SOME PROPERTIES OF SUBSETS OF FINITE GROUPS

By

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ABSTRACT

SOME PROPERTIES OF SUBSETS OF FINITE GROUPS

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This project is mainly concerned with various properties of subsets of finite groups. We first investigate the commutative and non-commutative generalized Latin squares of order 5. For the commutative squares, we list down the squares with 5, 13, 14 and 15 distinct elements. We then divide them into their equivalence classes and show that some of the squares are embeddable in groups. By using a similar approach, we generate the non-commutative generalized Latin squares of order 5 with 5, 24 and 25 distinct elements and determine which squares are embeddable in groups. Next, we investigate the classes of generalized Latin squares of order n with $n^2 - 1$ distinct elements. We determine the number of equivalence classes of these squares and show that all these squares are embeddable in groups. We then investigate the classes of non-commutative generalized Latin squares of order n with n distinct elements. We show the existence of at least three non-isomorphic non-commutative generalized Latin squares of order n with n distinct elements which are embeddable in groups when $n \geq 5$ is odd. By using a similar construction for the case when $n \geq 4$ is even, we show that certain non-commutative generalized Latin squares of order n are not embeddable in groups. Secondly, we investigate the exhaustion numbers of subsets of dihedral groups. Let $e(S)$ denote the exhaustion number of a subset $S \subseteq D_{2n}$. We first give some constructions of subsets with certain finite exhaustion numbers. We show that for any subset $S \subseteq D_{2n}$, $e(S) = 2$ if $n < |S| \leq 2n - 1$. Next, we show that

if $S = \{1, x, y, xy, x^2y, \ldots, x^iy\}$ where $i \in \{1, 3, 5, \ldots, n-3\}$ when n is even and $i \in \{2, 4, 6, \ldots, n-3\}$ when n is odd, then $e(S) \leq \frac{n+1-i}{2}$. We then classify the subsets $S \subseteq D_{2n}$ where $e(S) = \infty$ and finally, we show that there does not exist any subset S in D_{12} and D_{14} such that $e(S) = 5$. In addition, we also show that there does not exist any subset $S \subseteq D_{22}$ such that $e(S) = k$ for $k = 7, 8, 9$.

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APPROVAL SHEET

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DECLARATION

I hereby declare that the thesis is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UTAR or other institutions.

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CHAPTER 1

INTRODUCTION

This thesis is mainly concerned with various properties of subsets of finite groups. It consists of two main parts: embeddings of generalized Latin squares in finite groups and exhaustion numbers of subsets of dihedral groups. We first investigate the embeddings of generalized Latin squares in finite groups. Let n be a positive integer. A *generalized Latin square of order* n is an $n \times n$ matrix such that the elements in each row and each column are distinct. Hence, a generalized Latin square of order n has at least n distinct elements. A generalized Latin square of order n is said to be *commutative* if the $n \times n$ square is symmetric. Let G be an additive (or multiplicative) group and let S be an n-subset of G. Then the addition (or multiplication) table of S will form a generalized Latin square of order n. For instance, let $S = \{a, b, c, d, e\}$ be a 5-subset of G where G is a multiplicative group. Then we have the following multiplication table of S

	a	b	c	d	e
a	a^2	ba	ca	da	ea
b	ab	b^2	cb	db	eb
c	ac	bc	c^2	dc	ec
d	ad	bd	cd	d^2	ed
e	ae	be	ce	de	e^2

which is a generalized Latin square of order 5. It is clear that if a generalized Latin square of order n is commutative, it will have at most $\frac{n(n+1)}{2}$ distinct elements. On the other hand, if a generalized Latin square of order n is non-commutative, it will have at most n^2 distinct elements. Two generalized Latin squares L and L' are *isomorphic* if L' can be obtained by performing a permutation of rows and the same permutation of columns on L. We say that two squares are in the same equivalence class if they are isomorphic to one another.

A generalized Latin square of order n is said to be *embeddable* in a group G if it is isomorphic to the addition (or multiplication table) of a subset of the group.

A generalized Latin square with exactly n distinct elements is simply called a Latin square. The study of Latin squares is usually traced back to the 36 officers problem proposed by Leonhard Euler in 1782 [8]. The problem involves arranging 36 officers of six different ranks and six different regiments into a 6×6 square so that no rank or regiment will be repeated in any row or column. Such an arrangement would form a 6×6 square with each entry being an ordered pair. Euler called this square a Graeco-Latin square, simply because he used Greek and Latin characters as entries in the square. Nowadays, Graeco-Latin squares are called orthogonal squares.

According to Laywine and Mullen [12], statistics provided a major motivation for combinatorial results pertaining to Latin squares in the early and middle decades of the 20th century. In particular, one of the major statistical applications, experimental designs, was found early in the 20th century. Latin squares can be used in the design of experiments where one wishes to control the variation in two different directions.

Latin squares are also somewhat connected to graphs. According to the *Handbook of Combinatorial Design* [7], a Latin square of order n is equivalent to a $1-$factorisation of the complete bipartite graph $K_{n,n}$ or an edge partition of the complete tripartite graph $K_{n,n,n}$ into triangles. Various studies on bipartite graphs by using Latin squares have been done, see [1] and [17]. In [14], a construction of $t-$partite graphs from Latin squares was given and it was shown that the resulting $t-$partite graphs were more useful in certain network systems.

In algebra, Latin squares can be characterized as the multiplication tables of quasigroups [13]. The embeddings of generalized Latin squares of order 2 and 3 in groups have been investigated by Freiman, see [9] and [10]. For the commutative case, Freiman [10] showed that there are altogether 15 squares of order 3. These squares can be divided into seven equivalence classes and six of these classes are embeddable in groups. For the non-commutative case, it was shown that there are altogether 573 generalized Latin squares of order 3 and these squares can be divided into 118 equivalence classes, see [10]. Of the 118 classes, 45 classes are embeddable in groups.

The generalized Latin squares of order 4 have been studied by Tan in [16]. It was shown that there are altogether 996 commutative generalized Latin squares of order 4. These squares can be divided into 82 equivalence classes, and only 25 classes are embeddable in groups. For the non-commutative case, Tan showed that there are 20 non-commutative generalized Latin squares of order 4 with 4 distinct elements. These squares can be divided into four equivalence classes and only the squares from three equivalence classes are embeddable in groups. Tan also obtained the result that there are altogether 72 non-commutative generalized Latin squares of order 4 with 15 distinct elements. These 72 squares can be divided into five equivalence classes, all of which are embeddable in groups.

In Chapter 2, we investigate the commutative and non-commutative generalized Latin squares of order 5. For the commutative squares, we list down the squares with 5, 13, 14 and 15 distinct elements. We then divide them into their equivalence classes and show that some of the squares are embeddable in groups. We also generate the non-commutative generalized Latin squares of order 5 with 5, 24 and 25 distinct elements and divide them into their equivalence classes. Finally, we determine which squares are embeddable in groups.

In Chapter 3, we will focus on non-commutative generalized Latin squares of order n with $n^2 - 1$ and n distinct elements. As mentioned earlier, Latin squares have various applications in graph theory since they can be represented by graphs. For instance, generalized Latin squares of order n with $n^2 - 1$ distinct elements can be represented by Eulerian graphs. Hence, it is possible to determine whether two Eulerian graphs are isomorphic based on the isomorphism of the squares they represent. Therefore, we begin by investigating the classes of non-commutative generalized Latin squares of order n with $n^2 - 1$ distinct elements. We show that the number of equivalence classes of generalized Latin squares of order n with $n^2 - 1$ distinct elements is four if $n = 3$ and five if $n \geq 4$. It is also shown that all these squares are embeddable in groups regardless of whether n is odd or even.

Next, we investigate the classes of non-commutative generalized Latin squares of order n with n distinct elements. For this case, we show that certain constructions of non-commutative generalized Latin squares of order n with n distinct elements are only embeddable in groups when n is odd. We first show the existence of at least three non-isomorphic non-commutative generalized Latin squares of order n with n distinct elements which are embeddable in groups when $n \geq 5$ is odd. By using a similar construction for the case when $n \geq 4$ is even, we show that certain non-commutative generalized Latin squares of order n are not embeddable in groups.

It is important to note that there are different ways to generalize Latin squares. One such generalization was studied in [3] where the squares investigated are called perfect $\langle k, l\rangle$-Latin squares. A perfect $\langle k, l\rangle$-Latin square $A = (a_{i,j})$ of order n with m elements is an $n \times n$ matrix in which any row or column contains every distinct element and the element $a_{i,j}$ appears exactly k times in the i-th row and l times in the j-th column, or vice versa. Therefore, the elements are allowed to appear more than once in each row and each column. Throughout this thesis, we will investigate the generalized Latin squares with distinct elements in each row and each column as defined on Page 1.

In the second part of this thesis, we look at the exhaustion numbers of subsets of dihedral groups. Let G be a finite group. For k nonempty subsets $A_1, \ldots, A_k$ of G, the product $A_1 \cdots A_k$ is defined by

$$A_1 \cdots A_k = \{a_1 \cdots a_k | a_i \in A_i, 1 \leq i \leq k\}.$$

In the case $A_1 = A_2 = \cdots = A_k = A$, we use A^k to denote $A_1 \cdots A_k$. A nonempty subset A of G is called a *k-basis* of G if $A^k = G$. Let S be a nonempty subset of G. If there exists a positive integer k such that $S^k = G$, then S is said to be *exhaustive*. The minimal integer $k > 0$ such that S is exhaustive is called the *exhaustion number* of the set S and is denoted by $e(S)$. If $e(S) = \infty$, S is said to be *non-exhaustive*. If $e(S) = n$, then S is an n-basis of G. Note that if S is an n-basis of G, then $e(S) \leq n$.

The exhaustion numbers of various subsets of finite abelian groups have been investigated in [5] and [15]. In [5], the exhaustion numbers of subsets of cyclic groups which are in arithmetic progression were determined and some upper bounds for the exhaustion numbers of subsets of the cyclic group $\mathbb{Z}/p$ where p is an odd prime were also obtained.

Since the study of exhaustion numbers of subsets of abelian groups has been done extensively in [5] and [15], we are now interested in the exhaustion numbers of subsets of non-abelian groups. In the following table, we list some examples of non-abelian groups up to order 16 where D_{2n} is the dihedral group of order $2n$ for $n \geq 3$, Q_{2^n} is the generalized quaternion group of order 2^n for $n \geq 3$, A_4 is the alternating group of degree 4 and Z_2 is the cyclic group of order 2.

Table 1.1: Example of non-abelian groups

G	$\|G\|$
D_6	6
D_8, Q_8	8
D_{10}	10
D_{12}, A_4	12
D_{14}	14
D_{16}, $D_8 \times \mathbb{Z}_2$, Q_{16}, $Q_8 \times \mathbb{Z}_2$	16

From Table 1.1, we can deduce that for every even integer $k \geq 6$ there exist a non-abelian group of order k, which is the dihedral group of order k. Therefore we will investigate the exhaustion numbers of subsets of dihedral groups. The dihedral group of order $2n$ is often called the group of symmetries of a regular n-gon [11]. It has the presentation

$$D_{2n} = \langle x, y | x^n = y^2 = 1, yx = x^{n-1}y \rangle$$

and hence consists of the elements $\{1, x, x^2, \ldots, x^{n-1}, y, xy, \ldots, x^{n-1}y\}$. Geometrically, every element of D_{2n} is either a rotation or a reflection. In the presentation given here, the elements $\{1, x, x^2, \ldots, x^{n-1}\}$ are rotations and the elements $\{y, xy, \ldots, x^{n-1}y\}$ are reflections. It is important to note that the properties of D_{2n} depend on whether n is odd or even. For instance, when $n \geq 3$ is odd, the center of D_{2n} is trivial but when $n \geq 3$ is even, the center of D_{2n} is $\{1, x^{\frac{n}{2}}\}$ [2].

A generating set of a group G is a subset S that is not contained in any proper subgroups of G. Equivalently, a subset S of a group G is a generating set if every element of G can be written as a product of elements from S or their inverses. A minimal generating set of a group G is a generating set X such that no proper subset of X is a generating set of G. The exhaustion numbers of the non-minimal generating set $S = \{1, x, y\}$ of the dihedral group, quaternion group and semi-dihedral group have been determined by Tan in [16]. For the non-minimal generating set $S = \{1, x, y\}$ of the dihedral group, Tan showed that $e(S) = \frac{n+2}{2}$ when n is even and $e(S) = \frac{n+1}{2}$ when n is odd. It is clear that if a subset S is exhaustive, then it is a generating set. However, a generating set S is not necessarily exhaustive. For example, the subset $S = \{x, y\} \subseteq D_{2n}$ is a generating set of D_{2n} but $e(S) = \infty$ [16].

In Chapter 4 we study the exhaustion numbers of subsets of dihedral groups. We begin by investigating the subsets with finite exhaustion numbers. We first give some constructions of subsets with certain finite exhaustion numbers. We show that for any subset $S \subseteq D_{2n}$, $e(S) = 2$ if $n < |S| \leq 2n - 1$. Next, we show that if $S = \{1, x, y, xy, x^2y, \ldots, x^iy\}$ for $i \in \{1, 3, 5, \ldots, n-3\}$, then $e(S) \leq \frac{n+1-i}{2}$ when n is even. We also show that if $S = \{1, x, y, xy, x^2y, \ldots, x^iy\}$ for $i \in \{2, 4, 6, \ldots, n-3\}$, then $e(S) \leq \frac{n+1-i}{2}$ when n is odd. We then classify the subsets $S \subseteq D_{2n}$ where $e(S) = \infty$ and finally, we show that there does not exist any subset S in D_{12} and D_{14} such that $e(S) = 5$. In order to consider a wider range of $e(S)$ for which there does not exist any subset $S \subseteq D_{2n}$, we also show that there does not exist any subset $S \subseteq D_{22}$ such that $e(S) = k$ for $k = 7, 8, 9$.

CHAPTER 2

GENERALIZED LATIN SQUARES OF ORDER 5

2.1 Introduction

Let n be a positive integer. A *generalized Latin square of order* n is an $n \times n$ matrix such that the elements in each row and each column are distinct. If the $n \times n$ matrix is symmetric, then the generalized Latin square is said to be *commutative*. We use the matrix notation $L = (a_{ij})_n$ to denote the generalized Latin square L of order n where a_{ij} is the entry in the ith row and jth column ($i, j \in \{1, \ldots, n\}$).

Let G be a group with an n-subset S. By designating equal products by the same letter and unequal products by distinct letters in the addition (or multiplication) table of S, we obtain a generalized Latin square of order n. As the forms of squares representing addition (or multiplication) tables for a given set of elements of a group depend on the order in which these elements are taken, the following definition (see [10]) is necessary: Let $\mathcal{S}$ and $\mathcal{T}$ be generalized Latin squares of the same order with the same number of distinct elements, and let θ be a one-to-one mapping of the elements occurring in $\mathcal{S}$ onto those occurring in $\mathcal{T}$. Let $\theta[\mathcal{S}]$ denote the generalized Latin square obtained by applying θ to $\mathcal{S}$. If $\mathcal{T}$ can be obtained from $\theta[\mathcal{S}]$ by a permutation of rows and the same permutation of columns of $\theta[\mathcal{S}]$, then $\mathcal{S}$ and $\mathcal{T}$ are said to be isomorphic. It is not difficult to see that isomorphism defined in this manner is an equivalence relation. We may thus divide the squares into equivalence classes where two squares belong to the same equivalence class if they are isomorphic to one another. We say that a square is embeddable in a group G if it is isomorphic to the addition (or multiplication) table of a subset of G. In this chapter, we shall investigate the commutative and non-commutative generalized Latin squares of order 5. We will first generate the commutative generalized Latin squares of order 5 with 5, 13, 14 and 15 distinct elements. We then divide them into their equivalence classes and determine which squares are embeddable in groups by giving examples of finite abelian groups that contain these squares. By using a similar approach, we will classify the non-commutative generalized Latin squares of order 5 with 5, 24 and 25 distinct elements, and finally determine which squares are embeddable in groups.

2.2 Commutative Squares

In this section, we shall investigate the commutative generalized Latin squares of order 5 with 5, 13, 14 and 15 distinct elements. Note that a commutative generalized Latin square of order n has at least n distinct elements and at most $\frac{n(n+1)}{2}$ distinct elements. Let L be a commutative generalized Latin square of order 5. Then L will take the form of an addition table of a 5-subset of an abelian group G. We start by showing that for any integer $k \in \{5, 6, \ldots, 15\}$, there exist an abelian group G and a commutative 5-subset S of G such that $|S + S| = k$.

$k = 5$: $G = \mathbb{Z}_5$

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

$k = 6$: $G = \mathbb{Z}_6$

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	5
2	2	3	4	5	0
3	3	4	5	0	1
4	4	5	0	1	2

$k = 7$: $G = \mathbb{Z}_7$

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	5
2	2	3	4	5	6
3	3	4	5	6	0
4	4	5	6	0	1

$k = 8$: $G = \mathbb{Z}_8$

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	5
2	2	3	4	5	6
3	3	4	5	6	7
4	4	5	6	7	0

$k = 9$: $G = \mathbb{Z}_9$

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	5
2	2	3	4	5	6
3	3	4	5	6	7
4	4	5	6	7	8

$k = 10$: $G = \mathbb{Z}_6 \times \mathbb{Z}_2$

+	$(0,0)$	$(1,0)$	$(2,0)$	$(3,0)$	$(3,1)$
$(0,0)$	$(0,0)$	$(1,0)$	$(2,0)$	$(3,0)$	$(3,1)$
$(1,0)$	$(1,0)$	$(2,0)$	$(3,0)$	$(4,0)$	$(4,1)$
$(2,0)$	$(2,0)$	$(3,0)$	$(4,0)$	$(5,0)$	$(5,1)$
$(3,0)$	$(3,0)$	$(4,0)$	$(5,0)$	$(0,0)$	$(0,1)$
$(3,1)$	$(3,1)$	$(4,1)$	$(5,1)$	$(0,1)$	$(0,0)$

$k = 11$: $G = \mathbb{Z}_6 \times \mathbb{Z}_2$

+	$(0,0)$	$(1,0)$	$(2,0)$	$(3,1)$	$(0,1)$
$(0,0)$	$(0,0)$	$(1,0)$	$(2,0)$	$(3,1)$	$(0,1)$
$(1,0)$	$(1,0)$	$(2,0)$	$(3,0)$	$(4,1)$	$(1,1)$
$(2,0)$	$(2,0)$	$(3,0)$	$(4,0)$	$(5,1)$	$(2,1)$
$(3,1)$	$(3,1)$	$(4,1)$	$(5,1)$	$(0,0)$	$(3,0)$
$(0,1)$	$(0,1)$	$(1,1)$	$(2,1)$	$(3,0)$	$(0,0)$

$k = 12$: $G = \mathbb{Z}_6 \times \mathbb{Z}_2$

+	$(0,0)$	$(1,0)$	$(2,0)$	$(4,1)$	$(1,1)$
$(0,0)$	$(0,0)$	$(1,0)$	$(2,0)$	$(4,1)$	$(1,1)$
$(1,0)$	$(1,0)$	$(2,0)$	$(3,0)$	$(5,1)$	$(2,1)$
$(2,0)$	$(2,0)$	$(3,0)$	$(4,0)$	$(0,1)$	$(3,1)$
$(4,1)$	$(4,1)$	$(5,1)$	$(0,1)$	$(2,0)$	$(5,0)$
$(1,1)$	$(1,1)$	$(2,1)$	$(3,1)$	$(5,0)$	$(2,0)$

$k = 13$: $G = \mathbb{Z}_{31}$

+	0	1	2	4	9
0	0	1	2	4	9
1	1	2	3	5	10
2	2	3	4	6	11
4	4	5	6	8	13
9	9	10	11	13	18

$k = 14$: $G = \mathbb{Z}_{31}$

+	0	1	2	5	11
0	0	1	2	5	11
1	1	2	3	6	12
2	2	3	4	7	13
5	5	6	7	10	16
11	11	12	13	16	22

$k = 15$: $G = \mathbb{Z}_{31}$

+	0	1	3	7	15
0	0	1	3	7	15
1	1	2	4	8	16
3	3	4	6	10	18
7	7	8	10	14	22
15	15	16	18	22	30

In Sections 2.2.1, 2.2.2 and 2.2.3, we shall investigate the commutative generalized Latin squares of order 5 with 5, 13, 14 and 15 distinct elements and determine which squares are embeddable in groups. Recall that two Latin squares L and L' are isomorphic if L' can be obtained by performing a permutation of rows and the same permutation of columns on L. We will use Algorithm 1, which is based on the algorithm in [16], to generate the squares and divide them into their equivalence classes.

Algorithm 1: Generate and divide commutative generalized Latin squares of order 5 into their equivalence classes

1. Generate commutative generalized Latin square S_i for $i \geq 1$.
2. Assign ordinal number i to S_i.
3. If $i = 1$, create new equivalence class with S_1 as the representative square.
4. Else,
   - 4.1. Perform permutations on S_i.
   - 4.2. Rename elements in S_i.
   - 4.3. Compare with representative squares S_j for $j < i$.
     - 4.3.1. If S_i is isomorphic to S_j, add S_i to equivalence class represented by S_j.
     - 4.3.2. Else, create new equivalence class with S_i as the representative square.

We first summarize the number of commutative generalized Latin squares of order 5 in the following table.

Table 2.1: Number of commutative generalized Latin squares of order 5

Number of distinct elements	Number of squares
5	6
6	3066
7	54765

Table 2.1: (Continued)

Number of distinct elements	Number of squares
8	216085
9	311490
10	204697
11	68471
12	12235
13	1165
14	55
15	1
Total:	872036

2.2.1 Squares with 5 Distinct Elements

By using Algorithm 1, we find that there are altogether six commutative generalized Latin squares of order 5 with 5 distinct elements:

1	2	3	4	5	6
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B C D E A	B C E A D	B D A E C	B D E C A	B E A C D	B E D A C
C D E A B	C E D B A	C A E B D	C E B A D	C A D E B	C D B E A
D E A B C	D A B E C	D E B C A	D C A E B	D C E B A	D A E C B
E A B C D	E D A C B	E C D A B	E A D B C	E D B A C	E C A B D

It is straightforward to check that all six squares belong to the same equivalence class and are embeddable in groups. In the following, we show that Square 1 is embeddable in $\mathbb{Z}_5$.

Square 1: Embeddable in $\mathbb{Z}_5$.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

2.2.2 Squares with 13 Distinct Elements

We now look at the commutative generalized Latin squares of order 5 with 13 distinct elements. By using Algorithm 1, we find that there are altogether 1165 squares as listed in Appendix B. These squares can be divided into 21 equivalence classes as follows:

Table 2.2: Equivalence classes of commutative generalized Latin squares of order 5 with 13 distinct elements

Representative	Squares in the same equivalence class
1 A B C D E B A D F G C D H I J D F I K L E G J L M	2, 3, 4, 5, 6, 217, 236, 238, 259, 291, 293, 340, 351, 357, 390, 408, 412, 453, 477, 483, 504, 514, 522, 523, 524, 527, 528, 646, 648, 650, 652, 767, 770, 794, 815, 816, 852, 869, 870, 906, 909, 922, 929, 937, 967, 978, 982, 983, 984, 1041, 1043, 1085, 1087, 1121, 1124, 1130, 1136, 1152, 1154
7 A B C D E B A F G H C F A I J D G I K L E H J L M	21, 34, 531, 544, 657, 670, 988, 1049, 1095
8 A B C D E B A F G H C F B I J D G I K L E H J L M	22, 35, 50, 107, 163, 179, 315, 443, 533, 546, 592, 636, 658, 671, 688, 732, 747, 827, 890, 957, 991, 1031, 1051, 1078, 1096, 1110, 1120, 1147, 1165
9 A B C D E B A F G H C F D I J D G I K L E H J L M	10, 11, 12, 23, 24, 25, 26, 36, 37, 38, 39, 64, 70, 77, 83, 93, 113, 120, 128, 136, 150, 169, 171, 194, 196, 207, 209, 302, 318, 329, 333, 417, 431, 446, 448, 525, 529, 532, 534, 536, 538, 545, 547, 549, 551, 561, 565, 574, 578, 596, 605, 613, 623, 640, 644, 645, 649, 654, 655, 659, 660, 667, 668, 672, 673, 684, 701, 705, 719, 728, 736, 744, 761, 762, 821, 841, 842, 878, 887, 892, 945, 951, 959, 985, 987, 989, 990, 994, 996, 1002, 1006, 1016, 1023, 1035, 1038, 1040, 1044, 1046, 1048, 1050, 1053, 1063, 1066, 1074, 1081, 1083, 1086, 1090, 1091, 1093, 1094, 1103, 1107, 1115, 1118, 1139, 1145, 1157, 1161

Table 2.2: (Continued)

Representative	Squares in the same equivalence class
13 A B C D E B A F G H C F I A J D G A K L E H J L M	17, 29, 66, 79, 95, 123, 139, 153, 190, 203, 298, 325, 413, 427, 539, 562, 575, 608, 626, 664, 703, 721, 754, 838, 871, 942, 1003, 1018, 1064
14 A B C D E B A F G H C F I B J D G B K L E H J L M	18, 30, 220, 232, 262, 287, 336, 354, 393, 404, 449, 480, 500, 517, 541, 665, 773, 800, 808, 854, 866, 912, 915, 935, 964, 980, 1129, 1138, 1156
15 A B C D E B A F G H C F I E J D G E K L E H J L M	16, 19, 20, 31, 32, 245, 249, 251, 253, 274, 278, 280, 295, 364, 368, 370, 372, 376, 380, 396, 410, 464, 468, 470, 485, 486, 490, 506, 520, 526, 530, 540, 543, 647, 651, 662, 663, 780, 784, 787, 788, 797, 812, 848, 862, 893, 897, 900, 914, 919, 932, 960, 974, 986, 1042, 1084, 1127, 1133, 1148
27 A B C D E B A F G H C F I J K D G J I L E H K L M	40, 42, 535, 548, 552, 653, 666, 674, 992, 995, 1045, 1052, 1088, 1092
28 A B C D E B A F G H C F I J K D G J K L E H K L M	33, 41, 85, 126, 156, 211, 331, 433, 537, 542, 550, 582, 609, 627, 656, 661, 669, 697, 715, 758, 834, 875, 938, 993, 1009, 1020, 1047, 1059, 1089
43 A B C D E B C A F G C A H I J D F I K L E G J L M	51, 88, 108, 133, 164, 176, 312, 440, 583, 594, 618, 638, 678, 689, 713, 733, 742, 825, 885, 955, 1025, 1034, 1071, 1080, 1102, 1111, 1114, 1144, 1164

Table 2.2: (Continued)

Representative	Squares in the same equivalence class
44 A B C D E B C D F G C D H I J D F I K L E G J L M	45, 57, 61, 89, 90, 100, 116, 134, 135, 147, 159, 172, 174, 182, 186, 212, 218, 221, 223, 226, 233, 234, 256, 260, 264, 269, 272, 288, 289, 297, 304, 320, 337, 338, 342, 345, 352, 355, 373, 391, 395, 398, 401, 405, 407, 423, 435, 450, 452, 459, 462, 478, 481, 501, 503, 508, 511, 515, 519, 584, 585, 602, 619, 620, 633, 676, 679, 696, 711, 714, 727, 738, 750, 768, 771, 789, 796, 801, 802, 807, 810, 811, 820, 832, 846, 853, 855, 859, 861, 867, 868, 880, 907, 910, 917, 918, 923, 928, 931, 936, 949, 965, 966, 971, 973, 979, 981, 1026, 1027, 1069, 1072, 1099, 1101, 1123, 1126, 1132, 1137, 1153, 1155
46 A B C D E B C F A G C F H I J D A I K L E G J L M	48, 54, 55, 86, 91, 111, 112, 129, 131, 167, 168, 177, 178, 313, 314, 441, 442, 553, 555, 557, 559, 588, 597, 599, 614, 641, 643, 683, 686, 691, 709, 730, 735, 741, 743, 822, 826, 884, 886, 952, 956, 997, 999, 1011, 1015, 1037, 1039, 1057, 1058, 1077, 1082, 1106, 1109, 1112, 1113, 1140, 1143, 1159, 1162
47 A B C D E B C F E G C F H I J D E I K L E G J L M	49, 59, 63, 87, 92, 102, 118, 130, 132, 149, 161, 173, 175, 183, 187, 215, 219, 228, 237, 239, 243, 244, 254, 261, 265, 285, 286, 292, 294, 296, 305, 321, 334, 341, 350, 353, 358, 359, 363, 382, 389, 392, 397, 409, 411, 424, 436, 454, 458, 472, 476, 479, 484, 495, 496, 505, 507, 516, 521, 554, 556, 558, 560, 589, 591, 604, 615, 617, 635, 681, 682, 694, 707, 708, 725, 737, 749, 763, 765, 769, 772, 775, 777, 795, 799, 813, 814, 818, 830, 849, 851, 864, 865, 879, 901, 903, 908, 911, 920, 921, 930, 934, 947, 962, 963, 975, 977, 998, 1000, 1012, 1014, 1055, 1056, 1122, 1125, 1131, 1134, 1149, 1150
52 A B C D E B C F G H C F D I J D G I K L E H J L M	53, 56, 60, 65, 68, 78, 81, 94, 97, 99, 109, 110, 115, 121, 125, 137, 141, 146, 151, 155, 158, 165, 166, 180, 181, 184, 188, 191, 192, 204, 205, 299, 300, 306, 316, 317, 322, 326, 328, 414, 416, 425, 428, 430, 437, 444, 445, 563, 566, 576, 579, 586, 593, 595, 600, 607, 612, 621, 625, 630, 631, 637, 639, 675, 685, 690, 695, 702, 704, 710, 720, 722, 726, 729, 734, 745, 748, 753, 756, 757, 823, 828, 833, 839, 840, 873, 874, 883, 888, 891, 943, 944, 950, 953, 958, 1005, 1008, 1019, 1024, 1028, 1030, 1032, 1033, 1065, 1067, 1068, 1073, 1075, 1079, 1097, 1100, 1105, 1108, 1117, 1119, 1142, 1146, 1160, 1163
58 A B C D E B C F G H C F I E J D G E K L E H J L M	62, 101, 117, 148, 160, 185, 189, 246, 247, 250, 252, 275, 276, 279, 281, 307, 323, 365, 367, 369, 371, 377, 378, 381, 403, 426, 438, 465, 467, 469, 471, 487, 489, 491, 513, 601, 632, 693, 724, 740, 752, 782, 783, 785, 786, 792, 805, 819, 831, 843, 856, 882, 895, 896, 898, 899, 926, 948, 968

Table 2.2: (Continued)

Representative	Squares in the same equivalence class
67 A B C D E B C F G H C F I J K D G J E L E H K L M	71, 80, 84, 96, 119, 122, 127, 138, 152, 157, 162, 195, 197, 208, 210, 303, 324, 330, 332, 418, 432, 434, 439, 564, 567, 577, 580, 603, 606, 611, 624, 629, 634, 692, 698, 700, 716, 718, 723, 739, 751, 759, 760, 817, 829, 836, 837, 876, 877, 881, 940, 941, 946, 1004, 1007, 1017, 1021, 1060, 1061
69 A B C D E B C F G H C F I J K D G J H L E H K L M	72, 76, 82, 98, 103, 114, 124, 140, 142, 154, 170, 193, 201, 202, 206, 301, 311, 319, 327, 415, 422, 429, 447, 568, 569, 572, 581, 590, 598, 610, 616, 628, 642, 680, 687, 699, 706, 717, 731, 746, 755, 824, 835, 872, 889, 939, 954, 1001, 1010, 1013, 1022, 1036, 1054, 1062, 1076, 1104, 1116, 1141, 1158
73 A B C D E B C F G H C F I J K D G J L B E H K B M	75, 104, 106, 143, 145, 199, 200, 222, 230, 235, 240, 263, 267, 282, 290, 309, 310, 339, 347, 356, 361, 384, 386, 394, 406, 420, 421, 451, 455, 474, 482, 492, 498, 502, 518, 571, 573, 587, 622, 677, 712, 774, 779, 798, 809, 850, 863, 905, 913, 916, 933, 961, 976, 1029, 1070, 1098, 1128, 1135, 1151
74 A B C D E B C F G H C F I J K D G J L C E H K C M	105, 144, 198, 248, 277, 308, 366, 379, 419, 466, 488, 570, 781, 894
213 A B C D E B F D C G C D H I J D C I K L E G J L M	224, 257, 270, 343, 375, 399, 460, 509, 791, 804, 847, 860, 925, 972
214 A B C D E B F D E G C D H I J D E I K L E G J L M	216, 225, 227, 229, 231, 241, 242, 255, 258, 266, 268, 271, 273, 283, 284, 335, 344, 346, 348, 349, 360, 362, 374, 383, 385, 387, 388, 400, 402, 456, 457, 461, 463, 473, 475, 493, 494, 497, 499, 510, 512, 764, 766, 776, 778, 790, 793, 803, 806, 844, 845, 857, 858, 902, 904, 924, 927, 969, 970

In Proposition 2.1, we will see that some of the squares in Table 2.2 are not embeddable in any group.

Proposition 2.1. *Let L be a generalized Latin square of order 5 with 13 distinct elements. If L is isomorphic to Squares 1, 43, 44, 46, 47, 213 or 214, then L is not embeddable in any group.*

Proof. In the following, we will investigate each case separately:

(i) Let L be isomorphic to Square 1. Suppose that L is embeddable in a group G. Then there exist distinct elements $a,\ b,\ c,\ d,\ e \in G$ such that $2a = 2b$ and $b + c = a + d$. It follows from this that $2a + b + c = a + 2b + d$ and hence we have $a + c = b + d$, which is a contradiction.

(ii) Suppose that there exists a group G and elements $a,\ b,\ c,\ d,\ e \in G$ such that the addition table of the subset $\{a, b, c, d, e\}$ is Square 43. Then $2a = b + c$ and $2b = a + c$. It follows from this that $2a + 2b = a + b + 2c$ and hence we have $a + b = 2c$, which is a contradiction.

(iii) Suppose that there exists a group G and elements $a,\ b,\ c,\ d,\ e \in G$ such that the addition table of the subset $\{a, b, c, d, e\}$ is Square 44. Then we have $a + c = 2b$ and $b + c = a + d$, which gives us $a + b + 2c = a + 2b + d$. It follows that $2c = b + d$, which is a contradiction.

(iv) Suppose that there exists a group G and elements $a,\ b,\ c,\ d,\ e \in G$ such that the addition table of the subset $\{a, b, c, d, e\}$ is Square 46. Then $2a = b + d$ and $2b = a + c$. It follows from this that $2a + 2b = a + b + c + d$. Then we have $a + b = c + d$, which is a contradiction.

(v) Let L be isomorphic to Square 47. Suppose that L is embeddable in a group G. Then there exist distinct elements $a,\ b,\ c,\ d,\ e \in G$ such that $a + c = 2b$ and $b + d = a + e$. Then we obtain $a + b + c + d = a + 2b + e$ but this gives us $c + d = b + e$; a contradiction.

(vi) Suppose that there exists a group G and elements $a,\ b,\ c,\ d,\ e \in G$ such that the addition table of the subset $\{a, b, c, d, e\}$ is Square 213. Then $a + c =$

$b + d$ and $a + d = b + c$, which gives us $2a + c + d = 2b + c + d$. Hence, we have $2a = 2b$, which is a contradiction.

(vii) Suppose that there exists a group G and elements a, b, c, d, $e \in G$ such that the addition table of the subset $\{a, b, c, d, e\}$ is Square 214. Then we have $a + d = b + c$ and $b + d = a + e$. It follows from this that $a + b + 2d = a + b + c + e$. As a result, we have $2d = c + e$; a contradiction.

□

From Proposition 2.1, we see that the squares isomorphic to Squares 1, 43, 44, 46, 47, 213 and 214 are not embeddable in any group. The 14 remaining representative squares (Squares 7, 8, 9, 13, 14, 15, 27, 28, 52, 58, 67, 69, 73 and 74) are embeddable in finite abelian groups. In the following, we give examples of finite abelian groups which contain subsets S such that the addition table of S is isomorphic to the 14 remaining representative squares.

Square 7: Embeddable in $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_6$.

+	$(0,0,0)$	$(0,2,0)$	$(0,0,3)$	$(0,1,1)$	$(0,0,1)$
$(0,0,0)$	$(0,0,0)$	$(0,2,0)$	$(0,0,3)$	$(0,1,1)$	$(0,0,1)$
$(0,2,0)$	$(0,2,0)$	$(0,0,0)$	$(0,2,3)$	$(0,3,1)$	$(0,2,1)$
$(0,0,3)$	$(0,0,3)$	$(0,2,3)$	$(0,0,0)$	$(0,1,4)$	$(0,0,4)$
$(0,1,1)$	$(0,1,1)$	$(0,3,1)$	$(0,1,4)$	$(0,2,2)$	$(0,1,2)$
$(0,0,1)$	$(0,0,1)$	$(0,2,1)$	$(0,0,4)$	$(0,1,2)$	$(0,0,2)$

Square 8: Embeddable in $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3$.

+	$(0,0,0)$	$(0,2,0)$	$(0,1,0)$	$(1,1,1)$	$(0,0,1)$
$(0,0,0)$	$(0,0,0)$	$(0,2,0)$	$(0,1,0)$	$(1,1,1)$	$(0,0,1)$
$(0,2,0)$	$(0,2,0)$	$(0,0,0)$	$(0,3,0)$	$(1,3,1)$	$(0,2,1)$
$(0,1,0)$	$(0,1,0)$	$(0,3,0)$	$(0,2,0)$	$(1,2,1)$	$(0,1,1)$
$(1,1,1)$	$(1,1,1)$	$(1,3,1)$	$(1,2,1)$	$(0,2,2)$	$(1,1,2)$
$(0,0,1)$	$(0,0,1)$	$(0,2,1)$	$(0,1,1)$	$(1,1,2)$	$(0,0,2)$

Square 9: Embeddable in $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_6$.

+	$(0,0,0)$	$(0,2,0)$	$(1,2,2)$	$(0,0,4)$	$(0,1,1)$
$(0,0,0)$	$(0,0,0)$	$(0,2,0)$	$(1,2,2)$	$(0,0,4)$	$(0,1,1)$
$(0,2,0)$	$(0,2,0)$	$(0,0,0)$	$(1,0,2)$	$(0,2,4)$	$(0,3,1)$
$(1,2,2)$	$(1,2,2)$	$(1,0,2)$	$(0,0,4)$	$(1,2,0)$	$(1,3,3)$
$(0,0,4)$	$(0,0,4)$	$(0,2,4)$	$(1,2,0)$	$(0,0,2)$	$(0,1,5)$
$(0,1,1)$	$(0,1,1)$	$(0,3,1)$	$(1,3,3)$	$(0,1,5)$	$(0,2,2)$

Square 13: Embeddable in $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_6$.

+	$(0,0,0)$	$(0,2,0)$	$(0,1,1)$	$(0,3,5)$	$(1,0,4)$
$(0,0,0)$	$(0,0,0)$	$(0,2,0)$	$(0,1,1)$	$(0,3,5)$	$(1,0,4)$
$(0,2,0)$	$(0,2,0)$	$(0,0,0)$	$(0,3,1)$	$(0,1,5)$	$(1,2,4)$
$(0,1,1)$	$(0,1,1)$	$(0,3,1)$	$(0,2,2)$	$(0,0,0)$	$(1,1,5)$
$(0,3,5)$	$(0,3,5)$	$(0,1,5)$	$(0,0,0)$	$(0,2,4)$	$(1,3,3)$
$(1,0,4)$	$(1,0,4)$	$(1,2,4)$	$(1,1,5)$	$(1,3,3)$	$(0,0,2)$

Square 14: Embeddable in $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_6$.

+	$(0,0,0)$	$(0,2,0)$	$(0,1,2)$	$(0,1,4)$	$(1,0,2)$
$(0,0,0)$	$(0,0,0)$	$(0,2,0)$	$(0,1,2)$	$(0,1,4)$	$(1,0,2)$
$(0,2,0)$	$(0,2,0)$	$(0,0,0)$	$(0,3,2)$	$(0,3,4)$	$(1,2,2)$
$(0,1,2)$	$(0,1,2)$	$(0,3,2)$	$(0,2,4)$	$(0,2,0)$	$(1,1,4)$
$(0,1,4)$	$(0,1,4)$	$(0,3,4)$	$(0,2,0)$	$(0,2,2)$	$(1,1,0)$
$(1,0,2)$	$(1,0,2)$	$(1,2,2)$	$(1,1,4)$	$(1,1,0)$	$(0,0,4)$

Square 15: Embeddable in $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_7$.

+	$(0,0,0)$	$(0,2,0)$	$(1,0,1)$	$(1,0,3)$	$(0,0,4)$
$(0,0,0)$	$(0,0,0)$	$(0,2,0)$	$(1,0,1)$	$(1,0,3)$	$(0,0,4)$
$(0,2,0)$	$(0,2,0)$	$(0,0,0)$	$(1,2,1)$	$(1,2,3)$	$(0,2,4)$
$(1,0,1)$	$(1,0,1)$	$(1,2,1)$	$(0,0,2)$	$(0,0,4)$	$(1,0,5)$
$(1,0,3)$	$(1,0,3)$	$(1,2,3)$	$(0,0,4)$	$(0,0,6)$	$(1,0,0)$
$(0,0,4)$	$(0,0,4)$	$(0,2,4)$	$(1,0,5)$	$(1,0,0)$	$(0,0,1)$

Square 27: Embeddable in $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_5$.

+	$(0,0,0)$	$(0,2,0)$	$(0,0,1)$	$(1,0,1)$	$(1,1,2)$
$(0,0,0)$	$(0,0,0)$	$(0,2,0)$	$(0,0,1)$	$(1,0,1)$	$(1,1,2)$
$(0,2,0)$	$(0,2,0)$	$(0,0,0)$	$(0,2,1)$	$(1,2,1)$	$(1,3,2)$
$(0,0,1)$	$(0,0,1)$	$(0,2,1)$	$(0,0,2)$	$(1,0,2)$	$(1,1,3)$
$(1,0,1)$	$(1,0,1)$	$(1,2,1)$	$(1,0,2)$	$(0,0,2)$	$(0,1,3)$
$(1,1,2)$	$(1,1,2)$	$(1,3,2)$	$(1,1,3)$	$(0,1,3)$	$(0,2,4)$

Square 28: Embeddable in $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_7$.

+	$(0,0,0)$	$(0,2,0)$	$(1,0,2)$	$(0,0,3)$	$(1,0,4)$
$(0,0,0)$	$(0,0,0)$	$(0,2,0)$	$(1,0,2)$	$(0,0,3)$	$(1,0,4)$
$(0,2,0)$	$(0,2,0)$	$(0,0,0)$	$(1,2,2)$	$(0,2,3)$	$(1,2,4)$
$(1,0,2)$	$(1,0,2)$	$(1,2,2)$	$(0,0,4)$	$(1,0,5)$	$(0,0,6)$
$(0,0,3)$	$(0,0,3)$	$(0,2,3)$	$(1,0,5)$	$(0,0,6)$	$(1,0,0)$
$(1,0,4)$	$(1,0,4)$	$(1,2,4)$	$(0,0,6)$	$(1,0,0)$	$(0,0,1)$

Square 52: Embeddable in $\mathbb{Z}_{31}$.

+	0	1	2	4	9
0	0	1	2	4	9
1	1	2	3	5	10
2	2	3	4	6	11
4	4	5	6	8	13
9	9	10	11	13	18

Square 58: Embeddable in $\mathbb{Z}_{31}$.

+	0	1	2	5	7
0	0	1	2	5	7
1	1	2	3	6	8
2	2	3	4	7	9
5	5	6	7	10	12
7	7	8	9	12	14

Square 67: Embeddable in $\mathbb{Z}_{31}$.

+	0	1	2	5	10
0	0	1	2	5	10
1	1	2	3	6	11
2	2	3	4	7	12
5	5	6	7	10	15
10	10	11	12	15	20

Square 69: Embeddable in $\mathbb{Z}_{31}$.

+	0	1	2	5	9
0	0	1	2	5	9
1	1	2	3	6	10
2	2	3	4	7	11
5	5	6	7	10	14
9	9	10	11	14	18

Square 73: Embeddable in $\mathbb{Z}_{31}$.

+	0	1	2	5	27
0	0	1	2	5	27
1	1	2	3	6	28
2	2	3	4	7	29
5	5	6	7	10	1
27	27	28	29	1	23

Square 74: Embeddable in $\mathbb{Z}_{31}$.

+	0	1	2	5	28
0	0	1	2	5	28
1	1	2	3	6	29
2	2	3	4	7	30
5	5	6	7	10	2
28	28	29	30	2	25

2.2.3 Squares with 14 and 15 Distinct Elements

Next, we shall look at the commutative generalized Latin squares of order 5 with 14 distinct elements. There are altogether 55 such squares as follows:

1	2	3	4	5
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B A F G H	B C F G H	B D F G H	B E F G H	B F A G H
C F I J K	C F I J K	C F I J K	C F I J K	C A I J K
D G J L M	D G J L M	D G J L M	D G J L M	D G J L M
E H K M N	E H K M N	E H K M N	E H K M N	E H K M N
6	7	8	9	10
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F D G H	B F E G H	B F G A H	B F G C H	B F G E H
C D I J K	C E I J K	C G I J K	C G I J K	C G I J K
D G J L M	D G J L M	D A J L M	D C J L M	D E J L M
E H K M N	E H K M N	E H K M N	E H K M N	E H K M N
11	12	13	14	15
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H A	B F G H C	B F G H D	B F G H I	B F G H I
C G I J K	C G I J K	C G I J K	C G A J K	C G B J K
D H J L M	D H J L M	D H J L M	D H J L M	D H J L M
E A K M N	E C K M N	E D K M N	E I K M N	E I K M N

16	17	18	19	20
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G D J K	C G E J K	C G F J K	C G H J K	C G I J K
D H J L M	D H J L M	D H J L M	D H J L M	D H J L M
E I K M N	E I K M N	E I K M N	E I K M N	E I K M N

21	22	23	24	25
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J A K	C G J B K	C G J E K	C G J F K	C G J I K
D H A L M	D H B L M	D H E L M	D H F L M	D H I L M
E I K M N	E I K M N	E I K M N	E I K M N	E I K M N

26	27	28	29	30
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K A	C G J K B	C G J K D	C G J K F	C G J K H
D H K L M	D H K L M	D H K L M	D H K L M	D H K L M
E I A M N	E I B M N	E I D M N	E I F M N	E I H M N

31	32	33	34	35
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K A M	D H K B M	D H K C M	D H K E M	D H K F M
E I L M N	E I L M N	E I L M N	E I L M N	E I L M N

36	37	38	39	40
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K G M	D H K I M	D H K J M	D H K L M	D H K M A
E I L M N	E I L M N	E I L M N	E I L M N	E I L A N

41	42	43	44	45
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K M B	D H K M C	D H K M F	D H K M G	D H K M J
E I L B N	E I L C N	E I L F N	E I L G N	E I L J N

46	47	48	49	50
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K M N	D H K M N	D H K M N	D H K M N	D H K M N
E I L N A	E I L N B	E I L N C	E I L N D	E I L N F

51	52	53	54	55
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K M N	D H K M N	D H K M N	D H K M N	D H K M N
E I L N G	E I L N H	E I L N J	E I L N K	E I L N M

These squares can be divided into three equivalence classes as shown below:

Table 2.3: Equivalence classes of commutative generalized Latin squares of order 5 with 14 distinct elements

Representative	Squares in the same equivalence class
1 A B C D E B A F G H C F I J K D G J L M E H K M N	14, 18, 31, 35, 38, 46, 50, 53, 55
2 A B C D E B C F G H C F I J K D G J L M E H K M N	3, 4, 5, 8, 11, 15, 16, 17, 19, 20, 21, 24, 26, 29, 32, 33, 34, 36, 37, 39, 40, 43, 45, 47, 48, 49, 51, 52, 54
6 A B C D E B F D G H C D I J K D G J L M E H K M N	7, 9, 10, 12, 13, 22, 23, 25, 27, 28, 30, 41, 42, 44

Next, we see that all three representative squares are embeddable in abelian groups as shown below.

Square 1: Embeddable in $\mathbb{Z}_{22}$.

+	0	11	1	3	16
0	0	11	1	3	16
11	11	0	12	14	5
1	1	12	2	4	17
3	3	14	4	6	19
16	16	5	17	19	10

Square 2: Embeddable in $\mathbb{Z}_{31}$.

+	0	1	2	5	11
0	0	1	2	5	11
1	1	2	3	6	12
2	2	3	4	7	13
5	5	6	7	10	16
11	11	12	13	16	22

Square 6: Embeddable in $\mathbb{Z}_{31}$.

+	0	1	4	5	11
0	0	1	4	5	11
1	1	2	5	6	12
4	4	5	8	9	15
5	5	6	9	10	16
11	11	12	15	16	22

We shall now look at the commutative generalized Latin square of order 5 with 15 distinct elements:

A	B	C	D	E
B	F	G	H	I
C	G	J	K	L
D	H	K	M	N
E	I	L	N	O

This square is embeddable in $\mathbb{Z}_{31}$ as shown below:

+	0	1	3	7	15
0	0	1	3	7	15
1	1	2	4	8	16
3	3	4	6	10	18
7	7	8	10	14	22
15	15	16	18	22	30

2.3 Non-commutative Squares

In this section, we will investigate the non-commutative generalized Latin squares of order 5. We first show that for every integer $k = 5, 6, \ldots, 25$, there exist a group G and a non-commutative 5-subset S of G with the property

$|S^2| = k$ where $S^2 = \{xy|x, y \in S\}$. Next, we generate the non-commutative generalized Latin squares of order 5 with 5, 14 and 15 distinct elements. Finally, we divide these squares into their equivalence classes and determine which squares are embeddable in groups.

2.3.1 Some Embeddable Squares

It is clear that a non-commutative generalized Latin square of order n has at least n distinct elements and at most n^2 distinct elements. In this section, we show that for each integer $k = 5, 6, \ldots, 25$, there exist a group G and a non-commutative 5-subset S of G such that $|S^2| = k$ where $S^2 = \{xy|x, y \in S\}$.

$k = 5$: $D_{10} = \langle x, y | x^5 = y^2 = 1,\ yx = x^4y \rangle$

	y	xy	x^2y	x^3y	x^4y
y	1	x	x^2	x^3	x^4
xy	x^4	1	x	x^2	x^3
x^2y	x^3	x^4	1	x	x^2
x^3y	x^2	x^3	x^4	1	x
x^4y	x	x^2	x^3	x^4	1

$k = 6$: $D_{12} = \langle x, y | x^6 = y^2 = 1,\ yx = x^5y \rangle$

	xy	x^2y	x^3y	x^4y	x^5y
xy	1	x	x^2	x^3	x^4
x^2y	x^5	1	x	x^2	x^3
x^3y	x^4	x^5	1	x	x^2
x^4y	x^3	x^4	x^5	1	x
x^5y	x^2	x^3	x^4	x^5	1

$k = 7$: $D_{14} = \langle x, y | x^7 = y^2 = 1,\ yx = x^6y \rangle$

	x^2y	x^3y	x^4y	x^5y	x^6y
x^2y	1	x	x^2	x^3	x^4
x^3y	x^6	1	x	x^2	x^3
x^4y	x^5	x^6	1	x	x^2
x^5y	x^4	x^5	x^6	1	x
x^6y	x^3	x^4	x^5	x^6	1

$k = 8$: $D_{16} = \langle x, y | x^8 = y^2 = 1,\ yx = x^7 y \rangle$

	x^3y	x^4y	x^5y	x^6y	x^7y
x^3y	1	x	x^2	x^3	x^4
x^4y	x^7	1	x	x^2	x^3
x^5y	x^6	x^7	1	x	x^2
x^6y	x^5	x^6	x^7	1	x
x^7y	x^4	x^5	x^6	x^7	1

$k = 9$: $D_{10} = \langle x, y | x^5 = y^2 = 1,\ yx = x^4 y \rangle$

	1	x	x^4	y	xy
1	1	x	x^4	y	xy
x	x	x^2	1	x^4y	y
x^4	x^4	1	x^3	xy	x^2y
y	y	xy	x^4y	1	x
xy	xy	x^2y	y	x^4	1

$k = 10$: $D_{10} = \langle x, y | x^5 = y^2 = 1,\ yx = x^4 y \rangle$

	x	x^2	x^3	y	xy
x	x^2	x^3	x^4	x^4y	y
x^2	x^3	x^4	1	x^3y	x^4y
x^3	x^4	1	x	x^2y	x^3y
y	xy	x^2y	x^3y	1	x
xy	x^2y	x^3y	x^4y	x^4	1

$k = 11$: $D_{12} = \langle x, y | x^6 = y^2 = 1,\ yx = x^5 y \rangle$

	x^5	xy	x^3y	x^4y	x^5y
x^5	x^4	x^2y	x^4y	x^5y	y
xy	y	1	x^2	x^3	x^4
x^3y	x^2y	x^4	1	x	x^2
x^4y	x^3y	x^3	x^5	1	x
x^5y	x^4y	x^2	x^4	x^5	1

$k = 12$: $D_{12} = \langle x, y | x^6 = y^2 = 1,\ yx = x^5 y \rangle$

	1	x	x^2	x^3	y
1	1	x	x^2	x^3	y
x	x	x^2	x^3	x^4	x^5y
x^2	x^2	x^3	x^4	x^5	x^4y
x^3	x^3	x^4	x^5	1	x^3y
y	y	xy	x^2y	x^3y	1

$k = 13$: $D_{14} = \langle x, y | x^7 = y^2 = 1,\ yx = x^6 y \rangle$

	1	x	x^2	y	x^2y
1	1	x	x^2	y	x^2y
x	x	x^2	x^3	x^6y	xy
x^2	x^2	x^3	x^4	x^5y	y
y	y	xy	x^2y	1	x^2
x^2y	x^2y	x^3y	x^4y	x^5	1

$k = 14$: $D_{14} = \langle x, y | x^7 = y^2 = 1,\ yx = x^6 y \rangle$

	1	x	x^2	x^3	y
1	1	x	x^2	x^3	y
x	x	x^2	x^3	x^4	x^6y
x^2	x^2	x^3	x^4	x^5	x^5y
x^3	x^3	x^4	x^5	x^6	x^4y
y	y	xy	x^2y	x^3y	1

$k = 15$: $D_{16} = \langle x, y | x^8 = y^2 = 1,\ yx = x^7 y \rangle$

	1	x	x^2	x^5	y
1	1	x	x^2	x^5	y
x	x	x^2	x^3	x^6	x^7y
x^2	x^2	x^3	x^4	x^7	x^6y
x^5	x^5	x^6	x^7	x^2	x^3y
y	y	xy	x^2y	x^5y	1

$k = 16$: $D_{16} = \langle x, y | x^8 = y^2 = 1,\ yx = x^7 y \rangle$

	1	x	x^6	y	x^3y
1	1	x	x^6	y	x^3y
x	x	x^2	x^7	x^7y	x^2y
x^6	x^6	x^7	x^4	x^2y	x^5y
y	y	xy	x^6y	1	x^3
x^3y	x^3y	x^4y	xy	x^5	1

$k = 17$: $D_{18} = \langle x, y | x^9 = y^2 = 1,\ yx = x^8 y \rangle$

	x	x^2	x^3	x^5	y
x	x^2	x^3	x^4	x^6	x^8y
x^2	x^3	x^4	x^5	x^7	x^7y
x^3	x^4	x^5	x^6	x^8	x^6y
x^5	x^6	x^7	x^8	x	x^4y
y	xy	x^2y	x^3y	x^5y	1

$k = 18$: $D_{18} = \langle x, y | x^9 = y^2 = 1,\ yx = x^8 y \rangle$

	x	x^6	y	xy	x^4y
x	x^2	x^7	x^8y	y	x^3y
x^6	x^7	x^3	x^3y	x^4y	x^7y
y	xy	x^6y	1	x	x^4
xy	x^2y	x^7y	x^8	1	x^3
x^4y	x^5y	xy	x^5	x^6	1

$k = 19$: $D_{20} = \langle x, y | x^{10} = y^2 = 1,\ yx = x^9 y \rangle$

	x	x^2	x^4	y	xy
x	x^2	x^3	x^5	x^9y	y
x^2	x^3	x^4	x^6	x^8y	x^9y
x^4	x^5	x^6	x^8	x^6y	x^7y
y	xy	x^2y	x^4y	1	x
xy	x^2y	x^3y	x^5y	x^9	1

$k = 20$: $D_{20} = \langle x, y | x^{10} = y^2 = 1,\ yx = x^9 y \rangle$

	x	x^4	y	xy	x^4y
x	x^2	x^5	x^9y	y	x^3y
x^4	x^5	x^8	x^6y	x^7y	y
y	xy	x^4y	1	x	x^4
xy	x^2y	x^5y	x^9	1	x^3
x^4y	x^5y	x^8y	x^6	x^7	1

$k = 21$: $D_{30} = \langle x, y | x^{15} = y^2 = 1,\ yx = x^{14} y \rangle$

	x	x^2	x^6	y	x^6y
x	x^2	x^3	x^7	$x^{14}y$	x^5y
x^2	x^3	x^4	x^8	$x^{13}y$	x^4y
x^6	x^7	x^8	x^{12}	x^9y	y
y	xy	x^2y	x^6y	1	x^6
x^6y	x^7y	x^8y	$x^{12}y$	x^9	1

$k = 22$: $D_{30} = \langle x, y | x^{15} = y^2 = 1,\ yx = x^{14} y \rangle$

	x	x^3	y	x^3y	x^8y
x	x^2	x^4	$x^{14}y$	x^2y	x^7y
x^3	x^4	x^6	$x^{12}y$	y	x^5y
y	xy	x^3y	1	x^3	x^8
x^3y	x^4y	x^6y	x^{12}	1	x^5
x^8y	x^9y	$x^{11}y$	x^7	x^{10}	1

$k = 23$: $G = S_6$, the symmetric group of degree 6.

	(12)	(143)	(1435)	(136452)	(134)
(12)	(1)	(1243)	(12435)	(23645)	(1234)
(143)	(1432)	(134)	(1345)	(152)(46)	(1)
(1435)	(14352)	(1354)	(13)(45)	(1532)(46)	(35)
(136452)	(13645)	(245)(36)	(1524)(36)	(165)(234)	(145236)
(134)	(1342)	(1)	(15)	(164352)	(143)

$k = 24$: $G = S_6$, the symmetric group of degree 6.

	(12)	(143)	(1435)	(12345)	(136452)
(12)	(1)	(1243)	(12435)	(1345)	(23645)
(143)	(1432)	(134)	(1345)	(15)(23)	(152)(46)
(1435)	(14352)	(1354)	(13)(45)	(1523)	(1532)(46)
(12345)	(2345)	(12)(45)	(1254)	(13524)	(264)(35)
(136452)	(13645)	(245)(36)	(1524)(36)	(14)(365)	(165)(234)

$k = 25$: $G = S_6$, the symmetric group of degree 6.

	(12)	(143)	(1435)	(16345)	(136452)
(12)	(1)	(1243)	(12435)	(126345)	(23645)
(143)	(1432)	(134)	(1345)	(15)(36)	(152)(46)
(1435)	(14352)	(1354)	(13)(45)	(1563)	(1532)(46)
(16345)	(163452)	(16)(45)	(1654)	(13564)	(142)(35)
(136452)	(13645)	(245)(36)	(1524)(36)	(14)(265)	(165)(234)

2.3.2 Squares with 5 Distinct Elements

There are altogether 1338 non-commutative generalized Latin squares of order 5 with 5 distinct elements, which are listed lexicographically in Appendix C. These squares can be divided into 21 equivalence classes, where the square with the minimum ordinal number in a class is chosen to be the representative of that class. Recall that two Latin squares L and L' are isomorphic if L' can be obtained by performing a permutation of rows and the same permutation of columns on L. Based on the algorithm in [16], we obtain the following algorithm used to generate the squares and divide them into their equivalence classes:

Algorithm 2: Generate and divide non-commutative generalized Latin squares of order 5 into their equivalence classes

1. Generate non-commutative generalized Latin square S_i for $i \geq 1$.
2. Assign ordinal number i to S_i.
3. If $i = 1$, create new equivalence class with S_1 as the representative square.
4. Else,
   4.1. Perform permutations on S_i.
   4.2. Rename elements in S_i.
   4.3. Compare with representative squares S_j for $j < i$.
      4.3.1. If S_i is isomorphic to S_j, add S_i to equivalence class represented by S_j.
      4.3.2. Else, create new equivalence class with S_i as the representative square.

The equivalence classes obtained are shown in the following table:

Table 2.4: Equivalence classes of non-commutative generalized Latin squares of order 5 with 5 distinct elements

Representative	Squares in the same equivalence class
1 A B C D E B A D E C C D E A B D E B C A E C A B D	3, 29, 31, 51, 53, 59, 66, 74, 107, 108, 127, 144, 155, 178, 182, 185, 192, 208, 212, 239, 262, 278, 282, 307, 311, 314, 330, 453, 454, 573, 574, 866, 869, 895, 898, 1177, 1180, 1256, 1259

Table 2.4: (Continued)

Representative	Squares in the same equivalence class
2 A B C D E B A D E C C D E A B E C A B D D E B C A	4, 13, 15, 18, 20, 30, 32, 33, 35, 46, 48, 49, 55, 57, 68, 73, 80, 81, 98, 103, 105, 109, 114, 115, 121, 126, 129, 145, 148, 150, 154, 166, 167, 180, 183, 184, 190, 202, 204, 209, 211, 221, 224, 238, 243, 246, 250, 251, 263, 272, 275, 279, 281, 302, 305, 309, 312, 313, 328, 331, 337, 341, 348, 355, 386, 392, 409, 446, 458, 486, 519, 522, 525, 557, 577, 591, 609, 650, 653, 670, 685, 707, 711, 716, 717, 744, 749, 777, 778, 811, 838, 863, 868, 883, 897, 927, 949, 969, 972, 1013, 1016, 1055, 1058, 1080, 1083, 1086, 1088, 1111, 1112, 1146, 1175, 1182, 1207, 1228, 1244, 1260, 1282, 1285, 1322
5 A B C D E B A D E C C E A B D D C E A B E D B C A	21, 25, 38, 381, 403, 673, 742, 1007, 1051
6 A B C D E B A D E C C E A B D E D B C A D C E A B	8, 9, 11, 22, 24, 26, 28, 37, 39, 42, 44, 79, 82, 83, 113, 116, 117, 168, 169, 171, 230, 232, 236, 248, 249, 252, 285, 289, 291, 362, 397, 427, 493, 497, 529, 547, 613, 638, 641, 679, 741, 775, 776, 809, 835, 929, 955, 985, 988, 1026, 1052, 1113, 1114, 1150, 1206, 1230, 1298, 1301, 1328
7 A B C D E B A D E C C E B A D D C E B A E D A C B	23, 27, 40, 78, 101, 112, 123, 151, 170, 205, 222, 242, 253, 274, 301, 382, 384, 404, 407, 450, 533, 561, 626, 674, 677, 746, 750, 803, 836, 930, 943, 1008, 1010, 1053, 1057, 1138, 1205, 1229, 1316
10 A B C D E B A D E C D C E A B E D B C A C E A B D	12, 41, 43, 439, 455, 456, 498, 499, 530, 570, 575, 576, 614, 642, 658, 769, 794, 815, 823, 871, 875, 911, 914, 941, 956, 986, 991, 1106, 1133, 1154, 1185, 1188, 1196, 1242, 1272, 1275, 1297, 1306, 1327

Table 2.4: (Continued)

Representative	Squares in the same equivalence class
14 A B C D E B A D E C D E A C B E C B A D C D E B A	17, 34, 45, 344, 345, 352, 353, 369, 371, 379, 385, 405, 410, 418, 420, 675, 686, 693, 694, 703, 708, 712, 724, 728, 733, 735, 758, 1005, 1015, 1027, 1028, 1039, 1048, 1050, 1069, 1076, 1079, 1087, 1095
16 A B C D E B A D E C D E B C A E C A B D C D E A B	19, 36, 47, 374, 380, 406, 425, 445, 472, 477, 542, 558, 586, 603, 617, 676, 700, 732, 761, 812, 826, 864, 880, 884, 887, 925, 962, 1006, 1031, 1041, 1072, 1145, 1181, 1189, 1201, 1226, 1243, 1248, 1333
50 A B C D E B C A E D C D E A B E A D B C D E B C A	56, 58, 67, 76, 84, 86, 95, 102, 106, 110, 122, 128, 134, 136, 141, 146, 152, 157, 158, 160, 173, 189, 195, 199, 200, 203, 206, 219, 223, 229, 234, 241, 245, 254, 260, 267, 269, 273, 277, 286, 293, 304, 306, 315, 319, 322, 325, 359, 361, 398, 401, 429, 435, 444, 451, 467, 494, 507, 527, 537, 544, 549, 555, 564, 571, 600, 611, 619, 630, 637, 666, 680, 683, 743, 747, 763, 779, 801, 807, 814, 829, 834, 837, 856, 865, 900, 901, 919, 928, 931, 947, 951, 958, 979, 990, 1022, 1025, 1056, 1060, 1100, 1122, 1142, 1144, 1147, 1165, 1178, 1200, 1208, 1209, 1220, 1227, 1232, 1257, 1264, 1296, 1299, 1320, 1326, 1329
52 A B C D E B C A E D C D E B A E A D C B D E B A C	54, 60, 61, 64, 65, 69, 72, 179, 187, 188, 193, 194, 197, 198, 201, 308, 316, 317, 320, 321, 323, 326, 329, 357, 360, 370, 372, 375, 389, 394, 402, 412, 417, 419, 424, 433, 443, 460, 470, 478, 485, 515, 517, 523, 528, 551, 556, 579, 585, 592, 605, 607, 612, 646, 652, 672, 684, 688, 691, 692, 698, 730, 734, 736, 753, 756, 759, 764, 780, 813, 821, 828, 831, 849, 860, 870, 878, 888, 892, 896, 909, 933, 935, 957, 966, 971, 998, 1011, 1019, 1021, 1029, 1030, 1034, 1043, 1047, 1049, 1066, 1067, 1074, 1099, 1121, 1148, 1156, 1170, 1171, 1179, 1191, 1197, 1204, 1234, 1236, 1247, 1253, 1255, 1270, 1286, 1307, 1330, 1337

Table 2.4: (Continued)

Representative	Squares in the same equivalence class
62 A B C D E B C A E D D E B A C E A D C B C D E B A	63, 70, 71, 87, 92, 99, 100, 124, 125, 133, 138, 149, 153, 161, 176, 181, 186, 191, 196, 207, 214, 216, 220, 244, 247, 257, 266, 276, 295, 297, 303, 310, 318, 324, 327, 333, 335, 339, 350, 356, 390, 391, 411, 437, 448, 457, 484, 489, 504, 509, 520, 535, 545, 559, 566, 578, 593, 622, 628, 633, 649, 659, 664, 669, 687, 709, 713, 715, 718, 745, 748, 784, 786, 789, 790, 799, 806, 827, 832, 858, 859, 902, 910, 934, 936, 945, 954, 967, 974, 978, 980, 1014, 1020, 1054, 1059, 1082, 1084, 1085, 1090, 1117, 1119, 1123, 1124, 1136, 1139, 1163, 1172, 1198, 1203, 1233, 1235, 1263, 1269, 1280, 1283, 1289, 1295, 1318, 1323
75 A B C D E B C D E A C A E B D E D A C B D E B A C	93, 94, 104, 111, 130, 137, 142, 147, 156, 172, 177, 213, 215, 228, 235, 240, 255, 256, 261, 287, 292, 296, 298, 334, 336, 340, 343, 346, 349, 351, 354, 364, 376, 399, 423, 431, 436, 473, 480, 487, 490, 495, 516, 536, 546, 553, 563, 583, 602, 621, 627, 631, 634, 640, 645, 681, 696, 704, 705, 710, 714, 720, 721, 723, 726, 727, 757, 783, 785, 787, 788, 808, 820, 833, 844, 850, 867, 891, 899, 920, 923, 946, 953, 977, 987, 996, 1001, 1024, 1036, 1040, 1070, 1075, 1078, 1081, 1089, 1091, 1094, 1096, 1097, 1118, 1120, 1125, 1126, 1143, 1157, 1169, 1176, 1210, 1215, 1219, 1222, 1254, 1258, 1290, 1302, 1309, 1314, 1317, 1324
77 A B C D E B C D E A C D E A B E A B C D D E A B C	90, 97, 118, 119, 140, 143, 164, 175, 210, 217, 226, 237, 258, 271, 280, 283, 300, 365, 368, 387, 395, 414, 415, 461, 481, 491, 506, 581, 590, 635, 661, 668, 689, 702, 738, 739, 752, 781, 792, 854, 861, 905, 908, 976, 983, 1003, 1018, 1037, 1045, 1062, 1063, 1116, 1127, 1161, 1174, 1266, 1267, 1287, 1292
85 A B C D E B C D E A D A E B C E D A C B C E B A D	89, 131, 135, 159, 163, 227, 233, 264, 268, 288, 294, 363, 373, 400, 426, 430, 438, 441, 447, 452, 468, 469, 476, 496, 500, 503, 510, 531, 538, 541, 543, 550, 560, 565, 568, 572, 587, 599, 606, 618, 620, 624, 629, 639, 657, 660, 663, 682, 699, 729, 760, 770, 773, 796, 797, 800, 802, 805, 818, 825, 830, 839, 842, 847, 857, 873, 877, 890, 904, 913, 918, 926, 932, 938, 940, 948, 952, 960, 961, 968, 982, 989, 992, 1023, 1032, 1044, 1073, 1102, 1105, 1130, 1131, 1135, 1140, 1141, 1151, 1160, 1164, 1183, 1192, 1199, 1202, 1212, 1213, 1225, 1231, 1237, 1239, 1245, 1261, 1276, 1277, 1279, 1293, 1300, 1305, 1319, 1325, 1331, 1334

Table 2.4: (Continued)

Representative	Squares in the same equivalence class
88 A B C D E B C D E A D A E C B E D A B C C E B A D	132, 162, 231, 265, 290, 428, 463, 508, 540, 548, 598, 615, 665, 765, 810, 845, 855, 903, 921, 963, 981, 1110, 1149, 1166, 1218, 1224, 1262, 1294, 1336
91 A B C D E B C D E A D E A B C E A B C D C D E A B	96, 120, 139, 165, 174, 218, 225, 259, 270, 284, 299, 367, 388, 413, 416, 462, 465, 482, 492, 505, 512, 582, 589, 596, 636, 643, 662, 667, 690, 737, 740, 767, 782, 791, 853, 862, 881, 885, 906, 907, 975, 984, 999, 1004, 1017, 1046, 1061, 1108, 1115, 1128, 1162, 1173, 1194, 1250, 1265, 1268, 1288, 1291, 1312
332 A B C D E C A B E D B D E A C E C D B A D E A C B	338, 342, 347, 358, 378, 393, 421, 432, 434, 442, 459, 474, 479, 483, 488, 501, 502, 514, 518, 521, 524, 526, 532, 552, 554, 567, 580, 584, 594, 601, 608, 610, 623, 632, 647, 651, 654, 655, 656, 671, 695, 706, 719, 722, 725, 754, 755, 771, 772, 774, 793, 795, 798, 817, 819, 822, 840, 841, 843, 848, 852, 872, 874, 893, 915, 916, 917, 924, 937, 939, 950, 959, 965, 970, 973, 993, 994, 995, 1002, 1012, 1035, 1065, 1068, 1077, 1092, 1093, 1098, 1101, 1103, 1104, 1129, 1132, 1134, 1152, 1155, 1158, 1159, 1167, 1184, 1186, 1211, 1214, 1216, 1221, 1238, 1240, 1252, 1273, 1274, 1278, 1281, 1284, 1303, 1304, 1310, 1313, 1321, 1332, 1338
366 A B C D E C A D E B B E A C D E D B A C D C E B A	396, 701, 751, 1038, 1064
377 A B C D E C A D E B D E B C A E C A B D B D E A C	383, 408, 422, 440, 449, 464, 471, 475, 513, 534, 539, 562, 569, 588, 597, 604, 616, 625, 648, 678, 697, 731, 762, 766, 804, 816, 824, 846, 851, 876, 879, 889, 894, 912, 922, 942, 944, 964, 997, 1009, 1033, 1042, 1071, 1109, 1137, 1153, 1168, 1187, 1190, 1195, 1217, 1223, 1241, 1246, 1251, 1271, 1308, 1315, 1335

Table 2.4: (Continued)

Representative	Squares in the same equivalence class
466 A B C D E C D B E A D A E C B E C A B D B E D A C	595, 882, 886, 1193, 1249
511 A B C D E C D E A B E A B C D B C D E A D E A B C	644, 768, 1000, 1107, 1311

We next obtain a result which will help us determine which of the squares in Table 2.4 are not embeddable in any finite group.

Proposition 2.2. *Let G be a non-abelian group. There does not exist distinct elements $x,\ y,\ z \in G$ such that $x^2 = y^2$, $xy = yz$ and $y^2 \neq z^2$.*

Proof. If such $x,\ y,\ z$ exist in G, then $x = yzy^{-1}$ and hence, $x^2 = yz^2y^{-1}$. But then $y^2 = yz^2y^{-1}$ which gives us $y^2 = z^2$; a contradiction. □

By using Proposition 2.2, we easily have that the squares isomorphic to Squares 2, 6, 7, 16, 52, 62, 75, 77, 85, 91, 332 and 377 are not embeddable in any group. We also show below that Squares 1, 5, 10, 14, 50 and 88 are not embeddable in any group.

Proposition 2.3. *Let L be a generalized Latin square of order 5 with 5 distinct elements which is isomorphic to Square 1. Then L is not embeddable in any finite group.*

Proof. Suppose to the contrary that L is embeddable in a finite group G. Then there exist distinct elements $a, b, c, d, e \in G$ such that

$$dc = ed \tag{2.1}$$

$$c^2 = bd \tag{2.2}$$

$$e^2 = da \tag{2.3}$$

By (2.1), we have $c = d^{-1}ed$. Then $bd = (d^{-1}ed)^2 = d^{-1}e^2d$ (by (2.2)). That is, $b = d^{-1}e^2$. We then have $db = e^2 = da$ (by (2.3)). It follows that $b = a$, which is a contradiction. Hence, L is not embeddable in any finite group. □

Proposition 2.4. *Let L be a generalized Latin square of order 5 with 5 distinct elements which is isomorphic to Square 5. Then L is not embeddable in any finite group.*

Proof. Suppose to the contrary that L is embeddable in a finite group G. Then there exist distinct elements $a, b, c, d, e \in G$ such that

$$bd = eb \tag{2.4}$$

$$de = eb \tag{2.5}$$

$$e^2 = d^2 \tag{2.6}$$

By (2.4) and (2.5), we have $b = ebd^{-1}$ and $b = e^{-1}de$. Therefore

$$\begin{aligned} b = ebd^{-1} &= e^{-1}de \\ \Rightarrow \quad eb &= e^{-1}ded \\ \Rightarrow \quad e^2b &= ded \\ \Rightarrow \quad d^2b &= ded \qquad \text{(by (2.6))} \\ \Rightarrow \quad db &= ed \end{aligned}$$

which is a contradiction. Hence, L is not embeddable in any finite group. $\square$

Proposition 2.5. *Let L be a generalized Latin square of order 5 with 5 distinct elements which is isomorphic to Square 10. Then L is not embeddable in any finite group.*

Proof. Suppose to the contrary that L is embeddable in a finite group G. Then there exist distinct elements $a, b, c, d, e \in G$ such that

$$ae = eb \tag{2.7}$$

$$b^2 = ce \tag{2.8}$$

$$a^2 = dc \tag{2.9}$$

From (2.7), we have $b = e^{-1}ae$. Then $ce = (e^{-1}ae)^2 = e^{-1}a^2e$ (by (2.8)). and hence $c = e^{-1}a^2$. We then have $ec = a^2 = dc$ (by (2.9)). It follows that $e = d$, which is a contradiction. $\square$

Proposition 2.6. *Squares 14, 50 and 88 are not embeddable in any groups.*

Proof. Suppose that there exists a group G and elements $a,\ b,\ c,\ d,\ e \in G$ such that the multiplication table of G is Square 14. Then $ac = be = cb = da = ed$. It follows from this that

$$\begin{aligned} a &= cbc^{-1} \\ &= (beb^{-1})b(be^{-1}b^{-1}) \\ &= bebe^{-1}b^{-1} \\ &= b(dad^{-1})b(da^{-1}d^{-1})b^{-1} \\ &= bdad^{-1}bda^{-1}d^{-1}b^{-1}. \end{aligned}$$

Therefore

$$
\begin{aligned}
& abda = bdad^{-1}bd \\
\Rightarrow\ & ada = dad^{-1}bd \quad (\because ab = ba) \\
\Rightarrow\ & dba = dad^{-1}bd \quad (\because ad = db) \\
\Rightarrow\ & ba = ad^{-1}bd \\
\Rightarrow\ & b = d^{-1}bd \quad (\because ba = ab) \\
\Rightarrow\ & db = bd,
\end{aligned}
$$

which is a contradiction.

Now suppose that there exists a group G and elements $a,\ b,\ c,\ d,\ e \in G$ such that the multiplication table of G is Square 50. Then $bd = cb = dc$. Hence $b = dcd^{-1} = (cbc^{-1})c(cb^{-1}c^{-1}) = cbcb^{-1}c^{-1}$. It follows that $bcb = cbc$ and hence $cdb = cbc$ which implies that $db = bc$; a contradiction.

Next, suppose that there exists a group G and elements $a,\ b,\ c,\ d,\ e \in G$ such that the multiplication table of G is Square 88. Then $bc = de = eb$. Hence $d = ebe^{-1} = (bcb^{-1})b(bc^{-1}b^{-1}) = bcbc^{-1}b^{-1}$. It follows that $dbc = bcb$ and hence $bec = bcb$ which implies that $ec = cb$; a contradiction. □

Finally, we show that the three remaining representative squares are embeddable in finite groups.

Square 366: Embeddable in $D_{10} = \langle x, y | x^5 = y^2 = 1,\ yx = x^4y \rangle$.

	y	x^3y	x^2y	x^4y	xy
y	1	x^3	x^2	x^4	x
x^3y	x^2	1	x^4	x	x^3
x^2y	x^3	x	1	x^2	x^4
x^4y	x	x^4	x^3	1	x^2
xy	x^4	x^2	x	x^3	1

Square 466: Embeddable in

$$G = \langle x, y, z | x^2 = y, y^2 = 1, z^5 = 1, x^{-1}zx = z^2, y^{-1}zy = z^4 \rangle.$$

	xz^2	xz^4	xz^3	x	xz
xz^2	yz	y	yz^3	yz^2	yz^4
xz^4	yz^3	yz^2	y	yz^4	yz
xz^3	yz^2	yz	yz^4	yz^3	y
x	yz^4	yz^3	yz	y	yz^2
xz	y	yz^4	yz^2	yz	yz^3

Square 511: Embeddable in

$$G = \langle x, y, z | x^2 = y, y^2 = 1, z^5 = 1, x^{-1}zx = z^2, y^{-1}zy = z^4 \rangle.$$

	xyz^4	xyz^3	xyz^2	xyz	xy
xyz^4	yz	yz^3	y	yz^2	yz^4
xyz^3	y	yz^2	yz^4	yz	yz^3
xyz^2	yz^4	yz	yz^3	y	yz^2
xyz	yz^3	y	yz^2	yz^4	yz
xy	yz^2	yz^4	yz	yz^3	y

2.3.3 Squares with 24 and 25 Distinct Elements

In this section, we consider the generalized Latin squares of order 5 with 24 and 25 distinct elements, and show that all of the squares are embeddable in groups. We first look at the squares with 24 distinct elements. There are altogether 200 squares as follows:

A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
α F G H I	F β G H I	F G γ H I	F G H δ I	F G H I ϵ
J K L M N	J K L M N	J K L M N	J K L M N	J K L M N
O P Q R S	O P Q R S	O P Q R S	O P Q R S	O P Q R S
T U V W X	T U V W X	T U V W X	T U V W X	T U V W X

A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
F G H I J	F G H I J	F G H I J	F G H I J	F G H I J
ζ K L M N	K η L M N	K L θ M N	K L M ι N	K L M N κ
O P Q R S	O P Q R S	O P Q R S	O P Q R S	O P Q R S
T U V W X	T U V W X	T U V W X	T U V W X	T U V W X

A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
F G H I J	F G H I J	F G H I J	F G H I J	F G H I J
K L M N O	K L M N O	K L M N O	K L M N O	K L M N O
λ P Q R S	P μ Q R S	P Q ν R S	P Q R ξ S	P Q R S o
T U V W X	T U V W X	T U V W X	T U V W X	T U V W X

A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
F G H I J	F G H I J	F G H I J	F G H I J	F G H I J
K L M N O	K L M N O	K L M N O	K L M N O	K L M N O
P Q R S T	P Q R S T	P Q R S T	P Q R S T	P Q R S T
π U V W X	U ρ V W X	U V σ W X	U V W τ X	U V W X υ

where Square 1 is obtained by replacing α with B, Square 2 is obtained by replacing α with C, and so on as shown below:

α = B(1), C(2), D(3), or E(4);

β = A(5), C(6), D(7), or E(8);

γ = A(9), B(10), D(11), or E(12);

δ = A(13), B(14), C(15), or E(16);

ϵ = A(17), B(18), C(19), or D(20);

ζ = B(21), C(22), D(23), E(24), G(25), H(26), I(27), or J(28);

η = A(29), C(30), D(31), E(32), F(33), H(34), I(35), or J(36);

θ = A(37), B(38), D(39), E(40), F(41), G(42), I(43), or J(44);

ι = A(45), B(46), C(47), E(48), F(49), G(50), H(51), or J(52);

κ = A(53), B(54), C(55), D(56), F(57), G(58), H(59), or I(60);

λ = B(61), C(62), D(63), E(64), G(65), H(66), I(67), J(68), L(69), M(70), N(71), or O(72);

μ = A(73), C(74), D(75), E(76), F(77), H(78), I(79), J(80), K(81), M(82), N(83), or O(84);

ν = A(85), B(86), D(87), E(88), F(89), G(90), I(91), J(92), K(93), L(94), N(95), or O(96);

ξ = A(97), B(98), C(99), E(100), F(101), G(102), H(103), J(104), K(105), L(106), M(107), or O(108);

o = A(109), B(110), C(111), D(112), F(113), G(114), H(115), I(116), K(117), L(118), M(119), or N(120);

π = B(121), C(122), D(123), E(124), G(125), H(126), I(127), J(128), L(129), M(130), N(131), O(132), Q(133), R(134), S(135), or T(136);

ρ = A(137), C(138), D(139), E(140), F(141), H(142), I(143), J(144), K(145), M(146), N(147), O(148), P(149), R(150), S(151), or T(152);

σ = A(153), B(154), D(155), E(156), F(157), G(158), I(159), J(160), K(161), L(162), N(163), O(164), P(165), Q(166), S(167), or T(168);

τ = A(169), B(170), C(171), E(172), F(173), G(174), H(175), J(176), K(177), L(178), M(179(, O(180), P(181), Q(182), R(183), or T(184);

υ = A(185), B(186), C(187), D(188), F(189), G(190), H(191), I(192), K(193), L(194), M(195), N(196), P(197), Q(198), R(199), or S(200);

By using Algorithm 2, we divide these squares into 5 equivalence classes as shown below:

Table 2.5: Equivalence classes of non-commutative generalized Latin squares of order 5 with 24 distinct elements

Representative	Squares in the same equivalence class
1	22, 34, 63, 79, 95, 124, 144, 164, 184
2	3, 4, 10, 14, 18, 21, 23, 24, 26, 30, 33, 35, 36, 47, 51, 55, 59, 61, 62, 64, 67, 71, 75, 77, 78, 80, 83, 87, 91, 93, 94, 96, 112, 116, 120, 121, 122, 123, 128, 132, 136, 140, 141, 142, 143, 148, 152, 156, 160, 161, 162, 163, 168, 172, 176, 180, 181, 182, 183
5	37, 42, 97, 102, 107, 185, 190, 195, 200
6	7, 8, 9, 13, 17, 25, 29, 38, 39, 40, 41, 43, 44, 45, 50, 53, 58, 65, 70, 73, 82, 85, 90, 98, 99, 100, 101, 103, 104, 105, 106, 108, 109, 114, 119, 125, 130, 135, 137, 146, 151, 153, 158, 167, 169, 174, 179, 186, 187, 188, 189, 191, 192, 193, 194, 196, 197, 198, 199

11	12, 15, 16, 19, 20, 27, 28, 31, 32, 46, 48, 49, 52, 54, 56, 57, 60, 66, 68, 69, 72, 74, 76, 81, 84, 86, 88, 89, 92, 110, 111, 113, 115, 117, 118, 126, 127, 129, 131, 133, 134, 138, 139, 145, 147, 149, 150, 154, 155, 157, 159, 165, 166, 170, 171, 173, 175, 177, 178

In the following, we show that all the squares in Table 2.5 are embeddable in groups.

Square 1: Embeddable in S_5.

	(123)	(132)	(234)	(1342)	(125)
(123)	(132)	(1)	(13)(24)	(24)	(15)(23)
(132)	(1)	(123)	(142)	(1423)	(135)
(234)	(12)(34)	(134)	(243)	(1324)	(12345)
(1342)	(34)	(1234)	(1432)	(14)(23)	(1345)
(125)	(13)(25)	(253)	(13425)	(2534)	(152)

Square 2: Embeddable in S_5.

	(123)	(124)	(234)	(12)	(125)
(123)	(132)	(14)(23)	(13)(24)	(23)	(15)(23)
(124)	(13)(24)	(142)	(134)	(24)	(15)(24)
(234)	(12)(34)	(123)	(243)	(1234)	(12345)
(12)	(13)	(14)	(1342)	(1)	(15)
(125)	(13)(25)	(14)(25)	(13425)	(25)	(152)

Square 5: Embeddable in S_5.

	(12)	(13)	(124)	(1432)	(125)
(12)	(1)	(123)	(14)	(243)	(15)
(13)	(132)	(1)	(1324)	(12)(34)	(1325)
(124)	(24)	(1243)	(142)	(23)	(15)(24)
(1432)	(143)	(14)(23)	(34)	(13)(24)	(1435)
(125)	(25)	(1253)	(14)(25)	(2543)	(152)

Square 6: Embeddable in S_5.

	(12)	(143)	(1234)	(124)	(125)
(12)	(1)	(1243)	(134)	(14)	(15)
(143)	(1432)	(134)	(23)	(243)	(14325)
(1234)	(234)	(12)	(13)(24)	(1423)	(15)(234)
(124)	(24)	(123)	(1342)	(142)	(15)(24)
(125)	(25)	(12543)	(134)(25)	(14)(25)	(152)

Square 11: Embeddable in S_6.

	(12)	(143)	(1435)	(12345)	(136452)
(12)	(1)	(1243)	(12435)	(1345)	(23645)
(143)	(1432)	(134)	(1345)	(15)(23)	(152)(46)
(1435)	(14352)	(1354)	(13)(45)	(1523)	(1532)(46)
(12345)	(2345)	(12)(45)	(1254)	(13524)	(264)(35)
(136452)	(13645)	(245)(36)	(1524)(36)	(14)(365)	(165)(234)

There is only one non-commutative generalized Latin square of order 5 with 25 distinct elements:

A B C D E
F G H I J
K L M N O
P Q R S T
U V W X Y

This square is embeddable in S_6, the symmetric group of degree 6, as shown below:

	(12)	(143)	(1435)	(16345)	(136452)
(12)	(1)	(1243)	(12435)	(126345)	(23645)
(143)	(1432)	(134)	(1345)	(15)(36)	(152)(46)
(1435)	(14352)	(1354)	(13)(45)	(1563)	(1532)(46)
(16345)	(163452)	(16)(45)	(1654)	(13564)	(142)(35)
(136452)	(13645)	(245)(36)	(1524)(36)	(14)(265)	(165)(234)

CHAPTER 3

SOME CONSTRUCTIONS OF NON-COMMUTATIVE GENERALIZED LATIN SQUARES OF ORDER n

3.1 Introduction

In the previous chapter, we have investigated the non-commutative generalized Latin squares of order 5 with 5, 24 and 25 distinct elements. We now turn our attention to the non-commutative generalized Latin squares of order n. Note that the generalized Latin squares of order n with n^2 elements is a trivial case since only one such square exists for every $n \geq 3$. Therefore, in this chapter we will look at the classes of non-commutative generalized Latin squares of order n with $n^2 - 1$ and n distinct elements. We know that generalized Latin squares have various applications in graph theory since they can be represented by graphs. One of these applications is related to Eulerian graphs since these graphs can be used to represent generalized Latin squares of order n with $n^2 - 1$ distinct elements. Hence, we begin by investigating the generalized Latin squares of order n with $n^2 - 1$ distinct elements in Section 3.2. We show that the number of equivalence classes of generalized Latin squares of order n with $n^2 - 1$ distinct elements is four if $n = 3$ and five if $n \geq 4$. We then show that all of these squares are embeddable in groups regardless of whether n is odd or even.

In Section 3.3, we investigate the non-commutative generalized Latin squares of order n with n distinct elements. We start by showing the existence of at least three non-isomorphic non-commutative generalized Latin squares of order

n with n distinct elements which are embeddable in groups when $n \geq 5$ is odd. By using a similar construction for the case when $n \geq 4$ is even, we show that the non-commutative generalized Latin squares of order n are not embeddable in groups.

3.2 Constructions of Generalized Latin Squares of Order n with $n^2 - 1$ Distinct Elements

We first note that the number of generalized Latin squares of order n with $n^2 - 1$ distinct elements is $\frac{1}{2}n^2(n-1)^2$. Indeed, since there are $n^2 - 1$ distinct elements, exactly one of these elements (say X) will appear twice in the square. Suppose that X has appeared once before and appears for the second time in row r and column s. This means that X may be chosen from any one of the elements which appear in row i and column j where $i \in \{1, \ldots, r-1\}$ and $j \in \{1, \ldots, n\} \backslash \{s\}$. There are therefore $(r-1)(n-1)$ possible choices for X. Since X may occupy any one of the n positions in the rth row and the number of possible choices for X is $(r-1)(n-1)$ at each of these positions, it follows that the number of ways a repeated element may appear in row r is $n(r-1)(n-1)$. Then since the repeated element may appear from row 2 to row n, we have that the number of generalized Latin squares of order n with $n^2 - 1$ distinct elements is

$$\begin{aligned} & \sum_{r=2}^{n} n(r-1)(n-1) \\ = \; & n(n-1)(1 + 2 + \cdots + (n-1)) \\ = \; & \frac{1}{2}n^2(n-1)^2. \end{aligned}$$

Let $L = (a_{ij})$ be a generalized Latin square of order n with $n^2 - 1$ distinct elements. We consider the following cases and determine the number of generalized Latin squares in the same equivalence class as L in each case:

(i) Case 1: $a_{11} = a_{22}$

Let $L' = (b_{ij})_n$ be a generalized Latin square of order n with $n^2 - 1$ distinct elements such that $b_{ii} = b_{jj}$ for $1 \leq i < j \leq n$. Then L' is isomorphic to L and the total number of squares in this equivalence class is $\binom{n}{2} = \frac{n(n-1)}{2}$.

(ii) Case 2: $a_{12} = a_{21}$

Any generalized Latin square $L' = (b_{ij})_n$ with $n^2 - 1$ distinct elements such that $b_{ij} = b_{ji}$ for $1 \leq i < j \leq n$ is isomorphic to L and the total number of squares in this equivalence class is $\binom{n}{2} = \frac{n(n-1)}{2}$.

(iii) Case 3: $a_{12} = a_{23}$

Any generalized Latin square $L' = (b_{ij})_n$ with $n^2 - 1$ distinct elements such that $b_{ij} = b_{jk}$ for $1 \leq i, j, k \leq n$ where all i, j, k are distinct integers is isomorphic to L. The total number of squares satisfying this condition is $n(n-1)(n-2)$.

(iv) Case 4: $a_{11} = a_{23}$

Any generalized Latin square $L' = (b_{ij})_n$ with $n^2 - 1$ distinct elements such that $b_{ii} = b_{rs}$ for $1 \leq i, r, s \leq n$ where all i, r, s are distinct integers is isomorphic to L. The total number of squares satisfying this condition is $n(n-1)(n-2)$.

(v) Case 5: $a_{13} = a_{24}$

Let $L' = (b_{ij})_n$ be a generalized Latin square of order n with $n^2 - 1$ distinct elements such that $b_{ij} = b_{rs}$ for $1 \leq i, j \leq n$, $1 \leq r, s \leq n$ where all i, j, r, s are distinct integers is isomorphic to L. The total number of squares satisfying this condition is $\frac{1}{2}n(n-1)(n-2)(n-3)$.

By adding the number of squares in all the equivalence classes above we obtain the total number $\frac{1}{2}n^2(n-1)^2$. We have thus shown that the number of equivalence classes of generalized Latin squares of order n with $n^2 - 1$ distinct elements is 4 if $n = 3$ (Case 5 is excluded) and 5 if $n \geq 4$.

Next, we show that all the non-commutative generalized Latin squares of order n with $n^2 - 1$ distinct elements are embeddable in groups ($n \geq 3$). In what follows, let T_i be a subset of the group G_i such that the multiplication table of T_i is a generalized Latin square of order n with $n^2 - 1$ distinct elements isomorphic to Square i ($i = 1, \ldots, 5$). The notation S_n as usual denotes the symmetric group of degree n.

Square 1: $L = (a_{ij})_n$ where $a_{11} = a_{22}$.

For $n = 3$, take $G_1 = \langle x, y \mid x^4 = y^2 = 1, yx = x^3 y\rangle$ and $T_1 = \{xy, y, x\}$. For $n \geq 4$, take $G_1 = S_{n+1}$ and $T_1 = \{(1\,2), (1\,3)\} \cup \{(1\,2\,3\,a) \mid 4 \leq a \leq n+1\}$.

Square 2: $L = (a_{ij})_n$ where $a_{12} = a_{21}$.

For $n = 3$, take $G_2 = S_4$ and $T_2 = \{(1\,2\,3), (1\,3\,2), (1\,2\,3\,4)\}$. For $n \geq 4$, take $G_2 = S_{n+1}$ and $T_2 = \{(1\,2\,3), (1\,3\,2)\} \cup \{(1\,2\,3\,a) \mid 4 \leq a \leq n+1\}$.

Square 3: $L = (a_{ij})_n$ where $a_{12} = a_{23}$.

For $n = 3$, take $G_3 = S_4$ and $T_3 = \{(1\,2\,4), (1\,3\,4), (1\,3\,2)\}$. For $n \geq 4$, take $G_3 = S_{n+1}$ and $T_3 = \{(1\,2\,4), (1\,3\,4), (1\,3\,2)\} \cup \{(1\,2\,3\,4\,a) \mid 5 \leq a \leq n+1\}$.

Square 4: $L = (a_{ij})_n$ where $a_{11} = a_{23}$.

For $n = 3$, take $G_4 = S_4$ and $T_4 = \{(1\,4\,3), (1\,2), (1\,2\,3\,4)\}$. For $n \geq 4$, take $G_4 = S_{n+1}$ and $T_4 = \{(1\,4\,3), (1\,2), (1\,2\,3\,4)\} \cup \{(1\,2\,3\,4\,a) \mid 5 \leq a \leq n+1\}$.

Square 5: $L = (a_{ij})_n$ where $a_{13} = a_{24}$.

For $n = 4$, take $G_5 = S_5$ and $T_5 = \{(1\,2), (1\,4\,3), (1\,2\,3\,4\,5), (1\,4\,3\,5)\}$. For $n \geq 5$, take $G_5 = S_{n+1}$ and $T_5 = \{(1\,2), (1\,4\,3), (1\,2\,3\,4\,5), (1\,4\,3\,5)\} \cup$

$\{(1\,2\,3\,4\,5\,a) \mid 6 \leq a \leq n+1\}$.

Collecting the above results, we obtain the following:

Proposition 3.1. *The generalized Latin squares of order* n *with* n^2-1 *distinct elements are embeddable in groups* ($n \geq 2$).

We end this section by using the generalized Latin squares of order 6 with 35 distinct elements as an example of the results obtained previously. We first list down the five representative squares:

1	2	3	4	5
A B C D E F	A B C D E F	A B C D E F	A B C D E F	A B C D E F
G A H I J K	B G H I J K	G H B I J K	G H A I J K	G H I C J K
L M N O P Q	L M N O P Q	L M N O P Q	L M N O P Q	L M N O P Q
R S T U V W	R S T U V W	R S T U V W	R S T U V W	R S T U V W
X Y Z α β γ	X Y Z α β γ	X Y Z α β γ	X Y Z α β γ	X Y Z α β γ
δ ϵ ζ η θ ι	δ ϵ ζ η θ ι	δ ϵ ζ η θ ι	δ ϵ ζ η θ ι	δ ϵ ζ η θ ι

In the following, we show that all five squares are embeddable in groups.

Square 1: Embeddable in S_7 where

$$T_1 = \{(1\,2), (1\,3), (1\,2\,3\,4), (1\,2\,3\,5), (1\,2\,3\,6), (1\,2\,3\,7)\}.$$

	(12)	(13)	(1234)	(1235)	(1236)	(1237)
(12)	(1)	(123)	(134)	(135)	(136)	(137)
(13)	(132)	(1)	(14)(23)	(15)(23)	(16)(23)	(17)(23)
(1234)	(234)	(12)(34)	(13)(24)	(13425)	(13426)	(13427)
(1235)	(235)	(12)(35)	(13524)	(13)(25)	(13526)	(13527)
(1236)	(236)	(12)(36)	(13624)	(13625)	(13)(26)	(13627)
(1237)	(237)	(12)(37)	(13724)	(13725)	(13726)	(13)(27)

Square 2: Embeddable in S_7 where

$$T_2 = \{(1\,2\,3), (1\,3\,2), (1\,2\,3\,4), (1\,2\,3\,5), (1\,2\,3\,6), (1\,2\,3\,7)\}.$$

	(123)	(132)	(1234)	(1235)	(1236)	(1237)
(123)	(132)	(1)	(1324)	(1325)	(1326)	(1327)
(132)	(1)	(123)	(14)	(15)	(16)	(17)
(1234)	(1342)	(34)	(13)(24)	(13425)	(13426)	(13427)
(1235)	(1352)	(35)	(13524)	(13)(25)	(13526)	(13527)
(1236)	(1362)	(36)	(13624)	(13625)	(13)(26)	(13627)
(1237)	(1372)	(37)	(13724)	(13725)	(13726)	(13)(27)

Square 3: Embeddable in S_7 where

$$T_3 = \{(1\,2\,4), (1\,3\,4), (1\,3\,2), (1\,2\,3\,5), (1\,2\,3\,6), (1\,2\,3\,7)\}.$$

	(124)	(134)	(132)	(1235)	(1236)	(1237)
(124)	(142)	(12)(34)	(243)	(135)(24)	(136)(24)	(137)(24)
(134)	(13)(24)	(143)	(12)(34)	(15)(234)	(16)(234)	(17)(234)
(132)	(134)	(14)(23)	(123)	(15)	(16)	(17)
(1235)	(14)(235)	(124)(35)	(35)	(13)(25)	(13526)	(13527)
(1236)	(14)(236)	(124)(36)	(36)	(13625)	(13)(26)	(13627)
(1237)	(14)(237)	(124)(37)	(37)	(13725)	(13726)	(13)(27)

Square 4: Embeddable in S_7 where

$$T_4 = \{(1\,4\,3), (1\,2), (1\,2\,3\,4), (1\,2\,3\,5), (1\,2\,3\,6), (1\,2\,3\,7)\}.$$

	(143)	(12)	(1234)	(1235)	(1236)	(1237)
(143)	(134)	(1432)	(23)	(145)(23)	(146)(23)	(147)(23)
(12)	(1243)	(1)	(134)	(135)	(136)	(137)
(1234)	(12)	(234)	(13)(24)	(13425)	(13426)	(13427)
(1235)	(12)(354)	(235)	(13524)	(13)(25)	(13526)	(13527)
(1236)	(12)(364)	(236)	(13624)	(13625)	(13)(26)	(13627)
(1237)	(12)(374)	(237)	(13724)	(13725)	(13726)	(13)(27)

Square 5: Embeddable in S_7 where

$$T_5 = \{(1\,2), (1\,4\,3), (1\,2\,3\,4\,5), (1\,4\,3\,5), (1\,2\,3\,4\,5\,6), (1\,2\,3\,4\,5\,7)\}.$$

	(12)	(143)	(12345)	(1435)	(123456)	(123457)
(12)	(1)	(1243)	(1345)	(12435)	(13456)	(13457)
(143)	(1432)	(134)	(15)(23)	(1345)	(156)(23)	(157)(23)
(12345)	(2345)	(12)(45)	(13524)	(1254)	(135246)	(135247)
(1435)	(14352)	(1354)	(1523)	(13)(45)	(15236)	(15237)
(123456)	(23456)	(12)(456)	(135624)	(12564)	(135)(246)	(1356247)
(123457)	(23457)	(12)(457)	(135724)	(12574)	(1357246)	(135)(247)

3.3 Generalized Latin Squares of Order n with n Distinct Elements

Let n be a positive integer. Let $L = (a_{ij})_n$ be a non-commutative generalized Latin square of order n with n distinct elements $A_1, A_2, \ldots, A_n$. We shall fix the entries in the first row of L in the order $A_1, A_2, \ldots, A_n$. The entries in the succeeding rows are arranged cyclically in the same order. We look at three different patterns of L below.

(a) $L_1 = (a_{ij})_n$ where $a_{21} = A_n, a_{31} = A_{n-1}, \ldots, a_{n1} = A_2$.

$$L_1 = \begin{matrix} A_1 & A_2 & A_3 & \ldots & A_n \\ A_n & A_1 & A_2 & \ldots & A_{n-1} \\ A_{n-1} & A_n & A_1 & \ldots & A_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & A_4 & \ldots & A_1 \end{matrix}$$

(b) (i) $n = 2m+1$:

$L_2 = (a_{ij})_n$ where $a_{21} = A_{2m}$, $a_{31} = A_{2m-2}, \ldots, a_{m+1,1} = A_2$,

$a_{m+2,1} = A_{2m+1}, a_{m+3,1} = A_{2m-1}, \ldots, a_{2m+1,1} = A_3$.

$$L_2 = \begin{matrix} A_1 & A_2 & A_3 & \ldots & A_n \\ A_{2m} & A_{2m+1} & A_1 & \ldots & A_{2m-1} \\ A_{2m-2} & A_{2m-1} & A_{2m} & \ldots & A_{2m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & A_4 & \ldots & A_1 \\ A_{2m+1} & A_1 & A_2 & \ldots & A_{2m} \\ A_{2m-1} & A_{2m} & A_{2m+1} & \ldots & A_{2m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_3 & A_4 & A_5 & \ldots & A_2 \end{matrix}$$

(ii) $n = 2m$:

$L_3 = (a_{ij})_n$ where $a_{21} = A_{2m}$, $a_{31} = A_{2m-2}, \ldots, a_{m+1,1} = A_2$,

$a_{m+2,1} = A_{2m-1}, a_{m+3,1} = A_{2m-3}, \ldots, a_{2m,1} = A_3$.

$$L_3 = \begin{matrix} A_1 & A_2 & A_3 & \ldots & A_n \\ A_{2m} & A_1 & A_2 & \ldots & A_{2m-1} \\ A_{2m-2} & A_{2m-1} & A_{2m} & \ldots & A_{2m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_2 & A_3 & A_4 & \ldots & A_1 \\ A_{2m-1} & A_{2m} & A_1 & \ldots & A_{2m-2} \\ A_{2m-3} & A_{2m-2} & A_{2m-1} & \ldots & A_{2m-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_3 & A_4 & A_5 & \ldots & A_2 \end{matrix}$$

(c) (i) $n = 2m+1$:

$L_4 = (a_{ij})_n$ where $a_{21} = A_3$, $a_{31} = A_5, \ldots, a_{m+1,1} = A_{2m+1}$,

$a_{m+2,1} = A_2$, $a_{m+3,1} = A_4, \ldots, a_{2m+1,1} = A_{2m}$.

$$L_4 = \begin{matrix} A_1 & A_2 & A_3 & \ldots & A_n \\ A_3 & A_4 & A_5 & \ldots & A_2 \\ A_5 & A_6 & A_7 & \ldots & A_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{2m+1} & A_1 & A_2 & \ldots & A_{2m} \\ A_2 & A_3 & A_4 & \ldots & A_1 \\ A_4 & A_5 & A_6 & \ldots & A_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{2m} & A_{2m+1} & A_1 & \ldots & A_{2m-1} \end{matrix}$$

(ii) $n = 2m$:

$L_5 = (a_{ij})_n$ where $a_{21} = A_3$, $a_{31} = A_5, \ldots, a_{m,1} = A_{2m-1}$, $a_{m+1,1} = A_2$,

$a_{m+2,1} = A_4, \ldots, a_{2m,1} = A_{2m}$.

$$L_5 = \begin{matrix} A_1 & A_2 & A_3 & \ldots & A_n \\ A_3 & A_4 & A_5 & \ldots & A_2 \\ A_5 & A_6 & A_7 & \ldots & A_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{2m-1} & A_{2m} & A_1 & \ldots & A_{2m-2} \\ A_2 & A_3 & A_4 & \ldots & A_1 \\ A_4 & A_5 & A_6 & \ldots & A_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{2m} & A_1 & A_2 & \ldots & A_{2m-1} \end{matrix}$$

In [4], it was shown that for each integer $n \geq 3$, there exists a non-commutative generalized Latin square of order n which is embeddable in a group. Consider the dihedral group

$$D_{2n} = \langle x, y | x^n = y^2 = 1, yx = x^{n-1}y \rangle$$

of order $2n$ for $n \geq 3$. Let S be a subset of D_{2n} consisting of the elements $y, xy, \ldots, x^{n-1}y$. By considering the multiplication table of S with elements arranged in the order $y, xy, \ldots, x^{n-1}y$, we obtain the non-commutative generalized Latin square of order n as given in L_1. For example, when $n = 4$, we see that

$$L_1 = \begin{matrix} A_1 & A_2 & A_3 & A_4 \\ A_4 & A_1 & A_2 & A_3 \\ A_3 & A_4 & A_1 & A_2 \\ A_2 & A_3 & A_4 & A_1 \end{matrix}$$

and the corresponding multiplication table in D_8 is given as

	y	xy	x^2y	x^3y
y	1	x	x^2	x^3
xy	x^3	1	x	x^2
x^2y	x^2	x^3	1	x
x^3y	x	x^2	x^3	1

.

3.3.1 Some Generalized Latin Squares of Odd Order Which Are Embeddable in Groups

In this section we show that the squares L_2 and L_4 are embeddable in groups. To do this we consider the group

$$\begin{aligned} G &= \langle x, y, z | x^m = y, y^m = 1, z^{2m+1} = 1, x^{-1}zx = z^m, \\ &\quad y^{-1}zy = z^{2m} \rangle \\ &\cong C_{2m+1} \rtimes C_{m^2} \qquad (m = 2, 3, \dots) \end{aligned}$$

where C_n denotes the cyclic group of order n.

1. Let $S = \{x, xz^{n-1}, xz^{n-2}, \dots, xz\} \subseteq G$. The multiplication table of S when its elements are arranged in the order $x, xz^{n-1}, xz^{n-2}, \dots, xz$ takes the same form as L_2. For example, in the case $m = 2$, we have $G = \langle x, y, z | x^2 = y, y^2 = 1, z^5 = 1, x^{-1}zx = z^2, y^{-1}zy = z^4 \rangle$ and the multiplication table below:

	x	xz^4	xz^3	xz^2	xz
x	y	yz^3	yz	yz^4	yz^2
xz^4	yz^4	yz^2	y	yz^3	yz
xz^3	yz^3	yz	yz^4	yz^2	y
xz^2	yz^2	y	yz^3	yz	yz^4
xz	yz	yz^4	yz^2	y	yz^3

2. Let $S = \{xyz^{n-1}, xyz^{n-2}, \ldots, xyz, xy\} \subseteq G$. The multiplication table of S when its elements are arranged in the order $xyz^{n-1}, xyz^{n-2}, \ldots, xyz, xy$ takes the same form as L_4. For example, if $m = 3$, we have $G = \langle x, y, z | x^3 = y, y^3 = 1, z^7 = 1, x^{-1}zx = z^3, y^{-1}zy = z^6\rangle$ and the following multiplication table:

	xyz^6	xyz^5	xyz^4	xyz^3	xyz^2	xyz	xy
xyz^6	x^8z^2	x^8z^5	x^8z	x^8z^4	x^8	x^8z^3	x^8z^6
xyz^5	x^8z	x^8z^4	x^8	x^8z^3	x^8z^6	x^8z^2	x^8z^5
xyz^4	x^8	x^8z^3	x^8z^6	x^8z^2	x^8z^5	x^8z	x^8z^4
xyz^3	x^8z^6	x^8z^2	x^8z^5	x^8z	x^8z^4	x^8	x^8z^3
xyz^2	x^8z^5	x^8z	x^8z^4	x^8	x^8z^3	x^8z^6	x^8z^2
xyz	x^8z^4	x^8	x^8z^3	x^8z^6	x^8z^2	x^8z^5	x^8z
xy	x^8z^3	x^8z^6	x^8z^2	x^8z^5	x^8z	x^8z^4	x^8

The smallest odd order for the existence of a non-commutative generalized Latin square is 3. When $n = 3$, we see that

$$L_2 = \begin{matrix} A_1 & A_2 & A_3 \\ A_2 & A_3 & A_1 \\ A_3 & A_1 & A_2 \end{matrix}$$

is a commutative generalized Latin square which can be embedded in $\mathbb{Z}_3$ as shown below.

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

We also note that when $n = 3$,

$$L_4 = \begin{matrix} A_1 & A_2 & A_3 \\ A_3 & A_1 & A_2 \\ A_2 & A_3 & A_1 \end{matrix} = L_1$$

and hence L_4 is embeddable in the dihedral group of order 6 as shown below.

	y	xy	x^2y
y	1	x	x^2
xy	x^2	1	x
x^2y	x	x^2	1

3.3.2 Some Generalized Latin Squares of Even Order Which Are Not Embeddable in Groups

By using Proposition 2.2, we show that the squares L_3 and L_5 are not embeddable in any group.

1. Consider the square L_3. Suppose that there exists a group G with a subset S such that the multiplication table of S takes the same form as L_3. By taking x, y, z as the first, second and $(m+2)$-th elements respectively in the order of elements in the multiplication table of S, we then have that x, y, z satisfy the conditions in Proposition 2.2. But by Proposition 2.2, such a group does not exist. For example, when $n = 4$, we have

$$L_3 = \begin{matrix} A_1 & A_2 & A_3 & A_4 \\ A_4 & A_1 & A_2 & A_3 \\ A_2 & A_3 & A_4 & A_1 \\ A_3 & A_4 & A_1 & A_2 \end{matrix}.$$

   Choose x, y, z to be the 1st, 2nd and 4th elements in S. Then $x^2 = y^2 = A_1$, $xy = yz = A_4$ and $y^2 = A_1 \neq A_2 = z^2$.

2. Consider the square L_5. Suppose that there exists a group G with a subset S such that the multiplication table of S takes the same form as L_5. We consider two separate cases as follows:

   (a) Case 1: $3 \mid (n-1)$

   We choose x, y, z to be the $(\frac{n-1}{3}+1)$-th, $(\frac{2(n-1)}{3}+1)$-th and first elements respectively in the order of elements in the multiplication table of S. For example, when $n = 10$, we have

$$L_5 = \begin{matrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & A_8 & A_9 & A_{10} \\ A_3 & A_4 & A_5 & A_6 & A_7 & A_8 & A_9 & A_{10} & A_1 & A_2 \\ A_5 & A_6 & A_7 & A_8 & A_9 & A_{10} & A_1 & A_2 & A_3 & A_4 \\ A_7 & A_8 & A_9 & A_{10} & A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ A_9 & A_{10} & A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & A_8 \\ A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & A_8 & A_9 & A_{10} & A_1 \\ A_4 & A_5 & A_6 & A_7 & A_8 & A_9 & A_{10} & A_1 & A_2 & A_3 \\ A_6 & A_7 & A_8 & A_9 & A_{10} & A_1 & A_2 & A_3 & A_4 & A_5 \\ A_8 & A_9 & A_{10} & A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 \\ A_{10} & A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & A_8 & A_9 \end{matrix} .$$

Choose x, y, z to be the fourth, seventh and first elements respectively in the order of elements in the multiplication table of S. Then we see that $x^2 = y^2 = A_{10}$, $xy = yz = A_7$ and $y^2 = A_{10} \neq A_1 = z^2$. Again by Proposition 2.2, the elements x, y, z cannot be contained in any group.

(b) Case 2: $3 \nmid (n-1)$

(i) If $3 \mid n$, we choose x, y, z to be the first, $(\frac{n}{3}+1)$-th and $(\frac{n}{6}+1)$-th elements in S. For example, when $n = 6$, we obtain

$$L_5 = \begin{matrix} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ A_3 & A_4 & A_5 & A_6 & A_1 & A_2 \\ A_5 & A_6 & A_1 & A_2 & A_3 & A_4 \\ A_2 & A_3 & A_4 & A_5 & A_6 & A_1 \\ A_4 & A_5 & A_6 & A_1 & A_2 & A_3 \\ A_6 & A_1 & A_2 & A_3 & A_4 & A_5 \end{matrix} .$$

Choose x, y, z to be the first, third and second elements respectively in the order of elements in the multiplication table of S. Then we see that $x^2 = y^2 = A_1$, $xy = yz = A_5$ and $y^2 = A_1 \neq A_4 = z^2$. By Proposition 2.2, there does not exist any group containing x, y, z.

(ii) If $3 \nmid n$, we choose x, y, z to be the first, $(\frac{2(n+1)}{3})$-th and $(\frac{n+1}{3}+1)$-th elements respectively in the order of elements in the multiplication table of S. For example, when $n = 8$, we have

$$L_5 = \begin{array}{cccccccc} A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & A_8 \\ A_3 & A_4 & A_5 & A_6 & A_7 & A_8 & A_1 & A_2 \\ A_5 & A_6 & A_7 & A_8 & A_1 & A_2 & A_3 & A_4 \\ A_7 & A_8 & A_1 & A_2 & A_3 & A_4 & A_5 & A_6 \\ A_2 & A_3 & A_4 & A_5 & A_6 & A_7 & A_8 & A_1 \\ A_4 & A_5 & A_6 & A_7 & A_8 & A_1 & A_2 & A_3 \\ A_6 & A_7 & A_8 & A_1 & A_2 & A_3 & A_4 & A_5 \\ A_8 & A_1 & A_2 & A_3 & A_4 & A_5 & A_6 & A_7 \end{array}.$$

Choose x, y, z to be the first, sixth and fourth elements respectively in the order of elements in the multiplication table of S. Then we see that $x^2 = y^2 = A_1$, $xy = yz = A_4$ and $y^2 = A_1 \neq A_2 = z^2$. We again have by Proposition 2.2 that a group containing x, y, z cannot exist.

CHAPTER 4

EXHAUSTION NUMBERS OF SUBSETS OF DIHEDRAL GROUPS

4.1 Introduction

Let G be a finite non-abelian group. For a nonempty subset S of G, we say that S is *exhaustive* if there exists a positive integer n such that $S^n = G$. Let $e(S)$ denote the smallest positive integer n such that $S^n = G$. Then $e(S)$ is called the *exhaustion number* of the set S. If such a positive integer n does not exist, we write $e(S) = \infty$. It is clear that if S is a proper subgroup of G, then $e(S) = \infty$. Note that if $e(S) = n$, then S is an n-basis of G. Conversely, if S is an n-basis of G, then $e(S) \leq n$. In the case G is an abelian group, we write G additively and use S^n to denote $S + \cdots + S$ (n times) for any subset of G. The exhaustion numbers of various subsets of finite abelian groups have been investigated in [5] and [6].

In this chapter, we shall study the exhaustion numbers of various subsets of one of the finite non-abelian groups, which is the dihedral group. Recall that the dihedral group may be defined as follows:

$$D_{2n} = \langle x, y | x^n = y^2 = 1, xy = yx^{n-1} \rangle$$

where n is a positive integer and $n \geq 3$. The layout of this chapter is as follows: In Section 4.2, we shall determine the subsets with finite exhaustion numbers. Clearly, if $S = D_{2n}$, then $e(S) = 1$. So we shall focus on the case where $e(S) > 1$, if $e(S)$ exists. We first give some constructions of subsets with certain finite exhaustion numbers. We show that for any subset $S \subseteq D_{2n}$, $e(S) = 2$ if $n < |S| \leq 2n-1$. We then give some constructions of subsets S where $e(S) = k$ for $k \in \{2, 3\}$. Next, we show that if $S = \{1, x, y, xy, x^2y, \ldots, x^iy\}$ for $i \in \{1, 3, 5, \ldots, n-3\}$, then $e(S) \leq \frac{n+1-i}{2}$ when n is even. On the other hand, if $S = \{1, x, y, xy, x^2y, \ldots, x^iy\}$ for $i \in \{2, 4, 6, \ldots, n-3\}$, then $e(S) \leq \frac{n+1-i}{2}$ when n is odd. We also give some constructions of subsets $S \subseteq D_{2n}$ where $e(S) = n$ when n is even. Next, we give some constructions of subsets $S \subseteq D_{2n}$ where $e(S) \in \{n-1, n\}$ when n is odd. In Section 4.3, we shall classify the subsets $S \subseteq D_{2n}$ where $e(S) = \infty$. Finally, we show that there does not exist any subset S in both D_{12} and D_{14} such that $e(S) = 5$. In addition, we also show that there does not exist any subset $S \subseteq D_{22}$ such that $e(S) = k$ for $k = 7, 8, 9$.

Let x be a real number. Throughout this chapter, we shall use the notations $\lfloor x \rfloor$ and $\lceil x \rceil$ to denote the largest integer $\leq x$ and the smallest integer $\geq x$, respectively. It is clear that $\lceil x \rceil = \lfloor x \rfloor + 1$ if x is not an integer. We shall also use the notation $\omega(n)$ to denote the number of distinct prime factors of n for any positive integer n.

4.2 Subsets with Finite Exhaustion Numbers

In this section, we shall study some subsets $S \subseteq D_{2n}$ with finite exhaustion numbers. We begin by investigating the exhaustion number of the subset $S = \{1, x, xy\} \subseteq D_{2n}$.

Lemma 4.1. *Let $S = \{1, x, xy\} \subseteq D_{2n}$, $n \geq 3$ and let $r \geq 2$ be a positive integer. Then* *(i)* $\{1, x, \ldots, x^r\} \subseteq S^r$*;*

(ii) $\{x^{n-(r-2)}, x^{n-(r-3)}, \ldots, x^{n-1}\} \subseteq S^r$*;*

(iii) $\{y, xy, \ldots, x^r y\} \subseteq S^r$*;*

(iv) $\{x^{n-(r-2)}y, x^{n-(r-3)}y, \ldots, x^{n-1}y\} \subseteq S^r$*.*

Proof.

(i) If $a \in S^i (i \geq 1)$, then $a \in S^t$ for all $t \geq i$. Hence, for any positive integer $i \leq r$, we have $x^i = x \cdots x \in S^i \subseteq S^r$ since $1 \in S$.

(ii) Note that $y = (xy)x \in S \cdot S \subseteq S^i$ for all $i \geq 2$. It is clear that $xy \in S \subseteq S^i$ for all $i \geq 1$. Then

$$x^{n-(r-i)} = yx^{r-i-1}(xy) \in S^i \cdot S^{r-i-1} \cdot S = S^r \quad (i = 2, \ldots, r-1).$$

(iii) Since $\{1, x, x^2, \ldots, x^{r-1}\} \subseteq S^{r-1}$ and $xy \in S$, we have

$$\{xy, x^2y, \ldots, x^r y\} \subseteq S^r.$$

Note that $y = (xy)x \in S \cdot S \subseteq S^r$. Hence, $y \in S^r$.

(iv) Finally, it is clear that $x^{n-(r-i)}y = yx^{r-i} \in S^i \cdot S^{r-i} = S^r \quad (i = 2, \ldots, r-1)$.

□

Proposition 4.2. *Let* $S = \{1, x, xy\} \subseteq D_{2n}$*,* $n \geq 3$*. Then*

$$e(S) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n+2}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. We know that every element in D_{2n} can be written in the form $x^i y^j$ for some $i = 0, 1, \ldots, n-1$ and $j = 0, 1$. By Proposition 4.1, it is clear that

$S^r = D_{2n}$ if $n-(r-2) \leq r+1$. That is, $r \geq \frac{n+1}{2}$. Hence, if n is odd, $S^{\frac{n+1}{2}} = D_{2n}$ and therefore $e(S) \leq \frac{n+1}{2}$. If n is even, then $S^{\frac{n+2}{2}} = D_{2n}$ and so $e(S) \leq \frac{n+2}{2}$. If n is odd, note that

$$\{1, x, \ldots, x^{\frac{n-1}{2}}; x^{\frac{n+5}{2}}, x^{\frac{n+7}{2}}, \ldots, x^{n-1}\} \subseteq S^{\frac{n-1}{2}}.$$

Since $x^{\frac{n+1}{2}} = x^{n-(\frac{n-1}{2})} = yx^{\frac{n-1}{2}}y$, we have that $x^{\frac{n+1}{2}}$ cannot be written as a product of less than $\frac{n+1}{2}$ elements of S. Therefore, $e(S) = \frac{n+1}{2}$ if n is odd.

If n is even, note that

$$\{1, x, \ldots, x^{\frac{n}{2}}; x^{\frac{n}{2}+2}, x^{\frac{n}{2}+3}, \ldots, x^{n-1}\} \subseteq S^{\frac{n}{2}}.$$

Since $x^{\frac{n}{2}+1} = x^{n-(\frac{n}{2}-1)} = yx^{\frac{n}{2}-1}y$, we have that $x^{\frac{n}{2}+1}$ cannot be written as a product of less than $\frac{n}{2}+1$ elements of S. Therefore, $e(S) = \frac{n}{2}+1 = \frac{n+2}{2}$ if n is even. $\square$

By adding $y \in D_{2n}$ to the subset $S = \{1, x, xy\}$ above, we have the following results:

Lemma 4.3. *Let $S = \{1, x, y, xy\} \subseteq D_{2n}$, $n \geq 3$ and let $r \geq 2$ be a positive integer. Then*

(i) $\{1, x, \ldots, x^r\} \subseteq S^r$;

(ii) $\{y, xy, \ldots, x^r y\} \subseteq S^r$;

(iii) $\{x^{n-(r-1)}, x^{n-(r-2)}, \ldots, x^{n-1}\} \subseteq S^r$;

(iv) $\{x^{n-(r-1)}y, x^{n-(r-2)}y, \ldots, x^{n-1}y\} \subseteq S^r$.

Proof.

(i) If $a \in S^i (i \geq 1)$, then $a \in S^t$ for all $t \geq i$. Hence, for any positive integer $i \leq r$, we have

$$x^i = x \cdots x \in S^i \subseteq S^r.$$

(ii) Since $\{1, x, x^2, \ldots, x^{r-1}\} \subseteq S^{r-1}$ and $y, xy \in S$, we have

$$\{y, xy, x^2y, \ldots, x^ry\} \subseteq S^r.$$

(iii) Note that $y \in S \subseteq S^i$ for all $i \geq 1$. Then

$$x^{n-(r-i)} = yx^{r-i}y \in S^i \cdot S^{r-i} = S^r \quad (i = 1, 2, \ldots, r-1).$$

(iv) Finally, note that

$$x^{n-(r-i)}y = yx^{r-i} \in S^i \cdot S^{r-i} = S^r \quad (i = 1, 2, \ldots, r-1).$$

□

Proposition 4.4. *Let $S = \{1, x, y, xy\} \subseteq D_{2n}$, $n \geq 3$. Then*

$$e(S) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. We know that every element in D_{2n} can be written in the form x^iy^j for some $i = 0, 1, \ldots, n-1$ and $j = 0, 1$. By Lemma 4.3, it is clear that $S^r = D_{2n}$ if $n - (r-1) \leq r + 1$. That is, $r \geq \frac{n}{2}$. Hence, if n is odd, $S^{\frac{n+1}{2}} = D_{2n}$ and therefore $e(S) \leq \frac{n+1}{2}$. If n is even, then $S^{\frac{n}{2}} = D_{2n}$ and so $e(S) \leq \frac{n}{2}$. If n is odd, note that

$$\{1, x, \ldots, x^{\frac{n-1}{2}}; x^{\frac{n+5}{2}}, x^{\frac{n+7}{2}}, \ldots, x^{n-1}\} \subseteq S^{\frac{n-1}{2}}.$$

Since $x^{\frac{n+1}{2}} = yx^{\frac{n-1}{2}}y$ cannot be written as a product of less than $\frac{n+1}{2}$ elements of S. Therefore, $e(S) = \frac{n+1}{2}$ if n is odd. If n is even, note that

$$\{1, x, \ldots, x^{\frac{n}{2}}; x^{\frac{n}{2}+2}, x^{\frac{n}{2}+3}, \ldots, x^{n-1}\} \subseteq S^{\frac{n}{2}-1}.$$

Since $x^{\frac{n}{2}+1} = yx^{\frac{n}{2}-1}y$, we have that $x^{\frac{n}{2}+1}$ cannot be written as a product of less than $\frac{n}{2}$ elements of S. Therefore, $e(S) = \frac{n}{2}$ if n is even. □

From Propositions 4.2 and 4.4, the following result is obvious.

Proposition 4.5. *Let S, S' be subsets of D_{2n}. If $S \subseteq S'$ and $e(S)$ exists, then $e(S') \leq e(S)$.*

Proof. The result is clear since $S^m \subseteq S'^m$ for any positive integer m. □

Note that $|D_{2n}| = 2n$ and if $\phi \subsetneq S \subsetneq D_{2n}$, then $0 < |S| < 2n$. We shall begin with the subsets S where $n < |S| \leq 2n - 1$.

Proposition 4.6. *Let D_{2n} be the dihedral group of order $2n$ where $n \geq 4$. If S is a subset of D_{2n} where $n < |S| \leq 2n - 1$, then $e(S) = 2$.*

Proof. Let $S \subseteq D_{2n}$. Since $|S| > n$, we shall consider three cases as follows:

(i) Suppose that $\{1, x, x^2, \ldots, x^{n-1}\} \subseteq S$. Given that $|S| > n$, there exists $x^i y \in S$ for some $i \in \{0, 1, \ldots, n-1\}$. Note that for all $k \in \{0, 1, \ldots, n-1\}$,

$$x^{k+i}y = x^k \cdot x^i y \in S^2.$$

Hence, we have

$$\{x^i y, x^{1+i}y, \ldots, x^{(n-1)+i}y\} = \{y, xy, \ldots, x^{n-1}y\} \subseteq S^2. \qquad (4.1)$$

Given that $\{1, x, x^2, \ldots, x^{n-1}\} \subseteq S$ and $1 \in S$, we have

$$\{1, x, x^2, \ldots, x^{n-1}\} \subseteq S^2. \tag{4.2}$$

Combining (4.1) and (4.2), we have $S^2 = D_{2n}$.

(ii) Suppose that $\{y, xy, x^2y, \ldots, x^{n-1}y\} \subseteq S$. Given that $|S| > n$, there exists $x^i \in S$ for some $i \in \{0, 1, \ldots, n-1\}$. Note that for all $k \in \{0, 1, \ldots, n-1\}$,

$$x^{i+k}y = x^i \cdot x^k y \in S^2.$$

Hence, we have

$$\{x^iy, x^{1+i}y, \ldots, x^{(n-1)+i}y\} = \{y, xy, \ldots, x^{n-1}y\} \subseteq S^2. \tag{4.3}$$

Given that $\{y, xy, \ldots, x^{n-1}y\} \subseteq S$ and $y \in S$, we have

$$\{1, x, x^2, \ldots, x^{n-1}\} \subseteq S^2. \tag{4.4}$$

Combining (4.3) and (4.4), we have $S^2 = D_{2n}$.

(iii) Suppose that $\{1, x, x^2, \ldots, x^{n-1}\} \nsubseteq S$ and $\{y, xy, x^2y, \ldots, x^{n-1}y\} \nsubseteq S$. Let

$$A = S \cap \{1,\, x,\, x^2,\, \ldots,\, x^{n-1}\}$$

and

$$B = S \cap \{y, xy, x^2y, \ldots, x^{n-1}y\}.$$

Then $A, B \neq \emptyset$ (because $|S| \geq n+1$), $S = A \cup B$ and $A \cap B = \emptyset$. First, we show that $x^iy \in S^2$ for all $i \in \{0, 1, \ldots, n-1\}$. Let

$|A| = n - k$ where $k \in \{1, 2, \ldots, n-1\}$. Then $|B| = |S| - |A| \geq (n+1) - (n-k) = k+1$. Let $B' = \{x^{m_1}y, x^{m_2}y, \ldots, x^{m_{k+1}}y\} \subseteq B$ and let $A(x^{m_i}y) = \{x^j(x^{m_i}y) \mid x^j \in A\}$ for $i \in \{1, \ldots, k+1\}$. Then $A(x^{m_i}y) \neq A(x^{m_j}y)$ and $|A(x^{m_i}y)| = |A|$ for $i \neq j$, $i, j \in \{1, \ldots, k+1\}$. Thus $AB' = \{x^j(x^{m_l}y) \mid x^j \in A, x^{m_l}y \in B'\}$ has at least $|A| + k = n$ elements. But since there are only n elements of the form x^iy in D_{2n}, it follows that $x^iy \in AB' \subseteq S^2$ for all $i \in \{0, 1, \ldots, n-1\}$.

Next we show that $x^i \in S^2$ for all $i \in \{0, 1, \ldots, n-1\}$. Suppose first that $|A| \geq \lceil \frac{n+1}{2} \rceil$. Assume that $x^k \notin S^2$ for some $k \in \{0, 1, \ldots, n-1\}$. Since $x^k = (x)(x^{k-1}) = (x^2)(x^{k-2}) = (x^3)(x^{k-3}) = \cdots = (x^{n-1})(x^{k-(n-1)})$ and the x^i commute with one another, we have that the number of ways to represent x^k as a product of two elements of $\{1, x, \ldots, x^{n-1}\}$ is

$$\begin{cases} \lceil \frac{n}{2} \rceil, & \text{if } n \text{ is even} \\ \lceil \frac{n}{2} \rceil - 1, & \text{if } n \text{ is odd} \end{cases}.$$

Since $x^k \notin S^2$, so we must have

$$|A| \leq \begin{cases} \lceil \frac{n}{2} \rceil, & \text{if } n \text{ is even} \\ \lceil \frac{n}{2} \rceil - 1, & \text{if } n \text{ is odd} \end{cases},$$

which is a contradiction. Thus $x^i \in S^2$ for all $i \in \{0, 1, \ldots, n-1\}$. Now suppose that $|B| \geq \lceil \frac{n+1}{2} \rceil$. Assume that $x^k \notin S^2$ for some $k \in \{0, 1, \ldots, n-1\}$. Note that $x^k = (y)(x^{n-k}y) = (xy)(x^{n-k+1}y) = (x^2y)(x^{n-k+2}y) = \cdots = (x^{n-1}y)(x^{2n-k-1}y)$. If n is odd, the x^iy do not commute with one another. If n is even, then $(x^iy)(x^jy) = (x^jy)(x^iy)$ if and only if $i - j \equiv 0 \pmod{\frac{n}{2}}$. Thus, the number of ways to represent x^k

as a product of the $x^i y$ is

$$\begin{cases} \frac{n}{2}, & \text{if } n \text{ is even and } k = \frac{n}{2} \\ n, & \text{otherwise} \end{cases}.$$

It follows that $|B| < \lceil \frac{n+1}{2} \rceil$; a contradiction. We thus have that $x^i \in S^2$ for all $i \in \{0, 1, \ldots, n-1\}$.

Collecting the above results, we obtain that $S^2 = D_{2n}$.

□

From Proposition 4.6, we see that if $n < |S| \leq 2n$, then $e(S) = 2$. In the remaining of this section, we will focus on the subsets S where $|S| \leq n$. Firstly, we shall give some constructions of subsets S where $e(S) = k$ for $k \in \{2, 3\}$.

Proposition 4.7. *Let D_{2n} be the dihedral group of order $2n$ where $n \geq 6$. Let S be a subset of D_{2n} as follows:*

(i) $S = \{1, x, y, xy, x^2y, x^3y, \ldots, x^{n-3}y\}$, $|S| = n$;

(ii) $S = \{x, y, xy, x^2y, x^3y, \ldots, x^{n-3}y\}$, $|S| = n-1$;

(iii) $S = \{x, x^2, x^3, \ldots, x^{n-4}, y, xy\}$, $|S| = n-2$.

Then $e(S) = 2$.

Proof.

(i),(ii) Since $\{y, xy, x^2y, \ldots, x^{n-3}y\} \subseteq S$ and $x \in S$, we have

$$\{xy, x^2y, \ldots, x^{n-2}y\} \subseteq S^2.$$

Note that

$$x^{n-i} = y \cdot x^i y \in S \cdot S \quad \text{for all} \quad i = 1, 2, \ldots, n-3,$$

and hence, $\{x^{n-1}, x^{n-2}, \ldots, x^3\} \subseteq S^2$. It is clear that $1 = y \cdot y$, $x = xy \cdot y$, $x^2 = x \cdot x$, $y = x \cdot yx$, $x^{n-1}y = y \cdot x$ and hence, $1, x, x^2, y, x^{n-1}y \in S^2$. Thus, $S^2 = D_{2n}$ and we have $e(S) = 2$.

(iii) Since $\{x, x^2, x^3, \ldots, x^{n-4}\} \subseteq S$ and $x, xy \in S$, we have

$$\{x^2, x^3, x^4, \ldots, x^{n-3}\} \subseteq S^2$$

and

$$\{x^2y, x^3y, x^4y, \ldots, x^{n-3}y\} \subseteq S^2.$$

Next, we see that $x = xy \cdot y$, $x^{n-2} = x^2 \cdot x^{n-4}$, $x^{n-1} = y \cdot xy$, $y = xy \cdot x$, $xy = x \cdot y$, $x^{n-2}y = y \cdot x^2$, $x^{n-1}y = y \cdot x$ and $1 = y \cdot y$. Hence, $x, x^{n-2}, x^{n-1}, y, xy, x^{n-2}y, x^{n-1}y, 1 \in S^2$ and $e(S) = 2$.

□

In Table 4.1, we provide the total number of subsets S in both D_{12} and D_{14} where $|S| \leq n$ and $e(S) = 2$.

Table 4.1: Number of subsets S in D_{12} and D_{14} where $|S| \leq n$ and $e(S) = 2$

D_{12}		D_{14}	
$\|S\|$	Number of subsets	$\|S\|$	Number of subsets
4	12	5	434
5	312	6	2212
6	760	7	3332
Total:	1084	Total:	5978

Next, we shall proceed to obtain some subsets $S \subseteq D_{2n}$ where $e(S) = 3$.

Proposition 4.8. *Let D_{2n} be the dihedral group of order $2n$ where $n \geq 6$. Let S be a subset of D_{2n} as follows:*

(i) $S = \{1, xy, x^2y, x^3y, \ldots, x^{n-1}y\}$, $|S| = n$;

(ii) $S = \{1, x, y, xy, x^2y, x^3y, \ldots, x^{n-4}y\}$, $|S| = n - 1$;

(iii) $S = \{1, x, y, xy, x^2y, x^3y, \ldots, x^{n-5}y\}$, $|S| = n - 2$.

Then $e(S) = 3$.

Proof.

(i) Let $a \in S^i$ for any positive integer $i \geq 1$. Then $a \in S^{i+1}$ since $1 \in S$. Hence, we have

$$\{1, xy, x^2y, x^3y, \ldots, x^{n-1}y\} \subseteq S^3.$$

Note that

$$x^{n-i+1} = xy \cdot x^iy \in S \cdot S \quad \text{for } i = 2, 3, \ldots, n-1.$$

Therefore, $\{x^2, \ldots, x^{n-1}\} \subseteq S^2 \subseteq S^3$. It is clear that $x = x^2y \cdot xy \in S^2 \subseteq S^3$, $y = (xy) \cdot (x^2y) \cdot (xy) \in S^3$ and $S^3 = D_{2n}$. Since y cannot be written as a product of less than 3 elements of S, we have $e(S) = 3$.

(ii),(iii) Since

$$\{y, xy, x^2y, \ldots, x^{n-5}y\} \subseteq S \tag{4.5}$$

and $x^2 \in S^2$, we have

$$\{x^2y, x^3y, \ldots, x^{n-3}y\} \subseteq S^3. \tag{4.6}$$

Note that $y = y \cdot 1 \in S^2$, by multiplying y on the left with all the elements in (4.5), we see that

$$\{1, x^{n-1}, x^{n-2}, \ldots, x^5\} \subseteq S^3. \tag{4.7}$$

since $yx^iy = x^{n-i}$. By combining (4.6) and (4.7), we have

$$\{1, x^{n-1}, x^{n-2}, \ldots, x^5, x^2y, x^3y, \ldots, x^{n-3}y\} \subseteq S^3$$

It is clear that $x, x^2, x^3, y, xy \in S^3$, $x^4 = y \cdot x \cdot x^{n-5}y \in S^3$, $x^{n-2}y = y \cdot x \cdot x \in S^3$ and $x^{n-1}y = y \cdot x \cdot 1 \in S^3$. Thus, $S^3 = D_{2n}$. Since $x^{n-2}y$ cannot be written as a product of less than 3 elements of S, we have $e(S) = 3$.

□

From Propositions 4.7(i) and 4.8(iii), it is clear that $e(S)$ increases if we remove the last two elements from the subset S given in Proposition 4.7(i). In the following, we consider the subset $S = \{1, x, y, xy, x^2y, \ldots, x^iy\}$ for some integer i and obtain an upper bound for $e(S)$. In the next two propositions, we will consider the cases where n is even and n is odd separately.

Proposition 4.9. *Let D_{2n} be the dihedral group of order $2n$ where $n \geq 4$ is even. If $S = \{1, x, y, xy, x^2y, \ldots, x^iy\} \subseteq D_{2n}$, then $e(S) \leq \frac{n+1-i}{2}$ for $i = 1, 3, 5, \ldots, n-3$.*

Proof. Let $c \in S^k$ for $k \geq 1$. Then $c \in S^{k+1}$ since $1 \in S$. Given that i is odd and $i = 1, 3, 5, \ldots, n-3$. Since $1, x \in S$, we have

$$x^k \in S^k \subseteq S^{\frac{n+1-i}{2}} \quad \text{for } k = 1, 2, \ldots, \frac{n+1-i}{2}$$

and hence

$$\{1, x, x^2, \ldots, x^{\frac{n+1-i}{2}}\} \subseteq S^{\frac{n+1-i}{2}}. \tag{4.8}$$

Given that $\{y, xy, x^2y, \ldots, x^iy\} \subseteq S$, we have

$$\begin{aligned}
x^{n-j} &= y \cdot x^jy \in S \cdot S = S^2 \quad \text{for } j = 1, 2, \ldots, i. \\
x^{n-j+1} &= xy \cdot x^jy \in S \cdot S = S^2 \quad \text{for } j = 1, 2, \ldots, i. \\
x^{n-j+2} &= x^2y \cdot x^jy \in S \cdot S = S^2 \quad \text{for } j = 1, 2, \ldots, i. \\
&\vdots \\
x^{n-j+i} &= x^iy \cdot x^jy \in S \cdot S = S^2 \quad \text{for } j = 1, 2, \ldots, i.
\end{aligned} \tag{4.9}$$

Next, note that

$$\begin{aligned}
x^{n-j-u} &= y \cdot x^j \cdot x^uy \in S \cdot S^j \cdot S = S^{j+2} \\
x^{n-j-u+1} &= xy \cdot x^j \cdot x^uy \in S \cdot S^j \cdot S = S^{j+2} \\
x^{n-j-u+2} &= x^2y \cdot x^j \cdot x^uy \in S \cdot S^j \cdot S = S^{j+2} \\
&\vdots \\
x^{n-j-u+i} &= x^iy \cdot x^j \cdot x^uy \in S \cdot S^j \cdot S = S^{j+2}
\end{aligned} \tag{4.10}$$

for $j = 1, 2, \ldots, \frac{n-3-i}{2}$ and $u = 1, 2, \ldots, i$. Combining (4.8), (4.9) and (4.10), we have

$$\{1, x, x^2, \ldots, x^{n-1}\} \subseteq S^{\frac{n+1-i}{2}}.$$

Next, we note that

$$\{y, xy, x^2y, \ldots, x^iy\} \subseteq S \subseteq S^{\frac{n+1-i}{2}}. \tag{4.11}$$

Since $x^i y \in S$, we have

$$\begin{aligned} x^j y &= x^j \cdot y \in S^j \cdot S = S^{j+1} \quad \text{for } j = 1, 2, \ldots, \frac{n-1-i}{2}. \\ x^{j+1} y &= x^j \cdot xy \in S^j \cdot S = S^{j+1} \quad \text{for } j = 1, 2, \ldots, \frac{n-1-i}{2}. \\ x^{j+2} y &= x^j \cdot x^2 y \in S^j \cdot S = S^{j+1} \quad \text{for } j = 1, 2, \ldots, \frac{n-1-i}{2}. \\ &\vdots \\ x^{j+i} y &= x^j \cdot x^i y \in S^j \cdot S = S^{j+1} \quad \text{for } j = 1, 2, \ldots, \frac{n-1-i}{2}. \end{aligned} \tag{4.12}$$

Finally, we see that

$$\begin{aligned} x^{n-j} y &= y \cdot x^j \in S \cdot S^j = S^{j+1} \quad \text{for } j = 1, 2, \ldots, \frac{n-1-i}{2} \\ x^{n-j+1} y &= xy \cdot x^j \in S \cdot S^j = S^{j+1} \quad \text{for } j = 1, 2, \ldots, \frac{n-1-i}{2} \\ x^{n-j+2} y &= x^2 y \cdot x^j \in S \cdot S^j = S^{j+1} \quad \text{for } j = 1, 2, \ldots, \frac{n-1-i}{2} \\ &\vdots \\ x^{n-j+i} y &= x^i y \cdot x^j \in S \cdot S^j = S^{j+1} \quad \text{for } j = 1, 2, \ldots, \frac{n-1-i}{2} \end{aligned} \tag{4.13}$$

Combining (4.11), (4.12) and (4.13), we have

$$\{y, xy, x^2y, \ldots, x^{n-1}y\} \subseteq S^{\frac{n+1-i}{2}}.$$

Thus, $S^{\frac{n+1-i}{2}} = D_{2n}$ and hence $e(S) \leq \frac{n+1-i}{2}$. □

In Proposition 4.10, we will see that the case where n is odd for $S = \{1, x, y, xy, x^2y, \ldots, x^iy\} \subseteq D_{2n}$ can be proved similarly by considering even i where $i \in \{2, 4, 6, \ldots, n-3\}$.

Proposition 4.10. *Let D_{2n} be the dihedral group of order $2n$ where $n \geq 5$ is odd. If $S = \{1, x, y, xy, x^2y, \ldots, x^iy\} \subseteq D_{2n}$, then $e(S) \leq \frac{n+1-i}{2}$ for $i = 2, 4, 6, \ldots, n-3$.*

Proof. Let $c \in S^k$ for $k \geq 1$. Then $c \in S^{k+1}$ since $1 \in S$. Given that i is even and $i = 2, 4, 6, \ldots, n-3$. Since $1, x \in S$, we have

$$x^k \in S^k \subseteq S^{\frac{n+1-i}{2}} \quad \text{for } k = 1, 2, \ldots, \frac{n+1-i}{2}$$

and hence

$$\{1, x, x^2, \ldots, x^{\frac{n+1-i}{2}}\} \subseteq S^{\frac{n+1-i}{2}}. \tag{4.14}$$

Given that $\{y, xy, x^2y, \ldots, x^iy\} \subseteq S$, we have

$$\begin{aligned}
x^{n-j} &= y \cdot x^j y \in S \cdot S = S^2 \quad \text{for } j = 1, 2, \ldots, i. \\
x^{n-j+1} &= xy \cdot x^j y \in S \cdot S = S^2 \quad \text{for } j = 1, 2, \ldots, i. \\
x^{n-j+2} &= x^2y \cdot x^j y \in S \cdot S = S^2 \quad \text{for } j = 1, 2, \ldots, i. \\
&\vdots \\
x^{n-j+i} &= x^iy \cdot x^j y \in S \cdot S = S^2 \quad \text{for } j = 1, 2, \ldots, i.
\end{aligned} \tag{4.15}$$

Next, note that

$$\begin{aligned}
x^{n-j-u} &= y \cdot x^j \cdot x^u y \in S \cdot S^j \cdot S = S^{j+2} \\
x^{n-j-u+1} &= xy \cdot x^j \cdot x^u y \in S \cdot S^j \cdot S = S^{j+2} \\
x^{n-j-u+2} &= x^2y \cdot x^j \cdot x^u y \in S \cdot S^j \cdot S = S^{j+2} \\
&\vdots \\
x^{n-j-u+i} &= x^iy \cdot x^j \cdot x^u y \in S \cdot S^j \cdot S = S^{j+2}
\end{aligned} \tag{4.16}$$

for $j = 1, 2, \ldots, \frac{n-3-i}{2}$ and $u = 1, 2, \ldots, i$. Combining (4.14), (4.15) and (4.16), we have

$$\{1, x, x^2, \ldots, x^{n-1}\} \subseteq S^{\frac{n+1-i}{2}}.$$

Next, we note that

$$\{y, xy, x^2y, \ldots, x^iy\} \subseteq S \subseteq S^{\frac{n+1-i}{2}}. \tag{4.17}$$

Since $x^iy \in S$, we have

$$\begin{aligned}
x^jy = x^j \cdot y \in S^j \cdot S = S^{j+1} \quad &\text{for } j = 1, 2, \ldots, \frac{n-1-i}{2}. \\
x^{j+1}y = x^j \cdot xy \in S^j \cdot S = S^{j+1} \quad &\text{for } j = 1, 2, \ldots, \frac{n-1-i}{2}. \\
x^{j+2}y = x^j \cdot x^2y \in S^j \cdot S = S^{j+1} \quad &\text{for } j = 1, 2, \ldots, \frac{n-1-i}{2}. \\
\vdots \\
x^{j+i}y = x^j \cdot x^iy \in S^j \cdot S = S^{j+1} \quad &\text{for } j = 1, 2, \ldots, \frac{n-1-i}{2}.
\end{aligned} \tag{4.18}$$

Finally, we see that

$$\begin{aligned}
x^{n-j}y = y \cdot x^j \in S \cdot S^j = S^{j+1} \quad &\text{for } j = 1, 2, \ldots, \frac{n-1-i}{2} \\
x^{n-j+1}y = xy \cdot x^j \in S \cdot S^j = S^{j+1} \quad &\text{for } j = 1, 2, \ldots, \frac{n-1-i}{2} \\
x^{n-j+2}y = x^2y \cdot x^j \in S \cdot S^j = S^{j+1} \quad &\text{for } j = 1, 2, \ldots, \frac{n-1-i}{2} \\
\vdots \\
x^{n-j+i}y = x^iy \cdot x^j \in S \cdot S^j = S^{j+1} \quad &\text{for } j = 1, 2, \ldots, \frac{n-1-i}{2}
\end{aligned} \tag{4.19}$$

Combining (4.17), (4.18) and (4.19), we have

$$\{y, xy, x^2y, \ldots, x^{n-1}y\} \subseteq S^{\frac{n+1-i}{2}}.$$

Thus, $S^{\frac{n+1-i}{2}} = D_{2n}$ and hence $e(S) \leq \frac{n+1-i}{2}$. □

Now we turn our attention to the subsets S where $e(S) = n - 1$ when n is odd.

Proposition 4.11. *Let $S = \{x, y, x^2y\} \subseteq D_{2n}$ where $n \geq 5$ is odd. Then $e(S) = n - 1$.*

Proof. Let $a \in S^i$ for any positive integer $i \geq 1$. Then $a \in S^{i+2}$ since $y \cdot y = 1 \in S^2$. It is clear that for all $x^i \in S^i$, we have

$$\{x^2, x^4, \ldots, x^{n-1}\} \subseteq S^{n-1}. \tag{4.20}$$

and

$$\{x, x^3, \ldots, x^{n-2}\} \subseteq S^{n-2}. \tag{4.21}$$

Since $y \in S$, from (4.21), we note that

$$\{xy, x^3y, \ldots, x^{n-2}y\} \subseteq S^{n-1}. \tag{4.22}$$

and

$$\{yx, yx^3, \ldots, yx^{n-2}\} \subseteq S^{n-1}. \tag{4.23}$$

Since $x^{n-(2i+1)}y = yx^{2i+1}$ for $i = 0, 1, \ldots, \frac{n-3}{2}$, from (4.23) we have

$$\{x^{n-1}y, x^{n-3}y, \ldots, x^2y\} = \{yx, yx^3, \ldots, yx^{n-2}\} \subseteq S^{n-1}. \tag{4.24}$$

By combining (4.22) and (4.24), we see that

$$\{xy, x^2y, x^3y \ldots, x^{n-2}y, x^{n-1}y\} \subseteq S^{n-1}.$$

Note that $y = x^{n-2}(x^2y) \in S^{n-2} \cdot S = S^{n-1}$. Hence,

$$\{y, xy, x^2y, \ldots, x^{n-1}y\} \subseteq S^{n-1}.$$

Next from the equation (4.20), we have

$$\{x^2, x^4, \ldots, x^{n-5}, x^{n-3}\} \subseteq S^{n-3} \tag{4.25}$$

Since $y \in S$, from (4.25), we see that

$$\{x^2y, x^4y, \ldots, x^{n-5}y, x^{n-3}y\} \subseteq S^{n-2}. \tag{4.26}$$

By multiplying $x^2y \in S$ on the left hand side with all the elements in the subset $\{x^2y, x^4y, \ldots, x^{n-5}y, x^{n-3}y\}$, we have

$$\{(x^2y)x^2y, (x^2y)x^4y, \ldots, (x^2y)x^{n-5}y, (x^2y)x^{n-3}y\} \subseteq S^{n-1}. \tag{4.27}$$

Since $x^{n-2m+2} = x^2y \cdot x^{2m}y \quad$ for $m = 1, 2, \ldots, \frac{n-3}{2}$, from (4.27), we observe that

$$\{1, x^{n-2}, x^{n-4}, \ldots, x^7, x^5\} \subseteq S^{n-1}. \tag{4.28}$$

By combining (4.20) and (4.28), we see that

$$\{1, x^2, x^4, x^5, x^6, \ldots, x^{n-2}, x^{n-1}\} \subseteq S^{n-1}.$$

It is clear that $x = yx^{n-3}(x^2y) \in S \cdot S^{n-3} \cdot S = S^{n-1}$ and $x^3 = yx^{n-3}y \in S \cdot S^{n-3} \cdot S = S^{n-1}$. Thus we have shown that $S^{n-1} = D_{2n}$.

To show that $e(S) = n - 1$, we note that $xy \notin S^{n-2}$. Thus, $S^{n-2} \neq G$. This completes the proof. □

Remark 1. Let $S = \{x, y, x^2y\} \subseteq D_{2n}$ as described in Proposition 4.11. For $b \in D_{2n}$, b can be written as a product of a finite number of elements in S, say $Z(b)$. In the following table, we list down all the possible values of $Z(b)$ for some

$b \in D_{2n}$ where $Z(b) \leq n$. The two different representations of b as a product of elements in S will be given in the same row. By using the relations $yx^i = x^{n-i}y$ and $x^i = yx^{n-i}y$ and $x^{ka} = x^b yx^{(n-k)a} x^b y$, we obtain the following:

Table 4.2: $e(\{x, y, x^2y\}) = n - 1$, $n \geq 5$ is odd

b	$Z(b)$	b	$Z(b)$	Remark
1	$2, 4, 6, \ldots, n-1$			
x	$1, 3, 5, \ldots, n$	$yx^{n-1}y$	$n-1$	$x = yx^{n-1}y$
x^2	$2, 4, 6, \ldots, n-1$	$yx^{n-2}y$	$n-2, n$	$x^2 = yx^{n-2}y$
x^3	$3, 5, \ldots, n$	$yx^{n-3}y$	$n-3, n-1$	$x^3 = yx^{n-3}y$
x^4	$4, 6, \ldots, n-1$	$yx^{n-4}y$	$n-4, n-2, n$	$x^4 = yx^{n-4}y$
$\vdots$		$\vdots$		$\vdots$
x^{n-2}	$n-2, n$	yx^2y	$2, 4, 6, \ldots, n-1$	$x^{n-2} = yx^2y$
x^{n-1}	$n-1$	yxy	$3, 5, 7, \ldots, n$	$x^{n-1} = yxy$
y	$1, 3, 5, \ldots, n$	$x^{n-2}x^2y$	$n-1$	$y = x^{n-2}x^2y$
xy	$2, 4, 6, \ldots, n-1$	yx^{n-1}	n	$xy = yx^{n-1}$
x^2y	$1, 3, 5, \ldots, n$	yx^{n-2}	$n-1$	$x^2y = yx^{n-2}$
x^3y	$2, 4, 6, \ldots, n-1$	yx^{n-3}	$n-2, n$	$x^3y = yx^{n-3}$
x^4y	$3, 5, 7, \ldots, n$	yx^{n-4}	$n-3, n-1$	$x^4y = yx^{n-4}$
$\vdots$		$\vdots$		$\vdots$
$x^{n-2}y$	$n-3, n-1$	yx^2	$3, 5, 7, \ldots, n$	$x^{n-2}y = yx^2$
$x^{n-1}y$	$n-2, n$	yx	$2, 4, 6, \ldots, n-1$	$x^{n-1}y = yx$

In Tables 4.3 and 4.4, we list all the subsets $S \subseteq D_{10}$ and $S \subseteq D_{14}$ where $e(S) = n - 1$. We see that there are two sizes of $S \subseteq D_{10}$ and $S \subseteq D_{14}$ which are $|S| = 3$ and $|S| = 4$.

Table 4.3: Subsets $S \subseteq D_{10}$ where $e(S) = 4$

S				
$\|S\| = 3$			$\|S\| = 4$	
$\{x, y, x^2y\}$	$\{x^2, xy, x^2y\}$	$\{x^3, x^3y, x^4y\}$	$\{x, x^4, y, x^2y\}$	$\{x^2, x^3, y, xy\}$
$\{x, y, x^3y\}$	$\{x^2, x^2y, x^3y\}$	$\{x^4, y, x^2y\}$	$\{x, x^4, y, x^3y\}$	$\{x^2, x^3, y, x^4y\}$
$\{x, xy, x^3y\}$	$\{x^2, x^3y, x^4y\}$	$\{x^4, y, x^3y\}$	$\{x, x^4, xy, x^3y\}$	$\{x^2, x^3, xy, x^2y\}$
$\{x, xy, x^4y\}$	$\{x^3, y, xy\}$	$\{x^4, xy, x^3y\}$	$\{x, x^4, xy, x^4y\}$	$\{x^2, x^3, x^2y, x^3y\}$
$\{x, x^2y, x^4y\}$	$\{x^3, y, x^4y\}$	$\{x^4, xy, x^4y\}$	$\{x, x^4, x^2y, x^4y\}$	$\{x^2, x^3, x^3y, x^4y\}$
$\{x^2, y, xy\}$	$\{x^3, xy, x^2y\}$	$\{x^4, x^2y, x^4y\}$		
$\{x^2, y, x^4y\}$	$\{x^3, x^2y, x^3y\}$			

Table 4.4: Subsets $S \subseteq D_{14}$ where $e(S) = 6$

S				
$\|S\| = 3$			$\|S\| = 4$	
$\{x, y, x^2y\}$	$\{x^3, y, xy\}$	$\{x^5, y, x^3y\}$	$\{x, x^6, y, x^2y\}$	$\{x^3, x^4, y, xy\}$
$\{x, y, x^5y\}$	$\{x^3, y, x^6y\}$	$\{x^5, y, x^4y\}$	$\{x, x^6, y, x^5y\}$	$\{x^3, x^4, y, x^6y\}$
$\{x, xy, x^3y\}$	$\{x^3, xy, x^2y\}$	$\{x^5, xy, x^4y\}$	$\{x, x^6, xy, x^3y\}$	$\{x^3, x^4, xy, x^2y\}$
$\{x, xy, x^6y\}$	$\{x^3, x^2y, x^3y\}$	$\{x^5, xy, x^5y\}$	$\{x, x^6, xy, x^6y\}$	$\{x^3, x^4, x^2y, x^3y\}$
$\{x, x^2y, x^4y\}$	$\{x^3, x^3y, x^4y\}$	$\{x^5, x^2y, x^5y\}$	$\{x, x^6, x^2y, x^4y\}$	$\{x^3, x^4, x^3y, x^4y\}$
$\{x, x^3y, x^5y\}$	$\{x^3, x^4y, x^5y\}$	$\{x^5, x^2y, x^6y\}$	$\{x, x^6, x^3y, x^5y\}$	$\{x^3, x^4, x^4y, x^5y\}$
$\{x, x^4y, x^6y\}$	$\{x^3, x^5y, x^6y\}$	$\{x^5, x^3y, x^6y\}$	$\{x, x^6, x^4y, x^6y\}$	$\{x^3, x^4, x^5y, x^6y\}$
$\{x^2, y, x^3y\}$	$\{x^4, y, xy\}$	$\{x^6, y, x^2y\}$	$\{x^2, x^5, y, x^3y\}$	
$\{x^2, y, x^4y\}$	$\{x^4, y, x^6y\}$	$\{x^6, y, x^5y\}$	$\{x^2, x^5, y, x^4y\}$	
$\{x^2, xy, x^4y\}$	$\{x^4, xy, x^2y\}$	$\{x^6, xy, x^3y\}$	$\{x^2, x^5, xy, x^4y\}$	
$\{x^2, xy, x^5y\}$	$\{x^4, x^2y, x^3y\}$	$\{x^6, xy, x^6y\}$	$\{x^2, x^5, xy, x^5y\}$	
$\{x^2, x^2y, x^5y\}$	$\{x^4, x^3y, x^4y\}$	$\{x^6, x^2y, x^4y\}$	$\{x^2, x^5, x^2y, x^5y\}$	
$\{x^2, x^2y, x^6y\}$	$\{x^4, x^4y, x^5y\}$	$\{x^6, x^3y, x^5y\}$	$\{x^2, x^5, x^2y, x^6y\}$	
$\{x^2, x^3y, x^6y\}$	$\{x^4, x^5y, x^6y\}$	$\{x^6, x^4y, x^6y\}$	$\{x^2, x^5, x^3y, x^6y\}$	

From Table 4.3, there are altogether 30 subsets $S \subseteq D_{10}$ where $e(S) = 4$ and from Table 4.4, we see that there are altogether 63 subsets $S \subseteq D_{14}$ where $e(S) = 6$. When n is even, we conjecture that there does not exist any subset $S \subseteq D_{2n}$ such that $e(S) = n - 1$. This result will be discussed in more detail in Section 4.3.1 by using a numerical example.

Note that when n is even, the subset $S = \{x, y, x^2y\}$ considered in Proposition 4.11 gives us $e(S) = \infty$ as shown below.

Proposition 4.12. *Let $S = \{x, y, x^2y\}$ be a subset in D_{2n} where $n \geq 4$ is even. Then $e(S) = \infty$.*

Proof. If $\frac{n}{2}$ is even, we have

$$
\begin{aligned}
S^m &= \{x, x^3, x^5, \ldots, x^{n-1}, y, x^2y, x^4y, \ldots, x^{n-2}y\},\ m = \frac{n}{2} - 1, \frac{n}{2} + 1, \frac{n}{2} + 3, \ldots \\
S^m &= \{1, x^2, x^4, \ldots, xy, x^3y, x^5y, \ldots, x^{n-1}y\},\ m = \frac{n}{2}, \frac{n}{2} + 2, \frac{n}{2} + 4, \ldots.
\end{aligned}
$$

If $\frac{n}{2}$ is odd, we have

$$\begin{aligned} S^m &= \{1, x^2, x^4, \ldots, xy, x^3y, x^5y, \ldots, x^{n-1}y\},\ m = \frac{n}{2}-1, \frac{n}{2}+1, \frac{n}{2}+3, \ldots \\ S^m &= \{x, x^3, x^5, \ldots, x^{n-1}, y, x^2y, x^4y, \ldots, x^{n-2}y\},\ m = \frac{n}{2}, \frac{n}{2}+2, \frac{n}{2}+4, \ldots. \end{aligned}$$

Hence, $e(S) = \infty$. □

Next, we shall consider the subsets $S \subseteq D_{2n}$ where $e(S) = n$.

Proposition 4.13. *Let n, a and b be integers. Let $S = \{1, x^a y, x^b y\} \subseteq D_{2n}$ where $n \geq 4$ and $1 \leq a < b \leq n-1$. Let $p_1, \ldots, p_{\omega(n)}$ be the distinct prime factor(s) of n. If p_i does not divide $b-a$ for any $i \in \{1, \ldots, \omega(n)\}$, then $e(S) = n$.*

Proof. Let $c \in S^i$ for $i \geq 1$. Then $c \in S^{i+1}$ since $1 \in S$. We first note that $|\{x^{b-a}, x^{2(b-a)}, \ldots, x^{(n-1)(b-a)}, x^{n(b-a)}\}| = n$ since p_i does not divide $b-a$ for any $i \in \{1, \ldots, \omega(n)\}$. Let $q = \lfloor \frac{n}{2} \rfloor$ and let $r = \lceil \frac{n}{2} \rceil$. Then $r = q$ when n is even and $r = q+1$ when n is odd. Note that

$$\begin{aligned} x^{b-a} &= x^b y \cdot x^a y \in S^2 \\ x^{2(b-a)} &= (x^b y \cdot x^a y) \cdot (x^b y \cdot x^a y) \in S^4 \\ x^{3(b-a)} &= (x^b y \cdot x^a y)^3 \in S^6 \\ &\vdots \\ x^{q(b-a)} &= (x^b y \cdot x^a y)^q \in S^{2q} = \begin{cases} S^n, & \text{if } n \text{ is even;} \\ S^{n-1}, & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

and since $yx^k = x^{n-k}y$, we have

$$\begin{aligned}
x^{(n-1)(b-a)} &= x^a y \cdot x^b y \in S^2 \\
x^{(n-2)(b-a)} &= (x^a y \cdot x^b y) \cdot (x^a y \cdot x^b y) \in S^4 \\
x^{(n-3)(b-a)} &= (x^a y \cdot x^b y)^3 \in S^6 \\
&\vdots \\
x^{q(b-a)} &= (x^a y \cdot x^b y)^q \in S^{2q} = \begin{cases} S^n, & \text{if } n \text{ is even;} \\ S^{n-1}, & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Hence, $\{1, x, x^2, x^3, \ldots, x^{n-1}\} \subseteq S^n$ if n is even and

$$\{1, x, x^2, x^3, \ldots, x^{n-1}\} \subseteq S^{n-1} \subseteq S^n$$

if n is odd since $1 \in S$. Next, observe that

$$\begin{aligned}
x^{a+(b-a)}y &= x^b y \in S \\
x^{a+2(b-a)}y &= x^b y \cdot x^a y \cdot x^b y \in S^3 \\
x^{a+3(b-a)}y &= (x^b y \cdot x^a y)^2 \cdot x^b y \in S^5 \\
&\vdots \\
x^{a+r(b-a)}y &= (x^b y \cdot x^a y)^{r-1} \cdot xy \in S^{2r-1} = \begin{cases} S^{n-1}, & \text{if } n \text{ is even;} \\ S^n, & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

and since $yx^k = x^{n-k}y$, we see that

$$\begin{aligned}
x^{a+(n-1)(b-a)}y &= x^a y \cdot x^b y \cdot x^a y \in S^3 \\
x^{a+(n-2)(b-a)}y &= (x^a y \cdot x^b y) \cdot (x^a y \cdot x^b y) \cdot x^a y \in S^5 \\
x^{a+(n-3)(b-a)}y &= (x^a y \cdot x^b y)^3 \cdot x^a y \in S^7 \\
&\vdots \\
x^{a+(q+1)(b-a)}y &= (x^a y \cdot x^b y)^{r-1} \cdot y \in S^{2r-1} = \begin{cases} S^{n-1}, & \text{if } n \text{ is even;} \\ S^n, & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Hence, $\{y, xy, x^2y, x^3y, \ldots, x^{n-1}y\} \subseteq S^{n-1} \subseteq S^n$ if n is even and

$$\{y, xy, x^2y, x^3y, \ldots, x^{n-1}y\} \subseteq S^n$$

if n is odd since $1 \in S$.

Note that if n is even,

$$\{x^{b-a}, x^{2(b-a)}, \ldots, x^{(q-1)(b-a)}; x^{(q+1)(b-a)}, \ldots, x^{(n-1)(b-a)}\} \subseteq S^{n-1}.$$

It is clear that $x^{q(b-a)}$ cannot be written as a product of less than n elements of S. If n is odd, we have

$$\{x^{a+(b-a)}y, x^{a+2(b-a)}y, \ldots, x^{a+q(b-a)}y; x^{a+(q+2)(b-a)}y, \ldots, x^{a+(n-1)(b-a)}y\} \subseteq S^{n-1}.$$

It is clear that $x^{a+(q+1)(b-a)}y$ cannot be written as a product of less than n elements of S. Therefore, $e(S) = n$ for all $n \geq 4$. □

Now we shall look at other possible subsets S where $e(S) = n$. In Tables 4.5 and 4.6, we list all the subsets S in D_{12} and D_{14} where $e(S) = n$.

Table 4.5: Subsets $S \subseteq D_{12}$ where $e(S) = 6$

S		
$\{1, y, xy\}$	$\{1, x^3y, x^4y\}$	$\{x^3, xy, x^2y\}$
$\{1, y, x^5y\}$	$\{1, x^4y, x^5y\}$	$\{x^3, x^2y, x^3y\}$
$\{1, xy, x^2y\}$	$\{x^3, y, xy\}$	$\{x^3, x^3y, x^4y\}$
$\{1, x^2y, x^3y\}$	$\{x^3, y, x^5y\}$	$\{x^3, x^4y, x^5y\}$

Table 4.6: Subsets $S \subseteq D_{14}$ where $e(S) = 7$

S			
$\|S\| = 2$		$\|S\| = 3$	
$\{x, y\}$	$\{x^4, y\}$	$\{1, y, xy\}$	$\{x, x^6, y\}$
$\{x, xy\}$	$\{x^4, xy\}$	$\{1, y, x^2y\}$	$\{x, x^6, xy\}$
$\{x, x^2y\}$	$\{x^4, x^2y\}$	$\{1, y, x^3y\}$	$\{x, x^6, x^2y\}$
$\{x, x^3y\}$	$\{x^4, x^3y\}$	$\{1, y, x^4y\}$	$\{x, x^6, x^3y\}$
$\{x, x^4y\}$	$\{x^4, x^4y\}$	$\{1, y, x^5y\}$	$\{x, x^6, x^4y\}$
$\{x, x^5y\}$	$\{x^4, x^5y\}$	$\{1, y, x^6y\}$	$\{x, x^6, x^5y\}$
$\{x, x^6y\}$	$\{x^4, x^6y\}$	$\{1, xy, x^2y\}$	$\{x, x^6, x^6y\}$
$\{x^2, y\}$	$\{x^5, y\}$	$\{1, xy, x^3y\}$	$\{x^2, x^5, y\}$
$\{x^2, xy\}$	$\{x^5, xy\}$	$\{1, xy, x^4y\}$	$\{x^2, x^5, xy\}$
$\{x^2, x^2y\}$	$\{x^5, x^2y\}$	$\{1, xy, x^5y\}$	$\{x^2, x^5, x^2y\}$
$\{x^2, x^3y\}$	$\{x^5, x^3y\}$	$\{1, xy, x^6y\}$	$\{x^2, x^5, x^3y\}$
$\{x^2, x^4y\}$	$\{x^5, x^4y\}$	$\{1, x^2y, x^3y\}$	$\{x^2, x^5, x^4y\}$
$\{x^2, x^5y\}$	$\{x^5, x^5y\}$	$\{1, x^2y, x^4y\}$	$\{x^2, x^5, x^5y\}$
$\{x^2, x^6y\}$	$\{x^5, x^6y\}$	$\{1, x^2y, x^5y\}$	$\{x^2, x^5, x^6y\}$
$\{x^3, y\}$	$\{x^6, y\}$	$\{1, x^2y, x^6y\}$	$\{x^3, x^4, y\}$
$\{x^3, xy\}$	$\{x^6, xy\}$	$\{1, x^3y, x^4y\}$	$\{x^3, x^4, xy\}$
$\{x^3, x^2y\}$	$\{x^6, x^2y\}$	$\{1, x^3y, x^5y\}$	$\{x^3, x^4, x^2y\}$
$\{x^3, x^3y\}$	$\{x^6, x^3y\}$	$\{1, x^3y, x^6y\}$	$\{x^3, x^4, x^3y\}$
$\{x^3, x^4y\}$	$\{x^6, x^4y\}$	$\{1, x^4y, x^5y\}$	$\{x^3, x^4, x^4y\}$
$\{x^3, x^5y\}$	$\{x^6, x^5y\}$	$\{1, x^4y, x^6y\}$	$\{x^3, x^4, x^5y\}$
$\{x^3, x^6y\}$	$\{x^6, x^6y\}$	$\{1, x^5y, x^6y\}$	$\{x^3, x^4, x^6y\}$

In the following propositions, we show that there exist other subsets $S \subseteq D_{2n}$ such that $e(S) = n$. We begin by considering the smallest subset S where $e(S) = n$ when n is odd.

Proposition 4.14. *Let n, a and b be integers. Let $S = \{x^a, x^by\} \subseteq D_{2n}$ where $n \geq 5$ is odd and $0 \leq a, b \leq n-1$. Let $p_1, \ldots, p_{\omega(n)}$ be the distinct prime factor(s) of n. If p_i does not divide a for any $i \in \{1, \ldots, \omega(n)\}$, then $e(S) = n$.*

Proof. Since p_i does not divide a for any $i \in \{1, \ldots, \omega(n)\}$, then

$$|\{x^a, x^{2a}, \ldots, x^{(n-1)a}, x^{na}\}| = n.$$

Let $c \in S^k$ for some positive integer $k \geq 1$. Then $c \in S^{k+2}$ since $1 = x^b y \cdot x^b y \in S^2$. Now if k is odd, so is $k+2$. Then since n is odd and $x^a \in S$, we have

$$x^{ka} \in S^k \subseteq S^n \quad \text{for } k = 1, 3, 5, \ldots, n.$$

Hence,

$$\{x^a, x^{3a}, x^{5a}, \ldots, x^{(n-2)a}, x^{na}\} \subseteq S^n. \tag{4.29}$$

Next, observe that $x^{(n-k)a} = x^b y \cdot x^{ka} \cdot x^b y \in S \cdot S^k \cdot S = S^{k+2} \subseteq S^n$ for $k = 1, 3, 5, \ldots, n-2$ and hence,

$$\{x^{(n-1)a}, x^{(n-3)a}, \ldots, x^{4a}, x^{2a}\} \subseteq S^n. \tag{4.30}$$

By combining (4.29) and (4.30), we have

$$\{x^a, x^{2a}, x^{3a}, \ldots, x^{na}\} = \{1, x, x^2, \ldots, x^{n-1}\} \subseteq S^n. \tag{4.31}$$

Next, note that if $k+1$ is odd, then k is even. Given that n is odd, we have

$$x^{ka+b} y = x^{ka} \cdot x^b y \in S^k \cdot S = S^{k+1} \subseteq S^n \quad \text{for } k = 2, 4, \ldots, n-1$$

and hence

$$\{x^{2a+b} y, x^{4a+b} y, \ldots, x^{(n-3)a+b} y, x^{(n-1)a+b} y\} \subseteq S^n. \tag{4.32}$$

Finally, from the relation $x^{n-k}y = yx^k$, we have

$$x^{(n-k)a+b}y = x^b y \cdot x^{ka} \in S \cdot S^k = S^{k+1} \subseteq S^n \quad \text{for } k = 0, 2, 4, \ldots, n-1$$

and hence

$$\{x^{na+b}y, x^{(n-2)a+b}y, x^{(n-4)a+b}y, \ldots, x^{3a+b}y, x^{a+b}y\} \subseteq S^n. \tag{4.33}$$

By combining (4.32) and (4.33), we have

$$\{x^{a+b}y, x^{2a+b}y, \ldots, x^{na+b}y\} = \{y, xy, \ldots, x^{n-1}y\} \subseteq S^n. \tag{4.34}$$

Therefore, by (4.31) and (4.34), we have $S^n = D_{2n}$.

To show that $e(S) = n$, we note that $x^a \notin S^{n-1}$. Thus, $S^{n-1} \neq G$. This completes the proof. □

Remark 2. Let $S = \{x^a, x^b y\} \subseteq D_{2n}$ as described in Proposition 4.14. For $d \in D_{2n}$, d can be written as a product of a finite number of elements in S, say $Z(d)$. In the following table, we list down all the possible values of $Z(d)$ for some $d \in D_{2n}$ where $Z(d) \leq n$. The two different representations of d as a product of elements in S will be given in the same row. By using the relations $x^b y x^{ka} = x^{(n-k)a+b}y$ and $x^{ka} = x^b y x^{(n-k)a} x^b y$, we obtain the following:

Table 4.7: $e(\{x^a, x^b y\}) = n$, $n \geq 5$ is odd

d	$Z(d)$	d	$Z(d)$	Remark
1	$2, 4, 6, \ldots, n-1$	x^{na}	n	$1 = x^{na}$
x^a	$1, 3, 5, \ldots, n$			
x^{2a}	$2, 4, 6, \ldots, n-1$	$x^b y x^{(n-2)a} x^b y$	n	$x^{2a} = x^b y x^{(n-2)a} x^b y$
x^{3a}	$3, 5, \ldots, n$	$x^b y x^{(n-3)a} x^b y$	$n-1$	$x^{3a} = x^b y x^{(n-3)a} x^b y$
x^{4a}	$4, 6, \ldots, n-1$	$x^b y x^{(n-4)a} x^b y$	$n-2, n$	$x^{4a} = x^b y x^{(n-4)a} x^b y$
$\vdots$		$\vdots$		$\vdots$
$x^{(n-2)a}$	$n-2, n$	$x^b y x^{2a} x^b y$	$4, 6, \ldots, n-1$	$x^{(n-2)a} = x^b y x^{2a} x^b y$

Table 4.7: (Continued)

d	$Z(d)$	d	$Z(d)$	Remark
$x^{(n-1)a}$	$n-1$	$x^byx^ax^by$	$3,5,7,\ldots,n$	$x^{(n-1)a}=x^byx^ax^by$
x^by	$1,3,5,\ldots,n$			$x^{na+b}y=x^by$
$x^{a+b}y$	$2,4,6,\ldots,n-1$	$x^byx^{(n-1)a}$	n	$x^{a+b}y=x^byx^{(n-1)a}$
$x^{2a+b}y$	$3,5,7,\ldots,n$	$x^byx^{(n-2)a}$	$n-1$	$x^{2a+b}y=x^byx^{(n-2)a}$
$x^{3a+b}y$	$4,6,\ldots,n-1$	$x^byx^{(n-3)a}$	$n-2,n$	$x^{3a+b}y=x^byx^{(n-3)a}$
$x^{4a+b}y$	$5,7,\ldots,n$	$x^byx^{(n-4)a}$	$n-3,n-1$	$x^{4a+b}y=x^byx^{(n-4)a}$
$\vdots$		$\vdots$		$\vdots$
$x^{(n-2)a+b}y$	$n-1$	x^byx^{2a}	$3,5,7,\ldots,n$	$x^{(n-2)a+b}y=x^byx^{2a}$
$x^{(n-1)a+b}y$	n	x^byx^a	$2,4,6,\ldots,n-1$	$x^{(n-1)a+b}y=x^byx^a$

Proposition 4.15. *Let n, a, b, c be integers and let $S=\{x^a,x^b,x^cy\}\subseteq D_{2n}$, where $n\geq 5$ is odd, $1\leq a<b\leq n-1$, $0\leq c\leq n-1$ and $a+b=n$. Let $p_1,\ldots,p_{\omega(n)}$ be the distinct prime factor(s) of n. If p_i does not divide a for any $i\in\{1,\ldots,\omega(n)\}$, then $e(S)=n$.*

Proof. Let $e\in S^t$ for any positive integer $t\geq 1$. Then $e\in S^{t+2}$ since $x^cy\cdot x^cy=1\in S^2$. We first note that since p_i does not divide a for any $i\in\{1,\ldots,\omega(n)\}$, then $|\{x^a,x^{2a},\ldots,x^{na}\}|=n$. Given that $a+b=n$, we have

$$x^{ka}=x^{(n-k)b}\quad\text{for } k=1,2,\ldots,n-1.$$

Since x^a, $x^b\in S$, we have

$$x^{ka}\in S^k\subseteq S^n\quad\text{for } k=1,3,5,\ldots,n \tag{4.35}$$

and

$$x^{(n-k)a}=x^{kb}\in S^k\subseteq S^n\quad\text{for } k=1,3,5,\ldots,n-2. \tag{4.36}$$

Combining (4.35) and (4.36), we have

$$\{x^a,x^{2a},\ldots,x^{na}\}=\{1,x,x^2,\ldots,x^{n-1}\}\subseteq S^n.$$

Next, note that

$$x^{ka+c}y = x^{ka} \cdot x^{c}y \in S^{k} \cdot S = S^{k+1} \subseteq S^{n} \quad \text{for } k = 2, 4, \dots, n-1 \quad (4.37)$$

and

$$x^{(n-k)a+c}y = x^{kb} \cdot x^{c}y \in S^{k} \cdot S = S^{k+1} \subseteq S^{n} \quad \text{for } k = 0, 2, 4, \dots, n-1. \quad (4.38)$$

Combining (4.37) and (4.38), we have

$$\{x^{a+c}y, x^{2a+c}y, \dots, x^{(n-1)a+c}y, x^{na+c}y\} = \{y, xy, x^{2}y, \dots, x^{n-1}y\} \subseteq S^{n}.$$

Thus we have shown that $S^{n} = D_{2n}$.

To show that $e(S) = n$, we note that $x^{c}y \notin S^{n-1}$. Thus, $S^{n-1} \neq G$. This completes the proof. □

Remark 3. Let $S = \{x^{a}, x^{b}, x^{c}y\} \subseteq D_{2n}$ as described in Proposition 4.15. For $d \in D_{2n}$, d can be written as a product of a finite number of elements in S, say $Z(d)$. In the following table, we list down all the possible values of $Z(d)$ for some $d \in D_{2n}$ where $Z(d) \leq n$. The two different representations of d as a product of elements in S will be given in the same row. By using the relations $x^{ka} = x^{(n-k)b}$ and $x^{ka+c}y = x^{(n-k)b+c}y$, we obtain the following:

Table 4.8: $e(\{x^{a}, x^{b}, x^{c}y\}) = n$, $n \geq 5$ is odd

d	$Z(d)$	d	$Z(d)$	Remark
1	$2, 4, 6, \dots, n-1$	x^{na}	n	$1 = x^{na}$
x^{a}	$1, 3, 5, \dots, n$	$x^{(n-1)b}$	$n-1$	$x^{a} = x^{(n-1)b}$
x^{2a}	$2, 4, 6, \dots, n-1$	$x^{(n-2)b}$	$n-2, n$	$x^{2a} = x^{(n-2)b}$
x^{3a}	$3, 5, 7, \dots, n$	$x^{(n-3)b}$	$n-3, n-1$	$x^{3a} = x^{(n-3)b}$
x^{4a}	$4, 6, \dots, n-1$	$x^{(n-4)b}$	$n-4, n-2, n$	$x^{4a} = x^{(n-4)b}$
$\vdots$		$\vdots$		$\vdots$
$x^{(n-2)a}$	$n-2, n$	x^{2b}	$2, 4, 6, \dots, n-1$	$x^{(n-2)a} = x^{2b}$

Table 4.8: (Continued)

d	$Z(d)$	d	$Z(d)$	Remark
$x^c y$	$1,3,5,\ldots,n$			
$x^{a+c}y$	$2,4,6,\ldots,n-1$	$x^{(n-1)b+c}y$	n	$x^{a+c}y = x^{(n-1)b+c}y$
$x^{2a+c}y$	$3,5,7,\ldots,n$	$x^{(n-2)b+c}y$	$n-1$	$x^{2a+c}y = x^{(n-2)b+c}y$
$x^{3a+c}y$	$4,6,\ldots,n-1$	$x^{(n-3)b+c}y$	$n-2,n$	$x^{3a+c}y = x^{(n-3)b+c}y$
$\vdots$		$\vdots$		$\vdots$
$x^{(n-2)a+c}y$	$n-1$	$x^{2b+c}y$	$3,5,\ldots,n$	$x^{(n-2)a+c}y = x^{2b+c}y$
$x^{(n-1)a+c}y$	n	$x^{b+c}y$	$2,4,6,\ldots,n-1$	$x^{(n-1)a+c}y = x^{b+c}y$

By using the results obtained so far, we give a summary table of the subsets S where $e(S) = n$. Let a, b and c be integers. Let p_i be the distinct prime factor(s) of n for $i \in \{1, \ldots, \omega(n)\}$.

Table 4.9: Summary of subsets $S \subseteq D_{2n}$ where $e(S) = n$

S		Conditions
n is odd	n is even	
$\{x^a, x^b y\}$	-	$0 \leq a, b \leq n-1$ p_i does not divide a for any $i \in \{1, \ldots, \omega(n)\}$
$\{1, x^a y, x^b y\}$	$\{1, x^a y, x^b y\}$	$0 \leq a < b \leq n-1$ p_i does not divide a for any $i \in \{1, \ldots, \omega(n)\}$
$\{x^a, x^b, x^c y\}$	-	$1 \leq a < b \leq n-1, 0 \leq c \leq n-1, a+b=n$ p_i does not divide a for any $i \in \{1, \ldots, \omega(n)\}$

Finally, we end this section by giving examples of subsets $S \subseteq D_{2n}$ with finite exhaustion numbers in Table 4.10.

Table 4.10: Subsets $S \subseteq D_{2n}$ with certain finite exhaustion numbers

$e(S)$	S	
	n is even	n is odd
2	$\{1, x, y, xy, x^2y, \ldots, x^{n-3}y\}$	$\{1, x, y, xy, x^2y, \ldots, x^{n-3}y\}$
3	$\{1, x, y, xy, x^2y, \ldots, x^{n-5}y\}$	$\{1, x, y, xy, x^2y, \ldots, x^{n-5}y\}$
$\frac{n}{2}$	$\{1, x, y, xy\}$	-
$\frac{n+1}{2}$	-	$\{1, x, xy\}$
$\frac{n+2}{2}$	$\{1, x, xy\}$	-
$n-1$	-	$\{x, y, x^2y\}$
n	$\{1, y, xy\}$	$\{1, y, xy\}$

4.3 Non-exhaustive Subsets

In this section, we determine the biggest subsets $S \subseteq D_{2n}$ where $e(S) = \infty$. For convenience, we shall further divide this section into two subsections to discuss the cases where n is even and n is odd separately. Finally, we show that there does not exist any subset S in D_{12} and D_{14} such that $e(S) = 5$. We also show that there does not exist any subset S in D_{22} such that $e(S) = k$ for $k = 7, 8, 9$.

We begin with the following proposition:

Proposition 4.16. *Let S, S' be subsets in D_{2n}. If $S \subseteq S'$ and $e(S') = \infty$, then $e(S) = \infty$.*

Proof. If $e(S) < \infty$, then since $S \subseteq S'$, it follows by Proposition 4.5 that $e(S') \leq e(S)$ which contradicts the fact that $e(S') = \infty$. $\square$

In the next two subsections, we let S, S', S'', S''', S'''' $\subseteq D_{2n}$ where $S' \subsetneq S'' \subsetneq S''' \subsetneq S'''' \subsetneq S \subseteq D_{2n}$. Note that $S'' = S' \cup \{x^i y^j\}$, $S''' = S'' \cup \{x^i y^j\}$ and $S'''' = S''' \cup \{x^i y^j\}$ for some $i \in \{0, 1, \ldots, n-1\}$ and $j \in \{0, 1\}$. Since we begin by considering $S' \subsetneq S$ where $|S'| = 3$, we have $|S''| = 4$, $|S'''| = 5$ and $|S''''| = 6$. Firstly, we shall look at the case where n is even.

4.3.1 Non-exhaustive Subsets When n Is Even

In Section 4.2, we have obtained some general constructions for certain exhaustive subsets in D_{2n}. In Tables 4.11 and 4.12, we list down some exhaustive subsets $S \subseteq D_{2n}$ for the two smallest even integers n where $n \geq 4$.

Table 4.11: Subsets $S \subseteq D_8$ where $e(S) \in \{2, 3, 4\}$

$e(S)$	S
2	$\{1, x, y, xy\}$ (Proposition 4.4)
3	$\{1, x, xy\}$ (Proposition 4.2)
4	$\{1, y, xy\}$ (Proposition 4.13)

Table 4.12: Subsets $S \subseteq D_{12}$ where $e(S) \in \{2, 3, 4, 6\}$

$e(S)$	S
2	$\{1, x, y, xy, x^2y, x^3y\}$ (Proposition 4.7(i))
3	$\{1, x, y, xy\}$ (Proposition 4.4)
4	$\{1, x, xy\}$ (Proposition 4.2)
5	Does not exist.
6	$\{1, y, xy\}$ (Proposition 4.13)

From the two tables above, we see that there exist subsets $S \subseteq D_8$ such that $e(S) \in \{2, 3, 4\}$ but there does not exist any subset $S \subseteq D_{12}$ such that $e(S) = 5$. Therefore our main objective in Section 4.3.1 is to show that there does not exist any subset $S \subseteq D_{12}$ such that $e(S) = 5$. We conjecture that there does not exist any subset $S \subseteq D_{2n}$ such that $\frac{n+4}{2} \leq e(S) \leq n - 1$.

We first begin by classifying the subsets where $e(S) = \infty$. In order to show that there exist non-exhaustive subsets in D_{2n}, we shall investigate all the possible subsets of D_{2n}.

Proposition 4.17. *Let n be even where $n \geq 4$. Let $p_1, \ldots, p_{\omega(n)}$ be the distinct prime factor(s) of n. For $k = 1, 2, 3, 4, 5$, let $S_k \subseteq D_{2n}$ as follows:*

(a) $S_1 = \{1, x, x^2, \ldots, x^{n-1}\}$,

(b) $S_2 = \{y, xy, x^2y, \ldots, x^{n-1}y\}$,

(c) $S_3 = \{x, x^3, x^5, \ldots, x^{n-1}, y, x^2y, x^4y, \ldots, x^{n-2}y\}$,

(d) $S_4 = \{x, x^3, x^5, \ldots, x^{n-1}, xy, x^3y, x^5y, \ldots, x^{n-1}y\}$,

(e) $S_5 = S_{i_j} = \{1, x^{p_i}, x^{2p_i}, \ldots, x^{(n-p_i)}, x^j y, x^{j+p_i} y, x^{j+2p_i} y, \ldots, x^{j+(n-p_i)} y\}$, *for* $j = 0, 1, \ldots, p_i - 1$ *where* $i \in \{1, \ldots, \omega(n)\}$.

Then $e(S_k) = \infty$ *for* $k = 1, 2, 3, 4, 5$.

Proof.

(a) Since S_1 is a proper subgroup of D_{2n}, it follows that $e(S_1) = \infty$.

(b) Note that $S_2^{2m} = \{1, x, x^2, \ldots, x^{n-1}\}$ is a proper subgroup of D_{2n} for all positive integers m. Hence, $e(S_2) = \infty$.

(c) It is clear that $S_3^{2m} = \{1, x^2, x^4, \ldots, x^{n-2}, xy, x^3y, \ldots, x^{n-1}y\}$ is the subgroup of D_{2n} generated by x^2 and xy for all positive integer m. Hence, $e(S_3) = \infty$.

(d) Note that $S_4^{2m} = \{1, x^2, x^4, \ldots, x^{n-2}, y, x^2y, \ldots, x^{n-2}y\}$ is the subgroup of D_{2n} generated by x^2 and y for all positive integer m. Hence, $e(S_4) = \infty$.

(e) It is clear that $\{1, x^{p_i}, x^{2p_i}, \ldots, x^{(n-p_i)}\} \subseteq S_5^k$ for all $k \geq 2$ but

$$\{1, x, x^2, \ldots, x^{n-1}\} \setminus \{1, x^{p_i}, x^{2p_i}, \ldots, x^{(n-p_i)}\} \nsubseteq S_5^k$$

for all $k \geq 2$. Thus, $e(S_5) = \infty$.

□

Next, we show that $e(S) = \infty$ when $|S| = 2$.

Proposition 4.18. *Let* S *be a 2-subset of* D_{2n} *where* $n \geq 6$ *is even. Then* $e(S) = \infty$.

Proof. For any 2-subset S of D_{2n} ($n \geq 6$), note that there is a positive integer k such that S^k is contained in one of the following subsets of D_{2n}:

(a) $S_1 = \{1, x, x^2, \ldots, x^{n-1}\}$,

(b) $S_2 = \{y, xy, x^2y, \ldots, x^{n-1}y\}$,

(c) $S_3 = \{x, x^3, x^5, \ldots, x^{n-1}, y, x^2y, x^4y, \ldots, x^{n-2}y\}$,

(d) $S_4 = \{x, x^3, x^5, \ldots, x^{n-1}, xy, x^3y, x^5y, \ldots, x^{n-1}y\}$,

(e) $S_5 = S_{i_j} = \{1, x^{p_i}, x^{2p_i}, \ldots, x^{(n-p_i)}, x^jy, x^{j+p_i}y, x^{j+2p_i}y, \ldots, x^{j+(n-p_i)}y\}$ for $j = 0, 1, \ldots, p_i - 1$, $i \in \{1, \ldots, \omega(n)\}$.

It is clear that there are three types of 2-subsets in D_{2n}:

(i) If $S = \{x^i, x^j\}$ for $i, j \in \{0, 1, \ldots, n-1\}$ where $i \neq j$, then $S \subseteq S_1$.

(ii) If $S = \{x^i, x^jy\}$ for $i, j \in \{0, 1, \ldots, n-1\}$, then $S \subseteq S_3$, $S \subseteq S_4$ or $S \subseteq S_5$.

(iii) If $S = \{x^iy, x^jy\}$ for $i, j \in \{0, 1, \ldots, n-1\}$ where $i \neq j$, then $S \subseteq S_2$.

By Proposition 4.17, $e(S_i) = \infty$ for $i = 1, \ldots, 5$. Then by Proposition 4.16, it follows that $e(S) = \infty$. □

Since the subsets $S \subseteq D_{2n}$ give us $e(S) = \infty$ when $|S| = 2$, we see that there are no exhaustive 2-subsets in D_{2n} when n is even. Hence, there does not exist any 2-subset $S \subseteq D_{12}$ such that $e(S) = 5$ and so we move on to the subsets S where $|S| = 3$. In the next two propositions, we will consider two types of 3-subsets that will help us to verify that there does not exist any subset $S \subseteq D_{12}$ such that $e(S) = 5$:

(i) $S = \{1, x^a, x^by\}$ where $1 \leq a \leq n-1$ and $0 \leq b \leq n-1$

(ii) $S = \{1, x^ay, x^by\}$ where $1 \leq a < b \leq n-1$

Proposition 4.19. *Let n, a, b be integers and let $S = \{1, x^a, x^by\} \subseteq D_{2n}$ where $n \geq 4$ is even, $1 \leq a \leq n-1$ and $0 \leq b \leq n-1$. Let p_i be the distinct prime factor(s) of n for $i = 1, 2, \ldots, \omega(n)$.*

(i) If $a \neq mp_i$ for any positive integer m, then $e(S) \leq \frac{n}{2} + 1$.

(ii) If $a = mp_i$ for some positive integer m, then $e(S) = \infty$.

Proof.

(i) Let $c \in S^k$ for $k \geq 1$. Then $c \in S^{k+1}$ since $1 \in S$. We first note that if $a \neq mp_i$ for any positive integer m, then $|\{x^a, x^{2a}, \ldots, x^{(n-1)a}, x^{na}\}| = n$ and hence $|\{x^{b+a}y, x^{b+2a}y, \ldots, x^{b+(n-1)a}, x^{b+na}\}| = n$. Since $1, x^a \in S$, we have

$$x^{ka} \in S^{\frac{n}{2}+1} \quad \text{for } k = 1, 2, \ldots, \frac{n}{2} + 1$$

and hence

$$\{1, x^a, x^{2a}, \ldots, x^{(\frac{n}{2}+1)a}\} \subseteq S^{\frac{n}{2}+1}. \tag{4.39}$$

Since $\{1, x^a, x^{2a}, \ldots, x^{\frac{n}{2}a}\} \subseteq S^{\frac{n}{2}}$ and $x^b y \in S$, we have

$$\{x^b y, x^{b+a}y, x^{b+2a}y, \ldots, x^{b+\frac{n}{2}a}y\} \subseteq S^{\frac{n}{2}+1}. \tag{4.40}$$

Next, we have

$$x^{b+(n-k)a}y = x^{b+(n-ka)}y = x^b y \cdot x^{ka} \in S \cdot S^k \subseteq S^{k+1} \quad \text{for } k = 1, 2, \ldots, \frac{n}{2}$$

and hence

$$\{x^{b+(n-1)a}y, x^{b+(n-2)a}y, x^{b+(n-3)a}y, \ldots, x^{b+\frac{n}{2}a}y\} \subseteq S^{\frac{n}{2}+1}. \tag{4.41}$$

Combining (4.40) and (4.41), we see that

$$\{x^b y, x^{b+a}y, x^{b+2a}y, \ldots, x^{b+(n-1)a}y\} \subseteq S^{\frac{n}{2}+1}.$$

Finally, note that

$$x^{(n-k)a} = x^{n-ka} = x^b y \cdot x^{b+ka}y \in S \cdot S^k \subseteq S^{k+1} \quad \text{for } k = 1, 2, \ldots, \frac{n}{2} - 1$$

and hence

$$\{x^{(n-1)a}, x^{(n-2)a}, x^{(n-3)a}, \dots, x^{(\frac{n}{2}+1)a}\} \subseteq S^{\frac{n}{2}} \subseteq S^{\frac{n}{2}+1}. \tag{4.42}$$

Combining (4.39) and (4.42), we have

$$\{1, x^a, x^{2a}, \dots, x^{(n-1)a}\} \subseteq S^{\frac{n}{2}+1}.$$

Therefore, $S^{\frac{n}{2}+1} = D_{2n}$ and $e(S) \leq \frac{n}{2} + 1$.

(ii) If $a = mp_i$ for some positive integer m, then

$$\{1, x^a, x^b y\} \subseteq \{1, x^{p_i}, x^{2p_i}, \dots, x^{(n-p_i)}, x^j y, x^{j+p_i} y, x^{j+2p_i} y, \dots, x^{j+(n-p_i)} y\}$$

for some $j = 0, 1, \dots, p_i - 1$. By Propositions 4.17 and 4.16, we have $e(S) = \infty$.

□

Proposition 4.20. *Let* $S = \{1, x^a y, x^b y\} \subseteq D_{2n}$ *where* $n \geq 4$ *is even and* $1 \leq a < b \leq n-1$. *Let* p_i *be the distinct prime factor(s) of* n *for* $i = 1, 2, \dots, \omega(n)$.

(i) If $b - a \neq kp_i$ *for any positive integer* k, *then* $e(S) = n$.

(ii) If $b - a = kp_i$ *for some positive integer* k, *then* $e(S) = \infty$.

Proof.

(i) By Proposition 4.13, the result is obvious.

(ii) If $b - a = kp_i$ for some integer k, then

$$\{1, x^a y, x^b y\} \subseteq \{1, x^{p_i}, x^{2p_i}, \dots, x^{(n-p_i)}, x^j y, x^{j+p_i} y, x^{j+2p_i} y, \dots, x^{j(n-p_i)} y\}$$

for some $j = 0, 1, \ldots, p_i - 1$. By Propositions 4.17 and 4.16, we have $e(S) = \infty$.

□

In the following theorem, we are ready to show that there does not exist any subset $S \subseteq D_{12}$ such that $e(S) = 5$. Recall that we will be using the following notations: Let S, S', S'', S''', S'''' $\subseteq D_{2n}$ where $S' \subsetneq S'' \subsetneq S''' \subsetneq S'''' \subsetneq S \subseteq D_{2n}$. Note that $|S'| = 3$, $|S''| = 4$, $|S'''| = 5$ and $|S''''| = 6$.

Theorem 4.21. *Let D_{12} be the dihedral group of order 12. Then there does not exist any subset $S \subseteq D_{12}$ such that $e(S) = 5$.*

Proof. Given that D_{12} is the dihedral group of order 12, we have $n = 6$. Let p_i be the distinct prime factors of 6 for $i = 1, 2$. Then $p_1 = 2$ and $p_2 = 3$. Suppose that there exists a subset $S \subseteq D_{12}$ where $e(S) = 5$. By Propositions 4.6 and 4.18, it is clear that $3 \leq |S| \leq 6$. Firstly, suppose that $1 \notin S$. From the assumption that $e(S) = 5$, we have $x^u y \in S^5$ for all $u = 0, 1, 2, \ldots, 5$ and hence there exists $x^v y \in S$ for some $v = 0, 1, 2, \ldots, 5$. We consider three cases as follows:

(a) $\{x^a, x^b, x^c y\} \subseteq S$ for $1 \leq a < b \leq 5$ and $0 \leq c \leq 5$

(b) $\{x^a, x^b y, x^c y\} \subseteq S$ for $1 \leq a \leq 5$ and $0 \leq b < c \leq 5$

(c) $\{x^a y, x^b y, x^c y\} \subseteq S$ for $0 \leq a < b < c \leq 5$

where a, b, c are integers. In the following, we shall explain the three cases separately in detail.

(a) Let $S' = \{x^a, x^b, x^c y\} \subseteq S$ for $1 \leq a < b \leq 5$ and $0 \leq c \leq 5$. We first list down the subsets S' where $e(S') \leq 4$.

Table 4.13: $\{x^a, x^b, x^c y\} \subseteq S' \subseteq D_{12}$: $e(S') \leq 4$

S'	$e(S')$	S'	$e(S')$	S'	$e(S')$
$\{x, x^2, y\}$	3	$\{x^2, x^3, y\}$	4	$\{x^3, x^4, y\}$	4
$\{x, x^2, xy\}$	3	$\{x^2, x^3, xy\}$	4	$\{x^3, x^4, xy\}$	4
$\{x, x^2, x^2y\}$	3	$\{x^2, x^3, x^2y\}$	4	$\{x^3, x^4, x^2y\}$	4
$\{x, x^2, x^3y\}$	3	$\{x^2, x^3, x^3y\}$	4	$\{x^3, x^4, x^3y\}$	4
$\{x, x^2, x^4y\}$	3	$\{x^2, x^3, x^4y\}$	4	$\{x^3, x^4, x^4y\}$	4
$\{x, x^2, x^5y\}$	3	$\{x^2, x^3, x^5y\}$	4	$\{x^3, x^4, x^5y\}$	4
$\{x, x^4, y\}$	3	$\{x^2, x^5, y\}$	3	$\{x^4, x^5, y\}$	3
$\{x, x^4, xy\}$	3	$\{x^2, x^5, xy\}$	3	$\{x^4, x^5, xy\}$	3
$\{x, x^4, x^2y\}$	3	$\{x^2, x^5, x^2y\}$	3	$\{x^4, x^5, x^2y\}$	3
$\{x, x^4, x^3y\}$	3	$\{x^2, x^5, x^3y\}$	3	$\{x^4, x^5, x^3y\}$	3
$\{x, x^4, x^4y\}$	3	$\{x^2, x^5, x^4y\}$	3	$\{x^4, x^5, x^4y\}$	3
$\{x, x^4, x^5y\}$	3	$\{x^2, x^5, x^5y\}$	3	$\{x^4, x^5, x^5y\}$	3

By Proposition 4.5, we see that if $S' \subseteq S$, then $e(S) \leq e(S') \leq 4$. Next, we list down the subsets S' where $e(S') > 4$ or $e(S') = \infty$.

Table 4.14: $\{x^a, x^b, x^c y\} \subseteq S' \subseteq D_{12}$: $e(S') = \infty$

S'	$e(S')$	S'	$e(S')$	S'	$e(S')$
$\{x, x^3, y\}$	∞	$\{x, x^5, x^2y\}$	∞	$\{x^2, x^4, x^4y\}$	∞
$\{x, x^3, xy\}$	∞	$\{x, x^5, x^3y\}$	∞	$\{x^2, x^4, x^5y\}$	∞
$\{x, x^3, x^2y\}$	∞	$\{x, x^5, x^4y\}$	∞	$\{x^3, x^5, y\}$	∞
$\{x, x^3, x^3y\}$	∞	$\{x, x^5, x^5y\}$	∞	$\{x^3, x^5, xy\}$	∞
$\{x, x^3, x^4y\}$	∞	$\{x^2, x^4, y\}$	∞	$\{x^3, x^5, x^2y\}$	∞
$\{x, x^3, x^5y\}$	∞	$\{x^2, x^4, xy\}$	∞	$\{x^3, x^5, x^3y\}$	∞
$\{x, x^5, y\}$	∞	$\{x^2, x^4, x^2y\}$	∞	$\{x^3, x^5, x^4y\}$	∞
$\{x, x^5, xy\}$	∞	$\{x^2, x^4, x^3y\}$	∞	$\{x^3, x^5, x^5y\}$	∞

In Tables 4.15 and 4.16, we will list down the subsets S'' where $|S''| = 4$, $S'' = S' \cup \{x^i y^j\}$ where $e(S') = \infty$ for $i \in \{0, 1, \ldots, 5\}$ and $j \in \{0, 1\}$. Note that by Proposition 4.5, if $S' \subseteq S''$, then $e(S'') \leq e(S')$. Therefore, in the following we only list down the subsets S'' that do not contain any of the subsets S' listed in Table 4.13. We first list down the subsets S'' where $e(S'') \leq 4$.

Table 4.15: $\{x^a, x^b, x^c y\} \subseteq S'' \subseteq D_{12}$: $e(S'') \leq 4$

S''	$e(S'')$	S''	$e(S'')$	S''	$e(S'')$
$\{x, x^3, y, xy\}$	3	$\{x, x^5, xy, x^2y\}$	3	$\{x^2, x^4, x^2y, x^5y\}$	4
$\{x, x^3, y, x^3y\}$	3	$\{x, x^5, xy, x^4y\}$	4	$\{x^2, x^4, x^3y, x^4y\}$	3
$\{x, x^3, y, x^5y\}$	3	$\{x, x^5, x^2y, x^3y\}$	3	$\{x^2, x^4, x^4y, x^5y\}$	3
$\{x, x^3, xy, x^2y\}$	3	$\{x, x^5, x^2y, x^5y\}$	4	$\{x^3, x^5, y, xy\}$	3

Table 4.15: (Continued)

S''	$e(S'')$	S''	$e(S'')$	S''	$e(S'')$
$\{x, x^3, xy, x^4y\}$	3	$\{x, x^5, x^3y, x^4y\}$	3	$\{x^3, x^5, y, x^3y\}$	3
$\{x, x^3, x^2y, x^3y\}$	3	$\{x, x^5, x^4y, x^5y\}$	3	$\{x^3, x^5, y, x^5y\}$	3
$\{x, x^3, x^2y, x^5y\}$	3	$\{x^2, x^4, y, xy\}$	3	$\{x^3, x^5, xy, x^2y\}$	3
$\{x, x^3, x^3y, x^4y\}$	3	$\{x^2, x^4, y, x^3y\}$	4	$\{x^3, x^5, xy, x^4y\}$	3
$\{x, x^3, x^4y, x^5y\}$	3	$\{x^2, x^4, y, x^5y\}$	3	$\{x^3, x^5, x^2y, x^3y\}$	3
$\{x, x^5, y, xy\}$	3	$\{x^2, x^4, xy, x^2y\}$	3	$\{x^3, x^5, x^2y, x^5y\}$	3
$\{x, x^5, y, x^3y\}$	4	$\{x^2, x^4, xy, x^4y\}$	4	$\{x^3, x^5, x^3y, x^4y\}$	3
$\{x, x^5, y, x^5y\}$	3	$\{x^2, x^4, x^2y, x^3y\}$	3	$\{x^3, x^5, x^4y, x^5y\}$	3

Next, we list down the subsets S'' where $e(S'') > 4$ or $e(S'') = \infty$.

Table 4.16: $\{x^a, x^b, x^cy\} \subseteq S'' \subseteq D_{12}$: $e(S'') = \infty$

S''	$e(S'')$	S''	$e(S'')$	S''	$e(S'')$
$\{x, x^3, y, x^5\}$	∞	$\{x, x^3, x^4y, x^5\}$	∞	$\{x^2, x^4, xy, x^3y\}$	∞
$\{x, x^3, y, x^2y\}$	∞	$\{x, x^3, x^5y, x^5\}$	∞	$\{x^2, x^4, xy, x^5y\}$	∞
$\{x, x^3, y, x^4y\}$	∞	$\{x, x^5, y, x^2y\}$	∞	$\{x^2, x^4, x^2y, x^4y\}$	∞
$\{x, x^3, xy, x^5\}$	∞	$\{x, x^5, y, x^4y\}$	∞	$\{x^2, x^4, x^3y, x^5y\}$	∞
$\{x, x^3, xy, x^3y\}$	∞	$\{x, x^5, xy, x^3y\}$	∞	$\{x^3, x^5, y, x^2y\}$	∞
$\{x, x^3, xy, x^5y\}$	∞	$\{x, x^5, xy, x^5y\}$	∞	$\{x^3, x^5, y, x^4y\}$	∞
$\{x, x^3, x^2y, x^5\}$	∞	$\{x, x^5, x^2y, x^4y\}$	∞	$\{x^3, x^5, xy, x^3y\}$	∞
$\{x, x^3, x^2y, x^4y\}$	∞	$\{x, x^5, x^3y, x^5y\}$	∞	$\{x^3, x^5, xy, x^5y\}$	∞
$\{x, x^3, x^3y, x^5\}$	∞	$\{x^2, x^4, y, x^2y\}$	∞	$\{x^3, x^5, x^2y, x^4y\}$	∞
$\{x, x^3, x^3y, x^5y\}$	∞	$\{x^2, x^4, y, x^4y\}$	∞	$\{x^3, x^5, x^3y, x^5y\}$	∞

Since $e(S'') = \infty$ for the subsets S'' in Table 4.16, we consider the case where $|S'''| = 5$ and $S''' = S'' \cup \{x^iy^j\}$ for $i \in \{0, 1, \ldots, 5\}$ and $j \in \{0, 1\}$. Note that in the following we only list down the subsets S''' that do not contain any of the subsets listed in Tables 4.13 and 4.15.

Table 4.17: $\{x^a, x^b, x^cy\} \subseteq S''' \subseteq D_{12}$: $e(S''') = \infty$

S'''	$e(S''')$	S'''	$e(S''')$	S'''	$e(S''')$
$\{x, x^3, y, x^5, x^2y\}$	∞	$\{x, x^3, xy, x^3y, x^5y\}$	∞	$\{x^2, x^4, y, x^2y, x^4y\}$	∞
$\{x, x^3, y, x^5, x^4y\}$	∞	$\{x, x^3, x^2y, x^5, x^4y\}$	∞	$\{x^2, x^4, xy, x^3y, x^5y\}$	∞
$\{x, x^3, y, x^2y, x^4y\}$	∞	$\{x, x^3, x^3y, x^5, x^5y\}$	∞	$\{x^3, x^5, y, x^2y, x^4y\}$	∞
$\{x, x^3, xy, x^5, x^3y\}$	∞	$\{x, x^5, y, x^2y, x^4y\}$	∞	$\{x^3, x^5, xy, x^3y, x^5y\}$	∞
$\{x, x^3, xy, x^5, x^5y\}$	∞	$\{x, x^5, xy, x^3y, x^5y\}$	∞		

Since $e(S''') = \infty$ for the subsets S''' in Table 4.17, we consider the case where $|S''''| = 6$, $S'''' = S''' \cup \{x^iy^j\}$ for $i \in \{0, 1, \ldots, 5\}$ and $j \in \{0, 1\}$.

Note that in the following we only list down the subsets S'''' that do not contain any of the subsets listed in Tables 4.13 and 4.15.

Table 4.18: $\{x^a, x^b, x^c y\} \subseteq S'''' \subseteq D_{12}$: $e(S'''') = \infty$

S''''	$e(S'''')$
$\{x, x^3, x^5, y, x^2y, x^4y\}$	∞
$\{x, x^3, x^5, xy, x^3y, x^5y\}$	∞

By Proposition 4.6, if $|S| > 6$, then $e(S) = 2$, which is a contradiction. Hence, there does not exist any subset $S \subseteq D_{12}$ such that $e(S) = 5$.

(b) Let $S' = \{x^a, x^b y, x^c y\} \subseteq S$ for $1 \leq a \leq 5$ and $0 \leq b < c \leq 5$. We first list down the subsets S' where $e(S') \leq 4$.

Table 4.19: $\{x^a, x^b y, x^c y\} \subseteq S' \subseteq D_{12}$: $e(S') \leq 4$

S'	$e(S')$	S'	$e(S')$	S'	$e(S')$
$\{x, y, xy\}$	3	$\{x^2, xy, x^2y\}$	3	$\{x^4, x^2y, x^5y\}$	4
$\{x, y, x^3y\}$	4	$\{x^2, xy, x^4y\}$	4	$\{x^4, x^3y, x^4y\}$	3
$\{x, y, x^5y\}$	3	$\{x^2, x^2y, x^3y\}$	3	$\{x^4, x^4y, x^5y\}$	3
$\{x, xy, x^2y\}$	3	$\{x^2, x^2y, x^5y\}$	4	$\{x^5, y, xy\}$	3
$\{x, xy, x^4y\}$	4	$\{x^2, x^3y, x^4y\}$	3	$\{x^5, y, x^3y\}$	4
$\{x, x^2y, x^3y\}$	3	$\{x^2, x^4y, x^5y\}$	3	$\{x^5, y, x^5y\}$	3
$\{x, x^2y, x^5y\}$	4	$\{x^4, y, xy\}$	3	$\{x^5, xy, x^2y\}$	3
$\{x, x^3y, x^4y\}$	3	$\{x^4, y, x^3y\}$	4	$\{x^5, xy, x^4y\}$	4
$\{x, x^4y, x^5y\}$	3	$\{x^4, y, x^5y\}$	3	$\{x^5, x^2y, x^3y\}$	3
$\{x^2, y, xy\}$	3	$\{x^4, xy, x^2y\}$	3	$\{x^5, x^2y, x^5y\}$	4
$\{x^2, y, x^3y\}$	4	$\{x^4, xy, x^4y\}$	4	$\{x^5, x^3y, x^4y\}$	3
$\{x^2, y, x^5y\}$	3	$\{x^4, x^2y, x^3y\}$	3	$\{x^5, x^4y, x^5y\}$	3

Next, we list down the subsets S' where $e(S') > 4$ or $e(S') = \infty$.

Table 4.20: $\{x^a, x^b y, x^c y\} \subseteq S' \subseteq D_{12}$: $e(S') > 4$ or $e(S') = \infty$

S'	$e(S')$	S'	$e(S')$	S'	$e(S')$
$\{x, y, x^2y\}$	∞	$\{x^3, y, x^2y\}$	∞	$\{x^3, x^4y, x^5y\}$	6
$\{x, y, x^4y\}$	∞	$\{x^3, y, x^3y\}$	∞	$\{x^4, y, x^2y\}$	∞
$\{x, xy, x^3y\}$	∞	$\{x^3, y, x^4y\}$	∞	$\{x^4, y, x^4y\}$	∞
$\{x, xy, x^5y\}$	∞	$\{x^3, y, x^5y\}$	6	$\{x^4, xy, x^3y\}$	∞
$\{x, x^2y, x^4y\}$	∞	$\{x^3, xy, x^2y\}$	6	$\{x^4, xy, x^5y\}$	∞
$\{x, x^3y, x^5y\}$	∞	$\{x^3, xy, x^3y\}$	∞	$\{x^4, x^2y, x^4y\}$	∞
$\{x^2, y, x^2y\}$	∞	$\{x^3, xy, x^4y\}$	∞	$\{x^4, x^3y, x^5y\}$	∞
$\{x^2, y, x^4y\}$	∞	$\{x^3, xy, x^5y\}$	∞	$\{x^5, y, x^2y\}$	∞
$\{x^2, xy, x^3y\}$	∞	$\{x^3, x^2y, x^3y\}$	6	$\{x^5, y, x^4y\}$	∞
$\{x^2, xy, x^5y\}$	∞	$\{x^3, x^2y, x^4y\}$	∞	$\{x^5, xy, x^3y\}$	∞
$\{x^2, x^2y, x^4y\}$	∞	$\{x^3, x^2y, x^5y\}$	∞	$\{x^5, xy, x^5y\}$	∞

Table 4.20: (Continued)

S'	$e(S')$	S'	$e(S')$	S'	$e(S')$
$\{x^2, x^3y, x^5y\}$	∞	$\{x^3, x^3y, x^4y\}$	6	$\{x^5, x^2y, x^4y\}$	∞
$\{x^3, y, xy\}$	6	$\{x^3, x^3y, x^5y\}$	∞	$\{x^5, x^3y, x^5y\}$	∞

We see that $e(S') = \infty$ or $e(S') = 6$ for the subsets S' in Table 4.20. Hence, we consider the case where $|S''| = 4$, $S'' = S' \cup \{x^i y^j\}$ for $i \in \{0, 1, \dots, 5\}$ and $j \in \{0, 1\}$. Note that if $S'' = S' \cup \{x^d\}$ for $d \in \{1, 2, \dots, 5\}$, then S'' will contain one of the subsets S' listed in Tables 4.13 and 4.19. Hence, we will only list down the subsets $S'' = S' \cup \{x^d y\}$ for $d \in \{0, 1, \dots, 5\}$ that are not listed in Tables 4.15 and 4.16. We first list down the subsets S'' where $e(S'') \leq 4$.

Table 4.21: $\{x^a, x^b y, x^c y\} \subseteq S'' \subseteq D_{12}$: $e(S'') \leq 4$

S''	$e(S'')$	S''	$e(S'')$	S''	$e(S'')$
$\{x^3, y, xy, x^2y\}$	4	$\{x^3, y, x^3y, x^4y\}$	3	$\{x^3, xy, x^3y, x^4y\}$	3
$\{x^3, y, xy, x^3y\}$	3	$\{x^3, y, x^3y, x^5y\}$	3	$\{x^3, xy, x^4y, x^5y\}$	3
$\{x^3, y, xy, x^4y\}$	3	$\{x^3, y, x^4y, x^5y\}$	4	$\{x^3, x^2y, x^3y, x^4y\}$	4
$\{x^3, y, xy, x^5y\}$	4	$\{x^3, xy, x^2y, x^3y\}$	4	$\{x^3, x^2y, x^3y, x^5y\}$	3
$\{x^3, y, x^2y, x^3y\}$	3	$\{x^3, xy, x^2y, x^4y\}$	3	$\{x^3, x^2y, x^4y, x^5y\}$	3
$\{x^3, y, x^2y, x^5y\}$	3	$\{x^3, xy, x^2y, x^5y\}$	3	$\{x^3, x^3y, x^4y, x^5y\}$	4

Next, we list down the subsets S'' where $e(S'') > 4$ or $e(S'') = \infty$.

Table 4.22: $\{x^a, x^b y, x^c y\} \subseteq S'' \subseteq D_{12}$: $e(S'') = \infty$

S''	$e(S'')$	S''	$e(S'')$	S''	$e(S'')$
$\{x, y, x^2y, x^4y\}$	∞	$\{x^3, y, x^2y, x^4y\}$	∞	$\{x^5, y, x^2y, x^4y\}$	∞
$\{x, xy, x^3y, x^5y\}$	∞	$\{x^3, xy, x^3y, x^5y\}$	∞	$\{x^5, xy, x^3y, x^5y\}$	∞
$\{x^2, y, x^2y, x^4y\}$	∞	$\{x^4, y, x^2y, x^4y\}$	∞		
$\{x^2, xy, x^3y, x^5y\}$	∞	$\{x^4, xy, x^3y, x^5y\}$	∞		

We see that $e(S'') = \infty$ for the subsets S'' in Table 4.22. Therefore, we consider the case where $|S'''| = 5$, $S''' = S'' \cup \{x^i y^j\}$ for $i \in \{0, 1, \dots, 5\}$ and $j \in \{0, 1\}$. Note that we only need to consider the subsets S''' that are not listed in Table 4.17. It is clear that S''' will contain one of the subsets listed in Tables 4.13, 4.15, 4.19 and 4.21. Hence, $e(S''') < 5$, which is a contradiction.

(c) Let $S' = \{x^a y, x^b y, x^c y\} \subseteq S$ for $0 \leq a < b < c \leq 5$. By Proposition 4.17, $e(\{x^a y, x^b y, x^c y\}) = \infty$. Hence, we consider the case where $|S''| = 4$ and $S'' = S' \cup \{x^i y^j\}$ for $i \in \{0, 1, \ldots, 5\}$ and $j \in \{0, 1\}$. Note that by Proposition 4.17, $e(\{y, xy, x^2 y, x^3 y, x^4 y, x^5 y\}) = \infty$. If $|S| > 6$, then $e(S) = 2$. Therefore, we only consider adding the element x^d to S' for $1 \leq d \leq 5$. Then we have $S'' = \{x^a y, x^b y, x^c y, x^d\} \subseteq S$. It is clear from (b) that $e(S) < 5$.

From the three cases ((a), (b) and (c)) above, we see that if $1 \notin S$, there does not exist any subset $S \subseteq D_{12}$ such that $e(S) = 5$.

Secondly, suppose that $1 \in S$. From the assumption that $e(S) = 5$, we have $x^u y \in S^5$ for all $u = 0, 1, 2, \ldots, 5$ and hence there exists $x^v y \in S$ for some $v = 0, 1, 2, \ldots, 5$. We consider the following two cases:

(1) $\{1, x^a, x^b y\} \subseteq S$ for $1 \leq a \leq 5$ and $0 \leq b \leq 5$

(2) $\{1, x^a y, x^b y\} \subseteq S$ for $0 \leq a < b \leq 5$

where a, b are integers. We shall explain the two cases in detail. Let m be a positive integer. Recall that $p_1 = 2$ and $p_2 = 3$.

(1) Let $S' = \{1, x^a, x^b y\} \subseteq S$ for $1 \leq a \leq 5$ and $0 \leq b \leq 5$. Let m be a positive integer.

(1.1) If $a \neq mp_i$ for $i = 1, 2$, then $a \in \{1, 5\}$ and by Proposition 4.19, $e(S') \leq 4$. Hence, $e(S) \leq 4$, which is a contradiction.

(1.2) If $a = mp_i$, then $a \in \{2, 3, 4\}$ and by Proposition 4.19, $e(S') = \infty$. Hence, we consider the case where $|S''| = 4$ and $S'' = S' \cup \{x^i y^j\}$ for $i \in \{0, 1, \ldots, 5\}$ and $j \in \{0, 1\}$. Note that we only consider the subsets S'' that do not contain any of the subsets listed in Tables 4.13

and 4.19. We also note that $S'' = S' \cup \{1\}$ where $S' = \{x^d, x^e, x^f y\}$ in Table 4.14 or $S' = \{x^d, x^e y, x^f y\}$ in Table 4.20. Hence, we consider two cases as follows.

(i) Let $d \in \{1, 5\}$. Then S'' will contain one of the subsets in (1.1).

(ii) Let $d \in \{2, 3, 4\}$. We first list down the subsets S'' where $e(S'') \leq 4$.

Table 4.23: $\{1, x^a, x^b y\} \subseteq S'' \subseteq D_{12}$: $e(S'') \leq 4$

S''	$e(S'')$	S''	$e(S'')$	S''	$e(S'')$
$\{1, x^3, y, xy\}$	3	$\{1, x^3, xy, x^2y\}$	3	$\{1, x^3, x^2y, x^4y\}$	4
$\{1, x^3, y, x^2y\}$	4	$\{1, x^3, xy, x^3y\}$	4	$\{1, x^3, x^3y, x^4y\}$	3
$\{1, x^3, y, x^4y\}$	4	$\{1, x^3, xy, x^5y\}$	4	$\{1, x^3, x^3y, x^5y\}$	4
$\{1, x^3, y, x^5y\}$	3	$\{1, x^3, x^2y, x^3y\}$	3	$\{1, x^3, x^4y, x^5y\}$	3

Next we list down the subsets S'' where $e(S'') > 4$ or $e(S'') = \infty$.

Table 4.24: $\{1, x^a, x^b y\} \subseteq S'' \subseteq D_{12}$: $e(S'') = \infty$

S''	$e(S'')$	S''	$e(S'')$	S''	$e(S'')$
$\{1, x^2, y, x^4\}$	∞	$\{1, x^2, x^2y, x^4y\}$	∞	$\{1, x^3, x^2y, x^5y\}$	∞
$\{1, x^2, y, x^2y\}$	∞	$\{1, x^2, x^3y, x^4\}$	∞	$\{1, x^4, y, x^2y\}$	∞
$\{1, x^2, y, x^4y\}$	∞	$\{1, x^2, x^3y, x^5y\}$	∞	$\{1, x^4, y, x^4y\}$	∞
$\{1, x^2, xy, x^4\}$	∞	$\{1, x^2, x^4y, x^4\}$	∞	$\{1, x^4, xy, x^3y\}$	∞
$\{1, x^2, xy, x^3y\}$	∞	$\{1, x^2, x^5y, x^4\}$	∞	$\{1, x^4, xy, x^5y\}$	∞
$\{1, x^2, xy, x^5y\}$	∞	$\{1, x^3, y, x^3y\}$	∞	$\{1, x^4, x^2y, x^4y\}$	∞
$\{1, x^2, x^2y, x^4\}$	∞	$\{1, x^3, xy, x^4y\}$	∞	$\{1, x^4, x^3y, x^5y\}$	∞

Since $e(S'') = \infty$ for the subsets S'' in Table 4.24, we consider the case where $|S'''| = 5$ and $S''' = S'' \cup \{x^i y^j\}$ for $i \in \{0, 1, \ldots, 5\}$ and $j \in \{0, 1\}$. If $S''' = S'' \cup \{x^i\}$ for $i \in \{1, 2, \ldots, 5\}$, then S''' will contain one of the subsets S' in Table 4.13. Hence, in the following we will only consider the subsets $S''' = S'' \cup \{x^i y\}$ for $i \in \{0, 1, \ldots, 5\}$. Note that we only need to list down the subsets S''' that do not contain any of the subsets listed in Tables 4.15, 4.19 and 4.21.

Table 4.25: $\{1, x^a, x^b y\} \subseteq S''' \subseteq D_{12}$: $e(S''') = \infty$

S'''	$e(S''')$	S'''	$e(S''')$	S'''	$e(S''')$
$\{1, x^2, y, x^4, x^2y\}$	∞	$\{1, x^2, xy, x^4, x^5y\}$	∞	$\{1, x^4, y, x^2y, x^4y\}$	∞
$\{1, x^2, y, x^4, x^4y\}$	∞	$\{1, x^2, xy, x^3y, x^5y\}$	∞	$\{1, x^4, xy, x^3y, x^5y\}$	∞
$\{1, x^2, y, x^2y, x^4y\}$	∞	$\{1, x^2, x^2y, x^4, x^4y\}$	∞		
$\{1, x^2, xy, x^4, x^3y\}$	∞	$\{1, x^2, x^3y, x^4, x^5y\}$	∞		

Since $e(S''') = \infty$ for the subsets S''' in Table 4.25, we consider the case where $|S''''| = 6$ and $S'''' = S''' \cup \{x^i y^j\}$ for $i \in \{0, 1, \ldots, 5\}$ and $j \in \{0, 1\}$. Note that in the following we only list down the subsets S'''' that do not contain any of the subsets listed in Tables 4.13, 4.15, 4.19 and 4.21. Hence in Table 4.26 we obtain two of the largest non-exhaustive subsets in D_{12} which contain $\{1\}$.

Table 4.26: $\{1, x^a, x^b y\} \subseteq S'''' \subseteq D_{12}$: $e(S'''') = \infty$

S''''	$e(S'''')$
$\{1, x^2, x^4, y, x^2y, x^4y\}$	∞
$\{1, x^2, x^4, xy, x^3y, x^5y\}$	∞

By Proposition 4.6, if $|S| > 6$, then $e(S) = 2 < 5$. Hence, there does not exist any subset $S \subseteq D_{12}$ such that $e(S) = 5$.

(2) Let $S' = \{1, x^a y, x^b y\} \subseteq S$ for $0 \le a < b \le 5$. By Proposition 4.20, $e(S') = 6$ or $e(S') = \infty$. Hence, we consider the case where $|S''| = 4$, $S'' = S' \cup \{x^i y^j\}$ for $i \in \{0, 1, \ldots, 5\}$ and $j \in \{0, 1\}$. Note that these subsets S'' can be divided into two cases:

(i) Let $S'' = S' \cup \{x^i\}$ for $i \in \{1, 2, \ldots, 5\}$. If $i \in \{1, 5\}$, by Propositions 4.5 and 4.19 we have $e(S'') \le 4$. If $i \in \{2, 3, 4\}$, we see that these subsets have been discussed in part (1.2).

(ii) Let $S'' = S' \cup \{x^i y\}$ for $i \in \{0, 1, \ldots, 5\}$. We first list down the subsets S'' where $e(S'') \le 4$.

Table 4.27: $\{1, x^a y, x^b y\} \subseteq S'' \subseteq D_{12}$: $e(S'') \leq 4$

S''	$e(S'')$	S''	$e(S'')$	S''	$e(S'')$
$\{1, y, xy, x^2y\}$	4	$\{1, y, x^3y, x^4y\}$	3	$\{1, xy, x^3y, x^4y\}$	3
$\{1, y, xy, x^3y\}$	3	$\{1, y, x^3y, x^5y\}$	3	$\{1, xy, x^4y, x^5y\}$	3
$\{1, y, xy, x^4y\}$	3	$\{1, y, x^4y, x^5y\}$	4	$\{1, x^2y, x^3y, x^4y\}$	4
$\{1, y, xy, x^5y\}$	4	$\{1, xy, x^2y, x^3y\}$	4	$\{1, x^2y, x^3y, x^5y\}$	3
$\{1, y, x^2y, x^3y\}$	3	$\{1, xy, x^2y, x^4y\}$	3	$\{1, x^2y, x^4y, x^5y\}$	3
$\{1, y, x^2y, x^5y\}$	3	$\{1, xy, x^2y, x^5y\}$	3	$\{1, x^3y, x^4y, x^5y\}$	4

Next, we consider the subsets S'' where $e(S'') > 4$ or $e(S'') = \infty$.

Table 4.28: $\{1, x^a y, x^b y\} \subseteq S'' \subseteq D_{12}$: $e(S'') = \infty$

S''	$e(S'')$
$\{1, y, x^2y, x^4y\}$	∞
$\{1, xy, x^3y, x^5y\}$	∞

Since $e(S'') = \infty$ for the subsets S'' in Table 4.28, we consider the case where $|S'''| = 5$ and $S''' = S'' \cup \{x^i y^j\}$ for $i \in \{0, 1, \ldots, 5\}$ and $j \in \{0, 1\}$. We consider two cases:

(i) If $S''' = S'' \cup \{x^i\}$ for $i \in \{1, 2, \ldots, 5\}$, then from part (1) we see that $e(S) < 5$.

(ii) If $S''' = S'' \cup \{x^i y\}$ for $i \in \{0, 1, \ldots, 5\}$, then we see from Table 4.27 that $e(S) < 5$.

From cases (1) and (2) above, we see that when $1 \in S$, there does not exist any subset $S \subseteq D_{12}$ such that $e(S) = 5$.

Hence, we conclude that there does not exist any subset $S \subseteq D_{12}$ such that $e(S) = 5$. □

4.3.2 Non-exhaustive Subsets When n Is Odd

In Section 4.3.1, we have shown that D_{12} is the smallest case in which there does not exist any subset S such that $e(S) = 5$. We now turn our attention to the case where n is odd. In Tables 4.29 and 4.30, we list down some exhaustive subsets $S \subseteq D_{2n}$ for the two smallest odd integers n where $n \geq 5$.

Table 4.29: Subsets S in D_{10} where $e(S) \in \{2, 3, 4, 5\}$

$e(S)$	S
2	$\{1, x, y, xy, x^2y\}$ (Proposition 4.7(i))
3	$\{1, x, xy\}$ (Proposition 4.2)
4	$\{x, y, x^2y\}$ (Proposition 4.11)
5	$\{1, y, xy\}$ (Proposition 4.13)

Table 4.30: Subsets S in D_{14} where $e(S) \in \{2, 3, 4, 6, 7\}$

$e(S)$	S
2	$\{1, x, y, xy, x^2y, x^4y\}$ (Proposition 4.7(i))
3	$\{1, x, y, xy, x^2y\}$ (Proposition 4.8(ii))
4	$\{1, x, xy\}$ (Proposition 4.2)
5	Does not exist.
6	$\{x, y, x^2y\}$ (Proposition 4.11)
7	$\{1, y, xy\}$ (Proposition 4.13)

From Tables 4.29 and 4.30, we see that D_{14} is the smallest odd case in which there does not exist any subset S such that $e(S) = k$ for some integer $k < n$. In this section, our main objective is to show that there does not exist any subset S in D_{14} such that $e(S) = 5$. Note that the range of values of $e(S)$ for which there does not exist any subset $S \subseteq D_{2n}$ increases as n increases. Hence, we conjecture that there does not exist any subset $S \subseteq D_{2n}$ such that $\frac{n+3}{2} \leq e(S) \leq n-2$. At the end of the chapter, we shall verify our conjecture for $n = 11$. To do this, we first obtain some preliminary results. Similar to Section 4.3.1, we begin by classifying the subsets that are non-exhaustive.

Proposition 4.22. *Let n be odd where $n \geq 5$. Let $p_1, \ldots, p_{\omega(n)}$ be the distinct prime factor(s) of n. For $k = 1, 2, 3$, let $S_k \subseteq D_{2n}$ as follows:*

(a) $S_1 = \{1, x, x^2, \ldots, x^{n-1}\}$*;*

(b) $S_2 = \{y, xy, x^2y, \dots, x^{n-1}y\}$*;*

(c) $S_3 = S_{i_j} = \{1, x^{p_i}, x^{2p_i}, \dots, x^{(n-p_i)}, x^jy, x^{j+p_i}y, x^{j+2p_i}y, \dots, x^{j+(n-p_i)}y\}$ *for* $j = 0, 1, \dots, p_i - 1$.

Then $e(S_k) = \infty$ *for* $k = 1,\ 2,\ 3$.

Proof.

(a) Since S_1 is a proper subgroup of D_{2n}, it follows that $e(S_1) = \infty$.

(b) Note that $S_2^{2m} = \{1, x, x^2, \dots, x^{n-1}\}$ is a proper subgroup of D_{2n} for all positive integers m. Hence, $e(S_2) = \infty$.

(c) It is clear that $\{1, x^{p_i}, x^{2p_i}, \dots, x^{(n-1)p_i)}\} \subseteq S_3^k$ for all $k \geq 2$ but

$$\{1, x, x^2, \dots, x^{n-1}\} \setminus \{1, x^{p_i}, x^{2p_i}, \dots, x^{(n-1)p_i}\} \nsubseteq S^k$$

for all $k \geq 2$. Thus, $e(S_3) = \infty$.

□

In order to consider all the possible subsets, we will first look at the smallest case and determine the exhaustion numbers of S when $|S| = 2$.

Proposition 4.23. *Let* $S \subseteq D_{2n}$ *where* $n \geq 5$ *is odd. If* $|S| = 2$*, then* $e(S) = \infty$ *or* $e(S) = n$*.*

Proof. Let $S \subseteq D_{2n}$ where $|S| = 2$. We consider the three possible cases:

(1) Let $S = \{x^a, x^b\}$ for $0 \leq a < b \leq n-1$. Then $S \subseteq \{1, x, x^2, \dots, x^{n-1}\}$ and by Propositions 4.16 and 4.22, $e(S) = \infty$.

(2) Let $S = \{x^ay, x^by\}$ for $0 \leq a < b \leq n-1$. Then $S \subseteq \{y, xy, x^2y, \dots, x^{n-1}y\}$ and by Propositions 4.16 and 4.22, $e(S) = \infty$.

(3) Let $S = \{x^a, x^by\}$ for $0 \leq a, b \leq n-1$ and let $p_1, \dots, p_{\omega(n)}$ be the distinct prime factor(s) of n.

(3.1) If $a = mp_i$ for some positive integer m, then

$$S \subseteq \{1, x^{p_i}, x^{2p_i}, \ldots, x^{(n-1)p_i}, x^j y, x^{j+p_i} y, x^{j+2p_i} y, \ldots, x^{j+(n-p_i)} y\}$$

for some integer $j \in \{0, 1, \ldots, p_i - 1\}$. By Propositions 4.16 and 4.22, $e(S) = \infty$.

(3.2) Let $a \neq mp_i$ where m is a positive integer. The result is clear by Proposition 4.14.

□

Since we have $e(S) = n$ or $e(S) = \infty$ when $|S| = 2$, we proceed to investigate the subsets $S \subseteq D_{2n}$ where $|S| = 3$ when n is odd. In the next three propositions, we will look at the following three subsets.

(i) $S = \{1, x^a, x^b y\}$ where $1 \leq a \leq n-1$ and $0 \leq b \leq n-1$

(ii) $S = \{1, x^a y, x^b y\}$ where $1 \leq a < b \leq n-1$

(iii) $S = \{x^a, x^b, x^c y\}$ where $1 \leq a < b \leq n-1$, $0 \leq c \leq n-1$ and $a + b = n$

Proposition 4.24. *Let n, a, b be integers. Let $S = \{1, x^a, x^b y\} \subseteq D_{2n}$, $n \geq 5$ where n is odd, $1 \leq a \leq n-1$ and $0 \leq b \leq n-1$. Let p_i be the distinct prime factor(s) of n for $i = 1, 2, \ldots, \omega(n)$.*

(i) If $a \neq mp_i$ for any positive integer m, then $e(S) \leq \frac{n+1}{2}$.

(ii) If $a = mp_i$ for some positive integer m, then $e(S) = \infty$.

Proof.

(i) Let $c \in S^k$ for some positive integer $k \geq 1$. Then $c \in S^{k+1}$ since $1 \in S$. We first note that if $a \neq mp_i$ for any positive integer m, then $|\{x^a, x^{2a}, \ldots, x^{(n-1)a}, x^{na}\}| = n$ and hence

$$|\{x^{b+a} y, x^{b+2a} y, \ldots, x^{b+(n-1)a} y, x^{b+na} y\}| = n.$$

Since $1, x^a \in S$, we have

$$x^{ka} \in S^{\frac{n+1}{2}} \quad \text{for } k = 1, 2, \ldots, \frac{n+1}{2}$$

and hence

$$\{1, x^a, x^{2a}, \ldots, x^{(\frac{n+1}{2})a}\} \subseteq S^{\frac{n+1}{2}}. \tag{4.43}$$

Next, we have

$$x^{(n-k)a} = x^{n-ka} = x^b y \cdot x^{b+ka} y \in S \cdot S^{k+1} \subseteq S^{k+2} \quad \text{for } k = 1, 2, \ldots, \frac{n-3}{2}$$

and hence

$$\{x^{(n-1)a}, x^{(n-2)a}, x^{(n-3)a}, \ldots, x^{(\frac{n+3}{2})a}\} \subseteq S^{\frac{n+1}{2}}. \tag{4.44}$$

Combining (4.43) and (4.44), we have

$$\{1, x^a, x^{2a}, \ldots, x^{(n-1)a}\} = \{1, x, x^2, \ldots, x^{n-1}\} \subseteq S^{\frac{n+1}{2}}.$$

Since $\{1, x^a, x^{2a}, \ldots, x^{\frac{n-1}{2}a}\} \subseteq S^{\frac{n-1}{2}}$ and $x^b y \in S$, we have

$$\{x^b y, x^{b+a} y, x^{b+2a} y, \ldots, x^{b+\frac{n-1}{2}a} y\} \subseteq S^{\frac{n+1}{2}}. \tag{4.45}$$

Finally, note that

$$x^{b+(n-k)a} y = x^{b+(n-ka)} y = x^b y \cdot x^{ka} \in S \cdot S^k \subseteq S^{k+1}$$

for $k = 1, 2, \ldots, \frac{n-1}{2}$ and hence

$$\{x^{b+(n-1)a} y, x^{b+(n-2)a} y, x^{b+(n-3)a} y, \ldots, x^{b+(\frac{n+1}{2})a} y\} \subseteq S^{\frac{n}{2}+1}. \tag{4.46}$$

Combining (4.45) and (4.46), we see that

$$\{x^b y, x^{b+a} y, x^{b+2a} y, \ldots, x^{b+(n-1)a} y\} = \{y, xy, x^2 y, \ldots, x^{n-1} y\} \subseteq S^{\frac{n+1}{2}}.$$

Therefore, $S^{\frac{n+1}{2}} = D_{2n}$ and $e(S) \leq \frac{n+1}{2}$.

(ii) If $a = mp_i$ for some positive integer m, then

$$\{1, x^a, x^b y\} \subseteq \{1, x^{p_i}, x^{2p_i}, \ldots, x^{(n-p_i)}, x^j y, x^{j+p_i} y, x^{j+2p_i} y, \ldots, x^{j+(n-p_i)} y\}$$

for some integer $j = 0, 1, \ldots, p_i - 1$. By Propositions 4.22 and 4.16, we have $e(S) = \infty$.

□

Proposition 4.25. *Let $S = \{1, x^a y, x^b y\} \subseteq D_{2n}$, $n \geq 5$ where n is odd and $0 \leq a < b \leq n-1$. Let p_i be the distinct prime factor(s) of n for $i = 1, 2, \ldots, \omega(n)$.*

(i) If $b - a \neq mp_i$ for any positive integer m, then $e(S) = n$.

(ii) If $b - a = mp_i$ for some positive integer m, then $e(S) = \infty$.

Proof.

(i) By Proposition 4.13, we have $e(S) = n$.

(ii) If $b - a = mp_i$ for some positive integer m, then

$$\{1, x^a y, x^b y\} \subseteq \{1, x^{p_i}, x^{2p_i}, \ldots, x^{(n-p_i)}, y, x^{p_i} y, x^{2p_i} y, \ldots, x^{(n-p_i)} y\}.$$

By Propositions 4.22 and 4.16, we have $e(S) = \infty$.

□

Proposition 4.26. *Let $S = \{x^a, x^b, x^c y\} \subseteq D_{2n}$ where $n \geq 5$ is odd, $1 \leq a < b \leq n-1$, $0 \leq c \leq n-1$ and $a + b = n$. Let p_i be the distinct prime factor(s) of n for $i = 1, 2, \ldots, \omega(n)$.*

(i) If $a \neq mp_i$ for any positive integer m, then $e(S) = n$.

(ii) If $a = mp_i$ for some positive integer m, then $e(S) = \infty$.

Proof.

(i) By Proposition 4.15, we have $e(S) = n$.

(ii) If $a = mp_i$ for some positive integer m, then $b = qp_i$ for some positive integer q since $a + b = n$. Hence,

$$\{x^a, x^b, x^c y\} \subseteq \{1, x^{p_i}, x^{2p_i}, \ldots, x^{(n-p_i)}, x^j y, x^{j+p_i} y, x^{j+2p_i} y, \ldots, x^{j+(n-p_i)} y\}$$

for some integer $j = 0, 1, \ldots, p_i - 1$. By Propositions 4.22 and 4.16, we have $e(S) = \infty$.

$\square$

Now, we are ready to show that there does not exist any subset $S \subseteq D_{14}$ such that $e(S) = n - 2 = 7 - 2 = 5$. Once again, the following notations will be used: Let S, S', S'', S''', S'''' $\subseteq D_{2n}$ where $S' \subsetneq S'' \subsetneq S''' \subsetneq S'''' \subsetneq S \subseteq D_{2n}$. Note that $|S'| = 3$, $|S''| = 4$, $|S'''| = 5$ and $|S''''| = 6$.

Theorem 4.27. *Let D_{14} be the dihedral group of order 14. Then there does not exist any subset $S \subseteq D_{14}$ such that $e(S) = 5$.*

Proof. Given that D_{14} is the dihedral group of order 14, then we have $n = 7$. Suppose that there exists a subset $S \subseteq D_{14}$ where $e(S) = 5$. By Propositions 4.6 and 4.23, it is clear that $3 \leq |S| \leq 7$. Firstly, suppose that $1 \notin S$. From the assumption that $e(S) = 5$, we have $x^u y \in S^5$ for all $u = 0, 1, 2, \ldots, 6$ and hence there exists $x^v y \in S$ for some $v = 0, 1, 2, \ldots, 6$. Then we shall consider three cases as follows:

(a) $\{x^a, x^b, x^c y\} \subseteq S$ for $1 \leq a < b \leq 6$ and $0 \leq c \leq 6$

(b) $\{x^a, x^b y, x^c y\} \subseteq S$ for $1 \leq a \leq 6$ and $0 \leq b < c \leq 6$

(c) $\{x^a y, x^b y, x^c y\} \subseteq S$ for $0 \leq a < b < c \leq 6$.

where a, b, c are integers. In the following, we shall explain the three cases separately in detail.

(a) Let $S' = \{x^a, x^b, x^c y\} \subseteq S$ for $1 \leq a < b \leq 6$ and $0 \leq c \leq 6$. We first list down the subsets S' where $e(S') \leq 4$.

Table 4.31: $\{x^a, x^b, x^c y\} \subseteq S' \subseteq D_{14}$: $e(S') \leq 4$

S'	$e(S')$	S'	$e(S')$	S'	$e(S')$
$\{x, x^2, y\}$	4	$\{x^2, x^3, y\}$	3	$\{x^3, x^6, y\}$	4
$\{x, x^2, xy\}$	4	$\{x^2, x^3, xy\}$	3	$\{x^3, x^6, xy\}$	4
$\{x, x^2, x^2y\}$	4	$\{x^2, x^3, x^2y\}$	3	$\{x^3, x^6, x^2y\}$	4
$\{x, x^2, x^3y\}$	4	$\{x^2, x^3, x^3y\}$	3	$\{x^3, x^6, x^3y\}$	4
$\{x, x^2, x^4y\}$	4	$\{x^2, x^3, x^4y\}$	3	$\{x^3, x^6, x^4y\}$	4
$\{x, x^2, x^5y\}$	4	$\{x^2, x^3, x^5y\}$	3	$\{x^3, x^6, x^5y\}$	4
$\{x, x^2, x^6y\}$	4	$\{x^2, x^3, x^6y\}$	3	$\{x^3, x^6, x^6y\}$	4
$\{x, x^3, y\}$	3	$\{x^2, x^4, y\}$	4	$\{x^4, x^5, y\}$	3
$\{x, x^3, xy\}$	3	$\{x^2, x^4, xy\}$	4	$\{x^4, x^5, xy\}$	3
$\{x, x^3, x^2y\}$	3	$\{x^2, x^4, x^2y\}$	4	$\{x^4, x^5, x^2y\}$	3
$\{x, x^3, x^3y\}$	3	$\{x^2, x^4, x^3y\}$	4	$\{x^4, x^5, x^3y\}$	3
$\{x, x^3, x^4y\}$	3	$\{x^2, x^4, x^4y\}$	4	$\{x^4, x^5, x^4y\}$	3
$\{x, x^3, x^5y\}$	3	$\{x^2, x^4, x^5y\}$	4	$\{x^4, x^5, x^5y\}$	3
$\{x, x^3, x^6y\}$	3	$\{x^2, x^4, x^6y\}$	4	$\{x^4, x^5, x^6y\}$	3
$\{x, x^4, y\}$	4	$\{x^2, x^6, y\}$	3	$\{x^4, x^6, y\}$	3
$\{x, x^4, xy\}$	4	$\{x^2, x^6, xy\}$	3	$\{x^4, x^6, xy\}$	3
$\{x, x^4, x^2y\}$	4	$\{x^2, x^6, x^2y\}$	3	$\{x^4, x^6, x^2y\}$	3
$\{x, x^4, x^3y\}$	4	$\{x^2, x^6, x^3y\}$	3	$\{x^4, x^6, x^3y\}$	3
$\{x, x^4, x^4y\}$	4	$\{x^2, x^6, x^4y\}$	3	$\{x^4, x^6, x^4y\}$	3
$\{x, x^4, x^5y\}$	4	$\{x^2, x^6, x^5y\}$	3	$\{x^4, x^6, x^5y\}$	3
$\{x, x^4, x^6y\}$	4	$\{x^2, x^6, x^6y\}$	3	$\{x^4, x^6, x^6y\}$	3
$\{x, x^5, y\}$	3	$\{x^3, x^5, y\}$	4	$\{x^5, x^6, y\}$	4
$\{x, x^5, xy\}$	3	$\{x^3, x^5, xy\}$	4	$\{x^5, x^6, xy\}$	4
$\{x, x^5, x^2y\}$	3	$\{x^3, x^5, x^2y\}$	4	$\{x^5, x^6, x^2y\}$	4
$\{x, x^5, x^3y\}$	3	$\{x^3, x^5, x^3y\}$	4	$\{x^5, x^6, x^3y\}$	4
$\{x, x^5, x^4y\}$	3	$\{x^3, x^5, x^4y\}$	4	$\{x^5, x^6, x^4y\}$	4
$\{x, x^5, x^5y\}$	3	$\{x^3, x^5, x^5y\}$	4	$\{x^5, x^6, x^5y\}$	4
$\{x, x^5, x^6y\}$	3	$\{x^3, x^5, x^6y\}$	4	$\{x^5, x^6, x^6y\}$	4

By Proposition 4.5, we see that if $S' \subseteq S$, then $e(S) \leq e(S') \leq 4$. Next, we consider the subsets S' where $e(S') > 4$. Note that these subsets are of the form $S' = \{x^a, x^b, x^c y\}$ where $1 \leq a < b \leq 6$, $0 \leq c \leq 6$ and $a + b = 7$. By Proposition 4.15, we have $e(S') = 7$ as shown in Table 4.32 below.

Table 4.32: $\{x^a, x^b, x^c y\} \subseteq S' \subseteq D_{14}$: $e(S') = 7 > 4$

S'	$e(S')$	S'	$e(S')$	S'	$e(S')$
$\{x, x^6, y\}$	7	$\{x^2, x^5, y\}$	7	$\{x^3, x^4, y\}$	7
$\{x, x^6, xy\}$	7	$\{x^2, x^5, xy\}$	7	$\{x^3, x^4, xy\}$	7
$\{x, x^6, x^2y\}$	7	$\{x^2, x^5, x^2y\}$	7	$\{x^3, x^4, x^2y\}$	7
$\{x, x^6, x^3y\}$	7	$\{x^2, x^5, x^3y\}$	7	$\{x^3, x^4, x^3y\}$	7
$\{x, x^6, x^4y\}$	7	$\{x^2, x^5, x^4y\}$	7	$\{x^3, x^4, x^4y\}$	7
$\{x, x^6, x^5y\}$	7	$\{x^2, x^5, x^5y\}$	7	$\{x^3, x^4, x^5y\}$	7
$\{x, x^6, x^6y\}$	7	$\{x^2, x^5, x^6y\}$	7	$\{x^3, x^4, x^6y\}$	7

Since $e(S') = 7$ for the subsets S' in Table 4.32, we consider the case where $|S''| = 4$ and $S'' = S' \cup \{x^i y^j\}$ for $i \in \{0, 1, \dots, 6\}$ and $j \in \{0, 1\}$. Note that if $S'' = S' \cup \{x^d\}$ for $d \in \{1, 2, \dots, 6\}$, then S'' will contain one of the subsets listed in Table 4.31. Therefore, in the following we only consider the subsets $S'' = S' \cup \{x^d y\}$ for $d \in \{0, 1, \dots, 6\}$. We first list down the subsets S'' where $e(S'') \leq 4$.

Table 4.33: $\{x^a, x^b, x^c y\} \subseteq S' \subseteq D_{14}$: $e(S'') \leq 4$

S''	$e(S'')$	S''	$e(S'')$	S''	$e(S'')$
$\{x, x^6, y, xy\}$	4	$\{x^2, x^5, y, xy\}$	4	$\{x^3, x^4, y, x^2y\}$	4
$\{x, x^6, y, x^3y\}$	4	$\{x^2, x^5, y, x^2y\}$	4	$\{x^3, x^4, y, x^3y\}$	4
$\{x, x^6, y, x^4y\}$	4	$\{x^2, x^5, y, x^5y\}$	4	$\{x^3, x^4, y, x^4y\}$	4
$\{x, x^6, y, x^6y\}$	4	$\{x^2, x^5, y, x^6y\}$	4	$\{x^3, x^4, y, x^5y\}$	4
$\{x, x^6, xy, x^2y\}$	4	$\{x^2, x^5, xy, x^2y\}$	4	$\{x^3, x^4, xy, x^3y\}$	4
$\{x, x^6, xy, x^4y\}$	4	$\{x^2, x^5, xy, x^3y\}$	4	$\{x^3, x^4, xy, x^4y\}$	4
$\{x, x^6, xy, x^5y\}$	4	$\{x^2, x^5, xy, x^6y\}$	4	$\{x^3, x^4, xy, x^5y\}$	4
$\{x, x^6, x^2y, x^3y\}$	4	$\{x^2, x^5, x^2y, x^3y\}$	4	$\{x^3, x^4, xy, x^6y\}$	4
$\{x, x^6, x^2y, x^5y\}$	4	$\{x^2, x^5, x^2y, x^4y\}$	4	$\{x^3, x^4, x^2y, x^4y\}$	4
$\{x, x^6, x^2y, x^6y\}$	4	$\{x^2, x^5, x^3y, x^4y\}$	4	$\{x^3, x^4, x^2y, x^5y\}$	4
$\{x, x^6, x^3y, x^4y\}$	4	$\{x^2, x^5, x^3y, x^5y\}$	4	$\{x^3, x^4, x^2y, x^6y\}$	4
$\{x, x^6, x^3y, x^6y\}$	4	$\{x^2, x^5, x^4y, x^5y\}$	4	$\{x^3, x^4, x^3y, x^5y\}$	4
$\{x, x^6, x^4y, x^5y\}$	4	$\{x^2, x^5, x^4y, x^6y\}$	4	$\{x^3, x^4, x^3y, x^6y\}$	4
$\{x, x^6, x^5y, x^6y\}$	4	$\{x^2, x^5, x^5y, x^6y\}$	4	$\{x^3, x^4, x^4y, x^6y\}$	4

Next, we list down the subsets S'' where $e(S'') > 4$.

Table 4.34: $\{x^a, x^b, x^c y\} \subseteq S' \subseteq D_{14}$: $e(S'') = 6 > 4$

S''	$e(S'')$	S''	$e(S'')$	S''	$e(S'')$
$\{x, x^6, y, x^2y\}$	6	$\{x^2, x^5, y, x^3y\}$	6	$\{x^3, x^4, y, xy\}$	6
$\{x, x^6, y, x^5y\}$	6	$\{x^2, x^5, y, x^4y\}$	6	$\{x^3, x^4, y, x^6y\}$	6
$\{x, x^6, xy, x^3y\}$	6	$\{x^2, x^5, xy, x^4y\}$	6	$\{x^3, x^4, xy, x^2y\}$	6
$\{x, x^6, xy, x^6y\}$	6	$\{x^2, x^5, xy, x^5y\}$	6	$\{x^3, x^4, x^2y, x^3y\}$	6
$\{x, x^6, x^2y, x^4y\}$	6	$\{x^2, x^5, x^2y, x^5y\}$	6	$\{x^3, x^4, x^3y, x^4y\}$	6
$\{x, x^6, x^3y, x^5y\}$	6	$\{x^2, x^5, x^2y, x^6y\}$	6	$\{x^3, x^4, x^4y, x^5y\}$	6
$\{x, x^6, x^4y, x^6y\}$	6	$\{x^2, x^5, x^3y, x^6y\}$	6	$\{x^3, x^4, x^5y, x^6y\}$	6

Let $S''' = S'' \cup \{x^i y^j\}$ for $i \in \{0, 1, \ldots, 6\}$ and $j \in \{0, 1\}$ where S'' is in Table 4.34 and $|S'''| = 5$. Then we have the following cases:

(i) If $S''' = S'' \cup \{x^i\}$ for $i \in \{1, 2, \ldots, 6\}$, then S''' will contain one of the subsets in Table 4.31.

(ii) If $S''' = S'' \cup \{x^i y\}$ for $i \in \{0, 1, \ldots, 6\}$, then S''' will contain one of the subsets in Table 4.33.

By Proposition 4.5, we have $e(S''') < 5$ and hence there does not exist any subset $S \subseteq D_{14}$ where $e(S) = 5$.

(b) Let $S' = \{x^a, x^b y, x^c y\} \subseteq S$ for $1 \leq a \leq 6$ and $0 \leq b < c \leq 6$. We first list down the subsets S' where $e(S') \leq 4$.

Table 4.35: $\{x^a, x^b y, x^c y\} \subseteq S' \subseteq D_{14}$: $e(S') \leq 4$

S'	$e(S')$	S'	$e(S')$	S'	$e(S')$
$\{x, y, xy\}$	4	$\{x^3, y, x^2y\}$	4	$\{x^5, y, xy\}$	4
$\{x, y, x^3y\}$	4	$\{x^3, y, x^3y\}$	4	$\{x^5, y, x^2y\}$	4
$\{x, y, x^4y\}$	4	$\{x^3, y, x^4y\}$	4	$\{x^5, y, x^5y\}$	4
$\{x, y, x^6y\}$	4	$\{x^3, y, x^5y\}$	4	$\{x^5, y, x^6y\}$	4
$\{x, xy, x^2y\}$	4	$\{x^3, xy, x^3y\}$	4	$\{x^5, xy, x^2y\}$	4
$\{x, xy, x^4y\}$	4	$\{x^3, xy, x^4y\}$	4	$\{x^5, xy, x^3y\}$	4
$\{x, xy, x^5y\}$	4	$\{x^3, xy, x^5y\}$	4	$\{x^5, xy, x^6y\}$	4
$\{x, x^2y, x^3y\}$	4	$\{x^3, xy, x^6y\}$	4	$\{x^5, x^2y, x^3y\}$	4
$\{x, x^2y, x^5y\}$	4	$\{x^3, x^2y, x^4y\}$	4	$\{x^5, x^2y, x^4y\}$	4
$\{x, x^2y, x^6y\}$	4	$\{x^3, x^2y, x^5y\}$	4	$\{x^5, x^3y, x^4y\}$	4
$\{x, x^3y, x^4y\}$	4	$\{x^3, x^2y, x^6y\}$	4	$\{x^5, x^3y, x^5y\}$	4
$\{x, x^3y, x^6y\}$	4	$\{x^3, x^3y, x^5y\}$	4	$\{x^5, x^4y, x^5y\}$	4
$\{x, x^4y, x^5y\}$	4	$\{x^3, x^3y, x^6y\}$	4	$\{x^5, x^4y, x^6y\}$	4
$\{x, x^5y, x^6y\}$	4	$\{x^3, x^4y, x^6y\}$	4	$\{x^5, x^5y, x^6y\}$	4
$\{x^2, y, xy\}$	4	$\{x^4, y, x^2y\}$	4	$\{x^6, y, xy\}$	4
$\{x^2, y, x^2y\}$	4	$\{x^4, y, x^3y\}$	4	$\{x^6, y, x^3y\}$	4
$\{x^2, y, x^5y\}$	4	$\{x^4, y, x^4y\}$	4	$\{x^6, y, x^4y\}$	4
$\{x^2, y, x^6y\}$	4	$\{x^4, y, x^5y\}$	4	$\{x^6, y, x^6y\}$	4
$\{x^2, xy, x^2y\}$	4	$\{x^4, xy, x^3y\}$	4	$\{x^6, xy, x^2y\}$	4
$\{x^2, xy, x^3y\}$	4	$\{x^4, xy, x^4y\}$	4	$\{x^6, xy, x^4y\}$	4
$\{x^2, xy, x^6y\}$	4	$\{x^4, xy, x^5y\}$	4	$\{x^6, xy, x^5y\}$	4
$\{x^2, x^2y, x^3y\}$	4	$\{x^4, xy, x^6y\}$	4	$\{x^6, x^2y, x^3y\}$	4
$\{x^2, x^2y, x^4y\}$	4	$\{x^4, x^2y, x^4y\}$	4	$\{x^6, x^2y, x^5y\}$	4
$\{x^2, x^3y, x^4y\}$	4	$\{x^4, x^2y, x^5y\}$	4	$\{x^6, x^2y, x^6y\}$	4
$\{x^2, x^3y, x^5y\}$	4	$\{x^4, x^2y, x^6y\}$	4	$\{x^6, x^3y, x^4y\}$	4
$\{x^2, x^4y, x^5y\}$	4	$\{x^4, x^3y, x^5y\}$	4	$\{x^6, x^3y, x^6y\}$	4
$\{x^2, x^4y, x^6y\}$	4	$\{x^4, x^3y, x^6y\}$	4	$\{x^6, x^4y, x^5y\}$	4
$\{x^2, x^5y, x^6y\}$	4	$\{x^4, x^4y, x^6y\}$	4	$\{x^6, x^5y, x^6y\}$	4

Next, we list down the subsets S' where $e(S') > 4$.

Table 4.36: $\{x^a, x^by, x^cy\} \subseteq S' \subseteq D_{14}$: $e(S') = 6 > 4$

S'	$e(S')$	S'	$e(S')$	S'	$e(S')$
$\{x, y, x^2y\}$	6	$\{x^3, y, xy\}$	6	$\{x^5, y, x^3y\}$	6
$\{x, y, x^5y\}$	6	$\{x^3, y, x^6y\}$	6	$\{x^5, y, x^4y\}$	6
$\{x, xy, x^3y\}$	6	$\{x^3, xy, x^2y\}$	6	$\{x^5, xy, x^4y\}$	6
$\{x, xy, x^6y\}$	6	$\{x^3, x^2y, x^3y\}$	6	$\{x^5, xy, x^5y\}$	6
$\{x, x^2y, x^4y\}$	6	$\{x^3, x^3y, x^4y\}$	6	$\{x^5, x^2y, x^5y\}$	6
$\{x, x^3y, x^5y\}$	6	$\{x^3, x^4y, x^5y\}$	6	$\{x^5, x^2y, x^6y\}$	6
$\{x, x^4y, x^6y\}$	6	$\{x^3, x^5y, x^6y\}$	6	$\{x^5, x^3y, x^6y\}$	6
$\{x^2, y, x^3y\}$	6	$\{x^4, y, xy\}$	6	$\{x^6, y, x^2y\}$	6
$\{x^2, y, x^4y\}$	6	$\{x^4, y, x^6y\}$	6	$\{x^6, y, x^5y\}$	6
$\{x^2, xy, x^4y\}$	6	$\{x^4, xy, x^2y\}$	6	$\{x^6, xy, x^3y\}$	6
$\{x^2, xy, x^5y\}$	6	$\{x^4, x^2y, x^3y\}$	6	$\{x^6, xy, x^6y\}$	6
$\{x^2, x^2y, x^5y\}$	6	$\{x^4, x^3y, x^4y\}$	6	$\{x^6, x^2y, x^4y\}$	6
$\{x^2, x^2y, x^6y\}$	6	$\{x^4, x^4y, x^5y\}$	6	$\{x^6, x^3y, x^5y\}$	6
$\{x^2, x^3y, x^6y\}$	6	$\{x^4, x^5y, x^6y\}$	6	$\{x^6, x^4y, x^6y\}$	6

Since $e(S') = 6$ for the subsets S' in Table 4.36, we consider the case where $|S''| = 4$ and $S'' = S' \cup \{x^iy^j\}$ for $i \in \{0, 1, \ldots, 6\}$ and $j \in \{0, 1\}$. There are two cases to consider:

(i) If $S'' = S' \cup \{x^dy\}$ for $d \in \{0, 1, \ldots, 6\}$, then S'' will contain one of the subsets in Table 4.35 and hence $e(S'') \leq 4$.

(ii) If $S'' = S' \cup \{x^d\}$ for $d \in \{1, 2, \ldots, 6\}$, then S'' will either contain one of the subsets in Table 4.31 or S'' will have been listed in Tables 4.33 and 4.34. Hence, it is clear from (a) that there does not exist any subset $S \subseteq D_{14}$ where $e(S) = 5$.

(c) Let $S' = \{x^ay, x^by, x^cy\} \subseteq S$ for $0 \leq a < b < c \leq 6$. By Proposition 4.22, $e(\{x^ay, x^by, x^cy\}) = \infty$. Hence, we consider the case where $|S''| = 4$ and $S'' = S' \cup \{x^iy^j\}$ for $i \in \{1, 2, \ldots, 6\}$ and $j \in \{0, 1\}$. Note that by Proposition 4.22, $e(\{y, xy, x^2y, x^3y, x^4y, x^5y, x^6y\}) = \infty$. If $|S| > 7$, then $e(S) = 2$. Therefore, we only consider adding the element x^d to S' for some positive integer d where $1 \leq d \leq 6$. Then we have $S'' =$

$\{x^a y, x^b y, x^c y, x^d\} \subseteq S$ and $\{x^d, x^a y, x^b y\} \subseteq \{x^a y, x^b y, x^c y, x^d\}$. From (b), there does not exist any subset $S \subseteq D_{14}$ where $e(S) = 5$.

From the three cases ((a), (b) and (c)) above, we see that when $1 \notin S$, there does not exist any subset S in D_{14} such that $e(S) = 5$.

Secondly, suppose that $1 \in S$. From the assumption that $e(S) = 5$, we have $x^u y \in S^5$ for all $u = 0, 1, 2, \ldots, 6$ and hence there exists $x^v y \in S$ for some $v = 0, 1, 2, \ldots, 6$. We consider the following two cases:

(1) $\{1, x^a, x^b y\} \subseteq S$ for $1 \leq a \leq 6$ and $0 \leq b \leq 6$

(2) $\{1, x^a y, x^b y\} \subseteq S$ for $0 \leq a < b \leq 6$

where a and b are integers. We shall explain the two cases in detail. Let m be a positive integer.

(1) Let $S' = \{1, x^a, x^b y\} \subseteq S$ for $1 \leq a \leq 6$ and $0 \leq b \leq 6$. Since $a = 1, \ldots, 6$, then $a \neq 7m$ and by Proposition 4.24, $e(\{1, x^a, x^b y\}) \leq 4 < 5$.

(2) Let $S' = \{1, x^a y, x^b y\} \subseteq S$ for $0 \leq a < b \leq 6$. Since $0 \leq a < b \leq 6$, then $b - a \neq 7m$ and by Proposition 4.25, we have $e(\{1, x^a y, x^b y\}) = 7$. Let $S'' = S' \cup \{x^i y^j\}$ where $i \in \{0, 1, \ldots, 6\}$, $j \in \{0, 1\}$ and $|S''| = 4$. Note that if $x^c \in S''$ for $1 \leq c \leq 6$, then $\{1, x^c, x^a y\} \subseteq S''$ and from (1), we have $e(\{1, x^a y, x^b y, x^c\}) < 5$. Hence, we only consider $S'' = S' \cup \{x^d y\}$ for $0 \leq d \leq 6$ and $b < d$.

Table 4.37: $\{1, x^a y, x^b y\} \subseteq S'' \subseteq D_{14}$: $e(S'') \leq 4$

S''	$e(S'')$	S''	$e(S'')$	S''	$e(S'')$
$\{1, y, xy, x^2y\}$	4	$\{1, y, x^4y, x^5y\}$	3	$\{1, xy, x^5y, x^6y\}$	3
$\{1, y, xy, x^3y\}$	3	$\{1, y, x^4y, x^6y\}$	3	$\{1, x^2y, x^3y, x^4y\}$	4
$\{1, y, xy, x^4y\}$	4	$\{1, y, x^5y, x^6y\}$	4	$\{1, x^2y, x^3y, x^5y\}$	3
$\{1, y, xy, x^5y\}$	3	$\{1, xy, x^2y, x^3y\}$	4	$\{1, x^2y, x^3y, x^6y\}$	4
$\{1, y, xy, x^6y\}$	4	$\{1, xy, x^2y, x^4y\}$	3	$\{1, x^2y, x^4y, x^5y\}$	3
$\{1, y, x^2y, x^3y\}$	3	$\{1, xy, x^2y, x^5y\}$	4	$\{1, x^2y, x^4y, x^6y\}$	4
$\{1, y, x^2y, x^4y\}$	4	$\{1, xy, x^2y, x^6y\}$	3	$\{1, x^2y, x^5y, x^6y\}$	4
$\{1, y, x^2y, x^5y\}$	4	$\{1, xy, x^3y, x^4y\}$	3	$\{1, x^3y, x^4y, x^5y\}$	4
$\{1, y, x^2y, x^6y\}$	3	$\{1, xy, x^3y, x^5y\}$	4	$\{1, x^3y, x^4y, x^6y\}$	3
$\{1, y, x^3y, x^4y\}$	4	$\{1, xy, x^3y, x^6y\}$	4	$\{1, x^3y, x^5y, x^6y\}$	3
$\{1, y, x^3y, x^5y\}$	4	$\{1, xy, x^4y, x^5y\}$	4	$\{1, x^4y, x^5y, x^6y\}$	4
$\{1, y, x^3y, x^6y\}$	4	$\{1, xy, x^4y, x^6y\}$	4		

From Table 4.37, we have $e(S'') < 5$.

From the two cases above, we see that when $1 \in S$, there does not exist any subset $S \subseteq D_{14}$ such that $e(S) = 5$.

Hence, we conclude that there does not exist any subset $S \subseteq D_{14}$ such that $e(S) = 5$. □

We have shown that there does not exist any subset $S \subseteq D_{14}$ such that $e(S) = 5$. Next, we consider a wider range of integers k for which there does not exist any subset $S \subseteq D_{2n}$ such that $e(S) = k$. In the following, we show that there does not exist any subset $S \subseteq D_{22}$ such that $e(S) = k$ for $k = 7, 8, 9$.

Theorem 4.28. *Let D_{22} be the dihedral group of order 22. Then there does not exist any subset $S \subseteq D_{22}$ such that $e(S) = k$ for $k = 7, 8, 9$.*

Proof. Given that D_{22} is the dihedral group of order 22, then we have $n = 11$. Suppose that there exists a subset $S \subseteq D_{22}$ where $e(S) = k$ for $k \in \{7, 8, 9\}$. By Proposition 4.23, it is clear that $|S| \geq 3$. Firstly, suppose that $1 \notin S$. From the assumption that $e(S) = k$ for $k \in \{7, 8, 9\}$, we have $x^u y \in S^k$ for all $u = 0, 1, 2, \ldots, 10$ and hence there exists $x^v y \in S$ for some $v = 0, 1, 2, \ldots, 10$. Then we shall consider three cases as follows:

(a) $\{x^a, x^b, x^c y\} \subseteq S$ for $1 \leq a < b \leq 10$ and $0 \leq c \leq 10$

(b) $\{x^a, x^b y, x^c y\} \subseteq S$ for $1 \leq a \leq 10$ and $0 \leq b < c \leq 10$

(c) $\{x^a y, x^b y, x^c y\} \subseteq S$ for $0 \leq a < b < c \leq 10$.

where a, b, c are integers. In the following, we shall explain the three cases separately in detail.

(a) Let $S' = \{x^a, x^b, x^c y\} \subseteq S$ for $1 \leq a < b \leq 10$ and $0 \leq c \leq 10$. We first list down the subsets S' where $e(S') \leq 6$.

Table 4.38: $\{x^a, x^b, x^c y\} \subseteq S' \subseteq D_{22}$: $e(S') \leq 6$

S'	$e(S')$	S'	$e(S')$	S'	$e(S')$
$\{x, x^2, y\}$	4	$\{x^2, x^8, x^4y\}$	5	$\{x^5, x^7, x^8y\}$	5
$\{x, x^2, xy\}$	4	$\{x^2, x^8, x^5y\}$	5	$\{x^5, x^7, x^9y\}$	5
$\{x, x^2, x^2y\}$	4	$\{x^2, x^8, x^6y\}$	5	$\{x^5, x^7, x^{10}y\}$	5
$\{x, x^2, x^3y\}$	4	$\{x^2, x^8, x^7y\}$	5	$\{x^5, x^8, y\}$	4
$\{x, x^2, x^4y\}$	4	$\{x^2, x^8, x^8y\}$	5	$\{x^5, x^8, xy\}$	4
$\{x, x^2, x^5y\}$	4	$\{x^2, x^8, x^9y\}$	5	$\{x^5, x^8, x^2y\}$	4
$\{x, x^2, x^6y\}$	4	$\{x^2, x^8, x^{10}y\}$	5	$\{x^5, x^8, x^3y\}$	4
$\{x, x^2, x^7y\}$	4	$\{x^2, x^{10}, y\}$	5	$\{x^5, x^8, x^4y\}$	4
$\{x, x^2, x^8y\}$	4	$\{x^2, x^{10}, xy\}$	5	$\{x^5, x^8, x^5y\}$	4
$\{x, x^2, x^9y\}$	4	$\{x^2, x^{10}, x^2y\}$	5	$\{x^5, x^8, x^6y\}$	4
$\{x, x^2, x^{10}y\}$	4	$\{x^2, x^{10}, x^3y\}$	5	$\{x^5, x^8, x^7y\}$	4
$\{x, x^3, y\}$	5	$\{x^2, x^{10}, x^4y\}$	5	$\{x^5, x^8, x^8y\}$	4
$\{x, x^3, xy\}$	5	$\{x^2, x^{10}, x^5y\}$	5	$\{x^5, x^8, x^9y\}$	4
$\{x, x^3, x^2y\}$	5	$\{x^2, x^{10}, x^6y\}$	5	$\{x^5, x^8, x^{10}y\}$	4
$\{x, x^3, x^3y\}$	5	$\{x^2, x^{10}, x^7y\}$	5	$\{x^5, x^9, y\}$	5
$\{x, x^3, x^4y\}$	5	$\{x^2, x^{10}, x^8y\}$	5	$\{x^5, x^9, xy\}$	5
$\{x, x^3, x^5y\}$	5	$\{x^2, x^{10}, x^9y\}$	5	$\{x^5, x^9, x^2y\}$	5
$\{x, x^3, x^6y\}$	5	$\{x^2, x^{10}, x^{10}y\}$	5	$\{x^5, x^9, x^3y\}$	5
$\{x, x^3, x^7y\}$	5	$\{x^3, x^4, y\}$	5	$\{x^5, x^9, x^4y\}$	5
$\{x, x^3, x^8y\}$	5	$\{x^3, x^4, xy\}$	5	$\{x^5, x^9, x^5y\}$	5
$\{x, x^3, x^9y\}$	5	$\{x^3, x^4, x^2y\}$	5	$\{x^5, x^9, x^6y\}$	5
$\{x, x^3, x^{10}y\}$	5	$\{x^3, x^4, x^3y\}$	5	$\{x^5, x^9, x^7y\}$	5
$\{x, x^4, y\}$	5	$\{x^3, x^4, x^4y\}$	5	$\{x^5, x^9, x^8y\}$	5
$\{x, x^4, xy\}$	5	$\{x^3, x^4, x^5y\}$	5	$\{x^5, x^9, x^9y\}$	5
$\{x, x^4, x^2y\}$	5	$\{x^3, x^4, x^6y\}$	5	$\{x^5, x^9, x^{10}y\}$	5
$\{x, x^4, x^3y\}$	5	$\{x^3, x^4, x^7y\}$	5	$\{x^5, x^{10}, y\}$	4
$\{x, x^4, x^4y\}$	5	$\{x^3, x^4, x^8y\}$	5	$\{x^5, x^{10}, xy\}$	4
$\{x, x^4, x^5y\}$	5	$\{x^3, x^4, x^9y\}$	5	$\{x^5, x^{10}, x^2y\}$	4
$\{x, x^4, x^6y\}$	5	$\{x^3, x^4, x^{10}y\}$	5	$\{x^5, x^{10}, x^3y\}$	4
$\{x, x^4, x^7y\}$	5	$\{x^3, x^5, y\}$	5	$\{x^5, x^{10}, x^4y\}$	4
$\{x, x^4, x^8y\}$	5	$\{x^3, x^5, xy\}$	5	$\{x^5, x^{10}, x^5y\}$	4
$\{x, x^4, x^9y\}$	5	$\{x^3, x^5, x^2y\}$	5	$\{x^5, x^{10}, x^6y\}$	4
$\{x, x^4, x^{10}y\}$	5	$\{x^3, x^5, x^3y\}$	5	$\{x^5, x^{10}, x^7y\}$	4
$\{x, x^5, y\}$	5	$\{x^3, x^5, x^4y\}$	5	$\{x^5, x^{10}, x^8y\}$	4
$\{x, x^5, xy\}$	5	$\{x^3, x^5, x^5y\}$	5	$\{x^5, x^{10}, x^9y\}$	4
$\{x, x^5, x^2y\}$	5	$\{x^3, x^5, x^6y\}$	5	$\{x^5, x^{10}, x^{10}y\}$	4
$\{x, x^5, x^3y\}$	5	$\{x^3, x^5, x^7y\}$	5	$\{x^6, x^7, y\}$	5
$\{x, x^5, x^4y\}$	5	$\{x^3, x^5, x^8y\}$	5	$\{x^6, x^7, xy\}$	5
$\{x, x^5, x^5y\}$	5	$\{x^3, x^5, x^9y\}$	5	$\{x^6, x^7, x^2y\}$	5
$\{x, x^5, x^6y\}$	5	$\{x^3, x^5, x^{10}y\}$	5	$\{x^6, x^7, x^3y\}$	5
$\{x, x^5, x^7y\}$	5	$\{x^3, x^6, y\}$	4	$\{x^6, x^7, x^4y\}$	5
$\{x, x^5, x^8y\}$	5	$\{x^3, x^6, xy\}$	4	$\{x^6, x^7, x^5y\}$	5
$\{x, x^5, x^9y\}$	5	$\{x^3, x^6, x^2y\}$	4	$\{x^6, x^7, x^6y\}$	5
$\{x, x^5, x^{10}y\}$	5	$\{x^3, x^6, x^3y\}$	4	$\{x^6, x^7, x^7y\}$	5
$\{x, x^6, y\}$	4	$\{x^3, x^6, x^4y\}$	4	$\{x^6, x^7, x^8y\}$	5
$\{x, x^6, xy\}$	4	$\{x^3, x^6, x^5y\}$	4	$\{x^6, x^7, x^9y\}$	5
$\{x, x^6, x^2y\}$	4	$\{x^3, x^6, x^6y\}$	4	$\{x^6, x^7, x^{10}y\}$	5
$\{x, x^6, x^3y\}$	4	$\{x^3, x^6, x^7y\}$	4	$\{x^6, x^8, y\}$	5
$\{x, x^6, x^4y\}$	4	$\{x^3, x^6, x^8y\}$	4	$\{x^6, x^8, xy\}$	5
$\{x, x^6, x^5y\}$	4	$\{x^3, x^6, x^9y\}$	4	$\{x^6, x^8, x^2y\}$	5
$\{x, x^6, x^6y\}$	4	$\{x^3, x^6, x^{10}y\}$	4	$\{x^6, x^8, x^3y\}$	5
$\{x, x^6, x^7y\}$	4	$\{x^3, x^7, y\}$	4	$\{x^6, x^8, x^4y\}$	5
$\{x, x^6, x^8y\}$	4	$\{x^3, x^7, xy\}$	4	$\{x^6, x^8, x^5y\}$	5
$\{x, x^6, x^9y\}$	4	$\{x^3, x^7, x^2y\}$	4	$\{x^6, x^8, x^6y\}$	5
$\{x, x^6, x^{10}y\}$	4	$\{x^3, x^7, x^3y\}$	4	$\{x^6, x^8, x^7y\}$	5

Table 4.38: (Continued)

S'	$e(S')$	S'	$e(S')$	S'	$e(S')$
$\{x, x^7, y\}$	5	$\{x^3, x^7, x^4y\}$	4	$\{x^6, x^8, x^8y\}$	5
$\{x, x^7, xy\}$	5	$\{x^3, x^7, x^5y\}$	4	$\{x^6, x^8, x^9y\}$	5
$\{x, x^7, x^2y\}$	5	$\{x^3, x^7, x^6y\}$	4	$\{x^6, x^8, x^{10}y\}$	5
$\{x, x^7, x^3y\}$	5	$\{x^3, x^7, x^7y\}$	4	$\{x^6, x^9, y\}$	5
$\{x, x^7, x^4y\}$	5	$\{x^3, x^7, x^8y\}$	4	$\{x^6, x^9, xy\}$	5
$\{x, x^7, x^5y\}$	5	$\{x^3, x^7, x^9y\}$	4	$\{x^6, x^9, x^2y\}$	5
$\{x, x^7, x^6y\}$	5	$\{x^3, x^7, x^{10}y\}$	4	$\{x^6, x^9, x^3y\}$	5
$\{x, x^7, x^7y\}$	5	$\{x^3, x^9, y\}$	5	$\{x^6, x^9, x^4y\}$	5
$\{x, x^7, x^8y\}$	5	$\{x^3, x^9, xy\}$	5	$\{x^6, x^9, x^5y\}$	5
$\{x, x^7, x^9y\}$	5	$\{x^3, x^9, x^2y\}$	5	$\{x^6, x^9, x^6y\}$	5
$\{x, x^7, x^{10}y\}$	5	$\{x^3, x^9, x^3y\}$	5	$\{x^6, x^9, x^7y\}$	5
$\{x, x^8, y\}$	5	$\{x^3, x^9, x^4y\}$	5	$\{x^6, x^9, x^8y\}$	5
$\{x, x^8, xy\}$	5	$\{x^3, x^9, x^5y\}$	5	$\{x^6, x^9, x^9y\}$	5
$\{x, x^8, x^2y\}$	5	$\{x^3, x^9, x^6y\}$	5	$\{x^6, x^9, x^{10}y\}$	5
$\{x, x^8, x^3y\}$	5	$\{x^3, x^9, x^7y\}$	5	$\{x^6, x^{10}, y\}$	5
$\{x, x^8, x^4y\}$	5	$\{x^3, x^9, x^8y\}$	5	$\{x^6, x^{10}, xy\}$	5
$\{x, x^8, x^5y\}$	5	$\{x^3, x^9, x^9y\}$	5	$\{x^6, x^{10}, x^2y\}$	5
$\{x, x^8, x^6y\}$	5	$\{x^3, x^9, x^{10}y\}$	5	$\{x^6, x^{10}, x^3y\}$	5
$\{x, x^8, x^7y\}$	5	$\{x^3, x^{10}, y\}$	5	$\{x^6, x^{10}, x^4y\}$	5
$\{x, x^8, x^8y\}$	5	$\{x^3, x^{10}, xy\}$	5	$\{x^6, x^{10}, x^5y\}$	5
$\{x, x^8, x^9y\}$	5	$\{x^3, x^{10}, x^2y\}$	5	$\{x^6, x^{10}, x^6y\}$	5
$\{x, x^8, x^{10}y\}$	5	$\{x^3, x^{10}, x^3y\}$	5	$\{x^6, x^{10}, x^7y\}$	5
$\{x, x^9, y\}$	5	$\{x^3, x^{10}, x^4y\}$	5	$\{x^6, x^{10}, x^8y\}$	5
$\{x, x^9, xy\}$	5	$\{x^3, x^{10}, x^5y\}$	5	$\{x^6, x^{10}, x^9y\}$	5
$\{x, x^9, x^2y\}$	5	$\{x^3, x^{10}, x^6y\}$	5	$\{x^6, x^{10}, x^{10}y\}$	5
$\{x, x^9, x^3y\}$	5	$\{x^3, x^{10}, x^7y\}$	5	$\{x^7, x^8, y\}$	5
$\{x, x^9, x^4y\}$	5	$\{x^3, x^{10}, x^8y\}$	5	$\{x^7, x^8, xy\}$	5
$\{x, x^9, x^5y\}$	5	$\{x^3, x^{10}, x^9y\}$	5	$\{x^7, x^8, x^2y\}$	5
$\{x, x^9, x^6y\}$	5	$\{x^3, x^{10}, x^{10}y\}$	5	$\{x^7, x^8, x^3y\}$	5
$\{x, x^9, x^7y\}$	5	$\{x^4, x^5, y\}$	5	$\{x^7, x^8, x^4y\}$	5
$\{x, x^9, x^8y\}$	5	$\{x^4, x^5, xy\}$	5	$\{x^7, x^8, x^5y\}$	5
$\{x, x^9, x^9y\}$	5	$\{x^4, x^5, x^2y\}$	5	$\{x^7, x^8, x^6y\}$	5
$\{x, x^9, x^{10}y\}$	5	$\{x^4, x^5, x^3y\}$	5	$\{x^7, x^8, x^7y\}$	5
$\{x^2, x^3, y\}$	5	$\{x^4, x^5, x^4y\}$	5	$\{x^7, x^8, x^8y\}$	5
$\{x^2, x^3, xy\}$	5	$\{x^4, x^5, x^5y\}$	5	$\{x^7, x^8, x^9y\}$	5
$\{x^2, x^3, x^2y\}$	5	$\{x^4, x^5, x^6y\}$	5	$\{x^7, x^8, x^{10}y\}$	5
$\{x^2, x^3, x^3y\}$	5	$\{x^4, x^5, x^7y\}$	5	$\{x^7, x^9, y\}$	4
$\{x^2, x^3, x^4y\}$	5	$\{x^4, x^5, x^8y\}$	5	$\{x^7, x^9, xy\}$	4
$\{x^2, x^3, x^5y\}$	5	$\{x^4, x^5, x^9y\}$	5	$\{x^7, x^9, x^2y\}$	4
$\{x^2, x^3, x^6y\}$	5	$\{x^4, x^5, x^{10}y\}$	5	$\{x^7, x^9, x^3y\}$	4
$\{x^2, x^3, x^7y\}$	5	$\{x^4, x^6, y\}$	5	$\{x^7, x^9, x^4y\}$	4
$\{x^2, x^3, x^8y\}$	5	$\{x^4, x^6, xy\}$	5	$\{x^7, x^9, x^5y\}$	4
$\{x^2, x^3, x^9y\}$	5	$\{x^4, x^6, x^2y\}$	5	$\{x^7, x^9, x^6y\}$	4
$\{x^2, x^3, x^{10}y\}$	5	$\{x^4, x^6, x^3y\}$	5	$\{x^7, x^9, x^7y\}$	4
$\{x^2, x^4, y\}$	4	$\{x^4, x^6, x^4y\}$	5	$\{x^7, x^9, x^8y\}$	4
$\{x^2, x^4, xy\}$	4	$\{x^4, x^6, x^5y\}$	5	$\{x^7, x^9, x^9y\}$	4
$\{x^2, x^4, x^2y\}$	4	$\{x^4, x^6, x^6y\}$	5	$\{x^7, x^9, x^{10}y\}$	4
$\{x^2, x^4, x^3y\}$	4	$\{x^4, x^6, x^7y\}$	5	$\{x^7, x^{10}, y\}$	5
$\{x^2, x^4, x^4y\}$	4	$\{x^4, x^6, x^8y\}$	5	$\{x^7, x^{10}, xy\}$	5
$\{x^2, x^4, x^5y\}$	4	$\{x^4, x^6, x^9y\}$	5	$\{x^7, x^{10}, x^2y\}$	5
$\{x^2, x^4, x^6y\}$	4	$\{x^4, x^6, x^{10}y\}$	5	$\{x^7, x^{10}, x^3y\}$	5
$\{x^2, x^4, x^7y\}$	4	$\{x^4, x^8, y\}$	4	$\{x^7, x^{10}, x^4y\}$	5
$\{x^2, x^4, x^8y\}$	4	$\{x^4, x^8, xy\}$	4	$\{x^7, x^{10}, x^5y\}$	5
$\{x^2, x^4, x^9y\}$	4	$\{x^4, x^8, x^2y\}$	4	$\{x^7, x^{10}, x^6y\}$	5
$\{x^2, x^4, x^{10}y\}$	4	$\{x^4, x^8, x^3y\}$	4	$\{x^7, x^{10}, x^7y\}$	5

Table 4.38: (Continued)

S'	$e(S')$	S'	$e(S')$	S'	$e(S')$
$\{x^2, x^5, y\}$	5	$\{x^4, x^8, x^4y\}$	4	$\{x^7, x^{10}, x^8y\}$	5
$\{x^2, x^5, xy\}$	5	$\{x^4, x^8, x^5y\}$	4	$\{x^7, x^{10}, x^9y\}$	5
$\{x^2, x^5, x^2y\}$	5	$\{x^4, x^8, x^6y\}$	4	$\{x^7, x^{10}, x^{10}y\}$	5
$\{x^2, x^5, x^3y\}$	5	$\{x^4, x^8, x^7y\}$	4	$\{x^8, x^9, y\}$	5
$\{x^2, x^5, x^4y\}$	5	$\{x^4, x^8, x^8y\}$	4	$\{x^8, x^9, xy\}$	5
$\{x^2, x^5, x^5y\}$	5	$\{x^4, x^8, x^9y\}$	4	$\{x^8, x^9, x^2y\}$	5
$\{x^2, x^5, x^6y\}$	5	$\{x^4, x^8, x^{10}y\}$	4	$\{x^8, x^9, x^3y\}$	5
$\{x^2, x^5, x^7y\}$	5	$\{x^4, x^9, y\}$	5	$\{x^8, x^9, x^4y\}$	5
$\{x^2, x^5, x^8y\}$	5	$\{x^4, x^9, xy\}$	5	$\{x^8, x^9, x^5y\}$	5
$\{x^2, x^5, x^9y\}$	5	$\{x^4, x^9, x^2y\}$	5	$\{x^8, x^9, x^6y\}$	5
$\{x^2, x^5, x^{10}y\}$	5	$\{x^4, x^9, x^3y\}$	5	$\{x^8, x^9, x^7y\}$	5
$\{x^2, x^6, y\}$	5	$\{x^4, x^9, x^4y\}$	5	$\{x^8, x^9, x^8y\}$	5
$\{x^2, x^6, xy\}$	5	$\{x^4, x^9, x^5y\}$	5	$\{x^8, x^9, x^9y\}$	5
$\{x^2, x^6, x^2y\}$	5	$\{x^4, x^9, x^6y\}$	5	$\{x^8, x^9, x^{10}y\}$	5
$\{x^2, x^6, x^3y\}$	5	$\{x^4, x^9, x^7y\}$	5	$\{x^8, x^{10}, y\}$	5
$\{x^2, x^6, x^4y\}$	5	$\{x^4, x^9, x^8y\}$	5	$\{x^8, x^{10}, xy\}$	5
$\{x^2, x^6, x^5y\}$	5	$\{x^4, x^9, x^9y\}$	5	$\{x^8, x^{10}, x^2y\}$	5
$\{x^2, x^6, x^6y\}$	5	$\{x^4, x^9, x^{10}y\}$	5	$\{x^8, x^{10}, x^3y\}$	5
$\{x^2, x^6, x^7y\}$	5	$\{x^4, x^{10}, y\}$	5	$\{x^8, x^{10}, x^4y\}$	5
$\{x^2, x^6, x^8y\}$	5	$\{x^4, x^{10}, xy\}$	5	$\{x^8, x^{10}, x^5y\}$	5
$\{x^2, x^6, x^9y\}$	5	$\{x^4, x^{10}, x^2y\}$	5	$\{x^8, x^{10}, x^6y\}$	5
$\{x^2, x^6, x^{10}y\}$	5	$\{x^4, x^{10}, x^3y\}$	5	$\{x^8, x^{10}, x^7y\}$	5
$\{x^2, x^7, y\}$	5	$\{x^4, x^{10}, x^4y\}$	5	$\{x^8, x^{10}, x^8y\}$	5
$\{x^2, x^7, xy\}$	5	$\{x^4, x^{10}, x^5y\}$	5	$\{x^8, x^{10}, x^9y\}$	5
$\{x^2, x^7, x^2y\}$	5	$\{x^4, x^{10}, x^6y\}$	5	$\{x^8, x^{10}, x^{10}y\}$	5
$\{x^2, x^7, x^3y\}$	5	$\{x^4, x^{10}, x^7y\}$	5	$\{x^9, x^{10}, y\}$	4
$\{x^2, x^7, x^4y\}$	5	$\{x^4, x^{10}, x^8y\}$	5	$\{x^9, x^{10}, xy\}$	4
$\{x^2, x^7, x^5y\}$	5	$\{x^4, x^{10}, x^9y\}$	5	$\{x^9, x^{10}, x^2y\}$	4
$\{x^2, x^7, x^6y\}$	5	$\{x^4, x^{10}, x^{10}y\}$	5	$\{x^9, x^{10}, x^3y\}$	4
$\{x^2, x^7, x^7y\}$	5	$\{x^5, x^7, y\}$	5	$\{x^9, x^{10}, x^4y\}$	4
$\{x^2, x^7, x^8y\}$	5	$\{x^5, x^7, xy\}$	5	$\{x^9, x^{10}, x^5y\}$	4
$\{x^2, x^7, x^9y\}$	5	$\{x^5, x^7, x^2y\}$	5	$\{x^9, x^{10}, x^6y\}$	4
$\{x^2, x^7, x^{10}y\}$	5	$\{x^5, x^7, x^3y\}$	5	$\{x^9, x^{10}, x^7y\}$	4
$\{x^2, x^8, y\}$	5	$\{x^5, x^7, x^4y\}$	5	$\{x^9, x^{10}, x^8y\}$	4
$\{x^2, x^8, xy\}$	5	$\{x^5, x^7, x^5y\}$	5	$\{x^9, x^{10}, x^9y\}$	4
$\{x^2, x^8, x^2y\}$	5	$\{x^5, x^7, x^6y\}$	5	$\{x^9, x^{10}, x^{10}y\}$	4
$\{x^2, x^8, x^3y\}$	5	$\{x^5, x^7, x^7y\}$	5		

By Proposition 4.5, we see that if $S' \subseteq S$, then $e(S) \leq e(S') \leq 6$. Next, we list down the subsets S' where $e(S') > 6$. Note that these subsets are of the form $S' = \{x^a, x^b, x^cy\}$ where $1 \leq a < b \leq 10$, $0 \leq c \leq 10$ and $a + b = 11$. By Proposition 4.15, we have $e(S') = 11$ as shown in Table 4.39.

Table 4.39: $\{x^a, x^b, x^c y\} \subseteq S' \subseteq D_{22}$: $e(S') = 11$

S'	$e(S')$	S'	$e(S')$	S'	$e(S')$
$\{x, x^{10}, y\}$	11	$\{x^2, x^9, x^8y\}$	11	$\{x^4, x^7, x^5y\}$	11
$\{x, x^{10}, xy\}$	11	$\{x^2, x^9, x^9y\}$	11	$\{x^4, x^7, x^6y\}$	11
$\{x, x^{10}, x^2y\}$	11	$\{x^2, x^9, x^{10}y\}$	11	$\{x^4, x^7, x^7y\}$	11
$\{x, x^{10}, x^3y\}$	11	$\{x^3, x^8, y\}$	11	$\{x^4, x^7, x^8y\}$	11
$\{x, x^{10}, x^4y\}$	11	$\{x^3, x^8, xy\}$	11	$\{x^4, x^7, x^9y\}$	11
$\{x, x^{10}, x^5y\}$	11	$\{x^3, x^8, x^2y\}$	11	$\{x^4, x^7, x^{10}y\}$	11
$\{x, x^{10}, x^6y\}$	11	$\{x^3, x^8, x^3y\}$	11	$\{x^5, x^6, y\}$	11
$\{x, x^{10}, x^7y\}$	11	$\{x^3, x^8, x^4y\}$	11	$\{x^5, x^6, xy\}$	11
$\{x, x^{10}, x^8y\}$	11	$\{x^3, x^8, x^5y\}$	11	$\{x^5, x^6, x^2y\}$	11
$\{x, x^{10}, x^9y\}$	11	$\{x^3, x^8, x^6y\}$	11	$\{x^5, x^6, x^3y\}$	11
$\{x, x^{10}, x^{10}y\}$	11	$\{x^3, x^8, x^7y\}$	11	$\{x^5, x^6, x^4y\}$	11
$\{x^2, x^9, y\}$	11	$\{x^3, x^8, x^8y\}$	11	$\{x^5, x^6, x^5y\}$	11
$\{x^2, x^9, xy\}$	11	$\{x^3, x^8, x^9y\}$	11	$\{x^5, x^6, x^6y\}$	11
$\{x^2, x^9, x^2y\}$	11	$\{x^3, x^8, x^{10}y\}$	11	$\{x^5, x^6, x^7y\}$	11
$\{x^2, x^9, x^3y\}$	11	$\{x^4, x^7, y\}$	11	$\{x^5, x^6, x^8y\}$	11
$\{x^2, x^9, x^4y\}$	11	$\{x^4, x^7, xy\}$	11	$\{x^5, x^6, x^9y\}$	11
$\{x^2, x^9, x^5y\}$	11	$\{x^4, x^7, x^2y\}$	11	$\{x^5, x^6, x^{10}y\}$	11
$\{x^2, x^9, x^6y\}$	11	$\{x^4, x^7, x^3y\}$	11		
$\{x^2, x^9, x^7y\}$	11	$\{x^4, x^7, x^4y\}$	11		

Since $e(S') = 11$ for the subsets S' in Table 4.39, we consider the case where $|S''| = 4$ and $S'' = S' \cup \{x^i y^j\}$ for $i \in \{0, 1, \ldots, 10\}$ and $j \in \{0, 1\}$. Note that by Proposition 4.5, if $S' \subseteq S''$, then $e(S'') \leq e(S')$. Therefore, in the following we only consider the subsets S'' that do not contain any of the subsets S' listed in Table 4.38. Since $S' \cup \{x^i\}$ for $i \in \{1, 2, \ldots, 10\}$ contains one of the subsets in Table 4.38, we only consider the subsets $S'' = S' \cup \{x^i y\}$ for $i \in \{0, 1, \ldots, 10\}$. We first list down the subsets S'' where $e(S'') \leq 6$.

Table 4.40: $\{x^a, x^b, x^c y\} \subseteq S'' \subseteq D_{22}$: $e(S'') \leq 6$

S''	$e(S'')$	S''	$e(S'')$	S''	$e(S'')$
$\{x, x^{10}, y, xy\}$	6	$\{x^2, x^9, x^4y, x^{10}y\}$	4	$\{x^4, x^7, x^2y, x^4y\}$	4
$\{x, x^{10}, y, x^3y\}$	4	$\{x^2, x^9, x^5y, x^6y\}$	4	$\{x^4, x^7, x^2y, x^6y\}$	6
$\{x, x^{10}, y, x^4y\}$	6	$\{x^2, x^9, x^5y, x^7y\}$	6	$\{x^4, x^7, x^2y, x^7y\}$	6
$\{x, x^{10}, y, x^5y\}$	4	$\{x^2, x^9, x^5y, x^8y\}$	6	$\{x^4, x^7, x^2y, x^8y\}$	6
$\{x, x^{10}, y, x^6y\}$	4	$\{x^2, x^9, x^5y, x^{10}y\}$	4	$\{x^4, x^7, x^2y, x^9y\}$	6
$\{x, x^{10}, y, x^7y\}$	6	$\{x^2, x^9, x^6y, x^7y\}$	4	$\{x^4, x^7, x^3y, x^4y\}$	4
$\{x, x^{10}, y, x^8y\}$	4	$\{x^2, x^9, x^6y, x^8y\}$	6	$\{x^4, x^7, x^3y, x^5y\}$	4
$\{x, x^{10}, y, x^{10}y\}$	6	$\{x^2, x^9, x^6y, x^9y\}$	6	$\{x^4, x^7, x^3y, x^7y\}$	6
$\{x, x^{10}, xy, x^2y\}$	6	$\{x^2, x^9, x^7y, x^8y\}$	4	$\{x^4, x^7, x^3y, x^8y\}$	6
$\{x, x^{10}, xy, x^4y\}$	4	$\{x^2, x^9, x^7y, x^9y\}$	6	$\{x^4, x^7, x^3y, x^9y\}$	6
$\{x, x^{10}, xy, x^5y\}$	6	$\{x^2, x^9, x^7y, x^{10}y\}$	6	$\{x^4, x^7, x^3y, x^{10}y\}$	6

Table 4.40: (Continued)

S''	$e(S'')$	S''	$e(S'')$	S''	$e(S'')$
$\{x, x^{10}, xy, x^6y\}$	4	$\{x^2, x^9, x^8y, x^9y\}$	4	$\{x^4, x^7, x^4y, x^5y\}$	4
$\{x, x^{10}, xy, x^7y\}$	4	$\{x^2, x^9, x^8y, x^{10}y\}$	6	$\{x^4, x^7, x^4y, x^6y\}$	4
$\{x, x^{10}, xy, x^8y\}$	6	$\{x^2, x^9, x^9y, x^{10}y\}$	4	$\{x^4, x^7, x^4y, x^8y\}$	6
$\{x, x^{10}, xy, x^9y\}$	4	$\{x^3, x^8, y, xy\}$	6	$\{x^4, x^7, x^4y, x^9y\}$	6
$\{x, x^{10}, x^2y, x^3y\}$	6	$\{x^3, x^8, y, x^2y\}$	4	$\{x^4, x^7, x^4y, x^{10}y\}$	6
$\{x, x^{10}, x^2y, x^5y\}$	4	$\{x^3, x^8, y, x^3y\}$	6	$\{x^4, x^7, x^5y, x^6y\}$	4
$\{x, x^{10}, x^2y, x^6y\}$	6	$\{x^3, x^8, y, x^4y\}$	4	$\{x^4, x^7, x^5y, x^7y\}$	4
$\{x, x^{10}, x^2y, x^7y\}$	4	$\{x^3, x^8, y, x^7y\}$	4	$\{x^4, x^7, x^5y, x^9y\}$	6
$\{x, x^{10}, x^2y, x^8y\}$	4	$\{x^3, x^8, y, x^8y\}$	6	$\{x^4, x^7, x^5y, x^{10}y\}$	6
$\{x, x^{10}, x^2y, x^9y\}$	6	$\{x^3, x^8, y, x^9y\}$	4	$\{x^4, x^7, x^6y, x^7y\}$	4
$\{x, x^{10}, x^2y, x^{10}y\}$	4	$\{x^3, x^8, y, x^{10}y\}$	6	$\{x^4, x^7, x^6y, x^8y\}$	4
$\{x, x^{10}, x^3y, x^4y\}$	6	$\{x^3, x^8, xy, x^2y\}$	6	$\{x^4, x^7, x^6y, x^{10}y\}$	6
$\{x, x^{10}, x^3y, x^6y\}$	4	$\{x^3, x^8, xy, x^3y\}$	4	$\{x^4, x^7, x^7y, x^8y\}$	4
$\{x, x^{10}, x^3y, x^7y\}$	6	$\{x^3, x^8, xy, x^4y\}$	6	$\{x^4, x^7, x^7y, x^9y\}$	4
$\{x, x^{10}, x^3y, x^8y\}$	4	$\{x^3, x^8, xy, x^5y\}$	4	$\{x^4, x^7, x^8y, x^9y\}$	4
$\{x, x^{10}, x^3y, x^9y\}$	4	$\{x^3, x^8, xy, x^8y\}$	4	$\{x^4, x^7, x^8y, x^{10}y\}$	4
$\{x, x^{10}, x^3y, x^{10}y\}$	6	$\{x^3, x^8, xy, x^9y\}$	6	$\{x^4, x^7, x^9y, x^{10}y\}$	4
$\{x, x^{10}, x^4y, x^5y\}$	6	$\{x^3, x^8, xy, x^{10}y\}$	4	$\{x^5, x^6, y, x^2y\}$	6
$\{x, x^{10}, x^4y, x^7y\}$	4	$\{x^3, x^8, x^2y, x^3y\}$	6	$\{x^5, x^6, y, x^3y\}$	4
$\{x, x^{10}, x^4y, x^8y\}$	6	$\{x^3, x^8, x^2y, x^4y\}$	4	$\{x^5, x^6, y, x^4y\}$	4
$\{x, x^{10}, x^4y, x^9y\}$	4	$\{x^3, x^8, x^2y, x^5y\}$	6	$\{x^5, x^6, y, x^5y\}$	6
$\{x, x^{10}, x^4y, x^{10}y\}$	4	$\{x^3, x^8, x^2y, x^6y\}$	4	$\{x^5, x^6, y, x^6y\}$	6
$\{x, x^{10}, x^5y, x^6y\}$	6	$\{x^3, x^8, x^2y, x^9y\}$	4	$\{x^5, x^6, y, x^7y\}$	4
$\{x, x^{10}, x^5y, x^8y\}$	4	$\{x^3, x^8, x^2y, x^{10}y\}$	6	$\{x^5, x^6, y, x^8y\}$	4
$\{x, x^{10}, x^5y, x^9y\}$	6	$\{x^3, x^8, x^3y, x^4y\}$	6	$\{x^5, x^6, y, x^9y\}$	6
$\{x, x^{10}, x^5y, x^{10}y\}$	4	$\{x^3, x^8, x^3y, x^5y\}$	4	$\{x^5, x^6, xy, x^3y\}$	6
$\{x, x^{10}, x^6y, x^7y\}$	6	$\{x^3, x^8, x^3y, x^6y\}$	6	$\{x^5, x^6, xy, x^4y\}$	4
$\{x, x^{10}, x^6y, x^9y\}$	4	$\{x^3, x^8, x^3y, x^7y\}$	4	$\{x^5, x^6, xy, x^5y\}$	4
$\{x, x^{10}, x^6y, x^{10}y\}$	6	$\{x^3, x^8, x^3y, x^{10}y\}$	4	$\{x^5, x^6, xy, x^6y\}$	6
$\{x, x^{10}, x^7y, x^8y\}$	6	$\{x^3, x^8, x^4y, x^5y\}$	6	$\{x^5, x^6, xy, x^7y\}$	6
$\{x, x^{10}, x^7y, x^{10}y\}$	4	$\{x^3, x^8, x^4y, x^6y\}$	4	$\{x^5, x^6, xy, x^8y\}$	4
$\{x, x^{10}, x^8y, x^9y\}$	6	$\{x^3, x^8, x^4y, x^7y\}$	6	$\{x^5, x^6, xy, x^9y\}$	4
$\{x, x^{10}, x^9y, x^{10}y\}$	6	$\{x^3, x^8, x^4y, x^8y\}$	4	$\{x^5, x^6, xy, x^{10}y\}$	6
$\{x^2, x^9, y, xy\}$	4	$\{x^3, x^8, x^5y, x^6y\}$	6	$\{x^5, x^6, x^2y, x^4y\}$	6
$\{x^2, x^9, y, x^2y\}$	6	$\{x^3, x^8, x^5y, x^7y\}$	4	$\{x^5, x^6, x^2y, x^5y\}$	4
$\{x^2, x^9, y, x^3y\}$	6	$\{x^3, x^8, x^5y, x^8y\}$	6	$\{x^5, x^6, x^2y, x^6y\}$	4
$\{x^2, x^9, y, x^5y\}$	4	$\{x^3, x^8, x^5y, x^9y\}$	4	$\{x^5, x^6, x^2y, x^7y\}$	6
$\{x^2, x^9, y, x^6y\}$	4	$\{x^3, x^8, x^6y, x^7y\}$	6	$\{x^5, x^6, x^2y, x^8y\}$	6
$\{x^2, x^9, y, x^8y\}$	6	$\{x^3, x^8, x^6y, x^8y\}$	4	$\{x^5, x^6, x^2y, x^9y\}$	4
$\{x^2, x^9, y, x^9y\}$	6	$\{x^3, x^8, x^6y, x^9y\}$	6	$\{x^5, x^6, x^2y, x^{10}y\}$	4
$\{x^2, x^9, y, x^{10}y\}$	4	$\{x^3, x^8, x^6y, x^{10}y\}$	4	$\{x^5, x^6, x^3y, x^5y\}$	6
$\{x^2, x^9, xy, x^2y\}$	4	$\{x^3, x^8, x^7y, x^8y\}$	6	$\{x^5, x^6, x^3y, x^6y\}$	4
$\{x^2, x^9, xy, x^3y\}$	6	$\{x^3, x^8, x^7y, x^9y\}$	4	$\{x^5, x^6, x^3y, x^7y\}$	4
$\{x^2, x^9, xy, x^4y\}$	6	$\{x^3, x^8, x^7y, x^{10}y\}$	6	$\{x^5, x^6, x^3y, x^8y\}$	6
$\{x^2, x^9, xy, x^6y\}$	4	$\{x^3, x^8, x^8y, x^9y\}$	6	$\{x^5, x^6, x^3y, x^9y\}$	6
$\{x^2, x^9, xy, x^7y\}$	4	$\{x^3, x^8, x^8y, x^{10}y\}$	4	$\{x^5, x^6, x^3y, x^{10}y\}$	4
$\{x^2, x^9, xy, x^9y\}$	6	$\{x^3, x^8, x^9y, x^{10}y\}$	6	$\{x^5, x^6, x^4y, x^6y\}$	6
$\{x^2, x^9, xy, x^{10}y\}$	6	$\{x^4, x^7, y, xy\}$	4	$\{x^5, x^6, x^4y, x^7y\}$	4
$\{x^2, x^9, x^2y, x^3y\}$	4	$\{x^4, x^7, y, x^2y\}$	4	$\{x^5, x^6, x^4y, x^8y\}$	4
$\{x^2, x^9, x^2y, x^4y\}$	6	$\{x^4, x^7, y, x^4y\}$	6	$\{x^5, x^6, x^4y, x^9y\}$	6
$\{x^2, x^9, x^2y, x^5y\}$	6	$\{x^4, x^7, y, x^5y\}$	6	$\{x^5, x^6, x^4y, x^{10}y\}$	6
$\{x^2, x^9, x^2y, x^7y\}$	4	$\{x^4, x^7, y, x^6y\}$	6	$\{x^5, x^6, x^5y, x^7y\}$	6
$\{x^2, x^9, x^2y, x^8y\}$	4	$\{x^4, x^7, y, x^7y\}$	6	$\{x^5, x^6, x^5y, x^8y\}$	4
$\{x^2, x^9, x^2y, x^{10}y\}$	6	$\{x^4, x^7, y, x^9y\}$	4	$\{x^5, x^6, x^5y, x^9y\}$	4
$\{x^2, x^9, x^3y, x^4y\}$	4	$\{x^4, x^7, y, x^{10}y\}$	4	$\{x^5, x^6, x^5y, x^{10}y\}$	6

Table 4.40: (Continued)

S''	$e(S'')$	S''	$e(S'')$	S''	$e(S'')$
$\{x^2, x^9, x^3y, x^5y\}$	6	$\{x^4, x^7, xy, x^2y\}$	4	$\{x^5, x^6, x^6y, x^8y\}$	6
$\{x^2, x^9, x^3y, x^6y\}$	6	$\{x^4, x^7, xy, x^3y\}$	4	$\{x^5, x^6, x^6y, x^9y\}$	4
$\{x^2, x^9, x^3y, x^8y\}$	4	$\{x^4, x^7, xy, x^5y\}$	6	$\{x^5, x^6, x^6y, x^{10}y\}$	4
$\{x^2, x^9, x^3y, x^9y\}$	4	$\{x^4, x^7, xy, x^6y\}$	6	$\{x^5, x^6, x^7y, x^9y\}$	6
$\{x^2, x^9, x^4y, x^5y\}$	4	$\{x^4, x^7, xy, x^7y\}$	6	$\{x^5, x^6, x^7y, x^{10}y\}$	4
$\{x^2, x^9, x^4y, x^6y\}$	6	$\{x^4, x^7, xy, x^8y\}$	6	$\{x^5, x^6, x^8y, x^{10}y\}$	6
$\{x^2, x^9, x^4y, x^7y\}$	6	$\{x^4, x^7, xy, x^{10}y\}$	4		
$\{x^2, x^9, x^4y, x^9y\}$	4	$\{x^4, x^7, x^2y, x^3y\}$	4		

Next, we list down the subsets S'' where $e(S'') > 6$.

Table 4.41: $\{x^a, x^b, x^cy\} \subseteq S'' \subseteq D_{22}$: $e(S'') > 6$

S''	$e(S'')$	S''	$e(S'')$	S''	$e(S'')$
$\{x, x^{10}, y, x^2y\}$	10	$\{x^2, x^9, x^4y, x^8y\}$	10	$\{x^4, x^7, x^2y, x^{10}y\}$	10
$\{x, x^{10}, y, x^9y\}$	10	$\{x^2, x^9, x^5y, x^9y\}$	10	$\{x^4, x^7, x^3y, x^6y\}$	10
$\{x, x^{10}, xy, x^3y\}$	10	$\{x^2, x^9, x^6y, x^{10}y\}$	10	$\{x^4, x^7, x^4y, x^7y\}$	10
$\{x, x^{10}, xy, x^{10}y\}$	10	$\{x^3, x^8, y, x^5y\}$	10	$\{x^4, x^7, x^5y, x^8y\}$	10
$\{x, x^{10}, x^2y, x^4y\}$	10	$\{x^3, x^8, y, x^6y\}$	10	$\{x^4, x^7, x^6y, x^9y\}$	10
$\{x, x^{10}, x^3y, x^5y\}$	10	$\{x^3, x^8, xy, x^6y\}$	10	$\{x^4, x^7, x^7y, x^{10}y\}$	10
$\{x, x^{10}, x^4y, x^6y\}$	10	$\{x^3, x^8, xy, x^7y\}$	10	$\{x^5, x^6, y, xy\}$	10
$\{x, x^{10}, x^5y, x^7y\}$	10	$\{x^3, x^8, x^2y, x^7y\}$	10	$\{x^5, x^6, y, x^{10}y\}$	10
$\{x, x^{10}, x^6y, x^8y\}$	10	$\{x^3, x^8, x^2y, x^8y\}$	10	$\{x^5, x^6, xy, x^2y\}$	10
$\{x, x^{10}, x^7y, x^9y\}$	10	$\{x^3, x^8, x^3y, x^8y\}$	10	$\{x^5, x^6, x^2y, x^3y\}$	10
$\{x, x^{10}, x^8y, x^{10}y\}$	10	$\{x^3, x^8, x^3y, x^9y\}$	10	$\{x^5, x^6, x^3y, x^4y\}$	10
$\{x^2, x^9, y, x^4y\}$	10	$\{x^3, x^8, x^4y, x^9y\}$	10	$\{x^5, x^6, x^4y, x^5y\}$	10
$\{x^2, x^9, y, x^7y\}$	10	$\{x^3, x^8, x^4y, x^{10}y\}$	10	$\{x^5, x^6, x^5y, x^6y\}$	10
$\{x^2, x^9, xy, x^5y\}$	10	$\{x^3, x^8, x^5y, x^{10}y\}$	10	$\{x^5, x^6, x^6y, x^7y\}$	10
$\{x^2, x^9, xy, x^8y\}$	10	$\{x^4, x^7, y, x^3y\}$	10	$\{x^5, x^6, x^7y, x^8y\}$	10
$\{x^2, x^9, x^2y, x^6y\}$	10	$\{x^4, x^7, y, x^8y\}$	10	$\{x^5, x^6, x^8y, x^9y\}$	10
$\{x^2, x^9, x^2y, x^9y\}$	10	$\{x^4, x^7, xy, x^4y\}$	10	$\{x^5, x^6, x^9y, x^{10}y\}$	10
$\{x^2, x^9, x^3y, x^7y\}$	10	$\{x^4, x^7, xy, x^9y\}$	10		
$\{x^2, x^9, x^3y, x^{10}y\}$	10	$\{x^4, x^7, x^2y, x^5y\}$	10		

Let $S''' = S'' \cup \{x^iy^j\}$ for $i \in \{1, 2, \ldots, 10\}$ and $j \in \{0, 1\}$ where S'' is in Table 4.41 and $|S'''| = 5$. Then we have the following cases:

(i) If $S''' = S'' \cup \{x^i\}$ for $i \in \{1, 2, \ldots, 10\}$, then S''' will contain one of the subsets in Table 4.38.

(ii) If $S''' = S'' \cup \{x^iy\}$ for $i \in \{0, 1, \ldots, 10\}$, then S''' will contain one of the subsets in Table 4.40.

By Proposition 4.5, we have $e(S''') < k$ for $k \in \{7, 8, 9\}$ and hence there does not exist any subset S where $e(S) = k$ for $k \in \{7, 8, 9\}$.

(b) Let $S' = \{x^a, x^b y, x^c y\} \subseteq S$ for $1 \leq a \leq 10$ and $0 \leq b < c \leq 10$. We first list down the subsets S' where $e(S') \leq 6$.

Table 4.42: $\{x^a, x^b y, x^c y\} \subseteq S' \subseteq D_{22}$: $e(S') \leq 6$

S'	$e(S')$	S'	$e(S')$	S'	$e(S')$
$\{x, y, xy\}$	6	$\{x^4, x^2y, x^3y\}$	5	$\{x^7, x^4y, x^9y\}$	6
$\{x, y, x^3y\}$	5	$\{x^4, x^2y, x^4y\}$	4	$\{x^7, x^4y, x^{10}y\}$	6
$\{x, y, x^4y\}$	6	$\{x^4, x^2y, x^6y\}$	6	$\{x^7, x^5y, x^6y\}$	5
$\{x, y, x^5y\}$	4	$\{x^4, x^2y, x^7y\}$	6	$\{x^7, x^5y, x^7y\}$	4
$\{x, y, x^6y\}$	4	$\{x^4, x^2y, x^8y\}$	6	$\{x^7, x^5y, x^9y\}$	6
$\{x, y, x^7y\}$	6	$\{x^4, x^2y, x^9y\}$	6	$\{x^7, x^5y, x^{10}y\}$	6
$\{x, y, x^8y\}$	5	$\{x^4, x^3y, x^4y\}$	5	$\{x^7, x^6y, x^7y\}$	5
$\{x, y, x^{10}y\}$	6	$\{x^4, x^3y, x^5y\}$	4	$\{x^7, x^6y, x^8y\}$	4
$\{x, xy, x^2y\}$	6	$\{x^4, x^3y, x^7y\}$	6	$\{x^7, x^6y, x^{10}y\}$	6
$\{x, xy, x^4y\}$	5	$\{x^4, x^3y, x^8y\}$	6	$\{x^7, x^7y, x^8y\}$	5
$\{x, xy, x^5y\}$	6	$\{x^4, x^3y, x^9y\}$	6	$\{x^7, x^7y, x^9y\}$	4
$\{x, xy, x^6y\}$	4	$\{x^4, x^3y, x^{10}y\}$	6	$\{x^7, x^8y, x^9y\}$	5
$\{x, xy, x^7y\}$	4	$\{x^4, x^4y, x^5y\}$	5	$\{x^7, x^8y, x^{10}y\}$	4
$\{x, xy, x^8y\}$	6	$\{x^4, x^4y, x^6y\}$	4	$\{x^7, x^9y, x^{10}y\}$	5
$\{x, xy, x^9y\}$	5	$\{x^4, x^4y, x^8y\}$	6	$\{x^8, y, xy\}$	6
$\{x, x^2y, x^3y\}$	6	$\{x^4, x^4y, x^9y\}$	6	$\{x^8, y, x^2y\}$	5
$\{x, x^2y, x^5y\}$	5	$\{x^4, x^4y, x^{10}y\}$	6	$\{x^8, y, x^3y\}$	6
$\{x, x^2y, x^6y\}$	6	$\{x^4, x^5y, x^6y\}$	5	$\{x^8, y, x^4y\}$	4
$\{x, x^2y, x^7y\}$	4	$\{x^4, x^5y, x^7y\}$	4	$\{x^8, y, x^7y\}$	4
$\{x, x^2y, x^8y\}$	4	$\{x^4, x^5y, x^9y\}$	6	$\{x^8, y, x^8y\}$	6
$\{x, x^2y, x^9y\}$	6	$\{x^4, x^5y, x^{10}y\}$	6	$\{x^8, y, x^9y\}$	5
$\{x, x^2y, x^{10}y\}$	5	$\{x^4, x^6y, x^7y\}$	5	$\{x^8, y, x^{10}y\}$	6
$\{x, x^3y, x^4y\}$	6	$\{x^4, x^6y, x^8y\}$	4	$\{x^8, xy, x^2y\}$	6
$\{x, x^3y, x^6y\}$	5	$\{x^4, x^6y, x^{10}y\}$	6	$\{x^8, xy, x^3y\}$	5
$\{x, x^3y, x^7y\}$	6	$\{x^4, x^7y, x^8y\}$	5	$\{x^8, xy, x^4y\}$	6
$\{x, x^3y, x^8y\}$	4	$\{x^4, x^7y, x^9y\}$	4	$\{x^8, xy, x^5y\}$	4
$\{x, x^3y, x^9y\}$	4	$\{x^4, x^8y, x^9y\}$	5	$\{x^8, xy, x^8y\}$	4
$\{x, x^3y, x^{10}y\}$	6	$\{x^4, x^8y, x^{10}y\}$	4	$\{x^8, xy, x^9y\}$	6
$\{x, x^4y, x^5y\}$	6	$\{x^4, x^9y, x^{10}y\}$	5	$\{x^8, xy, x^{10}y\}$	5
$\{x, x^4y, x^7y\}$	5	$\{x^5, y, x^2y\}$	6	$\{x^8, x^2y, x^3y\}$	6
$\{x, x^4y, x^8y\}$	6	$\{x^5, y, x^3y\}$	4	$\{x^8, x^2y, x^4y\}$	5
$\{x, x^4y, x^9y\}$	4	$\{x^5, y, x^4y\}$	5	$\{x^8, x^2y, x^5y\}$	6
$\{x, x^4y, x^{10}y\}$	4	$\{x^5, y, x^5y\}$	6	$\{x^8, x^2y, x^6y\}$	4
$\{x, x^5y, x^6y\}$	6	$\{x^5, y, x^6y\}$	6	$\{x^8, x^2y, x^9y\}$	4
$\{x, x^5y, x^8y\}$	5	$\{x^5, y, x^7y\}$	5	$\{x^8, x^2y, x^{10}y\}$	6
$\{x, x^5y, x^9y\}$	6	$\{x^5, y, x^8y\}$	4	$\{x^8, x^3y, x^4y\}$	6
$\{x, x^5y, x^{10}y\}$	4	$\{x^5, y, x^9y\}$	6	$\{x^8, x^3y, x^5y\}$	5
$\{x, x^6y, x^7y\}$	6	$\{x^5, xy, x^3y\}$	6	$\{x^8, x^3y, x^6y\}$	6
$\{x, x^6y, x^9y\}$	5	$\{x^5, xy, x^4y\}$	4	$\{x^8, x^3y, x^7y\}$	4
$\{x, x^6y, x^{10}y\}$	6	$\{x^5, xy, x^5y\}$	5	$\{x^8, x^3y, x^{10}y\}$	4
$\{x, x^7y, x^8y\}$	6	$\{x^5, xy, x^6y\}$	6	$\{x^8, x^4y, x^5y\}$	6
$\{x, x^7y, x^{10}y\}$	5	$\{x^5, xy, x^7y\}$	6	$\{x^8, x^4y, x^6y\}$	5
$\{x, x^8y, x^9y\}$	6	$\{x^5, xy, x^8y\}$	5	$\{x^8, x^4y, x^7y\}$	6
$\{x, x^9y, x^{10}y\}$	6	$\{x^5, xy, x^9y\}$	4	$\{x^8, x^4y, x^8y\}$	4
$\{x^2, y, xy\}$	4	$\{x^5, xy, x^{10}y\}$	6	$\{x^8, x^5y, x^6y\}$	6
$\{x^2, y, x^2y\}$	6	$\{x^5, x^2y, x^4y\}$	6	$\{x^8, x^5y, x^7y\}$	5
$\{x^2, y, x^3y\}$	6	$\{x^5, x^2y, x^5y\}$	4	$\{x^8, x^5y, x^8y\}$	6
$\{x^2, y, x^5y\}$	5	$\{x^5, x^2y, x^6y\}$	5	$\{x^8, x^5y, x^9y\}$	4

Table 4.42: (Continued)

S'	$e(S')$	S'	$e(S')$	S'	$e(S')$
$\{x^2, y, x^6y\}$	5	$\{x^5, x^2y, x^7y\}$	6	$\{x^8, x^6y, x^7y\}$	6
$\{x^2, y, x^8y\}$	6	$\{x^5, x^2y, x^8y\}$	6	$\{x^8, x^6y, x^8y\}$	5
$\{x^2, y, x^9y\}$	6	$\{x^5, x^2y, x^9y\}$	5	$\{x^8, x^6y, x^9y\}$	6
$\{x^2, y, x^{10}y\}$	4	$\{x^5, x^2y, x^{10}y\}$	4	$\{x^8, x^6y, x^{10}y\}$	4
$\{x^2, xy, x^2y\}$	4	$\{x^5, x^3y, x^5y\}$	6	$\{x^8, x^7y, x^8y\}$	6
$\{x^2, xy, x^3y\}$	6	$\{x^5, x^3y, x^6y\}$	4	$\{x^8, x^7y, x^9y\}$	5
$\{x^2, xy, x^4y\}$	6	$\{x^5, x^3y, x^7y\}$	5	$\{x^8, x^7y, x^{10}y\}$	6
$\{x^2, xy, x^6y\}$	5	$\{x^5, x^3y, x^8y\}$	6	$\{x^8, x^8y, x^9y\}$	6
$\{x^2, xy, x^7y\}$	5	$\{x^5, x^3y, x^9y\}$	6	$\{x^8, x^8y, x^{10}y\}$	5
$\{x^2, xy, x^9y\}$	6	$\{x^5, x^3y, x^{10}y\}$	5	$\{x^8, x^9y, x^{10}y\}$	6
$\{x^2, xy, x^{10}y\}$	6	$\{x^5, x^4y, x^6y\}$	6	$\{x^9, y, xy\}$	4
$\{x^2, x^2y, x^3y\}$	4	$\{x^5, x^4y, x^7y\}$	4	$\{x^9, y, x^2y\}$	6
$\{x^2, x^2y, x^4y\}$	6	$\{x^5, x^4y, x^8y\}$	5	$\{x^9, y, x^3y\}$	6
$\{x^2, x^2y, x^5y\}$	6	$\{x^5, x^4y, x^9y\}$	6	$\{x^9, y, x^5y\}$	5
$\{x^2, x^2y, x^7y\}$	5	$\{x^5, x^4y, x^{10}y\}$	6	$\{x^9, y, x^6y\}$	5
$\{x^2, x^2y, x^8y\}$	5	$\{x^5, x^5y, x^7y\}$	6	$\{x^9, y, x^8y\}$	6
$\{x^2, x^2y, x^{10}y\}$	6	$\{x^5, x^5y, x^8y\}$	4	$\{x^9, y, x^9y\}$	6
$\{x^2, x^3y, x^4y\}$	4	$\{x^5, x^5y, x^9y\}$	5	$\{x^9, y, x^{10}y\}$	4
$\{x^2, x^3y, x^5y\}$	6	$\{x^5, x^5y, x^{10}y\}$	6	$\{x^9, xy, x^2y\}$	4
$\{x^2, x^3y, x^6y\}$	6	$\{x^5, x^6y, x^8y\}$	6	$\{x^9, xy, x^3y\}$	6
$\{x^2, x^3y, x^8y\}$	5	$\{x^5, x^6y, x^9y\}$	4	$\{x^9, xy, x^4y\}$	6
$\{x^2, x^3y, x^9y\}$	5	$\{x^5, x^6y, x^{10}y\}$	5	$\{x^9, xy, x^6y\}$	5
$\{x^2, x^4y, x^5y\}$	4	$\{x^5, x^7y, x^9y\}$	6	$\{x^9, xy, x^7y\}$	5
$\{x^2, x^4y, x^6y\}$	6	$\{x^5, x^7y, x^{10}y\}$	4	$\{x^9, xy, x^9y\}$	6
$\{x^2, x^4y, x^7y\}$	6	$\{x^5, x^8y, x^{10}y\}$	6	$\{x^9, xy, x^{10}y\}$	6
$\{x^2, x^4y, x^9y\}$	5	$\{x^6, y, x^2y\}$	6	$\{x^9, x^2y, x^3y\}$	4
$\{x^2, x^4y, x^{10}y\}$	5	$\{x^6, y, x^3y\}$	4	$\{x^9, x^2y, x^4y\}$	6
$\{x^2, x^5y, x^6y\}$	4	$\{x^6, y, x^4y\}$	5	$\{x^9, x^2y, x^5y\}$	6
$\{x^2, x^5y, x^7y\}$	6	$\{x^6, y, x^5y\}$	6	$\{x^9, x^2y, x^7y\}$	5
$\{x^2, x^5y, x^8y\}$	6	$\{x^6, y, x^6y\}$	6	$\{x^9, x^2y, x^8y\}$	5
$\{x^2, x^5y, x^{10}y\}$	5	$\{x^6, y, x^7y\}$	5	$\{x^9, x^2y, x^{10}y\}$	6
$\{x^2, x^6y, x^7y\}$	4	$\{x^6, y, x^8y\}$	4	$\{x^9, x^3y, x^4y\}$	4
$\{x^2, x^6y, x^8y\}$	6	$\{x^6, y, x^9y\}$	6	$\{x^9, x^3y, x^5y\}$	6
$\{x^2, x^6y, x^9y\}$	6	$\{x^6, xy, x^3y\}$	6	$\{x^9, x^3y, x^6y\}$	6
$\{x^2, x^7y, x^8y\}$	4	$\{x^6, xy, x^4y\}$	4	$\{x^9, x^3y, x^8y\}$	5
$\{x^2, x^7y, x^9y\}$	6	$\{x^6, xy, x^5y\}$	5	$\{x^9, x^3y, x^9y\}$	5
$\{x^2, x^7y, x^{10}y\}$	6	$\{x^6, xy, x^6y\}$	6	$\{x^9, x^4y, x^5y\}$	4
$\{x^2, x^8y, x^9y\}$	4	$\{x^6, xy, x^7y\}$	6	$\{x^9, x^4y, x^6y\}$	6
$\{x^2, x^8y, x^{10}y\}$	6	$\{x^6, xy, x^8y\}$	5	$\{x^9, x^4y, x^7y\}$	6
$\{x^2, x^9y, x^{10}y\}$	4	$\{x^6, xy, x^9y\}$	4	$\{x^9, x^4y, x^9y\}$	5
$\{x^3, y, xy\}$	6	$\{x^6, xy, x^{10}y\}$	6	$\{x^9, x^4y, x^{10}y\}$	5
$\{x^3, y, x^2y\}$	5	$\{x^6, x^2y, x^4y\}$	6	$\{x^9, x^5y, x^6y\}$	4
$\{x^3, y, x^3y\}$	6	$\{x^6, x^2y, x^5y\}$	4	$\{x^9, x^5y, x^7y\}$	6
$\{x^3, y, x^4y\}$	4	$\{x^6, x^2y, x^6y\}$	5	$\{x^9, x^5y, x^8y\}$	6
$\{x^3, y, x^7y\}$	4	$\{x^6, x^2y, x^7y\}$	6	$\{x^9, x^5y, x^{10}y\}$	5
$\{x^3, y, x^8y\}$	6	$\{x^6, x^2y, x^8y\}$	6	$\{x^9, x^6y, x^7y\}$	4
$\{x^3, y, x^9y\}$	5	$\{x^6, x^2y, x^9y\}$	5	$\{x^9, x^6y, x^8y\}$	6
$\{x^3, y, x^{10}y\}$	6	$\{x^6, x^2y, x^{10}y\}$	4	$\{x^9, x^6y, x^9y\}$	6
$\{x^3, xy, x^2y\}$	6	$\{x^6, x^3y, x^5y\}$	6	$\{x^9, x^7y, x^8y\}$	4
$\{x^3, xy, x^3y\}$	5	$\{x^6, x^3y, x^6y\}$	4	$\{x^9, x^7y, x^9y\}$	6
$\{x^3, xy, x^4y\}$	6	$\{x^6, x^3y, x^7y\}$	5	$\{x^9, x^7y, x^{10}y\}$	6
$\{x^3, xy, x^5y\}$	4	$\{x^6, x^3y, x^8y\}$	6	$\{x^9, x^8y, x^9y\}$	4
$\{x^3, xy, x^8y\}$	4	$\{x^6, x^3y, x^9y\}$	6	$\{x^9, x^8y, x^{10}y\}$	6
$\{x^3, xy, x^9y\}$	6	$\{x^6, x^3y, x^{10}y\}$	5	$\{x^9, x^9y, x^{10}y\}$	4
$\{x^3, xy, x^{10}y\}$	5	$\{x^6, x^4y, x^6y\}$	6	$\{x^{10}, y, xy\}$	6

Table 4.42: (Continued)

S'	$e(S')$	S'	$e(S')$	S'	$e(S')$
$\{x^3, x^2y, x^3y\}$	6	$\{x^6, x^4y, x^7y\}$	4	$\{x^{10}, y, x^3y\}$	5
$\{x^3, x^2y, x^4y\}$	5	$\{x^6, x^4y, x^8y\}$	5	$\{x^{10}, y, x^4y\}$	6
$\{x^3, x^2y, x^5y\}$	6	$\{x^6, x^4y, x^9y\}$	6	$\{x^{10}, y, x^5y\}$	4
$\{x^3, x^2y, x^6y\}$	4	$\{x^6, x^4y, x^{10}y\}$	6	$\{x^{10}, y, x^6y\}$	4
$\{x^3, x^2y, x^9y\}$	4	$\{x^6, x^5y, x^7y\}$	6	$\{x^{10}, y, x^7y\}$	6
$\{x^3, x^2y, x^{10}y\}$	6	$\{x^6, x^5y, x^8y\}$	4	$\{x^{10}, y, x^8y\}$	5
$\{x^3, x^3y, x^4y\}$	6	$\{x^6, x^5y, x^9y\}$	5	$\{x^{10}, y, x^{10}y\}$	6
$\{x^3, x^3y, x^5y\}$	5	$\{x^6, x^5y, x^{10}y\}$	6	$\{x^{10}, xy, x^2y\}$	6
$\{x^3, x^3y, x^6y\}$	6	$\{x^6, x^6y, x^8y\}$	6	$\{x^{10}, xy, x^4y\}$	5
$\{x^3, x^3y, x^7y\}$	4	$\{x^6, x^6y, x^9y\}$	4	$\{x^{10}, xy, x^5y\}$	6
$\{x^3, x^3y, x^{10}y\}$	4	$\{x^6, x^6y, x^{10}y\}$	5	$\{x^{10}, xy, x^6y\}$	4
$\{x^3, x^4y, x^5y\}$	6	$\{x^6, x^7y, x^9y\}$	6	$\{x^{10}, xy, x^7y\}$	4
$\{x^3, x^4y, x^6y\}$	5	$\{x^6, x^7y, x^{10}y\}$	4	$\{x^{10}, xy, x^8y\}$	6
$\{x^3, x^4y, x^7y\}$	6	$\{x^6, x^8y, x^{10}y\}$	6	$\{x^{10}, xy, x^9y\}$	5
$\{x^3, x^4y, x^8y\}$	4	$\{x^7, y, xy\}$	5	$\{x^{10}, x^2y, x^3y\}$	6
$\{x^3, x^5y, x^6y\}$	6	$\{x^7, y, x^2y\}$	4	$\{x^{10}, x^2y, x^5y\}$	5
$\{x^3, x^5y, x^7y\}$	5	$\{x^7, y, x^4y\}$	6	$\{x^{10}, x^2y, x^6y\}$	6
$\{x^3, x^5y, x^8y\}$	6	$\{x^7, y, x^5y\}$	6	$\{x^{10}, x^2y, x^7y\}$	4
$\{x^3, x^5y, x^9y\}$	4	$\{x^7, y, x^6y\}$	6	$\{x^{10}, x^2y, x^8y\}$	4
$\{x^3, x^6y, x^7y\}$	6	$\{x^7, y, x^7y\}$	6	$\{x^{10}, x^2y, x^9y\}$	6
$\{x^3, x^6y, x^8y\}$	5	$\{x^7, y, x^9y\}$	4	$\{x^{10}, x^2y, x^{10}y\}$	5
$\{x^3, x^6y, x^9y\}$	6	$\{x^7, y, x^{10}y\}$	5	$\{x^{10}, x^3y, x^4y\}$	6
$\{x^3, x^6y, x^{10}y\}$	4	$\{x^7, xy, x^2y\}$	5	$\{x^{10}, x^3y, x^6y\}$	5
$\{x^3, x^7y, x^8y\}$	6	$\{x^7, xy, x^3y\}$	4	$\{x^{10}, x^3y, x^7y\}$	6
$\{x^3, x^7y, x^9y\}$	5	$\{x^7, xy, x^5y\}$	6	$\{x^{10}, x^3y, x^8y\}$	4
$\{x^3, x^7y, x^{10}y\}$	6	$\{x^7, xy, x^6y\}$	6	$\{x^{10}, x^3y, x^9y\}$	4
$\{x^3, x^8y, x^9y\}$	6	$\{x^7, xy, x^7y\}$	6	$\{x^{10}, x^3y, x^{10}y\}$	6
$\{x^3, x^8y, x^{10}y\}$	5	$\{x^7, xy, x^8y\}$	6	$\{x^{10}, x^4y, x^5y\}$	6
$\{x^3, x^9y, x^{10}y\}$	6	$\{x^7, xy, x^{10}y\}$	4	$\{x^{10}, x^4y, x^7y\}$	5
$\{x^4, y, xy\}$	5	$\{x^7, x^2y, x^3y\}$	5	$\{x^{10}, x^4y, x^8y\}$	6
$\{x^4, y, x^2y\}$	4	$\{x^7, x^2y, x^4y\}$	4	$\{x^{10}, x^4y, x^9y\}$	4
$\{x^4, y, x^4y\}$	6	$\{x^7, x^2y, x^6y\}$	6	$\{x^{10}, x^4y, x^{10}y\}$	4
$\{x^4, y, x^5y\}$	6	$\{x^7, x^2y, x^7y\}$	6	$\{x^{10}, x^5y, x^6y\}$	6
$\{x^4, y, x^6y\}$	6	$\{x^7, x^2y, x^8y\}$	6	$\{x^{10}, x^5y, x^8y\}$	5
$\{x^4, y, x^7y\}$	6	$\{x^7, x^2y, x^9y\}$	6	$\{x^{10}, x^5y, x^9y\}$	6
$\{x^4, y, x^9y\}$	4	$\{x^7, x^3y, x^4y\}$	5	$\{x^{10}, x^5y, x^{10}y\}$	4
$\{x^4, y, x^{10}y\}$	5	$\{x^7, x^3y, x^5y\}$	4	$\{x^{10}, x^6y, x^7y\}$	6
$\{x^4, xy, x^2y\}$	5	$\{x^7, x^3y, x^7y\}$	6	$\{x^{10}, x^6y, x^9y\}$	5
$\{x^4, xy, x^3y\}$	4	$\{x^7, x^3y, x^8y\}$	6	$\{x^{10}, x^6y, x^{10}y\}$	6
$\{x^4, xy, x^5y\}$	6	$\{x^7, x^3y, x^9y\}$	6	$\{x^{10}, x^7y, x^8y\}$	6
$\{x^4, xy, x^6y\}$	6	$\{x^7, x^3y, x^{10}y\}$	6	$\{x^{10}, x^7y, x^{10}y\}$	5
$\{x^4, xy, x^7y\}$	6	$\{x^7, x^4y, x^5y\}$	5	$\{x^{10}, x^8y, x^9y\}$	6
$\{x^4, xy, x^8y\}$	6	$\{x^7, x^4y, x^6y\}$	4	$\{x^{10}, x^9y, x^{10}y\}$	6
$\{x^4, xy, x^{10}y\}$	4	$\{x^7, x^4y, x^8y\}$	6		

Next, we list down the subsets S' where $e(S') > 6$.

Table 4.43: $\{x^a, x^by, x^cy\} \subseteq S' \subseteq D_{22}$: $e(S') = 10 > 6$

S'	$e(S')$	S'	$e(S')$	S'	$e(S')$
$\{x, y, x^2y\}$	10	$\{x^4, x^2y, x^5y\}$	10	$\{x^7, x^5y, x^8y\}$	10
$\{x, y, x^9y\}$	10	$\{x^4, x^2y, x^{10}y\}$	10	$\{x^7, x^6y, x^9y\}$	10

Table 4.43: (Continued)

S'	$e(S')$	S'	$e(S')$	S'	$e(S')$
$\{x, xy, x^3y\}$	10	$\{x^4, x^3y, x^6y\}$	10	$\{x^7, x^7y, x^{10}y\}$	10
$\{x, xy, x^{10}y\}$	10	$\{x^4, x^4y, x^7y\}$	10	$\{x^8, y, x^5y\}$	10
$\{x, x^2y, x^4y\}$	10	$\{x^4, x^5y, x^8y\}$	10	$\{x^8, y, x^6y\}$	10
$\{x, x^3y, x^5y\}$	10	$\{x^4, x^6y, x^9y\}$	10	$\{x^8, xy, x^6y\}$	10
$\{x, x^4y, x^6y\}$	10	$\{x^4, x^7y, x^{10}y\}$	10	$\{x^8, xy, x^7y\}$	10
$\{x, x^5y, x^7y\}$	10	$\{x^5, y, xy\}$	10	$\{x^8, x^2y, x^7y\}$	10
$\{x, x^6y, x^8y\}$	10	$\{x^5, y, x^{10}y\}$	10	$\{x^8, x^2y, x^8y\}$	10
$\{x, x^7y, x^9y\}$	10	$\{x^5, xy, x^2y\}$	10	$\{x^8, x^3y, x^8y\}$	10
$\{x, x^8y, x^{10}y\}$	10	$\{x^5, x^2y, x^3y\}$	10	$\{x^8, x^3y, x^9y\}$	10
$\{x^2, y, x^4y\}$	10	$\{x^5, x^3y, x^4y\}$	10	$\{x^8, x^4y, x^9y\}$	10
$\{x^2, y, x^7y\}$	10	$\{x^5, x^4y, x^5y\}$	10	$\{x^8, x^4y, x^{10}y\}$	10
$\{x^2, xy, x^5y\}$	10	$\{x^5, x^5y, x^6y\}$	10	$\{x^8, x^5y, x^{10}y\}$	10
$\{x^2, xy, x^8y\}$	10	$\{x^5, x^6y, x^7y\}$	10	$\{x^9, y, x^4y\}$	10
$\{x^2, x^2y, x^6y\}$	10	$\{x^5, x^7y, x^8y\}$	10	$\{x^9, y, x^7y\}$	10
$\{x^2, x^2y, x^9y\}$	10	$\{x^5, x^8y, x^9y\}$	10	$\{x^9, xy, x^5y\}$	10
$\{x^2, x^3y, x^7y\}$	10	$\{x^5, x^9y, x^{10}y\}$	10	$\{x^9, xy, x^8y\}$	10
$\{x^2, x^3y, x^{10}y\}$	10	$\{x^6, y, xy\}$	10	$\{x^9, x^2y, x^6y\}$	10
$\{x^2, x^4y, x^8y\}$	10	$\{x^6, y, x^{10}y\}$	10	$\{x^9, x^2y, x^9y\}$	10
$\{x^2, x^5y, x^9y\}$	10	$\{x^6, xy, x^2y\}$	10	$\{x^9, x^3y, x^7y\}$	10
$\{x^2, x^6y, x^{10}y\}$	10	$\{x^6, x^2y, x^3y\}$	10	$\{x^9, x^3y, x^{10}y\}$	10
$\{x^3, y, x^5y\}$	10	$\{x^6, x^3y, x^4y\}$	10	$\{x^9, x^4y, x^8y\}$	10
$\{x^3, y, x^6y\}$	10	$\{x^6, x^4y, x^5y\}$	10	$\{x^9, x^5y, x^9y\}$	10
$\{x^3, xy, x^6y\}$	10	$\{x^6, x^5y, x^6y\}$	10	$\{x^9, x^6y, x^{10}y\}$	10
$\{x^3, xy, x^7y\}$	10	$\{x^6, x^6y, x^7y\}$	10	$\{x^{10}, y, x^2y\}$	10
$\{x^3, x^2y, x^7y\}$	10	$\{x^6, x^7y, x^8y\}$	10	$\{x^{10}, y, x^9y\}$	10
$\{x^3, x^2y, x^8y\}$	10	$\{x^6, x^8y, x^9y\}$	10	$\{x^{10}, xy, x^3y\}$	10
$\{x^3, x^3y, x^8y\}$	10	$\{x^6, x^9y, x^{10}y\}$	10	$\{x^{10}, xy, x^{10}y\}$	10
$\{x^3, x^3y, x^9y\}$	10	$\{x^7, y, x^3y\}$	10	$\{x^{10}, x^2y, x^4y\}$	10
$\{x^3, x^4y, x^9y\}$	10	$\{x^7, y, x^8y\}$	10	$\{x^{10}, x^3y, x^5y\}$	10
$\{x^3, x^4y, x^{10}y\}$	10	$\{x^7, xy, x^4y\}$	10	$\{x^{10}, x^4y, x^6y\}$	10
$\{x^3, x^5y, x^{10}y\}$	10	$\{x^7, xy, x^9y\}$	10	$\{x^{10}, x^5y, x^7y\}$	10
$\{x^4, y, x^3y\}$	10	$\{x^7, x^2y, x^5y\}$	10	$\{x^{10}, x^6y, x^8y\}$	10
$\{x^4, y, x^8y\}$	10	$\{x^7, x^2y, x^{10}y\}$	10	$\{x^{10}, x^7y, x^9y\}$	10
$\{x^4, xy, x^4y\}$	10	$\{x^7, x^3y, x^6y\}$	10	$\{x^{10}, x^8y, x^{10}y\}$	10
$\{x^4, xy, x^9y\}$	10	$\{x^7, x^4y, x^7y\}$	10		

Since $e(S') = 10$ for the subsets S' in Table 4.43, we consider the case where $|S''| = 4$ and $S'' = S' \cup \{x^i y^j\}$ for $i \in \{0, 1, \ldots, 10\}$ and $j \in \{0, 1\}$. There are two cases to consider:

(i) If $S'' = S' \cup \{x^d y\}$ for $d \in \{0, 1, \ldots, 10\}$, then S'' will contain one of the subsets in Table 4.42 and hence $e(S'') \leq 6$.

(ii) If $S'' = S' \cup \{x^d\}$ for $d \in \{1, 2, \ldots, 10\}$, then S'' will either contain one of the subsets in Table 4.38 or S'' will have been listed in Tables 4.40 and 4.41. Hence, it is clear from (a) that there does not exist any subset $S \subseteq D_{22}$ where $e(S) = k$ for $k \in \{7, 8, 9\}$.

(c) Let $S' = \{x^a y, x^b y, x^c y\} \subseteq S$ for $0 \leq a < b < c \leq 10$. By Proposition 4.22, $e(\{x^a y, x^b y, x^c y\}) = \infty$. Hence, we consider the case where $|S''| = 4$ and $S'' = S' \cup \{x^i y^j\}$ for $i \in \{0, 1, \ldots, 10\}$ and $j \in \{0, 1\}$. Note that by Proposition 4.22, $e(\{y, xy, x^2 y, x^3 y, \ldots, x^{10} y\}) = \infty$. If $|S| > 11$, then $e(S) = 2$. Therefore, we only consider $S'' = S' \cup \{x^d\}$ for some positive integer d where $1 \leq d \leq 10$. Then we have $S'' = \{x^a y, x^b y, x^c y, x^d\} \subseteq S$ and $\{x^d, x^a y, x^b y\} \subseteq \{x^a y, x^b y, x^c y, x^d\}$. From (b), there does not exist any subset $S \subseteq D_{22}$ where $e(S) = k$ for $k \in \{7, 8, 9\}$.

From the three cases ((a), (b) and (c)) above, we see that if $1 \in S$, there does not exist any subset $S \subseteq D_{22}$ such that $e(S) = k$ for $k \in \{7, 8, 9\}$.

Secondly, suppose that $1 \in S$. From the assumption that $e(S) = k$ for $k \in \{7, 8, 9\}$, we have $x^u y \in S^k$ for all $u = 0, 1, 2, \ldots, 10$ and hence there exists $x^v y \in S$ for some $v = 0, 1, 2, \ldots, 10$. We consider the following two cases:

(1) $\{1, x^a, x^b y\} \subseteq S$ for $1 \leq a \leq 10$ and $0 \leq b \leq 10$

(2) $\{1, x^a y, x^b y\} \subseteq S$ for $0 \leq a < b \leq 10$

where a and b are integers. We shall explain the two cases in detail. Let m be a positive integer.

(1) Let $S' = \{1, x^a, x^b y\} \subseteq S$ for $1 \leq a \leq 10$ and $0 \leq b \leq 10$. Since $a = 1, \ldots, 10$, then $a \neq 11m$ and by Proposition 4.24, $e(\{1, x^a, x^b y\}) \leq 6 < k$ for $k \in \{7, 8, 9\}$.

(2) Let $S' = \{1, x^a y, x^b y\} \subseteq S$ for $0 \leq a < b \leq 10$. Since $0 \leq a < b \leq 10$, then $b - a \neq 11m$ and by Proposition 4.25, we have $e(\{1, x^a y, x^b y\}) = 11$. Let $S'' = S' \cup \{x^i y^j\}$ for $i \in \{0, 1, \ldots, 10\}$ and $j \in \{0, 1\}$. Note that if $x^c \in S''$ for $1 \leq c \leq 10$, then $\{1, x^c, x^a y\} \subseteq S''$ and from (1), we

have $e(\{1, x^a y, x^b y, x^c\}) < k$ for $k \in \{7, 8, 9\}$. Hence, we only consider $S'' = S' \cup \{x^d y\}$ for $0 \le d \le 10$ and $b < d$.

Table 4.44: $\{1, x^a y, x^b y\} \subseteq S'' \subseteq D_{22}$: $e(S'') \le 6$

S''	$e(S'')$	S''	$e(S'')$	S''	$e(S'')$
$\{1, y, xy, x^2y\}$	6	$\{1, xy, x^3y, x^6y\}$	5	$\{1, x^3y, x^4y, x^6y\}$	5
$\{1, y, xy, x^3y\}$	5	$\{1, xy, x^3y, x^7y\}$	5	$\{1, x^3y, x^4y, x^7y\}$	5
$\{1, y, xy, x^4y\}$	5	$\{1, xy, x^3y, x^8y\}$	5	$\{1, x^3y, x^4y, x^8y\}$	5
$\{1, y, xy, x^5y\}$	5	$\{1, xy, x^3y, x^9y\}$	5	$\{1, x^3y, x^4y, x^9y\}$	6
$\{1, y, xy, x^6y\}$	6	$\{1, xy, x^3y, x^{10}y\}$	6	$\{1, x^3y, x^4y, x^{10}y\}$	5
$\{1, y, xy, x^7y\}$	5	$\{1, xy, x^4y, x^5y\}$	5	$\{1, x^3y, x^5y, x^6y\}$	5
$\{1, y, xy, x^8y\}$	5	$\{1, xy, x^4y, x^6y\}$	5	$\{1, x^3y, x^5y, x^7y\}$	6
$\{1, y, xy, x^9y\}$	5	$\{1, xy, x^4y, x^7y\}$	6	$\{1, x^3y, x^5y, x^8y\}$	5
$\{1, y, xy, x^{10}y\}$	6	$\{1, xy, x^4y, x^8y\}$	6	$\{1, x^3y, x^5y, x^9y\}$	5
$\{1, y, x^2y, x^3y\}$	5	$\{1, xy, x^4y, x^9y\}$	6	$\{1, x^3y, x^5y, x^{10}y\}$	5
$\{1, y, x^2y, x^4y\}$	6	$\{1, xy, x^4y, x^{10}y\}$	5	$\{1, x^3y, x^6y, x^7y\}$	5
$\{1, y, x^2y, x^5y\}$	5	$\{1, xy, x^5y, x^6y\}$	5	$\{1, x^3y, x^6y, x^8y\}$	5
$\{1, y, x^2y, x^6y\}$	5	$\{1, xy, x^5y, x^7y\}$	5	$\{1, x^3y, x^6y, x^9y\}$	6
$\{1, y, x^2y, x^7y\}$	5	$\{1, xy, x^5y, x^8y\}$	6	$\{1, x^3y, x^6y, x^{10}y\}$	6
$\{1, y, x^2y, x^8y\}$	5	$\{1, xy, x^5y, x^9y\}$	6	$\{1, x^3y, x^7y, x^8y\}$	5
$\{1, y, x^2y, x^9y\}$	6	$\{1, xy, x^5y, x^{10}y\}$	5	$\{1, x^3y, x^7y, x^9y\}$	5
$\{1, y, x^2y, x^{10}y\}$	5	$\{1, xy, x^6y, x^7y\}$	6	$\{1, x^3y, x^7y, x^{10}y\}$	6
$\{1, y, x^3y, x^4y\}$	5	$\{1, xy, x^6y, x^8y\}$	5	$\{1, x^3y, x^8y, x^9y\}$	6
$\{1, y, x^3y, x^5y\}$	5	$\{1, xy, x^6y, x^9y\}$	6	$\{1, x^3y, x^8y, x^{10}y\}$	5
$\{1, y, x^3y, x^6y\}$	6	$\{1, xy, x^6y, x^{10}y\}$	5	$\{1, x^3y, x^9y, x^{10}y\}$	5
$\{1, y, x^3y, x^7y\}$	6	$\{1, xy, x^7y, x^8y\}$	5	$\{1, x^4y, x^5y, x^6y\}$	6
$\{1, y, x^3y, x^8y\}$	6	$\{1, xy, x^7y, x^9y\}$	5	$\{1, x^4y, x^5y, x^7y\}$	5
$\{1, y, x^3y, x^9y\}$	5	$\{1, xy, x^7y, x^{10}y\}$	5	$\{1, x^4y, x^5y, x^8y\}$	5
$\{1, y, x^3y, x^{10}y\}$	5	$\{1, xy, x^8y, x^9y\}$	5	$\{1, x^4y, x^5y, x^9y\}$	5
$\{1, y, x^4y, x^5y\}$	5	$\{1, xy, x^8y, x^{10}y\}$	6	$\{1, x^4y, x^5y, x^{10}y\}$	6
$\{1, y, x^4y, x^6y\}$	5	$\{1, xy, x^9y, x^{10}y\}$	5	$\{1, x^4y, x^6y, x^7y\}$	5
$\{1, y, x^4y, x^7y\}$	6	$\{1, x^2y, x^3y, x^4y\}$	6	$\{1, x^4y, x^6y, x^8y\}$	6
$\{1, y, x^4y, x^8y\}$	6	$\{1, x^2y, x^3y, x^5y\}$	5	$\{1, x^4y, x^6y, x^9y\}$	5
$\{1, y, x^4y, x^9y\}$	5	$\{1, x^2y, x^3y, x^6y\}$	5	$\{1, x^4y, x^6y, x^{10}y\}$	5
$\{1, y, x^4y, x^{10}y\}$	5	$\{1, x^2y, x^3y, x^7y\}$	5	$\{1, x^4y, x^7y, x^8y\}$	5
$\{1, y, x^5y, x^6y\}$	6	$\{1, x^2y, x^3y, x^8y\}$	6	$\{1, x^4y, x^7y, x^9y\}$	5
$\{1, y, x^5y, x^7y\}$	5	$\{1, x^2y, x^3y, x^9y\}$	5	$\{1, x^4y, x^7y, x^{10}y\}$	6
$\{1, y, x^5y, x^8y\}$	6	$\{1, x^2y, x^3y, x^{10}y\}$	5	$\{1, x^4y, x^8y, x^9y\}$	5
$\{1, y, x^5y, x^9y\}$	5	$\{1, x^2y, x^4y, x^5y\}$	5	$\{1, x^4y, x^8y, x^{10}y\}$	5
$\{1, y, x^5y, x^{10}y\}$	6	$\{1, x^2y, x^4y, x^6y\}$	6	$\{1, x^4y, x^9y, x^{10}y\}$	6
$\{1, y, x^6y, x^7y\}$	5	$\{1, x^2y, x^4y, x^7y\}$	5	$\{1, x^5y, x^6y, x^7y\}$	6
$\{1, y, x^6y, x^8y\}$	5	$\{1, x^2y, x^4y, x^8y\}$	5	$\{1, x^5y, x^6y, x^8y\}$	5
$\{1, y, x^6y, x^9y\}$	5	$\{1, x^2y, x^4y, x^9y\}$	5	$\{1, x^5y, x^6y, x^9y\}$	5
$\{1, y, x^6y, x^{10}y\}$	5	$\{1, x^2y, x^4y, x^{10}y\}$	5	$\{1, x^5y, x^6y, x^{10}y\}$	5
$\{1, y, x^7y, x^8y\}$	5	$\{1, x^2y, x^5y, x^6y\}$	5	$\{1, x^5y, x^7y, x^8y\}$	5
$\{1, y, x^7y, x^9y\}$	6	$\{1, x^2y, x^5y, x^7y\}$	5	$\{1, x^5y, x^7y, x^9y\}$	6
$\{1, y, x^7y, x^{10}y\}$	5	$\{1, x^2y, x^5y, x^8y\}$	6	$\{1, x^5y, x^7y, x^{10}y\}$	5
$\{1, y, x^8y, x^9y\}$	5	$\{1, x^2y, x^5y, x^9y\}$	6	$\{1, x^5y, x^8y, x^9y\}$	5
$\{1, y, x^8y, x^{10}y\}$	5	$\{1, x^2y, x^5y, x^{10}y\}$	6	$\{1, x^5y, x^8y, x^{10}y\}$	5
$\{1, y, x^9y, x^{10}y\}$	6	$\{1, x^2y, x^6y, x^7y\}$	5	$\{1, x^5y, x^9y, x^{10}y\}$	5
$\{1, xy, x^2y, x^3y\}$	6	$\{1, x^2y, x^6y, x^8y\}$	5	$\{1, x^6y, x^7y, x^8y\}$	6
$\{1, xy, x^2y, x^4y\}$	5	$\{1, x^2y, x^6y, x^9y\}$	6	$\{1, x^6y, x^7y, x^9y\}$	5
$\{1, xy, x^2y, x^5y\}$	5	$\{1, x^2y, x^6y, x^{10}y\}$	6	$\{1, x^6y, x^7y, x^{10}y\}$	5
$\{1, xy, x^2y, x^6y\}$	5	$\{1, x^2y, x^7y, x^8y\}$	6	$\{1, x^6y, x^8y, x^9y\}$	5
$\{1, xy, x^2y, x^7y\}$	6	$\{1, x^2y, x^7y, x^9y\}$	5	$\{1, x^6y, x^8y, x^{10}y\}$	6
$\{1, xy, x^2y, x^8y\}$	5	$\{1, x^2y, x^7y, x^{10}y\}$	6	$\{1, x^6y, x^9y, x^{10}y\}$	5

Table 4.44: (Continued)

S''	$e(S'')$	S''	$e(S'')$	S''	$e(S'')$
$\{1, xy, x^2y, x^9y\}$	5	$\{1, x^2y, x^8y, x^9y\}$	5	$\{1, x^7y, x^8y, x^9y\}$	6
$\{1, xy, x^2y, x^{10}y\}$	5	$\{1, x^2y, x^8y, x^{10}y\}$	5	$\{1, x^7y, x^8y, x^{10}y\}$	5
$\{1, xy, x^3y, x^4y\}$	5	$\{1, x^2y, x^9y, x^{10}y\}$	5	$\{1, x^7y, x^9y, x^{10}y\}$	5
$\{1, xy, x^3y, x^5y\}$	6	$\{1, x^3y, x^4y, x^5y\}$	6	$\{1, x^8y, x^9y, x^{10}y\}$	6

From Table 4.44, we have $e(S'') \leq 6 < k$ for $k \in \{7, 8, 9\}$.

From cases (1) and (2) above, we see that if $1 \in S$, there does not exist any subset $S \subseteq D_{22}$ such that $e(S) = k$ for $k \in \{7, 8, 9\}$.

Hence, we conclude that there does not exist any subset $S \subseteq D_{22}$ such that $e(S) = k$ for $k \in \{7, 8, 9\}$. □

Remark 4. Note that the dihedral group is a metacyclic group. Some results on other metacyclic groups such as the generalized quaternion group,

$$Q_{2^n} = \langle x, y | x^{2n-1} = 1, x^{2n-2} = y^2, xy = yx^{2^{n-1}-1} \rangle$$

and the semi-dihedral group,

$$SD_{2^n} = \langle x, y | x^{2n-1} = y^2 = 1, xy = yx^{2^{n-2}-1} \rangle$$

have been obtained in [16]. It was shown that if $S = \{1, x, y\} \subseteq Q_{2^n}$, then $e(S) = 2^{n-2} + 1$. It was also shown that if $S = \{1, x, y\} \subseteq SD_{2^n}$, then $e(S) = 2^{n-2} + 2$. Hence, we notice that the results obtained in this chapter can be extended to the other metacyclic groups mentioned above.

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APPENDIX A

C CODE FOR ALGORITHMS 1 AND 2

```
#include <stdio.h>
#include <stdlib.h>

#define SIZE 5
#define DISTINCT 5 //number of distinct elements

int compare(char[SIZE][SIZE], int);

int main()
{
  FILE* spOut;
  FILE* spOut2;
  FILE* spOut3;
  FILE* spOut4;
  char list[25] = {'A','B','C', 'D', 'E', 'F', 'G', 'H', 'I', 'J', 'K', 'L', 'M', 'N', 'O', 'P',
                   'Q', 'R', 'S', 'T', 'U', 'V', 'W', 'X', 'Y'};
  char arr[SIZE][SIZE];
  int i, j, k, l, m, m1, n;
  int i2, j2, s, t, r;
  int x1, x2, x3, x4, x5;
  int y1, y2, y3, y4, y5;
  int z1, z2, z3, z4, z5;
  int x, y, z;
  int count = 0;
  int count2 = 0;
  int count3 = 0;
  int count4 = 0;
  int count5 = 0;
  int eval = 0;

  int input, input2;

  int is_in = 0;

  int temp[SIZE*SIZE];
  int temp2[SIZE*SIZE];
  char temp3[SIZE*SIZE];
  char temp4[SIZE*SIZE];
  int counter1 = 0;
  int counter2 = 5;
  int counter3 = 0;
  int counter4 = 0;
  bool check = false;
  bool check2 = false;
  bool check3 = false;
  bool check4 = false;

  bool commutative = true; //if generating non-commutative squares, commutative = false;
  int add = 0;

  for(i = 0; i < SIZE*SIZE; i++)
  {
    temp2[i] = 99;
    temp[i] = 88;
  }

  for(i = 0; i < SIZE; i++)
    temp2[i] = i;

  //assign Row 1 elements
  arr[0][0] = 'A';
  arr[0][1] = 'B';
  arr[0][2] = 'C';
  arr[0][3] = 'D';
  arr[0][4] = 'E';

  for(i = 1; i < DISTINCT; i++)
  {
    for(j = 0; j < DISTINCT; j++)
    {
      for(k = 0; k < DISTINCT; k++)
      {
        for(l = 0; l < DISTINCT; l++)
        {
          for(m = 0; m < DISTINCT; m++)
          {
            if(i != j && i != k && i != l && i != m && j != k && j != l
               && j != m && k != l && k != m && l != m && j != 1 &&
               k != 2 && l != 3 && m != 4)
            {
              //assign Row 2 elements
              arr[1][0] = list[i];
              arr[1][1] = list[j];
```

```
arr[1][2] = list[k];
arr[1][3] = list[l];
arr[1][4] = list[m];

for(x1 = 1; x1 < DISTINCT; x1++)
{
  for(x2 = 0; x2 < DISTINCT; x2++)
  {
    for(x3 = 0; x3 < DISTINCT; x3++)
    {
      for(x4 = 0; x4 < DISTINCT; x4++)
      {
        for(x5 = 0; x5 < DISTINCT; x5++)
        {
          if(x1 != x2 && x1 != x3 && x1 != x4 && x1 != x5
             && x2 != x3 && x2 != x4 && x2 != x5 && x3 != x4
             && x3 != x5 && x4 != x5 && x2 != 1 && x3 != 2
             && x4 != 3 && x5 != 4 && x1 != i && x2 != j
             && x3 != k && x4 != l && x5 != m)
          {
            //assign Row 3 elements
            arr[2][0] = list[x1];
            arr[2][1] = list[x2];
            arr[2][2] = list[x3];
            arr[2][3] = list[x4];
            arr[2][4] = list[x5];

            for(y1 = 1; y1 < DISTINCT; y1++)
            {
              for(y2 = 0; y2 < DISTINCT; y2++)
              {
                for(y3 = 0; y3 < DISTINCT; y3++)
                {
                  for(y4 = 0; y4 < DISTINCT; y4++)
                  {
                    for(y5 = 0; y5 < DISTINCT; y5++)
                    {
                      if(y1 != y2 && y1 != y3 && y1 != y4 && y1 != y5
                         && y2 != y3 && y2 != y4 && y2 != y5 && y3 != y4
                         && y3 != y5 && y4 != y5 && y2 != 1 && y3 != 2
                         && y4 != 3 && y5 != 4 && y1 != i && y1 != x1
                         && y2 != j && y2 != x2 && y3 != k && y3 != x3
                         && y4 != l && y4 != x4 && y5 != m && y5 !=x5)
                      {
                        //assign Row 4 elements
                        arr[3][0] = list[y1];
                        arr[3][1] = list[y2];
                        arr[3][2] = list[y3];
                        arr[3][3] = list[y4];
                        arr[3][4] = list[y5];

                        for(z1 = 1; z1 < DISTINCT; z1++)
                        {
                          for(z2 = 0; z2 < DISTINCT; z2++)
                          {
                            for(z3 = 0; z3 < DISTINCT; z3++)
                            {
                              for(z4 = 0; z4  < DISTINCT; z4++)
                              {
                                for(z5 = 0; z5 < DISTINCT; z5++)
                                {
                                  if(z1 != z2 && z1 != z3 && z1 != z4
                                     && z1 != z5 && z2 != z3 && z2 != z4
                                     && z2 != z5 && z3 != z4 && z3 != z5
                                     && z4 != z5 && z2 != 1 && z3 != 2
                                     && z4 != 3 && z5 != 4 && z1 != i
                                     && z1 != x1 && z1 != y1 && z2 != j
                                     && z2 != x2 && z2 != y2 && z3 != k
                                     && z3 != x3 && z3 != y3 && z4 != l
                                     && z4 != x4 && z4 != y4 && z5 != m
                                     && z5 != x5 && z5 != y5)
                                  {
                                    //assign Row 5 elements
                                    arr[4][0] = list[z1];
                                    arr[4][1] = list[z2];
                                    arr[4][2] = list[z3];
                                    arr[4][3] = list[z4];
                                    arr[4][4] = list[z5];

                                    //commutative squares
                                    if(arr[1][0] == 'B' && arr[2][0] == 'C'
                                       && arr[3][0] == 'D' && arr[4][0] == 'E'
                                       && arr[1][2] == arr[2][1]
                                       && arr[1][3] == arr[3][1]
                                       && arr[2][3] == arr[3][2]
                                       && arr[1][4] == arr[4][1]
                                       && arr[2][4] == arr[4][2]
                                       && arr[3][4] == arr[4][3])
                                    {
                                      if(commutative == true)
                                        check4 = true;
                                      else
                                        check4 = false;
                                    }
                                    else
                                    {
                                      if(commutative == false)
                                        check4 = true;
                                      else
```

```
    check4 = false;
}

if(check4 == true)
{
  count5 = 0;

  for(x = 0; x < SIZE; x++)
  {
    for(y = 0; y < SIZE; y++)
    {
      check3 = false;

      if(count5 == 0)
      {
        temp4[count5] = arr[x][y];
        count5++;
      }

      for(z = 0; z < count5; z++)
      {
        if(arr[x][y] == temp4[z])
          check3 = true;
      }

      if(check3 == false)
      {
        temp4[count5] = arr[x][y];
        count5++;
      }
    }
  }

  if(count5 == DISTINCT)
  {
    for(i2 = 0; i2 < SIZE*SIZE; i2++)
      temp3[i2] = 'X';

    for(i2 = 0; i2 < SIZE; i2++)
      temp3[i2] = arr[0][i2];

    counter2 = 5;

    //insert position of elements into
    //temp[2]
    for(i2 = 1; i2 < SIZE; i2++)
    {
      for(j2 = 0; j2 < SIZE; j2++)
      {
        check2 = false;

        for(m1 = 0; m1 < i2+1; m1++)
        {
          if(m1 == i2)
          {
            for(n = 0; n < j2; n++)
            {
              if(arr[i2][j2] == arr[m1][n])
              {
                for(r = 0; r < SIZE*SIZE; r++)
                {
                  if(temp3[r] == arr[i2][j2])
                  {
                    check2 = true;
                    add = r;
                  }
                }
              }
            }
          }
          else
          {
            for(n = 0; n < SIZE; n++)
            {
              if(arr[i2][j2] == arr[m1][n])
              {
                for(r = 0; r < SIZE*SIZE; r++)
                {
                  if(temp3[r] == arr[i2][j2])
                  {
                    check2 = true;
                    add = r;
                  }
                }
              }
            }
          }
        }

        if(check2 == true)
        {
          temp2[counter2] = temp2[add];
          counter2++;
        }
        else
        {
          temp3[counter2] = arr[i2][j2];
          temp2[counter2] = counter2;
          counter2++;
```

```
    }
  }
}

check = false;
spOut3 = fopen("temp_array.txt", "r");
count4 = 0;

if(count > 1)
{
  do
  {
    if(check == false)
    {
      counter3 = 0;

      for(t = 0; t < SIZE*SIZE; t++)
      {
        input2 = fscanf(spOut3, "%d",
                        &input);
        temp[count4] = input;
        count4++;

        if(count4 == SIZE*SIZE)
        {
          count4 = 0;

          for(s = 0; s < SIZE*SIZE; s++)
          {
            if(temp[s] == temp2[s])
              counter3++;

            if(counter3 == SIZE*SIZE)
              check = true;
          }
        }
      }
    }
    else
      break;
  }while(input2 != EOF);

  fclose(spOut3);
}

if(check == false)
{
  count++;

  if(count > 1)
    eval = compare(arr, count);

  spOut = fopen("Squares.txt", "a");
  spOut2 = fopen("Isomorphism.txt", "a");

  fprintf(spOut, "    (%d)\n", count);
  fprintf(spOut, " A B C D E\n");
  fprintf(spOut, "%2c%2c%2c%2c%2c\n",
          arr[1][0], arr[1][1], arr[1][2],
          arr[1][3], arr[1][4]);
  fprintf(spOut, "%2c%2c%2c%2c%2c\n",
          arr[2][0], arr[2][1], arr[2][2],
          arr[2][3], arr[2][4]);
  fprintf(spOut, "%2c%2c%2c%2c%2c\n",
          arr[3][0], arr[3][1], arr[3][2],
          arr[3][3], arr[3][4]);
  fprintf(spOut, "%2c%2c%2c%2c%2c\n\n",
          arr[4][0], arr[4][1], arr[4][2],
          arr[4][3], arr[4][4]);
  fprintf(spOut2, "%6d\t%6d\n",
          count, eval);

  fclose(spOut);
  fclose(spOut2);

  spOut3 = fopen("temp_array.txt", "a");

  for(s = 0; s < SIZE*SIZE; s++)
    fprintf(spOut3, "%d ", temp2[s]);

  fprintf(spOut3, "\n");
  fclose(spOut3);

  if(eval == 0)
  {
    spOut4 = fopen("temp_arrayEC.txt", "a");

    fprintf(spOut4, "%d ", count);

    for(s = 0; s < SIZE*SIZE; s++)
      fprintf(spOut4, "%d ", temp2[s]);

    fprintf(spOut4, "\n");
    fclose(spOut4);
  }
}

for(i2 = 0; i2 < SIZE*SIZE; i2++)
{
```

```
                                                        temp2[i2] = 99;
                                                        temp[i2] = 88;
                                                      }

                                                      for(i2 = 0; i2 < SIZE; i2++)
                                                        temp2[i2] = i2;

                                                        counter2 = 5;
                                                    }
                                                  }
                                                }
                                              }
                                            }
                                          }
                                        }
                                      }
                                    }
                                  }
                                }
                              }
                            }
                          }
                        }
                      }
                    }
                  }
                }
              }
            }
          }
        }
      }
    }
  }

  system("PAUSE");
  return 0;
}

/*
  Receive arr from main function
  Perform permutations on arr to check isomorphism with representative squares
*/

int compare(char arr[SIZE][SIZE], int count)
{
  FILE* spOut1;
  FILE* spOut5;

  int r1, s1, t, u;
  int r2 = 0;
  int s2 = 0;
  int r3 = 0;
  int s3 = 0;
  int r4 = 0;
  int s4 = 0;
  int r5 = 0;
  int s5 = 0;
  int count2 = 0;
  int count3 = 0;
  int count4 = 0;
  int count5 = 0;
  int count6 = 0;
  int count7 = 0;
  int EClass = 0; //return value(ordinal number of isomorphic square)
  int flag = 0;
  int temp[SIZE*SIZE+1];
  int temp2[SIZE*SIZE];
  int input, input2, i2, j2, m, n, r, s;
  int add = 0;
  int iso = 0;
  int counter2 = 5;
  int counter3 = 0;

  char list[5]={'A','B','C', 'D', 'E'};
  char tempS[SIZE]; //temporary array used for swapping
  char temp3[SIZE*SIZE];
  char arr3[SIZE][SIZE]; //store arr and perform 1st permutation
  char arr4[SIZE][SIZE]; //store arr3 and perform 2nd permutation
  char arr5[SIZE][SIZE]; //store arr4 and perform 3rd permutation
  char arr6[SIZE][SIZE]; //store arr5 and perform 4th permutation
  char arr7[SIZE][SIZE]; //store arr6 and perform 5th permutation

  bool check = false;
  bool check2 = false;

  for(i2 = 0; i2 < SIZE*SIZE; i2++)
  {
    temp2[i2] = 99;
    temp[i2] = 88;
  }

  for(i2 = 0; i2 < SIZE; i2++)
  {
    temp2[i2] = i2;
  }

  counter2 = 5;
```

```
for(r1 = 0; r1 < SIZE; r1++) //1st permutation
{
  for(s1 = 1; s1 < SIZE; s1++)
  {
    //set arr3 = arr
    for(t = 0; t < SIZE; t++)
    {
      for(u = 0; u < SIZE; u++)
      {
        arr3[t][u] = arr[t][u];
      }
    }

    if(r1 != s1) //do not swap same rows (redundant)
    {
      //swap rows (arr3)
      tempS[0] = arr3[r1][0];
      tempS[1] = arr3[r1][1];
      tempS[2] = arr3[r1][2];
      tempS[3] = arr3[r1][3];
      tempS[4] = arr3[r1][4];

      arr3[r1][0] = arr3[s1][0];
      arr3[r1][1] = arr3[s1][1];
      arr3[r1][2] = arr3[s1][2];
      arr3[r1][3] = arr3[s1][3];
      arr3[r1][4] = arr3[s1][4];

      arr3[s1][0] = tempS[0];
      arr3[s1][1] = tempS[1];
      arr3[s1][2] = tempS[2];
      arr3[s1][3] = tempS[3];
      arr3[s1][4] = tempS[4];

      //swap columns (arr3)
      tempS[0] = arr3[0][r1];
      tempS[1] = arr3[1][r1];
      tempS[2] = arr3[2][r1];
      tempS[3] = arr3[3][r1];
      tempS[4] = arr3[4][r1];

      arr3[0][r1] = arr3[0][s1];
      arr3[1][r1] = arr3[1][s1];
      arr3[2][r1] = arr3[2][s1];
      arr3[3][r1] = arr3[3][s1];
      arr3[4][r1] = arr3[4][s1];

      arr3[0][s1] = tempS[0];
      arr3[1][s1] = tempS[1];
      arr3[2][s1] = tempS[2];
      arr3[3][s1] = tempS[3];
      arr3[4][s1] = tempS[4];

      for(i2 = 0; i2 < SIZE*SIZE; i2++)
        temp3[i2] = 'X';

      for(i2 = 0; i2 < SIZE; i2++)
        temp3[i2] = arr3[0][i2];

      counter2 = 5;

      //insert position of elements into temp2
      for(i2 = 1; i2 < SIZE; i2++)
      {
        for(j2 = 0; j2 < SIZE; j2++)
        {
          check2 = false;

          for(m = 0; m < i2+1; m++)
          {
            if(m == i2)
            {
              for(n = 0; n < j2; n++)
              {
                if(arr3[i2][j2] == arr3[m][n])
                {
                  for(r = 0; r < SIZE*SIZE; r++)
                  {
                    if(temp3[r] == arr3[i2][j2])
                    {
                      check2 = true;
                      add = r;
                    }
                  }
                }
              }
            }
            else
            {
              for(n = 0; n < SIZE; n++)
              {
                if(arr3[i2][j2] == arr3[m][n])
                {
                  for(r = 0; r < SIZE*SIZE; r++)
                  {
                    if(temp3[r] == arr3[i2][j2])
                    {
                      check2 = true;
                      add = r;
```

```
              }
            }
          }
        }
      }
    }
    if(check2 == true)
    {
      temp2[counter2] = temp2[add];
      counter2++;
    }
    else
    {
      temp3[counter2] = arr3[i2][j2];
      temp2[counter2] = counter2;
      counter2++;
    }
  }
}

check = false;
spOut5 = fopen("temp_arrayEC.txt", "r");
count2 = 0;
count4 = 0;

do
{
  if(check == false)
  {
    counter3 = 0;

    for(t = 0; t < SIZE*SIZE+1; t++)
    {
      input2 = fscanf(spOut5, "%d", &input);
      temp[count4] = input;
      count4++;

      if(count4 == SIZE*SIZE+1)
      {
        count2++;
        count4 = 0;

        for(s = 0; s < SIZE*SIZE; s++)
        {
          if(temp[s+1] == temp2[s])
            counter3++;

          if(counter3 == SIZE*SIZE)
          {
            check = true;
            iso = temp[0];
          }
        }
      }
    }
  }
  else
    break;
}while(input2 != EOF);

fclose(spOut5);

if(check == true)
{
  EClass = iso;
  return EClass;
}
else //2nd permutation
{
  for(i2 = 0; i2 < SIZE*SIZE; i2++)
  {
    temp2[i2] = 99;
    temp[i2] = 88;
  }

  for(i2 = 0; i2 < SIZE; i2++)
    temp2[i2] = i2;

  counter2 = 5;

  //assign arr3 to arr4
  for(t = 0; t < SIZE; t++)
  {
    for(u = 0; u < SIZE; u++)
    {
      arr4[t][u] = arr3[t][u];
    }
  }

  for(r2 = 0; r2 < SIZE; r2++)
  {
    for(s2 = 1; s2 < SIZE; s2++)
    {
      for(t = 0; t < SIZE; t++)
      {
        for(u = 0; u < SIZE; u++)
        {
          arr4[t][u] = arr3[t][u];
        }
```

```
}

if(r2 != s2) //do not swap same rows (redundant)
{
  //swap rows (arr4)
  tempS[0] = arr4[r2][0];
  tempS[1] = arr4[r2][1];
  tempS[2] = arr4[r2][2];
  tempS[3] = arr4[r2][3];
  tempS[4] = arr4[r2][4];

  arr4[r2][0] = arr4[s2][0];
  arr4[r2][1] = arr4[s2][1];
  arr4[r2][2] = arr4[s2][2];
  arr4[r2][3] = arr4[s2][3];
  arr4[r2][4] = arr4[s2][4];

  arr4[s2][0] = tempS[0];
  arr4[s2][1] = tempS[1];
  arr4[s2][2] = tempS[2];
  arr4[s2][3] = tempS[3];
  arr4[s2][4] = tempS[4];

  //swap columns (arr4)
  tempS[0] = arr4[0][r2];
  tempS[1] = arr4[1][r2];
  tempS[2] = arr4[2][r2];
  tempS[3] = arr4[3][r2];
  tempS[4] = arr4[4][r2];

  arr4[0][r2] = arr4[0][s2];
  arr4[1][r2] = arr4[1][s2];
  arr4[2][r2] = arr4[2][s2];
  arr4[3][r2] = arr4[3][s2];
  arr4[4][r2] = arr4[4][s2];

  arr4[0][s2] = tempS[0];
  arr4[1][s2] = tempS[1];
  arr4[2][s2] = tempS[2];
  arr4[3][s2] = tempS[3];
  arr4[4][s2] = tempS[4];

  for(i2 = 0; i2 < SIZE*SIZE; i2++)
    temp3[i2] = 'X';

  for(i2 = 0; i2 < SIZE; i2++)
    temp3[i2] = arr4[0][i2];

  counter2 = 5;

  //insert position of elements into temp2
  for(i2 = 1; i2 < SIZE; i2++)
  {
    for(j2 = 0; j2 < SIZE; j2++)
    {
      check2 = false;

      for(m = 0; m < i2+1; m++)
      {
        if(m == i2)
        {
          for(n = 0; n < j2; n++)
          {
            if(arr4[i2][j2] == arr4[m][n])
            {
              for(r = 0; r < SIZE*SIZE; r++)
              {
                if(temp3[r] == arr4[i2][j2])
                {
                  check2 = true;
                  add = r;
                }
              }
            }
          }
        }
        else
        {
          for(n = 0; n < SIZE; n++)
          {
            if(arr4[i2][j2] == arr4[m][n])
            {
              for(r = 0; r < SIZE*SIZE; r++)
              {
                if(temp3[r] == arr4[i2][j2])
                {
                  check2 = true;
                  add = r;
                }
              }
            }
          }
        }
      }

      if(check2 == true)
      {
        temp2[counter2] = temp2[add];
        counter2++;
```

```
    }
    else
    {
      temp3[counter2] = arr4[i2][j2];
      temp2[counter2] = counter2;
      counter2++;
    }
  }
}

check = false;
spOut5 = fopen("temp_arrayEC.txt", "r");
count2 = 0;
count4 = 0;

do
{
  if(check == false)
  {
    counter3 = 0;

    for(t = 0; t < SIZE*SIZE+1; t++)
    {
      input2 = fscanf(spOut5, "%d", &input);
      temp[count4] = input;
      count4++;

      if(count4 == SIZE*SIZE+1)
      {
        count2++;
        count4 = 0;

        for(s = 0; s < SIZE*SIZE; s++)
        {
          if(temp[s+1] == temp2[s])
            counter3++;

          if(counter3 == SIZE*SIZE)
          {
            check = true;
            iso = temp[0];
          }
        }
      }
    }
  }
  else
    break;
}while(input2 != EOF);

fclose(spOut5);

if(iso == count)
  check = false;

//if isomorphic, assign equivalence class representative
if(check == true)
{
  EClass = iso;
  return EClass;
}
else //3rd permutation
{
  //assign arr4 to arr5
  for(t = 0; t < SIZE; t++)
  {
    for(u = 0; u < SIZE; u++)
    {
      arr5[t][u] = arr4[t][u];
    }
  }

  for(r3 = 0; r3 < SIZE; r3++)
  {
    for(s3 = 1; s3 < SIZE; s3++)
    {
      for(t = 0; t < SIZE; t++)
      {
        for(u = 0; u < SIZE; u++)
        {
          arr5[t][u] = arr4[t][u];
        }
      }
      if(r3 != s3) //do not swap same rows (redundant)
      {
        //swap rows (arr5)
        tempS[0] = arr5[r3][0];
        tempS[1] = arr5[r3][1];
        tempS[2] = arr5[r3][2];
        tempS[3] = arr5[r3][3];
        tempS[4] = arr5[r3][4];

        arr5[r3][0] = arr5[s3][0];
        arr5[r3][1] = arr5[s3][1];
        arr5[r3][2] = arr5[s3][2];
        arr5[r3][3] = arr5[s3][3];
        arr5[r3][4] = arr5[s3][4];

        arr5[s3][0] = tempS[0];
```

```
arr5[s3][1] = tempS[1];
arr5[s3][2] = tempS[2];
arr5[s3][3] = tempS[3];
arr5[s3][4] = tempS[4];

//swap columns (arr5)
tempS[0] = arr5[0][r3];
tempS[1] = arr5[1][r3];
tempS[2] = arr5[2][r3];
tempS[3] = arr5[3][r3];
tempS[4] = arr5[4][r3];

arr5[0][r3] = arr5[0][s3];
arr5[1][r3] = arr5[1][s3];
arr5[2][r3] = arr5[2][s3];
arr5[3][r3] = arr5[3][s3];
arr5[4][r3] = arr5[4][s3];

arr5[0][s3] = tempS[0];
arr5[1][s3] = tempS[1];
arr5[2][s3] = tempS[2];
arr5[3][s3] = tempS[3];
arr5[4][s3] = tempS[4];

for(i2 = 0; i2 < SIZE*SIZE; i2++)
  temp3[i2] = 'X';

for(i2 = 0; i2 < SIZE; i2++)
  temp3[i2] = arr5[0][i2];

counter2 = 5;

//insert position of elements into temp2
for(i2 = 1; i2 < SIZE; i2++)
{
  for(j2 = 0; j2 < SIZE; j2++)
  {
    check2 = false;

    for(m = 0; m < i2+1; m++)
    {
      if(m == i2)
      {
        for(n = 0; n < j2; n++)
        {
          if(arr5[i2][j2] == arr5[m][n])
          {
            for(r = 0; r < SIZE*SIZE; r++)
            {
              if(temp3[r] == arr5[i2][j2])
              {
                check2 = true;
                add = r;
              }
            }
          }
        }
      }
      else
      {
        for(n = 0; n < SIZE; n++)
        {
          if(arr5[i2][j2] == arr5[m][n])
          {
            for(r = 0; r < SIZE*SIZE; r++)
            {
              if(temp3[r] == arr5[i2][j2])
              {
                check2 = true;
                add = r;
              }
            }
          }
        }
      }
    }

    if(check2 == true)
    {
      temp2[counter2] = temp2[add];
      counter2++;
    }
    else
    {
      temp3[counter2] = arr5[i2][j2];
      temp2[counter2] = counter2;
      counter2++;
    }
  }
}

check = false;
spOut5 = fopen("temp_arrayEC.txt", "r");
count2 = 0;
count4 = 0;

do
{
  if(check == false)
```

```
  {
    counter3 = 0;

    for(t = 0; t < SIZE*SIZE+1; t++)
    {
      input2 = fscanf(spOut5, "%d", &input);
      temp[count4] = input;
      count4++;

      if(count4 == SIZE*SIZE+1)
      {
        count2++;
        count4 = 0;

        for(s = 0; s < SIZE*SIZE; s++)
        {
          if(temp[s+1] == temp2[s])
            counter3++;

          if(counter3 == SIZE*SIZE)
          {
            check = true;
            iso = temp[0];
          }
        }
      }
    }
  }
  else
    break;
}while(input2 != EOF);

fclose(spOut5);

if(iso == count)
  check = false;

//if isomorphic, assign equivalence class representative
if(check == true)
{
  EClass = iso;
  return EClass;
}
else //4th permutation
{
  //assign arr5 to arr6
  for(t = 0; t < SIZE; t++)
  {
    for(u = 0; u < SIZE; u++)
    {
      arr6[t][u] = arr5[t][u];
    }
  }

  for(r4 = 0; r4 < SIZE; r4++)
  {
    for(s4 = 1; s4 < SIZE; s4++)
    {
      for(t = 0; t < SIZE; t++)
      {
        for(u = 0; u < SIZE; u++)
        {
          arr6[t][u] = arr5[t][u];
        }
      }

      if(r4 != s4) //do not swap same rows (redundant)
      {
        //swap rows (arr6)
        tempS[0] = arr6[r4][0];
        tempS[1] = arr6[r4][1];
        tempS[2] = arr6[r4][2];
        tempS[3] = arr6[r4][3];
        tempS[4] = arr6[r4][4];

        arr6[r4][0] = arr6[s4][0];
        arr6[r4][1] = arr6[s4][1];
        arr6[r4][2] = arr6[s4][2];
        arr6[r4][3] = arr6[s4][3];
        arr6[r4][4] = arr6[s4][4];

        arr6[s4][0] = tempS[0];
        arr6[s4][1] = tempS[1];
        arr6[s4][2] = tempS[2];
        arr6[s4][3] = tempS[3];
        arr6[s4][4] = tempS[4];

        //swap columns (arr6)
        tempS[0] = arr6[0][r4];
        tempS[1] = arr6[1][r4];
        tempS[2] = arr6[2][r4];
        tempS[3] = arr6[3][r4];
        tempS[4] = arr6[4][r4];

        arr6[0][r4] = arr6[0][s4];
        arr6[1][r4] = arr6[1][s4];
        arr6[2][r4] = arr6[2][s4];
        arr6[3][r4] = arr6[3][s4];
        arr6[4][r4] = arr6[4][s4];
```

```
arr6[0][s4] = tempS[0];
arr6[1][s4] = tempS[1];
arr6[2][s4] = tempS[2];
arr6[3][s4] = tempS[3];
arr6[4][s4] = tempS[4];

for(i2 = 0; i2 < SIZE*SIZE; i2++)
  temp3[i2] = 'X';

for(i2 = 0; i2 < SIZE; i2++)
  temp3[i2] = arr6[0][i2];

counter2 = 5;

//insert position of elements into temp2
for(i2 = 1; i2 < SIZE; i2++)
{
  for(j2 = 0; j2 < SIZE; j2++)
  {
    check2 = false;

    for(m = 0; m < i2+1; m++)
    {
      if(m == i2)
      {
        for(n = 0; n < j2; n++)
        {
          if(arr6[i2][j2] == arr6[m][n])
          {
            for(r = 0; r < SIZE*SIZE; r++)
            {
              if(temp3[r] == arr6[i2][j2])
              {
                check2 = true;
                add = r;
              }
            }
          }
        }
      }
      else
      {
        for(n = 0; n < SIZE; n++)
        {
          if(arr6[i2][j2] == arr6[m][n])
          {
            for(r = 0; r < SIZE*SIZE; r++)
            {
              if(temp3[r] == arr6[i2][j2])
              {
                check2 = true;
                add = r;
              }
            }
          }
        }
      }
    }

    if(check2 == true)
    {
      temp2[counter2] = temp2[add];
      counter2++;
    }
    else
    {
      temp3[counter2] = arr6[i2][j2];
      temp2[counter2] = counter2;
      counter2++;
    }
  }
}

check = false;
spOut5 = fopen("temp_arrayEC.txt", "r");
count2 = 0;
count4 = 0;

do
{
  if(check == false)
  {
    counter3 = 0;

    for(t = 0; t < SIZE*SIZE+1; t++)
    {
      input2 = fscanf(spOut5, "%d", &input);
      temp[count4] = input;
      count4++;

      if(count4 == SIZE*SIZE+1)
      {
        count2++;
        count4 = 0;

        for(s = 0; s < SIZE*SIZE; s++)
        {
          if(temp[s+1] == temp2[s])
```

```
            counter3++;

          if(counter3 == SIZE*SIZE)
          {
            check = true;
            iso = temp[0];
          }
        }
      }
    }
  }
  else
    break;
}while(input2 != EOF);

fclose(spOut5);

if(iso == count)
  check = false;

//if isomorphic, assign equivalence class representative
if(check == true)
{
  EClass = iso;
  return EClass;
}
else //5th permutation
{
  //assign arr6 to arr7
  for(t = 0; t < SIZE; t++)
  {
    for(u = 0; u < SIZE; u++)
    {
      arr7[t][u] = arr6[t][u];
    }
  }

  for(r5 = 0; r5 < SIZE; r5++)
  {
    for(s5 = 1; s5 < SIZE; s5++)
    {
      for(t = 0; t < SIZE; t++)
      {
        for(u = 0; u < SIZE; u++)
        {
          arr7[t][u] = arr6[t][u];
        }
      }

      if(r5 != s5) //do not swap same rows (redundant)
      {
        //swap rows (arr7)
        tempS[0] = arr7[r5][0];
        tempS[1] = arr7[r5][1];
        tempS[2] = arr7[r5][2];
        tempS[3] = arr7[r5][3];
        tempS[4] = arr7[r5][4];

        arr7[r5][0] = arr7[s5][0];
        arr7[r5][1] = arr7[s5][1];
        arr7[r5][2] = arr7[s5][2];
        arr7[r5][3] = arr7[s5][3];
        arr7[r5][4] = arr7[s5][4];

        arr7[s5][0] = tempS[0];
        arr7[s5][1] = tempS[1];
        arr7[s5][2] = tempS[2];
        arr7[s5][3] = tempS[3];
        arr7[s5][4] = tempS[4];

        //swap columns (arr7)
        tempS[0] = arr7[0][r5];
        tempS[1] = arr7[1][r5];
        tempS[2] = arr7[2][r5];
        tempS[3] = arr7[3][r5];
        tempS[4] = arr7[4][r5];

        arr7[0][r5] = arr7[0][s5];
        arr7[1][r5] = arr7[1][s5];
        arr7[2][r5] = arr7[2][s5];
        arr7[3][r5] = arr7[3][s5];
        arr7[4][r5] = arr7[4][s5];

        arr7[0][s5] = tempS[0];
        arr7[1][s5] = tempS[1];
        arr7[2][s5] = tempS[2];
        arr7[3][s5] = tempS[3];
        arr7[4][s5] = tempS[4];

        for(i2 = 0; i2 < SIZE*SIZE; i2++)
          temp3[i2] = 'X';

        for(i2 = 0; i2 < SIZE; i2++)
          temp3[i2] = arr7[0][i2];

        counter2 = 5;

        //insert position of elements into temp2
        for(i2 = 1; i2 < SIZE; i2++)
```

```
{
  for(j2 = 0; j2 < SIZE; j2++)
  {
    check2 = false;

    for(m = 0; m < i2+1; m++)
    {
      if(m == i2)
      {
        for(n = 0; n < j2; n++)
        {
          if(arr7[i2][j2] == arr7[m][n])
          {
            for(r = 0; r < SIZE*SIZE; r++)
            {
              if(temp3[r] == arr7[i2][j2])
              {
                check2 = true;
                add = r;
              }
            }
          }
        }
      }
      else
      {
        for(n = 0; n < SIZE; n++)
        {
          if(arr7[i2][j2] == arr7[m][n])
          {
            for(r = 0; r < SIZE*SIZE; r++)
            {
              if(temp3[r] == arr7[i2][j2])
              {
                check2 = true;
                add = r;
              }
            }
          }
        }
      }
    }

    if(check2 == true)
    {
      temp2[counter2] = temp2[add];
      counter2++;
    }
    else
    {
      temp3[counter2] = arr7[i2][j2];
      temp2[counter2] = counter2;
      counter2++;
    }
  }
}

check = false;
spOut5 = fopen("temp_arrayEC.txt", "r");
count2 = 0;
count4 = 0;

do
{
  if(check == false)
  {
    counter3 = 0;

    for(t = 0; t < SIZE*SIZE+1; t++)
    {
      input2 = fscanf(spOut5, "%d", &input);
      temp[count4] = input;
      count4++;

      if(count4 == SIZE*SIZE+1)
      {
        count2++;
        count4 = 0;

        for(s = 0; s < SIZE*SIZE; s++)
        {
          if(temp[s+1] == temp2[s])
            counter3++;

          if(counter3 == SIZE*SIZE)
          {
            check = true;
            iso = temp[0];
          }
        }
      }
    }
  }
  else
    break;
}while(input2 != EOF);

fclose(spOut5);
```

```
                                        if(iso == count)
                                          check = false;

                                        //if isomorphic, assign equivalence class
                                        //representative
                                        if(check == true)
                                        {
                                          EClass = iso;
                                          return EClass;
                                        }
                                      }
                                    }
                                  }
                                }
                              }
                            }
                          }
                        }
                      }
                    }
                  }
                }
              }
            }
          }
        }
      }
    }
  }

  return EClass;
}
```

APPENDIX B

LIST OF COMMUTATIVE GENERALIZED LATIN SQUARES OF ORDER 5 WITH 13 DISTINCT ELEMENTS

1	2	3	4	5	6	7
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B A D F G	B A E F G	B A F C G	B A F E G	B A F G C	B A F G D	B A F G H
C D H I J	C E H I J	C F H I J	C F H I J	C F H I J	C F H I J	C F A I J
D F I K L	D F I K L	D C I K L	D E I K L	D G I K L	D G I K L	D G I K L
E G J L M	E G J L M	E G J L M	E G J L M	E C J L M	E D J L M	E H J L M

8	9	10	11	12	13	14
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B A F G H	B A F G H	B A F G H	B A F G H	B A F G H	B A F G H	B A F G H
C F B I J	C F D I J	C F E I J	C F G I J	C F H I J	C F I A J	C F I B J
D G I K L	D G I K L	D G I K L	D G I K L	D G I K L	D G A K L	D G B K L
E H J L M	E H J L M	E H J L M	E H J L M	E H J L M	E H J L M	E H J L M

15	16	17	18	19	20	21
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B A F G H	B A F G H	B A F G H	B A F G H	B A F G H	B A F G H	B A F G H
C F I E J	C F I H J	C F I J A	C F I J B	C F I J D	C F I J G	C F I J K
D G E K L	D G H K L	D G J K L	D G J K L	D G J K L	D G J K L	D G J A L
E H J L M	E H J L M	E H A L M	E H B L M	E H D L M	E H G L M	E H K L M

22	23	24	25	26	27	28
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B A F G H	B A F G H	B A F G H	B A F G H	B A F G H	B A F G H	B A F G H
C F I J K	C F I J K	C F I J K	C F I J K	C F I J K	C F I J K	C F I J K
D G J B L	D G J C L	D G J E L	D G J F L	D G J H L	D G J I L	D G J K L
E H K L M	E H K L M	E H K L M	E H K L M	E H K L M	E H K L M	E H K L M

29	30	31	32	33	34	35
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B A F G H	B A F G H	B A F G H	B A F G H	B A F G H	B A F G H	B A F G H
C F I J K	C F I J K	C F I J K	C F I J K	C F I J K	C F I J K	C F I J K
D G J L A	D G J L B	D G J L C	D G J L F	D G J L I	D G J L M	D G J L M
E H K A M	E H K B M	E H K C M	E H K F M	E H K I M	E H K M A	E H K M B

36	37	38	39	40	41	42
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B A F G H	B A F G H	B A F G H	B A F G H	B A F G H	B A F G H	B A F G H
C F I J K	C F I J K	C F I J K	C F I J K	C F I J K	C F I J K	C F I J K
D G J L M	D G J L M	D G J L M	D G J L M	D G J L M	D G J L M	D G J L M
E H K M C	E H K M D	E H K M F	E H K M G	E H K M I	E H K M J	E H K M L

43	44	45	46	47	48	49
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B C A F G	B C D F G	B C E F G	B C F A G	B C F E G	B C F G A	B C F G D
C A H I J	C D H I J	C E H I J	C F H I J	C F H I J	C F H I J	C F H I J
D F I K L	D F I K L	D F I K L	D A I K L	D E I K L	D G I K L	D G I K L
E G J L M	E G J L M	E G J L M	E G J L M	E G J L M	E A J L M	E D J L M

50	51	52	53	54	55	56
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B C F G H	B C F G H	B C F G H	B C F G H	B C F G H	B C F G H	B C F G H
C F A I J	C F B I J	C F D I J	C F E I J	C F G I J	C F H I J	C F I A J
D G I K L	D G I K L	D G I K L	D G I K L	D G I K L	D G I K L	D G A K L
E H J L M	E H J L M	E H J L M	E H J L M	E H J L M	E H J L M	E H J L M

57	58	59	60	61	62	63
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B C F G H	B C F G H	B C F G H	B C F G H	B C F G H	B C F G H	B C F G H
C F I B J	C F I E J	C F I H J	C F I J A	C F I J B	C F I J D	C F I J G
D G B K L	D G E K L	D G H K L	D G J K L	D G J K L	D G J K L	D G J K L
E H J L M	E H J L M	E H J L M	E H A L M	E H B L M	E H D L M	E H G L M

64	65	66	67	68	69	70
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B C F G H	B C F G H	B C F G H	B C F G H	B C F G H	B C F G H	B C F G H
C F I J K	C F I J K	C F I J K	C F I J K	C F I J K	C F I J K	C F I J K
D G J A L	D G J B L	D G J C L	D G J E L	D G J F L	D G J H L	D G J I L
E H K L M	E H K L M	E H K L M	E H K L M	E H K L M	E H K L M	E H K L M

71	72	73	74	75	76	77
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B C F G H	B C F G H	B C F G H	B C F G H	B C F G H	B C F G H	B C F G H
C F I J K	C F I J K	C F I J K	C F I J K	C F I J K	C F I J K	C F I J K
D G J K L	D G J L A	D G J L B	D G J L C	D G J L F	D G J L I	D G J L M
E H K L M	E H K A M	E H K B M	E H K C M	E H K F M	E H K I M	E H K M A

78	79	80	81	82	83	84
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B C F G H	B C F G H	B C F G H	B C F G H	B C F G H	B C F G H	B C F G H
C F I J K	C F I J K	C F I J K	C F I J K	C F I J K	C F I J K	C F I J K
D G J L M	D G J L M	D G J L M	D G J L M	D G J L M	D G J L M	D G J L M
E H K M B	E H K M C	E H K M D	E H K M F	E H K M G	E H K M I	E H K M J

85	86	87	88	89	90	91
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B C F G H	B D A F G	B D E F G	B D F A G	B D F C G	B D F E G	B D F G A
C F I J K	C A H I J	C E H I J	C F H I J	C F H I J	C F H I J	C F H I J
D G J L M	D F I K L	D F I K L	D A I K L	D C I K L	D E I K L	D G I K L
E H K M L	E G J L M	E G J L M	E G J L M	E G J L M	E G J L M	E A J L M

92	93	94	95	96	97	98
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B D F G C	B D F G H	B D F G H	B D F G H	B D F G H	B D F G H	B D F G H
C F H I J	C F A I J	C F B I J	C F D I J	C F E I J	C F G I J	C F H I J
D G I K L	D G I K L	D G I K L	D G I K L	D G I K L	D G I K L	D G I K L
E C J L M	E H J L M	E H J L M	E H J L M	E H J L M	E H J L M	E H J L M

99	100	101	102	103	104	105
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B D F G H	B D F G H	B D F G H	B D F G H	B D F G H	B D F G H	B D F G H
C F I A J	C F I B J	C F I E J	C F I H J	C F I J A	C F I J B	C F I J D
D G A K L	D G B K L	D G E K L	D G H K L	D G J K L	D G J K L	D G J K L
E H J L M	E H J L M	E H J L M	E H J L M	E H A L M	E H B L M	E H D L M

106	107	108	109	110	111	112
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B D F G H	B D F G H	B D F G H	B D F G H	B D F G H	B D F G H	B D F G H
C F I J G	C F I J K	C F I J K	C F I J K	C F I J K	C F I J K	C F I J K
D G J K L	D G J A L	D G J B L	D G J C L	D G J E L	D G J F L	D G J H L
E H G L M	E H K L M	E H K L M	E H K L M	E H K L M	E H K L M	E H K L M

113	114	115	116	117	118	119
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B D F G H	B D F G H	B D F G H	B D F G H	B D F G H	B D F G H	B D F G H
C F I J K	C F I J K	C F I J K	C F I J K	C F I J K	C F I J K	C F I J K
D G J I L	D G J K L	D G J L A	D G J L B	D G J L C	D G J L F	D G J L I
E H K L M	E H K L M	E H K A M	E H K B M	E H K C M	E H K F M	E H K I M

120	121	122	123	124	125	126
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B D F G H	B D F G H	B D F G H	B D F G H	B D F G H	B D F G H	B D F G H
C F I J K	C F I J K	C F I J K	C F I J K	C F I J K	C F I J K	C F I J K
D G J L M	D G J L M	D G J L M	D G J L M	D G J L M	D G J L M	D G J L M
E H K M A	E H K M B	E H K M C	E H K M D	E H K M F	E H K M G	E H K M I

127	128	129	130	131	132	133
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B D F G H	B D F G H	B E A F G	B E D F G	B E F A G	B E F C G	B E F G A
C F I J K	C F I J K	C A H I J	C D H I J	C F H I J	C F H I J	C F H I J
D G J L M	D G J L M	D F I K L	D F I K L	D A I K L	D C I K L	D G I K L
E H K M J	E H K M L	E G J L M	E G J L M	E G J L M	E G J L M	E A J L M

134	135	136	137	138	139	140
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B E F G C	B E F G D	B E F G H	B E F G H	B E F G H	B E F G H	B E F G H
C F H I J	C F H I J	C F A I J	C F B I J	C F D I J	C F E I J	C F G I J
D G I K L	D G I K L	D G I K L	D G I K L	D G I K L	D G I K L	D G I K L
E C J L M	E D J L M	E H J L M	E H J L M	E H J L M	E H J L M	E H J L M

141	142	143	144	145	146	147
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B E F G H	B E F G H	B E F G H	B E F G H	B E F G H	B E F G H	B E F G H
C F H I J	C F I A J	C F I B J	C F I E J	C F I H J	C F I J A	C F I J B
D G I K L	D G A K L	D G B K L	D G E K L	D G H K L	D G J K L	D G J K L
E H J L M	E H J L M	E H J L M	E H J L M	E H J L M	E H A L M	E H B L M

148	149	150	151	152	153	154
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B E F G H	B E F G H	B E F G H	B E F G H	B E F G H	B E F G H	B E F G H
C F I J D	C F I J G	C F I J K	C F I J K	C F I J K	C F I J K	C F I J K
D G J K L	D G J K L	D G J A L	D G J B L	D G J C L	D G J E L	D G J F L
E H D L M	E H G L M	E H K L M	E H K L M	E H K L M	E H K L M	E H K L M

155	156	157	158	159	160	161
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B E F G H	B E F G H	B E F G H	B E F G H	B E F G H	B E F G H	B E F G H
C F I J K	C F I J K	C F I J K	C F I J K	C F I J K	C F I J K	C F I J K
D G J H L	D G J I L	D G J K L	D G J L A	D G J L B	D G J L C	D G J L F
E H K L M	E H K L M	E H K L M	E H K A M	E H K B M	E H K C M	E H K F M

162	163	164	165	166	167	168
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B E F G H	B E F G H	B E F G H	B E F G H	B E F G H	B E F G H	B E F G H
C F I J K	C F I J K	C F I J K	C F I J K	C F I J K	C F I J K	C F I J K
D G J L I	D G J L M	D G J L M	D G J L M	D G J L M	D G J L M	D G J L M
E H K I M	E H K M A	E H K M B	E H K M C	E H K M D	E H K M F	E H K M G

169	170	171	172	173	174	175
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B E F G H	B E F G H	B E F G H	B F A C G	B F A E G	B F A G C	B F A G D
C F I J K	C F I J K	C F I J K	C A H I J	C A H I J	C A H I J	C A H I J
D G J L M	D G J L M	D G J L M	D C I K L	D E I K L	D G I K L	D G I K L
E H K M I	E H K M J	E H K M L	E G J L M	E G J L M	E C J L M	E D J L M

176	177	178	179	180	181	182
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F A G H	B F A G H	B F A G H	B F A G H	B F A G H	B F A G H	B F A G H
C A B I J	C A D I J	C A E I J	C A F I J	C A G I J	C A H I J	C A I B J
D G I K L	D G I K L	D G I K L	D G I K L	D G I K L	D G I K L	D G B K L
E H J L M	E H J L M	E H J L M	E H J L M	E H J L M	E H J L M	E H J L M

183	184	185	186	187	188	189
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F A G H	B F A G H	B F A G H	B F A G H	B F A G H	B F A G H	B F A G H
C A I E J	C A I F J	C A I H J	C A I J B	C A I J D	C A I J F	C A I J G
D G E K L	D G F K L	D G H K L	D G J K L	D G J K L	D G J K L	D G J K L
E H J L M	E H J L M	E H J L M	E H B L M	E H D L M	E H F L M	E H G L M

190	191	192	193	194	195	196
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F A G H	B F A G H	B F A G H	B F A G H	B F A G H	B F A G H	B F A G H
C A I J K	C A I J K	C A I J K	C A I J K	C A I J K	C A I J K	C A I J K
D G J A L	D G J B L	D G J C L	D G J E L	D G J F L	D G J H L	D G J I L
E H K L M	E H K L M	E H K L M	E H K L M	E H K L M	E H K L M	E H K L M

197	198	199	200	201	202	203
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F A G H	B F A G H	B F A G H	B F A G H	B F A G H	B F A G H	B F A G H
C A I J K	C A I J K	C A I J K	C A I J K	C A I J K	C A I J K	C A I J K
D G J K L	D G J L A	D G J L B	D G J L C	D G J L F	D G J L I	D G J L M
E H K L M	E H K A M	E H K B M	E H K C M	E H K F M	E H K I M	E H K M A

204	205	206	207	208	209	210
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F A G H	B F A G H	B F A G H	B F A G H	B F A G H	B F A G H	B F A G H
C A I J K	C A I J K	C A I J K	C A I J K	C A I J K	C A I J K	C A I J K
D G J L M	D G J L M	D G J L M	D G J L M	D G J L M	D G J L M	D G J L M
E H K M B	E H K M C	E H K M D	E H K M F	E H K M G	E H K M I	E H K M J

211	212	213	214	215	216	217
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F A G H	B F D A G	B F D C G	B F D E G	B F D G A	B F D G C	B F D G H
C A I J K	C D H I J	C D H I J	C D H I J	C D H I J	C D H I J	C D A I J
D G J L M	D A I K L	D C I K L	D E I K L	D G I K L	D G I K L	D G I K L
E H K M L	E G J L M	E G J L M	E G J L M	E A J L M	E C J L M	E H J L M

218	219	220	221	222	223	224
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F D G H	B F D G H	B F D G H	B F D G H	B F D G H	B F D G H	B F D G H
C D B I J	C D E I J	C D F I J	C D G I J	C D H I J	C D I A J	C D I B J
D G I K L	D G I K L	D G I K L	D G I K L	D G I K L	D G A K L	D G B K L
E H J L M	E H J L M	E H J L M	E H J L M	E H J L M	E H J L M	E H J L M

225	226	227	228	229	230	231
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F D G H	B F D G H	B F D G H	B F D G H	B F D G H	B F D G H	B F D G H
C D I E J	C D I F J	C D I H J	C D I J A	C D I J B	C D I J F	C D I J G
D G E K L	D G F K L	D G H K L	D G J K L	D G J K L	D G J K L	D G J K L
E H J L M	E H J L M	E H J L M	E H A L M	E H B L M	E H F L M	E H G L M

232	233	234	235	236	237	238
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F D G H	B F D G H	B F D G H	B F D G H	B F D G H	B F D G H	B F D G H
C D I J K	C D I J K	C D I J K	C D I J K	C D I J K	C D I J K	C D I J K
D G J A L	D G J B L	D G J C L	D G J E L	D G J F L	D G J H L	D G J I L
E H K L M	E H K L M	E H K L M	E H K L M	E H K L M	E H K L M	E H K L M

239	240	241	242	243	244	245
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F D G H	B F D G H	B F D G H	B F D G H	B F D G H	B F D G H	B F D G H
C D I J K	C D I J K	C D I J K	C D I J K	C D I J K	C D I J K	C D I J K
D G J K L	D G J L A	D G J L B	D G J L C	D G J L F	D G J L I	D G J L M
E H K L M	E H K A M	E H K B M	E H K C M	E H K F M	E H K I M	E H K M A

246	247	248	249	250	251	252
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F D G H	B F D G H	B F D G H	B F D G H	B F D G H	B F D G H	B F D G H
C D I J K	C D I J K	C D I J K	C D I J K	C D I J K	C D I J K	C D I J K
D G J L M	D G J L M	D G J L M	D G J L M	D G J L M	D G J L M	D G J L M
E H K M B	E H K M C	E H K M D	E H K M F	E H K M G	E H K M I	E H K M J

253	254	255	256	257	258	259
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F D G H	B F E A G	B F E C G	B F E G A	B F E G C	B F E G D	B F E G H
C D I J K	C E H I J	C E H I J	C E H I J	C E H I J	C E H I J	C E A I J
D G J L M	D A I K L	D C I K L	D G I K L	D G I K L	D G I K L	D G I K L
E H K M L	E G J L M	E G J L M	E A J L M	E C J L M	E D J L M	E H J L M

260	261	262	263	264	265	266
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F E G H	B F E G H	B F E G H	B F E G H	B F E G H	B F E G H	B F E G H
C E B I J	C E D I J	C E F I J	C E G I J	C E H I J	C E I A J	C E I B J
D G I K L	D G I K L	D G I K L	D G I K L	D G I K L	D G A K L	D G B K L
E H J L M	E H J L M	E H J L M	E H J L M	E H J L M	E H J L M	E H J L M

267	268	269	270	271	272	273
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F E G H	B F E G H	B F E G H	B F E G H	B F E G H	B F E G H	B F E G H
C E I F J	C E I H J	C E I J A	C E I J B	C E I J D	C E I J F	C E I J G
D G F K L	D G H K L	D G J K L	D G J K L	D G J K L	D G J K L	D G J K L
E H J L M	E H J L M	E H A L M	E H B L M	E H D L M	E H F L M	E H G L M

274	275	276	277	278	279	280
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F E G H	B F E G H	B F E G H	B F E G H	B F E G H	B F E G H	B F E G H
C E I J K	C E I J K	C E I J K	C E I J K	C E I J K	C E I J K	C E I J K
D G J A L	D G J B L	D G J C L	D G J E L	D G J F L	D G J H L	D G J I L
E H K L M	E H K L M	E H K L M	E H K L M	E H K L M	E H K L M	E H K L M

281	282	283	284	285	286	287
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F E G H	B F E G H	B F E G H	B F E G H	B F E G H	B F E G H	B F E G H
C E I J K	C E I J K	C E I J K	C E I J K	C E I J K	C E I J K	C E I J K
D G J K L	D G J L A	D G J L B	D G J L C	D G J L F	D G J L I	D G J L M
E H K L M	E H K A M	E H K B M	E H K C M	E H K F M	E H K I M	E H K M A

288	289	290	291	292	293	294
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F E G H	B F E G H	B F E G H	B F E G H	B F E G H	B F E G H	B F E G H
C E I J K	C E I J K	C E I J K	C E I J K	C E I J K	C E I J K	C E I J K
D G J L M	D G J L M	D G J L M	D G J L M	D G J L M	D G J L M	D G J L M
E H K M B	E H K M C	E H K M D	E H K M F	E H K M G	E H K M I	E H K M J

295	296	297	298	299	300	301
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F E G H	B F G A C	B F G A D	B F G A H	B F G A H	B F G A H	B F G A H
C E I J K	C G H I J	C G H I J	C G A I J	C G B I J	C G D I J	C G E I J
D G J L M	D A I K L	D A I K L	D A I K L	D A I K L	D A I K L	D A I K L
E H K M L	E C J L M	E D J L M	E H J L M	E H J L M	E H J L M	E H J L M

302	303	304	305	306	307	308
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G A H	B F G A H	B F G A H	B F G A H	B F G A H	B F G A H	B F G A H
C G F I J	C G H I J	C G I B J	C G I E J	C G I F J	C G I H J	C G I J A
D A I K L	D A I K L	D A B K L	D A E K L	D A F K L	D A H K L	D A J K L
E H J L M	E H J L M	E H J L M	E H J L M	E H J L M	E H J L M	E H A L M

309	310	311	312	313	314	315
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G A H	B F G A H	B F G A H	B F G A H	B F G A H	B F G A H	B F G A H
C G I J B	C G I J D	C G I J F	C G I J K	C G I J K	C G I J K	C G I J K
D A J K L	D A J K L	D A J K L	D A J B L	D A J C L	D A J E L	D A J F L
E H B L M	E H D L M	E H F L M	E H K L M	E H K L M	E H K L M	E H K L M

316	317	318	319	320	321	322
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G A H	B F G A H	B F G A H	B F G A H	B F G A H	B F G A H	B F G A H
C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K
D A J G L	D A J H L	D A J I L	D A J K L	D A J L B	D A J L C	D A J L F
E H K L M	E H K L M	E H K L M	E H K L M	E H K B M	E H K C M	E H K F M

323	324	325	326	327	328	329
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G A H	B F G A H	B F G A H	B F G A H	B F G A H	B F G A H	B F G A H
C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K
D A J L G	D A J L I	D A J L M	D A J L M	D A J L M	D A J L M	D A J L M
E H K G M	E H K I M	E H K M A	E H K M B	E H K M C	E H K M D	E H K M F

330	331	332	333	334	335	336
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G A H	B F G A H	B F G A H	B F G A H	B F G C A	B F G C D	B F G C H
C G I J K	C G I J K	C G I J K	C G I J K	C G H I J	C G H I J	C G A I J
D A J L M	D A J L M	D A J L M	D A J L M	D C I K L	D C I K L	D C I K L
E H K M G	E H K M I	E H K M J	E H K M L	E A J L M	E D J L M	E H J L M

337	338	339	340	341	342	343
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G C H	B F G C H	B F G C H	B F G C H	B F G C H	B F G C H	B F G C H
C G B I J	C G D I J	C G E I J	C G F I J	C G H I J	C G I A J	C G I B J
D C I K L	D C I K L	D C I K L	D C I K L	D C I K L	D C A K L	D C B K L
E H J L M	E H J L M	E H J L M	E H J L M	E H J L M	E H J L M	E H J L M

344	345	346	347	348	349	350
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G C H	B F G C H	B F G C H	B F G C H	B F G C H	B F G C H	B F G C H
C G I E J	C G I F J	C G I H J	C G I J A	C G I J B	C G I J D	C G I J F
D C E K L	D C F K L	D C H K L	D C J K L	D C J K L	D C J K L	D C J K L
E H J L M	E H J L M	E H J L M	E H A L M	E H B L M	E H D L M	E H F L M

351	352	353	354	355	356	357
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G C H	B F G C H	B F G C H	B F G C H	B F G C H	B F G C H	B F G C H
C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K
D C J A L	D C J B L	D C J E L	D C J F L	D C J G L	D C J H L	D C J I L
E H K L M	E H K L M	E H K L M	E H K L M	E H K L M	E H K L M	E H K L M

358	359	360	361	362	363	364
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G C H	B F G C H	B F G C H	B F G C H	B F G C H	B F G C H	B F G C H
C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K
D C J K L	D C J L A	D C J L B	D C J L F	D C J L G	D C J L I	D C J L M
E H K L M	E H K A M	E H K B M	E H K F M	E H K G M	E H K I M	E H K M A

365	366	367	368	369	370	371
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G C H	B F G C H	B F G C H	B F G C H	B F G C H	B F G C H	B F G C H
C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K
D C J L M	D C J L M	D C J L M	D C J L M	D C J L M	D C J L M	D C J L M
E H K M B	E H K M C	E H K M D	E H K M F	E H K M G	E H K M I	E H K M J

372	373	374	375	376	377	378
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G C H	B F G E A	B F G E C	B F G E D	B F G E H	B F G E H	B F G E H
C G I J K	C G H I J	C G H I J	C G H I J	C G A I J	C G B I J	C G D I J
D C J L M	D E I K L	D E I K L	D E I K L	D E I K L	D E I K L	D E I K L
E H K M L	E A J L M	E C J L M	E D J L M	E H J L M	E H J L M	E H J L M

379	380	381	382	383	384	385
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G E H	B F G E H	B F G E H	B F G E H	B F G E H	B F G E H	B F G E H
C G E I J	C G F I J	C G H I J	C G I A J	C G I B J	C G I F J	C G I H J
D E I K L	D E I K L	D E I K L	D E A K L	D E B K L	D E F K L	D E H K L
E H J L M	E H J L M	E H J L M	E H J L M	E H J L M	E H J L M	E H J L M

386	387	388	389	390	391	392
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G E H	B F G E H	B F G E H	B F G E H	B F G E H	B F G E H	B F G E H
C G I J A	C G I J B	C G I J D	C G I J F	C G I J K	C G I J K	C G I J K
D E J K L	D E J K L	D E J K L	D E J K L	D E J A L	D E J B L	D E J C L
E H A L M	E H B L M	E H D L M	E H F L M	E H K L M	E H K L M	E H K L M

393	394	395	396	397	398	399
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G E H	B F G E H	B F G E H	B F G E H	B F G E H	B F G E H	B F G E H
C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K
D E J F L	D E J G L	D E J H L	D E J I L	D E J K L	D E J L A	D E J L B
E H K L M	E H K L M	E H K L M	E H K L M	E H K L M	E H K A M	E H K B M

400	401	402	403	404	405	406
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G E H	B F G E H	B F G E H	B F G E H	B F G E H	B F G E H	B F G E H
C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K
D E J L C	D E J L F	D E J L G	D E J L I	D E J L M	D E J L M	D E J L M
E H K C M	E H K F M	E H K G M	E H K I M	E H K M A	E H K M B	E H K M C

407	408	409	410	411	412	413
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G E H	B F G E H	B F G E H	B F G E H	B F G E H	B F G E H	B F G H A
C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G A I J
D E J L M	D E J L M	D E J L M	D E J L M	D E J L M	D E J L M	D H I K L
E H K M D	E H K M F	E H K M G	E H K M I	E H K M J	E H K M L	E A J L M

414	415	416	417	418	419	420
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H A	B F G H A	B F G H A	B F G H A	B F G H A	B F G H A	B F G H A
C G B I J	C G D I J	C G E I J	C G F I J	C G H I J	C G I A J	C G I B J
D H I K L	D H I K L	D H I K L	D H I K L	D H I K L	D H A K L	D H B K L
E A J L M	E A J L M	E A J L M	E A J L M	E A J L M	E A J L M	E A J L M

421	422	423	424	425	426	427
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H A	B F G H A	B F G H A	B F G H A	B F G H A	B F G H A	B F G H A
C G I E J	C G I F J	C G I J B	C G I J D	C G I J F	C G I J H	C G I J K
D H E K L	D H F K L	D H J K L	D H J K L	D H J K L	D H J K L	D H J A L
E A J L M	E A J L M	E A B L M	E A D L M	E A F L M	E A H L M	E A K L M

428	429	430	431	432	433	434
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H A	B F G H A	B F G H A	B F G H A	B F G H A	B F G H A	B F G H A
C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K
D H J B L	D H J C L	D H J E L	D H J F L	D H J G L	D H J I L	D H J K L
E A K L M	E A K L M	E A K L M	E A K L M	E A K L M	E A K L M	E A K L M

435	436	437	438	439	440	441
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H A	B F G H A	B F G H A	B F G H A	B F G H A	B F G H A	B F G H A
C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K
D H J L B	D H J L C	D H J L F	D H J L G	D H J L I	D H J L M	D H J L M
E A K B M	E A K C M	E A K F M	E A K G M	E A K I M	E A K M B	E A K M C

442	443	444	445	446	447	448
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H A	B F G H A	B F G H A	B F G H A	B F G H A	B F G H A	B F G H A
C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K
D H J L M	D H J L M	D H J L M	D H J L M	D H J L M	D H J L M	D H J L M
E A K M D	E A K M F	E A K M G	E A K M H	E A K M I	E A K M J	E A K M L

449	450	451	452	453	454	455
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H C	B F G H C	B F G H C	B F G H C	B F G H C	B F G H C	B F G H C
C G A I J	C G B I J	C G D I J	C G E I J	C G F I J	C G H I J	C G I A J
D H I K L	D H I K L	D H I K L	D H I K L	D H I K L	D H I K L	D H A K L
E C J L M	E C J L M	E C J L M	E C J L M	E C J L M	E C J L M	E C J L M

456	457	458	459	460	461	462
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H C	B F G H C	B F G H C	B F G H C	B F G H C	B F G H C	B F G H C
C G I B J	C G I E J	C G I F J	C G I J A	C G I J B	C G I J D	C G I J F
D H B K L	D H E K L	D H F K L	D H J K L	D H J K L	D H J K L	D H J K L
E C J L M	E C J L M	E C J L M	E C A L M	E C B L M	E C D L M	E C F L M

463	464	465	466	467	468	469
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H C	B F G H C	B F G H C	B F G H C	B F G H C	B F G H C	B F G H C
C G I J H	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K
D H J K L	D H J A L	D H J B L	D H J C L	D H J E L	D H J F L	D H J G L
E C H L M	E C K L M	E C K L M	E C K L M	E C K L M	E C K L M	E C K L M

470	471	472	473	474	475	476
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H C	B F G H C	B F G H C	B F G H C	B F G H C	B F G H C	B F G H C
C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K
D H J I L	D H J K L	D H J L A	D H J L B	D H J L F	D H J L G	D H J L I
E C K L M	E C K L M	E C K A M	E C K B M	E C K F M	E C K G M	E C K I M

477	478	479	480	481	482	483
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H C	B F G H C	B F G H C	B F G H C	B F G H C	B F G H C	B F G H C
C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K
D H J L M	D H J L M	D H J L M	D H J L M	D H J L M	D H J L M	D H J L M
E C K M A	E C K M B	E C K M D	E C K M F	E C K M G	E C K M H	E C K M I

484	485	486	487	488	489	490
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H C	B F G H C	B F G H D	B F G H D	B F G H D	B F G H D	B F G H D
C G I J K	C G I J K	C G A I J	C G B I J	C G D I J	C G E I J	C G F I J
D H J L M	D H J L M	D H I K L	D H I K L	D H I K L	D H I K L	D H I K L
E C K M J	E C K M L	E D J L M	E D J L M	E D J L M	E D J L M	E D J L M

491	492	493	494	495	496	497
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H D	B F G H D	B F G H D	B F G H D	B F G H D	B F G H D	B F G H D
C G H I J	C G I A J	C G I B J	C G I E J	C G I F J	C G I J A	C G I J B
D H I K L	D H A K L	D H B K L	D H E K L	D H F K L	D H J K L	D H J K L
E D J L M	E D J L M	E D J L M	E D J L M	E D J L M	E D A L M	E D B L M

498	499	500	501	502	503	504
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H D	B F G H D	B F G H D	B F G H D	B F G H D	B F G H D	B F G H D
C G I J F	C G I J H	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K
D H J K L	D H J K L	D H J A L	D H J B L	D H J C L	D H J E L	D H J F L
E D F L M	E D H L M	E D K L M	E D K L M	E D K L M	E D K L M	E D K L M

505	506	507	508	509	510	511
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H D	B F G H D	B F G H D	B F G H D	B F G H D	B F G H D	B F G H D
C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K
D H J G L	D H J I L	D H J K L	D H J L A	D H J L B	D H J L C	D H J L F
E D K L M	E D K L M	E D K L M	E D K A M	E D K B M	E D K C M	E D K F M

512	513	514	515	516	517	518
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H D	B F G H D	B F G H D	B F G H D	B F G H D	B F G H D	B F G H D
C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K
D H J L G	D H J L I	D H J L M	D H J L M	D H J L M	D H J L M	D H J L M
E D K G M	E D K I M	E D K M A	E D K M B	E D K M C	E D K M F	E D K M G

519	520	521	522	523	524	525
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H D	B F G H D	B F G H D	B F G H D	B F G H I	B F G H I	B F G H I
C G I J K	C G I J K	C G I J K	C G I J K	C G A B J	C G A E J	C G A F J
D H J L M	D H J L M	D H J L M	D H J L M	D H B K L	D H E K L	D H F K L
E D K M H	E D K M I	E D K M J	E D K M L	E I J L M	E I J L M	E I J L M

526	527	528	529	530	531	532
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G A I J	C G A J B	C G A J D	C G A J F	C G A J H	C G A J K	C G A J K
D H I K L	D H J K L	D H J K L	D H J K L	D H J K L	D H J A L	D H J B L
E I J L M	E I B L M	E I D L M	E I F L M	E I H L M	E I K L M	E I K L M

533	534	535	536	537	538	539
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G A J K	C G A J K	C G A J K	C G A J K	C G A J K	C G A J K	C G A J K
D H J C L	D H J E L	D H J F L	D H J G L	D H J I L	D H J K L	D H J L A
E I K L M	E I K L M	E I K L M	E I K L M	E I K L M	E I K L M	E I K A M

540	541	542	543	544	545	546
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G A J K	C G A J K	C G A J K	C G A J K	C G A J K	C G A J K	C G A J K
D H J L B	D H J L C	D H J L F	D H J L G	D H J L M	D H J L M	D H J L M
E I K B M	E I K C M	E I K F M	E I K G M	E I K M A	E I K M B	E I K M C

547	548	549	550	551	552	553
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G A J K	C G A J K	C G A J K	C G A J K	C G A J K	C G A J K	C G B A J
D H J L M	D H J L M	D H J L M	D H J L M	D H J L M	D H J L M	D H A K L
E I K M D	E I K M F	E I K M G	E I K M H	E I K M J	E I K M L	E I J L M

554	555	556	557	558	559	560
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G B E J	C G B F J	C G B I J	C G B J A	C G B J D	C G B J F	C G B J H
D H E K L	D H F K L	D H I K L	D H J K L	D H J K L	D H J K L	D H J K L
E I J L M	E I J L M	E I J L M	E I A L M	E I D L M	E I F L M	E I H L M

561	562	563	564	565	566	567
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G B J K	C G B J K	C G B J K	C G B J K	C G B J K	C G B J K	C G B J K
D H J A L	D H J B L	D H J C L	D H J E L	D H J F L	D H J G L	D H J I L
E I K L M	E I K L M	E I K L M	E I K L M	E I K L M	E I K L M	E I K L M

568	569	570	571	572	573	574
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G B J K	C G B J K	C G B J K	C G B J K	C G B J K	C G B J K	C G B J K
D H J K L	D H J L A	D H J L B	D H J L C	D H J L F	D H J L G	D H J L M
E I K L M	E I K A M	E I K B M	E I K C M	E I K F M	E I K G M	E I K M A

575	576	577	578	579	580	581
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G B J K	C G B J K	C G B J K	C G B J K	C G B J K	C G B J K	C G B J K
D H J L M	D H J L M	D H J L M	D H J L M	D H J L M	D H J L M	D H J L M
E I K M B	E I K M C	E I K M D	E I K M F	E I K M G	E I K M H	E I K M J

582	583	584	585	586	587	588
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G B J K	C G D A J	C G D B J	C G D E J	C G D F J	C G D I J	C G D J A
D H J L M	D H A K L	D H B K L	D H E K L	D H F K L	D H I K L	D H J K L
E I K M L	E I J L M	E I J L M	E I J L M	E I J L M	E I J L M	E I A L M

589	590	591	592	593	594	595
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G D J B	C G D J F	C G D J H	C G D J K	C G D J K	C G D J K	C G D J K
D H J K L	D H J K L	D H J K L	D H J A L	D H J B L	D H J C L	D H J E L
E I B L M	E I F L M	E I H L M	E I K L M	E I K L M	E I K L M	E I K L M

596	597	598	599	600	601	602
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G D J K	C G D J K	C G D J K	C G D J K	C G D J K	C G D J K	C G D J K
D H J F L	D H J G L	D H J I L	D H J K L	D H J L A	D H J L B	D H J L C
E I K L M	E I K L M	E I K L M	E I K L M	E I K A M	E I K B M	E I K C M

603	604	605	606	607	608	609
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G D J K	C G D J K	C G D J K	C G D J K	C G D J K	C G D J K	C G D J K
D H J L F	D H J L G	D H J L M	D H J L M	D H J L M	D H J L M	D H J L M
E I K F M	E I K G M	E I K M A	E I K M B	E I K M C	E I K M D	E I K M F

610	611	612	613	614	615	616
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G D J K	C G D J K	C G D J K	C G D J K	C G E A J	C G E B J	C G E F J
D H J L M	D H J L M	D H J L M	D H J L M	D H A K L	D H B K L	D H F K L
E I K M G	E I K M H	E I K M J	E I K M L	E I J L M	E I J L M	E I J L M

617	618	619	620	621	622	623
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G E I J	C G E J A	C G E J B	C G E J D	C G E J F	C G E J H	C G E J K
D H I K L	D H J K L	D H J K L	D H J K L	D H J K L	D H J K L	D H J A L
E I J L M	E I A L M	E I B L M	E I D L M	E I F L M	E I H L M	E I K L M

624	625	626	627	628	629	630
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G E J K	C G E J K	C G E J K	C G E J K	C G E J K	C G E J K	C G E J K
D H J B L	D H J C L	D H J E L	D H J F L	D H J G L	D H J I L	D H J K L
E I K L M	E I K L M	E I K L M	E I K L M	E I K L M	E I K L M	E I K L M

631	632	633	634	635	636	637
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G E J K	C G E J K	C G E J K	C G E J K	C G E J K	C G E J K	C G E J K
D H J L A	D H J L B	D H J L C	D H J L F	D H J L G	D H J L M	D H J L M
E I K A M	E I K B M	E I K C M	E I K F M	E I K G M	E I K M A	E I K M B

638	639	640	641	642	643	644
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G E J K	C G E J K	C G E J K	C G E J K	C G E J K	C G E J K	C G E J K
D H J L M	D H J L M	D H J L M	D H J L M	D H J L M	D H J L M	D H J L M
E I K M C	E I K M D	E I K M F	E I K M G	E I K M H	E I K M J	E I K M L

645	646	647	648	649	650	651
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G F A J	C G F B J	C G F E J	C G F I J	C G F J A	C G F J B	C G F J D
D H A K L	D H B K L	D H E K L	D H I K L	D H J K L	D H J K L	D H J K L
E I J L M	E I J L M	E I J L M	E I J L M	E I A L M	E I B L M	E I D L M

652	653	654	655	656	657	658
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G F J H	C G F J K	C G F J K	C G F J K	C G F J K	C G F J K	C G F J K
D H J K L	D H J A L	D H J B L	D H J C L	D H J E L	D H J F L	D H J G L
E I H L M	E I K L M	E I K L M	E I K L M	E I K L M	E I K L M	E I K L M

659	660	661	662	663	664	665
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G F J K	C G F J K	C G F J K	C G F J K	C G F J K	C G F J K	C G F J K
D H J I L	D H J K L	D H J L A	D H J L B	D H J L C	D H J L F	D H J L G
E I K L M	E I K L M	E I K A M	E I K B M	E I K C M	E I K F M	E I K G M

666	667	668	669	670	671	672
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G F J K	C G F J K	C G F J K	C G F J K	C G F J K	C G F J K	C G F J K
D H J L M	D H J L M	D H J L M	D H J L M	D H J L M	D H J L M	D H J L M
E I K M A	E I K M B	E I K M C	E I K M D	E I K M F	E I K M G	E I K M H

673	674	675	676	677	678	679
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G F J K	C G F J K	C G H A J	C G H B J	C G H E J	C G H F J	C G H I J
D H J L M	D H J L M	D H A K L	D H B K L	D H E K L	D H F K L	D H I K L
E I K M J	E I K M L	E I J L M	E I J L M	E I J L M	E I J L M	E I J L M

680	681	682	683	684	685	686
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G H J A	C G H J B	C G H J D	C G H J F	C G H J K	C G H J K	C G H J K
D H J K L	D H J K L	D H J K L	D H J K L	D H J A L	D H J B L	D H J C L
E I A L M	E I B L M	E I D L M	E I F L M	E I K L M	E I K L M	E I K L M

687	688	689	690	691	692	693
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G H J K	C G H J K	C G H J K	C G H J K	C G H J K	C G H J K	C G H J K
D H J E L	D H J F L	D H J G L	D H J I L	D H J K L	D H J L A	D H J L B
E I K L M	E I K L M	E I K L M	E I K L M	E I K L M	E I K A M	E I K B M

694	695	696	697	698	699	700
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G H J K	C G H J K	C G H J K	C G H J K	C G H J K	C G H J K	C G H J K
D H J L C	D H J L F	D H J L G	D H J L M	D H J L M	D H J L M	D H J L M
E I K C M	E I K F M	E I K G M	E I K M A	E I K M B	E I K M C	E I K M D

701	702	703	704	705	706	707
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G H J K	C G H J K	C G H J K	C G H J K	C G H J K	C G I A J	C G I B J
D H J L M	D H J L M	D H J L M	D H J L M	D H J L M	D H A K L	D H B K L
E I K M F	E I K M G	E I K M H	E I K M J	E I K M L	E I J L M	E I J L M

708	709	710	711	712	713	714
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G I E J	C G I F J	C G I J A	C G I J B	C G I J D	C G I J F	C G I J H
D H E K L	D H F K L	D H J K L	D H J K L	D H J K L	D H J K L	D H J K L
E I J L M	E I J L M	E I A L M	E I B L M	E I D L M	E I F L M	E I H L M

715	716	717	718	719	720	721
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K
D H J A L	D H J B L	D H J C L	D H J E L	D H J F L	D H J G L	D H J I L
E I K L M	E I K L M	E I K L M	E I K L M	E I K L M	E I K L M	E I K L M

722	723	724	725	726	727	728
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K
D H J K L	D H J L A	D H J L B	D H J L C	D H J L F	D H J L G	D H J L M
E I K L M	E I K A M	E I K B M	E I K C M	E I K F M	E I K G M	E I K M A

729	730	731	732	733	734	735
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K	C G I J K
D H J L M	D H J L M	D H J L M	D H J L M	D H J L M	D H J L M	D H J L M
E I K M B	E I K M C	E I K M D	E I K M F	E I K M G	E I K M H	E I K M J

736	737	738	739	740	741	742
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G I J K	C G J A B	C G J A D	C G J A F	C G J A H	C G J A K	C G J A K
D H J L M	D H A K L	D H A K L	D H A K L	D H A K L	D H A B L	D H A C L
E I K M L	E I B L M	E I D L M	E I F L M	E I H L M	E I K L M	E I K L M

743	744	745	746	747	748	749
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J A K	C G J A K	C G J A K	C G J A K	C G J A K	C G J A K	C G J A K
D H A E L	D H A F L	D H A G L	D H A I L	D H A J L	D H A K L	D H A L B
E I K L M	E I K L M	E I K L M	E I K L M	E I K L M	E I K L M	E I K B M

750	751	752	753	754	755	756
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J A K	C G J A K	C G J A K	C G J A K	C G J A K	C G J A K	C G J A K
D H A L C	D H A L F	D H A L G	D H A L J	D H A L M	D H A L M	D H A L M
E I K C M	E I K F M	E I K G M	E I K J M	E I K M A	E I K M B	E I K M C

757	758	759	760	761	762	763
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J A K	C G J A K	C G J A K	C G J A K	C G J A K	C G J A K	C G J B A
D H A L M	D H A L M	D H A L M	D H A L M	D H A L M	D H A L M	D H B K L
E I K M D	E I K M F	E I K M G	E I K M H	E I K M J	E I K M L	E I A L M

764	765	766	767	768	769	770
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J B D	C G J B F	C G J B H	C G J B K	C G J B K	C G J B K	C G J B K
D H B K L	D H B K L	D H B K L	D H B A L	D H B C L	D H B E L	D H B F L
E I D L M	E I F L M	E I H L M	E I K L M	E I K L M	E I K L M	E I K L M

771	772	773	774	775	776	777
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J B K	C G J B K	C G J B K	C G J B K	C G J B K	C G J B K	C G J B K
D H B G L	D H B I L	D H B J L	D H B K L	D H B L A	D H B L C	D H B L F
E I K L M	E I K L M	E I K L M	E I K L M	E I K A M	E I K C M	E I K F M

778	779	780	781	782	783	784
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J B K	C G J B K	C G J B K	C G J B K	C G J B K	C G J B K	C G J B K
D H B L G	D H B L J	D H B L M	D H B L M	D H B L M	D H B L M	D H B L M
E I K G M	E I K J M	E I K M A	E I K M B	E I K M C	E I K M D	E I K M F

785	786	787	788	789	790	791
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J B K	C G J B K	C G J B K	C G J B K	C G J E A	C G J E B	C G J E D
D H B L M	D H B L M	D H B L M	D H B L M	D H E K L	D H E K L	D H E K L
E I K M G	E I K M H	E I K M J	E I K M L	E I A L M	E I B L M	E I D L M

792	793	794	795	796	797	798
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J E F	C G J E H	C G J E K	C G J E K	C G J E K	C G J E K	C G J E K
D H E K L	D H E K L	D H E A L	D H E B L	D H E C L	D H E F L	D H E G L
E I F L M	E I H L M	E I K L M	E I K L M	E I K L M	E I K L M	E I K L M

799	800	801	802	803	804	805
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J E K	C G J E K	C G J E K	C G J E K	C G J E K	C G J E K	C G J E K
D H E I L	D H E J L	D H E K L	D H E L A	D H E L B	D H E L C	D H E L F
E I K L M	E I K L M	E I K L M	E I K A M	E I K B M	E I K C M	E I K F M

806	807	808	809	810	811	812
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J E K	C G J E K	C G J E K	C G J E K	C G J E K	C G J E K	C G J E K
D H E L G	D H E L J	D H E L M	D H E L M	D H E L M	D H E L M	D H E L M
E I K G M	E I K J M	E I K M A	E I K M B	E I K M C	E I K M D	E I K M F

813	814	815	816	817	818	819
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J E K	C G J E K	C G J E K	C G J E K	C G J F A	C G J F B	C G J F D
D H E L M	D H E L M	D H E L M	D H E L M	D H F K L	D H F K L	D H F K L
E I K M G	E I K M H	E I K M J	E I K M L	E I A L M	E I B L M	E I D L M

820	821	822	823	824	825	826
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J F H	C G J F K	C G J F K	C G J F K	C G J F K	C G J F K	C G J F K
D H F K L	D H F A L	D H F B L	D H F C L	D H F E L	D H F G L	D H F I L
E I H L M	E I K L M	E I K L M	E I K L M	E I K L M	E I K L M	E I K L M

827	828	829	830	831	832	833
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J F K	C G J F K	C G J F K	C G J F K	C G J F K	C G J F K	C G J F K
D H F J L	D H F K L	D H F L A	D H F L B	D H F L C	D H F L G	D H F L J
E I K L M	E I K L M	E I K A M	E I K B M	E I K C M	E I K G M	E I K J M

834	835	836	837	838	839	840
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J F K	C G J F K	C G J F K	C G J F K	C G J F K	C G J F K	C G J F K
D H F L M	D H F L M	D H F L M	D H F L M	D H F L M	D H F L M	D H F L M
E I K M A	E I K M B	E I K M C	E I K M D	E I K M F	E I K M G	E I K M H

841	842	843	844	845	846	847
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J F K	C G J F K	C G J I A	C G J I B	C G J I D	C G J I F	C G J I H
D H F L M	D H F L M	D H I K L	D H I K L	D H I K L	D H I K L	D H I K L
E I K M J	E I K M L	E I A L M	E I B L M	E I D L M	E I F L M	E I H L M

848	849	850	851	852	853	854
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J I K	C G J I K	C G J I K	C G J I K	C G J I K	C G J I K	C G J I K
D H I A L	D H I B L	D H I C L	D H I E L	D H I F L	D H I G L	D H I J L
E I K L M	E I K L M	E I K L M	E I K L M	E I K L M	E I K L M	E I K L M

855	856	857	858	859	860	861
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J I K	C G J I K	C G J I K	C G J I K	C G J I K	C G J I K	C G J I K
D H I K L	D H I L A	D H I L B	D H I L C	D H I L F	D H I L G	D H I L J
E I K L M	E I K A M	E I K B M	E I K C M	E I K F M	E I K G M	E I K J M

862	863	864	865	866	867	868
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J I K	C G J I K	C G J I K	C G J I K	C G J I K	C G J I K	C G J I K
D H I L M	D H I L M	D H I L M	D H I L M	D H I L M	D H I L M	D H I L M
E I K M A	E I K M B	E I K M C	E I K M D	E I K M F	E I K M G	E I K M H

869	870	871	872	873	874	875
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J I K	C G J I K	C G J K A	C G J K A	C G J K A	C G J K A	C G J K A
D H I L M	D H I L M	D H K A L	D H K B L	D H K C L	D H K E L	D H K F L
E I K M J	E I K M L	E I A L M	E I A L M	E I A L M	E I A L M	E I A L M

876	877	878	879	880	881	882
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K A	C G J K A	C G J K A	C G J K A	C G J K A	C G J K A	C G J K A
D H K G L	D H K I L	D H K J L	D H K L B	D H K L C	D H K L F	D H K L G
E I A L M	E I A L M	E I A L M	E I A B M	E I A C M	E I A F M	E I A G M

883	884	885	886	887	888	889
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K A	C G J K A	C G J K A	C G J K A	C G J K A	C G J K A	C G J K A
D H K L J	D H K L M	D H K L M	D H K L M	D H K L M	D H K L M	D H K L M
E I A J M	E I A M B	E I A M C	E I A M D	E I A M F	E I A M G	E I A M H

890	891	892	893	894	895	896
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K A	C G J K A	C G J K A	C G J K B	C G J K B	C G J K B	C G J K B
D H K L M	D H K L M	D H K L M	D H K A L	D H K B L	D H K C L	D H K E L
E I A M J	E I A M K	E I A M L	E I B L M	E I B L M	E I B L M	E I B L M

897	898	899	900	901	902	903
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K B	C G J K B	C G J K B	C G J K B	C G J K B	C G J K B	C G J K B
D H K F L	D H K G L	D H K I L	D H K J L	D H K L A	D H K L C	D H K L F
E I B L M	E I B L M	E I B L M	E I B L M	E I B A M	E I B C M	E I B F M

904	905	906	907	908	909	910
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K B	C G J K B	C G J K B	C G J K B	C G J K B	C G J K B	C G J K B
D H K L G	D H K L J	D H K L M	D H K L M	D H K L M	D H K L M	D H K L M
E I B G M	E I B J M	E I B M A	E I B M C	E I B M D	E I B M F	E I B M G

911	912	913	914	915	916	917
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K B	C G J K B	C G J K B	C G J K B	C G J K D	C G J K D	C G J K D
D H K L M	D H K L M	D H K L M	D H K L M	D H K A L	D H K B L	D H K C L
E I B M H	E I B M J	E I B M K	E I B M L	E I D L M	E I D L M	E I D L M

918	919	920	921	922	923	924
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K D	C G J K D	C G J K D	C G J K D	C G J K D	C G J K D	C G J K D
D H K E L	D H K F L	D H K G L	D H K I L	D H K J L	D H K L A	D H K L B
E I D L M	E I D L M	E I D L M	E I D L M	E I D L M	E I D A M	E I D B M

925	926	927	928	929	930	931
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K D	C G J K D	C G J K D	C G J K D	C G J K D	C G J K D	C G J K D
D H K L C	D H K L F	D H K L G	D H K L J	D H K L M	D H K L M	D H K L M
E I D C M	E I D F M	E I D G M	E I D J M	E I D M A	E I D M B	E I D M C

932	933	934	935	936	937	938
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K D	C G J K D	C G J K D	C G J K D	C G J K D	C G J K D	C G J K F
D H K L M	D H K L M	D H K L M	D H K L M	D H K L M	D H K L M	D H K A L
E I D M F	E I D M G	E I D M H	E I D M J	E I D M K	E I D M L	E I F L M

939	940	941	942	943	944	945
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K F	C G J K F	C G J K F	C G J K F	C G J K F	C G J K F	C G J K F
D H K B L	D H K C L	D H K E L	D H K F L	D H K G L	D H K I L	D H K J L
E I F L M	E I F L M	E I F L M	E I F L M	E I F L M	E I F L M	E I F L M

946	947	948	949	950	951	952
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K F	C G J K F	C G J K F	C G J K F	C G J K F	C G J K F	C G J K F
D H K L A	D H K L B	D H K L C	D H K L G	D H K L J	D H K L M	D H K L M
E I F A M	E I F B M	E I F C M	E I F G M	E I F J M	E I F M A	E I F M B

953	954	955	956	957	958	959
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K F	C G J K F	C G J K F	C G J K F	C G J K F	C G J K F	C G J K F
D H K L M	D H K L M	D H K L M	D H K L M	D H K L M	D H K L M	D H K L M
E I F M C	E I F M D	E I F M G	E I F M H	E I F M J	E I F M K	E I F M L

960	961	962	963	964	965	966
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K H	C G J K H	C G J K H	C G J K H	C G J K H	C G J K H	C G J K H
D H K A L	D H K B L	D H K C L	D H K E L	D H K F L	D H K G L	D H K I L
E I H L M	E I H L M	E I H L M	E I H L M	E I H L M	E I H L M	E I H L M

967	968	969	970	971	972	973
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K H	C G J K H	C G J K H	C G J K H	C G J K H	C G J K H	C G J K H
D H K J L	D H K L A	D H K L B	D H K L C	D H K L F	D H K L G	D H K L J
E I H L M	E I H A M	E I H B M	E I H C M	E I H F M	E I H G M	E I H J M

974	975	976	977	978	979	980
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K H	C G J K H	C G J K H	C G J K H	C G J K H	C G J K H	C G J K H
D H K L M	D H K L M	D H K L M	D H K L M	D H K L M	D H K L M	D H K L M
E I H M A	E I H M B	E I H M C	E I H M D	E I H M F	E I H M G	E I H M J

981	982	983	984	985	986	987
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K H	C G J K H	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K L M	D H K L M	D H K A B	D H K A C	D H K A F	D H K A G	D H K A J
E I H M K	E I H M L	E I L B M	E I L C M	E I L F M	E I L G M	E I L J M

988	989	990	991	992	993	994
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K A M	D H K A M	D H K A M	D H K A M	D H K A M	D H K A M	D H K A M
E I L M A	E I L M B	E I L M C	E I L M D	E I L M F	E I L M G	E I L M H

995	996	997	998	999	1000	1001
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K A M	D H K A M	D H K B A	D H K B C	D H K B F	D H K B G	D H K B J
E I L M J	E I L M K	E I L A M	E I L C M	E I L F M	E I L G M	E I L J M

1002	1003	1004	1005	1006	1007	1008
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K B M	D H K B M	D H K B M	D H K B M	D H K B M	D H K B M	D H K B M
E I L M A	E I L M B	E I L M C	E I L M D	E I L M F	E I L M G	E I L M H

1009	1010	1011	1012	1013	1014	1015
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K B M	D H K B M	D H K C A	D H K C B	D H K C F	D H K C G	D H K C J
E I L M J	E I L M K	E I L A M	E I L B M	E I L F M	E I L G M	E I L J M

1016	1017	1018	1019	1020	1021	1022
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K C M	D H K C M	D H K C M	D H K C M	D H K C M	D H K C M	D H K C M
E I L M A	E I L M B	E I L M C	E I L M D	E I L M F	E I L M G	E I L M H

1023	1024	1025	1026	1027	1028	1029
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K C M	D H K C M	D H K E A	D H K E B	D H K E C	D H K E F	D H K E G
E I L M J	E I L M K	E I L A M	E I L B M	E I L C M	E I L F M	E I L G M

1030	1031	1032	1033	1034	1035	1036
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K E J	D H K E M	D H K E M	D H K E M	D H K E M	D H K E M	D H K E M
E I L J M	E I L M A	E I L M B	E I L M C	E I L M D	E I L M F	E I L M G

1037	1038	1039	1040	1041	1042	1043
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K E M	D H K E M	D H K E M	D H K F A	D H K F B	D H K F C	D H K F G
E I L M H	E I L M J	E I L M K	E I L A M	E I L B M	E I L C M	E I L G M

1044	1045	1046	1047	1048	1049	1050
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K F J	D H K F M	D H K F M	D H K F M	D H K F M	D H K F M	D H K F M
E I L J M	E I L M A	E I L M B	E I L M C	E I L M D	E I L M F	E I L M G

1051	1052	1053	1054	1055	1056	1057
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K F M	D H K F M	D H K F M	D H K G A	D H K G B	D H K G C	D H K G F
E I L M H	E I L M J	E I L M K	E I L A M	E I L B M	E I L C M	E I L F M

1058	1059	1060	1061	1062	1063	1064
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K G J	D H K G M	D H K G M	D H K G M	D H K G M	D H K G M	D H K G M
E I L J M	E I L M A	E I L M B	E I L M C	E I L M D	E I L M F	E I L M G

1065	1066	1067	1068	1069	1070	1071
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K G M	D H K G M	D H K G M	D H K I A	D H K I B	D H K I C	D H K I F
E I L M H	E I L M J	E I L M K	E I L A M	E I L B M	E I L C M	E I L F M

1072	1073	1074	1075	1076	1077	1078
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K I G	D H K I J	D H K I M	D H K I M	D H K I M	D H K I M	D H K I M
E I L G M	E I L J M	E I L M A	E I L M B	E I L M C	E I L M D	E I L M F

1079	1080	1081	1082	1083	1084	1085
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K I M	D H K I M	D H K I M	D H K I M	D H K J A	D H K J B	D H K J C
E I L M G	E I L M H	E I L M J	E I L M K	E I L A M	E I L B M	E I L C M

1086	1087	1088	1089	1090	1091	1092
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K J F	D H K J G	D H K J M	D H K J M	D H K J M	D H K J M	D H K J M
E I L F M	E I L G M	E I L M A	E I L M B	E I L M C	E I L M D	E I L M F

1093	1094	1095	1096	1097	1098	1099
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K J M	D H K J M	D H K J M	D H K J M	D H K L A	D H K L B	D H K L C
E I L M G	E I L M H	E I L M J	E I L M K	E I L A M	E I L B M	E I L C M

1100	1101	1102	1103	1104	1105	1106
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K L F	D H K L G	D H K L J	D H K L M	D H K L M	D H K L M	D H K L M
E I L F M	E I L G M	E I L J M	E I L M A	E I L M B	E I L M C	E I L M D

1107	1108	1109	1110	1111	1112	1113
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K L M	D H K L M	D H K L M	D H K L M	D H K L M	D H K M A	D H K M A
E I L M F	E I L M G	E I L M H	E I L M J	E I L M K	E I L A B	E I L A C

1114	1115	1116	1117	1118	1119	1120
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K M A	D H K M A	D H K M A	D H K M A	D H K M A	D H K M A	D H K M A
E I L A D	E I L A F	E I L A G	E I L A H	E I L A J	E I L A K	E I L A M

1121	1122	1123	1124	1125	1126	1127
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K M B	D H K M B	D H K M B	D H K M B	D H K M B	D H K M B	D H K M B
E I L B A	E I L B C	E I L B D	E I L B F	E I L B G	E I L B H	E I L B J

1128	1129	1130	1131	1132	1133	1134
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K M B	D H K M B	D H K M C	D H K M C	D H K M C	D H K M C	D H K M C
E I L B K	E I L B M	E I L C A	E I L C B	E I L C D	E I L C F	E I L C G

1135	1136	1137	1138	1139	1140	1141
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K M C	D H K M C	D H K M C	D H K M C	D H K M F	D H K M F	D H K M F
E I L C H	E I L C J	E I L C K	E I L C M	E I L F A	E I L F B	E I L F C

1142	1143	1144	1145	1146	1147	1148
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K M F	D H K M F	D H K M F	D H K M F	D H K M F	D H K M F	D H K M G
E I L F D	E I L F G	E I L F H	E I L F J	E I L F K	E I L F M	E I L G A

1149	1150	1151	1152	1153	1154	1155
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K M G	D H K M G	D H K M G	D H K M G	D H K M G	D H K M G	D H K M G
E I L G B	E I L G C	E I L G D	E I L G F	E I L G H	E I L G J	E I L G K

1156	1157	1158	1159	1160	1161	1162
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L	C G J K L
D H K M G	D H K M J	D H K M J	D H K M J	D H K M J	D H K M J	D H K M J
E I L G M	E I L J A	E I L J B	E I L J C	E I L J D	E I L J F	E I L J G

1163	1164	1165
A B C D E	A B C D E	A B C D E
B F G H I	B F G H I	B F G H I
C G J K L	C G J K L	C G J K L
D H K M J	D H K M J	D H K M J
E I L J H	E I L J K	E I L J M

APPENDIX C

LIST OF NON-COMMUTATIVE GENERALIZED LATIN SQUARES OF ORDER 5 WITH 5 DISTINCT ELEMENTS

1	2	3	4	5	6	7
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B A D E C	B A D E C	B A D E C	B A D E C	B A D E C	B A D E C	B A D E C
C D E A B	C D E A B	C D E B A	C D E B A	C E A B D	C E A B D	C E B A D
D E B C A	E C A B D	D E A C B	E C B A D	D C E A B	E D B C A	D C E B A
E C A B D	D E B C A	E C B A D	D E A C B	E D B C A	D C E A B	E D A C B

8	9	10	11	12	13	14
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B A D E C	B A D E C	B A D E C	B A D E C	B A D E C	B A D E C	B A D E C
C E B A D	D C E A B	D C E A B	D C E B A	D C E B A	D E A C B	D E A C B
E D A C B	C E A B D	E D B C A	C E B A D	E D A C B	C D E B A	E C B A D
D C E B A	E D B C A	C E A B D	E D A C B	C E B A D	E C B A D	C D E B A

15	16	17	18	19	20	21
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B A D E C	B A D E C	B A D E C	B A D E C	B A D E C	B A D E C	B A D E C
D E B C A	D E B C A	E C A B D	E C A B D	E C B A D	E C B A D	E D A C B
C D E A B	E C A B D	C D E A B	D E B C A	C D E B A	D E A C B	C E B A D
E C A B D	C D E A B	D E B C A	C D E A B	D E A C B	C D E B A	D C E B A

22	23	24	25	26	27	28
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B A D E C	B A D E C	B A D E C	B A E C D	B A E C D	B A E C D	B A E C D
E D A C B	E D B C A	E D B C A	C D A E B	C D A E B	C D B E A	C D B E A
D C E B A	C E A B D	D C E A B	D E B A C	E C D B A	D E A B C	E C D A B
C E B A D	D C E A B	C E A B D	E C D B A	D E B A C	E C D A B	D E A B C

29	30	31	32	33	34	35
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B A E C D	B A E C D	B A E C D	B A E C D	B A E C D	B A E C D	B A E C D
C E D A B	C E D A B	C E D B A	C E D B A	D C A E B	D C A E B	D C B E A
D C B E A	E D A B C	D C A E B	E D B A C	C E D B A	E D B A C	C E D A B
E D A B C	D C B E A	E D B A C	D C A E B	E D B A C	C E D B A	E D A B C

36	37	38	39	40	41	42
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B A E C D	B A E C D	B A E C D	B A E C D	B A E C D	B A E C D	B A E C D
D C B E A	D E A B C	D E A B C	D E B A C	D E B A C	E C D A B	E C D A B
E D A B C	C D B E A	E C D A B	C D A E B	E C D B A	C D B E A	D E A B C
C E D A B	E C D A B	C D B E A	E C D B A	C D A E B	D E A B C	C D B E A

43	44	45	46	47	48	49
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B A E C D	B A E C D	B A E C D	B A E C D	B A E C D	B A E C D	B C A E D
E C D B A	E C D B A	E D A B C	E D A B C	E D B A C	E D B A C	C D E A B
C D A E B	D E B A C	C E D A B	D C B E A	C E D B A	D C A E B	D E B C A
D E B A C	C D A E B	D C B E A	C E D A B	D C A E B	C E D B A	E A D B C

50	51	52	53	54	55	56
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B C A E D	B C A E D	B C A E D	B C A E D	B C A E D	B C A E D	B C A E D
C D E A B	C D E B A	C D E B A	C E D A B	C E D A B	C E D B A	C E D B A
E A D B C	D E B A C	E A D C B	D A E B C	E D B C A	D A E C B	E D B A C
D E B C A	E A D C B	D E B A C	E D B C A	D A E B C	E D B A C	D A E C B

57	58	59	60	61	62	63
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B C A E D	B C A E D	B C A E D	B C A E D	B C A E D	B C A E D	B C A E D
D A E B C	D A E B C	D A E C B	D A E C B	D E B A C	D E B A C	D E B C A
C E D A B	E D B C A	C E D B A	E D B A C	C D E B A	E A D C B	C D E A B
E D B C A	C E D A B	E D B A C	C E D B A	E A D C B	C D E B A	E A D B C

64	65	66	67	68	69	70
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B C A E D	B C A E D	B C A E D	B C A E D	B C A E D	B C A E D	B C A E D
D E B C A	E A D B C	E A D B C	E A D C B	E A D C B	E D B A C	E D B A C
E A D B C	C D E A B	D E B C A	C D E B A	D E B A C	C E D B A	D A E C B
C D E A B	D E B C A	C D E A B	D E B A C	C D E B A	D A E C B	C E D B A
71	72	73	74	75	76	77
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B C A E D	B C A E D	B C D E A	B C D E A	B C D E A	B C D E A	B C D E A
E D B C A	E D B C A	C A E B D	C A E B D	C A E B D	C A E B D	C D E A B
C E D A B	D A E B C	D E A C B	D E B A C	E D A C B	E D B A C	E A B C D
D A E B C	C E D A B	E D B A C	E D A C B	D E B A C	D E A C B	D E A B C
78	79	80	81	82	83	84
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B C D E A	B C D E A	B C D E A	B C D E A	B C D E A	B C D E A	B C D E A
C E A B D	C E A B D	C E B A D	C E B A D	C E B A D	C E B A D	D A E B C
D A E C B	E D B A C	D A E B C	D A E C B	E D A B C	E D A C B	C E B A D
E D B A C	D A E C B	E D A C B	E D A B C	D A E C B	D A E B C	E D A C B
85	86	87	88	89	90	91
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B C D E A	B C D E A	B C D E A	B C D E A	B C D E A	B C D E A	B C D E A
D A E B C	D A E C B	D A E C B	D A E C B	D A E C B	D E A B C	D E A B C
E D A C B	C E A B D	C E B A D	E D A B C	E D B A C	C D E A B	E A B C D
C E B A D	E D B A C	E D A B C	C E B A D	C E A B D	E A B C D	C D E A B
92	93	94	95	96	97	98
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B C D E A	B C D E A	B C D E A	B C D E A	B C D E A	B C D E A	B C D E A
D E A C B	D E A C B	D E B A C	D E B A C	E A B C D	E A B C D	E D A B C
C A E B D	E D B A C	C A E B D	E D A C B	C D E A B	D E A B C	C E B A D
E D B A C	C A E B D	E D A C B	C A E B D	D E A B C	C D E A B	D A E C B
99	100	101	102	103	104	105
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B C D E A	B C D E A	B C D E A	B C D E A	B C D E A	B C D E A	B C D E A
E D A B C	E D A C B	E D A C B	E D A C B	E D A C B	E D B A C	E D B A C
D A E C B	C A E B D	C E B A D	D A E B C	D E B A C	C A E B D	C E A B D
C E B A D	D E B A C	D A E B C	C E B A D	C A E B D	D E A C B	D A E C B
106	107	108	109	110	111	112
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B C D E A	B C D E A	B C E A D	B C E A D	B C E A D	B C E A D	B C E A D
E D B A C	E D B A C	C A D E B	C A D E B	C A D E B	C A D E B	C D A E B
D A E C B	D E A C B	D E A B C	D E B C A	E D A B C	E D B C A	D E B C A
C E A B D	C A E B D	E D B C A	E D A B C	D E B C A	D E A B C	E A D B C
113	114	115	116	117	118	119
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B C E A D	B C E A D	B C E A D	B C E A D	B C E A D	B C E A D	B C E A D
C D A E B	C D B E A	C D B E A	C D B E A	C D B E A	C E D B A	D A B E C
E A D B C	D E A B C	D E A C B	E A D B C	E A D C B	E D A C B	C E D B A
D E B C A	E A D C B	E A D B C	D E A C B	D E A B C	D A B E C	E D A C B
120	121	122	123	124	125	126
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B C E A D	B C E A D	B C E A D	B C E A D	B C E A D	B C E A D	B C E A D
D A B E C	D E A B C	D E A B C	D E A B C	D E A B C	D E A C B	D E A C B
E D A C B	C A D E B	C D B E A	E A D C B	E D B C A	C D B E A	E A D B C
C E D B A	E D B C A	E A D C B	C D B E A	C A D E B	E A D B C	C D B E A
127	128	129	130	131	132	133
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B C E A D	B C E A D	B C E A D	B C E A D	B C E A D	B C E A D	B C E A D
D E B C A	D E B C A	D E B C A	D E B C A	E A D B C	E A D B C	E A D B C
C A D E B	C D A E B	E A D B C	E D A B C	C D A E B	C D B E A	D E A C B
E D A B C	E A D B C	C D A E B	C A D E B	D E B C A	D E A C B	C D B E A
134	135	136	137	138	139	140
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B C E A D	B C E A D	B C E A D	B C E A D	B C E A D	B C E A D	B C E A D
E A D B C	E A D C B	E A D C B	E D A B C	E D A B C	E D A C B	E D A C B
D E B C A	C D B E A	D E A B C	C A D E B	D E B C A	C E D B A	D A B E C
C D A E B	D E A B C	C D B E A	D E B C A	C A D E B	D A B E C	C E D B A
141	142	143	144	145	146	147
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B C E A D	B C E A D	B D A E C	B D A E C	B D A E C	B D A E C	B D A E C
E D B C A	E D B C A	C A E B D	C E B A D	C E B A D	C E B A D	C E B A D
C A D E B	D E A B C	E C D A B	D A E C B	D C E B A	E A D C B	E C D B A
D E A B C	C A D E B	D E B C A	E C D B A	E A D C B	D C E B A	D A E C B

148	149	150	151	152	153	154
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B D A E C	B D A E C	B D A E C	B D A E C	B D A E C	B D A E C	B D A E C
C E D A B	C E D A B	C E D B A	C E D B A	C E D B A	C E D B A	D A E C B
D C E B A	E A B C D	D A E C B	D C E A B	E A B C D	E C B A D	C E B A D
E A B C D	D C E B A	E C B A D	E A B C D	D C E A B	D A E C B	E C D B A

155	156	157	158	159	160	161
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B D A E C	B D A E C	B D A E C	B D A E C	B D A E C	B D A E C	B D A E C
D A E C B	D A E C B	D A E C B	D C E A B	D C E A B	D C E B A	D C E B A
C E D B A	E C B A D	E C D B A	C E D B A	E A B C D	C E B A D	C E D A B
E C B A D	C E D B A	C E B A D	E A B C D	C E D B A	E A D C B	E A B C D

162	163	164	165	166	167	168
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B D A E C	B D A E C	B D A E C	B D A E C	B D A E C	B D A E C	B D A E C
D C E B A	D C E B A	D E B C A	D E B C A	E A B C D	E A B C D	E A B C D
E A B C D	E A D C B	C A E B D	E C D A B	C E D A B	C E D B A	D C E A B
C E D A B	C E B A D	E C D A B	C A E B D	D C E B A	D C E A B	C E D B A

169	170	171	172	173	174	175
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B D A E C	B D A E C	B D A E C	B D A E C	B D A E C	B D A E C	B D A E C
E A B C D	E A D C B	E A D C B	E C B A D	E C B A D	E C D A B	E C D A B
D C E B A	C E B A D	D C E B A	C E D B A	D A E C B	C A E B D	D E B C A
C E D A B	D C E B A	C E B A D	D A E C B	C E D B A	D E B C A	C A E B D

176	177	178	179	180	181	182
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B D A E C	B D A E C	B D E A C	B D E A C	B D E A C	B D E A C	B D E A C
E C D B A	E C D B A	C A B E D	C A B E D	C A D E B	C A D E B	C E A B D
C E B A D	D A E C B	D E A C B	E C D B A	D E B C A	E C A B D	D C B E A
D A E C B	C E B A D	E C D B A	D E A C B	E C A B D	D E B C A	E A D C B

183	184	185	186	187	188	189
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B D E A C	B D E A C	B D E A C	B D E A C	B D E A C	B D E A C	B D E A C
C E A B D	C E D B A	C E D B A	D C A E B	D C A E B	D C B E A	D C B E A
E A D C B	D C A E B	E A B C D	C E D B A	E A B C D	C E A B D	E A D C B
D C B E A	E A B C D	D C A E B	E A B C D	C E D B A	E A D C B	C E A B D

190	191	192	193	194	195	196
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B D E A C	B D E A C	B D E A C	B D E A C	B D E A C	B D E A C	B D E A C
D E A C B	D E A C B	D E B C A	D E B C A	E A B C D	E A B C D	E A D C B
C A B E D	E C D B A	C A D E B	E C A B D	C E D B A	D C A E B	C E A B D
E C D B A	C A B E D	E C A B D	C A D E B	D C A E B	C E D B A	D C B E A

197	198	199	200	201	202	203
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B D E A C	B D E A C	B D E A C	B D E A C	B D E A C	B D E C A	B D E C A
E A D C B	E C A B D	E C A B D	E C D B A	E C D B A	C A B E D	C A B E D
D C B E A	C A D E B	D E B C A	C A B E D	D E A C B	D E A B C	E C D A B
C E A B D	D E B C A	C A D E B	D E A C B	C A B E D	E C D A B	D E A B C

204	205	206	207	208	209	210
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B D E C A	B D E C A	B D E C A	B D E C A	B D E C A	B D E C A	B D E C A
C A D E B	C A D E B	C A D E B	C A D E B	C E A B D	C E A B D	C E B A D
D E A B C	D E B A C	E C A B D	E C B A D	D A B E C	E C D A B	E A D B C
E C B A D	E C A B D	D E B A C	D E A B C	E C D A B	D A B E C	D C A E B

211	212	213	214	215	216	217
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B D E C A	B D E C A	B D E C A	B D E C A	B D E C A	B D E C A	B D E C A
C E D A B	C E D A B	D A B E C	D A B E C	D A B E C	D A B E C	D C A E B
D A B E C	E C A B D	C E A B D	C E D A B	E C A B D	E C D A B	C E B A D
E C A B D	D A B E C	E C D A B	E C A B D	C E D A B	C E A B D	E A D B C

218	219	220	221	222	223	224
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B D E C A	B D E C A	B D E C A	B D E C A	B D E C A	B D E C A	B D E C A
D C A E B	D E A B C	D E A B C	D E A B C	D E A B C	D E B A C	D E B A C
E A D B C	C A B E D	C A D E B	E C B A D	E C D A B	C A D E B	E C A B D
C E B A D	E C D A B	E C B A D	C A D E B	C A B E D	E C A B D	C A D E B

225	226	227	228	229	230	231
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B D E C A	B D E C A	B D E C A	B D E C A	B D E C A	B D E C A	B D E C A
E A D B C	E A D B C	E C A B D	E C A B D	E C A B D	E C A B D	E C B A D
C E B A D	D C A E B	C A D E B	C E D A B	D A B E C	D E B A C	C A D E B
D C A E B	C E B A D	D E B A C	D A B E C	C E D A B	C A D E B	D E A B C

232	233	234	235	236	237	238
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B D E C A	B D E C A	B D E C A	B D E C A	B D E C A	B E A C D	B E A C D
E C B A D	E C D A B	E C D A B	E C D A B	E C D A B	C A D E B	C D B E A
D E A B C	C A B E D	C E A B D	D A B E C	D E A B C	E D B A C	D A E B C
C A D E B	D E A B C	D A B E C	C E A B D	C A B E D	D C E B A	E C D A B

239	240	241	242	243	244	245
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B E A C D	B E A C D	B E A C D	B E A C D	B E A C D	B E A C D	B E A C D
C D B E A	C D B E A	C D B E A	C D E A B	C D E A B	C D E A B	C D E A B
D C E A B	E A D B C	E C D A B	D A B E C	D C B E A	E A D B C	E C D B A
E A D B C	D C E A B	D A E B C	E C D B A	E A D B C	D C B E A	D A B E C

246	247	248	249	250	251	252
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B E A C D	B E A C D	B E A C D	B E A C D	B E A C D	B E A C D	B E A C D
C D E B A	C D E B A	D A B E C	D A B E C	D A B E C	D A B E C	D A E B C
D A B E C	E C D A B	C D E A B	C D E B A	E C D A B	E C D B A	C D B E A
E C D A B	D A B E C	E C D B A	E C D A B	C D E B A	C D E A B	E C D A B

253	254	255	256	257	258	259
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B E A C D	B E A C D	B E A C D	B E A C D	B E A C D	B E A C D	B E A C D
D A E B C	D C B E A	D C B E A	D C E A B	D C E A B	D C E B A	D C E B A
E C D A B	C D E A B	E A D B C	C D B E A	E A D B C	C A D E B	E D B A C
C D B E A	E A D B C	C D E A B	E A D B C	C D B E A	E D B A C	C A D E B

260	261	262	263	264	265	266
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B E A C D	B E A C D	B E A C D	B E A C D	B E A C D	B E A C D	B E A C D
E A D B C	E A D B C	E A D B C	E A D B C	E C D A B	E C D A B	E C D A B
C D B E A	C D E A B	D C B E A	D C E A B	C D B E A	C D E B A	D A B E C
D C E A B	D C B E A	C D E A B	C D B E A	D A E B C	D A B E C	C D E B A

267	268	269	270	271	272	273
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B E A C D	B E A C D	B E A C D	B E A C D	B E A C D	B E D A C	B E D A C
E C D A B	E C D B A	E C D B A	E D B A C	E D B A C	C A B E D	C A B E D
D A E B C	C D E A B	D A B E C	C A D E B	D C E B A	D C E B A	E D A C B
C D B E A	D A B E C	C D E A B	D C E B A	C A D E B	E D A C B	D C E B A

274	275	276	277	278	279	280
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B E D A C	B E D A C	B E D A C	B E D A C	B E D A C	B E D A C	B E D A C
C A E B D	C A E B D	C A E B D	C A E B D	C D A E B	C D A E B	C D B E A
D C A E B	D C B E A	E D A C B	E D B C A	D C E B A	E A B C D	E C A B D
E D B C A	E D A C B	D C B E A	D C A E B	E A B C D	D C E B A	D A E C B

281	282	283	284	285	286	287
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B E D A C	B E D A C	B E D A C	B E D A C	B E D A C	B E D A C	B E D A C
C D E B A	C D E B A	D A E C B	D A E C B	D C A E B	D C A E B	D C A E B
D C A E B	E A B C D	C D B E A	E C A B D	C A E B D	C D E B A	E A B C D
E A B C D	D C A E B	E C A B D	C D B E A	E D B C A	E A B C D	C D E B A

288	289	290	291	292	293	294
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B E D A C	B E D A C	B E D A C	B E D A C	B E D A C	B E D A C	B E D A C
D C A E B	D C B E A	D C B E A	D C E B A	D C E B A	D C E B A	D C E B A
E D B C A	C A E B D	E D A C B	C A B E D	C D A E B	E A B C D	E D A C B
C A E B D	E D A C B	C A E B D	E D A C B	E A B C D	C D A E B	C A B E D

295	296	297	298	299	300	301
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B E D A C	B E D A C	B E D A C	B E D A C	B E D A C	B E D A C	B E D A C
E A B C D	E A B C D	E A B C D	E A B C D	E C A B D	E C A B D	E D A C B
C D A E B	C D E B A	D C A E B	D C E B A	C D B E A	D A E C B	C A B E D
D C E B A	D C A E B	C D E B A	C D A E B	D A E C B	C D B E A	D C E B A

302	303	304	305	306	307	308
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B E D A C	B E D A C	B E D A C	B E D A C	B E D A C	B E D C A	B E D C A
E D A C B	E D A C B	E D A C B	E D B C A	E D B C A	C A B E D	C A B E D
C A E B D	D C B E A	D C E B A	C A E B D	D C A E B	D C E A B	E D A B C
D C B E A	C A E B D	C A B E D	D C A E B	C A E B D	E D A B C	D C E A B

309	310	311	312	313	314	315
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B E D C A	B E D C A	B E D C A	B E D C A	B E D C A	B E D C A	B E D C A
C A E B D	C A E B D	C D A E B	C D A E B	C D E A B	C D E A B	D A B E C
D C A E B	E D B A C	D A E B C	E C B A D	D A B E C	E C A B D	C D E A B
E D B A C	D C A E B	E C B A D	D A E B C	E C A B D	D A B E C	E C A B D

316	317	318	319	320	321	322
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B E D C A	B E D C A	B E D C A	B E D C A	B E D C A	B E D C A	B E D C A
D A B E C	D A E B C	D A E B C	D C A E B	D C A E B	D C E A B	D C E A B
E C A B D	C D A E B	E C B A D	C A E B D	E D B A C	C A B E D	E D A B C
C D E A B	E C B A D	C D A E B	E D B A C	C A E B D	E D A B C	C A B E D

323	324	325	326	327	328	329
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B E D C A	B E D C A	B E D C A	B E D C A	B E D C A	B E D C A	B E D C A
E C A B D	E C A B D	E C B A D	E C B A D	E D A B C	E D A B C	E D B A C
C D E A B	D A B E C	C D A E B	D A E B C	C A B E D	D C E A B	C A E B D
D A B E C	C D E A B	D A E B C	C D A E B	D C E A B	C A B E D	D C A E B

330	331	332	333	334	335	336
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
B E D C A	C A B E D	C A B E D	C A B E D	C A B E D	C A B E D	C A B E D
E D B A C	B D E A C	B D E A C	B D E C A	B D E C A	B E D A C	B E D A C
D C A E B	D E A C B	E C D B A	D E A B C	E C D A B	D C E B A	E D A C B
C A E B D	E C D B A	D E A C B	E C D A B	D E A B C	E D A C B	D C E B A

337	338	339	340	341	342	343
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C A B E D	C A B E D	C A B E D	C A B E D	C A B E D	C A B E D	C A B E D
B E D C A	B E D C A	D C E A B	D C E A B	D C E B A	D C E B A	D E A B C
D C E A B	E D A B C	B E D C A	E D A B C	B E D A C	E D A C B	B D E C A
E D A B C	D C E A B	E D A B C	B E D C A	E D A C B	B E D A C	E C D A B

344	345	346	347	348	349	350
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C A B E D	C A B E D	C A B E D	C A B E D	C A B E D	C A B E D	C A B E D
D E A B C	D E A C B	D E A C B	E C D A B	E C D A B	E C D B A	E C D B A
E C D A B	B D E A C	E C D B A	B D E C A	D E A B C	B D E A C	D E A C B
B D E C A	E C D B A	B D E A C	D E A B C	B D E C A	D E A C B	B D E A C

351	352	353	354	355	356	357
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C A B E D	C A B E D	C A B E D	C A B E D	C A D E B	C A D E B	C A D E B
E D A B C	E D A B C	E D A C B	E D A C B	B C E A D	B C E A D	B C E A D
B E D C A	D C E A B	B E D A C	D C E B A	D E A B C	D E B C A	E D A B C
D C E A B	B E D C A	D C E B A	B E D A C	E D B C A	E D A B C	D E B C A

358	359	360	361	362	363	364
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C A D E B	C A D E B	C A D E B	C A D E B	C A D E B	C A D E B	C A D E B
B C E A D	B D E A C	B D E A C	B D E C A	B D E C A	B D E C A	B D E C A
E D B C A	D E B C A	E C A B D	D E A B C	D E B A C	E C A B D	E C B A D
D E A B C	E C A B D	D E B C A	E C B A D	E C A B D	D E B A C	D E A B C

365	366	367	368	369	370	371
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C A D E B	C A D E B	C A D E B	C A D E B	C A D E B	C A D E B	C A D E B
B E A C D	B E A C D	D C E B A	D C E B A	D E A B C	D E A B C	D E A B C
D C E B A	E D B A C	B E A C D	E D B A C	B C E A D	B D E C A	E C B A D
E D B A C	D C E B A	E D B A C	B E A C D	E D B C A	E C B A D	B D E C A

372	373	374	375	376	377	378
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C A D E B	C A D E B	C A D E B	C A D E B	C A D E B	C A D E B	C A D E B
D E A B C	D E B A C	D E B A C	D E B C A	D E B C A	D E B C A	D E B C A
E D B C A	B D E C A	E C A B D	B C E A D	B D E A C	E C A B D	E D A B C
B C E A D	E C A B D	B D E C A	E D A B C	E C A B D	B D E A C	B C E A D

379	380	381	382	383	384	385
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C A D E B	C A D E B	C A D E B	C A D E B	C A D E B	C A D E B	C A D E B
E C A B D	E C A B D	E C A B D	E C A B D	E C B A D	E C B A D	E D A B C
B D E A C	B D E C A	D E B A C	D E B C A	B D E C A	D E A B C	B C E A D
D E B C A	D E B A C	B D E C A	B D E A C	D E A B C	B D E C A	D E B C A

386	387	388	389	390	391	392
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C A D E B	C A D E B	C A D E B	C A D E B	C A D E B	C A E B D	C A E B D
E D A B C	E D B A C	E D B A C	E D B C A	E D B C A	B C D E A	B C D E A
D E B C A	B E A C D	D C E B A	B C E A D	D E A B C	D E A C B	D E B A C
B C E A D	D C E B A	B E A C D	D E A B C	B C E A D	E D B A C	E D A C B

393	394	395	396	397	398	399
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C A E B D	C A E B D	C A E B D	C A E B D	C A E B D	C A E B D	C A E B D
B C D E A	B C D E A	B D A E C	B D A E C	B E D A C	B E D A C	B E D A C
E D A C B	E D B A C	D E B C A	E C D A B	D C A E B	D C B E A	E D A C B
D E B A C	D E A C B	E C D A B	D E B C A	E D B C A	E D A C B	D C B E A

400	401	402	403	404	405	406
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C A E B D	C A E B D	C A E B D	C A E B D	C A E B D	C A E B D	C A E B D
B E D A C	B E D C A	B E D C A	D C A E B	D C A E B	D C A E B	D C A E B
E D B C A	D C A E B	E D B A C	B E D A C	B E D C A	E D B A C	E D B C A
D C A E B	E D B A C	D C A E B	E D B C A	E D B A C	B E D C A	B E D A C
407	408	409	410	411	412	413
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C A E B D	C A E B D	C A E B D	C A E B D	C A E B D	C A E B D	C A E B D
D C B E A	D C B E A	D E A C B	D E A C B	D E B A C	D E B A C	D E B C A
B E D A C	E D A C B	B C D E A	E D B A C	B C D E A	E D A C B	B D A E C
E D A C B	B E D A C	E D B A C	B C D E A	E D A C B	B C D E A	E C D A B
414	415	416	417	418	419	420
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C A E B D	C A E B D	C A E B D	C A E B D	C A E B D	C A E B D	C A E B D
D E B C A	E C D A B	E C D A B	E D A C B	E D A C B	E D A C B	E D A C B
E C D A B	B D A E C	D E B C A	B C D E A	B E D A C	D C B E A	D E B A C
B D A E C	D E B C A	B D A E C	D E B A C	D C B E A	B E D A C	B C D E A
421	422	423	424	425	426	427
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C A E B D	C A E B D	C A E B D	C A E B D	C A E B D	C A E B D	C D A E B
E D B A C	E D B A C	E D B A C	E D B A C	E D B C A	E D B C A	B A E C D
B C D E A	B E D C A	D C A E B	D E A C B	B E D A C	D C A E B	D E B A C
D E A C B	D C A E B	B E D C A	B C D E A	D C A E B	B E D A C	E C D B A
428	429	430	431	432	433	434
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C D A E B	C D A E B	C D A E B	C D A E B	C D A E B	C D A E B	C D A E B
B A E C D	B C E A D	B C E A D	B E D A C	B E D A C	B E D C A	B E D C A
E C D B A	D E B C A	E A D B C	D C E B A	E A B C D	D A E B C	E C B A D
D E B A C	E A D B C	D E B C A	E A B C D	D C E B A	E C B A D	D A E B C
435	436	437	438	439	440	441
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C D A E B	C D A E B	C D A E B	C D A E B	C D A E B	C D A E B	C D A E B
D A E B C	D A E B C	D C E B A	D C E B A	D E B A C	D E B A C	D E B C A
B E D C A	E C B A D	B E D A C	E A B C D	B A E C D	E C D B A	B C E A D
E C B A D	B E D C A	E A B C D	B E D A C	E C D B A	B A E C D	E A D B C
442	443	444	445	446	447	448
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C D A E B	C D A E B	C D A E B	C D A E B	C D A E B	C D A E B	C D A E B
D E B C A	E A B C D	E A B C D	E A D B C	E A D B C	E C B A D	E C B A D
E A D B C	B E D A C	D C E B A	B C E A D	D E B C A	B E D C A	D A E B C
B C E A D	D C E B A	B E D A C	D E B C A	B C E A D	D A E B C	B E D C A
449	450	451	452	453	454	455
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C D A E B	C D A E B	C D B E A	C D B E A	C D B E A	C D B E A	C D B E A
E C D B A	E C D B A	B A E C D	B A E C D	B C E A D	B C E A D	B C E A D
B A E C D	D E B A C	D E A B C	E C D A B	D E A B C	D E A C B	E A D B C
D E B A C	B A E C D	E C D A B	D E A B C	E A D C B	E A D B C	D E A C B
456	457	458	459	460	461	462
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C D B E A	C D B E A	C D B E A	C D B E A	C D B E A	C D B E A	C D B E A
B C E A D	B E A C D	B E A C D	B E A C D	B E A C D	B E D A C	B E D A C
E A D C B	D A E B C	D C E A B	E A D B C	E C D A B	D A E C B	E C A B D
D E A B C	E C D A B	E A D B C	D C E A B	D A E B C	E C A B D	D A E C B
463	464	465	466	467	468	469
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C D B E A	C D B E A	C D B E A	C D B E A	C D B E A	C D B E A	C D B E A
D A E B C	D A E B C	D A E C B	D A E C B	D C E A B	D C E A B	D E A B C
B E A C D	E C D A B	B E D A C	E C A B D	B E A C D	E A D B C	B A E C D
E C D A B	B E A C D	E C A B D	B E D A C	E A D B C	B E A C D	E C D A B
470	471	472	473	474	475	476
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C D B E A	C D B E A	C D B E A	C D B E A	C D B E A	C D B E A	C D B E A
D E A B C	D E A B C	D E A B C	D E A C B	D E A C B	E A D B C	E A D B C
B C E A D	E A D C B	E C D A B	B C E A D	E A D B C	B C E A D	B E A C D
E A D C B	B C E A D	B A E C D	E A D B C	B C E A D	D E A C B	D C E A B
477	478	479	480	481	482	483
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C D B E A	C D B E A	C D B E A	C D B E A	C D B E A	C D B E A	C D B E A
E A D B C	E A D B C	E A D C B	E A D C B	E C A B D	E C A B D	E C D A B
D C E A B	D E A C B	B C E A D	D E A B C	B E D A C	D A E C B	B A E C D
B E A C D	B C E A D	D E A B C	B C E A D	D A E C B	B E D A C	D E A B C

484	485	486	487	488	489	490
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C D B E A	C D B E A	C D B E A	C D E A B	C D E A B	C D E A B	C D E A B
E C D A B	E C D A B	E C D A B	B A D E C	B A D E C	B C A E D	B C A E D
B E A C D	D A E B C	D E A B C	D E B C A	E C A B D	D E B C A	E A D B C
D A E B C	B E A C D	B A E C D	E C A B D	D E B C A	E A D B C	D E B C A
491	492	493	494	495	496	497
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C D E A B	C D E A B	C D E A B	C D E A B	C D E A B	C D E A B	C D E A B
B C D E A	B C D E A	B E A C D	B E A C D	B E A C D	B E A C D	B E D C A
D E A B C	E A B C D	D A B E C	D C B E A	E A D B C	E C D B A	D A B E C
E A B C D	D E A B C	E C D B A	E A D B C	D C B E A	D A B E C	E C A B D
498	499	500	501	502	503	504
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C D E A B	C D E A B	C D E A B	C D E A B	C D E A B	C D E A B	C D E A B
B E D C A	D A B E C	D A B E C	D A B E C	D A B E C	D C B E A	D C B E A
E C A B D	B E A C D	B E D C A	E C A B D	E C D B A	B E A C D	E A D B C
D A B E C	E C D B A	E C A B D	B E D C A	B E A C D	E A D B C	B E A C D
505	506	507	508	509	510	511
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C D E A B	C D E A B	C D E A B	C D E A B	C D E A B	C D E A B	C D E A B
D E A B C	D E A B C	D E B C A	D E B C A	D E B C A	D E B C A	E A B C D
B C D E A	E A B C D	B A D E C	B C A E D	E A D B C	E C A B D	B C D E A
E A B C D	B C D E A	E C A B D	E A D B C	B C A E D	B A D E C	D E A B C
512	513	514	515	516	517	518
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C D E A B	C D E A B	C D E A B	C D E A B	C D E A B	C D E A B	C D E A B
E A B C D	E A D B C	E A D B C	E A D B C	E A D B C	E C A B D	E C A B D
D E A B C	B C A E D	B E A C D	D C B E A	D E B C A	B A D E C	B E D C A
B C D E A	D E B C A	D C B E A	B E A C D	B C A E D	D E B C A	D A B E C
519	520	521	522	523	524	525
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C D E A B	C D E A B	C D E A B	C D E A B	C D E B A	C D E B A	C D E B A
E C A B D	E C A B D	E C D B A	E C D B A	B A D E C	B A D E C	B C A E D
D A B E C	D E B C A	B E A C D	D A B E C	D E A C B	E C B A D	D E B A C
B E D C A	B A D E C	D A B E C	B E A C D	E C B A D	D E A C B	E A D C B
526	527	528	529	530	531	532
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C D E B A	C D E B A	C D E B A	C D E B A	C D E B A	C D E B A	C D E B A
B C A E D	B E A C D	B E A C D	B E D A C	B E D A C	D A B E C	D A B E C
E A D C B	D A B E C	E C D A B	D C A E B	E A B C D	B E A C D	E C D A B
D E B A C	E C D A B	D A B E C	E A B C D	D C A E B	E C D A B	B E A C D
533	534	535	536	537	538	539
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C D E B A	C D E B A	C D E B A	C D E B A	C D E B A	C D E B A	C D E B A
D C A E B	D C A E B	D E A C B	D E A C B	D E B A C	D E B A C	E A B C D
B E D A C	E A B C D	B A D E C	E C B A D	B C A E D	E A D C B	B E D A C
E A B C D	B E D A C	E C B A D	B A D E C	E A D C B	B C A E D	D C A E B
540	541	542	543	544	545	546
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C D E B A	C D E B A	C D E B A	C D E B A	C D E B A	C D E B A	C D E B A
E A B C D	E A D C B	E A D C B	E C B A D	E C B A D	E C D A B	E C D A B
D C A E B	B C A E D	D E B A C	B A D E C	D E A C B	B E A C D	D A B E C
B E D A C	D E B A C	B C A E D	D E A C B	B A D E C	D A B E C	B E A C D
547	548	549	550	551	552	553
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C E A B D	C E A B D	C E A B D	C E A B D	C E A B D	C E A B D	C E A B D
B A D E C	B A D E C	B C D E A	B C D E A	B D E A C	B D E A C	B D E C A
D C E A B	E D B C A	D A E C B	E D B A C	D C B E A	E A D C B	D A B E C
E D B C A	D C E A B	E D B A C	D A E C B	E A D C B	D C B E A	E C D A B
554	555	556	557	558	559	560
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C E A B D	C E A B D	C E A B D	C E A B D	C E A B D	C E A B D	C E A B D
B D E C A	D A B E C	D A B E C	D A E C B	D A E C B	D C B E A	D C B E A
E C D A B	B D E C A	E C D A B	B C D E A	E D B A C	B D E A C	E A D C B
D A B E C	E C D A B	B D E C A	E D B A C	B C D E A	E A D C B	B D E A C
561	562	563	564	565	566	567
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C E A B D	C E A B D	C E A B D	C E A B D	C E A B D	C E A B D	C E A B D
D C E A B	D C E A B	E A D C B	E A D C B	E C D A B	E C D A B	E D B A C
B A D E C	E D B C A	B D E A C	D C B E A	B D E C A	D A B E C	B C D E A
E D B C A	B A D E C	D C B E A	B D E A C	D A B E C	B D E C A	D A E C B

568	569	570	571	572	573	574
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C E A B D	C E A B D	C E A B D	C E B A D	C E B A D	C E B A D	C E B A D
E D B A C	E D B C A	E D B C A	B A D E C	B A D E C	B C D E A	B C D E A
D A E C B	B A D E C	D C E A B	D C E B A	E D A C B	D A E B C	D A E C B
B C D E A	D C E A B	B A D E C	E D A C B	D C E B A	E D A C B	E D A B C

575	576	577	578	579	580	581
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C E B A D	C E B A D	C E B A D	C E B A D	C E B A D	C E B A D	C E B A D
B C D E A	B C D E A	B D A E C	B D A E C	B D A E C	B D A E C	B D E C A
E D A B C	E D A C B	D A E C B	D C E B A	E A D C B	E C D B A	D C A E B
D A E C B	D A E B C	E C D B A	E A D C B	D C E B A	D A E C B	E A D B C

582	583	584	585	586	587	588
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C E B A D	C E B A D	C E B A D	C E B A D	C E B A D	C E B A D	C E B A D
B D E C A	D A E B C	D A E B C	D A E C B	D A E C B	D A E C B	D A E C B
E A D B C	B C D E A	E D A C B	B C D E A	B D A E C	E C D B A	E D A B C
D C A E B	E D A C B	B C D E A	E D A B C	E C D B A	B D A E C	B C D E A

589	590	591	592	593	594	595
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C E B A D	C E B A D	C E B A D	C E B A D	C E B A D	C E B A D	C E B A D
D C A E B	D C A E B	D C E B A	D C E B A	D C E B A	D C E B A	E A D B C
B D E C A	E A D B C	B A D E C	B D A E C	E A D C B	E D A C B	B D E C A
E A D B C	B D E C A	E D A C B	E A D C B	B D A E C	B A D E C	D C A E B

596	597	598	599	600	601	602
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C E B A D	C E B A D	C E B A D	C E B A D	C E B A D	C E B A D	C E B A D
E A D B C	E A D C B	E A D C B	E C D B A	E C D B A	E D A B C	E D A B C
D C A E B	B D A E C	D C E B A	B D A E C	D A E C B	B C D E A	D A E C B
B D E C A	D C E B A	B D A E C	D A E C B	B D A E C	D A E C B	B C D E A

603	604	605	606	607	608	609
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C E B A D	C E B A D	C E B A D	C E B A D	C E D A B	C E D A B	C E D A B
E D A C B	E D A C B	E D A C B	E D A C B	B A E C D	B A E C D	B C A E D
B A D E C	B C D E A	D A E B C	D C E B A	D C B E A	E D A B C	D A E B C
D C E B A	D A E B C	B C D E A	B A D E C	E D A B C	D C B E A	E D B C A

610	611	612	613	614	615	616
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C E D A B	C E D A B	C E D A B	C E D A B	C E D A B	C E D A B	C E D A B
B C A E D	B D A E C	B D A E C	B D E C A	B D E C A	D A B E C	D A B E C
E D B C A	D C E B A	E A B C D	D A B E C	E C A B D	B D E C A	E C A B D
D A E B C	E A B C D	D C E B A	E C A B D	D A B E C	E C A B D	B D E C A

617	618	619	620	621	622	623
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C E D A B	C E D A B	C E D A B	C E D A B	C E D A B	C E D A B	C E D A B
D A E B C	D A E B C	D C B E A	D C B E A	D C E B A	D C E B A	E A B C D
B C A E D	E D B C A	B A E C D	E D A B C	B D A E C	E A B C D	B D A E C
E D B C A	B C A E D	E D A B C	B A E C D	E A B C D	B D A E C	D C E B A

624	625	626	627	628	629	630
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C E D A B	C E D A B	C E D A B	C E D A B	C E D A B	C E D A B	C E D A B
E A B C D	E C A B D	E C A B D	E D A B C	E D A B C	E D B C A	E D B C A
D C E B A	B D E C A	D A B E C	B A E C D	D C B E A	B C A E D	D A E B C
B D A E C	D A B E C	B D E C A	D C B E A	B A E C D	D A E B C	B C A E D

631	632	633	634	635	636	637
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C E D B A	C E D B A	C E D B A	C E D B A	C E D B A	C E D B A	C E D B A
B A E C D	B A E C D	B C A E D	B C A E D	B C E A D	B C E A D	B D A E C
D C A E B	E D B A C	D A E C B	E D B A C	D A B E C	E D A C B	D A E C B
E D B A C	D C A E B	E D B A C	D A E C B	E D A C B	D A B E C	E C B A D

638	639	640	641	642	643	644
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C E D B A	C E D B A	C E D B A	C E D B A	C E D B A	C E D B A	C E D B A
B D A E C	B D A E C	B D A E C	B D E A C	B D E A C	D A B E C	D A B E C
D C E A B	E A B C D	E C B A D	D C A E B	E A B C D	B C E A D	E D A C B
E A B C D	D C E A B	D A E C B	E A B C D	D C A E B	E D A C B	B C E A D

645	646	647	648	649	650	651
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C E D B A	C E D B A	C E D B A	C E D B A	C E D B A	C E D B A	C E D B A
D A E C B	D A E C B	D A E C B	D A E C B	D C A E B	D C A E B	D C A E B
B C A E D	B D A E C	E C B A D	E D B A C	B A E C D	B D E A C	E A B C D
E D B A C	E C B A D	B D A E C	B C A E D	E D B A C	E A B C D	B D E A C

652	653	654	655	656	657	658
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C E D B A	C E D B A	C E D B A	C E D B A	C E D B A	C E D B A	C E D B A
D C A E B	D C E A B	D C E A B	E A B C D	E A B C D	E A B C D	E A B C D
E D B A C	B D A E C	E A B C D	B D A E C	B D E A C	D C A E B	D C E A B
B A E C D	E A B C D	B D A E C	D C E A B	D C A E B	B D E A C	B D A E C
659	660	661	662	663	664	665
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C E D B A	C E D B A	C E D B A	C E D B A	C E D B A	C E D B A	C E D B A
E C B A D	E C B A D	E D A C B	E D A C B	E D B A C	E D B A C	E D B A C
B D A E C	D A E C B	B C E A D	D A B E C	B A E C D	B C A E D	D A E C B
D A E C B	B D A E C	D A B E C	B C E A D	D C A E B	D A E C B	B C A E D
666	667	668	669	670	671	672
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
C E D B A	D A B E C	D A B E C	D A B E C	D A B E C	D A B E C	D A B E C
E D B A C	B C E A D	B C E A D	B D E C A	B D E C A	B D E C A	B D E C A
D C A E B	C E D B A	E D A C B	C E A B D	C E D A B	E C A B D	E C D A B
B A E C D	E D A C B	C E D B A	E C D A B	E C A B D	C E D A B	C E A B D
673	674	675	676	677	678	679
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D A B E C	D A B E C	D A B E C	D A B E C	D A B E C	D A B E C	D A B E C
B E A C D	B E A C D	B E A C D	B E A C D	B E D C A	B E D C A	C D E A B
C D E A B	C D E B A	E C D A B	E C D B A	C D E A B	E C A B D	B E A C D
E C D B A	E C D A B	C D E B A	C D E A B	E C A B D	C D E A B	E C D B A
680	681	682	683	684	685	686
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D A B E C	D A B E C	D A B E C	D A B E C	D A B E C	D A B E C	D A B E C
C D E A B	C D E A B	C D E A B	C D E B A	C D E B A	C E A B D	C E A B D
B E D C A	E C A B D	E C D B A	B E A C D	E C D A B	B D E C A	E C D A B
E C A B D	B E D C A	B E A C D	E C D A B	B E A C D	E C D A B	B D E C A
687	688	689	690	691	692	693
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D A B E C	D A B E C	D A B E C	D A B E C	D A B E C	D A B E C	D A B E C
C E D A B	C E D A B	C E D B A	C E D B A	E C A B D	E C A B D	E C A B D
B D E C A	E C A B D	B C E A D	E D A C B	B D E C A	B E D C A	C D E A B
E C A B D	B D E C A	E D A C B	B C E A D	C E D A B	C D E A B	B E D C A
694	695	696	697	698	699	700
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D A B E C	D A B E C	D A B E C	D A B E C	D A B E C	D A B E C	D A B E C
E C A B D	E C D A B	E C D A B	E C D A B	E C D A B	E C D B A	E C D B A
C E D A B	B D E C A	B E A C D	C D E B A	C E A B D	B E A C D	C D E A B
B D E C A	C E A B D	C D E B A	B E A C D	B D E C A	C D E A B	B E A C D
701	702	703	704	705	706	707
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D A B E C	D A B E C	D A E B C	D A E B C	D A E B C	D A E B C	D A E B C
E D A C B	E D A C B	B C A E D	B C A E D	B C D E A	B C D E A	B E A C D
B C E A D	C E D B A	C E D A B	E D B C A	C E B A D	E D A C B	C D B E A
C E D B A	B C E A D	E D B C A	C E D A B	E D A C B	C E B A D	E C D A B
708	709	710	711	712	713	714
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D A E B C	D A E B C	D A E B C	D A E B C	D A E B C	D A E B C	D A E B C
B E A C D	B E D C A	B E D C A	C D A E B	C D A E B	C D B E A	C D B E A
E C D A B	C D A E B	E C B A D	B E D C A	E C B A D	B E A C D	E C D A B
C D B E A	E C B A D	C D A E B	E C B A D	B E D C A	E C D A B	B E A C D
715	716	717	718	719	720	721
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D A E B C	D A E B C	D A E B C	D A E B C	D A E B C	D A E B C	D A E B C
C E B A D	C E B A D	C E D A B	C E D A B	E C B A D	E C B A D	E C D A B
B C D E A	E D A C B	B C A E D	E D B C A	B E D C A	C D A E B	B E A C D
E D A C B	B C D E A	E D B C A	B C A E D	C D A E B	B E D C A	C D B E A
722	723	724	725	726	727	728
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D A E B C	D A E B C	D A E B C	D A E B C	D A E B C	D A E C B	D A E C B
E C D A B	E D A C B	E D A C B	E D B C A	E D B C A	B C A E D	B C A E D
C D B E A	B C D E A	C E B A D	B C A E D	C E D A B	C E D B A	E D B A C
B E A C D	C E B A D	B C D E A	C E D A B	B C A E D	E D B A C	C E D B A
729	730	731	732	733	734	735
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D A E C B	D A E C B	D A E C B	D A E C B	D A E C B	D A E C B	D A E C B
B C D E A	B C D E A	B C D E A	B C D E A	B D A E C	B D A E C	B D A E C
C E A B D	C E B A D	E D A B C	E D B A C	C E B A D	C E D B A	E C B A D
E D B A C	E D A B C	C E B A D	C E A B D	E C D B A	E C B A D	C E D B A

736	737	738	739	740	741	742
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D A E C B	D A E C B	D A E C B	D A E C B	D A E C B	D A E C B	D A E C B
B D A E C	B E D A C	B E D A C	C D B E A	C D B E A	C E A B D	C E A B D
E C D B A	C D B E A	E C A B D	B E D A C	E C A B D	B C D E A	E D B A C
C E B A D	E C A B D	C D B E A	E C A B D	B E D A C	E D B A C	B C D E A

743	744	745	746	747	748	749
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D A E C B	D A E C B	D A E C B	D A E C B	D A E C B	D A E C B	D A E C B
C E B A D	C E B A D	C E B A D	C E B A D	C E D B A	C E D B A	C E D B A
B C D E A	B D A E C	E C D B A	E D A B C	B C A E D	B D A E C	E C B A D
E D A B C	E C D B A	B D A E C	B C D E A	E D B A C	E C B A D	B D A E C

750	751	752	753	754	755	756
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D A E C B	D A E C B	D A E C B	D A E C B	D A E C B	D A E C B	D A E C B
C E D B A	E C A B D	E C A B D	E C B A D	E C B A D	E C D B A	E C D B A
E D B A C	B E D A C	C D B E A	B D A E C	C E D B A	B D A E C	C E B A D
B C A E D	C D B E A	B E D A C	C E D B A	B D A E C	C E B A D	B D A E C

757	758	759	760	761	762	763
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D A E C B	D A E C B	D A E C B	D A E C B	D A E C B	D A E C B	D C A E B
E D A B C	E D A B C	E D B A C	E D B A C	E D B A C	E D B A C	B A E C D
B C D E A	C E B A D	B C A E D	B C D E A	C E A B D	C E D B A	C E D B A
C E B A D	B C D E A	C E D B A	C E A B D	B C D E A	B C A E D	E D B A C

764	765	766	767	768	769	770
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D C A E B	D C A E B	D C A E B	D C A E B	D C A E B	D C A E B	D C A E B
B A E C D	B D E A C	B D E A C	B D E C A	B D E C A	B E D A C	B E D A C
E D B A C	C E D B A	E A B C D	C E B A D	E A D B C	C A E B D	C D E B A
C E D B A	E A B C D	C E D B A	E A D B C	C E B A D	E D B C A	E A B C D

771	772	773	774	775	776	777
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D C A E B	D C A E B	D C A E B	D C A E B	D C A E B	D C A E B	D C A E B
B E D A C	B E D A C	B E D C A	B E D C A	C A E B D	C A E B D	C A E B D
E A B C D	E D B C A	C A E B D	E D B A C	B E D A C	B E D C A	E D B A C
C D E B A	C A E B D	E D B A C	C A E B D	E D B C A	E D B A C	B E D C A

778	779	780	781	782	783	784
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D C A E B	D C A E B	D C A E B	D C A E B	D C A E B	D C A E B	D C A E B
C A E B D	C D E B A	C D E B A	C E B A D	C E B A D	C E D B A	C E D B A
E D B C A	B E D A C	E A B C D	B D E C A	E A D B C	B A E C D	B D E A C
B E D A C	E A B C D	B E D A C	E A D B C	B D E C A	E D B A C	E A B C D

785	786	787	788	789	790	791
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D C A E B	D C A E B	D C A E B	D C A E B	D C A E B	D C A E B	D C A E B
C E D B A	C E D B A	E A B C D	E A B C D	E A B C D	E A B C D	E A D B C
E A B C D	E D B A C	B D E A C	B E D A C	C D E B A	C E D B A	B D E C A
B D E A C	B A E C D	C E D B A	C D E B A	B E D A C	B D E A C	C E B A D

792	793	794	795	796	797	798
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D C A E B	D C A E B	D C A E B	D C A E B	D C A E B	D C A E B	D C A E B
E A D B C	E D B A C	E D B A C	E D B A C	E D B A C	E D B C A	E D B C A
C E B A D	B A E C D	B E D C A	C A E B D	C E D B A	B E D A C	C A E B D
B D E C A	C E D B A	C A E B D	B E D C A	B A E C D	C A E B D	B E D A C

799	800	801	802	803	804	805
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D C B E A	D C B E A	D C B E A	D C B E A	D C B E A	D C B E A	D C B E A
B A E C D	B A E C D	B D E A C	B D E A C	B E A C D	B E A C D	B E D A C
C E D A B	E D A B C	C E A B D	E A D C B	C D E A B	E A D B C	C A E B D
E D A B C	C E D A B	E A D C B	C E A B D	E A D B C	C D E A B	E D A C B

806	807	808	809	810	811	812
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D C B E A	D C B E A	D C B E A	D C B E A	D C B E A	D C B E A	D C B E A
B E D A C	C A E B D	C A E B D	C D E A B	C D E A B	C E A B D	C E A B D
E D A C B	B E D A C	E D A C B	B E A C D	E A D B C	B D E A C	E A D C B
C A E B D	E D A C B	B E D A C	E A D B C	B E A C D	E A D C B	B D E A C

813	814	815	816	817	818	819
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D C B E A	D C B E A	D C B E A	D C B E A	D C B E A	D C B E A	D C B E A
C E D A B	C E D A B	E A D B C	E A D B C	E A D C B	E A D C B	E D A B C
B A E C D	E D A B C	B E A C D	C D E A B	B D E A C	C E A B D	B A E C D
E D A B C	B A E C D	C D E A B	B E A C D	C E A B D	B D E A C	C E D A B

820	821	822	823	824	825	826
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D C B E A	D C B E A	D C B E A	D C E A B	D C E A B	D C E A B	D C E A B
E D A B C	E D A C B	E D A C B	B A D E C	B A D E C	B D A E C	B D A E C
C E D A B	B E D A C	C A E B D	C E A B D	E D B C A	C E D B A	E A B C D
B A E C D	C A E B D	B E D A C	E D B C A	C E A B D	E A B C D	C E D B A

827	828	829	830	831	832	833
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D C E A B	D C E A B	D C E A B	D C E A B	D C E A B	D C E A B	D C E A B
B E A C D	B E A C D	B E D C A	B E D C A	C A B E D	C A B E D	C D B E A
C D B E A	E A D B C	C A B E D	E D A B C	B E D C A	E D A B C	B E A C D
E A D B C	C D B E A	E D A B C	C A B E D	E D A B C	B E D C A	E A D B C

834	835	836	837	838	839	840
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D C E A B	D C E A B	D C E A B	D C E A B	D C E A B	D C E A B	D C E A B
C D B E A	C E A B D	C E A B D	C E D B A	C E D B A	E A B C D	E A B C D
E A D B C	B A D E C	E D B C A	B D A E C	E A B C D	B D A E C	C E D B A
B E A C D	E D B C A	B A D E C	E A B C D	B D A E C	C E D B A	B D A E C

841	842	843	844	845	846	847
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D C E A B	D C E A B	D C E A B	D C E A B	D C E A B	D C E A B	D C E B A
E A D B C	E A D B C	E D A B C	E D A B C	E D B C A	E D B C A	B A D E C
B E A C D	C D B E A	B E D C A	C A B E D	B A D E C	C E A B D	C E B A D
C D B E A	B E A C D	C A B E D	B E D C A	C E A B D	B A D E C	E D A C B

848	849	850	851	852	853	854
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D C E B A	D C E B A	D C E B A	D C E B A	D C E B A	D C E B A	D C E B A
B A D E C	B D A E C	B D A E C	B D A E C	B D A E C	B E A C D	B E A C D
E D A C B	C E B A D	C E D A B	E A B C D	E A D C B	C A D E B	E D B A C
C E B A D	E A D C B	E A B C D	C E D A B	C E B A D	E D B A C	C A D E B

855	856	857	858	859	860	861
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D C E B A	D C E B A	D C E B A	D C E B A	D C E B A	D C E B A	D C E B A
B E D A C	B E D A C	B E D A C	B E D A C	C A B E D	C A B E D	C A D E B
C A B E D	C D A E B	E A B C D	E D A C B	B E D A C	E D A C B	B E A C D
E D A C B	E A B C D	C D A E B	C A B E D	E D A C B	B E D A C	E D B A C

862	863	864	865	866	867	868
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D C E B A	D C E B A	D C E B A	D C E B A	D C E B A	D C E B A	D C E B A
C A D E B	C D A E B	C D A E B	C E B A D	C E B A D	C E B A D	C E B A D
E D B A C	B E D A C	E A B C D	B A D E C	B D A E C	E A D C B	E D A C B
B E A C D	E A B C D	B E D A C	E D A C B	E A D C B	B D A E C	B A D E C

869	870	871	872	873	874	875
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D C E B A	D C E B A	D C E B A	D C E B A	D C E B A	D C E B A	D C E B A
C E D A B	C E D A B	E A B C D	E A B C D	E A B C D	E A B C D	E A D C B
B D A E C	E A B C D	B D A E C	B E D A C	C D A E B	C E D A B	B D A E C
E A B C D	B D A E C	C E D A B	C D A E B	B E D A C	B D A E C	C E B A D

876	877	878	879	880	881	882
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D C E B A	D C E B A	D C E B A	D C E B A	D C E B A	D C E B A	D C E B A
E A D C B	E D A C B	E D A C B	E D A C B	E D A C B	E D B A C	E D B A C
C E B A D	B A D E C	B E D A C	C A B E D	C E B A D	B E A C D	C A D E B
B D A E C	C E B A D	C A B E D	B E D A C	B A D E C	C A D E B	B E A C D

883	884	885	886	887	888	889
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D E A B C	D E A B C	D E A B C	D E A B C	D E A B C	D E A B C	D E A B C
B A E C D	B A E C D	B C D E A	B C D E A	B C E A D	B C E A D	B C E A D
C D B E A	E C D A B	C D E A B	E A B C D	C A D E B	C D B E A	E A D C B
E C D A B	C D B E A	E A B C D	C D E A B	E D B C A	E A D C B	C D B E A

890	891	892	893	894	895	896
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D E A B C	D E A B C	D E A B C	D E A B C	D E A B C	D E A B C	D E A B C
B C E A D	B D E C A	B D E C A	B D E C A	B D E C A	C A B E D	C A B E D
E D B C A	C A B E D	C A D E B	E C B A D	E C D A B	B D E C A	E C D A B
C A D E B	E C D A B	E C B A D	C A D E B	C A B E D	E C D A B	B D E C A

897	898	899	900	901	902	903
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D E A B C	D E A B C	D E A B C	D E A B C	D E A B C	D E A B C	D E A B C
C A D E B	C A D E B	C A D E B	C A D E B	C D B E A	C D B E A	C D B E A
B C E A D	B D E C A	E C B A D	E D B C A	B A E C D	B C E A D	E A D C B
E D B C A	E C B A D	B D E C A	B C E A D	E C D A B	E A D C B	B C E A D

904	905	906	907	908	909	910
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D E A B C	D E A B C	D E A B C	D E A B C	D E A B C	D E A B C	D E A B C
C D B E A	C D E A B	C D E A B	E A B C D	E A B C D	E A D C B	E A D C B
E C D A B	B C D E A	E A B C D	B C D E A	C D E A B	B C E A D	C D B E A
B A E C D	E A B C D	B C D E A	C D E A B	B C D E A	C D B E A	B C E A D

911	912	913	914	915	916	917
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D E A B C	D E A B C	D E A B C	D E A B C	D E A B C	D E A B C	D E A B C
E C B A D	E C B A D	E C D A B	E C D A B	E C D A B	E C D A B	E D B C A
B D E C A	C A D E B	B A E C D	B D E C A	C A B E D	C D B E A	B C E A D
C A D E B	B D E C A	C D B E A	C A B E D	B D E C A	B A E C D	C A D E B

918	919	920	921	922	923	924
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D E A B C	D E A C B	D E A C B	D E A C B	D E A C B	D E A C B	D E A C B
E D B C A	B A D E C	B A D E C	B C D E A	B C D E A	B C E A D	B C E A D
C A D E B	C D E B A	E C B A D	C A E B D	E D B A C	C D B E A	E A D B C
B C E A D	E C B A D	C D E B A	E D B A C	C A E B D	E A D B C	C D B E A

925	926	927	928	929	930	931
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D E A C B	D E A C B	D E A C B	D E A C B	D E A C B	D E A C B	D E A C B
B D E A C	B D E A C	C A B E D	C A B E D	C A E B D	C A E B D	C D B E A
C A B E D	E C D B A	B D E A C	E C D B A	B C D E A	E D B A C	B C E A D
E C D B A	C A B E D	E C D B A	B D E A C	E D B A C	B C D E A	E A D B C

932	933	934	935	936	937	938
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D E A C B	D E A C B	D E A C B	D E A C B	D E A C B	D E A C B	D E A C B
C D B E A	C D E B A	C D E B A	E A D B C	E A D B C	E C B A D	E C B A D
E A D B C	B A D E C	E C B A D	B C E A D	C D B E A	B A D E C	C D E B A
B C E A D	E C B A D	B A D E C	C D B E A	B C E A D	C D E B A	B A D E C

939	940	941	942	943	944	945
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D E A C B	D E A C B	D E A C B	D E A C B	D E B A C	D E B A C	D E B A C
E C D B A	E C D B A	E D B A C	E D B A C	B A E C D	B A E C D	B C A E D
B D E A C	C A B E D	B C D E A	C A E B D	C D A E B	E C D B A	C D E B A
C A B E D	B D E A C	C A E B D	B C D E A	E C D B A	C D A E B	E A D C B

946	947	948	949	950	951	952
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D E B A C	D E B A C	D E B A C	D E B A C	D E B A C	D E B A C	D E B A C
B C A E D	B C D E A	B C D E A	B D E C A	B D E C A	C A D E B	C A D E B
E A D C B	C A E B D	E D A C B	C A D E B	E C A B D	B D E C A	E C A B D
C D E B A	E D A C B	C A E B D	E C A B D	C A D E B	E C A B D	B D E C A

953	954	955	956	957	958	959
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D E B A C	D E B A C	D E B A C	D E B A C	D E B A C	D E B A C	D E B A C
C A E B D	C A E B D	C D A E B	C D A E B	C D E B A	C D E B A	E A D C B
B C D E A	E D A C B	B A E C D	E C D B A	B C A E D	E A D C B	B C A E D
E D A C B	B C D E A	E C D B A	B A E C D	E A D C B	B C A E D	C D E B A

960	961	962	963	964	965	966
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D E B A C	D E B A C	D E B A C	D E B A C	D E B A C	D E B A C	D E B A C
E A D C B	E C A B D	E C A B D	E C D B A	E C D B A	E D A C B	E D A C B
C D E B A	B D E C A	C A D E B	B A E C D	C D A E B	B C D E A	C A E B D
B C A E D	C A D E B	B D E C A	C D A E B	B A E C D	C A E B D	B C D E A

967	968	969	970	971	972	973
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D E B C A	D E B C A	D E B C A	D E B C A	D E B C A	D E B C A	D E B C A
B A D E C	B A D E C	B C A E D	B C A E D	B C E A D	B C E A D	B C E A D
C D E A B	E C A B D	C D E A B	E A D B C	C A D E B	C D A E B	E A D B C
E C A B D	C D E A B	E A D B C	C D E A B	E D A B C	E A D B C	C D A E B

974	975	976	977	978	979	980
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D E B C A	D E B C A	D E B C A	D E B C A	D E B C A	D E B C A	D E B C A
B C E A D	B D A E C	B D A E C	B D E A C	B D E A C	C A D E B	C A D E B
E D A B C	C A E B D	E C D A B	C A D E B	E C A B D	B C E A D	B D E A C
C A D E B	E C D A B	C A E B D	E C A B D	C A D E B	E D A B C	E C A B D

981	982	983	984	985	986	987
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D E B C A	D E B C A	D E B C A	D E B C A	D E B C A	D E B C A	D E B C A
C A D E B	C A D E B	C A E B D	C A E B D	C D A E B	C D A E B	C D E A B
E C A B D	E D A B C	B D A E C	E C D A B	B C E A D	E A D B C	B A D E C
B D E A C	B C E A D	E C D A B	B D A E C	E A D B C	B C E A D	E C A B D

988	989	990	991	992	993	994
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D E B C A	D E B C A	D E B C A	D E B C A	D E B C A	D E B C A	D E B C A
C D E A B	C D E A B	C D E A B	E A D B C	E A D B C	E A D B C	E A D B C
B C A E D	E A D B C	E C A B D	B C A E D	B C E A D	C D A E B	C D E A B
E A D B C	B C A E D	B A D E C	C D E A B	C D A E B	B C E A D	B C A E D

995	996	997	998	999	1000	1001
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D E B C A	D E B C A	D E B C A	D E B C A	D E B C A	D E B C A	D E B C A
E C A B D	E C A B D	E C A B D	E C A B D	E C D A B	E C D A B	E D A B C
B A D E C	B D E A C	C A D E B	C D E A B	B D A E C	C A E B D	B C E A D
C D E A B	C A D E B	B D E A C	B A D E C	C A E B D	B D A E C	C A D E B

1002	1003	1004	1005	1006	1007	1008
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
D E B C A	E A B C D	E A B C D	E A B C D	E A B C D	E A B C D	E A B C D
E D A B C	B C D E A	B C D E A	B D A E C	B D A E C	B D A E C	B D A E C
C A D E B	C D E A B	D E A B C	C E D A B	C E D B A	D C E A B	D C E B A
B C E A D	D E A B C	C D E A B	D C E B A	D C E A B	C E D B A	C E D A B

1009	1010	1011	1012	1013	1014	1015
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E A B C D	E A B C D	E A B C D	E A B C D	E A B C D	E A B C D	E A B C D
B D E A C	B D E A C	B E D A C	B E D A C	B E D A C	B E D A C	C D A E B
C E D B A	D C A E B	C D A E B	C D E B A	D C A E B	D C E B A	B E D A C
D C A E B	C E D B A	D C E B A	D C A E B	C D E B A	C D A E B	D C E B A

1016	1017	1018	1019	1020	1021	1022
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E A B C D	E A B C D	E A B C D	E A B C D	E A B C D	E A B C D	E A B C D
C D A E B	C D E A B	C D E A B	C D E B A	C D E B A	C E D A B	C E D A B
D C E B A	B C D E A	D E A B C	B E D A C	D C A E B	B D A E C	D C E B A
B E D A C	D E A B C	B C D E A	D C A E B	B E D A C	D C E B A	B D A E C

1023	1024	1025	1026	1027	1028	1029
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E A B C D	E A B C D	E A B C D	E A B C D	E A B C D	E A B C D	E A B C D
C E D B A	C E D B A	C E D B A	C E D B A	D C A E B	D C A E B	D C A E B
B D A E C	B D E A C	D C A E B	D C E A B	B D E A C	B E D A C	C D E B A
D C E A B	D C A E B	B D E A C	B D A E C	C E D B A	C D E B A	B E D A C

1030	1031	1032	1033	1034	1035	1036
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E A B C D	E A B C D	E A B C D	E A B C D	E A B C D	E A B C D	E A B C D
D C A E B	D C E A B	D C E A B	D C E B A	D C E B A	D C E B A	D C E B A
C E D B A	B D A E C	C E D B A	B D A E C	B E D A C	C D A E B	C E D A B
B D E A C	C E D B A	B D A E C	C E D A B	C D A E B	B E D A C	B D A E C

1037	1038	1039	1040	1041	1042	1043
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E A B C D	E A B C D	E A D B C	E A D B C	E A D B C	E A D B C	E A D B C
D E A B C	D E A B C	B C A E D	B C A E D	B C E A D	B C E A D	B C E A D
B C D E A	C D E A B	C D E A B	D E B C A	C D A E B	C D B E A	D E A C B
C D E A B	B C D E A	D E B C A	C D E A B	D E B C A	D E A C B	C D B E A

1044	1045	1046	1047	1048	1049	1050
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E A D B C	E A D B C	E A D B C	E A D B C	E A D B C	E A D B C	E A D B C
B C E A D	B D E C A	B D E C A	B E A C D	B E A C D	B E A C D	B E A C D
D E B C A	C E B A D	D C A E B	C D B E A	C D E A B	D C B E A	D C E A B
C D A E B	D C A E B	C E B A D	D C E A B	D C B E A	C D E A B	C D B E A

1051	1052	1053	1054	1055	1056	1057
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E A D B C	E A D B C	E A D B C	E A D B C	E A D B C	E A D B C	E A D B C
C D A E B	C D A E B	C D B E A	C D B E A	C D B E A	C D B E A	C D E A B
B C E A D	D E B C A	B C E A D	B E A C D	D C E A B	D E A C B	B C A E D
D E B C A	B C E A D	D E A C B	D C E A B	B E A C D	B C E A D	D E B C A

1058	1059	1060	1061	1062	1063	1064
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E A D B C	E A D B C	E A D B C	E A D B C	E A D B C	E A D B C	E A D B C
C D E A B	C D E A B	C D E A B	C E B A D	C E B A D	D C A E B	D C A E B
B E A C D	D C B E A	D E B C A	B D E C A	D C A E B	B D E C A	C E B A D
D C B E A	B E A C D	B C A E D	D C A E B	B D E C A	C E B A D	B D E C A

1065	1066	1067	1068	1069	1070	1071
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E A D B C	E A D B C	E A D B C	E A D B C	E A D B C	E A D B C	E A D B C
D C B E A	D C B E A	D C E A B	D C E A B	D E A C B	D E A C B	D E B C A
B E A C D	C D E A B	B E A C D	C D B E A	B C E A D	C D B E A	B C A E D
C D E A B	B E A C D	C D B E A	B E A C D	C D B E A	B C E A D	C D E A B

1072	1073	1074	1075	1076	1077	1078
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E A D B C	E A D B C	E A D B C	E A D C B	E A D C B	E A D C B	E A D C B
D E B C A	D E B C A	D E B C A	B C A E D	B C A E D	B C E A D	B C E A D
B C E A D	C D A E B	C D E A B	C D E B A	D E B A C	C D B E A	D E A B C
C D A E B	B C E A D	B C A E D	D E B A C	C D E B A	D E A B C	C D B E A
1079	1080	1081	1082	1083	1084	1085
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E A D C B	E A D C B	E A D C B	E A D C B	E A D C B	E A D C B	E A D C B
B D A E C	B D A E C	B D E A C	B D E A C	C D B E A	C D B E A	C D E B A
C E B A D	D C E B A	C E A B D	D C B E A	B C E A D	D E A B C	B C A E D
D C E B A	C E B A D	D C B E A	C E A B D	D E A B C	B C E A D	D E B A C
1086	1087	1088	1089	1090	1091	1092
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E A D C B	E A D C B	E A D C B	E A D C B	E A D C B	E A D C B	E A D C B
C D E B A	C E A B D	C E A B D	C E B A D	C E B A D	D C B E A	D C B E A
D E B A C	B D E A C	D C B E A	B D A E C	D C E B A	B D E A C	C E A B D
B C A E D	D C B E A	B D E A C	D C E B A	B D A E C	C E A B D	B D E A C
1093	1094	1095	1096	1097	1098	1099
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E A D C B	E A D C B	E A D C B	E A D C B	E A D C B	E A D C B	E C A B D
D C E B A	D C E B A	D E A B C	D E A B C	D E B A C	D E B A C	B A D E C
B D A E C	C E B A D	B C E A D	C D B E A	B C A E D	C D E B A	C D E A B
C E B A D	B D A E C	C D B E A	B C E A D	C D E B A	B C A E D	D E B C A
1100	1101	1102	1103	1104	1105	1106
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E C A B D	E C A B D	E C A B D	E C A B D	E C A B D	E C A B D	E C A B D
B A D E C	B D E A C	B D E A C	B D E C A	B D E C A	B D E C A	B D E C A
D E B C A	C A D E B	D E B C A	C A D E B	C E D A B	D A B E C	D E B A C
C D E A B	D E B C A	C A D E B	D E B A C	D A B E C	C E D A B	C A D E B
1107	1108	1109	1110	1111	1112	1113
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E C A B D	E C A B D	E C A B D	E C A B D	E C A B D	E C A B D	E C A B D
B E D A C	B E D A C	B E D C A	B E D C A	C A D E B	C A D E B	C A D E B
C D B E A	D A E C B	C D E A B	D A B E C	B D E A C	B D E C A	D E B A C
D A E C B	C D B E A	D A B E C	C D E A B	D E B C A	D E B A C	B D E C A
1114	1115	1116	1117	1118	1119	1120
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E C A B D	E C A B D	E C A B D	E C A B D	E C A B D	E C A B D	E C A B D
C A D E B	C D B E A	C D B E A	C D E A B	C D E A B	C D E A B	C D E A B
D E B C A	B E D A C	D A E C B	B A D E C	B E D C A	D A B E C	D E B C A
B D E A C	D A E C B	B E D A C	D E B C A	D A B E C	B E D C A	B A D E C
1121	1122	1123	1124	1125	1126	1127
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E C A B D	E C A B D	E C A B D	E C A B D	E C A B D	E C A B D	E C A B D
C E D A B	C E D A B	D A B E C	D A B E C	D A B E C	D A B E C	D A E C B
B D E C A	D A B E C	B D E C A	B E D C A	C D E A B	C E D A B	B E D A C
D A B E C	B D E C A	C E D A B	C D E A B	B E D C A	B D E C A	C D B E A
1128	1129	1130	1131	1132	1133	1134
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E C A B D	E C A B D	E C A B D	E C A B D	E C A B D	E C A B D	E C A B D
D A E C B	D E B A C	D E B A C	D E B C A	D E B C A	D E B C A	D E B C A
C D B E A	B D E C A	C A D E B	B A D E C	B D E A C	C A D E B	C D E A B
B E D A C	C A D E B	B D E C A	C D E A B	C A D E B	B D E A C	B A D E C
1135	1136	1137	1138	1139	1140	1141
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E C B A D	E C B A D	E C B A D	E C B A D	E C B A D	E C B A D	E C B A D
B A D E C	B A D E C	B D A E C	B D A E C	B D E C A	B D E C A	B E D C A
C D E B A	D E A C B	C E D B A	D A E C B	C A D E B	D E A B C	C D A E B
D E A C B	C D E B A	D A E C B	C E D B A	D E A B C	C A D E B	D A E B C
1142	1143	1144	1145	1146	1147	1148
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E C B A D	E C B A D	E C B A D	E C B A D	E C B A D	E C B A D	E C B A D
B E D C A	C A D E B	C A D E B	C D A E B	C D A E B	C D E B A	C D E B A
D A E B C	B D E C A	D E A B C	B E D C A	D A E B C	B A D E C	D E A C B
C D A E B	D E A B C	B D E C A	D A E B C	B E D C A	D E A C B	B A D E C
1149	1150	1151	1152	1153	1154	1155
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E C B A D	E C B A D	E C B A D	E C B A D	E C B A D	E C B A D	E C B A D
C E D B A	C E D B A	D A E B C	D A E B C	D A E C B	D A E C B	D E A B C
B D A E C	D A E C B	B E D C A	C D A E B	B D A E C	C E D B A	B D E C A
D A E C B	B D A E C	C D A E B	B E D C A	C E D B A	B D A E C	C A D E B

1156	1157	1158	1159	1160	1161	1162
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E C B A D	E C B A D	E C B A D	E C D A B	E C D A B	E C D A B	E C D A B
D E A B C	D E A C B	D E A C B	B A E C D	B A E C D	B D A E C	B D A E C
C A D E B	B A D E C	C D E B A	C D B E A	D E A B C	C A E B D	D E B C A
B D E C A	C D E B A	B A D E C	D E A B C	C D B E A	D E B C A	C A E B D

1163	1164	1165	1166	1167	1168	1169
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E C D A B	E C D A B	E C D A B	E C D A B	E C D A B	E C D A B	E C D A B
B D E C A	B D E C A	B D E C A	B D E C A	B E A C D	B E A C D	B E A C D
C A B E D	C E A B D	D A B E C	D E A B C	C D B E A	C D E B A	D A B E C
D E A B C	D A B E C	C E A B D	C A B E D	D A E B C	D A B E C	C D E B A

1170	1171	1172	1173	1174	1175	1176
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E C D A B	E C D A B	E C D A B	E C D A B	E C D A B	E C D A B	E C D A B
B E A C D	C A B E D	C A B E D	C A E B D	C A E B D	C D B E A	C D B E A
D A E B C	B D E C A	D E A B C	B D A E C	D E B C A	B A E C D	B E A C D
C D B E A	D E A B C	B D E C A	D E B C A	B D A E C	D E A B C	D A E B C

1177	1178	1179	1180	1181	1182	1183
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E C D A B	E C D A B	E C D A B	E C D A B	E C D A B	E C D A B	E C D A B
C D B E A	C D B E A	C D E B A	C D E B A	C E A B D	C E A B D	D A B E C
D A E B C	D E A B C	B E A C D	D A B E C	B D E C A	D A B E C	B D E C A
B E A C D	B A E C D	D A B E C	B E A C D	D A B E C	B D E C A	C E A B D

1184	1185	1186	1187	1188	1189	1190
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E C D A B	E C D A B	E C D A B	E C D A B	E C D A B	E C D A B	E C D A B
D A B E C	D A B E C	D A B E C	D A E B C	D A E B C	D E A B C	D E A B C
B E A C D	C D E B A	C E A B D	B E A C D	C D B E A	B A E C D	B D E C A
C D E B A	B E A C D	B D E C A	C D B E A	B E A C D	C D B E A	C A B E D

1191	1192	1193	1194	1195	1196	1197
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E C D A B	E C D A B	E C D A B	E C D A B	E C D B A	E C D B A	E C D B A
D E A B C	D E A B C	D E B C A	D E B C A	B A E C D	B A E C D	B D A E C
C A B E D	C D B E A	B D A E C	C A E B D	C D A E B	D E B A C	C E B A D
B D E C A	B A E C D	C A E B D	B D A E C	D E B A C	C D A E B	D A E C B

1198	1199	1200	1201	1202	1203	1204
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E C D B A	E C D B A	E C D B A	E C D B A	E C D B A	E C D B A	E C D B A
B D A E C	B D E A C	B D E A C	B E A C D	B E A C D	C A B E D	C A B E D
D A E C B	C A B E D	D E A C B	C D E A B	D A B E C	B D E A C	D E A C B
C E B A D	D E A C B	C A B E D	D A B E C	C D E A B	D E A C B	B D E A C

1205	1206	1207	1208	1209	1210	1211
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E C D B A	E C D B A	E C D B A	E C D B A	E C D B A	E C D B A	E C D B A
C D A E B	C D A E B	C D E A B	C D E A B	C E B A D	C E B A D	D A B E C
B A E C D	D E B A C	B E A C D	D A B E C	B D A E C	D A E C B	B E A C D
D E B A C	B A E C D	D A B E C	B E A C D	D A E C B	B D A E C	C D E A B

1212	1213	1214	1215	1216	1217	1218
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E C D B A	E C D B A	E C D B A	E C D B A	E C D B A	E C D B A	E C D B A
D A B E C	D A E C B	D A E C B	D E A C B	D E A C B	D E B A C	D E B A C
C D E A B	B D A E C	C E B A D	B D E A C	C A B E D	B A E C D	C D A E B
B E A C D	C E B A D	B D A E C	C A B E D	B D E A C	C D A E B	B A E C D

1219	1220	1221	1222	1223	1224	1225
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E D A B C	E D A B C	E D A B C	E D A B C	E D A B C	E D A B C	E D A B C
B A E C D	B A E C D	B C D E A	B C D E A	B C E A D	B C E A D	B E D C A
C E D A B	D C B E A	C E B A D	D A E C B	C A D E B	D E B C A	C A B E D
D C B E A	C E D A B	D A E C B	C E B A D	D E B C A	C A D E B	D C E A B

1226	1227	1228	1229	1230	1231	1232
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E D A B C	E D A B C	E D A B C	E D A B C	E D A B C	E D A B C	E D A B C
B E D C A	C A B E D	C A B E D	C A D E B	C A D E B	C E B A D	C E B A D
D C E A B	B E D C A	D C E A B	B C E A D	D E B C A	B C D E A	D A E C B
C A B E D	D C E A B	B E D C A	D E B C A	B C E A D	D A E C B	B C D E A

1233	1234	1235	1236	1237	1238	1239
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E D A B C	E D A B C	E D A B C	E D A B C	E D A B C	E D A B C	E D A B C
C E D A B	C E D A B	D A E C B	D A E C B	D C B E A	D C B E A	D C E A B
B A E C D	D C B E A	B C D E A	C E B A D	B A E C D	C E D A B	B E D C A
D C B E A	B A E C D	C E B A D	B C D E A	C E D A B	B A E C D	C A B E D

1240	1241	1242	1243	1244	1245	1246
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E D A B C	E D A B C	E D A B C	E D A C B	E D A C B	E D A C B	E D A C B
D C E A B	D E B C A	D E B C A	B A D E C	B A D E C	B C D E A	B C D E A
C A B E D	B C E A D	C A D E B	C E B A D	D C E B A	C A E B D	C E B A D
B E D C A	C A D E B	B C E A D	D C E B A	C E B A D	D E B A C	D A E B C

1247	1248	1249	1250	1251	1252	1253
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E D A C B	E D A C B	E D A C B	E D A C B	E D A C B	E D A C B	E D A C B
B C D E A	B C D E A	B C E A D	B C E A D	B E D A C	B E D A C	B E D A C
D A E B C	D E B A C	C E D B A	D A B E C	C A B E D	C A E B D	D C B E A
C E B A D	C A E B D	D A B E C	C E D B A	D C E B A	D C B E A	C A E B D

1254	1255	1256	1257	1258	1259	1260
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E D A C B	E D A C B	E D A C B	E D A C B	E D A C B	E D A C B	E D A C B
B E D A C	C A B E D	C A B E D	C A E B D	C A E B D	C A E B D	C A E B D
D C E B A	B E D A C	D C E B A	B C D E A	B E D A C	D C B E A	D E B A C
C A B E D	D C E B A	B E D A C	D E B A C	D C B E A	B E D A C	B C D E A

1261	1262	1263	1264	1265	1266	1267
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E D A C B	E D A C B	E D A C B	E D A C B	E D A C B	E D A C B	E D A C B
C E B A D	C E B A D	C E B A D	C E B A D	C E D B A	C E D B A	D A B E C
B A D E C	B C D E A	D A E B C	D C E B A	B C E A D	D A B E C	B C E A D
D C E B A	D A E B C	B C D E A	B A D E C	D A B E C	B C E A D	C E D B A

1268	1269	1270	1271	1272	1273	1274
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E D A C B	E D A C B	E D A C B	E D A C B	E D A C B	E D A C B	E D A C B
D A B E C	D A E B C	D A E B C	D C B E A	D C B E A	D C E B A	D C E B A
C E D B A	B C D E A	C E B A D	B E D A C	C A E B D	B A D E C	B E D A C
B C E A D	C E B A D	B C D E A	C A E B D	B E D A C	C E B A D	C A B E D

1275	1276	1277	1278	1279	1280	1281
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E D A C B	E D A C B	E D A C B	E D A C B	E D B A C	E D B A C	E D B A C
D C E B A	D C E B A	D E B A C	D E B A C	B A E C D	B A E C D	B C A E D
C A B E D	C E B A D	B C D E A	C A E B D	C E D B A	D C A E B	C E D B A
B E D A C	B A D E C	C A E B D	B C D E A	D C A E B	C E D B A	D A E C B

1282	1283	1284	1285	1286	1287	1288
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E D B A C	E D B A C	E D B A C	E D B A C	E D B A C	E D B A C	E D B A C
B C A E D	B C D E A	B C D E A	B C D E A	B C D E A	B E A C D	B E A C D
D A E C B	C A E B D	C E A B D	D A E C B	D E A C B	C A D E B	D C E B A
C E D B A	D E A C B	D A E C B	C E A B D	C A E B D	D C E B A	C A D E B

1289	1290	1291	1292	1293	1294	1295
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E D B A C	E D B A C	E D B A C	E D B A C	E D B A C	E D B A C	E D B A C
B E D C A	B E D C A	C A D E B	C A D E B	C A E B D	C A E B D	C A E B D
C A E B D	D C A E B	B E A C D	D C E B A	B C D E A	B E D C A	D C A E B
D C A E B	C A E B D	D C E B A	B E A C D	D E A C B	D C A E B	B E D C A

1296	1297	1298	1299	1300	1301	1302
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E D B A C	E D B A C	E D B A C	E D B A C	E D B A C	E D B A C	E D B A C
C A E B D	C E A B D	C E A B D	C E D B A	C E D B A	C E D B A	C E D B A
D E A C B	B C D E A	D A E C B	B A E C D	B C A E D	D A E C B	D C A E B
B C D E A	D A E C B	B C D E A	D C A E B	D A E C B	B C A E D	B A E C D

1303	1304	1305	1306	1307	1308	1309
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E D B A C	E D B A C	E D B A C	E D B A C	E D B A C	E D B A C	E D B A C
D A E C B	D A E C B	D A E C B	D A E C B	D C A E B	D C A E B	D C A E B
B C A E D	B C D E A	C E A B D	C E D B A	B A E C D	B E D C A	C A E B D
C E D B A	C E A B D	B C D E A	B C A E D	C E D B A	C A E B D	B E D C A

1310	1311	1312	1313	1314	1315	1316
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E D B A C	E D B A C	E D B A C	E D B A C	E D B A C	E D B C A	E D B C A
D C A E B	D C E B A	D C E B A	D E A C B	D E A C B	B A D E C	B A D E C
C E D B A	B E A C D	C A D E B	B C D E A	C A E B D	C E A B D	D C E A B
B A E C D	C A D E B	B E A C D	C A E B D	B C D E A	D C E A B	C E A B D

1317	1318	1319	1320	1321	1322	1323
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E D B C A	E D B C A	E D B C A	E D B C A	E D B C A	E D B C A	E D B C A
B C A E D	B C A E D	B C E A D	B C E A D	B E D A C	B E D A C	C A D E B
C E D A B	D A E B C	C A D E B	D E A B C	C A E B D	D C A E B	B C E A D
D A E B C	C E D A B	D E A B C	C A D E B	D C A E B	C A E B D	D E A B C

1324	1325	1326	1327	1328	1329	1330
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E D B C A	E D B C A	E D B C A	E D B C A	E D B C A	E D B C A	E D B C A
C A D E B	C A E B D	C A E B D	C E A B D	C E A B D	C E D A B	C E D A B
D E A B C	B E D A C	D C A E B	B A D E C	D C E A B	B C A E D	D A E B C
B C E A D	D C A E B	B E D A C	D C E A B	B A D E C	D A E B C	B C A E D

1331	1332	1333	1334	1335	1336	1337
A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E	A B C D E
E D B C A	E D B C A	E D B C A	E D B C A	E D B C A	E D B C A	E D B C A
D A E B C	D A E B C	D C A E B	D C A E B	D C E A B	D C E A B	D E A B C
B C A E D	C E D A B	B E D A C	C A E B D	B A D E C	C E A B D	B C E A D
C E D A B	B C A E D	C A E B D	B E D A C	C E A B D	B A D E C	C A D E B

1338
A B C D E
E D B C A
D E A B C
C A D E B
B C E A D

APPENDIX D

LIST OF PUBLICATIONS

1. H. V. Chen, A. Y. M. Chin, and S. Sharmini, "Constructions of non-commutative generalized latin squares of order 5," *Proceedings of the 6th IMT-GT Conference on Mathematics, Statistics and its Applications (ICMSA2010)*, pp. 120–130, 2010.
2. H. V. Chen, A. Y. M. Chin, and S. Sharmini, "Generalized Latin squares of order n with $n^2 - 1$ distinct elements," *Periodica Math. Hungarica* (to appear).
3. H. V. Chen, A. Y. M. Chin, and S. Sharmini, "Some constructions of non-commutative Latin squares of order n," *Proceedings of World Academy of Science, Engineering and Technology*, Issue 79, pp. 272–274, 2011.
4. H. V. Chen, A. Y. M. Chin, and S. Sharmini, "Exhaustion Numbers of 2-Subsets of Dihedral Groups," *Proceedings of World Academy of Science, Engineering and Technology*, Issue 62, pp. 174–175, 2012.