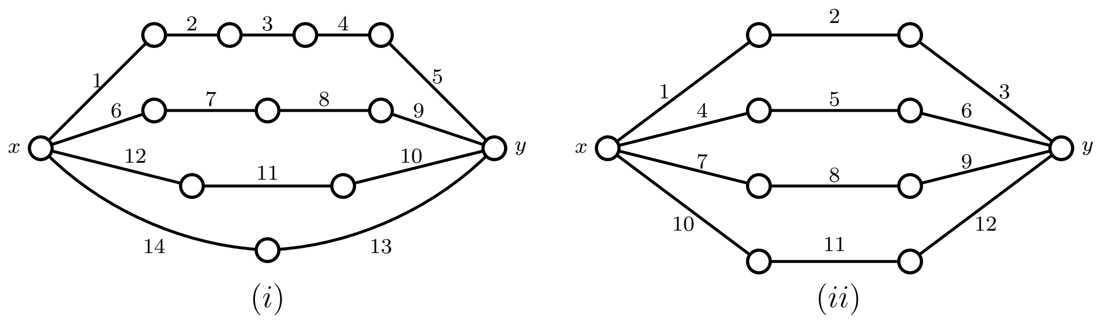


Figure 4.4: Two anti-magic labelings on 4-bridges.

Note that the vertex sums $w(x)$ and $w(y)$ of x and y are given by $3m_1 + 2m_2 + 2m_3 + m_4 + 2$ and $4m_1 + 3m_2 + m_3 + 2$ respectively. Note that the vertex sums of the degree-2 vertices include distinct natural odd numbers and they are all less than either of $w(x)$ and $w(y)$.

This means that φ_1 is an anti-magic labeling of the 4-bridge.

Subcase II.2: All paths have the same length m.

In this case, an anti-magic labeling is obtained by labeling the edges of the i-th path with the labels $(i-1)m+1, (i-1)m+2, \ldots, im$ successively all starting from x to y. In this case $w(x) = 6m + 4$ and $w(y) = 10m$. The rest of the vertex sums consist of distinct odd natural numbers.

Figure 4.4(ii) illustrates the case $m = 3$.

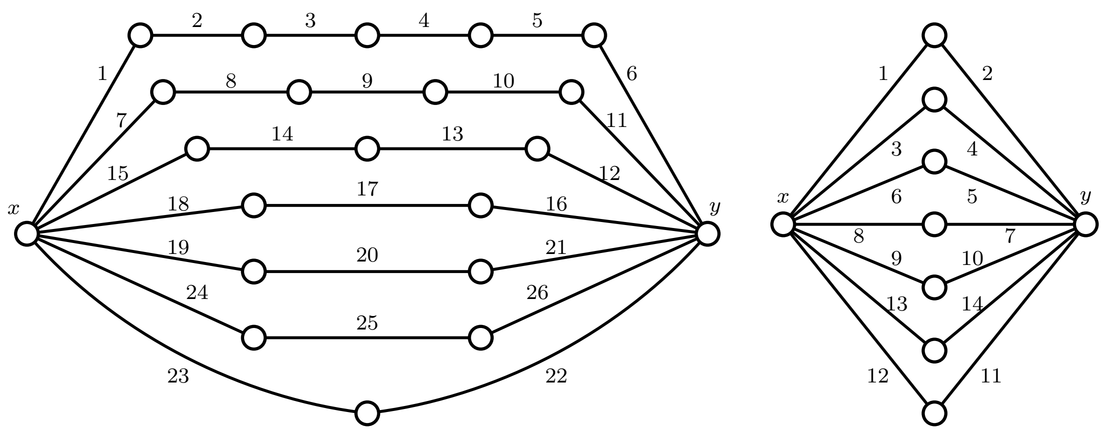


Figure 4.5: Two anti-magic labelings on 7-bridges.

Now suppose $k \geq 2$.

Let H_1 denote the 4-bridge subgraph induced by the m_j-path, $j = 1, 2, 3, 4$. Also, for each $i = 2, \ldots, k$, let H_i denote the 3-bridge subgraph induced by the m_{3i-1}-path, m_{3i}-path and the m_{3i+1}-path.

Define $p_0 = 0$, $p_1 = m_1 + m_2 + m_3 + m_4$ and $p_i = p_{i-1} + m_{3i-1} + m_{3i} + m_{3i+1}$ for $i \geq 2$.

Label H_1 using φ_1 first. Then for each $i = 2, \ldots, k$, label the edges of H_i so that

(i) the edges of the m_{3i-1}-path receive the labels $p_{i-1}+1, p_{i-1}+2, \ldots, p_{i-1}+ m_{3i-1}$ successively starting from the vertex x, and

(ii) label the edges of the m_{3i+1}-path with $p_{i-1}+m_{3i-1}+1, p_{i-1}+m_{3i-1}+ 2, \ldots, p_{i-1} + m_{3i-1} + m_{3i+1}$ successively starting from the vertex y.

(iii) Finally, label the edges of the m_{3i}-path with $p_{i-1} + m_{3i-1} + m_{3i+1} + 1, p_{i-1} + m_{3i-1} + m_{3i+1} + 2, \ldots, p_{i-1} + m_{3i-1} + m_{3i+1} + m_{3i}$ starting from the vertex x.

Figure 4.5 illustrates the cases $(m_1, m_2, \ldots, m_7) = (6, 5, 4, 3, 3, 3, 2)$ and $(m_1, m_2, \ldots, m_7) = (2, 2, \ldots, 2)$.

It is routine to check that the vertex sums of x and y are given by

$$w(x) = 2p_k + 2k + m_1 - m_4 + \sum_{i=2}^{k}(3p_{i-1} - 2m_{3i})$$

and

$$w(y) = k + 1 + 4m_1 + 3m_2 + m_3 + 2(p_1 - p_k) + \sum_{i=2}^{k}(3p_i + 2m_{3i-1})$$

respectively.

Also, note that the vertex sums of the degree-2 vertices consist of distinct odd natural numbers each of which is less than either of $w(x)$ and $w(y)$.

This completes the proof for Case II.

Case III: $r = 3k + 2$.

Suppose $k = 1$.

Let φ_2 denote the following edge labeling on the 5-bridge graph.

(i) Label the edges of the m_1-path with $1, 2, \ldots, m_1$ successively starting from the vertex x.

(ii) Label the edges of the m_2-path with $m_1+1, m_1+2, \ldots, m_1+m_2$ successively starting from the vertex y.

(iii) For each $i \in \{3,4,5\}$, label the edges of the m_i-path with $q_i+1, q_i+2, \ldots, q_i+m_i$ successively all starting from x to y. Here $q_3 = m_1+m_2$ and $q_j = q_{j-1}+m_{j-1}$ for $j \in \{4,5\}$.

Figure 4.6 illustrates the case $(m_1, m_2, m_3, m_4, m_5) = (6, 5, 4, 3, 2)$.

Note that the vertex sums of x and y are given by $w(x) = 4(m_1+m_2)+2m_3+m_4+4$ and $w(y) = 5m_1+3(m_2+m_3)+2m_4+m_5+1$ respectively.

Clearly the vertex sums of the degree-2 vertices in φ_2 consist of odd distinct natural numbers and each is less than either of $w(x)$ and $w(y)$.

Hence φ_2 is an anti-magic labeling of the 5-bridge.

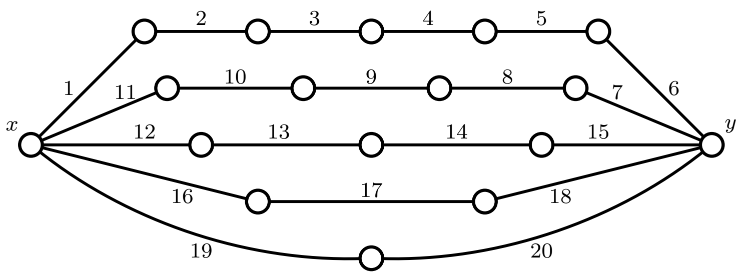


Figure 4.6: Anti-magic labeling of a 5-bridge.

Now suppose $k \geq 2$.

Let H_1 denote the 5-bridge induced by the m_j-path, $j = 1, 2, \ldots, 5$. Also, for each $i = 2, \ldots, k$, let H_i denote the 3-bridge subgraph induced by the m_{3i}-path, m_{3i+1}-path and the m_{3i+2}-path.

Define $p_0 = 0$, $p_1 = m_1+m_2+\cdots+m_5$ and $p_i = p_{i-1}+m_{3i}+m_{3i+1}+m_{3i+2}$ for $i \geq 2$.

Label H_1 using φ_2 first. Then for each $i = 2, \ldots, k$, label the edges of H_i so that

(i) the edges of the m_{3i}-path receive the labels $p_{i-1}+1, p_{i-1}+2, \ldots, p_{i-1}+m_{3i}$ successively starting from the vertex x, and

(ii) label the edges of the m_{3i+2}-path with $p_{i-1} + m_{3i} + 1, p_{i-1} + m_{3i} + 2, \ldots, p_{i-1} + m_{3i} + m_{3i+2}$ successively starting from the vertex y.

(iii) Finally, label the edges of the m_{3i+1}-path with $p_{i-1} + m_{3i} + m_{3i+2} + 1, p_{i-1} + m_{3i} + m_{3i+2} + 2, \ldots, p_{i-1} + m_{3i} + m_{3i+2} + m_{3i+1}$ starting from the vertex x.

Figure 4.7 illustrates the case $(m_1, m_2, \ldots, m_8) = (6, 5, 4, 3, 3, 3, 2, 2)$.

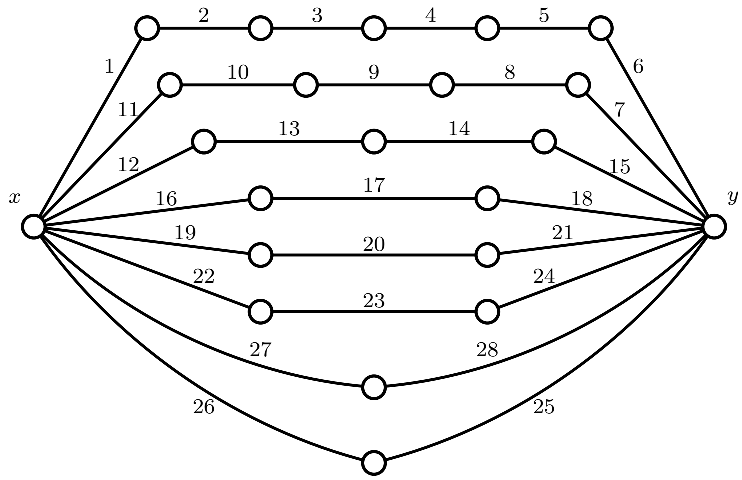


Figure 4.7: Anti-magic labeling of an 8-bridge.

It is routine to check that the vertex sums of x and y are given by

$$w(x) = 2(p_k + k + 1 + m_1 + m_2 - m_5) - m_4 + \sum_{i=2}^{k}(3p_{i-1} - 2m_{3i+1})$$

and

$$w(y) = 2(2m_1 + m_2 + m_3) + m_4 + k + p_k + \sum_{i=2}^{k}(3p_{i-1} + 2m_{3i})$$

respectively.

Also, note that the vertex sums of the degree-2 vertices consist of distinct odd natural numbers each of which is less than either of $w(x)$ and $w(y)$.

This completes the proof for Case III and so is the proof for Theorem 4.1.

□

CHAPTER 5

FUTURE WORK AND DISCUSSION

5.1 Introduction

A *friendship graph*, denoted by f_n is constructed by overlapping a vertex from n copies of cycles with 3 vertices C_3. Figure 5.1 illustrates an example of a friendship graph f_4.

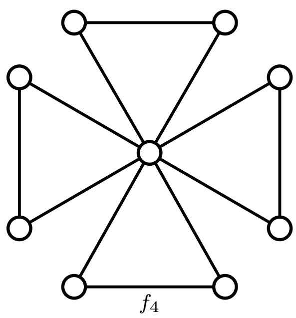


Figure 5.1: f_4.

In this chapter, we study the anti-magicness of extended friendship graphs, a class of graphs derived from friendship graphs. Then, we investigate the anti-magicness of $G(r, s)$, a class of sparse graphs constructed by joining an r-bridge graph with an extended friendship graph. Lastly, we introduce $H(r, s)$, a class of sparse graphs constructed by joining two r-bridge graphs and present a proposition to show that $H(r, s)$ is anti-magic for some values of r and s.

5.2 G(r,s)

Definition 5.1. *Consider the friendship with s cycles, where $s \geq 2$. Subdivide the edges of the s cycles arbitrarily resulting in a graph with s cycles having lengths $n_1, n_2, \ldots, n_s$. Call such a graph an extended friendship graph. Let F_s denote any extended friendship graph with s cycles.*

Remark 5.1. *We know that when $s = 1$, F_1 is a cycle with n_1 vertices C_{n_1}, where $n_1 \geq 3$. We will include the case $s = 1$ in F_s in the following discussions.*

We prove the following proposition.

Proposition 5.1. *F_s is anti-magic for every natural number $s \geq 1$.*

Proof: Recall that in Theorem 2.19, cycles are anti-magic. Therefore, we only need to consider the case $s \geq 2$.

Throughout this section, we shall assume that in the graph F_s, the lengths of the cycles in F_s satisfy the condition $n_1 \geq n_2 \geq \cdots \geq n_s$. Also, we shall call the cycles in F_s the n_j-cycle, $j = 1, 2, \ldots, s$.

Let z denote the vertex of degree $2s$ in the F_s and let $w(z)$ denote the vertex sum of z.

Let φ_3 denote the following edge labeling on the F_s.

For each $j = 1, 2, \ldots, s$, label the edge of the n_j-cycle with $p_{j-1}+1, p_{j-1}+2, \ldots, p_j$, all successively starting from the vertex z. Here $p_0 = 0$ and $p_k = p_{k-1} + n_k$ for $k \in \{1, 2, \ldots, s\}$.

Figure 5.2 illustrates the case F_3 with $(n_1, n_2, n_3) = (6, 5, 3)$.

The vertex sum $w(z)$ of z is $\sum_{i=1}^{s}[2(s-i)+1]n_i + s$. Note that the vertex sums of the degree-2 vertices consist of distinct natural odd numbers and they are all less than $w(z)$.

This means that φ_3 is an anti-magic labeling of the F_s.

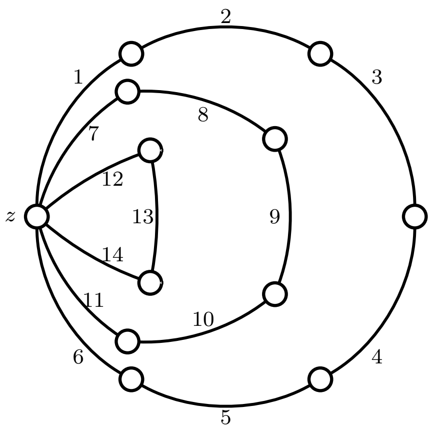


Figure 5.2: The anti-magic labeling of a F_3.

Based on Definition 5.1, the definition of $G(r, s)$ is given as follows.

Definition 5.2. *Let $G(r, s)$ denote any graph obtained by overlapping a vertex of degree r in an r-bridge graph with the vertex of degree 2s in F_s. Let $(m_1, m_2, \cdots, m_r)$ be the lengths of the paths in the r-bridge graph. Also, let $(n_1, n_2, \cdots, n_s)$ be the lengths of the cycles in the F_s.*

Throughout this chapter, we shall assume that in the graph $G(r, s)$, the path lengths in the r-bridge graph satisfy the condition $m_1 \geq m_2 \geq \cdots \geq m_r$. Also, we shall call the paths in $G(r, s)$ the m_i-path, $i = 1, 2, \ldots, r$ and the cycles in $G(r, s)$ the n_j-cycle, $j = 1, 2, \ldots, s$.

Let x denote the vertex of degree r and y denote the vertex of degree $r + 2s$ in $G(r, s)$. Then, let $w(x), w(y)$ denote the vertex-sums of x and y respectively.

Suppose $M = \sum_{i=1}^{r} m_i$ and $N = \sum_{j=1}^{s} n_j$.

Remark 5.2. *In Chapter 4, we assume that $r \geq 3$ when we studied the anti-magicness of r-bridge graphs. For $G(r, s)$, we may include the cases $r = 1, 2$. If r = 1, an r-bridge graph is just a path whereas if r = 2, an r-bridge graph is just a cycle.*

We are interested in studying the problem below.

Problem 5.1. *Is $G(r,s)$ anti-magic for all natural numbers r and s ?*

We believe that Problem 5.1 is true for all natural numbers r and s.

In what follows, we shall provide some supporting evidence to the above claim by showing that Problem 5.1 is true for several values of r and s.

Proposition 5.2. *$G(1,s)$ is anti-magic.*

Proof: Assume $s = 1$.

Case (I): When $M \neq N - 2i$, for any $i = 1, 2, \ldots, \frac{n-1}{2}$.

Let φ_4 represent the following edge labeling on $G(1,1)$.

(i) First, label the edges of the path with $1, 2, \ldots, M$ successively starting from the vertex x.

(ii) Next, label the edges of the cycle with $M+1, M+2, \ldots, M+N$ successively starting from the vertex y.

Figure 5.3(i) illustrates the case $(m_1, n_1) = (4, 5)$ while Figure 5.3(ii) illustrates the case $(m_1, n_1) = (4, 4)$.

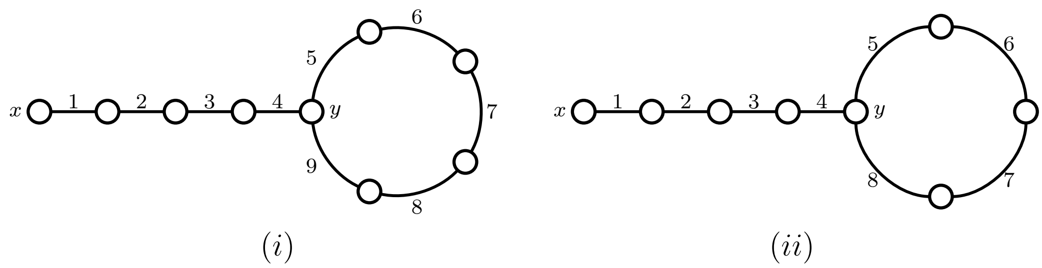


Figure 5.3: Two anti-magic labelings on $G(1,1)$

Note that the vertex sums $w(x)$ and $w(y)$ of x and y are given by 1 and $3M + N + 1$ respectively. Note that the vertex sums of the degree-2 vertices consist of distinct natural odd numbers. Note that $w(y)$ is even. Therefore, they are greater than $w(x)$, but lesser than $w(y)$.

This means that φ_4 is an anti-magic labeling of $G(1, 1)$.

Case (II): When $M = N - 2i$, for some $i \in 1, 2, \ldots, \frac{n-1}{2}$

Let φ_5 represent the following edge labeling on $G(1, 1)$ graph.

(i) First, label the edges of the cycle with $1, 2, \ldots, N$ successively starting from the vertex y.

(ii) Next, label the edges of the path with $N + 1, N + 2, \ldots, N + M$ successively starting from the vertex y.

The steps of the case $(m_1, n_1) = (4, 6)$ are shown in Figure 5.4. Note that the vertex sums $w(x)$ and $w(y)$ of x and y are both even, and are given by $M + N$ and $2N + 2$ respectively. Note that the vertex sums of the degree-2 vertices include distinct natural odd numbers. Therefore, they are not equal to $w(x)$ and $w(y)$.

This indicates that φ_5 is an anti-magic labeling of $G(1, 1)$.

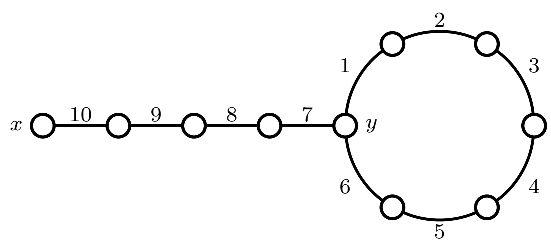


Figure 5.4: Another anti-magic labeling of $G(1, 1)$.

Now assume $s \geq 2$.

We may modify φ_4 to label $G(1, s)$.

(i) First, label the edges of the path with $1, 2, \ldots, M$ successively starting from the vertex x.

(ii) For each $j = 1, 2, \ldots, s$, label the edges of the n_j-cycle with $M + p_{i-1} + 1, M + p_{i-1} + 2, \ldots, M + p_j$, all successively starting from the vertex y. Here $p_0 = 0$ and $p_k = p_{k-1} + n_k$ for $k \in \{1, 2, \ldots, s\}$.

Figure 5.5 illustrates the case $(m_1, n_1, n_2) = (3, 6, 3)$.

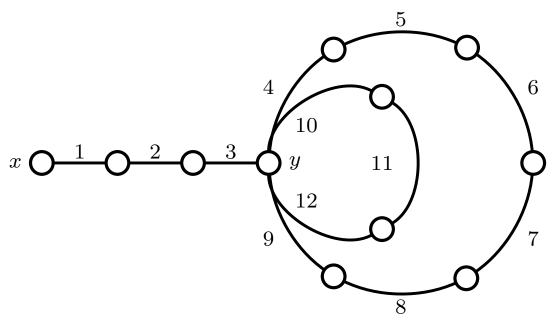


Figure 5.5: The anti-magic labeling of a $G(1, 2)$.

To verify the anti-magicness of the labeling, we need to calculate the vertex-sums of x and y :

$i)$ $w(x) = 1$;

$ii)$ $w(y) = (2s + 1)M + \sum_{i=1}^{s} [2(s - i) + 1]n_i + s$.

Besides, note that the vertex sums of degree-2 vertices include odd distinct natural numbers. By comparison, we observe that they are greater than $w(x)$, but lesser than $w(y)$.

This completes the proof for Proposition 5.2. □

Proposition 5.3. *$G(2, s)$ is anti-magic for any natural number s.*

Proof: Since 2-bridge graph is just a cycle, $G(2, s)$ is isomorphic to F_{s+1} and the proof follows from Proposition 5.1. □

Proposition 5.4. $G(3,1)$ *is anti-magic.*

Proof: We divide the proof into two cases.

Case (I): When $M \geq N$.

Let φ_6 represent the following edge labeling on $G(3,1)$ graph.

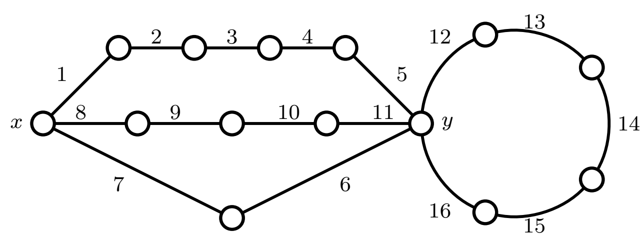
(i) First, label the edges of the 3-bridge part with $1, 2, \ldots, M$ using φ_0.

(ii) Next, label the edges of the cycle with $M+1, M+2, \ldots, M+N$ successively starting from the vertex y.

Figure 5.6 illustrates an example of an anti-magic labeling of a $G(3,1)$ where the lengths of the paths $(m_1, m_2, m_3) = (5, 4, 2)$ and the length of the cycle $n_1 = 5$.

Note that the vertex sums $w(x)$ and $w(y)$ of x and y are given by $2(m_1 + m_3 + 1)$ and $3M + 2m_1 + N + 2$ respectively. Note that the vertex sums of the degree-2 vertices include distinct odd natural numbers. By comparison, we observe that they are lesser than $w(y)$. Note that $w(x)$ is even and $w(x) < w(y)$.

This indicates that φ_6 is an anti-magic labeling of $G(3,1)$.



(ii)

Figure 5.6: The anti-magic labeling of a $G(3,1)$.

Case (II): When $M < N$.

Let φ_7 represent the following edge labeling on $G(3,1)$ graph.

(i) First, label the edges of the cycle with $1, 2, \dots, N$ successively starting from the vertex y.

(ii) Next, label the edges of the 3-bridge part with $N+1, N+2, \dots, M+N$ using φ_0.

Figure 5.7 illustrates the case $(m_1, m_2, m_3, n_1) = (3, 2, 2, 8)$.

Note that the vertex sums $w(x)$ and $w(y)$ of x and y are given by $2(m_1 + m_3 + 1) + 3N$ and $M + 2m_1 + 4N + 2$ respectively. Note that the vertex sums of the degree-2 vertices include distinct odd natural numbers. By comparison, we observe that they are lesser than either of $w(x)$ and $w(y)$ and $w(x) \neq w(y)$.

This indicates that φ_7 is an anti-magic labeling of $G(3, 1)$.

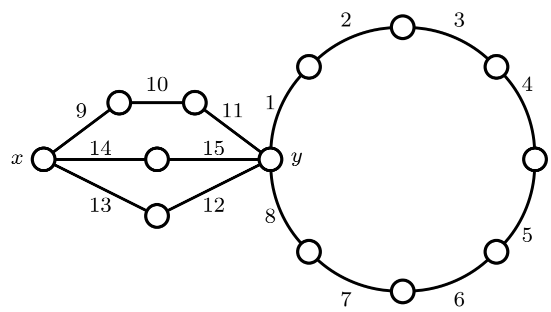


Figure 5.7: The anti-magic labeling of another $G(3, 1)$.

5.3 Conclusion and Future Work

In Chapter 4, we proved that multi-bridge graphs are anti-magic. In Chapter 5, we showed that Problem 5.1 is true for some natural numbers r and s. One of our future research direction is to prove that this is true.

Another future research direction is related to another class of graphs $H(r, s)$, which is defined as follows.

Definition 5.3. *Let $H(r, s)$ denote any graph obtained from an r-bridge graph $G(r)$ and an s-bridge graph $G(s)$ by overlapping a vertex of degree r of $G(r)$ with a vertex of degree s of $G(s)$.*

Let x and y denote the vertex of degree r and the vertex of degree s in $H(r, s)$ respectively. Also, let z denote the vertex of degree $r + s$ in $H(r, s)$. Figure 5.8 illustrates an example of $H(3, 3)$ using two 3-bridge graphs shown in Figure 4.2.

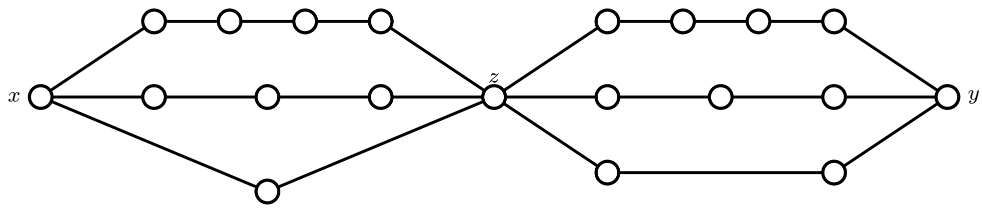


Figure 5.8: An example of $H(3, 3)$.

Remark 5.3. *We extend the definiton of $G(r, s)$ to include the cases $r, s \in \{1, 2\}$.*

We are also interested in studying the following problem.

Problem 5.2. *Is $H(r, s)$ anti-magic for all natural numbers r and s ?*

We believe that Problem 5.2 is true for all natural numbers r and s. In support of this claim, we have the following.

Proposition 5.5. $H(3,1)$ *is anti-magic.*

Proof: As $H(3,1)$ is isomorphic to $G(3,1)$, this follows from Proposition 5.4.

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