RABI OSCILLATIONS IN THE JAYNES-CUMMINGS MODEL

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A project report submitted in partial fulfilment of the requirements for the award of Bachelor of Science (Hons.) Physics

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DECLARATION

I hereby declare that this project report is based on my original work except for citations and quotations which have been duly acknowledged. I also declare that it has not been previously and concurrently submitted for any other degree or award at UTAR or other institutions.

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APPROVAL FOR SUBMISSION

I certify that this project report entitled **"RABI OSCILLATIONS IN THE JAYNES-CUMMINGS MODEL"** was prepared by **TING HOCK KING** has met the required standard for submission in partial fulfilment of the requirements for the award of Bachelor of Science (Hons.) Physics at Universiti Tunku Abdul Rahman.

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Date : _________________________

Specially dedicated to

my beloved mother, brother and sisters

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ABSTRACT

In this thesis, the interaction between a two-level atom and single-mode quantized light sources is studied quantum mechanically. This is also known as the Jaynes-Cummings Model (JCM). This model is studied theoretically by first deriving the total atom-field Hamiltonian in Schrodinger Picture, which is the sum of Hamiltonian of the two-level atom, field Hamiltonian and atom-field interaction Hamiltonian. The field Hamiltonian is obtained through the field quantization while the atom-field interaction Hamiltonian is derived by using the Electric Dipole Approximation. The Schrodinger Picture total atom-field Hamiltonian is then changed into the Interaction Picture by performing the necessary unitary transformation. The non-energy conserving terms in the Interaction Picture JCM Hamiltonian are omitted by applying Rotating Wave Approximation (RWA). This Hamiltonian is then used to derive the corresponding unitary operator. The quantum state of the combined system of atom and field at any time is derived by performing unitary transformation on the initial quantum state. From the quantum state of the system, the relationship between the probability that the atom is in the ground state $P(t)$ and time t can be established. We consider two major field states, namely coherent state and thermal state, which serve as different initial field statistics for the computation of $P(t)$. From $P(t)$, we see the striking collapse and revival feature of the Rabi Oscillations and emphasis will be given on discussing this interesting feature. We then discuss the effects of interaction strength λ, initial field states and detuning Δ on the collapse and revival behavior of the Rabi Oscillations. In this thesis, the derivations are initiated with the single- photon JCM case, then the whole process is extended to two-photon JCM, three-photon JCM and it is finally generalized to k-photon JCM.

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

1.1 Background

In physics, the interaction between light and atom can be studied classically, semiclassically and quantum mechanically. In Classical Optics, the light is treated as electromagnetic waves while atom is modelled as a Hertzian dipole. When the interaction between light and atom is analyzed semiclassically, only the energy of the atom is quantized but the light is still treated as waves. Lastly, when the interaction is studied quantum mechanically, the energy of the atom is quantized and at the same time, the light is considered as photons (quantized energy packets). Quantum Optics is the branch of physics which studies the optical phenomena by using Quantum Mechanics. Jaynes Cummings Model is proposed by E.T.Jaynes and F.W.Cummings in 1963 and it is one of the fundamental problems in Quantum Optics. This model is important because there are optical phenomena which could not be explained classically or semiclassically, for example the spontaneous emission of light by an excited atom.

In Jaynes Cummings Model, a two-level atom is allowed to interact with single mode (frequency) quantized light sources. The combined system of the atom and field is in an ideal case in which there is no energy dissipation. The quantized light sources will serve as the initial field states in the light-atom interaction. Coherent states and thermal states are some examples of the initial field states. When the two-level atom interacts with the coherent field state, beautiful collapse and revival of the Rabi Oscillations could be observed. Conversely, chaotic results are observed if thermal state is prepared as an initial field state interacting with the atom.

1.2 Aims and Objectives

The Jaynes-Cummings Model (JCM) will be studied theoretically by using quantum mechanics. The quantization of electromagnetic field, quantization of the energy of two-level atom and modelling of atom-field interaction by using Electric Dipole Approximation will be studied. Then, Interaction Picture representation of the JCM Hamiltonian and Rotating Wave Approximation (RWA) will be used to simplify the JCM Hamiltonian. After that, the unitary operator method is used to derive the probability of the atom in ground state as a function of time. The probability function will then be plotted by using coherent and thermal initial field states. The field statistics of the coherent and thermal states are going to be studied as well in this final year project. Finally, the collapses and revivals of the Rabi Oscillations are investigated. At the same time, the effect of detuning, interaction strength and mean photon number on the Rabi Oscillations will also be discussed. The probability functions will be derived first for single-photon JCM. Then, the derivations of the probability function are extended to two-photon JCM, three-photon JCM and they are finally generalized to k-photon JCM.

CHAPTER 2

LITERATURE REVIEW

2.1 Coherent State

The coherent state of the radiation field, $|\alpha\rangle$ is defined mathematically as follows:

$$|\alpha\rangle = exp(-|\alpha|^2/2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{2.1}$$

where $|n\rangle$ is the number state. A number state represents the monochromatic quantized field of angular frequency ω containing n number of photons. α is a complex eigenvalue which satisfies

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \tag{2.2}$$

and $\hat{a}$ is the annihilation operator.

Equation (2.1) shows that the coherent state can be expressed as the linear superposition of the number state $|n\rangle$. According to Fox (2006), the photon stream of the coherent state is random as shown in Figure 2.1 below.

Figure 2.1: Photon stream of the coherent state

The mean number of photons in the coherent state $|n\rangle$ can be calculated as

$$\begin{aligned}\bar{n} &= \langle\alpha|\hat{n}|\alpha\rangle \\ &= \langle\alpha|\hat{a}^{\dagger}\hat{a}|\alpha\rangle. \qquad (2.3) \\ &= |\alpha|^2\end{aligned}$$

Derivation of Coherent State's Photon-number Probability Distribution Function

The following derivation is completed mainly following the approach in Fox (2006). Consider a monochromatic light of angular frequency ω and constant intensity I. In quantum mechanical picture of light, a light beam is considered to consist of a stream of discrete energy packets known as photons. The photon flux Φ is defined as the average number of photons passing through the cross section of a light beam in a unit time. The photon flux Φ can be calculated by using the following method:

$$\Phi = \frac{IA}{\hbar\omega} = \frac{\boldsymbol{P}}{\hbar\omega} \text{ photons/s},$$

where A is the cross sectional area of the light beam, and $\boldsymbol{P}$ is the power of the beam .

Then, the average number of photons $\bar{n}$ within a beam segment of length L is

$$\bar{n} = \frac{\Phi \mathrm{L}}{c},$$

where c is the speed of light in vacuum.

Assuming that L is large enough that $\bar{n}$ is a well defined integer value. Then, the light beam is subdivided into N subsegments of length L/N each. The value of N is so large that only a very small probability of p = $\bar{n}$/N of finding a photon within any particular subsegment, and a negligibly small probability of finding two or more photons. Then, let $P(n)$ to be the probability of finding n photons within a beam of length L containing N subsegments.

$P(n)$ could be found by using Binomial distribution. Since it was explained earlier that the probability of finding two or more photons within a beam of length L containing N subsegments is negligible, it is equivalent to say that $P(n)$ is the probability of finding n subsegments containing 1 photon and (N$-n$) subsegments containing no photons within a beam of length L containing N subsegments. By using Binomial distribution,

$$P(n) = \frac{N!}{n!\,(N-n)!} p^n (1-p)^{N-n}.$$

Since $p = \dfrac{\bar{n}}{N}$ as explained earlier,

$$P(n) = \frac{N!}{n!\,(N-n)!}\left(\frac{\bar{n}}{N}\right)^{n}\left(1-\frac{\bar{n}}{N}\right)^{N-n}$$

$$= \frac{1}{n!}\left(\frac{N!}{(N-n)!\,N^{n}}\right)(\bar{n})^{n}\left(1-\frac{\bar{n}}{N}\right)^{N-n}. \tag{2.4}$$

To simplify $\left(\frac{N!}{(N-n)!\,N^{n}}\right)$ in (2.4), the Stirling's Formula $\lim_{N\to\infty}(\ln N!) = N\ln N - N$ is used.

Since N is a very large number as explained earlier,

$$\lim_{N\to\infty}\left[\ln\left(\frac{N!}{(N-n)!\,N^{n}}\right)\right] = \lim_{N\to\infty}[\ln N! - \ln(N-n)! - n\mathrm{lnN}]$$

$$= \lim_{N\to\infty}\ln N! - \lim_{N\to\infty}\ln(N-n)! - \lim_{N\to\infty} n\mathrm{lnN}.$$

By using the Stirling's Formula above,

$$\lim_{N\to\infty}\left[\ln\left(\frac{N!}{(N-n)!\,N^{n}}\right)\right] = N\ln N - N - [(N-n)\ln(N-n) - (N-n)] - \lim_{N\to\infty} n\mathrm{lnN}$$

$$= N\ln N - N - [(N-n)\ln N - N] - \lim_{N\to\infty} n\mathrm{lnN}$$

$$= n\mathrm{lnN} - \lim_{N\to\infty} n\mathrm{lnN}$$

$$= 0,$$

since $\mathrm{N} \gg \mathrm{n}, (\mathrm{N}-\mathrm{n}) \approx \mathrm{N}$.

Then,

$$\ln\left(\frac{N!}{(N-n)!\,N^{n}}\right) = 0$$

$$\frac{N!}{(N-n)!\,N^{n}} = e^{0} = 1. \tag{2.5}$$

Next, to simplify $\left(1-\frac{\bar{n}}{N}\right)^{N-n}$ in (2.4), by using binomial expansion,

$$\left(1-\frac{\bar{n}}{N}\right)^{N-n} = 1 - \frac{N-n}{1!}\left(\frac{\bar{n}}{N}\right) + \frac{(N-n)(N-n-1)}{2!}\left(\frac{\bar{n}}{N}\right)^{2} - \frac{(N-n)(N-n-1)(N-n-2)}{3!}\left(\frac{\bar{n}}{N}\right)^{3} \pm \dots\dots \tag{2.6}$$

Since N is a very large number and $\mathrm{N} \gg \mathrm{n}$, then $(\mathrm{N}-\mathrm{n}) \approx \mathrm{N}, (\mathrm{N}-\mathrm{n}-1) \approx \mathrm{N}$,

$(\mathrm{N}-\mathrm{n}-2) \approx \mathrm{N} \ldots \ldots$. Then, (2.6) can be approximated as

$$\left(1-\frac{\bar{n}}{N}\right)^{N-n} \approx 1-\frac{N}{1!}\left(\frac{\bar{n}}{N}\right)+\frac{(N)(N)}{2!}\left(\frac{\bar{n}}{N}\right)^{2}-\frac{(N)(N)(N)}{3!}\left(\frac{\bar{n}}{N}\right)^{3} \pm \ldots \ldots$$

$$=1-\left(\frac{\bar{n}}{1!}\right)+\frac{\bar{n}}{2!}^{2}-\frac{\bar{n}}{3!}^{3} \pm \ldots \ldots$$

$$=\sum_{r=0}^{\infty} \frac{(-\bar{n})^{r}}{r!}$$

$$=exp(-\bar{n}). \tag{2.7}$$

By substituting (2.5) and (2.7) into (2.4), (2.4) becomes

$$P(n)=\frac{1}{n!}(1)(\bar{n})^{n} exp(-\bar{n})$$

$$=\frac{\bar{n}^{n}}{n!} exp(-\bar{n}), \qquad where\; n=0,1,2,3 \ldots \ldots. \tag{2.8}$$

Equation (2.8) is the probability distribution function of the coherent state, which is also a Poisson distribution. The photon probability distribution for the laser approaches the Poisson distribution. Hence, laser is an example for coherent state.

For coherent state, the variance of the photon number will be equal to the mean photon number $\bar{n}$.

2.2 Thermal State

According to Fox (2006), the electromagnetic field emitted by a hot body is called thermal light(state) or blackbody radiation. For a thermal photons stream, the thermal photons are bunched as shown in Figure 2.2 below:

Figure 2.2: Photon stream of thermal state

The quantized energy E_n of a single-mode electromagnetic field is given by

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega,$$

where $n = 0,1,2 \ldots \ldots$, and ω is the angular frequency. (2.9)

The expression in (2.9) can be proven as follows:

Let $|n\rangle$ be the number state. A number state represents the monochromatic quantized field of angular frequency ω containing n number of photons. Then, the Electromagnetic Field Hamiltonian $\hat{H}_R$ eigenvalue function is

$$\hat{H}_R|n\rangle = E_n|n\rangle, \tag{2.10}$$

where $\hat{H}_R$ is given by

$$\hat{H}_R = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \tag{2.11}$$

The expression (2.11) will be proven later in later section.

By substituting (2.11) into (2.10), the left-hand side (L.H.S) of (2.10) becomes

$$\hat{H}_R|n\rangle = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|n\rangle.$$

Since $\hat{a}^\dagger\hat{a} = \hat{n}\ and\ \hat{n}|n\rangle = n|n\rangle$,

$$\hat{H}_R|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle.$$

By comparing with the right-hand side (R.H.S) of (2.10),

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right). \qquad \text{(Proven)}$$

If a single mode electromagnetic field of angular frequency ω is considered, from Statistical Mechanics, the probability $P(n)$ that n photons are thermally excited at angular frequency ω is given by the Boltzmann's Law,

$$P(n) = \frac{exp\left(-\frac{E_n}{k_BT}\right)}{\sum_{n=0}^{\infty} exp\left(-\frac{E_n}{k_BT}\right)}. \tag{2.12}$$

By substituting (2.9) into (2.12), (2.12) becomes

$$P(n) = \frac{exp\left(-\frac{\left(n+\frac{1}{2}\right)\hbar\omega}{k_B T}\right)}{\sum_{n=0}^{\infty} exp\left(-\frac{\left(n+\frac{1}{2}\right)\hbar\omega}{k_B T}\right)}$$

$$= \frac{exp\left(-\frac{n\hbar\omega}{k_B T}\right)}{\sum_{n=0}^{\infty} exp\left(-\frac{n\hbar\omega}{k_B T}\right)}$$

$$= \frac{\left[exp\left(-\frac{\hbar\omega}{k_B T}\right)\right]^n}{\sum_{n=0}^{\infty}\left[exp\left(-\frac{\hbar\omega}{k_B T}\right)\right]^n} . \tag{2.13}$$

The denominator in (2.13) can be expanded using geometrical series as

$$\sum_{n=0}^{\infty}\left[exp\left(-\frac{\hbar\omega}{k_B T}\right)\right]^n = 1 + exp\left(-\frac{\hbar\omega}{k_B T}\right) + \left[exp\left(-\frac{\hbar\omega}{k_B T}\right)\right]^2 + \ \dots\dots$$

$$= \frac{1}{1 - exp\left(-\frac{\hbar\omega}{k_B T}\right)} , \tag{2.14}$$

where we have made use of the identity $\sum\limits_{n=0}^{\infty} x^n = \frac{1}{1-x}$, $|x| < 1$.

By substituting (2.14) into (2.13), (2.13) becomes

$$P(n) = \frac{\left[exp\left(-\frac{\hbar\omega}{k_B T}\right)\right]^n}{\left[1 - exp\left(-\frac{\hbar\omega}{k_B T}\right)\right]^{-1}}$$

$$= \left[1 - exp\left(-\frac{\hbar\omega}{k_B T}\right)\right]\left[exp\left(-\frac{\hbar\omega}{k_B T}\right)\right]^n$$

$$= \left[1 - exp\left(-\frac{\hbar\omega}{k_B T}\right)\right] exp\left(-\frac{n\hbar\omega}{k_B T}\right). \tag{2.15}$$

To simplify (2.15) further, if $\bar{n}$ is the mean thermal photon number,

$$\bar{n} = \sum_{n=0}^{\infty} n[P(n)].$$

Let $a = exp\left(-\frac{\hbar\omega}{k_B T}\right)$ and by substituting (2.15) into it,

$$\bar{n} = \textstyle\sum_{n=0}^{\infty} n[(1-a)a^n]$$

$$= (1-a)\sum_{n=0}^{\infty} n[a^n]$$

$$= (1-a)\sum_{n=0}^{\infty} a\frac{d[a^n]}{da}$$

$$= a(1-a)\frac{d}{da}\sum_{n=0}^{\infty} a^n.$$

By using $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$,

$$\bar{n} = a(1-a)\frac{d}{da}\left(\frac{1}{1-a}\right),$$

$$\Rightarrow \bar{n} = \frac{a}{1-a}.$$

So, $a = exp\left(-\frac{\hbar\omega}{k_B T}\right) = \frac{\bar{n}}{\bar{n}+1}$. (2.16)

By substituting (2.16) into (2.15), (2.15) becomes

$$P(n) = \left[1 - \frac{\bar{n}}{\bar{n}+1}\right]\left(\frac{\bar{n}}{\bar{n}+1}\right)^n$$

$$= \frac{1}{\bar{n}+1}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n \quad . \tag{2.17}$$

Equation (2.7) is the probability distribution function for the thermal state and it is also a Bose-Einstein distribution. For the thermal photon number probability distribution, the variance is greater than the mean photon number $\bar{n}$.

2.3 Quantization of the Electromagnetic Field

Most of the following derivation is completed by referring to Schleich (2001). The free radiation field will be quantized under the absence of charges and currents in vacuum. Starting from the Maxwell's Equations, we have

$$\vec{\nabla} \cdot \vec{D} = \rho, \tag{2.18}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{2.19}$$

$$\vec{\nabla} \cdot \vec{B} = 0\,, \tag{2.20}$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}\,, \tag{2.21}$$

At the same time, we also have

$$\vec{D} = \varepsilon_0 \vec{E}, \tag{2.22}$$

$$\vec{B} = \mu_0 \vec{H}, \tag{2.23}$$

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}, \tag{2.24}$$

where $\vec{E}$ and $\vec{D}$ are the electric field and displacement current respectively, $\vec{H}$ and $\vec{B}$ are the magnetic field and flux density respectively, t is time, c is the speed of light in vacuum, ε_0 and μ_0 are the electric permittivity and magnetic permeability in vacuum, respectively. Then,

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}\ and \tag{2.25}$$

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{2.26}$$

where Φ is the scalar potential and $\vec{A}$ is the vector potential.

It can be shown that both definitions (2.25) & (2.26) satisfy Maxwell's Equations (2.19) & (2.20).

By making use of (2.25) and (2.26),

$$\vec{\nabla} \times \vec{E} = \vec{\nabla} \times \left(-\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t} \right)$$

$$= -\vec{\nabla} \times \vec{\nabla}\Phi - \frac{\partial\left(\vec{\nabla} \times \vec{A}\right)}{\partial t}$$

$$= -\frac{\partial \vec{B}}{\partial t},$$

where we have made use of $\vec{\nabla} \times \vec{\nabla}\mathrm{F} = 0$ and F is any scalar field.

Similarly,

$$\vec{\nabla} \cdot \vec{B} = \vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{A}\right)$$

$$= 0$$

where we have made use of $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} = 0$ and $\vec{F}$ is any vector field.

Now, we would like to solve the Maxwell's Equations.

From (2.21), we have

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J}$$

$$\vec{\nabla} \times \frac{\vec{B}}{\mu_0} = \frac{\partial\left(\varepsilon_0 \vec{E}\right)}{\partial t} + \vec{J}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}. \tag{2.27}$$

By substituting (2.25) & (2.26) into (2.27), (2.27) becomes

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \frac{1}{c^2} \frac{\partial\left(-\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}\right)}{\partial t} + \mu_0 \vec{J}$$

$$= \frac{1}{c^2}\left[-\frac{\partial\left(\vec{\nabla}\Phi\right)}{\partial t} - \frac{\partial^2 \vec{A}}{\partial t^2}\right] + \mu_0 \vec{J}. \tag{2.28}$$

By substituting $\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \vec{\nabla}^2 \vec{A}$ into (2.28),

$$\vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \vec{\nabla}^2 \vec{A} = \frac{1}{c^2}\left[-\frac{\partial\left(\vec{\nabla}\Phi\right)}{\partial t} - \frac{\partial^2 \vec{A}}{\partial t^2}\right] + \mu_0 \vec{J}$$

$$\Rightarrow \vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) + \frac{1}{c^2}\vec{\nabla}\left(\frac{\partial\Phi}{\partial t}\right) - \mu_0\vec{J}$$

$$\Rightarrow \vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \vec{\nabla}\left[(\vec{\nabla}\cdot\vec{A}) + \frac{1}{c^2}\left(\frac{\partial\Phi}{\partial t}\right)\right] - \mu_0\vec{J}\,. \tag{2.29}$$

Similarly, by substituting (2.22) and (2.25) into (2.18), we have

$$\vec{\nabla}\cdot\vec{D} = \rho$$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\Rightarrow \vec{\nabla}\cdot\left(-\vec{\nabla}\Phi - \frac{\partial\vec{A}}{\partial t}\right) = \frac{\rho}{\varepsilon_0}$$

$$\Rightarrow -\vec{\nabla}\cdot\vec{\nabla}\Phi - \frac{\partial(\vec{\nabla}\cdot\vec{A})}{\partial t} = \frac{\rho}{\varepsilon_0}. \tag{2.30}$$

So far, the Maxwell's Equation have been reduced to the following two equations,

$$\vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \vec{\nabla}\left[(\vec{\nabla}\cdot\vec{A}) + \frac{1}{c^2}\left(\frac{\partial\Phi}{\partial t}\right)\right] - \mu_0\vec{J} \ and \tag{2.29}$$

$$-\vec{\nabla}\cdot\vec{\nabla}\Phi - \frac{\partial(\vec{\nabla}\cdot\vec{A})}{\partial t} = \frac{\rho}{\varepsilon_0}\,. \tag{2.30}$$

In Quantum Optics, the Lorentz Gauge and the Coulomb Gauge are used very frequently. So, in order to solve (2.29) and (2.30), the Coulomb Gauge has been chosen. The Coulomb Gauge is defined by the constraint

$$\vec{\nabla}\cdot\vec{A} = 0. \tag{2.31}$$

By substituting (2.31) into (2.29) and (2.30), the two equations can be further reduced to

$$\vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \vec{\nabla}\left[\frac{1}{c^2}\left(\frac{\partial\Phi}{\partial t}\right)\right] - \mu_0\vec{J} \text{ and} \tag{2.32}$$

$$-\vec{\nabla}\cdot\vec{\nabla}\Phi = \frac{\rho}{\varepsilon_0}\,. \tag{2.33}$$

Equation (2.33) is also known as the Poisson Equation.

By taking into consideration that the charges and currents are absent, i.e., $\rho = 0\ and\ \vec{J} = 0$, (2.33) becomes

$$\vec{\nabla} \cdot \vec{\nabla}\Phi = 0. \tag{2.34}$$

In fact, there are many possible solutions to this Laplace's Equation. Since we have the freedom to choose the scalar potential Φ to simplify our calculation, we can choose $\Phi = 0$ to be our solution. The following is the explanation of why we are free to choose the vector potential $\vec{A}$ and scalar potential Φ.

It is defined earlier that $\vec{B} = \vec{\nabla} \times \vec{A}$. Now, if there is a transformed vector potential $\overrightarrow{A'} = \vec{A} + \vec{\nabla}\Lambda$, in which the difference is given by $\vec{\nabla}\Lambda$. Then,

$$\overrightarrow{B'} = \vec{\nabla} \times \overrightarrow{A'}$$

$$= \vec{\nabla} \times (\vec{A} + \vec{\nabla}\Lambda)$$

$$= \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{\nabla}\Lambda$$

$$= \vec{\nabla} \times \vec{A}$$

$$= \vec{B},$$

where we have made use of $\vec{\nabla} \times \vec{\nabla}\mathrm{F} = 0$ and F is any scalar field.

It is also defined earlier that $\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$. Now, if there is a transformed vector potential $\Phi' = \Phi - \frac{\partial \Lambda}{\partial t}$, in which the difference is given by $(-\frac{\partial \Lambda}{\partial t})$. Then,

$$\vec{E}' = -\vec{\nabla}\Phi' - \frac{\partial \vec{A}'}{\partial t}$$

$$= -\vec{\nabla}\left(\Phi - \frac{\partial \Lambda}{\partial t}\right) - \frac{\partial\left(\vec{A} + \vec{\nabla}\Lambda\right)}{\partial t}$$

$$= -\vec{\nabla}\Phi + \frac{\partial\left(\vec{\nabla}\Lambda\right)}{\partial t} - \frac{\partial \vec{A}}{\partial t} - \frac{\partial\left(\vec{\nabla}\Lambda\right)}{\partial t}$$

$$= -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$$

$$= \vec{E}.$$

These show that we are free to choose the vector potential $\vec{A}$ and scalar potential Φ and they will all give the same electric field $\vec{E}$ and magnetic flux density $\vec{B}$. This explains why we can choose $\Phi = 0$ as the solution of (2.34). A great advantage of choosing $\Phi = 0$ is that it can simplify further the differential equation (2.32). So, by substituting $\Phi = 0$ and $\vec{J} = 0$ into (2.32), the latter becomes

$$\vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = \vec{\nabla}\left[\frac{1}{c^2}\left(\frac{\partial\Phi}{\partial t}\right)\right] - \mu_0\vec{J}$$

$$\Rightarrow \vec{\nabla}^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = 0. \qquad (2.35)$$

Equation (2.35) could be solved by using the method of separable variables.

Let $\vec{A}(\vec{r},t) = \alpha q(t)\vec{v}(\vec{r})$. (2.36)

By substituting (2.36) into (2.35),

$$\vec{\nabla}^2[\alpha q(t)\vec{v}(\vec{r})] - \frac{1}{c^2}\frac{\partial^2[\alpha q(t)\vec{v}(\vec{r})]}{\partial t^2} = 0$$

$$\Rightarrow q(t)\vec{\nabla}^2[\vec{v}(\vec{r})] - \frac{1}{c^2}\alpha\vec{v}(\vec{r})\ddot{q}(t) = 0.$$

By decomposing it into x, y and z components, we have

$$\begin{cases} q(t)\vec{\nabla}^2[v_x(\vec{r})] = \frac{1}{c^2}\alpha v_x(\vec{r})\ddot{q}(t) \\ q(t)\vec{\nabla}^2[v_y(\vec{r})] = \frac{1}{c^2}\alpha v_y(\vec{r})\ddot{q}(t) \\ q(t)\vec{\nabla}^2[v_z(\vec{r})] = \frac{1}{c^2}\alpha v_z(\vec{r})\ddot{q}(t) \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\vec{\nabla}^2[v_x(\vec{r})]}{v_x(\vec{r})} = \frac{1}{c^2}\frac{\ddot{q}(t)}{q(t)}. & (2.37) \\ \frac{\vec{\nabla}^2[v_y(\vec{r})]}{v_y(\vec{r})} = \frac{1}{c^2}\frac{\ddot{q}(t)}{q(t)}. & (2.38) \\ \frac{\vec{\nabla}^2[v_z(\vec{r})]}{v_z(\vec{r})} = \frac{1}{c^2}\frac{\ddot{q}(t)}{q(t)}. & (2.39) \end{cases}$$

Since the left-hand side and right-hand side of (2.37) - (2.39) depend only on position $\vec{r}$ and time t respectively, both sides are constant. This constant will be equal to $-k^2$, where k is the wave number. Then, equations (2.37) - (2.39) become

$$\begin{cases} \dfrac{\vec{\nabla}^2[v_x(\vec{r})]}{v_x(\vec{r})} = \dfrac{1}{c^2}\dfrac{\ddot{q}(t)}{q(t)} = -k^2. \\ \dfrac{\vec{\nabla}^2\left[v_y(\vec{r})\right]}{v_y(\vec{r})} = \dfrac{1}{c^2}\dfrac{\ddot{q}(t)}{q(t)} = -k^2. \\ \dfrac{\vec{\nabla}^2[v_z(\vec{r})]}{v_z(\vec{r})} = \dfrac{1}{c^2}\dfrac{\ddot{q}(t)}{q(t)} = -k^2. \end{cases}$$

Then, we have

$$\begin{cases} \vec{\nabla}^2[\vec{v}(\vec{r})] + k^2\vec{v}(\vec{r}) = 0\,. & (2.40) \\ \ddot{q}(t) + c^2k^2q(t) = 0\,. & (2.41) \end{cases}$$

Since $\Omega = ck,$ where Ω is the frequency of the electromagnetic field, (2.41) becomes

$$\ddot{q}(t) + \Omega^2 q(t) = 0. \tag{2.42}$$

Now, equation (2.40) will be solved for the case of a box-shaped resonator as shown in Figure 2.3 below:

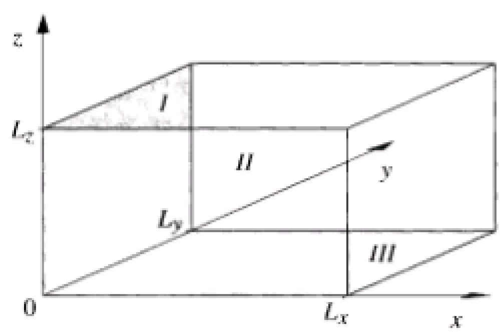


Figure 2.3: Box-shaped resonator.

Boundary Conditions on the Electric and Magnetic Field

The boundary conditions for equation (2.40) could be determined as follows:

Since $\Phi = 0$,

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$$

$$= -\frac{\partial \vec{A}}{\partial t} \ . \tag{2.43}$$

Additionally, the tangential component (parallel to box surfaces) of $\vec{E}$ and the normal component (perpendicular to box surfaces) of $\vec{B}$ should vanish. If we let $\vec{e}_p(\vec{r})$ to be a unit vector parallel to the boundary surface at position $\vec{r}$ on the boundary, then we have

$$\vec{e}_p(\vec{r}) \cdot \vec{E}(\vec{r}, t)|_{boundary} = 0$$

Substitution of (2.36) and (2.43) into it yields

$$\vec{e}_p(\vec{r}) \cdot \left(-\frac{\partial \vec{A}}{\partial t}\right)|_{boundary} = 0$$

$$\Rightarrow \vec{e}_p(\vec{r}) \cdot \frac{\partial\big(\alpha q(t)\vec{v}(\vec{r})\big)}{\partial t}|_{boundary} = 0$$

$$\Rightarrow \vec{e}_p(\vec{r}) \cdot \alpha \dot{q}(t)\vec{v}(\vec{r})|_{boundary} = 0$$

$$\Rightarrow \vec{e}_p(\vec{r}) \cdot \vec{v}(\vec{r})|_{boundary} = 0. \tag{2.44}$$

Let $\vec{e}_n(\vec{r})$ be a unit vector perpendicular to the boundary surface at position $\vec{r}$ on the boundary. Then,

$$\vec{e}_n(\vec{r}) \cdot \vec{B}(\vec{r}, t)|_{boundary} = 0.$$

By substituting (2.26) and (2.36) into it,

$$\vec{e}_n(\vec{r}) \cdot (\vec{\nabla} \times \vec{A})|_{boundary} = 0$$

$$\Rightarrow \vec{e}_n(\vec{r}) \cdot (\vec{\nabla} \times \alpha q(t)\vec{v}(\vec{r}))|_{boundary} = 0$$

$$\Rightarrow \vec{e}_n(\vec{r}) \cdot \left(\vec{\nabla} \times \vec{v}(\vec{r})\right)|_{boundary} = 0. \tag{2.45}$$

From (2.44), the boundary conditions for side I of Figure 1 are

$$v_y(x=0,y,z) = v_z(x=0,y,z) = 0. \tag{2.46}$$

For side II, the boundary conditions are $v_x(x, y=0, z) = v_z(x, y=0, z) = 0.$ (2.47)

For side III, the boundary conditions are $v_x(x, y, z=0) = v_y(x, y, z=0) = 0.$ (2.48)

So, the solutions of Helmholtz equation (2.40) after applying the boundary conditions are

$$v_x(x,y,z) = Nn_x cos(k_x x) sin(k_y y) sin(k_z z), \tag{2.49}$$

$$v_y(x,y,z) = Nn_y sin(k_x x) cos(k_y y) sin(k_z z), \tag{2.50}$$

$$v_z(x,y,z) = Nn_z sin(k_x x) sin(k_y y) cos(k_z z), \tag{2.51}$$

where N is the normalization factor and $\vec{e} = (n_x, n_y, n_z)$ is a unit vector.

Under Coulomb Gauge condition, $\vec{e}$ is the polarization vector which is orthogonal to the wave propagation direction. The orthogonality property is shown as follows:

Under Coulomb Gauge condition,

$$\vec{\nabla} \cdot \vec{A} = 0$$

$$\Rightarrow \vec{\nabla} \cdot [\alpha q(t) \vec{v}(\vec{r})] = 0$$

$$\Rightarrow \vec{\nabla} \cdot \vec{v}(\vec{r}) = 0 \tag{2.52}$$

$$\Rightarrow \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0$$

$$\Rightarrow -(n_x k_x + n_y k_y + n_z k_z) sin(k_x x) sin(k_y y) sin(k_z z) = 0. \tag{2.53}$$

Case 1: If $k_x \neq 0$ and $k_y \neq 0$ and $k_z \neq 0$, then

$$n_x k_x + n_y k_y + n_z k_z = 0$$

$$\Rightarrow \vec{e} \cdot \vec{k} = 0 \tag{2.54}$$

at any position $\vec{r}$.

This shows that the polarization vector $\vec{e}$ is orthogonal to the wave vector $\vec{k}$ (propagation direction). This also depicts that Coulomb Gauge condition reflects the transversality of the wave.

For a certain propagation direction, there are two linearly independent orthogonal directions. Therefore, each wave vector $\vec{k}$ will correspond to two linearly independent polarization vectors $\vec{e}_1$ and $\vec{e}_2$.

Case 2: If $k_x = 0 \; or \; k_y = 0 \; or \; k_z = 0$ (any one component of $\vec{k}$ is zero), then (2.53) is satisfied and this means Coulomb Gauge condition is satisfied automatically. In this case, there will be only one polarization vector $\vec{e}$.

In conclusion, there are two polarization directions generally. There will be only one polarization direction when one of the wave numbers becomes zero.

Next, the discrete values of the mode numbers k_x, k_y and k_z could be determined from another set of boundary conditions in Figure 1. Let I', II' and III' be the surfaces directly opposite of I, II and III, respectively, and the following conditions are fulfilled for these surfaces:

Surface I': $v_y(x = L_x, y, z) = v_z(x = L_x, y, z) = 0.$ (2.55)

Surface II': $v_x\left(x, y = L_y, z\right) = v_z\left(x, y = L_y, z\right) = 0.$ (2.56)

Surface III': $v_x(x, y, z = L_z) = v_y(x, y, z = L_z) = 0.$ (2.57)

By substituting (2.55) into (2.50), (2.50) becomes

$$0 = Nn_y sin(k_x L_x) cos\left(k_y y\right) sin(k_z z)$$

$$sin(k_x L_x) = 0$$

$$k_x L_x = l_x \pi$$

$$k_x = \frac{l_x \pi}{L_x}. \tag{2.58}$$

By substituting (2.56) into (2.51), (2.51) becomes

$$0 = Nn_z sin(k_x x) sin\left(k_y L_y\right) cos(k_z z)$$

$$sin\left(k_y L_y\right) = 0$$

$$k_y L_y = l_y \pi$$

$$k_y = \frac{l_y \pi}{L_y}. \tag{2.59}$$

By substituting (2.57) into (2.49), (2.49) becomes

$$0 = Nn_x cos(k_x x) sin(k_y y) sin(k_z L_z)$$

$$sin(k_z L_z) = 0$$

$$k_z L_z = l_z \pi$$

$$k_z = \frac{l_z \pi}{L_z} \ , \tag{2.60}$$

where l_x, l_y and l_z are integers.

Boundary Conditions on the Magnetic Field

Since the components of the magnetic flux density $\vec{B}$ normal to the surfaces of the rectangular box vanish, from (2.45) derived earlier, we have

$$\vec{e}_n(\vec{r}) \cdot \left(\vec{\nabla} \times \vec{v}(\vec{r})\right) |_{boundary} = 0, \tag{2.45}$$

$$\Rightarrow \left(\vec{\nabla} \times \vec{v}(\vec{r})\right) = \begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x(\vec{r}) & v_y(\vec{r}) & v_z(\vec{r}) \end{matrix}$$

$$= \left[\frac{\partial v_z(\vec{r})}{\partial y} - \frac{\partial v_y(\vec{r})}{\partial z}\right]\hat{\imath} - \left[\frac{\partial v_z(\vec{r})}{\partial x} - \frac{\partial v_x(\vec{r})}{\partial z}\right]\hat{\jmath} + \left[\frac{\partial v_y(\vec{r})}{\partial x} - \frac{\partial v_x(\vec{r})}{\partial y}\right]\hat{k}.$$

So, at surfaces I and I' in which $v_y(x = 0, y, z) = v_z(x = 0, y, z) = v_y(x = L_x, y, z) = v_z(x = L_x, y, z) = 0$,

$$\left(\vec{\nabla} \times \vec{v}(\vec{r})\right)_x |_{x=0 \ or \ x=L_x} = \left[\frac{\partial v_z(\vec{r})}{\partial y} - \frac{\partial v_y(\vec{r})}{\partial z}\right]_{x=0 \ or \ x=L_x} = 0.$$

At surfaces II and II' in which $v_x(x, y = 0, z) = v_z(x, y = 0, z) = v_x\left(x, y = L_y, z\right) = v_z\left(x, y = L_y, z\right) = 0$,

$$\left(\vec{\nabla} \times \vec{v}(\vec{r})\right)_y |_{y=0 \ or \ y=L_y} = \left[\frac{\partial v_z(\vec{r})}{\partial x} - \frac{\partial v_x(\vec{r})}{\partial z}\right]_{y=0 \ or \ y=L_y} = 0.$$

At surfaces III and III' in which $v_x(x, y, z = 0) = v_y(x, y, z = 0) = v_x(x, y, z = L_z) = v_y(x, y, z = L_z) = 0$,

$$\left(\vec{\nabla} \times \vec{v}(\vec{r})\right)_z |_{z=0 \ or \ z=L_z} = \left[\frac{\partial v_y(\vec{r})}{\partial x} - \frac{\partial v_x(\vec{r})}{\partial y}\right]_{z=0 \ or \ z=L_z} = 0.$$

Therefore, equation (2.45) is satisfied automatically.

Determination of Normalization Constant N by Using Orthonormality of Mode Functions

From (2.54), it was explained earlier that under the Coulomb Gauge condition, each wave vector $\vec{k}$ corresponds to two polarization vectors $\vec{e}_1$ and $\vec{e}_2$. There will be only one polarization vector when one of the wave vector components becomes zero. From (2.58) until (2.60), it is also proven that the mode components k_x, k_y and k_z are integers multiples l_x, l_y and l_z of $\frac{\pi}{L_x}, \frac{\pi}{L_y}$ and $\frac{\pi}{L_z}$ respectively. Therefore, the polarization vector $\vec{e}_{l,1}$ and $\vec{e}_{l,2}$ is denoted as $\vec{e}_\ell$, where ℓ represents a set of four numbers, that is the polarization index (1 or 2) and a set of l_x, l_y and l_z. Similarly, the mode functions are also denoted as $\vec{v}_\ell(\vec{r})$. Now, two different mode functions are orthonormal, that is,

$$\int [\vec{v}_\ell(\vec{r}) \cdot \vec{v}_{\ell'}(\vec{r})]\, d^3r = \delta_{\ell,\ell'} = \begin{cases} 1, & if\ \ell = \ell' \\ 0, & if\ \ell \neq \ell' \end{cases} . \qquad (2.61)$$

Next, consider two different mode functions of the same polarization index but with at least 1 different l_x, l_y and l_z. Then, by evaluating the integral I over the entire volume of the rectangular box:

$$I = \int [\vec{v}_l(\vec{r}) \cdot \vec{v}_{l'}(\vec{r})]\, d^3r$$

$$= \int \left\{[\vec{v}_l(\vec{r})]_x[\vec{v}_{l'}(\vec{r})]_x + [\vec{v}_l(\vec{r})]_y[\vec{v}_{l'}(\vec{r})]_y + [\vec{v}_l(\vec{r})]_z[\vec{v}_{l'}(\vec{r})]_z\right\} d^3r$$

$$= \int \Big\{\left[Nn_x cos(k_x x) sin\left(k_y y\right) sin(k_z z)\right]\left[N'n_x' cos(k_x' x) sin\left(k_y' y\right) sin(k_z' z)\right]$$
$$+ \left[Nn_y sin(k_x x) cos\left(k_y y\right) sin(k_z z)\right]\left[N'n_y' sin(k_x' x) cos\left(k_y' y\right) sin(k_z' z)\right]$$
$$+ \left[Nn_z sin(k_x x) sin\left(k_y y\right) cos(k_z z)\right]\left[N'n_z' sin(k_x' x) sin\left(k_y' y\right) cos(k_z' z)\right]\Big\}\, d^3r$$

$$= NN'\left\{n_x n_x{}' \int_0^{L_x} cos(k_x x)\, cos(k_x{}'x)dx \int_0^{L_y} sin(k_y y) sin(k_y{}'y)dy \int_0^{L_z} sin(k_z z)\, sin(k_z{}'z)dz\right.$$
$$+ n_y n_y{}' \int_0^{L_x} sin(k_x x)\, sin(k_x{}'x)dx \int_0^{L_y} cos(k_y y) cos(k_y{}'y)dy \int_0^{L_z} sin(k_z z)\, sin(k_z{}'z)dz$$
$$\left.+ n_z n_z{}' \int_0^{L_x} sin(k_x x)\, sin(k_x{}'x)dx \int_0^{L_y} sin(k_y y) sin(k_y{}'y)dy \int_0^{L_z} cos(k_z z)\, cos(k_z{}'z)dz\right\}$$

Case 1: If $l = l'$

$$I = N^2 \left\{n_x{}^2 \int_0^{L_x} cos^2(k_x x)\, dx \int_0^{L_y} sin^2(k_y y)dy \int_0^{L_z} sin^2(k_z z)\, dz\right.$$
$$+ n_y{}^2 \int_0^{L_x} sin^2(k_x x)\, dx \int_0^{L_y} cos^2(k_y y)dy \int_0^{L_z} sin^2(k_z z)\, dz$$
$$\left.+ n_z{}^2 \int_0^{L_x} sin^2(k_x x)\, dx \int_0^{L_y} sin^2(k_y y)dy \int_0^{L_z} cos^2(k_z z)\, dz\right\}. \quad (2.62)$$

Now, $\int_0^{L_x} cos^2(k_x x)\, dx$

$$= \int_0^{L_x} \frac{1 + cos(2k_x x)}{2} dx$$

$$= \frac{1}{2}\left[x + \frac{sin(2k_x x)}{2k_x}\right]_0^{L_x}$$

$$= \frac{1}{2}\left[x + \frac{sin\left(2\left(\frac{l_x \pi}{L_x}\right)x\right)}{2\left(\frac{l_x \pi}{L_x}\right)}\right]_0^{L_x}$$

$$= \frac{L_x}{2}$$

and

$$\int_0^{L_x} sin^2(k_x x)\, dx = \int_0^{L_x} \frac{1 - cos(2k_x x)}{2} dx$$

$$= \frac{1}{2}\left[x - \frac{sin(2k_x x)}{2k_x}\right]_0^{L_x}$$

$$= \frac{1}{2}\left[x - \frac{sin\left(2\left(\frac{l_x\pi}{L_x}\right)x\right)}{2\left(\frac{l_x\pi}{L_x}\right)}\right]_0^{L_x}$$

$$= \frac{L_x}{2}.$$

Similar results applied for integration with respect to y and z. Then, (2.62) is simplified to:

$$I = N^2\left\{{n_x}^2\left(\frac{L_x}{2}\right)\left(\frac{L_y}{2}\right)\left(\frac{L_z}{2}\right) + {n_y}^2\left(\frac{L_x}{2}\right)\left(\frac{L_y}{2}\right)\left(\frac{L_z}{2}\right) + {n_z}^2\left(\frac{L_x}{2}\right)\left(\frac{L_y}{2}\right)\left(\frac{L_z}{2}\right)\right\}$$

$$= N^2\left(\frac{L_xL_yL_z}{8}\right)\left({n_x}^2 + {n_y}^2 + {n_z}^2\right).$$

Since unit vector $\vec{e} = \left(n_x, n_y, n_z\right)$, ${n_x}^2 + {n_y}^2 + {n_z}^2 = 1$. Therefore,

$$I = N^2\left(\frac{L_xL_yL_z}{8}\right)$$

$$= N^2\left(\frac{V}{8}\right)$$

where V is the volume of the rectangular box.

Case 2: If $l \neq l'$

$$\int_0^{L_x} cos(k_x x)\, cos(k_x{}'x)dx = \int_0^{L_x} \frac{cos(k_x x + k_x{}'x) + cos(k_x x - k_x{}'x)}{2} dx$$

$$= \frac{1}{2}\left[\frac{sin(k_x x + k_x{}'x)}{(k_x + k_x{}')} + \frac{sin(k_x x - k_x{}'x)}{(k_x - k_x{}')}\right]_0^{L_x}$$

$$= 0$$

and

$$\int_0^{L_x} sin(k_x x)\, sin(k_x{}'x)dx = \int_0^{L_x} \frac{cos(k_x x - k_x{}'x) - cos(k_x x + k_x{}'x)}{2} dx$$

$$= \frac{1}{2}\left[\frac{sin(k_x x - k_x{}'x)}{(k_x - k_x{}')} - \frac{sin(k_x x + k_x{}'x)}{(k_x + k_x{}')}\right]_0^{L_x}$$

$$= 0.$$

Similar results applied for integration with respect to y and z. Then, (2.62) is simplified to

$$I = 0\ .$$

Therefore, by combining case 1 and case 2,

$$I = N^2 \left(\frac{V}{8}\right) \delta_{l,l'}\ . \tag{2.63}$$

By comparing (2.63) with (2.61), we have

$$N^2 \left(\frac{V}{8}\right) = 1$$

$$N = \sqrt{\frac{8}{V}}\ . \tag{2.64}$$

By defining effective mode volume $V_\ell = \frac{V}{8}$,

$$N = \sqrt{\frac{1}{V_\ell}}\ . \tag{2.65}$$

The Energy of the Electromagnetic Field

The energy of the electromagnetic field can be calculated by using

$$H_R = \int \left[\frac{1}{2}\varepsilon_0 \vec{E}^2(\vec{r},t) + \frac{1}{2}\mu_0 \vec{H}^2(\vec{r},t)\right] d^3r\ . \tag{2.66}$$

To solve (2.66), the electric field $\vec{E}$ and the magnetic field $\vec{H}$ must first be determined. In order to separate the effective mode volume V_ℓ from the mode function, it is defined that

$$\vec{v}_\ell(\vec{r}) = \sqrt{\frac{1}{V_\ell}}\vec{u}_\ell(\vec{r})\ . \tag{2.67}$$

Then, the orthonormality condition (2.61) becomes

$$\int [\vec{v}_\ell(\vec{r}) \cdot \vec{v}_{\ell'}(\vec{r})]\, d^3r = \delta_{\ell,\ell'}$$

$$\Rightarrow \int \left[\sqrt{\frac{1}{V_\ell}} \vec{u}_\ell(\vec{r}) \cdot \sqrt{\frac{1}{V_\ell}} \vec{u}_{\ell'}(\vec{r}) \right] d^3r = \delta_{\ell,\ell'}$$

$$\Rightarrow \frac{1}{\sqrt{V_\ell V_{\ell'}}} \int [\vec{u}_\ell(\vec{r}) \cdot \vec{u}_{\ell'}(\vec{r})]\, d^3r = \delta_{\ell,\ell'}$$

$$\Rightarrow \int [\vec{u}_\ell(\vec{r}) \cdot \vec{u}_{\ell'}(\vec{r})]\, d^3r = \sqrt{V_\ell V_{\ell'}} \delta_{\ell,\ell'} \,. \tag{2.68}$$

By using the definition in (2.67), (2.36) becomes

$$\vec{A}(\vec{r},t) = \alpha q(t) \vec{v}(\vec{r})$$

$$= \frac{1}{\sqrt{V_\ell \varepsilon_0}} q_\ell(t) \vec{u}_\ell(\vec{r}), \tag{2.69}$$

where constant α is determined by the mode volume and the electric permittivity.

Equation (2.69) gives the form of vector potential $\vec{A}(\vec{r},t)$ of single mode. For the multimode case,

$$\vec{A}(\vec{r},t) = \sum_\ell \frac{1}{\sqrt{V_\ell \varepsilon_0}} q_\ell(t) \vec{u}_\ell(\vec{r}) \,. \tag{2.70}$$

Now, from (2.43),

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} \,. \tag{2.43}$$

By substituting (2.70) into (2.43), (2.43) becomes

$$\vec{E} = -\frac{\partial}{\partial t} \sum_\ell \frac{1}{\sqrt{V_\ell \varepsilon_0}} q_\ell(t) \vec{u}_\ell(\vec{r})$$

$$= -\sum_\ell \frac{1}{\sqrt{V_\ell \varepsilon_0}} \dot{q}_\ell(t) \vec{u}_\ell(\vec{r}) \,. \tag{2.71}$$

From (2.26),

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{2.26}$$

$$\Rightarrow \vec{H} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A} \,. \tag{2.72}$$

By substituting (2.70) into (2.72), (2.72) becomes

$$\vec{H} = \frac{1}{\mu_0}\vec{\nabla}\times\left[\sum_\ell \frac{1}{\sqrt{V_\ell\varepsilon_0}} q_\ell(t)\vec{u}_\ell(\vec{r})\right]$$
$$= \sum_\ell \frac{1}{\mu_0\sqrt{V_\ell\varepsilon_0}} q_\ell(t)\left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right). \tag{2.73}$$

By substituting (2.71) and (2.73) into (2.66), (2.66) becomes

$$H_R = \int\left[\frac{1}{2}\varepsilon_0\vec{E}^2(\vec{r},t) + \frac{1}{2}\mu_0\vec{H}^2(\vec{r},t)\right] d^3r$$
$$= \int\left\{\frac{1}{2}\varepsilon_0\left[\sum_\ell \frac{1}{\sqrt{V_\ell\varepsilon_0}}\dot{q}_\ell(t)\vec{u}_\ell(\vec{r})\right]\left[\sum_\ell \frac{1}{\sqrt{V_\ell\varepsilon_0}}\dot{q}_\ell(t)\vec{u}_\ell(\vec{r})\right] + \frac{1}{2}\mu_0\left[\sum_\ell \frac{1}{\mu_0\sqrt{V_\ell\varepsilon_0}} q_\ell(t)\left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right)\right]\left[\sum_\ell \frac{1}{\mu_0\sqrt{V_\ell\varepsilon_0}} q_\ell(t)\left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right)\right]\right\} d^3r$$
$$= \frac{1}{2}\sum_\ell\sum_{\ell'}\frac{\dot{q}_\ell(t)\dot{q}_{\ell'}(t)}{\sqrt{V_\ell V_{\ell'}}}\int \vec{u}_\ell(\vec{r})\vec{u}_{\ell'}(\vec{r})\, d^3r + \frac{c^2}{2}\sum_\ell\sum_{\ell'}\frac{q_\ell(t)q_{\ell'}(t)}{\sqrt{V_\ell V_{\ell'}}}\int\left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right)\cdot\left(\vec{\nabla}\times\vec{u}_{\ell'}(\vec{r})\right) d^3r, \tag{2.74}$$

where $c^2 = \dfrac{1}{\mu_0\varepsilon_0}$.

By using $\vec{\nabla}\cdot\left(\vec{f}\times\vec{g}\right) = \vec{g}\cdot\left(\vec{\nabla}\times\vec{f}\right) - \vec{f}\cdot\left(\vec{\nabla}\times\vec{g}\right)$ and let $\vec{g} = \vec{\nabla}\times\vec{u}_\ell(\vec{r})$ *and* $\vec{f} = \vec{u}_{\ell'}(\vec{r})$, then,

$$\vec{\nabla}\cdot\left[\vec{u}_{\ell'}(\vec{r})\times\left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right)\right] = \left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right)\cdot\left(\vec{\nabla}\times\vec{u}_{\ell'}(\vec{r})\right) - \vec{u}_{\ell'}(\vec{r})\cdot\left[\vec{\nabla}\times\left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right)\right]$$
$$\left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right)\cdot\left(\vec{\nabla}\times\vec{u}_{\ell'}(\vec{r})\right) = \vec{\nabla}\cdot\left[\vec{u}_{\ell'}(\vec{r})\times\left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right)\right] + \vec{u}_{\ell'}(\vec{r})\cdot\left[\vec{\nabla}\times\left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right)\right]. \tag{2.75}$$

Now, by substituting (2.75) into the second part of the integral in (2.74), we have

$$\int\left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right)\cdot\left(\vec{\nabla}\times\vec{u}_{\ell'}(\vec{r})\right) d^3r = \int\left\{\vec{\nabla}\cdot\left[\vec{u}_{\ell'}(\vec{r})\times\left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right)\right]\right\} d^3r + \int\left\{\vec{u}_{\ell'}(\vec{r})\cdot\left[\vec{\nabla}\times\left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right)\right]\right\} d^3r. \tag{2.76}$$

According to Gauss Theorem,

$$\int \vec{\nabla}\cdot\vec{F}d^3r = \int_{surface} \vec{F}\cdot d\vec{S} \text{ , where } \vec{F} \text{ is any vector field.} \tag{2.77}$$

So, by applying Gauss Theorem on (2.76), (2.76) becomes

$$\int \left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right)\cdot\left(\vec{\nabla}\times\vec{u}_{\ell'}(\vec{r})\right)d^3r = \int_{surface}\left[\vec{u}_{\ell'}(\vec{r})\times\left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right)\right]\cdot d\vec{S} + \int\left\{\vec{u}_{\ell'}(\vec{r})\cdot\left[\vec{\nabla}\times\left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right)\right]\right\}d^3r. \tag{2.78}$$

For the integral $\int_{surface}\left[\vec{u}_{\ell'}(\vec{r})\times\left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right)\right]\cdot d\vec{S}$, $d\vec{S}$ is a vector perpendicular to the rectangular box surface. Now, from (2.71), electric field $\vec{E}$ is proportional to $\vec{u}_{\ell'}(\vec{r})$ and from (2.73), magnetic field $\vec{H}$ is proportional to $\vec{\nabla}\times\vec{u}_\ell(\vec{r})$. Therefore, $\left[\vec{u}_{\ell'}(\vec{r})\times\left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right)\right]$ has the same direction as vector product of $\vec{E}$ and $\vec{H}$. As discussed earlier, at boundary surfaces, the electric field $\vec{E}$ is perpendicular to the surfaces while magnetic field $\vec{H}$ is parallel to the surfaces. Therefore, the vector product $\vec{E}\times\vec{H}$ has a direction parallel to the surfaces. This implies that the direction of $\left[\vec{u}_{\ell'}(\vec{r})\times\left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right)\right]$ is parallel to the boundary surfaces. Hence, $\left[\vec{u}_{\ell'}(\vec{r})\times\left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right)\right]$ will be perpendicular to $d\vec{S}$ and their dot product $\left[\vec{u}_{\ell'}(\vec{r})\times\left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right)\right]\cdot d\vec{S}$ will be equal to zero. The integral $\int_{surface}\left[\vec{u}_{\ell'}(\vec{r})\times\left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right)\right]\cdot d\vec{S}$ vanishes. Then, (2.78) becomes

$$\int \left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right)\cdot\left(\vec{\nabla}\times\vec{u}_{\ell'}(\vec{r})\right)d^3r = \int\left\{\vec{u}_{\ell'}(\vec{r})\cdot\left[\vec{\nabla}\times\left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right)\right]\right\}d^3r. \tag{2.79}$$

To solve (2.79), the identity $\vec{\nabla}\times\left(\vec{\nabla}\times\vec{F}\right) = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{F}\right) - \vec{\nabla}^2\vec{F}$, where $\vec{F}$ is any vector field, is used. So, (2.79) becomes

$$\int \left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right)\cdot\left(\vec{\nabla}\times\vec{u}_{\ell'}(\vec{r})\right)d^3r = \int\left\{\vec{u}_{\ell'}(\vec{r})\cdot\left[\vec{\nabla}\left(\vec{\nabla}\cdot\vec{u}_\ell(\vec{r})\right) - \vec{\nabla}^2\vec{u}_\ell(\vec{r})\right]\right\}d^3r. \tag{2.80}$$

From (2.52), due to Coulomb Gauge condition,

$$\vec{\nabla}\cdot\vec{v}_\ell(\vec{r}) = 0\,. \tag{2.52}$$

Since it was defined earlier that $\vec{v}_\ell(\vec{r}) = \sqrt{\frac{1}{V_\ell}}\vec{u}_\ell(\vec{r})$ and by substituting it into (2.52), then

$$\vec{\nabla} \cdot \vec{u}_\ell(\vec{r}) = 0. \tag{2.81}$$

From (2.40) as well,

$$\vec{\nabla}^2[\vec{v}_\ell(\vec{r})] + {k_\ell}^2\vec{v}_\ell(\vec{r}) = 0\ . \tag{2.40}$$

Then,

$$\vec{\nabla}^2[\vec{u}_\ell(\vec{r})] = -{k_\ell}^2\vec{u}_\ell(\vec{r})\ . \tag{2.82}$$

.By substituting (2.81) and (2.82) into (2.80), (2.80) becomes

$$\int \left(\vec{\nabla} \times \vec{u}_\ell(\vec{r})\right) \cdot \left(\vec{\nabla} \times \vec{u}_{\ell'}(\vec{r})\right) d^3r = \int \left\{\vec{u}_{\ell'}(\vec{r}) \cdot \left[{k_\ell}^2\vec{u}_\ell(\vec{r})\right]\right\} d^3r\ .$$

By substituting (2.68) into it,

$$\int \left(\vec{\nabla} \times \vec{u}_\ell(\vec{r})\right) \cdot \left(\vec{\nabla} \times \vec{u}_{\ell'}(\vec{r})\right) d^3r = {k_\ell}^2\sqrt{V_\ell V_{\ell'}}\delta_{\ell,\ell'}\ . \tag{2.83}$$

So, by substituting (2.68) and (2.83) into (2.74), (2.74) becomes

$$H_R = \frac{1}{2}\sum_\ell \sum_{\ell'} \dot{q}_\ell(t)\dot{q}_{\ell'}(t)\delta_{\ell,\ell'} + \frac{c^2}{2}\sum_\ell \sum_{\ell'} q_\ell(t)q_{\ell'}(t){k_\ell}^2\delta_{\ell,\ell'}$$

$$= \frac{1}{2}\sum_\ell {\dot{q}_\ell}^2(t) + \frac{c^2}{2}\sum_\ell {q_\ell}^2(t){k_\ell}^2.$$

Since $k_\ell = \dfrac{\Omega_\ell}{c}$, where Ω_ℓ is the mode frequency,

$$H_R = \frac{1}{2}\sum_\ell \left[{\dot{q}_\ell}^2(t) + {q_\ell}^2(t){\Omega_\ell}^2\right]\ . \tag{2.84}$$

From (2.84), it can be seen that the energy of the electromagnetic field (classically) is the sum of the energy of unit mass harmonic oscillators of different modes with displacement$= q_\ell$ and momentum$= \dot{q}_\ell$. Hence, the concept of energy quantization of harmonic oscillators could be used to quantize the energy of field oscillators.

Now, consider an ℓth mode unit mass harmonic oscillator. The Hamiltonian H_ℓ is

$$H_\ell = \frac{1}{2}\dot{q}_\ell{}^2 + \frac{1}{2}q_\ell{}^2\Omega_\ell{}^2 \, . \tag{2.85}$$

Let p_ℓ be the momentum of the unit mass harmonic oscillator. Then,

$$p_\ell = \dot{q}_\ell \, .$$

From (2.71) and (2.73), it can be seen that the electric field is proportional to the $\dot{q}_\ell$ while magnetic field is proportional to q_ℓ. Hence, the electric field and magnetic field are analogous to the displacement q_ℓ and momentum $\dot{q}_\ell$ of a unit mass oscillator. Therefore, these field oscillators could be quantized in the same way as the unit mass harmonic oscillator conveniently by introducing the complex-valued amplitudes a_ℓ and a_ℓ^*. Then, the classical a_ℓ and a_ℓ^* will be converted into the quantum mechanical annihilation and creation operators $\hat{a}_\ell$ and $\hat{a}_\ell^\dagger$ of mode ℓ.

Now, the amplitudes a_ℓ and a_ℓ^* are defined as

$$a_\ell = \frac{1}{\sqrt{2\hbar\Omega_\ell}}(\Omega_\ell q_\ell + ip_\ell) \tag{2.86}$$

$$a_\ell^* = \frac{1}{\sqrt{2\hbar\Omega_\ell}}(\Omega_\ell q_\ell - ip_\ell) \, . \tag{2.87}$$

By adding (2.86) and (2.87), we have

$$q_\ell = \sqrt{\frac{\hbar}{2\Omega_\ell}}(a_\ell + a_\ell^*) \, . \tag{2.88}$$

By subtracting (2.87) from (2.86),

$$p_\ell = \dot{q}_\ell = \frac{1}{i}\sqrt{\frac{\hbar\Omega_\ell}{2}}(a_\ell - a_\ell^*) \, . \tag{2.89}$$

By substituting (2.88) and (2.89) into (2.85), (2.85) becomes

$$\begin{aligned} H_\ell &= \frac{1}{2}\left[\frac{1}{i}\sqrt{\frac{\hbar\Omega_\ell}{2}}(a_\ell - a_\ell^*)\right]^2 + \frac{1}{2}\left[\sqrt{\frac{\hbar}{2\Omega_\ell}}(a_\ell + a_\ell^*)\right]^2 \Omega_\ell{}^2 \\ &= -\frac{1}{2}\frac{\hbar\Omega_\ell}{2}\left(a_\ell^2 - a_\ell a_\ell^* - a_\ell^* a_\ell + a_\ell^{*2}\right) + \frac{1}{2}\frac{\hbar\Omega_\ell}{2}\left(a_\ell^2 + a_\ell a_\ell^* + a_\ell^* a_\ell + a_\ell^{*2}\right) \end{aligned}$$

$$= \frac{\hbar\Omega_\ell}{2}(a_\ell a_\ell^* + a_\ell^* a_\ell) \ . \tag{2.90}$$

In (2.90), the order of the complex amplitudes is remained in order to facilitate the process of conversion into the operators form.

<u>Quantization of the Oscillator</u>

The electromagnetic field is quantized by quantizing each mode of the unit mass harmonic oscillator. To achieve this, the following commutation relation is postulated:

$$[\hat{q}_\ell \, , \hat{p}_{\ell'}] = i\hbar\delta_{\ell,\ell'} \ . \tag{2.91}$$

Then, by using the correspondence principle to convert the classical forms into quantum mechanical operators, from (2.86), (2.87), (2.88) and (2.89),

$$\hat{a}_\ell = \frac{1}{\sqrt{2\hbar\Omega_\ell}}(\Omega_\ell\hat{q}_\ell + i\hat{p}_\ell) \, , \tag{2.92}$$

$$\hat{a}_\ell^\dagger = \frac{1}{\sqrt{2\hbar\Omega_\ell}}(\Omega_\ell\hat{q}_\ell - i\hat{p}_\ell) \, , \tag{2.93}$$

$$\hat{q}_\ell = \sqrt{\frac{\hbar}{2\Omega_\ell}}\left(\hat{a}_\ell + \hat{a}_\ell^\dagger\right) \ , \tag{2.94}$$

$$\hat{p}_\ell = \frac{1}{i}\sqrt{\frac{\hbar\Omega_\ell}{2}}\left(\hat{a}_\ell - \hat{a}_\ell^\dagger\right). \tag{2.95}$$

By substituting (2.94) and (2.95) into (2.91), (2.91) becomes

$$[\hat{q}_\ell \, , \hat{p}_{\ell'}] = i\hbar\delta_{\ell,\ell'} \tag{2.91}$$

$$\hat{q}_\ell\hat{p}_{\ell'} - \hat{p}_{\ell'}\hat{q}_\ell = i\hbar\delta_{\ell,\ell'}$$

$$\left[\sqrt{\frac{\hbar}{2\Omega_\ell}}\left(\hat{a}_\ell + \hat{a}_\ell^\dagger\right)\right]\left[\frac{1}{i}\sqrt{\frac{\hbar\Omega_{\ell'}}{2}}\left(\hat{a}_{\ell'} - \hat{a}_{\ell'}^\dagger\right)\right] - \left[\frac{1}{i}\sqrt{\frac{\hbar\Omega_{\ell'}}{2}}\left(\hat{a}_{\ell'} - \hat{a}_{\ell'}^\dagger\right)\right]\left[\sqrt{\frac{\hbar}{2\Omega_\ell}}\left(\hat{a}_\ell + \hat{a}_\ell^\dagger\right)\right]$$
$$= i\hbar\delta_{\ell,\ell'}$$

$$\frac{\hbar}{2i}\left[\left(\hat{a}_\ell\hat{a}_{\ell'} - \hat{a}_\ell\hat{a}_{\ell'}^\dagger + \hat{a}_\ell^\dagger\hat{a}_{\ell'} - \hat{a}_\ell^\dagger\hat{a}_{\ell'}^\dagger\right) - \left(\hat{a}_{\ell'}\hat{a}_\ell + \hat{a}_{\ell'}\hat{a}_\ell^\dagger - \hat{a}_{\ell'}^\dagger\hat{a}_\ell - \hat{a}_{\ell'}^\dagger\hat{a}_\ell^\dagger\right)\right] = i\hbar\delta_{\ell,\ell'} \ .$$

Since $[\hat{a}_\ell\,, \hat{a}_{\ell'}] = [\hat{a}_\ell^\dagger\,, \hat{a}_{\ell'}^\dagger] = 0$ or $(\hat{a}_\ell \hat{a}_{\ell'} - \hat{a}_{\ell'} \hat{a}_\ell) = (\hat{a}_\ell^\dagger \hat{a}_{\ell'}^\dagger - \hat{a}_{\ell'}^\dagger \hat{a}_\ell^\dagger) = 0$,

$$\frac{\hbar}{2i}\left[\left(-\hat{a}_\ell \hat{a}_{\ell'}^\dagger + \hat{a}_\ell^\dagger \hat{a}_{\ell'}\right) - \left(\hat{a}_{\ell'} \hat{a}_\ell^\dagger - \hat{a}_{\ell'}^\dagger \hat{a}_\ell\right)\right] = i\hbar\delta_{\ell,\ell'}$$

$$i\hbar\left[\hat{a}_\ell\,, \hat{a}_{\ell'}^\dagger\right] = i\hbar\delta_{\ell,\ell'}$$

$$\left[\hat{a}_\ell\,, \hat{a}_{\ell'}^\dagger\right] = \delta_{\ell,\ell'}\,. \tag{2.96}$$

By converting (2.90) into quantum mechanical operator form, (2.90) becomes

$$\hat{H}_\ell = \frac{\hbar\Omega_\ell}{2}\left(\hat{a}_\ell \hat{a}_\ell^\dagger + \hat{a}_\ell^\dagger \hat{a}_\ell\right). \tag{2.97}$$

So, to consider all the modes of electromagnetic field, the Hamiltonian $\hat{H}_R$ is

$$\hat{H}_R = \sum_\ell \frac{\hbar\Omega_\ell}{2}\left(\hat{a}_\ell \hat{a}_\ell^\dagger + \hat{a}_\ell^\dagger \hat{a}_\ell\right). \tag{2.98}$$

From (2.96),

$$\hat{a}_\ell \hat{a}_\ell^\dagger - \hat{a}_\ell^\dagger \hat{a}_\ell = 1$$

$$\hat{a}_\ell \hat{a}_\ell^\dagger = \hat{a}_\ell^\dagger \hat{a}_\ell + 1\,. \tag{2.99}$$

By substituting (2.99) into (2.98), (2.98) becomes

$$\hat{H}_R = \sum_\ell \hbar\Omega_\ell \left(\hat{a}_\ell^\dagger \hat{a}_\ell + \frac{1}{2}\right). \tag{2.100}$$

For the case of single mode electromagnetic field, the Hamiltonian $\hat{H}_R$ reduces to

$$\hat{H}_F = \hbar\omega\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right), \tag{2.101}$$

where $\Omega = \omega$ is the single mode frequency and $\hat{H}_F$ is the single-mode field Hamiltonian.

From (2.71), the electric field operator $\hat{\vec{E}}$ of multimode electromagnetic field can be evaluated as

$$\vec{E} = -\sum_\ell \frac{1}{\sqrt{V_\ell \varepsilon_0}}\,\dot{q}_\ell(t)\vec{u}_\ell(\vec{r}). \tag{2.71}$$

In operator form,

$$\hat{\vec{E}} = -\sum_{\ell} \frac{1}{\sqrt{V_\ell \varepsilon_0}} \hat{p}_\ell(t) \vec{u}_\ell(\vec{r}) \ .$$

By substituting (2.95) into it,

$$\hat{\vec{E}} = i \sum_{\ell} \vec{u}_\ell(\vec{r}) \sqrt{\frac{\hbar \Omega_\ell}{2 V_\ell \varepsilon_0}} \left(\hat{a}_\ell - \hat{a}_\ell^\dagger \right) \tag{2.102}$$

$$\hat{\vec{E}} = i \sum_{\ell} \vec{u}_\ell(\vec{r}) \sqrt{\frac{\hbar \Omega_\ell}{2 V_\ell \varepsilon_0}} \hat{a}_\ell - i \sum_{\ell} \vec{u}_\ell(\vec{r}) \sqrt{\frac{\hbar \Omega_\ell}{2 V_\ell \varepsilon_0}} \hat{a}_\ell^\dagger$$

$$\hat{\vec{E}} = \hat{\vec{E}}^{(+)} + \hat{\vec{E}}^{(-)},$$

where $\hat{\vec{E}}^{(+)} = i \sum_{\ell} \vec{u}_\ell(\vec{r}) \sqrt{\frac{\hbar \Omega_\ell}{2 V_\ell \varepsilon_0}} \hat{a}_\ell$ and $\hat{\vec{E}}^{(-)} = -i \sum_{\ell} \vec{u}_\ell(\vec{r}) \sqrt{\frac{\hbar \Omega_\ell}{2 V_\ell \varepsilon_0}} \hat{a}_\ell^\dagger$.

For single mode case, (2.102) reduces to

$$\hat{\vec{E}} = i \vec{u}(\vec{r}) \sqrt{\frac{\hbar \omega}{2 V \varepsilon_0}} \left(\hat{a} - \hat{a}^\dagger \right) , \tag{2.103}$$

where V is the effective mode volume .

From (2.73), the magnetic field operator $\hat{\vec{H}}$ of multimode electromagnetic field can be evaluated as

$$\vec{H} = \sum_{\ell} \frac{1}{\mu_0 \sqrt{V_\ell \varepsilon_0}} q_\ell(t) \left(\vec{\nabla} \times \vec{u}_\ell(\vec{r}) \right) . \tag{2.73}$$

In operator form,

$$\hat{\vec{H}} = \sum_{\ell} \frac{1}{\mu_0 \sqrt{V_\ell \varepsilon_0}} \hat{q}_\ell(t) \left(\vec{\nabla} \times \vec{u}_\ell(\vec{r}) \right) . \tag{2.104}$$

By substituting (2.94) into (2.104), (2.104) becomes

$$\hat{\vec{H}} = \sum_{\ell} \frac{1}{\mu_0 \sqrt{V_\ell \varepsilon_0}} \left(\vec{\nabla} \times \vec{u}_\ell(\vec{r}) \right) \sqrt{\frac{\hbar}{2 \Omega_\ell}} \left(\hat{a}_\ell + \hat{a}_\ell^\dagger \right)$$

$$\Rightarrow \ \widehat{\vec{H}} = \frac{1}{\mu_0}\sum_{\ell}\sqrt{\frac{\hbar}{2\Omega_\ell V_\ell \varepsilon_0}}\left(\vec{\nabla}\times\vec{u}_\ell(\vec{r})\right)\left(\hat{a}_\ell + \hat{a}_\ell^\dagger\right) . \qquad (2.105)$$

For single mode case, (2.105) reduces to

$$\widehat{\vec{H}} = \frac{1}{\mu_0}\sqrt{\frac{\hbar}{2\omega V\varepsilon_0}}\left(\vec{\nabla}\times\vec{u}(\vec{r})\right)(\hat{a} + \hat{a}^\dagger), \qquad (2.106)$$

where V is the effective mode volume.

2.4 Quantization of the Energy (Hamiltonian) of the Two-Level Atom

Let $\hat{H}_A$ be the Hamiltonian of the two-level atom. Then, according to Scully and Zubairy (1997), the Hamiltonian eigenvalue equation is

$$\hat{H}_A|i\rangle = E_i|i\rangle \ , \qquad (2.107)$$

where $|i\rangle$ represents the energy eigenstates of the atom, E_i is the energy eigenvalues of the atom.

From (2.107), by multiplying both sides with $\langle i|$, (2.107) becomes

$$\hat{H}_A|i\rangle\langle i| = E_i|i\rangle\langle i|.$$

Then, by taking summations on both sides,

$$\textstyle\sum_i \hat{H}_A|i\rangle\langle i| = \sum_i E_i|i\rangle\langle i|. \qquad (2.108)$$

Since the two-level atom has only two energy levels, there are two energy eigenvalues (E_1 and E_2) and eigenstates ($|1\rangle$ and $|2\rangle$). Then, (2.108) becomes

$$\hat{H}_A(|1\rangle\langle 1| + |2\rangle\langle 2|) = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2| \ . \qquad (2.109)$$

Since $|1\rangle$ and $|2\rangle$ form a complete set of energy eigenstates, $|1\rangle\langle 1| + |2\rangle\langle 2| = \hat{I}$. Then, (2.109) becomes

$$\hat{H}_A = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2| \, . \qquad (2.110)$$

Here $E_1 = \hbar\omega_1$ and $E_2 = \hbar\omega_2$, where ω_1 is the angular frequency associated with the ground energy level and $\omega_2$is the angular frequency associated with the excited energy level. Then, (2.110) becomes

$$\hat{H}_A = \hbar\omega_1|1\rangle\langle 1| + \hbar\omega_2|2\rangle\langle 2|$$

$$= \begin{pmatrix} \hbar\omega_2 & 0 \\ 0 & \hbar\omega_1 \end{pmatrix}. \tag{2.111}$$

By making $\hat{H}_A = a\hat{I} + b\hat{\sigma}_3$, (2.112)

$$\begin{pmatrix} \hbar\omega_2 & 0 \\ 0 & \hbar\omega_1 \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and}$$

therefore, $a + b = \hbar\omega_2$, $a - b = \hbar\omega_1$.

After solving the simultaneous equations,

$$a = \frac{\hbar}{2}(\omega_1 + \omega_2) \text{ and } b = \frac{\hbar}{2}(\omega_2 - \omega_1)\,. \tag{2.113}$$

Therefore, by substituting (2.113) into (2.112),

$$\hat{H}_A = \frac{\hbar}{2}(\omega_1 + \omega_2)\hat{I} + \frac{\hbar}{2}(\omega_2 - \omega_1)\hat{\sigma}_3\,. \tag{2.114}$$

Therefore, the Hamiltonian for two-level atom, $\hat{H}_A$ has been determined.

2.5 Jaynes-Cummings Model (JCM) and the micromaser

In Jaynes-Cummings Model, the interaction between a two-level atom and a single-mode quantized electromagnetic field is studied. The system of the two-level atom and photons are located inside an ideal cavity in which there is no energy dissipation. For single mode quantized electromagnetic field, examples are abundant. Some of the common ones for example, are coherent state, thermal state, number state etc. The coherent and thermal states have been explained in part 2.1 and 2.2 while a number state $|n\rangle$ represents a single mode quantized electromagnetic field containing n photons. One of the field states actually serves as the initial field state to interact with the two-level atom. When they interact, the electron will absorb the photon energy and make a transition to the higher energy level. Then, the atom is in the excited state. In other way, the electron may also make a transition from high to ground energy levels. Under this situation, the atom is in the ground state.

In single-photon Jaynes-Cummings Model, one photon will be absorbed or released by the two-level atom during its interaction with the single mode photons. Then, for two-photon Jaynes-Cummings Model, two photons will be absorbed or released by the atom. Then, the Jaynes-Cummings Model can then be generalized to k-photon transition case. For the single-photon JCM Hamiltonian $\hat{H}$, it involves the summation of Hamiltonian of the two-level atom $\hat{H}_A$, Hamiltonian of the quantized electromagnetic field $\hat{H}_F$ and the Hamiltonian due to the interaction between atom and photons $\hat{H}_{int}$, i.e., $\hat{H} = \hat{H}_A + \hat{H}_F + \hat{H}_{int}$.

The single-photon JCM Hamiltonian is used to derive the probability that the atom is in the ground state as a function of parameter λt, where λ is the interaction strength and t is the time. Interesting results can be seen with different initial field states. For instance, in single-photon JCM, when the atom is allowed to interact with a quantized field in coherent state, the energy of the atom will actually oscillate between the high and low energy values with time. This oscillation will collapse after a certain period of time. After the collapse of the oscillation, there is a stage in which there is no information about the atom's energy state. Then, the oscillation revives again and the same process is repeated. The oscillation mentioned is also known as the Rabi Oscillations.

The Jaynes-Cummings Model can be demonstrated experimentally through the interaction between a Rydberg atom (two-level atom) and the fundamental mode of a microwave cavity as shown in next page.

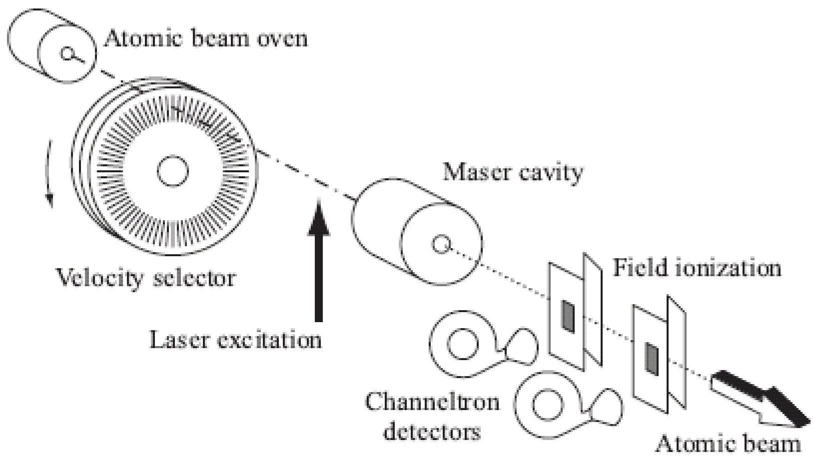


Figure 2.4: The micromaser

The configuration shown in Figure 2.4 above is called a micromaser. In a micromaser, the situation in which at most one atom is present in the cavity at any time can be created.

As shown in the figure above, Rydberg atoms (e.g. Rubidium atoms) produced from the atomic beam oven pass through the velocity selector. Then, the velocity-selected beam of atoms is sent into the laser excitation region. In the laser excitation region, the atoms are promoted to excited states. Due to the fact that the Rydberg atom has very long lifetime in the upper energy level, the spontaneous emission effect can be neglected. Afterwards, only a single Rydberg atom will enter and leave the superconducting microwave cavity through the small holes on the opposite sides of the cavity. During the time in which the atom is inside the cavity, the Rydberg atom can emit a single photon or reabsorbs a photon during its interaction with the photons which are already present into the cavity. This is actually a JCM-type interaction. In order to reduce the noise caused by the thermal photons emitted by the cavity itself, the temperature of the cavity is reduced to as low as 2.5K. This is to make the interaction environment as ideal as possible, as required in the JCM-type interaction.

When the Rydberg atom leaves the cavity, its energy state can be determined through the ease of its ionization by the DC electric field. In Figure 2.6, the electric field in the first ionization region is strong enough to ionize the Rydberg atom in its excited state, but it is still too weak to ionize the ground state Rydberg atom. Hence, the atom in excited state will be detected by the channeltron detector in the first ionization region. Conversely, if the leaving Rydberg atom is in its ground state, it will be ionized by the stronger electric field and detected in the second region.

2.6 Electric Dipole Approximation

In Jaynes-Cummings Model, the atom studied is a two-level atom with only one proton and one electron. When the latter interacts with a single mode electromagnetic field, the electric field component of the light wave does not change considerably over the size of the atom. This is due to the fact that the diameter d of the atom ($d \approx 0.1$nm) is very much smaller than the range of wavelength λ of interest (λ>100nm) (Garrison and Chiao, 2008). So, on the scale of optical wavelength, the electron only occupies a small region surrounding the proton. According to the derivations by Schleich (2001) using Taylor's expansions of vector potential, when $d \ll \lambda$, it implies that the electric field at the position of the electron is approximately equal to the electric field at the position of the proton. Therefore, the two-level atom with a proton and an electron can be modeled as a dipole while the electric field of the light wave across the atom can be well approximated as a uniform electric field. This is known as the Electric Dipole Approximation and it helps to simplify the problem of calculating the interaction Hamiltonian. With Electric Dipole Approximation, the interaction energy (Hamiltonian) between the atom (dipole) and the light (uniform electric field) can be derived as shown in next page.

Derivation of the Electric Dipole Energy in a Uniform Electric Field

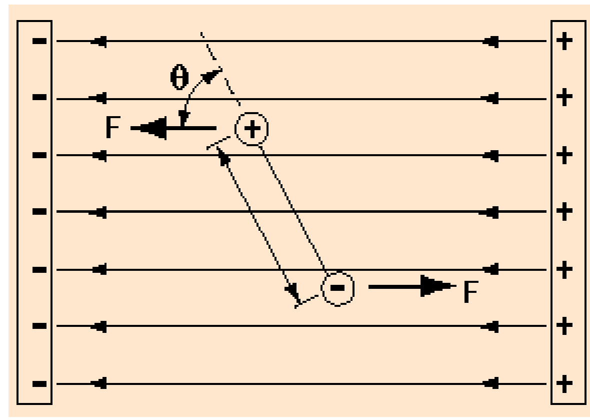


Figure 2.5: Electric dipole in a uniform electric field

Consider the torque acting on the dipole charges q caused by the uniform electric field $\vec{E}$ as shown in Figure 2.5 above. Then, the torque $\vec{\tau}$ is

$$\vec{\tau} = \vec{r}_{+} \times \vec{F}_{+} + \vec{r}_{-} \times \vec{F}_{-}$$

$$\vec{\tau} = \frac{\vec{d}}{2} \times q\vec{E} + \left(-\frac{\vec{d}}{2}\right) \times \left(-q\vec{E}\right),$$

where $\vec{\mathrm{d}}$ is the vector pointing from negative to the positve charge. Then,

$$\vec{\tau} = q\vec{d} \times \vec{E}.$$

Let $\vec{p} = q\vec{d}$, where $\vec{p}$ is the electric dipole moment. Then,

$$\vec{\tau} = \vec{p} \times \vec{E}$$

$$|\vec{\tau}| = \left|\vec{p} \times \vec{E}\right|$$

$$= pE|sin\theta\hat{e}|,$$

where $p = |\vec{p}|, E = \left|\vec{E}\right|, and\ \hat{e} =$ unit vector perpendicular to $\vec{p}$ and $\vec{E}$.

Then, $|\vec{\tau}| = pE|sin\theta|$.

Now, by choosing dipole energy U_{int}= 0 at $\theta = \frac{\pi}{2}$, the energy U_{int} of the dipole configuration in Figure 2.5 is

$$U_{int} = \int_{\frac{\pi}{2}}^{\theta} |\vec{\tau}| d\theta , \quad where\ |\vec{\tau}| = pE|sin\theta|$$

$$= \int_{\frac{\pi}{2}}^{\theta} pE|sin\theta| d\theta$$

$$= \int_{\frac{\pi}{2}}^{\theta} pE sin\theta d\theta \ .$$

$|sin\theta| = sin\theta$ because sinθ is positive for $0 \leq \theta \leq \frac{\pi}{2}$. When θ changes from $\frac{\pi}{2}$ to 0, the dipole will be in the low energy configuration, in which it becomes stable as illustrated below.

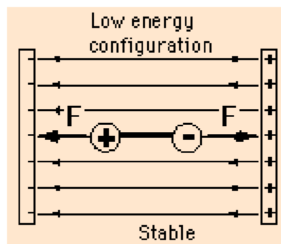


Figure 2.6: Low dipole energy configuration

Then,

$$U_{int} = [-pEcos\theta]_{\frac{\pi}{2}}^{\theta}$$

$$= -pEcos\theta$$

$$= -\vec{p} \cdot \vec{E} \ . \qquad (2.115)$$

Equation (2.115) gives the classical form of the Interaction Energy (Hamiltonian) U_{int} between the dipole and the uniform electric field $\vec{E}$. As explained in the part of Electric Dipole Approximation before, the Interaction Hamiltonian $\widehat{H}_{int}$ between light (electromagnetic field) and two-level atom can be approximated as the Interaction Hamiltonian U_{int} between a dipole and the uniform electric field $\vec{E}$. Then, by using the Correspondence Principle, the quantum mechanical operator form of (2.115) is

$$\widehat{H}_{int} = \widehat{U}_{int} = -\hat{\vec{p}} \cdot \hat{\vec{E}} \ , \qquad (2.116)$$

where $\hat{\vec{p}}$ is the electric dipole operator.

According to Barnett and Radmore (1997),

$$\hat{\vec{p}} = \vec{p}^{*}\hat{\sigma}_{+} + \vec{p}\hat{\sigma}_{-} \ .$$

Since $\vec{p}$ is a real vector, $\vec{p}^{*} = \vec{p}$.

$$\hat{\vec{p}} = \vec{p}(\hat{\sigma}_{+} + \hat{\sigma}_{-}) \, . \tag{2.117}$$

From equation (2.103) of the part of Quantization of Electromagnetic Field, the electric field operator $\hat{\vec{E}}$ for a single-mode electromagnetic field is

$$\hat{\vec{E}} = i\vec{u}(\vec{r})\sqrt{\frac{\hbar\omega}{2V\varepsilon_0}}(\hat{a} - \hat{a}^{\dagger}) \, , \tag{2.103}$$

where V is the effective mode volume .

By substituting (2.117) and (2.103) into (2.116), (2.116) becomes

$$\hat{H}_{int} = -\vec{p}(\hat{\sigma}_{+} + \hat{\sigma}_{-}) \cdot i\vec{u}(\vec{r})\sqrt{\frac{\hbar\omega}{2V\varepsilon_0}}(\hat{a} - \hat{a}^{\dagger})$$

$$= -\vec{p} \cdot \vec{u}(\vec{r})i\sqrt{\frac{\hbar\omega}{2V\varepsilon_0}}(\hat{\sigma}_{+} + \hat{\sigma}_{-})(\hat{a} - \hat{a}^{\dagger})$$

$$= -i\hbar\vec{p} \cdot \vec{u}(\vec{r})\sqrt{\frac{\omega}{2\hbar V\varepsilon_0}}(\hat{\sigma}_{+} + \hat{\sigma}_{-})(\hat{a} - \hat{a}^{\dagger}) \, . \tag{2.118}$$

Let $\lambda = \vec{p} \cdot \vec{u}(\vec{r})\sqrt{\frac{\omega}{2\hbar V\varepsilon_0}}$, then (2.118) becomes

$$\hat{H}_{int} = -i\hbar\lambda(\hat{\sigma}_{+} + \hat{\sigma}_{-})(\hat{a} - \hat{a}^{\dagger}) \, . \tag{2.119}$$

where λ is also known as the interaction strength.

2.7 Schrodinger, Heisenberg and Interaction Pictures

Schrodinger Picture

In Quantum Mechanics, observables are defined as any physical quantities which can be measured. The observables in Quantum Optics can be expressed in Schrodinger, Heisenberg or Interaction Pictures. For Schrodinger Picture, the observables are represented by time independent Hermitian Operators $\hat{X}^{(S)}$ (Garrison and Chiao, 2008). The quantum state is then described by a time dependent ket vector $|\Psi^{(\mathrm{S})}(\mathrm{t})\rangle$, which satisfies the Schrodinger equation

$$i\hbar\frac{\partial}{\partial t}|\Psi^{(\mathrm{S})}(\mathrm{t})\rangle = \hat{H}^{(S)}|\Psi^{(\mathrm{S})}(\mathrm{t})\rangle\,. \tag{2.120}$$

Heisenberg Picture

In Heisenberg original formulation of Quantum Mechanics, which is 1 year before Schrodinger's, there is no wave function or wave equation. The observables are represented by matrices that evolve in time according to a quantum version of Hamilton's equations of classical mechanics. This type of quantum theory is known as the Heisenberg Picture (Garrison and Chiao, 2008).

In contrast with Schrodinger Picture, the quantum states in Heisenberg Picture, $|\Psi^{(\mathrm{H})}\rangle$ are time independent while the observables are represented by time dependent Hermitian Operators, $\hat{X}^{(H)}(t)$. The mathematical relationship between Schrodinger and Heisenberg Picture quantum state is given by

$$|\Psi^{(\mathrm{S})}(\mathrm{t})\rangle = \hat{U}(t-t_0)|\Psi^{(\mathrm{H})}\rangle\,, \tag{2.121}$$

where $\hat{U}(t-t_0)$ is a Unitary Operator.

The two pictures quantum states coincide at time $t=t_0$. At $t=t_0$, $|\Psi^{(\mathrm{S})}(\mathrm{t})\rangle = |\Psi^{(\mathrm{H})}\rangle$. Therefore, $\hat{U}(0)=\hat{I}$. To solve $\hat{U}(t-t_0)$, (2.121) is substituted into (2.120),

$$i\hbar\frac{\partial}{\partial t}\left[\hat{U}(t-t_0)|\Psi^{(\mathrm{H})}\rangle\right] = \hat{H}^{(S)}\left[\hat{U}(t-t_0)|\Psi^{(\mathrm{H})}\rangle\right]$$

$$\Rightarrow i\hbar \int \frac{d\left[\hat{U}(t-t_0)\right]}{\hat{U}(t-t_0)} = \int \hat{H}^{(S)}\, dt$$

$$\Rightarrow\ i\hbar \ln\left[\hat{U}(t-t_0)\right] = \hat{H}^{(S)} t + C$$

$$\Rightarrow\ \hat{U}(t-t_0) = Aexp\left(-i\,\hat{H}^{(S)} t/\hbar\right). \qquad (2.122)$$

By applying the condition $\hat{U}(0) = \hat{I}$,

$$\hat{I} = Aexp\left(-i\,\hat{H}^{(S)} t_0/\hbar\right)$$

$$\Rightarrow A = exp\left(i\,\hat{H}^{(S)} t_0/\hbar\right).$$

By substituting $A = exp\left(i\,\hat{H}^{(S)} t_0/\hbar\right)$ into (2.122),

$$\hat{U}(t-t_0) = exp\left[-i\,\hat{H}^{(S)}(t-t_0)/\hbar\right].$$

The choice of the value of t_0 depends on the convenience for the problems in Quantum Optics. Most of the time, t_0 is set to $t_0 = 0$.

Besides that, the mathematical relationship between the Operators in Schrodinger and Heisenberg Pictures could also be established. This is achieved by enforcing the condition that expectation values given by observables in both pictures must be equivalent, which means

$$\left\langle \Psi^{(\mathrm{H})}\middle|\hat{X}^{(H)}(t)\middle|\Psi^{(\mathrm{H})}\right\rangle = \left\langle \Psi^{(\mathrm{S})}(\mathrm{t})\middle|\hat{X}^{(S)}\middle|\Psi^{(\mathrm{S})}(\mathrm{t})\right\rangle. \qquad (2.123)$$

By substituting (2.121) into (2.123), (2.123) becomes

$$\left\langle \Psi^{(\mathrm{H})}\middle|\hat{X}^{(H)}(t)\middle|\Psi^{(\mathrm{H})}\right\rangle = \langle \Psi^{(\mathrm{H})}\middle|\hat{U}^{\dagger}(t-t_0)\hat{X}^{(S)}\hat{U}(t-t_0)\middle|\Psi^{(\mathrm{H})}\rangle,$$

where $\langle \Psi^{(\mathrm{S})}\big| = \langle \Psi^{(\mathrm{H})}\big|\hat{U}^{\dagger}(t-t_0)$.

Then, by comparing left and right-hand sides,

$$\hat{X}^{(H)}(t) = \hat{U}^{\dagger}(t-t_0)\hat{X}^{(S)}\hat{U}(t-t_0). \qquad (2.124)$$

By setting $t_0 = 0$,

$$\hat{X}^{(H)}(t) = \hat{U}^{\dagger}(t)\hat{X}^{(S)}\hat{U}(t).$$

Interaction Picture

In the study of interaction between electromagnetic field and atom (Jaynes-Cummings Model), the interaction energy between the atom and field is much smaller than the energies of individual photons. It is therefore useful to rewrite the Schrodinger Picture Hamiltonian,

$$\hat{H}^{(S)} = \hat{H}_F^{(S)} + \hat{H}_A^{(S)} + \hat{H}_{Int}^{(S)} \text{ as } \hat{H}^{(S)} = \hat{H}_0^{(S)} + \hat{H}_{Int}^{(S)}, \quad \text{where } \hat{H}_0^{(S)} = \hat{H}_F^{(S)} + \hat{H}_A^{(S)},$$

$\hat{H}_F^{(S)}$ *is the* Field Hamiltonian, $\hat{H}_A^{(S)}$ is the atom Hamiltonian and $\hat{H}_{Int}^{(S)}$ is the Interaction Hamiltonian. Then, $\hat{H}_0^{(S)}$ is the unperturbed Hamiltonian while $\hat{H}_{Int}^{(S)}$ is considered as the perturbation (small disturbance).

Interaction Picture is useful in the theoretical study of Jaynes-Cummings Model (JCM) because by performing unitary transformation on the JCM Hamiltonian from Schrodinger to Interaction Picture JCM Hamiltonian $\hat{H}^{(I)}$, the form of Hamiltonian will become simpler. The significance of transforming from Schrodinger Picture into Interaction Picture is that the unperturbed Hamiltonian $\hat{H}_0^{(S)}$ is removed from the transformed JCM Hamiltonian since it does not affect the JCM system under studied. The Interaction Picture state vector $|\Psi^{(I)}(t)\rangle$ is defined by the following unitary transformation:

$$|\Psi^{(I)}(t)\rangle = \hat{U}_0^{\dagger}(t)|\Psi^{(S)}(t)\rangle, \tag{2.125}$$

where $\hat{U}_0^{\dagger}(t) = exp\left[i\,\hat{H}_0^{(S)}(t - t_0)/\hbar\right].$ (2.126)

The expression in (2.126) could be proven as follows:

From Schrodinger's Equation,

$$\hat{H}_0^{(S)}|\Psi^{(S)}(t)\rangle = i\hbar\frac{\partial}{\partial t}|\Psi^{(S)}(t)\rangle. \tag{2.127}$$

Since by definition, $|\Psi^{(S)}(t)\rangle = \hat{U}_0(t)|\Psi^{(H)}\rangle$ (setting $t_0 = 0$), then (2.127) becomes

$$\hat{H}_0^{(S)}\left[\hat{U}_0(t)|\Psi^{(H)}\rangle\right] = i\hbar\frac{\partial}{\partial t}\left[\hat{U}_0(t)|\Psi^{(H)}\rangle\right]$$

$$\hat{H}_0^{(S)}\hat{U}_0(t) = i\hbar\frac{\partial}{\partial t}\hat{U}_0(t)$$

$$\Rightarrow i\hbar \int \frac{d\left[\hat{U}_0(t)\right]}{\hat{U}_0(t)} = \int \hat{H}_0^{(S)}\, dt$$

$$\Rightarrow \; i\hbar \ln\left[\hat{U}_0(t)\right] = \hat{H}_0^{(S)} t + C$$

$$\Rightarrow \hat{U}_0(t) = A exp\left(-i\, \hat{H}_0^{(S)} t/\hbar\right). \qquad (2.128)$$

By applying the condition $\hat{U}_0(0) = \hat{I}$,

$$A = \hat{I}\,. \qquad (2.129)$$

By substituting (2.129) into (2.128),

$$\hat{U}_0(t) = exp\left[-i\, \hat{H}_0^{(S)} t/\hbar\right]. \qquad (2.130)$$

Then,

$$\hat{U}_0^{\dagger}(t) = exp\left[i\, \hat{H}_0^{(S)} t/\hbar\right]. \qquad (2.131)$$

<u>Derivation of Transformation Formula from Schrodinger Picture Hamiltonian into Interaction Picture Hamiltonian</u>

The following derivation is completed by referring to Barnett and Radmore (1997). The Time Dependent Schrodinger Equation in Interaction Picture is given by

$$i\hbar \frac{\partial}{\partial t}\left|\Psi^{(\mathrm{I})}(\mathrm{t})\right\rangle = \hat{H}^{(I)}\left|\Psi^{(\mathrm{I})}(\mathrm{t})\right\rangle. \qquad (2.132)$$

From (2.132),

$$\mathrm{L.H.S} = i\hbar \frac{\partial}{\partial t}\left|\Psi^{(\mathrm{I})}(\mathrm{t})\right\rangle. \qquad (2.133)$$

By substituting (2.125) into (2.133), (2.133) becomes

$$\mathrm{L.H.S} = i\hbar \frac{\partial}{\partial t}\left[\hat{U}_0^{\dagger}(t)\left|\Psi^{(\mathrm{S})}(\mathrm{t})\right\rangle\right].$$

Let $\hat{U}(t) = \hat{U}_0^{\dagger}(t) = exp\left[i\, \hat{H}_0^{(S)} t/\hbar\right]$, then,

$$\mathrm{L.H.S} = i\hbar \frac{\partial}{\partial t}\left[\hat{U}(t)\left|\Psi^{(\mathrm{S})}(\mathrm{t})\right\rangle\right]$$

$$= i\hbar\left(\frac{\partial\hat{U}}{\partial t}\left|\Psi^{(S)}\right\rangle + \hat{U}\frac{\partial\left|\Psi^{(S)}\right\rangle}{\partial t}\right)$$

$$= i\hbar\dot{\hat{U}}\left|\Psi^{(S)}\right\rangle + i\hbar\hat{U}\frac{\partial\left|\Psi^{(S)}\right\rangle}{\partial t}\,. \qquad (2.134)$$

The Time Dependent Schrodinger Equation in Schrodinger Picture is given by

$$i\hbar\frac{\partial}{\partial t}\left|\Psi^{(S)}(\mathrm{t})\right\rangle = \hat{H}^{(S)}\left|\Psi^{(S)}(\mathrm{t})\right\rangle\,. \qquad (2.135)$$

By substituting (2.135) into (2.134), (2.134) becomes

$$\mathrm{L.H.S} = i\hbar\dot{\hat{U}}\left|\Psi^{(S)}\right\rangle + \hat{U}\hat{H}^{(S)}\left|\Psi^{(S)}\right\rangle\,. \qquad (2.136)$$

From (2.132),

$$\mathrm{R.H.S} = \hat{H}^{(I)}\left|\Psi^{(I)}(\mathrm{t})\right\rangle$$

$$= \hat{H}^{(I)}\hat{U}\left|\Psi^{(S)}\right\rangle\,. \qquad (2.137)$$

Since L.H.S = R.H.S, by comparing (2.136) and (2.137),

$$\left(i\hbar\dot{\hat{U}} + \hat{U}\hat{H}^{(S)}\right)\left|\Psi^{(S)}\right\rangle = \hat{H}^{(I)}\hat{U}\left|\Psi^{(S)}\right\rangle$$

$$\hat{H}^{(I)}\hat{U} = i\hbar\dot{\hat{U}} + \hat{U}\hat{H}^{(S)}$$

$$\hat{H}^{(I)} = i\hbar\dot{\hat{U}}\hat{U}^{\dagger} + \hat{U}\hat{H}^{(S)}\hat{U}^{\dagger}\,. \qquad (2.138)$$

CHAPTER 3

METHODOLOGY

3.1 Derivation process

In this thesis, the interaction between a two-level atom with coherent and thermal initial field states (Jaynes-Cummings Model) is studied. The main objective is to derive the relationship between the probability that the atom is in the ground state, P_1 and the parameter λt, where λ is the interaction strength, t is the time. The derivation is initiated with single-photon Jaynes-Cummings Model, then it is extended to two-photon and three-photon Jaynes-Cummings Model, and finally generalized to k-photon Jaynes-Cummings Model. In each case, the coherent and thermal states will serve as initial field states for interaction with the two-level atom and they give different results.

The derivation is started with the quantization of the single-mode electromagnetic field. During the quantization of the electromagnetic field, the Maxwell's Equations are modified by defining scalar and vector potentials. Then, the modified Maxwell's Equations are simplified by using Coulomb Gauge and applying the condition that the charges and currents are absent. The simplified wave equations are solved by using method of separable variables with an appropriate set of boundary conditions. After the solutions of electric and magnetic fields are obtained, they are used to derive the classical Hamiltonian for generalized multimode electromagnetic field. Since the classical Hamiltonian for electromagnetic field has the same form as the Hamiltonian for unit mass harmonic oscillator, the electromagnetic field is treated as field oscillators and they are quantized in exactly the same way as the unit mass harmonic oscillators. This is done by rewriting the Hamiltonian in terms of the complex-valued amplitudes and finally converting it into the field Hamiltonian operator, $\widehat{H}_F$ by using the correspondence principle. The detailed derivation is presented in part 2.3, Literature Review.

Next, the energy of the two-level atom is quantized by defining the ground state energy equals to $\hbar\omega_1$ and excited state energy equals to $\hbar\omega_2$. The Hamiltonian of the two-level atom, $\widehat{H}_A$ is found and its full derivation is given in part 2.4 in Literature Review. After the quantization of the single mode electromagnetic field and energy of two-level atom, the Hamiltonian due to the interaction between the two-level atom and photons, $\widehat{H}_{int}$ is determined by using the Electric Dipole Approximation. Under the Electric Dipole Approximation, the two-level atom is modeled as a dipole while the electromagnetic field interacting with the atom is treated as uniform electric field across the dipole. This approximated model is valid because the size of the atom is very small compared with the electromagnetic field wavelength. This has been discussed further in part 2.6 in Literature Review.

So far, the Hamiltonian of the single mode electromagnetic field $\widehat{H}_F$, two-level atom Hamiltonian $\widehat{H}_A$ and the interaction Hamiltonian between atom and field $\widehat{H}_{int}$ are found. Then, the Jaynes-Cummings Model (JCM) Hamiltonian, $\widehat{H}$ is obtained as $\widehat{H} = \widehat{H}_A + \widehat{H}_F + \widehat{H}_{int}$. This expression of JCM Hamiltonian is in the Schrodinger Picture and it will then be converted into Interaction Picture in order to simplify the form of the JCM Hamiltonian. This is achieved by applying the transformation equation $\widehat{H}^{(I)} = i\hbar\dot{\widehat{U}}\widehat{U}^{\dagger} + \widehat{U}\widehat{H}^{(S)}\widehat{U}^{\dagger}$, where $\widehat{H}^{(I)}$ is the Interaction Picture JCM Hamiltonian, $\widehat{H}^{(S)}$ is the Schrodinger Picture JCM Hamiltonian and $\widehat{U}$ is the Unitary operator. The proof of the transformation equation and details of the transformation are presented in Literature Review part 2.7 and Methodology part 3.2.

After that, the Interaction Picture JCM Hamiltonian is used to derive the expression for single-photon JCM unitary operator. This unitary operator is different from the $\widehat{U}$ mentioned above because this new unitary operator will perform a unitary transformation on the composite quantum state of JCM and describes the evolution of the quantum state. The expression of this single-photon JCM unitary operator is given by $\widehat{U} = \exp\left(-\frac{i\widehat{H}^{(I)}t}{\hbar}\right)$. Before deriving the single-photon JCM unitary operator, the Rotating Wave Approximation (RWA) is made. Under the RWA, the expression of $\widehat{H}^{(I)}$ (it is $\widehat{H}_I$ in 3.2) is simplified by discarding the non-energy conserving terms in $\widehat{H}^{(I)}$. This is explained in more detailed in Methodology 3.2. Then, the simplified $\widehat{H}^{(I)}$ is used to derive the expression for single-photon JCM unitary operator, as shown in Results 4.1.1. Besides that, the original form of $\widehat{H}^{(I)}$ is also used to derive the single-photon

JCM unitary operator and this will only serve for future study (in Results 4.1.2). This is because there are effects shown by the results obtained from single-photon JCM unitary operator without RWA, which are worth to study and this is out of the scope of this Final Year Project.

The single-photon JCM unitary operator derived with RWA is then used to compute the probability that the two-level atom is in the ground state as a function of time t. The detailed calculations are shown in Results 4.2. Next, the photon number probability distribution for coherent state (equation (2.8) in Literature Review 2.1) is substituted into the probability function derived and the graph of probability that the two-level atom is in the ground state against time t is plotted by using Matlab. Similar process applied for the thermal initial state which has the photon number probability distribution given by equation (2.17) in Literature Review 2.2. The graphical results are then studied and discussed.

For the case of two-photon JCM, it has a slightly different form of Interaction Picture JCM Hamiltonian compared with the single-photon case, which is given by equation (3.22) in Methodology 3.3. Then, similar process is repeated in which the two-photon JCM unitary operator is first derived and then followed by the calculation of probability function mentioned above (shown in Results 4.3.1). By introducing different initial field states into the probability function, more interesting graphical results are obtained. These results are then studied and compared with single-photon JCM case. The same process as above is repeated again for three-photon JCM case (shown in Results 4.3.2). Finally, general expressions for k-photon JCM unitary operator and probability function are derived as well in Results 4.3.3.

3.2 Derivation of Jaynes-Cummings Model (JCM) Hamiltonian in Interaction Picture

Let $\hat{H}_A$ be the Hamiltonian of the two-level atom, $\hat{H}_F$ be the Hamiltonian of the single mode electromagnetic field, $\hat{H}_{int}$ be the Hamiltonian of the atom-field interaction, $\hat{H}$ be the total Hamiltonian in Schrodinger picture and $\hat{H}_I$ be the total Hamiltonian in Interaction picture.

Then, let ω be the frequency of the incoming photons, ω_1 be the frequency corresponds to the ground level of the atom, ω_2 be the frequency corresponds to the excited level of the atom and $\Delta = \omega_2 - \omega_1 - \omega$ be the detuning.

Lastly, δ and $\hat{H}_0$ are given by

$\delta = \dfrac{\Delta}{2}$ and

$\hat{H}_0 = \hat{H}_A + \hat{H}_F$.

From Literature Review part, the Hamiltonian derivations for atom, field and atom-field interaction in Schrodinger picture are

$$\hat{H}_A = \frac{\hbar}{2}(\omega_1 + \omega_2)\hat{I} + \frac{\hbar}{2}(\omega_2 - \omega_1)\hat{\sigma}_3 \qquad (2.114)$$

$$\hat{H}_F = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \qquad (2.101)$$

$$\hat{H}_{int} = -i\hbar\lambda(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} - \hat{a}^\dagger). \qquad (2.119)$$

Then, as discussed in Literature Review, the JCM Hamiltonian $\hat{H}$ is the sum of (2.114), (2.101) and (2.119).

$$\begin{aligned}\hat{H} &= \hat{H}_A + \hat{H}_F + \hat{H}_{int} \\ &= \frac{\hbar}{2}(\omega_1 + \omega_2)\hat{I} + \frac{\hbar}{2}(\omega_2 - \omega_1)\hat{\sigma}_3 + \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) - i\hbar\lambda(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} - \hat{a}^\dagger) \\ &= \hat{H}_0 - i\hbar\lambda(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} - \hat{a}^\dagger) \qquad (3.1)\end{aligned}$$

Now,

$$\begin{aligned}\hat{H}_0 &= \frac{\hbar}{2}(\omega_1 + \omega_2)\hat{I} + \frac{\hbar}{2}(\omega_2 - \omega_1)\hat{\sigma}_3 + \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \\ &= \frac{\hbar}{2}(\omega_1 + \omega_2 + \Delta - 2\delta)\hat{I} + \frac{\hbar}{2}(\omega_2 - \omega_1)\hat{\sigma}_3 + \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \qquad (3.2)\end{aligned}$$

Since

$$\Delta = \omega_2 - \omega_1 - \omega$$

$$\Rightarrow \omega = \omega_2 - \omega_1 - \Delta.$$

Considering the detuning Δ is very small compared with $(\omega_2 - \omega_1)$, then,

$$\omega \approx \omega_2 - \omega_1 \ . \tag{3.3}$$

By substituting (3.3) into (3.2), (3.2) becomes

$$\begin{aligned}
\hat{H}_0 &= \frac{\hbar}{2}(\omega_1 + \omega_2 + \Delta - 2\delta)\hat{I} + \frac{\hbar}{2}\omega\hat{\sigma}_3 + \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \\
&= \frac{\hbar}{2}(2\omega_2 - \omega - 2\delta)\hat{I} + \frac{\hbar}{2}\omega\hat{\sigma}_3 + \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \\
&= \frac{\hbar}{2}\begin{pmatrix} 2\omega_2 - \omega - 2\delta & 0 \\ 0 & 2\omega_2 - \omega - 2\delta \end{pmatrix} + \frac{\hbar}{2}\begin{pmatrix} \omega & 0 \\ 0 & -\omega \end{pmatrix} + \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) \\
&= \hbar\begin{pmatrix} \omega_2 - \delta & 0 \\ 0 & \omega_2 - \omega - \delta \end{pmatrix} + \hbar\omega\left(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}\right).
\end{aligned} \tag{3.4}$$

Now, to change the JCM Hamiltonian $\hat{H}$ from Schrodinger to Interaction Picture, the equation

$$\hat{H}_I = i\hbar\dot{\hat{U}}\hat{U}^\dagger + \hat{U}\hat{H}\hat{U}^\dagger \tag{3.5}$$

is used, as derived in Literature Review, where

$$\begin{aligned}
\hat{U} &= exp\left(i\hat{H}_0 t/\hbar\right) \\
&= exp\left\{i\left[\hbar\begin{pmatrix} \omega_2 - \delta & 0 \\ 0 & \omega_2 - \omega - \delta \end{pmatrix} + \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\right]t/\hbar\right\} \\
&= exp\left[it\begin{pmatrix} \omega_2 - \delta & 0 \\ 0 & \omega_2 - \omega - \delta \end{pmatrix} + i\omega t\left(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}\right)\right] \\
&= exp\left[it\begin{pmatrix} \omega_2 - \delta & 0 \\ 0 & \omega_2 - \omega - \delta \end{pmatrix}\right] exp\left[i\omega t\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)\right] .
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
\dot{\hat{U}} &= \frac{d\hat{U}}{dt} \\
&= \left[i\hat{H}_0/\hbar\right]exp\left[i\hat{H}_0 t/\hbar\right]
\end{aligned} \tag{3.7}$$

$$\hat{U}^\dagger = exp\left[-i\hat{H}_0 t/\hbar\right] \tag{3.8}$$

In order to solve (3.5), we have

$$i\hbar\dot{\hat{U}}\hat{U}^{\dagger} = i\hbar\left[i\hat{H}_0/\hbar\right]exp\left[i\hat{H}_0 t/\hbar\right]exp\left[-i\hat{H}_0 t/\hbar\right]$$

$$= -\hat{H}_0$$

$$= -\hbar\begin{pmatrix}\omega_2-\delta & 0\\ 0 & \omega_2-\omega-\delta\end{pmatrix} - \hbar\omega\left(\hat{a}^{\dagger}\hat{a}+\frac{1}{2}\right), \tag{3.9}$$

and

$$\hat{U}\hat{H}\hat{U}^{\dagger} = \hat{U}\left(\hat{H}_0+\hat{H}_{int}\right)\hat{U}^{\dagger}$$

$$= \hat{U}\hat{H}_0\hat{U}^{\dagger} + \hat{U}\hat{H}_{int}\hat{U}^{\dagger}. \tag{3.10}$$

Then, to solve (3.10), we have

$$\hat{U}\hat{H}_0\hat{U}^{\dagger} = exp\left(i\hat{H}_0 t/\hbar\right)\hat{H}_0 exp\left[-i\hat{H}_0 t/\hbar\right]$$

$$= \hat{H}_0$$

$$= \frac{\hbar}{2}(\omega_1+\omega_2)\hat{I} + \frac{\hbar}{2}(\omega_2-\omega_1)\hat{\sigma}_3 + \hbar\omega\left(\hat{a}^{\dagger}\hat{a}+\frac{1}{2}\right)$$

$$= \begin{pmatrix}\hbar\omega_2 & 0\\ 0 & \hbar\omega_1\end{pmatrix} + \hbar\omega\left(\hat{a}^{\dagger}\hat{a}+\frac{1}{2}\right), \tag{3.11}$$

and

$$\hat{U}\hat{H}_{int}\hat{U}^{\dagger} = exp\left(i\hat{H}_0 t/\hbar\right)[-i\hbar\lambda(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a}-\hat{a}^{\dagger})]exp\left[-i\hat{H}_0 t/\hbar\right]$$

$$= exp\left[it\begin{pmatrix}\omega_2-\delta & 0\\ 0 & \omega_2-\omega-\delta\end{pmatrix}\right] exp\left[i\omega t\left(\hat{a}^{\dagger}\hat{a}+\frac{1}{2}\right)\right][-i\hbar\lambda(\hat{\sigma}_+ + \hat{\sigma}_-)(\hat{a} - \hat{a}^{\dagger})]exp\left[-it\begin{pmatrix}\omega_2-\delta & 0\\ 0 & \omega_2-\omega-\delta\end{pmatrix}\right] exp\left[-i\omega t\left(\hat{a}^{\dagger}\hat{a}+\frac{1}{2}\right)\right]$$

$$= -i\hbar\lambda\left[\left(\begin{matrix}exp[it(\omega_2-\delta)] & 0\\ 0 & exp[it(\omega_2-\omega-\delta)]\end{matrix}\right)(\hat{\sigma}_+ + \hat{\sigma}_-)\left(\begin{matrix}exp[-it(\omega_2-\delta)] & 0\\ 0 & exp[-it(\omega_2-\omega-\delta)]\end{matrix}\right)\right]\left[\left(exp\left(\frac{i\omega t}{2}\right)\sum_{m=0}^{\infty} exp[i\omega t(m+1)]\,|m+1\rangle\langle m+1|\right)(\hat{a}-\hat{a}^{\dagger})exp\left(-\frac{i\omega t}{2}\right)\sum_{k=0}^{\infty} exp[-i\omega t(k+1)]\,|k+1\rangle\langle k+1|\right]. \tag{3.12}$$

To simplify (3.12),

$$-i\hbar\lambda\left[\begin{pmatrix} exp[it(\omega_2-\delta)] & 0 \\ 0 & exp[it(\omega_2-\omega-\delta)] \end{pmatrix}(\hat{\sigma}_+ + \hat{\sigma}_-)\begin{pmatrix} exp[-it(\omega_2-\delta)] & 0 \\ 0 & exp[-it(\omega_2-\omega-\delta)] \end{pmatrix}\right]$$

$$= -i\hbar\lambda\left[\begin{pmatrix} exp[it(\omega_2-\delta)] & 0 \\ 0 & exp[it(\omega_2-\omega-\delta)] \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} exp[-it(\omega_2-\delta)] & 0 \\ 0 & exp[-it(\omega_2-\omega-\delta)] \end{pmatrix}\right]$$

$$= \begin{pmatrix} 0 & exp(i\omega t) \\ exp(-i\omega t) & 0 \end{pmatrix}, \qquad (3.13)$$

and

$$\left[\left(exp\left(\frac{i\omega t}{2}\right)\sum_{m=0}^{\infty} exp[i\omega t(m+1)]\,|m+1\rangle\langle m+1|\right)\hat{a}exp\left(-\frac{i\omega t}{2}\right)\sum_{k=0}^{\infty} exp[-i\omega t(k+1)]\,|k+1\rangle\langle k+1|\right]$$

$$= \left\{\left(exp\left(\frac{i\omega t}{2}\right)\sum_{m=0}^{\infty} exp[i\omega t(m+1)]\,|m+1\rangle\langle m+1|\right)\left[\sum_{n=0}^{\infty}(n+1)^{1/2}\,|n\rangle\langle n+1|\right]exp\left(-\frac{i\omega t}{2}\right)\sum_{k=0}^{\infty} exp[-i\omega t(k+1)]\,|k+1\rangle\langle k+1|\right\}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left\{exp[i\omega t(m+1)]|m+1\rangle\langle m+1|(n+1)^{1/2}|n\rangle\langle n+1|exp[-i\omega t(k+1)]|k+1\rangle\langle k+1|\right\}$$

$$= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\left\{exp[i\omega t(m+1-k-1)](m+2)^{1/2}|m+1\rangle\langle m+2|k+1\rangle\langle k+1|\right\}$$

$$= \sum_{k=0}^{\infty}\left\{exp[i\omega t(k-k-1)](k+1)^{1/2}|k\rangle\langle k+1|k+1\rangle\langle k+1|\right\}$$

$$= exp[-i\omega t]\sum_{k=0}^{\infty}\left\{(k+1)^{1/2}|k\rangle\langle k+1|\right\}$$

$$= \hat{a}exp[-i\omega t]\ , \qquad (3.14)$$

and

$$\left[\left(exp\left(\frac{i\omega t}{2}\right)\sum_{m=0}^{\infty} exp[i\omega t(m+1)]\,|m+1\rangle\langle m+1|\right)\hat{a}^{\dagger} exp\left(-\frac{i\omega t}{2}\right)\sum_{k=0}^{\infty} exp[-i\omega t(k+1)]\,|k+1\rangle\langle k+1|\right]$$

$$=\left[\left(exp\left(\frac{i\omega t}{2}\right)\sum_{r=0}^{\infty} exp[i\omega t(r+1)]\,|r+1\rangle\langle r+1|\right)\hat{a} exp\left(-\frac{i\omega t}{2}\right)\sum_{s=0}^{\infty} exp[-i\omega t(s+1)]\,|s+1\rangle\langle s+1|\right]^{\dagger}$$

$$=\{\hat{a} exp[-i\omega t]\}^{\dagger}$$

$$=\hat{a}^{\dagger} exp[i\omega t]\ . \qquad (3.15)$$

By substituting (3.13), (3.14) and (3.15) into (3.12), (3.12) becomes

$$\hat{U}\hat{H}_{int}\hat{U}^{\dagger} = -i\hbar\lambda\begin{pmatrix} 0 & exp(i\omega t) \\ exp(-i\omega t) & 0 \end{pmatrix}\{\hat{a} exp[-i\omega t] - \hat{a}^{\dagger} exp[i\omega t]\}$$

$$= -i\hbar\lambda\begin{pmatrix} 0 & \{\hat{a} exp[-i\omega t] - \hat{a}^{\dagger} exp[i\omega t]\} exp(i\omega t) \\ \{\hat{a} exp[-i\omega t] - \hat{a}^{\dagger} exp[i\omega t]\} exp(-i\omega t) & 0 \end{pmatrix}$$

$$= -i\hbar\lambda[\hat{\sigma}_{+} exp(i\omega t) + \hat{\sigma}_{-} exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^{\dagger} exp(i\omega t)]\ . \qquad (3.16)$$

By substituting (3.16) and (3.11) into (3.10), (3.10) becomes

$$\hat{U}\hat{H}\hat{U}^{\dagger} = \begin{pmatrix} \hbar\omega_2 & 0 \\ 0 & \hbar\omega_1 \end{pmatrix} + \hbar\omega\left(\hat{a}^{\dagger}\hat{a} + \frac{1}{2}\right) - i\hbar\lambda[\hat{\sigma}_{+} exp(i\omega t) + \hat{\sigma}_{-} exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^{\dagger} exp(i\omega t)]\ . \qquad (3.17)$$

Then, by substituting (3.9) and (3.17) into (3.5), (3.5) becomes

$$\hat{H}_I = i\hbar\dot{\hat{U}}\hat{U}^{\dagger} + \hat{U}\hat{H}\hat{U}^{\dagger}$$

$$= -\hbar\begin{pmatrix}\omega_2-\delta & 0\\ 0 & \omega_2-\omega-\delta\end{pmatrix} - \hbar\omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right) + \begin{pmatrix}\hbar\omega_2 & 0\\ 0 & \hbar\omega_1\end{pmatrix} + \hbar\omega\left(\hat{a}^\dagger\hat{a}+\frac{1}{2}\right)$$
$$- i\hbar\lambda[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]$$

$$= \begin{pmatrix}\hbar\delta & 0\\ 0 & -\hbar\omega_2+\hbar\omega+\hbar\delta+\hbar\omega_1\end{pmatrix} - i\hbar\lambda[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]$$

$$= \begin{pmatrix}\hbar\delta & 0\\ 0 & -\hbar\Delta+\hbar\delta\end{pmatrix} - i\hbar\lambda[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]$$

$$= \begin{pmatrix}\frac{\hbar\Delta}{2} & 0\\ 0 & -\frac{\hbar\Delta}{2}\end{pmatrix} - i\hbar\lambda[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]$$

$$= \frac{\hbar\Delta}{2}\hat{\sigma}_3 - i\hbar\lambda[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)] \qquad (3.18)$$

$$= \frac{\hbar\Delta}{2}\hat{\sigma}_3 - i\hbar\lambda[\hat{\sigma}_+\hat{a} - \hat{\sigma}_+\hat{a}^\dagger exp(2i\omega t) + \hat{\sigma}_-\hat{a} exp(-2i\omega t) - \hat{\sigma}_-\hat{a}^\dagger]\,. \qquad (3.19)$$

By applying the Rotating Wave Approximation,

$$\hat{H}_I = \frac{\hbar\Delta}{2}\hat{\sigma}_3 - i\hbar\lambda[\hat{\sigma}_+\hat{a} - \hat{\sigma}_-\hat{a}^\dagger]\,. \qquad (3.20)$$

<u>Rotating Wave Approximation</u>

From (3.19), the total Hamiltonian of single photon Jaynes Cummings Model in Interaction Picture is

$$\hat{H}_I = \frac{\hbar\Delta}{2}\hat{\sigma}_3 - i\hbar\lambda[\hat{\sigma}_+\hat{a} - \hat{\sigma}_+\hat{a}^\dagger exp(2i\omega t) + \hat{\sigma}_-\hat{a} exp(-2i\omega t) - \hat{\sigma}_-\hat{a}^\dagger]$$

In the interaction expression $-i\hbar\lambda[\hat{\sigma}_+\hat{a} - \hat{\sigma}_+\hat{a}^\dagger exp(2i\omega t) + \hat{\sigma}_-\hat{a} exp(-2i\omega t) - \hat{\sigma}_-\hat{a}^\dagger]$,

according to Garrison and Chiao (2008), $\hat{\sigma}_+\hat{a}$ represents the energy excitation of the atom accompanied with absorption of a photon, $\hat{\sigma}_-\hat{a}^\dagger$ means the de-excitation of the atom with the release of a photon, $\hat{\sigma}_+\hat{a}^\dagger$ denotes the energy excitation of the atom and release of a photon (field excitation) at the same time, and $\hat{\sigma}_-\hat{a}$ represents the de-excitation of the atom accompanied with absorption of a photon.

Therefore, it can be seen that the two terms $\hat{\sigma}_+\hat{a}^\dagger$ and $\hat{\sigma}_-\hat{a}$ violate the principle of conservation of energy. Hence, the two terms $\hat{\sigma}_+\hat{a}^\dagger exp(2i\omega t)$ and $\hat{\sigma}_-\hat{a}exp(-2i\omega t)$ can be omitted and the total Hamiltonian becomes

$$\hat{H}_I = \frac{\hbar\Delta}{2}\hat{\sigma}_3 - i\hbar\lambda[\hat{\sigma}_+\hat{a} - \hat{\sigma}_-\hat{a}^\dagger]\,.$$

The approximation made by omitting the two terms mentioned is known as Rotating Wave Approximation.

3.3 General Expression for k-photon JCM Hamiltonian in Interaction Picture

From Methodology 3.2, the single-photon JCM Hamiltonian is given by

$$\hat{H}_I = \frac{\hbar\Delta}{2}\hat{\sigma}_3 - i\hbar\lambda[\hat{\sigma}_+\hat{a} - \hat{\sigma}_-\hat{a}^\dagger]\,. \tag{3.20}$$

The term $\hat{\sigma}_+\hat{a}$ is physically interpreted as the transition of the two-level atom from ground to excited state with the absorption of one photon. Conversely, the term $\hat{\sigma}_-\hat{a}^\dagger$ denotes the transition of the atom from excited to ground state followed by the release of a photon.

Then, according to Sukumar and Buck (1981), for k-photon JCM Hamiltonian, it can be expressed generally as

$$\hat{H}_I = \frac{\hbar\Delta}{2}\hat{\sigma}_3 - i\hbar\lambda[\hat{\sigma}_+\hat{a}^k - \hat{\sigma}_-\hat{a}^{\dagger k}]. \tag{3.21}$$

When k = 2,

$$\hat{H}_I = \frac{\hbar\Delta}{2}\hat{\sigma}_3 - i\hbar\lambda[\hat{\sigma}_+\hat{a}^2 - \hat{\sigma}_-\hat{a}^{\dagger 2}]. \tag{3.22}$$

In this case, the term $\hat{\sigma}_+\hat{a}^2$ will mean the transition of the two-level atom from ground to excited state with the absorption of two photons. Then, the physical meaning of the term $\hat{\sigma}_-\hat{a}^{\dagger 2}$ is the de-excitation of the atom from excited to ground state accompanied by the release of two photons.

Then, for three-photon JCM Hamiltonian in Interaction Picture, it is given by

$$\hat{H}_I = \frac{\hbar\Delta}{2}\hat{\sigma}_3 - i\hbar\lambda[\hat{\sigma}_+\hat{a}^3 - \hat{\sigma}_-\hat{a}^{\dagger 3}]\,. \quad (3.23)$$

The expressions in (3.22), (3.23) and (3.21) will then be used to derive the two-photon, three-photon and k-photon JCM unitary operator in Results part.

CHAPTER 4

RESULTS AND DISCUSSIONS

4.1.1 Derivation of Jaynes Cummings Model-type Unitary Operator

From previous derivation, we have the Hamiltonian in interaction picture as follows:

$$\widehat{H}_I = \frac{\hbar\Delta}{2}\hat{\sigma}_3 - i\hbar\lambda(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-) \tag{4.1}$$

Now, the Unitary Operator $\widehat{U}$ is

$$\begin{aligned}
\widehat{U} &= \exp\left(-\frac{i\widehat{H}_I t}{\hbar}\right) \\
&= \exp\ \{-i\left[\tfrac{\hbar\Delta}{2}\hat{\sigma}_3 - i\hbar\lambda(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]t\,/\hbar\} \\
&= \exp\{-i\left[\tfrac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]\} \\
&= \cos\hat{\theta} - i\sin\hat{\theta},\ \text{ where } \hat{\theta} = \tfrac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)
\end{aligned} \tag{4.2}$$

Let $\hat{c} = \cos\hat{\theta}\ \ and\ \hat{s} = \sin\hat{\theta}$

Therefore, $\widehat{U} = \hat{c} - i\hat{s}$. Now,

$$\hat{c} = \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}\left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]^{2m} \tag{4.3}$$

$$\hat{s} = \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}\left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]^{2m+1} \tag{4.4}$$

Now,

$$\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-) = \frac{\Delta \mathrm{t}}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} - i\lambda t\left[\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\hat{a} - \hat{a}^\dagger\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\right]$$

$$= \begin{pmatrix}\frac{\Delta \mathrm{t}}{2} & -i\lambda t\hat{a}\\ i\lambda t\hat{a}^\dagger & -\frac{\Delta \mathrm{t}}{2}\end{pmatrix} \tag{4.5}$$

Consider the even power expansion, we have

$$\left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]^2 = \left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]\left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]$$

$$= \begin{pmatrix}\frac{\Delta \mathrm{t}}{2} & -i\lambda t\hat{a}\\ i\lambda t\hat{a}^\dagger & -\frac{\Delta \mathrm{t}}{2}\end{pmatrix}\begin{pmatrix}\frac{\Delta \mathrm{t}}{2} & -i\lambda t\hat{a}\\ i\lambda t\hat{a}^\dagger & -\frac{\Delta \mathrm{t}}{2}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}\hat{a}^\dagger & 0\\ 0 & \frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^\dagger\hat{a}\end{pmatrix} \tag{4.6}$$

$$\left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]^4 = \left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]^2\left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]^2$$

$$= \begin{pmatrix}\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}\hat{a}^\dagger & 0\\ 0 & \frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^\dagger\hat{a}\end{pmatrix}\begin{pmatrix}\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}\hat{a}^\dagger & 0\\ 0 & \frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^\dagger\hat{a}\end{pmatrix}$$

$$= \begin{pmatrix}\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}\hat{a}^\dagger\right)^2 & 0\\ 0 & \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^\dagger\hat{a}\right)^2\end{pmatrix} \tag{4.7}$$

$$\left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]^6 = \left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]^4\left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]^2$$

$$= \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}\hat{a}^\dagger\right)^2 & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^\dagger\hat{a}\right)^2 \end{pmatrix} \begin{pmatrix} \frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}\hat{a}^\dagger & 0 \\ 0 & \frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^\dagger\hat{a} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}\hat{a}^\dagger\right)^3 & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^\dagger\hat{a}\right)^3 \end{pmatrix} \tag{4.8}$$

From here, we can actually deduce that for general even power 2m,

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]^{2m} = \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}\hat{a}^\dagger\right)^m & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^\dagger\hat{a}\right)^m \end{pmatrix}$$

However, this could be proven by using mathematical induction as follows:

Let P(m) be the statement that

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]^{2m} = \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}\hat{a}^\dagger\right)^m & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^\dagger\hat{a}\right)^m \end{pmatrix}$$

where m are positive integers.

Basis case: For m = 1,

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]^{2} = \begin{pmatrix} \frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}\hat{a}^\dagger & 0 \\ 0 & \frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^\dagger\hat{a} \end{pmatrix}.$$

This basis case has been proven in (4.6).

Therefore, P(1) is true.

Inductive step: Consider when m = k, we suppose that P(k) is true, which means we suppose

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]^{2k} = \begin{pmatrix} (\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}\hat{a}^\dagger)^k & 0 \\ 0 & (\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^\dagger\hat{a})^k \end{pmatrix} \text{ is true.}$$

Then, for m = k+1,

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]^{2(k+1)} = \left[\frac{\Delta t}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]^{2k+2}$$

$$= \left[\frac{\Delta t}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]^{2k}\left[\frac{\Delta t}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]^{2}$$

$$= \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}\hat{a}^\dagger\right)^k & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^\dagger\hat{a}\right)^k \end{pmatrix}\begin{pmatrix} \frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}\hat{a}^\dagger & 0 \\ 0 & \frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^\dagger\hat{a} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}\hat{a}^\dagger\right)^{k+1} & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^\dagger\hat{a}\right)^{k+1} \end{pmatrix}.$$

Therefore, if we suppose that P(k) is true, then P(k+1) is true.

By mathematical induction, we can conclude that P(m) is true for all m belongs to positive integers.

As a result, it is proven that

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]^{2m} = \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}\hat{a}^\dagger\right)^m & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^\dagger\hat{a}\right)^m \end{pmatrix}, \qquad (4.9)$$

where m are positive integers.

By substituting (4.9) into (4.3), we have

$$\hat{c} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \begin{pmatrix} \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger\right)^m & 0 \\ 0 & \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger \hat{a}\right)^m \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \left(\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger}\right)^{2m} & 0 \\ 0 & \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \left(\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger \hat{a}}\right)^{2m} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger} & 0 \\ 0 & \cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger \hat{a}} \end{pmatrix}. \qquad (4.10)$$

Now,

$$\left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]^{2m+1} = \left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]^{2m} \left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]. \qquad (4.11)$$

By substituting (4.5) and (4.9) into (4.11), we have

$$\left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t(\hat{\sigma}_+\hat{a} - \hat{a}^\dagger\hat{\sigma}_-)\right]^{2m+1}$$

$$= \begin{pmatrix} \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger\right)^m & 0 \\ 0 & \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger \hat{a}\right)^m \end{pmatrix} \begin{pmatrix} \frac{\Delta \mathrm{t}}{2} & -i\lambda t\hat{a} \\ i\lambda t\hat{a}^\dagger & -\frac{\Delta \mathrm{t}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger\right)^m & -i\lambda t\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger\right)^m \hat{a} \\ i\lambda t\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger \hat{a}\right)^m \hat{a}^\dagger & -\frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger \hat{a}\right)^m \end{pmatrix}. \qquad (4.12)$$

By substituting (4.12) into (4.4), we have

$$\hat{s} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \begin{pmatrix} \frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger\right)^m & -i\lambda t\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger\right)^m \hat{a} \\ i\lambda t\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger\hat{a}\right)^m \hat{a}^\dagger & -\frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger\hat{a}\right)^m \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[\frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger\right)^m\right] & \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[-i\lambda t\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger\right)^m \hat{a}\right] \\ \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[i\lambda t\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger\hat{a}\right)^m \hat{a}^\dagger\right] & \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[-\frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger\hat{a}\right)^m\right] \end{pmatrix}.$$

Then,

$-i\hat{s}$

$$= -i \begin{pmatrix} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[\frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger\right)^m\right] & \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[-i\lambda t\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger\right)^m \hat{a}\right] \\ \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[i\lambda t\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger\hat{a}\right)^m \hat{a}^\dagger\right] & \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[-\frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger\hat{a}\right)^m\right] \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[\frac{-i\Delta \mathrm{t}}{2}\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger\right)^m\right] & \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[-\lambda t\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger\right)^m \hat{a}\right] \\ \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[\lambda t\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger\hat{a}\right)^m \hat{a}^\dagger\right] & \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[\frac{i\Delta \mathrm{t}}{2}\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger\hat{a}\right)^m\right] \end{pmatrix}$$

$= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where

$$A_{11} = \frac{-i\Delta \mathrm{t}}{2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \frac{\left(\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger}\right)^{2m+1}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger}}$$

$$= \frac{-i\Delta \mathrm{t}}{2} \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger}},$$

$$A_{12} = -\lambda t \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \frac{\left(\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger}\right)^{2m+1}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger}} \hat{a}$$

$$= -\lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger}} \hat{a},$$

$$A_{21} = \lambda t \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \frac{\left(\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger\hat{a}}\right)^{2m+1}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger\hat{a}}} \hat{a}^\dagger$$

$$= \lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger\hat{a}}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger\hat{a}}} \hat{a}^\dagger,$$

$$A_{22} = \frac{i\Delta \mathrm{t}}{2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \frac{\left(\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger\hat{a}}\right)^{2m+1}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger\hat{a}}}$$

$$= \frac{i\Delta \mathrm{t}}{2} \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger\hat{a}}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger\hat{a}}}.$$

Therefore,

$$-i\hat{s} = \begin{pmatrix} \frac{-i\Delta \mathrm{t}}{2} \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger}} & -\lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}\hat{a}^\dagger}} \hat{a} \\ \lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger\hat{a}}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger\hat{a}}} \hat{a}^\dagger & \frac{i\Delta \mathrm{t}}{2} \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger\hat{a}}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^\dagger\hat{a}}} \end{pmatrix}. \tag{4.13}$$

By substituting (4.10) and (4.13) into (4.2), we have

$$\widehat{U} = \begin{pmatrix} \cos\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}\hat{a}^\dagger} & 0 \\ 0 & \cos\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^\dagger\hat{a}} \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{-i\Delta t}{2} \frac{\sin\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}\hat{a}^\dagger}}{\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}\hat{a}^\dagger}} & -\lambda t \frac{\sin\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}\hat{a}^\dagger}}{\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}\hat{a}^\dagger}} \hat{a} \\ \lambda t \frac{\sin\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^\dagger\hat{a}}}{\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^\dagger\hat{a}}} \hat{a}^\dagger & \frac{i\Delta t}{2} \frac{\sin\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^\dagger\hat{a}}}{\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^\dagger\hat{a}}} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}\hat{a}^\dagger} - \frac{i\Delta t}{2} \frac{\sin\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}\hat{a}^\dagger}}{\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}\hat{a}^\dagger}} & -\lambda t \frac{\sin\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}\hat{a}^\dagger}}{\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}\hat{a}^\dagger}} \hat{a} \\ \lambda t \frac{\sin\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^\dagger\hat{a}}}{\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^\dagger\hat{a}}} \hat{a}^\dagger & \cos\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^\dagger\hat{a}} + \frac{i\Delta t}{2} \frac{\sin\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^\dagger\hat{a}}}{\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^\dagger\hat{a}}} \end{pmatrix}.$$

Since $\hat{n} = \hat{a}^\dagger\hat{a}$ *and* $\hat{a}\,\hat{a}^\dagger = \hat{n} + 1$,

$$\widehat{U} = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \text{, where} \qquad (4.14)$$

$$B_{11} = \cos\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2(\hat{n}+1)} - \frac{i\Delta t}{2} \frac{\sin\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2(\hat{n}+1)}}{\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2(\hat{n}+1)}},$$

$$B_{12} = -\lambda t \frac{\sin\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2(\hat{n}+1)}}{\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2(\hat{n}+1)}} \hat{a},$$

$$B_{21} = \lambda t \frac{\sin\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}}}{\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}}} \hat{a}^\dagger,$$

$$B_{22} = \cos\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}} + \frac{i\Delta t}{2} \frac{\sin\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}}}{\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}}} \ .$$

4.1.2 Derivation of Jaynes Cummings Model-type Unitary Operator Without Rotating Wave Approximation (RWA)

From previous derivation, we have the Hamiltonian in interaction picture (without Rotating Wave Approximation) as follows:

$$\hat{H}_I = \frac{\hbar\Delta}{2}\hat{\sigma}_3 - i\hbar\lambda[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]. \qquad (3.18)$$

Now, the Unitary Operator $\hat{U}$ is

$$\hat{U} = \exp\left(-\frac{i\hat{H}_I t}{\hbar}\right)$$

$$= \cos\hat{\theta} - i\sin\hat{\theta}\,, \qquad (4.15)$$

where

$$\hat{\theta} = \frac{\hat{H}_I t}{\hbar}$$

$$= \frac{\Delta t}{2}\hat{\sigma}_3 - i\lambda t[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]. \qquad (4.16)$$

Let $\hat{c} = \cos\hat{\theta}\ \ and\ \hat{s} = \sin\hat{\theta},$

then, $\hat{U} = \hat{c} - i\hat{s}.$ (4.17)

Now,

$$\hat{c} = \cos\left\{\frac{\Delta t}{2}\hat{\sigma}_3 - i\lambda t[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]\right\} \quad (4.18)$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!}\left\{\frac{\Delta t}{2}\hat{\sigma}_3 - i\lambda t[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]\right\}^{2m} \quad (4.19)$$

$$\hat{s} = \sin\left\{\frac{\Delta t}{2}\hat{\sigma}_3 - i\lambda t[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]\right\} \quad (4.20)$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left\{\frac{\Delta t}{2}\hat{\sigma}_3 - i\lambda t[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]\right\}^{2m+1}. \quad (4.21)$$

To simplify (4.19) and (4.21), consider

$$\left\{\frac{\Delta t}{2}\hat{\sigma}_3 - i\lambda t[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]\right\}^2$$

$$\begin{aligned} &= \left(\frac{\Delta t}{2}\right)^2 \hat{\sigma}_3{}^2 - \frac{i\lambda\Delta t^2}{2}\hat{\sigma}_3[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)] \\ &\quad - \frac{i\lambda\Delta t^2}{2}[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]\hat{\sigma}_3 \\ &\quad - \lambda^2 t^2[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)]^2[\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2 \end{aligned}$$

$$\begin{aligned} &= \left(\frac{\Delta t}{2}\right)^2 \hat{I} - \frac{i\lambda\Delta t^2}{2}[\hat{\sigma}_+ exp(i\omega t) - \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)] \\ &\quad - \frac{i\lambda\Delta t^2}{2}[-\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)] \\ &\quad - \lambda^2 t^2[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)]^2[\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2 \end{aligned}$$

$$= \left(\frac{\Delta t}{2}\right)^2 \hat{I} - \lambda^2 t^2[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)]^2[\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2 \quad (4.22)$$

$$
= \begin{pmatrix} \left(\frac{\Delta \mathrm{t}}{2}\right)^2 - \lambda^2 \mathrm{t}^2 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2 & 0 \\ 0 & \left(\frac{\Delta \mathrm{t}}{2}\right)^2 - \lambda^2 \mathrm{t}^2 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2 \end{pmatrix}.
$$

Next, consider

$$
\left\{\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]\right\}^3
$$

$$
= \left\{\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]\right\}^2 \left\{\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]\right\}
$$

By substituting (4.22) into it, it becomes

$$
\left\{\left(\frac{\Delta \mathrm{t}}{2}\right)^2 \hat{I} - \lambda^2 \mathrm{t}^2 [\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)]^2 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2\right\}\left\{\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]\right\}
$$

$$
\begin{aligned}
= &\left(\frac{\Delta \mathrm{t}}{2}\right)^3 \hat{\sigma}_3 - i\lambda t\left(\frac{\Delta \mathrm{t}}{2}\right)^2 [\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)] \\
&- \frac{\Delta \mathrm{t}}{2}\lambda^2 \mathrm{t}^2 [\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)]^2 \hat{\sigma}_3 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2 \\
&+ i\lambda^3 \mathrm{t}^3 [\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)]^3 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^3
\end{aligned}
$$

$$
\begin{aligned}
= &\left(\frac{\Delta \mathrm{t}}{2}\right)^3 \hat{\sigma}_3 - i\lambda t\left(\frac{\Delta \mathrm{t}}{2}\right)^2 [\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)] \\
&- \frac{\Delta \mathrm{t}}{2}\lambda^2 \mathrm{t}^2 [\hat{\sigma}_+ exp(i\omega t) \\
&+ \hat{\sigma}_- exp(-i\omega t)][-\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2 \\
&+ i\lambda^3 \mathrm{t}^3 [\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)]^3 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^3
\end{aligned}
$$

$$
\begin{aligned}
= &\left(\frac{\Delta \mathrm{t}}{2}\right)^3 \hat{\sigma}_3 - i\lambda t\left(\frac{\Delta \mathrm{t}}{2}\right)^2 [\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)] \\
&- \frac{\Delta \mathrm{t}}{2}\lambda^2 \mathrm{t}^2 \hat{\sigma}_3 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2 \\
&+ i\lambda^3 \mathrm{t}^3 [\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)]^3 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^3 \qquad (4.23)
\end{aligned}
$$

$$
= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ ,where}
$$

$$a_{11} = \left(\frac{\Delta \mathrm{t}}{2}\right)^3 - \left(\frac{\Delta \mathrm{t}}{2}\right)\lambda^2 \mathrm{t}^2[\hat{a}exp(-i\omega t) - \hat{a}^{\dagger}exp(i\omega t)]^2$$

$$= \left(\frac{\Delta \mathrm{t}}{2}\right)\left\{\left(\frac{\Delta \mathrm{t}}{2}\right)^2 - \lambda^2 \mathrm{t}^2[\hat{a}exp(-i\omega t) - \hat{a}^{\dagger}exp(i\omega t)]^2\right\},$$

$$a_{12} = -i\lambda texp(i\omega t)\left\{\left(\frac{\Delta \mathrm{t}}{2}\right)^2 [\hat{a}exp(-i\omega t) - \hat{a}^{\dagger}exp(i\omega t)] - \lambda^2 \mathrm{t}^2[\hat{a}exp(-i\omega t) - \hat{a}^{\dagger}exp(i\omega t)]^3\right\}$$

$$= -i\lambda texp(i\omega t)\left\{\left(\frac{\Delta \mathrm{t}}{2}\right)^2 - \lambda^2 \mathrm{t}^2[\hat{a}exp(-i\omega t) - \hat{a}^{\dagger}exp(i\omega t)]^2\right\}[\hat{a}exp(-i\omega t) - \hat{a}^{\dagger}exp(i\omega t)],$$

$$a_{21} = -i\lambda texp(-i\omega t)\left\{\left(\frac{\Delta \mathrm{t}}{2}\right)^2 [\hat{a}exp(-i\omega t) - \hat{a}^{\dagger}exp(i\omega t)] - \lambda^2 \mathrm{t}^2[\hat{a}exp(-i\omega t) - \hat{a}^{\dagger}exp(i\omega t)]^3\right\}$$

$$= -i\lambda texp(-i\omega t)\left\{\left(\frac{\Delta \mathrm{t}}{2}\right)^2 - \lambda^2 \mathrm{t}^2[\hat{a}exp(-i\omega t) - \hat{a}^{\dagger}exp(i\omega t)]^2\right\}[\hat{a}exp(-i\omega t) - \hat{a}^{\dagger}exp(i\omega t)],$$

$$a_{22} = -\frac{\Delta \mathrm{t}}{2}\left\{\left(\frac{\Delta \mathrm{t}}{2}\right)^2 - \lambda^2 \mathrm{t}^2[\hat{a}exp(-i\omega t) - \hat{a}^{\dagger}exp(i\omega t)]^2\right\}.$$

Next, consider

$$\left\{\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a}exp(-i\omega t) - \hat{a}^{\dagger}exp(i\omega t)]\right\}^4$$

$$= \left\{\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a}exp(-i\omega t) - \hat{a}^{\dagger}exp(i\omega t)]\right\}^3 \left\{\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a}exp(-i\omega t) - \hat{a}^{\dagger}exp(i\omega t)]\right\}$$

By substituting (4.23) into it, it becomes

$$\left\{\left(\frac{\Delta \mathrm{t}}{2}\right)^3 \hat{\sigma}_3 - i\lambda t\left(\frac{\Delta \mathrm{t}}{2}\right)^2 [\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)] \right.$$
$$- \frac{\Delta \mathrm{t}}{2}\lambda^2 \mathrm{t}^2 \hat{\sigma}_3 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2$$
$$\left. + i\lambda^3 \mathrm{t}^3 [\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)]^3 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^3 \right\} \left\{\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 \right.$$
$$\left. - i\lambda t [\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)] \right\}$$

$$= \left(\frac{\Delta \mathrm{t}}{2}\right)^4 \hat{I} - 2\left(\frac{\Delta \mathrm{t}}{2}\right)^2 \lambda^2 \mathrm{t}^2 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2$$
$$+ \lambda^4 \mathrm{t}^4 [\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)]^4 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^4 \quad (4.24)$$

$$= \left\{\left(\frac{\Delta \mathrm{t}}{2}\right)^2 \hat{I} - \lambda^2 \mathrm{t}^2 [\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)]^2 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2 \right\}^2 \quad (4.25)$$

$$= \begin{pmatrix} \left\{\left(\frac{\Delta \mathrm{t}}{2}\right)^2 - \lambda^2 \mathrm{t}^2 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2 \right\}^2 & 0 \\ 0 & \left\{\left(\frac{\Delta \mathrm{t}}{2}\right)^2 - \lambda^2 \mathrm{t}^2 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2 \right\}^2 \end{pmatrix}.$$

Next, consider

$$\left\{\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t [\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)] \right\}^5$$

$$= \left\{\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t [\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)] \right\}^4 \left\{\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 \right.$$
$$\left. - i\lambda t [\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)] \right\}$$

By substituting (4.24) into it, it becomes

$$\left\{\left(\frac{\Delta \mathrm{t}}{2}\right)^4 \hat{I} - 2\left(\frac{\Delta \mathrm{t}}{2}\right)^2 \lambda^2 \mathrm{t}^2 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2 \right.$$
$$\left. + \lambda^4 \mathrm{t}^4 [\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)]^4 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^4 \right\} \left\{\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 \right.$$
$$\left. - i\lambda t [\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)] \right\}$$

$$= \left(\frac{\Delta \mathrm{t}}{2}\right)^5 \hat{\sigma}_3 - i\lambda t\left(\frac{\Delta \mathrm{t}}{2}\right)^4 [\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]$$
$$- 2\left(\frac{\Delta \mathrm{t}}{2}\right)^3 \lambda^2 \mathrm{t}^2 \hat{\sigma}_3 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2$$
$$+ 2i\lambda^3 \mathrm{t}^3 \left(\frac{\Delta \mathrm{t}}{2}\right)^2 [\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^3$$
$$+ \frac{\Delta \mathrm{t}}{2} \lambda^4 \mathrm{t}^4 [\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)]^4 \hat{\sigma}_3 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^4$$
$$- i\lambda^5 \mathrm{t}^5 [\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)]^5 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^5 \quad (4.26)$$

$= \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$,where

$$b_{11} = \left(\frac{\Delta \mathrm{t}}{2}\right)^5 - 2\left(\frac{\Delta \mathrm{t}}{2}\right)^3 \lambda^2 \mathrm{t}^2 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2$$
$$+ \frac{\Delta \mathrm{t}}{2} \lambda^4 \mathrm{t}^4 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^4$$

$$= \left(\frac{\Delta \mathrm{t}}{2}\right)\left\{\left(\frac{\Delta \mathrm{t}}{2}\right)^4 - 2\left(\frac{\Delta \mathrm{t}}{2}\right)^2 \lambda^2 \mathrm{t}^2 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2 \right.$$
$$\left. + \lambda^4 \mathrm{t}^4 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^4 \right\}$$

$$= \left(\frac{\Delta \mathrm{t}}{2}\right)\left\{\left(\frac{\Delta \mathrm{t}}{2}\right)^2 - \lambda^2 \mathrm{t}^2 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2 \right\}^2,$$

$$b_{12} = -i\lambda t\left(\frac{\Delta \mathrm{t}}{2}\right)^4 exp(i\omega t)[\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]$$
$$+ 2i\lambda^3 \mathrm{t}^3 \left(\frac{\Delta \mathrm{t}}{2}\right)^2 exp(i\omega t)[\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^3$$
$$- i\lambda^5 \mathrm{t}^5 exp(i\omega t)[\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^5$$

$$= -i\lambda t exp(i\omega t)\left\{\left(\frac{\Delta \mathrm{t}}{2}\right)^4 - 2\lambda^2 \mathrm{t}^2 \left(\frac{\Delta \mathrm{t}}{2}\right)^2 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2 \right.$$
$$\left. + \lambda^4 \mathrm{t}^4 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^4 \right\}[\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]$$

$$= -i\lambda t exp(i\omega t)\left\{\left(\frac{\Delta \mathrm{t}}{2}\right)^2 - \lambda^2 \mathrm{t}^2 [\hat{a} exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2 \right\}^2 [\hat{a} exp(-i\omega t)$$
$$- \hat{a}^\dagger exp(i\omega t)],$$

$$b_{21} = -i\lambda t\left(\frac{\Delta t}{2}\right)^4 exp(-i\omega t)[\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]$$
$$+ 2i\lambda^3 t^3 \left(\frac{\Delta t}{2}\right)^2 exp(-i\omega t)[\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^3$$
$$- i\lambda^5 t^5 exp(-i\omega t)[\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^5$$

$$= -i\lambda t exp(-i\omega t)\left\{\left(\frac{\Delta t}{2}\right)^4 - 2\lambda^2 t^2 \left(\frac{\Delta t}{2}\right)^2 [\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2\right.$$
$$\left. + \lambda^4 t^4 [\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^4\right\}[\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]$$

$$= -i\lambda t exp(-i\omega t)\left\{\left(\frac{\Delta t}{2}\right)^2 - \lambda^2 t^2 [\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2\right\}^2 [\hat{a}exp(-i\omega t)$$
$$- \hat{a}^\dagger exp(i\omega t)],$$

$$b_{22} = -\left(\frac{\Delta t}{2}\right)^5 + 2\left(\frac{\Delta t}{2}\right)^3 \lambda^2 t^2 [\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2$$
$$- \frac{\Delta t}{2}\lambda^4 t^4 [\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^4$$

$$= -\left(\frac{\Delta t}{2}\right)\left\{\left(\frac{\Delta t}{2}\right)^4 - 2\left(\frac{\Delta t}{2}\right)^2 \lambda^2 t^2 [\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2\right.$$
$$\left. + \lambda^4 t^4 [\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^4\right\}$$

$$= -\left(\frac{\Delta t}{2}\right)\left\{\left(\frac{\Delta t}{2}\right)^2 - \lambda^2 t^2 [\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2\right\}^2.$$

From the calculations above, it can be deduced that for any even power expansion 2m,

$$\left\{\frac{\Delta t}{2}\hat{\sigma}_3 - i\lambda t[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]\right\}^{2m}$$

$$= \begin{pmatrix} \left\{\left(\frac{\Delta t}{2}\right)^2 - \lambda^2 t^2 [\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2\right\}^m & 0 \\ 0 & \left\{\left(\frac{\Delta t}{2}\right)^2 - \lambda^2 t^2 [\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2\right\}^m \end{pmatrix}$$

where m is a positive integer. (4.27)

For any odd power expansion (2m+1),

$$\left\{\frac{\Delta\mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]\right\}^{2m+1}$$

$$= \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}, \text{ where} \tag{4.28}$$

$$d_{11} = \left(\frac{\Delta\mathrm{t}}{2}\right)\left\{\left(\frac{\Delta\mathrm{t}}{2}\right)^2 - \lambda^2\mathrm{t}^2[\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2\right\}^m,$$

$$d_{12} = -i\lambda t exp(i\omega t)\left\{\left(\frac{\Delta\mathrm{t}}{2}\right)^2 - \lambda^2\mathrm{t}^2[\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2\right\}^m [\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)],$$

$$d_{21} = -i\lambda t exp(-i\omega t)\left\{\left(\frac{\Delta\mathrm{t}}{2}\right)^2 - \lambda^2\mathrm{t}^2[\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2\right\}^m [\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)],$$

$$d_{22} = -\left(\frac{\Delta\mathrm{t}}{2}\right)\left\{\left(\frac{\Delta\mathrm{t}}{2}\right)^2 - \lambda^2\mathrm{t}^2[\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2\right\}^m.$$

By substituting (4.27) into (4.19), (4.19) becomes

$$\hat{c} = \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}\left\{\left(\frac{\Delta\mathrm{t}}{2}\right)^2 - \lambda^2\mathrm{t}^2[\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2\right\}^m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \left\{\cos\sqrt{\left\{\left(\frac{\Delta\mathrm{t}}{2}\right)^2 - \lambda^2\mathrm{t}^2[\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2\right\}}\right\}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.29}$$

Then, by substituting (4.28) into (4.21), (4.21) becomes

$$\hat{s} = \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}\left\{\frac{\Delta\mathrm{t}}{2}\hat{\sigma}_3 - i\lambda t[\hat{\sigma}_+ exp(i\omega t) + \hat{\sigma}_- exp(-i\omega t)][\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]\right\}^{2m+1}$$

$$= \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}, \text{ where} \tag{4.30}$$

$$s_{11} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \left(\frac{\Delta t}{2}\right) \left\{ \left(\frac{\Delta t}{2}\right)^2 - \lambda^2 t^2 [\hat{a} exp(-i\omega t) - \hat{a}^{\dagger} exp(i\omega t)]^2 \right\}^m$$

$$= \left(\frac{\Delta t}{2}\right) \frac{\sin \sqrt{\left\{ \left(\frac{\Delta t}{2}\right)^2 - \lambda^2 t^2 [\hat{a} exp(-i\omega t) - \hat{a}^{\dagger} exp(i\omega t)]^2 \right\}}}{\sqrt{\left\{ \left(\frac{\Delta t}{2}\right)^2 - \lambda^2 t^2 [\hat{a} exp(-i\omega t) - \hat{a}^{\dagger} exp(i\omega t)]^2 \right\}}},$$

$$s_{12} = -i\lambda t exp(i\omega t) \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \left\{ \left(\frac{\Delta t}{2}\right)^2 - \lambda^2 t^2 [\hat{a} exp(-i\omega t) - \hat{a}^{\dagger} exp(i\omega t)]^2 \right\}^m [\hat{a} exp(-i\omega t) - \hat{a}^{\dagger} exp(i\omega t)]$$

$$= -i\lambda t exp(i\omega t) \frac{\sin \sqrt{\left\{ \left(\frac{\Delta t}{2}\right)^2 - \lambda^2 t^2 [\hat{a} exp(-i\omega t) - \hat{a}^{\dagger} exp(i\omega t)]^2 \right\}}}{\sqrt{\left\{ \left(\frac{\Delta t}{2}\right)^2 - \lambda^2 t^2 [\hat{a} exp(-i\omega t) - \hat{a}^{\dagger} exp(i\omega t)]^2 \right\}}} [\hat{a} exp(-i\omega t) - \hat{a}^{\dagger} exp(i\omega t)],$$

$$s_{21} = -i\lambda t exp(-i\omega t) \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \left\{ \left(\frac{\Delta t}{2}\right)^2 - \lambda^2 t^2 [\hat{a} exp(-i\omega t) - \hat{a}^{\dagger} exp(i\omega t)]^2 \right\}^m [\hat{a} exp(-i\omega t) - \hat{a}^{\dagger} exp(i\omega t)]$$

$$= -i\lambda t exp(-i\omega t) \frac{\sin \sqrt{\left\{ \left(\frac{\Delta t}{2}\right)^2 - \lambda^2 t^2 [\hat{a} exp(-i\omega t) - \hat{a}^{\dagger} exp(i\omega t)]^2 \right\}}}{\sqrt{\left\{ \left(\frac{\Delta t}{2}\right)^2 - \lambda^2 t^2 [\hat{a} exp(-i\omega t) - \hat{a}^{\dagger} exp(i\omega t)]^2 \right\}}} [\hat{a} exp(-i\omega t) - \hat{a}^{\dagger} exp(i\omega t)],$$

$$s_{22} = -\left(\frac{\Delta t}{2}\right) \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \left\{ \left(\frac{\Delta t}{2}\right)^2 - \lambda^2 t^2 [\hat{a} exp(-i\omega t) - \hat{a}^{\dagger} exp(i\omega t)]^2 \right\}^m$$

$$= -\left(\frac{\Delta t}{2}\right)\frac{\sin\sqrt{\left\{\left(\frac{\Delta t}{2}\right)^2 - \lambda^2 t^2[\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2\right\}}}{\sqrt{\left\{\left(\frac{\Delta t}{2}\right)^2 - \lambda^2 t^2[\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2\right\}}}.$$

Finally, by substituting (4.29) and (4.30) into (4.17), (4.17) becomes

$$\hat{U} = \hat{c} - i\hat{s}$$

$$= \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}, \text{ where}$$

$$U_{11} = \cos\sqrt{\left\{\left(\frac{\Delta t}{2}\right)^2 - \lambda^2 t^2[\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2\right\}}$$
$$- \left(\frac{i\Delta t}{2}\right)\frac{\sin\sqrt{\left\{\left(\frac{\Delta t}{2}\right)^2 - \lambda^2 t^2[\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2\right\}}}{\sqrt{\left\{\left(\frac{\Delta t}{2}\right)^2 - \lambda^2 t^2[\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2\right\}}},$$

$$U_{12} = -\lambda t exp(i\omega t)\frac{\sin\sqrt{\left\{\left(\frac{\Delta t}{2}\right)^2 - \lambda^2 t^2[\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2\right\}}}{\sqrt{\left\{\left(\frac{\Delta t}{2}\right)^2 - \lambda^2 t^2[\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2\right\}}}[\hat{a}exp(-i\omega t)$$
$$- \hat{a}^\dagger exp(i\omega t)]$$

$$U_{21} = -\lambda t exp(-i\omega t)\frac{\sin\sqrt{\left\{\left(\frac{\Delta t}{2}\right)^2 - \lambda^2 t^2[\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2\right\}}}{\sqrt{\left\{\left(\frac{\Delta t}{2}\right)^2 - \lambda^2 t^2[\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2\right\}}}[\hat{a}exp(-i\omega t)$$
$$- \hat{a}^\dagger exp(i\omega t)]$$

$$U_{22} = \cos\sqrt{\left\{\left(\frac{\Delta \mathrm{t}}{2}\right)^2 - \lambda^2\mathrm{t}^2[\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2\right\}}$$
$$+\left(\frac{i\Delta \mathrm{t}}{2}\right)\frac{\sin\sqrt{\left\{\left(\frac{\Delta \mathrm{t}}{2}\right)^2 - \lambda^2\mathrm{t}^2[\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2\right\}}}{\sqrt{\left\{\left(\frac{\Delta \mathrm{t}}{2}\right)^2 - \lambda^2\mathrm{t}^2[\hat{a}exp(-i\omega t) - \hat{a}^\dagger exp(i\omega t)]^2\right\}}}$$

4.2 Probability Function for Single-photon Jaynes-Cummings Model

Let $|\varphi(t)\rangle$ be the quantum state of the system containing 1 two-level atom and photons at time t. Suppose initially the atom is in the ground state. Then, the initial state is

$$|\varphi(0)\rangle = \sum_{n=0}^{\infty} c_{1,n}(0)|1\rangle\,|n\rangle$$
$$= \sum_{n=0}^{\infty} a_n|1\rangle\,|n\rangle, \text{ where } a_n = c_{1,n}(0). \tag{4.31}$$

From the Second Postulate of Quantum Mechanics, the evolution of the quantum state is described by the unitary operation of the initial state. Then,

$$|\varphi(t)\rangle = \hat{\mathrm{U}}(\mathrm{t})|\varphi(0)\rangle\,. \tag{4.32}$$

From (4.14), $\hat{\mathrm{U}}(\mathrm{t})$ in its outer product form is

$$\hat{\mathrm{U}}(\mathrm{t}) = B_{11}|2\rangle\langle 2| + B_{12}|2\rangle\langle 1| + B_{21}|1\rangle\langle 2| + B_{22}|1\rangle\langle 1|. \tag{4.33}$$

By substituting (4.31) and (4.33) into (4.32),

$$|\varphi(t)\rangle = \sum_{n=0}^{\infty} a_n\{|2\rangle \otimes [B_{12}|n\rangle] + |1\rangle \otimes [B_{22}|n\rangle]\} \tag{4.34}$$

where

$$B_{12} = -\lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2(\hat{n}+1)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2(\hat{n}+1)}} \hat{a},$$

$$B_{22} = \cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}} + \frac{i\Delta \mathrm{t}}{2} \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}}}.$$

To simplify $B_{12}|n\rangle$,

$$B_{12}|n\rangle = -\lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2(\hat{n}+1)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2(\hat{n}+1)}} \hat{a}|n\rangle$$

$$= -\lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2(\hat{n}+1)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2(\hat{n}+1)}} \sqrt{n}\,|n-1\rangle\,. \tag{4.35}$$

From (4.35), consider

$$\frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2(\hat{n}+1)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2(\hat{n}+1)}} |n-1\rangle = \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2(\hat{n}+1)\right)^m}{(2m+1)!} |n-1\rangle. \tag{4.36}$$

To simplify (4.36), when m=1,

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2(\hat{n}+1)\right)|n-1\rangle = \frac{\Delta^2 \mathrm{t}^2}{4}|n-1\rangle + \lambda^2 \mathrm{t}^2(\hat{n}+1)|n-1\rangle$$

$$= \frac{\Delta^2 \mathrm{t}^2}{4}|n-1\rangle + \lambda^2 t^2 n|n-1\rangle$$

$$= \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n\right)|n-1\rangle. \tag{4.37}$$

When m=2,

$$\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(\hat{n}+1)\right)^2|n-1\rangle=\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(\hat{n}+1)\right)\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(\hat{n}+1)\right)|n-1\rangle. \tag{4.38}$$

By substituting (4.37) into (4.38), RHS of (4.38) becomes

$$\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(\hat{n}+1)\right)\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2 n\right)|n-1\rangle$$

$$=\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2 n\right)\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(\hat{n}+1)\right)|n-1\rangle. \tag{4.39}$$

Again, by substituting (4.37) into (4.39), RHS of (4.39) becomes

$$\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2 n\right)\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2 n\right)|n-1\rangle.$$

So,

$$\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(\hat{n}+1)\right)^2|n-1\rangle=\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2 n\right)^2|n-1\rangle. \tag{4.40}$$

Now, consider when m=3,

$$\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(\hat{n}+1)\right)^3|n-1\rangle$$

$$=\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(\hat{n}+1)\right)\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(\hat{n}+1)\right)^2|n-1\rangle. \tag{4.41}$$

By substituting (4.40) into (4.41), RHS of (4.41) becomes

$$\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(\hat{n}+1)\right)\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2 n\right)^2|n-1\rangle$$

$$=\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2 n\right)^2\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(\hat{n}+1)\right)|n-1\rangle. \tag{4.42}$$

By substituting (4.37) into (4.42), RHS of (4.42) becomes

$$\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n\right)^2\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n\right)|n-1\rangle=\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n\right)^3|n-1\rangle.$$

So,

$$\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2(\hat{n}+1)\right)^3|n-1\rangle=\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n\right)^3|n-1\rangle\,.$$

By using the similar method as before, it can be deduced that for any integer m,

$$\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2(\hat{n}+1)\right)^m|n-1\rangle=\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n\right)^m|n-1\rangle. \tag{4.43}$$

By substituting (4.43) into (4.36), (4.36) becomes

$$\frac{\sin\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2(\hat{n}+1)}}{\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2(\hat{n}+1)}}|n-1\rangle=\sum_{m=0}^{\infty}(-1)^m\frac{\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n\right)^m}{(2m+1)!}|n-1\rangle$$

$$=\frac{\sin\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n}}{\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n}}|n-1\rangle. \tag{4.44}$$

By substituting (4.44) into (4.35), (4.35) becomes

$$B_{12}|n\rangle=-\lambda t\,\frac{\sin\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n}}{\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n}}\sqrt{n}\,|n-1\rangle\,. \tag{4.45}$$

Next, to simplify $B_{22}|n\rangle$,

$$B_{22}|n\rangle=\left(\cos\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}}+\frac{i\Delta t}{2}\frac{\sin\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}}}{\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}}}\right)|n\rangle$$

$$=\left(\cos\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}}\right)|n\rangle+\left(\frac{i\Delta t}{2}\frac{\sin\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}}}{\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}}}\right)|n\rangle. \tag{4.46}$$

From (4.46), consider

$$\left(\cos\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}}\right)|n\rangle = \sum_{m=0}^{\infty}(-1)^m \frac{\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}\right)^m}{(2m)!}|n\rangle\,. \tag{4.47}$$

To simplify (4.47), when m=1,

$$\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}\right)|n\rangle = \frac{\Delta^2 t^2}{4}|n\rangle + \lambda^2 t^2 n|n\rangle$$

$$= \left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n\right)|n\rangle\,. \tag{4.48}$$

When m=2,

$$\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}\right)^2 |n\rangle = \left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}\right)\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}\right)|n\rangle. \tag{4.49}$$

By substituting (4.48) into (4.49), RHS of (4.49) becomes

$$\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}\right)\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n\right)|n\rangle = \left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n\right)\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}\right)|n\rangle. \tag{4.50}$$

Again, by substituting (4.48) into (4.50), RHS of (4.50) becomes

$$\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n\right)\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n\right)|n\rangle.$$

So,

$$\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}\right)^2 |n\rangle = \left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n\right)^2 |n\rangle. \tag{4.51}$$

Now, consider when m=3,

$$\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}\right)^3 |n\rangle = \left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}\right)\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}\right)^2 |n\rangle. \tag{4.52}$$

By substituting (4.51) into (4.52), RHS of (4.52) becomes

$$\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}\right)\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n\right)^2 |n\rangle = \left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n\right)^2\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}\right)|n\rangle. \tag{4.53}$$

By substituting (4.48) into (4.53), RHS of (4.53) becomes

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2 n\right)^2\left(\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2 n\right)|n\rangle=\left(\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2 n\right)^3|n\rangle.$$

So,

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2 \hat{n}\right)^3|n\rangle=\left(\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2 n\right)^3|n\rangle . \qquad (4.54)$$

By using the similar method as before, it can be deduced that for any integer m,

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2 \hat{n}\right)^m|n\rangle=\left(\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2 n\right)^m|n\rangle . \qquad (4.55)$$

By substituting (4.55) into (4.47), (4.47) becomes

$$\begin{aligned}\left(\cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2 \hat{n}}\right)|n\rangle&=\sum_{m=0}^{\infty}(-1)^m\frac{\left(\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2 n\right)^m}{(2m)!}|n\rangle\\&=\left(\cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2 n}\right)|n\rangle . \qquad (4.56)\end{aligned}$$

From (4.46) also, consider

$$\frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2 \hat{n}}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2 \hat{n}}}|n\rangle=\sum_{m=0}^{\infty}(-1)^m\frac{\left(\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2 \hat{n}\right)^m}{(2m+1)!}|n\rangle.$$

By substituting (4.55) into it, it becomes

$$\sum_{m=0}^{\infty}(-1)^m\frac{\left(\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2 n\right)^m}{(2m+1)!}|n\rangle=\frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2 n}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2 n}}|n\rangle. \qquad (4.57)$$

By substituting (4.56) and (4.57) into (4.46), (4.46) becomes

$$B_{22}|n\rangle=\left(\cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2 \hat{n}}+\frac{i\Delta \mathrm{t}}{2}\frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2 \hat{n}}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2 \hat{n}}}\right)|n\rangle$$

$$= \left(\cos\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n} + \frac{i\Delta t}{2} \frac{\sin\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n}}{\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n}} \right) |n\rangle. \tag{4.58}$$

Finally, by substituting (4.45) and (4.58) into (4.34), (4.34) becomes

$$|\varphi(t)\rangle = \sum_{n=0}^{\infty} \{a_n r|2\rangle|n-1\rangle + a_n s|1\rangle|n\rangle\}$$

$$= \sum_{n=0}^{\infty} \{C_{2,n-1}(t)|2\rangle|n-1\rangle + C_{1,n}(t)|1\rangle|n\rangle\},$$

where

$$r = -\lambda t \frac{\sin\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n}}{\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n}} \sqrt{n},$$

$$s = \cos\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n} + \frac{i\Delta t}{2} \frac{\sin\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n}}{\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n}}.$$

Let $P_1(t)$ be the probability in which the atom is in the ground state. Then,

$$P_1(t) = \sum_{n=0}^{\infty} \left|C_{1,n}(t)\right|^2 = \sum_{n=0}^{\infty} \{|a_n|^2 |s|^2\}$$

$$= \sum_{n=0}^{\infty} \left\{ |a_n|^2 \left[\cos^2\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n} + \frac{\Delta^2 t^2}{4} \frac{\sin^2\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n}}{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n} \right] \right\}. \tag{4.59}$$

4.3.1 Derivation of Two-photon Jaynes-Cummings Model Unitary Operator

From Methodology 3.3, we have the Hamiltonian in interaction picture as follows:

$$\hat{H}_I = \frac{\hbar\Delta}{2}\hat{\sigma}_3 - i\hbar\lambda[\hat{\sigma}_+\hat{a}^2 - \hat{\sigma}_-\hat{a}^{\dagger 2}] \qquad (3.22)$$

$$= \frac{\hbar\Delta}{2}\hat{\sigma}_3 + i\hbar\lambda(\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^2) \qquad (4.60)$$

Now, the Unitary Operator $\hat{U}$ is

$$\hat{U} = \exp\left(-\frac{i\hat{H}_I t}{\hbar}\right)$$

$$= \exp\ \{-i\left[\tfrac{\hbar\Delta}{2}\hat{\sigma}_3 + i\hbar\lambda(\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^2)\right] t\,/\hbar\}$$

$$= \exp\{-i\left[\tfrac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^2)\right]\}$$

$$= \cos\hat{\theta} - i\sin\hat{\theta},\ \text{where } \hat{\theta} = \tfrac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^2). \qquad (4.61)$$

Let $\hat{c} = \cos\hat{\theta}$ *and* $\hat{s} = \sin\hat{\theta}$,

then, $\hat{U} = \hat{c} - i\hat{s}$. Now,

$$\hat{c} = \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^2)\right]^{2m}, \qquad (4.62)$$

$$\hat{s} = \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^2)\right]^{2m+1}. \qquad (4.63)$$

Now,

$$\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^2) = \frac{\Delta t}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + i\lambda t\left[\hat{a}^{\dagger 2}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} - \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\hat{a}^2\right]$$

$$= \begin{pmatrix}\frac{\Delta t}{2} & -i\lambda t\hat{a}^2\\ i\lambda t\hat{a}^{\dagger 2} & -\frac{\Delta t}{2}\end{pmatrix}. \qquad (4.64)$$

Consider the even power expansion, we have

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^2)\right]^2 = \left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^2)\right]\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^2)\right]$$

$$= \begin{pmatrix} \frac{\Delta t}{2} & -i\lambda t\hat{a}^2 \\ i\lambda t\hat{a}^{\dagger 2} & -\frac{\Delta t}{2} \end{pmatrix}\begin{pmatrix} \frac{\Delta t}{2} & -i\lambda t\hat{a}^2 \\ i\lambda t\hat{a}^{\dagger 2} & -\frac{\Delta t}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^2\hat{a}^{\dagger 2} & 0 \\ 0 & \frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 2}\hat{a}^2 \end{pmatrix}, \qquad (4.65)$$

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^2)\right]^4 = \left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^2)\right]^2\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^2)\right]^2$$

$$= \begin{pmatrix} \frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^2\hat{a}^{\dagger 2} & 0 \\ 0 & \frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 2}\hat{a}^2 \end{pmatrix}\begin{pmatrix} \frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^2\hat{a}^{\dagger 2} & 0 \\ 0 & \frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 2}\hat{a}^2 \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^2\hat{a}^{\dagger 2}\right)^2 & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 2}\hat{a}^2\right)^2 \end{pmatrix} \qquad (4.66)$$

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^2)\right]^6 = \left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^2)\right]^4\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^2)\right]^2$$

$$= \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^2\hat{a}^{\dagger 2}\right)^2 & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 2}\hat{a}^2\right)^2 \end{pmatrix}\begin{pmatrix} \frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^2\hat{a}^{\dagger 2} & 0 \\ 0 & \frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 2}\hat{a}^2 \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^2 \hat{a}^{\dagger 2}\right)^3 & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 2} \hat{a}^2\right)^3 \end{pmatrix} \quad (4.67)$$

From here, we can actually deduce that for general even power 2m,

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^2)\right]^{2m} = \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^2 \hat{a}^{\dagger 2}\right)^m & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 2} \hat{a}^2\right)^m \end{pmatrix}$$

This could be proven by using mathematical induction as follows:

Let P(m) be the statement that

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^2)\right]^{2m} = \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^2 \hat{a}^{\dagger 2}\right)^m & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 2} \hat{a}^2\right)^m \end{pmatrix},$$

where m are positive integers.

Basis case: For m = 1,

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^2)\right]^{2} = \begin{pmatrix} \frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^2 \hat{a}^{\dagger 2} & 0 \\ 0 & \frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 2} \hat{a}^2 \end{pmatrix}.$$

This basis case has been proven in (4.65).

Therefore, P(1) is true.

Inductive step: Consider when m = k, we suppose that P(k) is true, which means we suppose

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^2)\right]^{2k} = \begin{pmatrix} (\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^2 \hat{a}^{\dagger 2})^k & 0 \\ 0 & (\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 2} \hat{a}^2)^k \end{pmatrix} \text{ is true.}$$

Then, for m = k+1,

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^2)\right]^{2(k+1)} = \left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^2)\right]^{2k+2}$$

$$= \left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^2)\right]^{2k}\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^2)\right]^{2}$$

$$= \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^2 \hat{a}^{\dagger 2}\right)^k & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 2} \hat{a}^2\right)^k \end{pmatrix} \begin{pmatrix} \frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^2 \hat{a}^{\dagger 2} & 0 \\ 0 & \frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 2} \hat{a}^2 \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^2 \hat{a}^{\dagger 2}\right)^{k+1} & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 2} \hat{a}^2\right)^{k+1} \end{pmatrix}$$

Therefore, if we suppose that P(k) is true, then P(k+1) is true.

By mathematical induction, we can conclude that P(m) is true for all m belongs to positive integers.

As a result, it is proven that

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 2}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^2)\right]^{2m} = \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^2 \hat{a}^{\dagger 2}\right)^m & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 2} \hat{a}^2\right)^m \end{pmatrix} \quad (4.68)$$

where m are positive integers.

By substituting (4.68) into (4.62), we have

$$\hat{c} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^2 \hat{a}^{\dagger 2}\right)^m & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 2} \hat{a}^2\right)^m \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \left(\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^2 \hat{a}^{\dagger 2}} \right)^{2m} & 0 \\ 0 & \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \left(\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2} \right)^{2m} \end{pmatrix}$$

$$= \begin{pmatrix} \cos \sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^2 \hat{a}^{\dagger 2}} & 0 \\ 0 & \cos \sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2} \end{pmatrix}. \qquad (4.69)$$

Now,

$$\left[\frac{\Delta \mathrm{t}}{2} \hat{\sigma}_3 + i\lambda t (\hat{a}^{\dagger 2} \hat{\sigma}_- - \hat{\sigma}_+ \hat{a}^2) \right]^{2m+1}$$
$$= \left[\frac{\Delta \mathrm{t}}{2} \hat{\sigma}_3 + i\lambda t (\hat{a}^{\dagger 2} \hat{\sigma}_- - \hat{\sigma}_+ \hat{a}^2) \right]^{2m} \left[\frac{\Delta \mathrm{t}}{2} \hat{\sigma}_3 + i\lambda t (\hat{a}^{\dagger 2} \hat{\sigma}_- - \hat{\sigma}_+ \hat{a}^2) \right]. \qquad (4.70)$$

By substituting (4.64) and (4.68) into (4.70), (4.70) becomes

$$\left[\frac{\Delta \mathrm{t}}{2} \hat{\sigma}_3 + i\lambda t (\hat{a}^{\dagger 2} \hat{\sigma}_- - \hat{\sigma}_+ \hat{a}^2) \right]^{2m+1}$$
$$= \begin{pmatrix} \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^2 \hat{a}^{\dagger 2} \right)^m & 0 \\ 0 & \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2 \right)^m \end{pmatrix} \begin{pmatrix} \frac{\Delta \mathrm{t}}{2} & -i\lambda t \hat{a}^2 \\ i\lambda t \hat{a}^{\dagger 2} & -\frac{\Delta \mathrm{t}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\Delta \mathrm{t}}{2} \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^2 \hat{a}^{\dagger 2} \right)^m & -i\lambda t \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^2 \hat{a}^{\dagger 2} \right)^m \hat{a}^2 \\ i\lambda t \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2 \right)^m \hat{a}^{\dagger 2} & -\frac{\Delta \mathrm{t}}{2} \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2 \right)^m \end{pmatrix}. \qquad (4.71)$$

By substituting (4.71) into (4.63), we have

$$\hat{s} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \begin{pmatrix} \frac{\Delta \mathrm{t}}{2} \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^2 \hat{a}^{\dagger 2} \right)^m & -i\lambda t \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^2 \hat{a}^{\dagger 2} \right)^m \hat{a}^2 \\ i\lambda t \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2 \right)^m \hat{a}^{\dagger 2} & -\frac{\Delta \mathrm{t}}{2} \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2 \right)^m \end{pmatrix}.$$

$$= \begin{pmatrix} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[\frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^2\hat{a}^{\dagger 2}\right)^m\right] & \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[-i\lambda t\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^2\hat{a}^{\dagger 2}\right)^m \hat{a}^2\right] \\ \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[i\lambda t\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^{\dagger 2}\hat{a}^2\right)^m \hat{a}^{\dagger 2}\right] & \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[-\frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^{\dagger 2}\hat{a}^2\right)^m\right] \end{pmatrix}.$$

Then, $-i\hat{s}$

$$= -i \begin{pmatrix} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[\frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^2\hat{a}^{\dagger 2}\right)^m\right] & \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[-i\lambda t\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^2\hat{a}^{\dagger 2}\right)^m \hat{a}^2\right] \\ \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[i\lambda t\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^{\dagger 2}\hat{a}^2\right)^m \hat{a}^{\dagger 2}\right] & \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[-\frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^{\dagger 2}\hat{a}^2\right)^m\right] \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[\frac{-i\Delta \mathrm{t}}{2}\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^2\hat{a}^{\dagger 2}\right)^m\right] & \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[-\lambda t\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^2\hat{a}^{\dagger 2}\right)^m \hat{a}^2\right] \\ \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[\lambda t\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^{\dagger 2}\hat{a}^2\right)^m \hat{a}^{\dagger 2}\right] & \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[\frac{i\Delta \mathrm{t}}{2}\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^{\dagger 2}\hat{a}^2\right)^m\right] \end{pmatrix}$$

$= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where

$$A_{11} = \frac{-i\Delta \mathrm{t}}{2}\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\frac{\left(\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^2\hat{a}^{\dagger 2}}\right)^{2m+1}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^2\hat{a}^{\dagger 2}}}$$

$$= \frac{-i\Delta \mathrm{t}}{2}\frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^2\hat{a}^{\dagger 2}}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^2\hat{a}^{\dagger 2}}},$$

$$A_{12} = -\lambda t\sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\frac{\left(\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^2\hat{a}^{\dagger 2}}\right)^{2m+1}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^2\hat{a}^{\dagger 2}}}\hat{a}^2$$

$$= -\lambda t\frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^2\hat{a}^{\dagger 2}}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^2\hat{a}^{\dagger 2}}}\hat{a}^2,$$

$$A_{21} = \lambda t \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \frac{\left(\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2}\right)^{2m+1}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2}} \hat{a}^{\dagger 2}$$

$$= \lambda t \frac{\sin \sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2}} \hat{a}^{\dagger 2},$$

$$A_{22} = \frac{i\Delta \mathrm{t}}{2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \frac{\left(\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2}\right)^{2m+1}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2}}$$

$$= \frac{i\Delta \mathrm{t}}{2} \frac{\sin \sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2}}.$$

Therefore,

$$-i\hat{s} = \begin{pmatrix} \frac{-i\Delta \mathrm{t}}{2} \frac{\sin \sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^2 \hat{a}^{\dagger 2}}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^2 \hat{a}^{\dagger 2}}} & -\lambda t \frac{\sin \sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^2 \hat{a}^{\dagger 2}}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^2 \hat{a}^{\dagger 2}}} \hat{a}^2 \\ \lambda t \frac{\sin \sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2}} \hat{a}^{\dagger 2} & \frac{i\Delta \mathrm{t}}{2} \frac{\sin \sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2}} \end{pmatrix}. \qquad (4.72)$$

By substituting (4.69) and (4.72) into (4.61), we have

$$\hat{U} = \begin{pmatrix} \cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^2 \hat{a}^{\dagger 2}} & 0 \\ 0 & \cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2} \end{pmatrix}$$

$$+ \begin{pmatrix} \frac{-i\Delta \mathrm{t}}{2} \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^2 \hat{a}^{\dagger 2}}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^2 \hat{a}^{\dagger 2}}} & -\lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^2 \hat{a}^{\dagger 2}}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^2 \hat{a}^{\dagger 2}}} \hat{a}^2 \\ \lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2}} \hat{a}^{\dagger 2} & \frac{i\Delta \mathrm{t}}{2} \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2}} \end{pmatrix}$$

$$= \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \text{ where} \tag{4.73}$$

$$B_{11} = \cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^2 \hat{a}^{\dagger 2}} - \frac{i\Delta \mathrm{t}}{2} \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^2 \hat{a}^{\dagger 2}}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^2 \hat{a}^{\dagger 2}}},$$

$$B_{12} = -\lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^2 \hat{a}^{\dagger 2}}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^2 \hat{a}^{\dagger 2}}} \hat{a}^2,$$

$$B_{21} = \lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2}} \hat{a}^{\dagger 2},$$

$$B_{22} = \cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2} + \frac{i\Delta \mathrm{t}}{2} \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 2} \hat{a}^2}}.$$

Given $\hat{n} = \hat{a}^{\dagger}\hat{a}$ *and* $\hat{a}\,\hat{a}^{\dagger} = \hat{n} + 1$, now,

$$\hat{a}^2\hat{a}^{\dagger 2} = \hat{a}\,\hat{a}\hat{a}^{\dagger}\hat{a}^{\dagger}$$

$$= \hat{a}(\hat{a}^{\dagger}\hat{a} + 1)\hat{a}^{\dagger}$$

$$= \hat{a}\hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger} + \hat{a}\hat{a}^{\dagger}$$

$$= (\hat{n} + 1)(\hat{n} + 1) + (\hat{n} + 1)$$

$$= (\hat{n} + 1)(\hat{n} + 1 + 1)$$

$$= (\hat{n} + 1)(\hat{n} + 2), \tag{4.74}$$

$$\hat{a}^{\dagger 2}\hat{a}^2 = \hat{a}^{\dagger}\hat{a}^{\dagger}\hat{a}\,\hat{a}$$

$$= \hat{a}^{\dagger}(\hat{a}\hat{a}^{\dagger} - 1)\hat{a}$$

$$= \hat{a}^{\dagger}\hat{a}\hat{a}^{\dagger}\hat{a} - \hat{a}^{\dagger}\hat{a}$$

$$= \hat{n}\hat{n} - \hat{n}$$

$$= \hat{n}(\hat{n} - 1). \tag{4.75}$$

Therefore, by substituting (4.74) and (4.75) into (4.73), (4.73) becomes

$$\hat{U} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \text{ where} \tag{4.76}$$

$$C_{11} = \cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2(\hat{n} + 1)(\hat{n} + 2)} - \frac{i\Delta \mathrm{t}}{2}\frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2(\hat{n} + 1)(\hat{n} + 2)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2(\hat{n} + 1)(\hat{n} + 2)}},$$

$$C_{12} = -\lambda t\frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2(\hat{n} + 1)(\hat{n} + 2)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2(\hat{n} + 1)(\hat{n} + 2)}}\hat{a}^2,$$

$$C_{21} = \lambda t\frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2\hat{n}(\hat{n} - 1)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2\hat{n}(\hat{n} - 1)}}\hat{a}^{\dagger 2},$$

$$C_{22} = \cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)} + \frac{i\Delta \mathrm{t}}{2} \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)}}.$$

Probability Function for Two-photon Jaynes-Cummings Model

Let $|\varphi(t)\rangle$ be the quantum state of the system containing 1 two-level atom and photons at time t.

Suppose initially the atom is in the ground state. Then, the initial state is

$$|\varphi(0)\rangle = \sum_{n=0}^{\infty} c_{1,n}(0)|1\rangle\,|n\rangle$$

$$= \sum_{n=0}^{\infty} a_n|1\rangle\,|n\rangle, \;\; where \;\; a_n = c_{1,n}(0). \tag{4.77}$$

Then,

$$|\varphi(t)\rangle \; = \hat{\mathrm{U}}(\mathrm{t})|\varphi(0)\rangle. \tag{4.78}$$

From (4.76), $\hat{\mathrm{U}}(\mathrm{t})$ in its outer product form is

$$\hat{\mathrm{U}}(\mathrm{t}) = C_{11}|2\rangle\langle 2| + C_{12}|2\rangle\langle 1| + C_{21}|1\rangle\langle 2| + C_{22}|1\rangle\langle 1|. \tag{4.79}$$

By substituting (4.77) and (4.79) into (4.78),

$$|\varphi(t)\rangle \; = \sum_{n=0}^{\infty} a_n\{|2\rangle \otimes [C_{12}|n\rangle] + |1\rangle \otimes [C_{22}|n\rangle]\}, \tag{4.80}$$

where

$$C_{12} = -\lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)}} \hat{a}^2,$$

$$C_{22} = \cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)} + \frac{i\Delta \mathrm{t}}{2} \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)}}.$$

To simplify $C_{12}|n\rangle$,

$$C_{12}|n\rangle = -\lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2(\hat{n}+1)(\hat{n}+2)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2(\hat{n}+1)(\hat{n}+2)}} \hat{a}^2|n\rangle$$

$$= -\lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2(\hat{n}+1)(\hat{n}+2)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2(\hat{n}+1)(\hat{n}+2)}} \sqrt{n}\sqrt{n-1}\,|n-2\rangle. \quad (4.81)$$

From (4.81), consider

$$\frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2(\hat{n}+1)(\hat{n}+2)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2(\hat{n}+1)(\hat{n}+2)}} |n-2\rangle$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2(\hat{n}+1)(\hat{n}+2)\right)^m}{(2m+1)!} |n-2\rangle. \quad (4.82)$$

To simplify (4.82), when m=1,

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2(\hat{n}+1)(\hat{n}+2)\right)|n-2\rangle = \frac{\Delta^2 \mathrm{t}^2}{4}|n-2\rangle + \lambda^2 \mathrm{t}^2(\hat{n}+1)(\hat{n}+2)|n-2\rangle$$

$$= \frac{\Delta^2 \mathrm{t}^2}{4}|n-2\rangle + \lambda^2 \mathrm{t}^2(\hat{n}+1)n|n-2\rangle$$

$$= \frac{\Delta^2 \mathrm{t}^2}{4}|n-2\rangle + \lambda^2 t^2 n(n-1)|n-2\rangle$$

$$= \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 t^2 n(n-1)\right)|n-2\rangle. \quad (4.83)$$

When m=2,

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2(\hat{n}+1)(\hat{n}+2)\right)^2 |n-2\rangle$$

$$= \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)\right)\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)\right)|n-2\rangle. \tag{4.84}$$

By substituting (4.83) into (4.84), RHS of (4.84) becomes

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)\right)\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 t^2 n(n-1)\right)|n-2\rangle$$

$$= \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 t^2 n(n-1)\right)\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)\right)|n-2\rangle. \tag{4.85}$$

Again, by substituting (4.83) into (4.85), RHS of (4.85) becomes

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 t^2 n(n-1)\right)\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 t^2 n(n-1)\right)|n-2\rangle.$$

So,

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)\right)^2 |n-2\rangle = \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 t^2 n(n-1)\right)^2 |n-2\rangle. \tag{4.86}$$

Now, consider when m=3,

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)\right)^3 |n-2\rangle$$

$$= \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)\right)\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)\right)^2 |n-2\rangle. \tag{4.87}$$

By substituting (4.86) into (4.87), RHS of (4.87) becomes

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)\right)\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 t^2 n(n-1)\right)^2 |n-2\rangle$$

$$= \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 t^2 n(n-1)\right)^2 \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)\right)|n-2\rangle. \tag{4.88}$$

By substituting (4.83) into (4.88), RHS of (4.88) becomes

$$\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n(n-1)\right)^2\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n(n-1)\right)|n-2\rangle.$$

So,

$$\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2(\hat{n}+1)(\hat{n}+2)\right)^3|n-2\rangle=\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n(n-1)\right)^3|n-2\rangle.$$

By using the similar method as before, it can be deduced that for any integer m,

$$\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2(\hat{n}+1)(\hat{n}+2)\right)^m|n-2\rangle=\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n(n-1)\right)^m|n-2\rangle. \tag{4.89}$$

By substituting (4.89) into (4.82), (4.82) becomes

$$\frac{\sin\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2(\hat{n}+1)(\hat{n}+2)}}{\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2(\hat{n}+1)(\hat{n}+2)}}|n-2\rangle=\sum_{m=0}^{\infty}(-1)^m\frac{\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n(n-1)\right)^m}{(2m+1)!}|n-2\rangle$$

$$=\frac{\sin\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n(n-1)}}{\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n(n-1)}}|n-2\rangle. \tag{4.90}$$

By substituting (4.90) into (4.81), (4.81) becomes

$$C_{12}|n\rangle=-\lambda t\frac{\sin\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n(n-1)}}{\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n(n-1)}}\sqrt{n}\sqrt{n-1}\,|n-2\rangle. \tag{4.91}$$

Next, to simplify $C_{22}|n\rangle$,

$$C_{22}|n\rangle=\left(\cos\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}(\hat{n}-1)}+\frac{i\Delta t}{2}\frac{\sin\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}(\hat{n}-1)}}{\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}(\hat{n}-1)}}\right)|n\rangle$$

$$= \left(\cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)}\right)|n\rangle + \left(\frac{i\Delta \mathrm{t}}{2}\frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)}}\right)|n\rangle. \quad (4.92)$$

From (4.92), consider

$$\left(\cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)}\right)|n\rangle = \sum_{m=0}^{\infty}(-1)^m \frac{\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)\right)^m}{(2m)!}|n\rangle. \quad (4.93)$$

To simplify (4.93), when m=1,

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)\right)|n\rangle = \frac{\Delta^2 \mathrm{t}^2}{4}|n\rangle + \lambda^2 \mathrm{t}^2 \hat{n}(n-1)|n\rangle$$

$$= \frac{\Delta^2 \mathrm{t}^2}{4}|n\rangle + \lambda^2 \mathrm{t}^2 (n-1)n|n\rangle$$

$$= \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (n-1)n\right)|n\rangle. \quad (4.94)$$

When m=2,

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)\right)^2 |n\rangle = \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)\right)\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)\right)|n\rangle. \quad (4.95)$$

By substituting (4.94) into (4.95), RHS of (4.95) becomes

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)\right)\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (n-1)n\right)|n\rangle$$

$$= \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (n-1)n\right)\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)\right)|n\rangle. \quad (4.96)$$

Again, by substituting (4.94) into (4.96), RHS of (4.96) becomes

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (n-1)n\right)\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (n-1)n\right)|n\rangle.$$

So,

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)\right)^2 |n\rangle = \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (n-1)n\right)^2 |n\rangle. \tag{4.97}$$

Now, consider when m=3,

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)\right)^3 |n\rangle = \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)\right)\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)\right)^2 |n\rangle. \tag{4.98}$$

By substituting (4.97) into (4.98), RHS of (4.98) becomes

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)\right)\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (n-1)n\right)^2 |n\rangle$$

$$= \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (n-1)n\right)^2 \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)\right)|n\rangle. \tag{4.99}$$

By substituting (4.94) into (4.99), RHS of (4.99) becomes

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (n-1)n\right)^2 \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (n-1)n\right)|n\rangle = \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (n-1)n\right)^3 |n\rangle.$$

So,

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)\right)^3 |n\rangle = \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (n-1)n\right)^3 |n\rangle. \tag{4.100}$$

By using the similar method as before, it can be deduced that for any integer m,

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)\right)^m |n\rangle = \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (n-1)n\right)^m |n\rangle$$

$$= \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1)\right)^m |n\rangle. \tag{4.101}$$

By substituting (4.101) into (4.93), (4.93) becomes

$$\left(\cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)}\right)|n\rangle = \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1)\right)^m}{(2m)!}|n\rangle$$

$$= \left(\cos \sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1)} \right) |n\rangle. \tag{4.102}$$

From (4.92) also, consider

$$\frac{\sin \sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)}} |n\rangle = \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1) \right)^m}{(2m+1)!} |n\rangle.$$

By substituting (4.101) into it, it becomes

$$\sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1) \right)^m}{(2m+1)!} |n\rangle = \frac{\sin \sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1)}} |n\rangle. \tag{4.103}$$

By substituting (4.102) and (4.103) into (4.92), (4.92) becomes

$$C_{22}|n\rangle = \left(\cos \sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)} + \frac{i\Delta \mathrm{t}}{2} \frac{\sin \sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}(\hat{n}-1)}} \right) |n\rangle.$$

$$C_{22}|n\rangle = \left(\cos \sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1)} + \frac{i\Delta \mathrm{t}}{2} \frac{\sin \sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1)}} \right) |n\rangle. \tag{4.104}$$

Finally, by substituting (4.91) and (4.104) into (4.80), (4.80) becomes

$$|\varphi(t)\rangle = \textstyle\sum_{n=0}^{\infty} \{a_n r|2\rangle|n-2\rangle + a_n s|1\rangle|n\rangle\}$$

$$= \sum_{n=0}^{\infty} \{C_{2,n-1}(t)|2\rangle|n-2\rangle + C_{1,n}(t)|1\rangle|n\rangle\},$$

where

$$r = -\lambda t \frac{\sin \sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 t^2 n(n-1)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 t^2 n(n-1)}} \sqrt{n} \sqrt{n-1}\,,$$

$$s = \cos\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n(n-1)} + \frac{i\Delta t}{2}\frac{\sin\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n(n-1)}}{\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n(n-1)}}.$$

Let $P_1(t)$ be the probability in which the atom is in the ground state. Then,

$$P_1(t) = \sum_{n=0}^{\infty} \left|C_{1,n}(t)\right|^2$$

$$= \sum_{n=0}^{\infty} \{|a_n|^2 |s|^2\}$$

$$= \sum_{n=0}^{\infty} \left\{|a_n|^2 \left[\cos^2\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n(n-1)} + \frac{\Delta^2 t^2}{4}\frac{\sin^2\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n(n-1)}}{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n(n-1)}\right]\right\}. \quad (4.105)$$

4.3.2 Derivation of Three-photon Jaynes-Cummings Model Unitary Operator

From Methodology (3.3), we have the Hamiltonian in interaction picture as follows:

$$\hat{H}_I = \frac{\hbar\Delta}{2}\hat{\sigma}_3 - i\hbar\lambda[\hat{\sigma}_+\hat{a}^3 - \hat{\sigma}_-\hat{a}^{\dagger 3}] \quad (3.23)$$

$$= \frac{\hbar\Delta}{2}\hat{\sigma}_3 + i\hbar\lambda(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3). \quad (4.106)$$

Now, the Unitary Operator $\hat{U}$ is

$$\hat{U} = \exp\left(-\frac{i\hat{H}_I t}{\hbar}\right)$$

$$= \exp\left\{-i\left[\frac{\hbar\Delta}{2}\hat{\sigma}_3 + i\hbar\lambda(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]t/\hbar\right\}$$

$$= \exp\left\{-i\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]\right\}$$

$$= \cos\hat{\theta} - i\sin\hat{\theta}, \text{ where } \hat{\theta} = \frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3). \quad (4.107)$$

Let $\hat{c} = \cos\hat{\theta}$ *and* $\hat{s} = \sin\hat{\theta}$,

then, $\hat{U} = \hat{c} - i\hat{s}$. Now,

$$\hat{c} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!}\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]^{2m} \quad (4.108)$$

$$\hat{s} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]^{2m+1}. \quad (4.109)$$

Now,

$$\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3) = \frac{\Delta t}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + i\lambda t\left[\hat{a}^{\dagger 3}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} - \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\hat{a}^3\right]$$

$$= \begin{pmatrix}\frac{\Delta t}{2} & -i\lambda t\hat{a}^3\\ i\lambda t\hat{a}^{\dagger 3} & -\frac{\Delta t}{2}\end{pmatrix}. \quad (4.110)$$

Consider the even power expansion, we have

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]^2 = \left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]$$

$$= \begin{pmatrix}\frac{\Delta t}{2} & -i\lambda t\hat{a}^3\\ i\lambda t\hat{a}^{\dagger 3} & -\frac{\Delta t}{2}\end{pmatrix}\begin{pmatrix}\frac{\Delta t}{2} & -i\lambda t\hat{a}^3\\ i\lambda t\hat{a}^{\dagger 3} & -\frac{\Delta t}{2}\end{pmatrix}$$

$$= \begin{pmatrix}\frac{\Delta^2 t^2}{4} + \lambda^2 t^2\hat{a}^3\hat{a}^{\dagger 3} & 0\\ 0 & \frac{\Delta^2 t^2}{4} + \lambda^2 t^2\hat{a}^{\dagger 3}\hat{a}^3\end{pmatrix}, \quad (4.111)$$

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]^4 = \left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]^2\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]^2$$

$$= \begin{pmatrix}\frac{\Delta^2 t^2}{4} + \lambda^2 t^2\hat{a}^3\hat{a}^{\dagger 3} & 0\\ 0 & \frac{\Delta^2 t^2}{4} + \lambda^2 t^2\hat{a}^{\dagger 3}\hat{a}^3\end{pmatrix}\begin{pmatrix}\frac{\Delta^2 t^2}{4} + \lambda^2 t^2\hat{a}^3\hat{a}^{\dagger 3} & 0\\ 0 & \frac{\Delta^2 t^2}{4} + \lambda^2 t^2\hat{a}^{\dagger 3}\hat{a}^3\end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^3 \hat{a}^{\dagger 3}\right)^2 & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 3} \hat{a}^3\right)^2 \end{pmatrix}, \quad (4.112)$$

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]^6 = \left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]^4 \left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]^2$$

$$= \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^3 \hat{a}^{\dagger 3}\right)^2 & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 3} \hat{a}^3\right)^2 \end{pmatrix} \begin{pmatrix} \frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^3 \hat{a}^{\dagger 3} & 0 \\ 0 & \frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 3} \hat{a}^3 \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^3 \hat{a}^{\dagger 3}\right)^3 & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 3} \hat{a}^3\right)^3 \end{pmatrix}. \quad (4.113)$$

From here, we can actually deduce that for general even power 2m,

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]^{2m} = \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^3 \hat{a}^{\dagger 3}\right)^m & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 3} \hat{a}^3\right)^m \end{pmatrix}$$

However, this could be proven by using mathematical induction as follows:

Let P(m) be the statement that

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]^{2m} = \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^3 \hat{a}^{\dagger 3}\right)^m & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 3} \hat{a}^3\right)^m \end{pmatrix},$$

where m are positive integers.

Basis case: For m = 1,

$$\left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]^2 = \begin{pmatrix} \frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3} & 0 \\ 0 & \frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^{\dagger 3}\hat{a}^3 \end{pmatrix}.$$

This basis case has been proven in (4.111).

Therefore, P(1) is true.

Inductive step: Consider when m = k, we suppose that P(k) is true, which means we suppose

$$\left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]^{2k} = \begin{pmatrix} (\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3})^k & 0 \\ 0 & (\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^{\dagger 3}\hat{a}^3)^k \end{pmatrix} \text{ is true.}$$

Then, for m = k+1,

$$\left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]^{2(k+1)} = \left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]^{2k+2}$$

$$= \left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]^{2k}\left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]^2$$

$$= \begin{pmatrix} \left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}\right)^k & 0 \\ 0 & \left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^{\dagger 3}\hat{a}^3\right)^k \end{pmatrix}\begin{pmatrix} \frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3} & 0 \\ 0 & \frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^{\dagger 3}\hat{a}^3 \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}\right)^{k+1} & 0 \\ 0 & \left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^{\dagger 3}\hat{a}^3\right)^{k+1} \end{pmatrix}.$$

Therefore, if we suppose that P(k) is true, then P(k+1) is true.

By mathematical induction, we can conclude that P(m) is true for all m belongs to positive integers.

As a result, it is proven that

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]^{2m} = \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^3\hat{a}^{\dagger 3}\right)^m & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 3}\hat{a}^3\right)^m \end{pmatrix} \quad (4.114)$$

where m are positive integers.

By substituting (4.114) into (4.108), we have

$$\hat{c} = \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}\begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^3\hat{a}^{\dagger 3}\right)^m & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 3}\hat{a}^3\right)^m \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}\left(\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^3\hat{a}^{\dagger 3}}\right)^{2m} & 0 \\ 0 & \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}\left(\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 3}\hat{a}^3}\right)^{2m} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^3\hat{a}^{\dagger 3}} & 0 \\ 0 & \cos\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 3}\hat{a}^3} \end{pmatrix}. \quad (4.115)$$

Now,

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]^{2m+1}$$
$$= \left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]^{2m}\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]. \quad (4.116)$$

By substituting (4.110) and (4.114) into (4.116), (4.116) becomes

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger 3}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^3)\right]^{2m+1}$$
$$= \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^3\hat{a}^{\dagger 3}\right)^m & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger 3}\hat{a}^3\right)^m \end{pmatrix}\begin{pmatrix} \frac{\Delta t}{2} & -i\lambda t\hat{a}^3 \\ i\lambda t\hat{a}^{\dagger 3} & -\frac{\Delta t}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}\right)^m & -i\lambda t\left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}\right)^m \hat{a}^3 \\ i\lambda t\left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^{\dagger 3}\hat{a}^3\right)^m \hat{a}^{\dagger 3} & -\frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^{\dagger 3}\hat{a}^3\right)^m \end{pmatrix}. \quad (4.117)$$

By substituting (4.117) into (4.109), we have

$$\hat{s} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \begin{pmatrix} \frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}\right)^m & -i\lambda t\left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}\right)^m \hat{a}^3 \\ i\lambda t\left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^{\dagger 3}\hat{a}^3\right)^m \hat{a}^{\dagger 3} & -\frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^{\dagger 3}\hat{a}^3\right)^m \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[\frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}\right)^m\right] & \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[-i\lambda t\left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}\right)^m \hat{a}^3\right] \\ \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[i\lambda t\left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^{\dagger 3}\hat{a}^3\right)^m \hat{a}^{\dagger 3}\right] & \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[-\frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^{\dagger 3}\hat{a}^3\right)^m\right] \end{pmatrix}.$$

Then, $-i\hat{s}$

$$= -i \begin{pmatrix} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[\frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}\right)^m\right] & \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[-i\lambda t\left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}\right)^m \hat{a}^3\right] \\ \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[i\lambda t\left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^{\dagger 3}\hat{a}^3\right)^m \hat{a}^{\dagger 3}\right] & \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[-\frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^{\dagger 3}\hat{a}^3\right)^m\right] \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[\frac{-i\Delta \mathrm{t}}{2}\left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}\right)^m\right] & \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[-\lambda t\left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}\right)^m \hat{a}^3\right] \\ \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[\lambda t\left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^{\dagger 3}\hat{a}^3\right)^m \hat{a}^{\dagger 3}\right] & \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[\frac{i\Delta \mathrm{t}}{2}\left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^{\dagger 3}\hat{a}^3\right)^m\right] \end{pmatrix}$$

$= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where

$$A_{11} = \frac{-i\Delta \mathrm{t}}{2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \frac{\left(\sqrt{\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}}\right)^{2m+1}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}}}$$

$$= \frac{-i\Delta \mathrm{t}}{2} \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^3 \hat{a}^{\dagger 3}}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^3 \hat{a}^{\dagger 3}}},$$

$$A_{12} = -\lambda t \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \frac{\left(\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^3 \hat{a}^{\dagger 3}}\right)^{2m+1}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^3 \hat{a}^{\dagger 3}}} \hat{a}^3$$

$$= -\lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^3 \hat{a}^{\dagger 3}}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^3 \hat{a}^{\dagger 3}}} \hat{a}^3,$$

$$A_{21} = \lambda t \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \frac{\left(\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 3} \hat{a}^3}\right)^{2m+1}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 3} \hat{a}^3}} \hat{a}^{\dagger 3}$$

$$= \lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 3} \hat{a}^3}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 3} \hat{a}^3}} \hat{a}^{\dagger 3},$$

$$A_{22} = \frac{i\Delta \mathrm{t}}{2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \frac{\left(\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 3} \hat{a}^3}\right)^{2m+1}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 3} \hat{a}^3}}$$

$$= \frac{i\Delta \mathrm{t}}{2} \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 3} \hat{a}^3}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 3} \hat{a}^3}}$$

Therefore,

$$-i\hat{s}=\begin{pmatrix} \frac{-i\Delta\mathrm{t}}{2}\frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}}} & -\lambda t\frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}}}\hat{a}^3 \\ \lambda t\frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^{\dagger 3}\hat{a}^3}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^{\dagger 3}\hat{a}^3}}\hat{a}^{\dagger 3} & \frac{i\Delta\mathrm{t}}{2}\frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^{\dagger 3}\hat{a}^3}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^{\dagger 3}\hat{a}^3}} \end{pmatrix}. \quad (4.118)$$

By substituting (4.115) and (4.118) into (4.107), we have

$$\hat{U}=\begin{pmatrix} \cos\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}} & 0 \\ 0 & \cos\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^{\dagger 3}\hat{a}^3} \end{pmatrix}$$

$$+\begin{pmatrix} \frac{-i\Delta\mathrm{t}}{2}\frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}}} & -\lambda t\frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}}}\hat{a}^3 \\ \lambda t\frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^{\dagger 3}\hat{a}^3}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^{\dagger 3}\hat{a}^3}}\hat{a}^{\dagger 3} & \frac{i\Delta\mathrm{t}}{2}\frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^{\dagger 3}\hat{a}^3}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^{\dagger 3}\hat{a}^3}} \end{pmatrix}$$

$$=\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \text{where} \quad (4.119)$$

$$B_{11}=\cos\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}}-\frac{i\Delta\mathrm{t}}{2}\frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}}},$$

$$B_{12}=-\lambda t\frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^3\hat{a}^{\dagger 3}}}\hat{a}^3,$$

$$B_{21} = \lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 3}\hat{a}^3}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 3}\hat{a}^3}} \hat{a}^{\dagger 3},$$

$$B_{22} = \cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 3}\hat{a}^3} + \frac{i\Delta \mathrm{t}}{2} \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 3}\hat{a}^3}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger 3}\hat{a}^3}}.$$

Given $\hat{n} = \hat{a}^\dagger \hat{a}$ *and* $\hat{a}\,\hat{a}^\dagger = \hat{n} + 1$, now,

$$\begin{aligned}
\hat{a}^3 \hat{a}^{\dagger 3} &= \hat{a}\,\hat{a}^2 \hat{a}^{\dagger 2} \hat{a}^\dagger \\
&= \hat{a}\,(\hat{n}+1)(\hat{n}+2)\hat{a}^\dagger \quad \text{, (since } \hat{a}^2\hat{a}^{\dagger 2} = (\hat{n}+1)(\hat{n}+2).\text{)} \\
&= \hat{a}(\hat{a}^\dagger\hat{a}+1)(\hat{a}^\dagger\hat{a}+2)\hat{a}^\dagger \\
&= \hat{a}(\hat{a}^\dagger\hat{a}+1)(\hat{a}^\dagger\hat{a}\hat{a}^\dagger+2\hat{a}^\dagger) \\
&= \hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger + 2\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger + \hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger + 2\hat{a}\hat{a}^\dagger \\
&= (\hat{n}+1)(\hat{n}+1)(\hat{n}+1) + 3(\hat{n}+1)(\hat{n}+1) + 2(\hat{n}+1) \\
&= (\hat{n}+1)[(\hat{n}+1)^2 + 3(\hat{n}+1) + 2] \\
&= (\hat{n}+1)(\hat{n}+2)(\hat{n}+3),
\end{aligned} \tag{4.120}$$

$$\begin{aligned}
\hat{a}^{\dagger 3}\hat{a}^3 &= \hat{a}^\dagger \hat{a}^{\dagger 2}\hat{a}^2\,\hat{a} \\
&= \hat{a}^\dagger \hat{n}(\hat{n}-1)\hat{a} \\
&= \hat{a}^\dagger(\hat{a}^\dagger\hat{a})(\hat{a}^\dagger\hat{a}-1)\hat{a} \\
&= \hat{a}^\dagger(\hat{a}\hat{a}^\dagger-1)(\hat{a}\hat{a}^\dagger-2)\hat{a} \\
&= \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} - 2\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} + 2\hat{a}^\dagger\hat{a} \\
&= \hat{n}\hat{n}\hat{n} - 3\hat{n}\hat{n} + 2\hat{n} \\
&= \hat{n}^3 - 3\hat{n}^2 + 2\hat{n} \\
&= \hat{n}\,(\hat{n}-1)(\hat{n}-2).
\end{aligned} \tag{4.121}$$

Therefore, by substituting (4.120) and (4.121) into (4.119), (4.119) becomes

$$\hat{U} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \tag{4.122}$$

where

$$C_{11} = \cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)(\hat{n}+3)} - \frac{i\Delta \mathrm{t}}{2} \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)(\hat{n}+3)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)(\hat{n}+3)}},$$

$$C_{12} = -\lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)(\hat{n}+3)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)(\hat{n}+3)}} \hat{a}^3,$$

$$C_{21} = \lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)}} \hat{a}^{\dagger 3},$$

$$C_{22} = \cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)} + \frac{i\Delta \mathrm{t}}{2} \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)}}.$$

Probability Function for Three-photon Jaynes-Cummings Model

Let $|\varphi(t)\rangle$ be the quantum state of the system containing 1 two-level atom and photons time t.

Suppose initially the atom is in the ground state. Then, the initial state is

$$|\varphi(0)\rangle = \sum_{n=0}^{\infty} c_{1,n}(0)|1\rangle\,|n\rangle$$

$$= \sum_{n=0}^{\infty} a_n|1\rangle\,|n\rangle, \;\; where \;\; a_n = c_{1,n}(0). \tag{4.123}$$

Then,

$$|\varphi(t)\rangle = \hat{\mathrm{U}}(\mathrm{t})|\varphi(0)\rangle. \tag{4.124}$$

From (4.122), $\hat{\mathrm{U}}(\mathrm{t})$ in its outer product form is

$$\hat{\mathrm{U}}(\mathrm{t}) = C_{11}|2\rangle\langle 2| + C_{12}|2\rangle\langle 1| + C_{21}|1\rangle\langle 2| + C_{22}|1\rangle\langle 1|. \tag{4.125}$$

By substituting (4.123) and (4.125) into (4.124),

$$|\varphi(t)\rangle = \sum_{n=0}^{\infty} a_n\{|2\rangle \otimes [C_{12}|n\rangle] + |1\rangle \otimes [C_{22}|n\rangle]\}, \tag{4.126}$$

where

$$C_{12} = -\lambda t \frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)}}\hat{a}^3,$$

$$C_{22} = \cos\sqrt{\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{n}\,(\hat{n}-1)(\hat{n}-2)} + \frac{i\Delta\mathrm{t}}{2}\frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{n}\,(\hat{n}-1)(\hat{n}-2)}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{n}\,(\hat{n}-1)(\hat{n}-2)}}.$$

To simplify $C_{12}|n\rangle$,

$$\begin{aligned} C_{12}|n\rangle &= -\lambda t \frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)}}\hat{a}^3|n\rangle \\ &= -\lambda t \frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)}}\sqrt{n}\,\sqrt{n-1}\,\sqrt{n-2}\,|n-3\rangle. \end{aligned} \tag{4.127}$$

From (4.127), consider

$$\begin{aligned} &\frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)}}|n-3\rangle \\ &= \sum_{m=0}^{\infty}(-1)^m \frac{\left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\right)^m}{(2m+1)!}|n-3\rangle. \end{aligned} \tag{4.128}$$

To simplify (4.128), when m=1,

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\right)|n-3\rangle$$

$$= \frac{\Delta^2 \mathrm{t}^2}{4}|n-3\rangle + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)(\hat{n}+3)|n-3\rangle$$

$$= \frac{\Delta^2 \mathrm{t}^2}{4}|n-3\rangle + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)n|n-3\rangle$$

$$= \frac{\Delta^2 \mathrm{t}^2}{4}|n-3\rangle + \lambda^2 t^2 n(n-1)(\hat{n}+1)|n-3\rangle$$

$$= \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 t^2 n(n-1)(n-2)\right)|n-3\rangle. \quad (4.129)$$

When m=2,

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\right)^2 |n-3\rangle$$

$$= \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\right)\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\right)|n-3\rangle. \quad (4.130)$$

By substituting (4.129) into (4.130), RHS of (4.130) becomes

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\right)\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 t^2 n(n-1)(n-2)\right)|n-3\rangle$$

$$= \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 t^2 n(n-1)(n-2)\right)\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\right)|n-3\rangle. \quad (4.131)$$

Again, by substituting (4.129) into (4.131), RHS of (4.131) becomes

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 t^2 n(n-1)(n-2)\right)\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 t^2 n(n-1)(n-2)\right)|n-3\rangle.$$

So,

$$\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\right)^2|n-3\rangle=\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2t^2n(n-1)(n-2)\right)^2|n-3\rangle. \tag{4.132}$$

Now, consider when m=3,

$$\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\right)^3|n-3\rangle$$

$$=\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\right)\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\right)^2|n-3\rangle. \tag{4.133}$$

By substituting (4.132) into (4.133), RHS of (4.133) becomes

$$\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\right)\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2t^2n(n-1)(n-2)\right)^2|n-3\rangle$$

$$=\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2t^2n(n-1)(n-2)\right)^2\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\right)|n-3\rangle. \tag{4.134}$$

By substituting (4.129) into (4.134), RHS of (4.134) becomes

$$\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2t^2n(n-1)(n-2)\right)^2\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2t^2n(n-1)(n-2)\right)|n-3\rangle.$$

So,

$$\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\right)^3|n-3\rangle=\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2t^2n(n-1)(n-2)\right)^3|n-3\rangle.$$

By using the similar method as before, it can be deduced that for any integer m,

$$\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\right)^m|n-3\rangle=\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2t^2n(n-1)(n-2)\right)^m|n-3\rangle. \tag{4.135}$$

By substituting (4.135) into (4.128), (4.128) becomes

$$\frac{\sin\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)}}{\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)}}|n-3\rangle$$

$$=\sum_{m=0}^{\infty}(-1)^m\frac{\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n(n-1)(n-2)\right)^m}{(2m+1)!}|n-3\rangle$$

$$=\frac{\sin\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n(n-1)(n-2)}}{\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n(n-1)(n-2)}}|n-3\rangle. \qquad (4.136)$$

By substituting (4.136) into (4.127), (4.127) becomes

$$C_{12}|n\rangle=-\lambda t\frac{\sin\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n(n-1)(n-2)}}{\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n(n-1)(n-2)}}\sqrt{n}\,\sqrt{n-1}\,\sqrt{n-2}|n-3\rangle. \qquad (4.137)$$

Next, to simplify $C_{22}|n\rangle$,

$$C_{22}|n\rangle=\left(\cos\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}\,(\hat{n}-1)(\hat{n}-2)}\right.$$
$$\left.+\frac{i\Delta t}{2}\frac{\sin\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}\,(\hat{n}-1)(\hat{n}-2)}}{\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}\,(\hat{n}-1)(\hat{n}-2)}}\right)|n\rangle$$

$$=\left(\cos\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}\,(\hat{n}-1)(\hat{n}-2)}\right)|n\rangle$$
$$+\left(\frac{i\Delta t}{2}\frac{\sin\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}\,(\hat{n}-1)(\hat{n}-2)}}{\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{n}\,(\hat{n}-1)(\hat{n}-2)}}\right)|n\rangle. \qquad (4.138)$$

From (4.138), consider

$$\left(\cos\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)}\right)|n\rangle=\sum_{m=0}^{\infty}(-1)^m\frac{\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\right)^m}{(2m)!}|n\rangle. \tag{4.139}$$

To simplify (4.139), when m=1,

$$\begin{aligned}\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\right)|n\rangle &= \frac{\Delta^2 t^2}{4}|n\rangle+\lambda^2 t^2 \hat{n}\,(\hat{n}-1)(n-2)|n\rangle\\ &= \frac{\Delta^2 t^2}{4}|n\rangle+\lambda^2 t^2 (n-2)\hat{n}\,(\hat{n}-1)|n\rangle\\ &= \frac{\Delta^2 t^2}{4}|n\rangle+\lambda^2 t^2 (n-2)\hat{n}\,(n-1)|n\rangle\\ &= \frac{\Delta^2 t^2}{4}|n\rangle+\lambda^2 t^2 (n-2)(n-1)\hat{n}\,|n\rangle\\ &= \frac{\Delta^2 t^2}{4}|n\rangle+\lambda^2 t^2 (n-2)(n-1)n\,|n\rangle\\ &= \left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 (n-2)(n-1)n\right)|n\rangle. \end{aligned} \tag{4.140}$$

When m=2,

$$\begin{aligned}&\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\right)^2|n\rangle\\ &=\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\right)\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\right)|n\rangle. \end{aligned} \tag{4.141}$$

By substituting (4.140) into (4.141), RHS of (4.141) becomes

$$\begin{aligned}&\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\right)\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 (n-2)(n-1)n\right)|n\rangle\\ &=\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 (n-2)(n-1)n\right)\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\right)|n\rangle. \end{aligned} \tag{4.142}$$

Again, by substituting (4.140) into (4.142), RHS of (4.142) becomes

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (n-2)(n-1)n\right)\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (n-2)(n-1)n\right)|n\rangle.$$

So,

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\right)^2 |n\rangle = \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (n-2)(n-1)n\right)^2 |n\rangle. \qquad (4.143)$$

Now, consider when m=3,

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\right)^3 |n\rangle$$

$$= \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\right)\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\right)^2 |n\rangle. \qquad (4.144)$$

By substituting (4.143) into (4.144), RHS of (4.144) becomes

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\right)\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (n-2)(n-1)n\right)^2 |n\rangle$$

$$= \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (n-2)(n-1)n\right)^2 \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\right)|n\rangle. \qquad (4.145)$$

By substituting (4.140) into (4.145), RHS of (4.145) becomes

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (n-2)(n-1)n\right)^2 \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (n-2)(n-1)n\right)|n\rangle$$

$$= \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (n-2)(n-1)n\right)^3 |n\rangle$$

So,

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\right)^3 |n\rangle = \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (n-2)(n-1)n\right)^3 |n\rangle. \qquad (4.146)$$

By using the similar method as before, it can be deduced that for any integer m,

$$\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{n}\,(\hat{n}-1)(\hat{n}-2)\right)^m|n\rangle=\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(n-2)(n-1)n\right)^m|n\rangle$$

$$=\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2n(n-1)(n-2)\right)^m|n\rangle. \qquad (4.147)$$

By substituting (4.147) into (4.139), (4.139) becomes

$$\left(\cos\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{n}\,(\hat{n}-1)(\hat{n}-2)}\right)|n\rangle=\sum_{m=0}^{\infty}(-1)^m\frac{\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2n(n-1)(n-2)\right)^m}{(2m)!}|n\rangle$$

$$=\left(\cos\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2n(n-1)(n-2)}\right)|n\rangle. \quad (4.148)$$

From (4.138) also, consider

$$\frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{n}\,(\hat{n}-1)(\hat{n}-2)}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{n}\,(\hat{n}-1)(\hat{n}-2)}}|n\rangle=\sum_{m=0}^{\infty}(-1)^m\frac{\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{n}\,(\hat{n}-1)(\hat{n}-2)\right)^m}{(2m+1)!}|n\rangle$$

By substituting (4.147) into it, it becomes

$$\sum_{m=0}^{\infty}(-1)^m\frac{\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2n(n-1)(n-2)\right)^m}{(2m+1)!}|n\rangle=\frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2n(n-1)(n-2)}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2n(n-1)(n-2)}}|n\rangle$$

$$(4.149)$$

By substituting (4.148) and (4.149) into (4.138), (4.138) becomes

$$C_{22}|n\rangle=\left(\cos\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{n}\,(\hat{n}-1)(\hat{n}-2)}+\frac{i\Delta\mathrm{t}}{2}\frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{n}\,(\hat{n}-1)(\hat{n}-2)}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{n}\,(\hat{n}-1)(\hat{n}-2)}}\right)|n\rangle$$

$$= \left(\cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1)(n-2)} + \frac{i\Delta \mathrm{t}}{2} \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1)(n-2)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1)(n-2)}} \right) |n\rangle \quad (4.150)$$

Finally, by substituting (4.137) and (4.150) into (4.126), (4.126) becomes

$$|\varphi(t)\rangle = \sum_{n=0}^{\infty} \{a_n r |2\rangle |n-3\rangle + a_n s |1\rangle |n\rangle\}$$

$$= \sum_{n=0}^{\infty} \left\{C_{2,n-1}(t)|2\rangle|n-3\rangle + C_{1,n}(t)|1\rangle|n\rangle\right\},$$

where

$$r = -\lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 t^2 n(n-1)(n-2)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 t^2 n(n-1)(n-2)}} \sqrt{n}\,\sqrt{n-1}\,\sqrt{n-2}\,,$$

$$s = \cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1)(n-2)} + \frac{i\Delta \mathrm{t}}{2} \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1)(n-2)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1)(n-2)}}.$$

Let $P_1(t)$ be the probability in which the atom is in the ground state. Then,

$$P_1(t) = \sum_{n=0}^{\infty} \left|C_{1,n}(t)\right|^2$$

$$= \sum_{n=0}^{\infty} \{|a_n|^2 |s|^2\}$$

$$= \sum_{n=0}^{\infty} \left\{ |a_n|^2 \left[\cos^2\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1)(n-2)} + \frac{\Delta^2 \mathrm{t}^2}{4} \frac{\sin^2\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1)(n-2)}}{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1)(n-2)} \right] \right\}. \quad (4.151)$$

4.3.3 Derivation of k-photon Jaynes-Cummings Model Unitary Operator

From Methodology 3.3, we have the Hamiltonian in interaction picture as follows:

$$\hat{H}_I = \frac{\hbar\Delta}{2}\hat{\sigma}_3 - i\hbar\lambda[\hat{\sigma}_+\hat{a}^k - \hat{\sigma}_-\hat{a}^{\dagger k}] \qquad (3.21)$$

$$= \frac{\hbar\Delta}{2}\hat{\sigma}_3 + i\hbar\lambda(\hat{a}^{\dagger k}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^k). \qquad (4.152)$$

Now, the Unitary Operator $\hat{U}$ is

$$\hat{U} = \exp\left(-\frac{i\hat{H}_I t}{\hbar}\right)$$

$$= \exp\ \{-i\left[\frac{\hbar\Delta}{2}\hat{\sigma}_3 + i\hbar\lambda(\hat{a}^{\dagger k}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^k)\right]t\,/\hbar\}$$

$$= \exp\{-i\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^k)\right]\}$$

$$= \cos\hat{\theta} - i\sin\hat{\theta},\ \text{ where } \hat{\theta} = \frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^k). \qquad (4.153)$$

Let $\hat{c} = \cos\hat{\theta}\ \ and\ \hat{s} = \sin\hat{\theta},$

then, $\hat{U} = \hat{c} - i\hat{s}$. Now,

$$\hat{c} = \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^k)\right]^{2m} \qquad (4.154)$$

$$\hat{s} = \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^k)\right]^{2m+1}. \qquad (4.155)$$

Now,

$$\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^k) = \frac{\Delta t}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + i\lambda t\left[\hat{a}^{\dagger k}\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} - \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\hat{a}^k\right]$$

$$= \begin{pmatrix}\frac{\Delta t}{2} & -i\lambda t\hat{a}^k\\ i\lambda t\hat{a}^{\dagger k} & -\frac{\Delta t}{2}\end{pmatrix}. \qquad (4.156)$$

Consider the even power expansion, we have

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^k)\right]^2 = \left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^k)\right]\left[\frac{\Delta t}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^k)\right]$$

$$=\begin{pmatrix}\frac{\Delta t}{2} & -i\lambda t\hat{a}^{k}\\ i\lambda t\hat{a}^{\dagger k} & -\frac{\Delta t}{2}\end{pmatrix}\begin{pmatrix}\frac{\Delta t}{2} & -i\lambda t\hat{a}^{k}\\ i\lambda t\hat{a}^{\dagger k} & -\frac{\Delta t}{2}\end{pmatrix}$$

$$=\begin{pmatrix}\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{a}^{k}\hat{a}^{\dagger k} & 0\\ 0 & \frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{a}^{\dagger k}\hat{a}^{k}\end{pmatrix}, \qquad (4.157)$$

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3+i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- -\hat{\sigma}_+\hat{a}^{k})\right]^4=\left[\frac{\Delta t}{2}\hat{\sigma}_3+i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- -\hat{\sigma}_+\hat{a}^{k})\right]^2\left[\frac{\Delta t}{2}\hat{\sigma}_3+i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- -\hat{\sigma}_+\hat{a}^{k})\right]^2$$

$$=\begin{pmatrix}\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{a}^{k}\hat{a}^{\dagger k} & 0\\ 0 & \frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{a}^{\dagger k}\hat{a}^{k}\end{pmatrix}\begin{pmatrix}\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{a}^{k}\hat{a}^{\dagger k} & 0\\ 0 & \frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{a}^{\dagger k}\hat{a}^{k}\end{pmatrix}$$

$$=\begin{pmatrix}\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{a}^{k}\hat{a}^{\dagger k}\right)^2 & 0\\ 0 & \left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{a}^{\dagger k}\hat{a}^{k}\right)^2\end{pmatrix}, \qquad (4.158)$$

$$\left[\frac{\Delta t}{2}\hat{\sigma}_3+i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- -\hat{\sigma}_+\hat{a}^{k})\right]^6=\left[\frac{\Delta t}{2}\hat{\sigma}_3+i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- -\hat{\sigma}_+\hat{a}^{k})\right]^4\left[\frac{\Delta t}{2}\hat{\sigma}_3+i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- -\hat{\sigma}_+\hat{a}^{k})\right]^2$$

$$=\begin{pmatrix}\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{a}^{k}\hat{a}^{\dagger k}\right)^2 & 0\\ 0 & \left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{a}^{\dagger k}\hat{a}^{k}\right)^2\end{pmatrix}\begin{pmatrix}\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{a}^{k}\hat{a}^{\dagger k} & 0\\ 0 & \frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{a}^{\dagger k}\hat{a}^{k}\end{pmatrix}$$

$$=\begin{pmatrix}\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{a}^{k}\hat{a}^{\dagger k}\right)^3 & 0\\ 0 & \left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2\hat{a}^{\dagger k}\hat{a}^{k}\right)^3\end{pmatrix}. \qquad (4.159)$$

Therefore, we can actually deduce that for general even power 2m,

$$\left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^k)\right]^{2m} = \begin{pmatrix} \left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^k\hat{a}^{\dagger k}\right)^m & 0 \\ 0 & \left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^{\dagger k}\hat{a}^k\right)^m \end{pmatrix}.$$

However, this could be proven by using mathematical induction as follows:

Let P(m) be the statement that

$$\left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^k)\right]^{2m} = \begin{pmatrix} \left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^k\hat{a}^{\dagger k}\right)^m & 0 \\ 0 & \left(\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^{\dagger k}\hat{a}^k\right)^m \end{pmatrix},$$

where m are positive integers.

Basis case: For m = 1,

$$\left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^k)\right]^{2} = \begin{pmatrix} \frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^k\hat{a}^{\dagger k} & 0 \\ 0 & \frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^{\dagger k}\hat{a}^k \end{pmatrix}.$$

This basis case has been proven in (4.157).

Therefore, P(1) is true.

Inductive step: Consider when m = q, we suppose that P(q) is true, which means we suppose

$$\left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^k)\right]^{2q} = \begin{pmatrix} (\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^k\hat{a}^{\dagger k})^q & 0 \\ 0 & (\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{a}^{\dagger k}\hat{a}^k)^q \end{pmatrix} \text{ is true.}$$

Then, for m = q+1,

$$\left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^k)\right]^{2(q+1)} = \left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^k)\right]^{2q+2}$$

$$= \left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^k)\right]^{2q}\left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^k)\right]^{2}$$

$$= \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^k \hat{a}^{\dagger k}\right)^q & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger k} \hat{a}^k\right)^q \end{pmatrix} \begin{pmatrix} \frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^k \hat{a}^{\dagger k} & 0 \\ 0 & \frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger k} \hat{a}^k \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^k \hat{a}^{\dagger k}\right)^{q+1} & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger k} \hat{a}^k\right)^{q+1} \end{pmatrix}.$$

Therefore, if we suppose that P(q) is true, then P(q+1) is true.

By mathematical induction, we can conclude that P(m) is true for all m belongs to positive integers.

As a result, it is proven that

$$\left[\frac{\Delta t}{2} \hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger k} \hat{\sigma}_- - \hat{\sigma}_+ \hat{a}^k)\right]^{2m} = \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^k \hat{a}^{\dagger k}\right)^m & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger k} \hat{a}^k\right)^m \end{pmatrix}. \tag{4.160}$$

By substituting (4.160) into (4.154), we have

$$\hat{c} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \begin{pmatrix} \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^k \hat{a}^{\dagger k}\right)^m & 0 \\ 0 & \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger k} \hat{a}^k\right)^m \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \left(\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^k \hat{a}^{\dagger k}}\right)^{2m} & 0 \\ 0 & \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \left(\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{a}^{\dagger k} \hat{a}^k}\right)^{2m} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^k \hat{a}^{\dagger k}} & 0 \\ 0 & \cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k} \end{pmatrix}. \tag{4.161}$$

Now,

$$\left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^k)\right]^{2m+1}$$

$$= \left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^k)\right]^{2m} \left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^k)\right]. \tag{4.162}$$

By substituting (4.156) and (4.160) into (4.162), (4.162) becomes

$$\left[\frac{\Delta \mathrm{t}}{2}\hat{\sigma}_3 + i\lambda t(\hat{a}^{\dagger k}\hat{\sigma}_- - \hat{\sigma}_+\hat{a}^k)\right]^{2m+1}$$

$$= \begin{pmatrix} \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^k \hat{a}^{\dagger k}\right)^m & 0 \\ 0 & \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k\right)^m \end{pmatrix} \begin{pmatrix} \frac{\Delta \mathrm{t}}{2} & -i\lambda t\hat{a}^k \\ i\lambda t\hat{a}^{\dagger k} & -\frac{\Delta \mathrm{t}}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^k \hat{a}^{\dagger k}\right)^m & -i\lambda t\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^k \hat{a}^{\dagger k}\right)^m \hat{a}^k \\ i\lambda t\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k\right)^m \hat{a}^{\dagger k} & -\frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k\right)^m \end{pmatrix}. \tag{4.163}$$

By substituting (4.163) into (4.155), we have

$$\hat{s} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \begin{pmatrix} \frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^k \hat{a}^{\dagger k}\right)^m & -i\lambda t\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^k \hat{a}^{\dagger k}\right)^m \hat{a}^k \\ i\lambda t\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k\right)^m \hat{a}^{\dagger k} & -\frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k\right)^m \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[\frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^k \hat{a}^{\dagger k}\right)^m\right] & \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[-i\lambda t\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^k \hat{a}^{\dagger k}\right)^m \hat{a}^k\right] \\ \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[i\lambda t\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k\right)^m \hat{a}^{\dagger k}\right] & \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\left[-\frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k\right)^m\right] \end{pmatrix}$$

Then, $-i\hat{s}$

$$= -i\begin{pmatrix} \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}\left[\frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^k\hat{a}^{\dagger k}\right)^m\right] & \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}\left[-i\lambda t\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^k\hat{a}^{\dagger k}\right)^m\hat{a}^k\right] \\ \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}\left[i\lambda t\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^{\dagger k}\hat{a}^k\right)^m\hat{a}^{\dagger k}\right] & \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}\left[-\frac{\Delta \mathrm{t}}{2}\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^{\dagger k}\hat{a}^k\right)^m\right] \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}\left[\frac{-i\Delta \mathrm{t}}{2}\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^k\hat{a}^{\dagger k}\right)^m\right] & \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}\left[-\lambda t\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^k\hat{a}^{\dagger k}\right)^m\hat{a}^k\right] \\ \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}\left[\lambda t\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^{\dagger k}\hat{a}^k\right)^m\hat{a}^{\dagger k}\right] & \sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}\left[\frac{i\Delta \mathrm{t}}{2}\left(\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^{\dagger k}\hat{a}^k\right)^m\right] \end{pmatrix}$$

$= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where

$$A_{11} = \frac{-i\Delta \mathrm{t}}{2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}\frac{\left(\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^k\hat{a}^{\dagger k}}\right)^{2m+1}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^k\hat{a}^{\dagger k}}}$$

$$= \frac{-i\Delta \mathrm{t}}{2}\frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^k\hat{a}^{\dagger k}}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^k\hat{a}^{\dagger k}}},$$

$$A_{12} = -\lambda t\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}\frac{\left(\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^k\hat{a}^{\dagger k}}\right)^{2m+1}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^k\hat{a}^{\dagger k}}}\hat{a}^k$$

$$= -\lambda t\frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^k\hat{a}^{\dagger k}}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^k\hat{a}^{\dagger k}}}\hat{a}^k,$$

$$A_{21} = \lambda t\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}\frac{\left(\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^{\dagger k}\hat{a}^k}\right)^{2m+1}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2\hat{a}^{\dagger k}\hat{a}^k}}\hat{a}^{\dagger k}$$

$$= \lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k}} \hat{a}^{\dagger k},$$

$$A_{22} = \frac{i\Delta \mathrm{t}}{2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \frac{\left(\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k}\right)^{2m+1}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k}}$$

$$= \frac{i\Delta \mathrm{t}}{2} \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k}}.$$

Therefore,

$$-i\hat{s} = \begin{pmatrix} \frac{-i\Delta \mathrm{t}}{2} \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{k} \hat{a}^{\dagger k}}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{k} \hat{a}^{\dagger k}}} & -\lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{k} \hat{a}^{\dagger k}}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{k} \hat{a}^{\dagger k}}} \hat{a}^k \\ \lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k}} \hat{a}^{\dagger k} & \frac{i\Delta \mathrm{t}}{2} \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k}} \end{pmatrix}. \qquad (4.164)$$

By substituting (4.161) and (4.164) into (4.153), we have

$$\hat{U} = \begin{pmatrix} \cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{k} \hat{a}^{\dagger k}} & 0 \\ 0 & \cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k} \end{pmatrix}$$
$$+ \begin{pmatrix} \frac{-i\Delta \mathrm{t}}{2} \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{k} \hat{a}^{\dagger k}}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{k} \hat{a}^{\dagger k}}} & -\lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{k} \hat{a}^{\dagger k}}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{k} \hat{a}^{\dagger k}}} \hat{a}^k \\ \lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k}} \hat{a}^{\dagger k} & \frac{i\Delta \mathrm{t}}{2} \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k}} \end{pmatrix}$$

$$= \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \text{ , where} \tag{4.165}$$

$$B_{11} = \cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^k \hat{a}^{\dagger k}} - \frac{i\Delta \mathrm{t}}{2} \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^k \hat{a}^{\dagger k}}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^k \hat{a}^{\dagger k}}},$$

$$B_{12} = -\lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^k \hat{a}^{\dagger k}}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^k \hat{a}^{\dagger k}}} \hat{a}^k,$$

$$B_{21} = \lambda t \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k}} \hat{a}^{\dagger k},$$

$$B_{22} = \cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k} + \frac{i\Delta \mathrm{t}}{2} \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{a}^{\dagger k} \hat{a}^k}},$$

Now, from the previous derivations for one, two and three photons cases, we have $\hat{a}\,\hat{a}^\dagger = \hat{n} + 1$, $\hat{a}^2 \hat{a}^{\dagger 2} = (\hat{n}+1)(\hat{n}+2)$ and $\hat{a}^3 \hat{a}^{\dagger 3} = (\hat{n}+1)(\hat{n}+2)(\hat{n}+3)$.

Hence, it can be deduced that $\hat{a}^k \hat{a}^{\dagger k} = (\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\ldots\ldots\ldots(\hat{n}+k)$. (4.166)

From previous derivations, we also have $\hat{a}^\dagger \hat{a} = \hat{n}$, $\hat{a}^{\dagger 2}\hat{a}^2 = \hat{n}(\hat{n}-1)$ and $\hat{a}^{\dagger 3}\hat{a}^3 = \hat{n}\,(\hat{n}-1)(\hat{n}-2)$. So, it can be deduced that

$$\hat{a}^{\dagger k}\hat{a}^k = \hat{n}\,(\hat{n}-1)(\hat{n}-2)\ldots\ldots\ldots[\hat{n}-(k-1)]. \tag{4.167}$$

Therefore, by substituting (4.166) and (4.167) into (4.165), (4.165) becomes

$$\hat{U} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \text{ , where} \tag{4.168}$$

$$C_{11} = \cos\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 (\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\ldots\ldots\ldots(\hat{n}+k)}$$

$$- \frac{i\Delta t}{2} \frac{\sin\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 (\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\ldots\ldots\ldots(\hat{n}+k)}}{\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 (\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\ldots\ldots\ldots(\hat{n}+k)}},$$

$$C_{12} = -\lambda t \frac{\sin\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 (\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\ldots\ldots\ldots(\hat{n}+k)}}{\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 (\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\ldots\ldots\ldots(\hat{n}+k)}} \hat{a}^k,$$

$$C_{21} = \lambda t \frac{\sin\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\ldots\ldots\ldots[\hat{n}-(k-1)]}}{\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\ldots\ldots\ldots[\hat{n}-(k-1)]}} \hat{a}^{\dagger k},$$

$$C_{22} = \cos\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\ldots\ldots\ldots[\hat{n}-(k-1)]}$$

$$+ \frac{i\Delta t}{2} \frac{\sin\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\ldots\ldots\ldots[\hat{n}-(k-1)]}}{\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\ldots\ldots\ldots[\hat{n}-(k-1)]}}.$$

Probability Function for k-Photon Jaynes-Cummings Model

Let $|\varphi(t)\rangle$ be the quantum state of the system containing 1 two-level atom and photons at time t.

Suppose initially the atom is in the ground state. Then, the initial state is

$$|\varphi(0)\rangle = \sum_{n=0}^{\infty} c_{1,n}(0)|1\rangle\,|n\rangle$$

$$= \sum_{n=0}^{\infty} a_n|1\rangle\,|n\rangle, \quad \text{where } a_n = c_{1,n}(0). \tag{4.169}$$

Then,

$$|\varphi(t)\rangle = \widehat{\mathrm{U}}(\mathrm{t})|\varphi(0)\rangle. \tag{4.170}$$

From (4.168), $\widehat{\mathrm{U}}(\mathrm{t})$ in its outer product form is

$$\widehat{\mathrm{U}}(\mathrm{t}) = C_{11}|2\rangle\langle 2| + C_{12}|2\rangle\langle 1| + C_{21}|1\rangle\langle 2| + C_{22}|1\rangle\langle 1| \tag{4.171}$$

By substituting (4.169) and (4.171) into (4.170),

$$|\varphi(t)\rangle = \sum_{n=0}^{\infty} a_n\{|2\rangle \otimes [C_{12}|n\rangle] + |1\rangle \otimes [C_{22}|n\rangle]\}, \tag{4.172}$$

where

$$C_{12} = -\lambda t \frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\ldots\ldots\ldots(\hat{n}+k)}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\ldots\ldots\ldots(\hat{n}+k)}}\hat{a}^k,$$

$$C_{22} = \cos\sqrt{\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{n}\,(\hat{n}-1)(\hat{n}-2)\ldots\ldots\ldots[\hat{n}-(k-1)]}$$
$$+\frac{i\Delta\mathrm{t}}{2}\frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{n}\,(\hat{n}-1)(\hat{n}-2)\ldots\ldots\ldots[\hat{n}-(k-1)]}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2\hat{n}\,(\hat{n}-1)(\hat{n}-2)\ldots\ldots\ldots[\hat{n}-(k-1)]}}.$$

To simplify $C_{12}|n\rangle$,

$$C_{12}|n\rangle = -\lambda t \frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\ldots\ldots\ldots(\hat{n}+k)}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\ldots\ldots\ldots(\hat{n}+k)}}\hat{a}^k|n\rangle$$

$$= -\lambda t \frac{\sin\sqrt{\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\ldots\ldots\ldots(\hat{n}+k)}}{\sqrt{\frac{\Delta^2\mathrm{t}^2}{4} + \lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\ldots\ldots\ldots(\hat{n}+k)}}\sqrt{n}\sqrt{n-1}\ldots\ldots\sqrt{n-(k-1)}\,|n - k\rangle. \tag{4.173}$$

From (4.173), consider

$$\frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\dots\dots\dots(\hat{n}+k)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\dots\dots\dots(\hat{n}+k)}}|n-k\rangle$$

$$=\sum_{m=0}^{\infty}(-1)^m\frac{\left(\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\dots\dots\dots(\hat{n}+k)\right)^m}{(2m+1)!}|n-k\rangle. \qquad (4.174)$$

To simplify (4.174), when m=1,

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\dots\dots\dots(\hat{n}+k)\right)|n-k\rangle$$

$$=\frac{\Delta^2 \mathrm{t}^2}{4}|n-k\rangle+\lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\dots\dots\dots(\hat{n}+k)|n-k\rangle$$

$$=\frac{\Delta^2 \mathrm{t}^2}{4}|n-k\rangle+\lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\dots\dots\dots[\hat{n}+(k-1)]n|n-k\rangle$$

$$=\frac{\Delta^2 \mathrm{t}^2}{4}|n-k\rangle+\lambda^2\mathrm{t}^2 n(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\dots\dots\dots[\hat{n}+(k-2)](\mathrm{n}-1)|n-k\rangle$$

$$=\frac{\Delta^2 \mathrm{t}^2}{4}|n-k\rangle+\lambda^2\mathrm{t}^2 n(\mathrm{n}-1)(\mathrm{n}-2)\dots\dots\dots[n-(k-1)]|n-k\rangle$$

$$=\left(\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2 n(\mathrm{n}-1)(\mathrm{n}-2)\dots\dots\dots[n-(k-1)]\right)|n-k\rangle. \qquad (4.175)$$

When m=2,

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\dots\dots\dots(\hat{n}+k)\right)^2|n-k\rangle$$

$$=\left(\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\dots\dots\dots(\hat{n}+k)\right)\Bigg(\frac{\Delta^2 \mathrm{t}^2}{4}$$
$$+\lambda^2\mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\dots\dots\dots(\hat{n}+k)\Bigg)|n-k\rangle. \qquad (4.176)$$

By substituting (4.175) into (4.176), RHS of (4.176) becomes

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)(\hat{n}+3) \dots \dots \dots (\hat{n}+k)\right)\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(\mathrm{n}-1)(\mathrm{n}-2) \dots \dots \dots [n-(k-1)]\right)|n-3\rangle$$

$$= \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(\mathrm{n}-1)(\mathrm{n}-2) \dots \dots \dots [n-(k-1)]\right)\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)(\hat{n}+3) \dots \dots \dots (\hat{n}+k)\right)|n-k\rangle. \qquad (4.177)$$

Again, by substituting (4.175) into (4.177), RHS of (4.177) becomes

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(\mathrm{n}-1)(\mathrm{n}-2) \dots \dots \dots [n-(k-1)]\right)\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(\mathrm{n}-1)(\mathrm{n}-2) \dots \dots \dots [n-(k-1)]\right)|n-k\rangle.$$

So,

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)(\hat{n}+3) \dots \dots \dots (\hat{n}+k)\right)^2 |n-k\rangle$$

$$= \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(\mathrm{n}-1)(\mathrm{n}-2) \dots \dots \dots [n-(k-1)]\right)^2 |n-k\rangle. \qquad (4.178)$$

Now, consider when m=3,

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)(\hat{n}+3) \dots \dots \dots (\hat{n}+k)\right)^3 |n-k\rangle$$

$$= \left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)(\hat{n}+3) \dots \dots \dots (\hat{n}+k)\right)\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 (\hat{n}+1)(\hat{n}+2)(\hat{n}+3) \dots \dots \dots (\hat{n}+k)\right)^2 |n-k\rangle. \qquad (4.179)$$

By substituting (4.178) into (4.179), RHS of (4.179) becomes

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\ldots\ldots\ldots(\hat{n}+k)\right)\left(\frac{\Delta^2 \mathrm{t}^2}{4}\right.$$
$$\left.+\lambda^2 \mathrm{t}^2 n(\mathrm{n}-1)(\mathrm{n}-2)\ldots\ldots\ldots[n-(k-1)]\right)^2|n-k\rangle$$

$$=\left(\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2 n(\mathrm{n}-1)(\mathrm{n}-2)\ldots\ldots\ldots[n-(k-1)]\right)^2\left(\frac{\Delta^2 \mathrm{t}^2}{4}\right.$$
$$\left.+\lambda^2 \mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\ldots\ldots\ldots(\hat{n}+k)\right)|n-k\rangle. \tag{4.180}$$

By substituting (4.175) into (4.180), RHS of (4.180) becomes

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2 n(\mathrm{n}-1)(\mathrm{n}-2)\ldots\ldots\ldots[n-(k-1)]\right)^2\left(\frac{\Delta^2 \mathrm{t}^2}{4}\right.$$
$$\left.+\lambda^2 \mathrm{t}^2 n(\mathrm{n}-1)(\mathrm{n}-2)\ldots\ldots\ldots[n-(k-1)]\right)|n-k\rangle.$$

So,

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\ldots\ldots\ldots(\hat{n}+k)\right)^3|n-k\rangle$$

$$=\left(\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2 n(\mathrm{n}-1)(\mathrm{n}-2)\ldots\ldots\ldots[n-(k-1)]\right)^3|n-k\rangle.$$

By using the similar method as before, it can be deduced that for any integer m,

$$\left(\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\ldots\ldots\ldots(\hat{n}+k)\right)^m|n-k\rangle$$

$$=\left(\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2 n(\mathrm{n}-1)(\mathrm{n}-2)\ldots\ldots\ldots[n-(k-1)]\right)^m|n-k\rangle. \tag{4.181}$$

By substituting (4.181) into (4.174), (4.174) becomes

$$\frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\ldots\ldots\ldots(\hat{n}+k)}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4}+\lambda^2 \mathrm{t}^2(\hat{n}+1)(\hat{n}+2)(\hat{n}+3)\ldots\ldots\ldots(\hat{n}+k)}}|n-k\rangle$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(\mathrm{n}-1)(\mathrm{n}-2) \dots \dots \dots [n-(k-1)]\right)^m}{(2m+1)!} |n-k\rangle$$

$$= \frac{\sin \sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(\mathrm{n}-1)(\mathrm{n}-2) \dots \dots \dots [n-(k-1)]}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(\mathrm{n}-1)(\mathrm{n}-2) \dots \dots \dots [n-(k-1)]}} |n-k\rangle. \qquad (4.182)$$

By substituting (4.182) into (4.173), (4.173) becomes

$$C_{12}|n\rangle = -\lambda t \left[\sqrt{n}\, \sqrt{n-1} \dots \dots \sqrt{n-(k-1)}\right] \cdot$$

$$\frac{\sin \sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(\mathrm{n}-1)(\mathrm{n}-2) \dots \dots \dots [n-(k-1)]}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(\mathrm{n}-1)(\mathrm{n}-2) \dots \dots \dots [n-(k-1)]}} |n-k\rangle. \qquad (4.183)$$

Next, to simplify $C_{22}|n\rangle$,

$$C_{22}|n\rangle = \left(\cos \sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2) \dots \dots \dots [\hat{n}-(k-1)]} \right.$$

$$\left. + \frac{i\Delta \mathrm{t}}{2} \frac{\sin \sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2) \dots \dots \dots [\hat{n}-(k-1)]}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2) \dots \dots \dots [\hat{n}-(k-1)]}}\right) |n\rangle.$$

$$= \left(\cos \sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2) \dots \dots \dots [\hat{n}-(k-1)]}\right) |n\rangle$$

$$+ \left(\frac{i\Delta \mathrm{t}}{2} \frac{\sin \sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2) \dots \dots \dots [\hat{n}-(k-1)]}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2) \dots \dots \dots [\hat{n}-(k-1)]}}\right) |n\rangle. \quad (4.184)$$

From (4.184), consider

$$\left(\cos \sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2) \dots \dots \dots [\hat{n}-(k-1)]}\right) |n\rangle$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\ldots\ldots\ldots[\hat{n}-(k-1)]\right)^m}{(2m)!} |n\rangle. \qquad (4.185)$$

To simplify (4.185), when m=1,

$$\left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\ldots\ldots\ldots[\hat{n}-(k-1)]\right)|n\rangle$$

$$= \frac{\Delta^2 t^2}{4}|n\rangle + \lambda^2 t^2 \hat{n}\,(\hat{n}-1)(n-2)\ldots\ldots\ldots[n-(k-1)]|n\rangle$$

$$= \frac{\Delta^2 t^2}{4}|n\rangle + \lambda^2 t^2 [n-(k-1)]\hat{n}\,(\hat{n}-1)(n-2)\ldots\ldots\ldots[\hat{n}-(k-2)]|n\rangle$$

$$= \frac{\Delta^2 t^2}{4}|n\rangle + \lambda^2 t^2 [n-(k-1)][n-(k-2)]\hat{n}\,(\hat{n}-1)(n-2)\ldots\ldots\ldots[\hat{n}-(k-3)]|n\rangle$$

$$= \frac{\Delta^2 t^2}{4}|n\rangle + \lambda^2 t^2 [n-(k-1)][n-(k-2)]\ldots\ldots\ldots(n-1)n|n\rangle$$

$$= \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n(n-1)\ldots\ldots\ldots[n-(k-1)]\right)|n\rangle. \qquad (4.186)$$

When m=2,

$$\left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\ldots\ldots\ldots[\hat{n}-(k-1)]\right)^2 |n\rangle$$

$$= \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\ldots\ldots\ldots[\hat{n}-(k-1)]\right)\left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\ldots\ldots\ldots[\hat{n}-(k-1)]\right)|n\rangle. \qquad (4.187)$$

By substituting (4.186) into (4.187), RHS of (4.187) becomes

$$\left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\ldots\ldots\ldots[\hat{n}-(k-1)]\right)\left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n(n-1)\ldots\ldots\ldots[n-(k-1)]\right)|n\rangle$$

$$= \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n(n-1) \dots \dots \dots [n-(k-1)]\right)\left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}\, (\hat{n}-1)(\hat{n}-2) \dots \dots \dots [\hat{n} - (k-1)]\right)|n\rangle. \quad (4.188)$$

Again, by substituting (4.186) into (4.188), RHS of (4.188) becomes

$$\left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n(n-1) \dots \dots \dots [n-(k-1)]\right)\left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n(n-1) \dots \dots \dots [n-(k-1)]\right)|n\rangle.$$

So,

$$\left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}\, (\hat{n}-1)(\hat{n}-2) \dots \dots \dots [\hat{n} - (k-1)]\right)^2 |n\rangle$$

$$= \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n(n-1) \dots \dots \dots [n-(k-1)]\right)^2 |n\rangle. \quad (4.189)$$

Now, consider when m=3,

$$\left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}\, (\hat{n}-1)(\hat{n}-2) \dots \dots \dots [\hat{n} - (k-1)]\right)^3 |n\rangle$$

$$= \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}\, (\hat{n}-1)(\hat{n}-2) \dots \dots \dots [\hat{n} - (k-1)]\right)\left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}\, (\hat{n}-1)(\hat{n}-2) \dots \dots \dots [\hat{n} - (k-1)]\right)^2 |n\rangle. \quad (4.190)$$

By substituting (4.189) into (4.190), RHS of (4.190) becomes

$$\left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}\, (\hat{n}-1)(\hat{n}-2) \dots \dots \dots [\hat{n} - (k-1)]\right)\left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n(n-1) \dots \dots \dots [n-(k-1)]\right)^2 |n\rangle.$$

$$= \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n(n-1) \dots \dots \dots [n-(k-1)]\right)^2 \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}\, (\hat{n}-1)(\hat{n}-2) \dots \dots \dots [\hat{n} - (k-1)]\right)|n\rangle. \quad (4.191)$$

By substituting (4.186) into (4.191), RHS of (4.191) becomes

$$\left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n(n-1) \ldots \ldots \ldots [n-(k-1)]\right)^2 \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n(n-1) \ldots \ldots \ldots [n-(k-1)]\right) |n\rangle$$

$$= \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n(n-1) \ldots \ldots \ldots [n-(k-1)]\right)^3 |n\rangle.$$

So,

$$\left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2) \ldots \ldots \ldots [\hat{n}-(k-1)]\right)^3 |n\rangle$$

$$= \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n(n-1) \ldots \ldots \ldots [n-(k-1)]\right)^3 |n\rangle. \qquad (4.192)$$

By using the similar method as before, it can be deduced that for any integer m,

$$\left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2) \ldots \ldots \ldots [\hat{n}-(k-1)]\right)^m |n\rangle$$

$$= \left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n(n-1) \ldots \ldots \ldots [n-(k-1)]\right)^m |n\rangle. \qquad (4.193)$$

By substituting (4.193) into (4.185), (4.185) becomes

$$\left(\cos\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2) \ldots \ldots \ldots [\hat{n}-(k-1)]}\right)|n\rangle$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n(n-1) \ldots \ldots \ldots [n-(k-1)]\right)^m}{(2m)!} |n\rangle$$

$$= \left(\cos\sqrt{\frac{\Delta^2 t^2}{4} + \lambda^2 t^2 n(n-1) \ldots \ldots \ldots [n-(k-1)]}\right)|n\rangle. \qquad (4.194)$$

From (4.184) also, consider

$$\frac{\sin\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\,\dots\dots\dots\,[\hat{n}-(k-1)]}}{\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\,\dots\dots\dots\,[\hat{n}-(k-1)]}}|n\rangle$$

$$=\sum_{m=0}^{\infty}(-1)^m\frac{\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\,\dots\dots\dots\,[\hat{n}-(k-1)]\right)^m}{(2m+1)!}|n\rangle.$$

By substituting (4.193) into it, it becomes

$$\sum_{m=0}^{\infty}(-1)^m\frac{\left(\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n(n-1)\,\dots\dots\dots\,[n-(k-1)]\right)^m}{(2m+1)!}|n\rangle$$

$$=\frac{\sin\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n(n-1)\,\dots\dots\dots\,[n-(k-1)]}}{\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n(n-1)\,\dots\dots\dots\,[n-(k-1)]}}|n\rangle. \tag{4.195}$$

By substituting (4.194) and (4.195) into (4.184), (4.184) becomes

$$C_{22}|n\rangle=\left(\cos\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\,\dots\dots\dots\,[\hat{n}-(k-1)]}\right.$$
$$\left.+\frac{i\Delta t}{2}\frac{\sin\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\,\dots\dots\dots\,[\hat{n}-(k-1)]}}{\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 \hat{n}\,(\hat{n}-1)(\hat{n}-2)\,\dots\dots\dots\,[\hat{n}-(k-1)]}}\right)|n\rangle$$

$$=\left(\cos\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n(n-1)\,\dots\dots\dots\,[n-(k-1)]}\right.$$
$$\left.+\frac{i\Delta t}{2}\frac{\sin\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n(n-1)\,\dots\dots\dots\,[n-(k-1)]}}{\sqrt{\frac{\Delta^2 t^2}{4}+\lambda^2 t^2 n(n-1)\,\dots\dots\dots\,[n-(k-1)]}}\right)|n\rangle. \tag{4.196}$$

Finally, by substituting (4.183) and (4.196) into (4.172), (4.172) becomes

$$
\begin{aligned}
|\varphi(t)\rangle &= \sum_{n=0}^{\infty}\{a_n r|2\rangle|n-k\rangle + a_n s|1\rangle|n\rangle\} \\
&= \sum_{n=0}^{\infty}\left\{C_{2,n-1}(t)|2\rangle|n-k\rangle + C_{1,n}(t)|1\rangle|n\rangle\right\},
\end{aligned}
$$

where

$$
r = -\lambda t\left[\sqrt{n}\,\sqrt{n-1}\ldots\ldots\sqrt{n-(k-1)}\right]\cdot \frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(\mathrm{n}-1)(\mathrm{n}-2)\ldots\ldots\ldots[n-(k-1)]}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(\mathrm{n}-1)(\mathrm{n}-2)\ldots\ldots\ldots[n-(k-1)]}},
$$

$$
s = \cos\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1)\ldots\ldots\ldots[n-(k-1)]} + \frac{i\Delta \mathrm{t}}{2}\frac{\sin\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1)\ldots\ldots\ldots[n-(k-1)]}}{\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1)\ldots\ldots\ldots[n-(k-1)]}}.
$$

Let $P_1(t)$ be the probability in which the atom is in the ground state. Then,

$$
\begin{aligned}
P_1(t) &= \sum_{n=0}^{\infty}\left|C_{1,n}(t)\right|^2 \\
&= \sum_{n=0}^{\infty}\{|a_n|^2|s|^2\} \\
&= \sum_{n=0}^{\infty}\left\{|a_n|^2\left[\cos^2\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1)\ldots\ldots\ldots[n-(k-1)]} + \frac{\Delta^2 \mathrm{t}^2}{4}\frac{\sin^2\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1)\ldots\ldots\ldots[n-(k-1)]}}{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1)\ldots\ldots\ldots[n-(k-1)]}\right]\right\}
\end{aligned}
\tag{4.197}
$$

4.4 Discussions of single-photon and two-photon JCM results

Single-photon JCM probability functions

From (4.59), the probability that the atom is in the ground state is given by

$$P_1(t) = \sum_{n=0}^{\infty}\left\{|a_n|^2\left[\cos^2\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2 n}+\frac{\Delta^2\mathrm{t}^2}{4}\frac{\sin^2\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2 n}}{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2 n}\right]\right\} \tag{4.59}$$

$$= \sum_{n=0}^{\infty}\left\{|a_n|^2\left[\frac{\Delta^2+2\lambda^2 n}{\Delta^2+4\lambda^2 n}+\frac{2\lambda^2 n}{\Delta^2+4\lambda^2 n}cos\left(2\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2 n}\right)\right]\right\}. \tag{4.198}$$

Case 1: Non-resonant case

For coherent state, the photon number probability distribution $P(n)$ is given by

$$P(n) = \frac{\bar{n}^n}{n!}exp(-\bar{n}). \tag{2.8}$$

Now, $|a_n|^2 = P(n)$. By substituting (1.5) into (4.198), we have

$$P_1(t) = \sum_{n=0}^{\infty}\left\{\frac{\bar{n}^n}{n!}exp(-\bar{n})\left[\frac{\Delta^2+2\lambda^2 n}{\Delta^2+4\lambda^2 n}+\frac{2\lambda^2 n}{\Delta^2+4\lambda^2 n}cos\left(2\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2 n}\right)\right]\right\}. \tag{4.199}$$

For thermal state, the photon number probability distribution $P(n)$ is given by

$$P(n) = \frac{1}{\bar{n}+1}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n. \tag{2.17}$$

Again, $|a_n|^2 = P(n)$. By substituting (2.17) into (4.198), we have

$$P_1(t) = \sum_{n=0}^{\infty}\left\{\frac{1}{\bar{n}+1}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n\left[\frac{\Delta^2+2\lambda^2 n}{\Delta^2+4\lambda^2 n}+\frac{2\lambda^2 n}{\Delta^2+4\lambda^2 n}cos\left(2\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2 n}\right)\right]\right\}. \tag{4.200}$$

Case 2: Resonant case

For resonant case, Δ=0. So, for coherent state, (4.199) becomes

$$P_1(t) = \frac{1}{2}\sum_{n=0}^{\infty}\left\{\frac{\bar{n}^n}{n!}exp(-\bar{n})\left[1+cos2\lambda\mathrm{t}\sqrt{n}\right]\right\}. \quad (4.201)$$

Similarly, for thermal state, by substituting Δ=0 into (4.200), we have

$$P_1(t) = \frac{1}{2}\sum_{n=0}^{\infty}\left\{\frac{1}{\bar{n}+1}\left(\frac{\bar{n}}{\bar{n}+1}\right)^n\left[1+cos2\lambda\mathrm{t}\sqrt{n}\right]\right\}. \quad (4.202)$$

Two-photon JCM probability functions

From (4.105), the probability that the atom is in the ground state is given by

$$P_1(t) = \sum_{n=0}^{\infty}\left\{|a_n|^2\left[\cos^2\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2 n(n-1)} + \frac{\Delta^2\mathrm{t}^2}{4}\frac{\sin^2\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2 n(n-1)}}{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2 n(n-1)}\right]\right\} \quad (4.105)$$

$$= \sum_{n=0}^{\infty}\left\{|a_n|^2\left[\frac{\Delta^2+2\lambda^2 n(n-1)}{\Delta^2+4\lambda^2 n(n-1)} + \frac{2\lambda^2 n(n-1)}{\Delta^2+4\lambda^2 n(n-1)}cos\left(2\sqrt{\frac{\Delta^2\mathrm{t}^2}{4}+\lambda^2\mathrm{t}^2 n(n-1)}\right)\right]\right\}. \quad (4.203)$$

<u>Case 1: Non-resonant case</u>

For coherent state, the photon number probability distribution $P(n)$ is given by

$$P(n) = \frac{\bar{n}^n}{n!}exp(-\bar{n}). \quad (2.8)$$

Now, $|a_n|^2 = P(n)$. By substituting (2.8) into (4.203), we have

$$P_1(t) = \sum_{n=0}^{\infty} \left\{ \frac{\bar{n}^n}{n!} exp(-\bar{n}) \left[\frac{\Delta^2 + 2\lambda^2 n(n-1)}{\Delta^2 + 4\lambda^2 n(n-1)} \right.\right.$$
$$\left.\left. + \frac{2\lambda^2 n(n-1)}{\Delta^2 + 4\lambda^2 n(n-1)} cos\left(2\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1)} \right) \right] \right\}. \quad (4.204)$$

For thermal state, the photon number probability distribution $P(n)$ is given by

$$P(n) = \frac{1}{\bar{n}+1} \left(\frac{\bar{n}}{\bar{n}+1} \right)^n. \quad (2.17)$$

Again, $|a_n|^2 = P(n)$. By substituting (2.17) into (4.203), we have

$$P_1(t) = \sum_{n=0}^{\infty} \left\{ \frac{1}{\bar{n}+1} \left(\frac{\bar{n}}{\bar{n}+1} \right)^n \left[\frac{\Delta^2 + 2\lambda^2 n(n-1)}{\Delta^2 + 4\lambda^2 n(n-1)} \right.\right.$$
$$\left.\left. + \frac{2\lambda^2 n(n-1)}{\Delta^2 + 4\lambda^2 n(n-1)} cos\left(2\sqrt{\frac{\Delta^2 \mathrm{t}^2}{4} + \lambda^2 \mathrm{t}^2 n(n-1)} \right) \right] \right\}. \quad (4.205)$$

Case 2: Resonant case

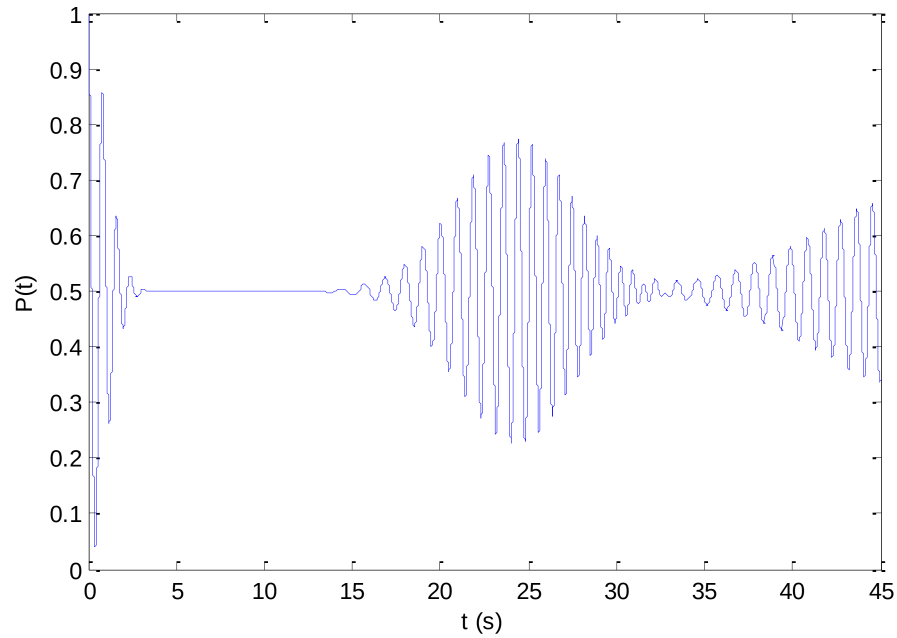
For resonant case, Δ=0. So, for coherent state, (4.204) becomes

$$P_1(t) = \frac{1}{2} \sum_{n=0}^{\infty} \left\{ \frac{\bar{n}^n}{n!} exp(-\bar{n}) \left[1 + cos 2\lambda \mathrm{t} \sqrt{n(n-1)} \right] \right\}. \quad (4.206)$$

Similarly, for thermal state, by substituting Δ=0 into (4.205), we have

$$P_1(t) = \frac{1}{2} \sum_{n=0}^{\infty} \left\{ \frac{1}{\bar{n}+1} \left(\frac{\bar{n}}{\bar{n}+1} \right)^n \left[1 + cos 2\lambda \mathrm{t} \sqrt{n(n-1)} \right] \right\}. \quad (4.207)$$

In the above calculations, $P_1(t)$ denotes the probability that the atom is in the ground state. Next, the graphs of probability $P_1(t)$ (i.e. P(t) in y-axis of the graphs) against time t will be plotted for single-photon and two-photon JCM with different mean photon number $\bar{n}$, detuning Δ and interaction strength λ. The effects of these parameters on the Rabi oscillations will be discussed. Figure 4.1 to 4.12 in the next part give the graphical results for single-photon JCM while Figure 4.13 to 4.21 give the graphical results for two-photon JCM.



Graph of probability that the atom is in the ground state P(t) against time t(s).

Figure 4.1: Coherent state, $\lambda = 1s^{-1}$, $\bar{n} = 15$, Δ= 0.

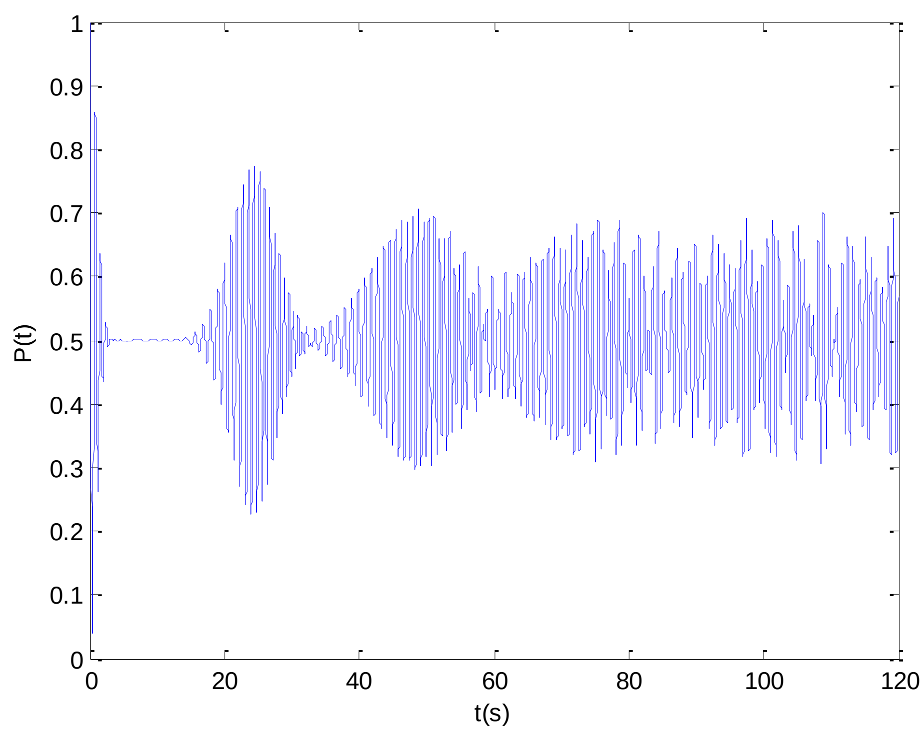


Figure 4.2: Coherent state, $\lambda = 1s^{-1}$, $\bar{n} = 15$, Δ= 0.

According to the initial condition, the atom is in the ground state (lower energy level) at t=0. Therefore, in Figure 4.1, the probability P is equals to 1 at t=0. Then, the atom oscillates between the upper and lower energy levels and reaches a period of quiescence (roughly between t=5s and t=15s). During the period of quiescence, there is no information whether the atom is in the lower or upper energy level. The atom has equal probability (P(t)=0.5) to be in the upper and lower energy level. After the period of quiescence, there is a revival of oscillation of the atom between its upper and lower energy level. This oscillation is also known as the Rabi Oscillation. Then, the Rabi Oscillations collapse and followed by another revival of the oscillations. It can be seen from Figure 4.2 that as time increases, the collapses and revivals features become less prominent.

The collapse or damping of the Rabi Oscillations has nothing to do with energy dissipation. It is due to the destructive interference of the oscillating terms. The probability $P(t)$ that the atom is in the ground state is given by

$$P(t) = \frac{1}{2}\sum_{n=0}^{\infty}\left[\frac{\bar{n}^n}{n!}exp(-\bar{n})\right]\left(1 + cos2\lambda\sqrt{n}t\right) .$$

Therefore, $P(t)$ is the sum of the oscillating cosinusoidal terms, $cos2\lambda\sqrt{n}t$. If there are terms oscillating by 180^{o} out of phase with each other, then there will be an approximate cancellation of these terms. This will then result in destructive interference which causes the collapse of Rabi Oscillations. Conversely, if there are terms which are in phase with each other, a constructive interference which results in the revival of Rabi Oscillations is observed.

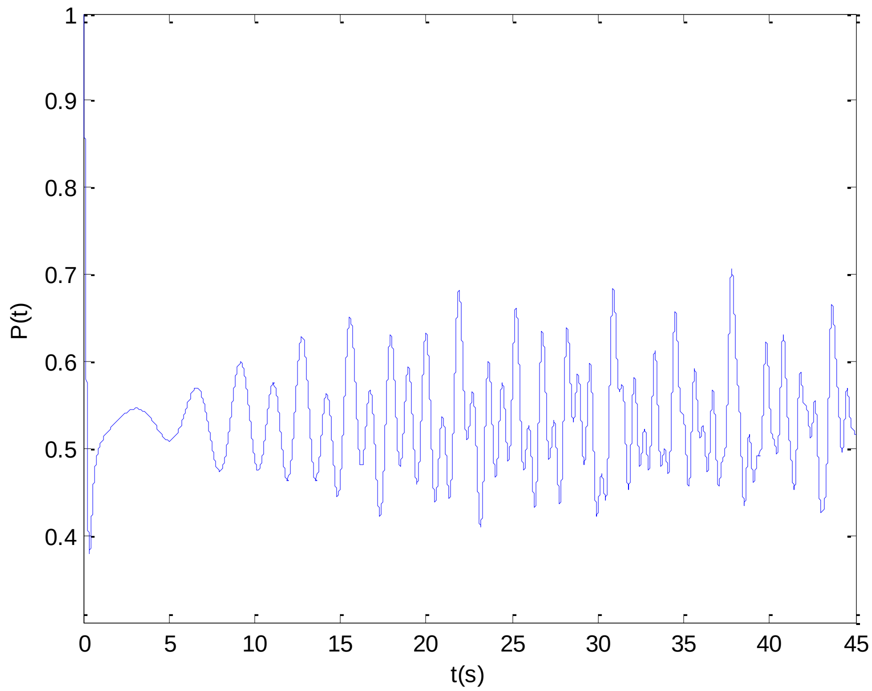


Figure 4.3: Thermal state, $\lambda = 1s^{-1}$, $\bar{n} = 15$, Δ= 0.

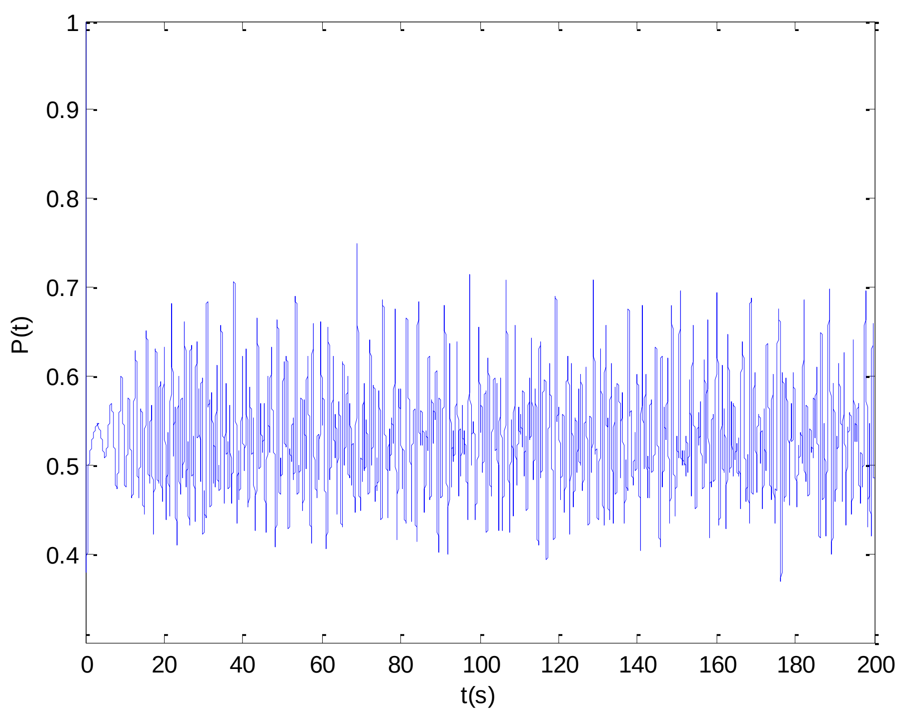


Figure 4.4: Thermal state, $\lambda = 1s^{-1}$, $\bar{n} = 15$, Δ= 0.

Figure 4.3 and 4.4 show the results obtained by using thermal field as the initial field state. There is no pattern and no collapse and revival features observed. The behavior of the energy states of the atom is completely chaotic as shown in Figure 4.3. This chaotic behavior persists even though time t increases to a large value t=200s, as illustrated in Figure 4.4. This could be explained by the fact that thermal state is a statistical mixture which has only minimum information. Unlike coherent states (which its wave function can be defined as the linear superposition of number states), we do not have enough information to form the wave function of a thermal state. Hence, it is physically logical that when a thermal initial field state interacts with the atom, it gives a chaotic behavior of the probability that the atom is in its ground state against time. This is in contrast with the case of using coherent initial field state discussed earlier, which shows a nice collapses and revivals of the Rabi oscillations.

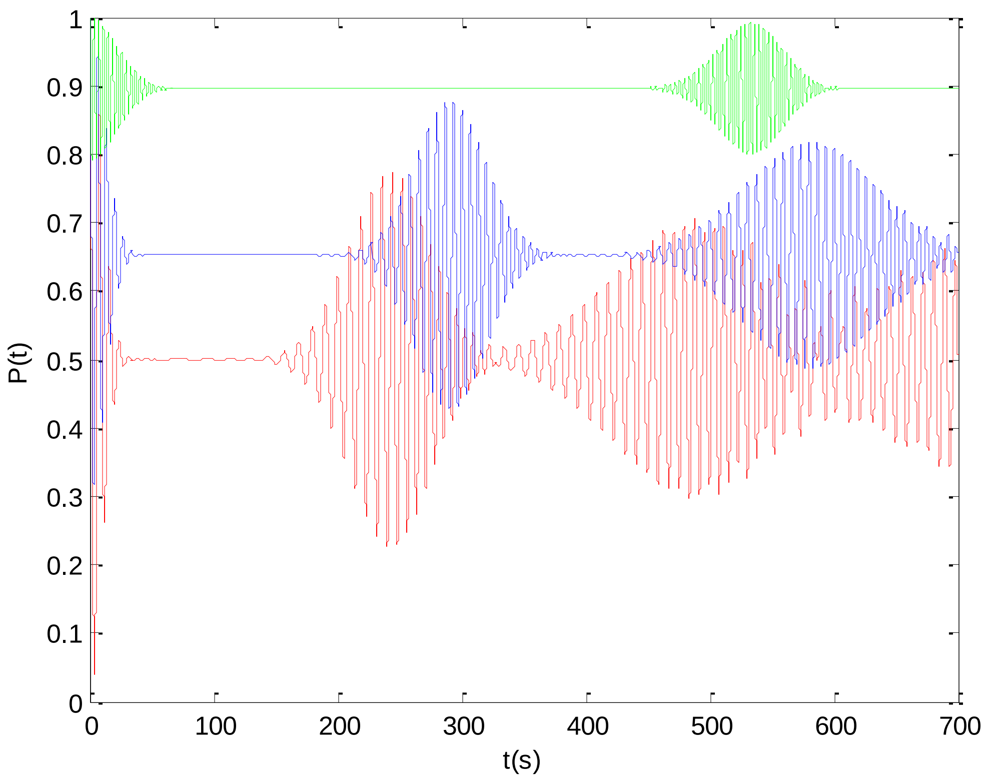


Figure 4.5: Coherent state, $\lambda = 0.1s^{-1}$, $\bar{n} = 15$, $\Delta = 0$ rad/s (red), $\Delta = 0.5$ rad/s (blue), $\Delta = 1.5$ rad/s (green)

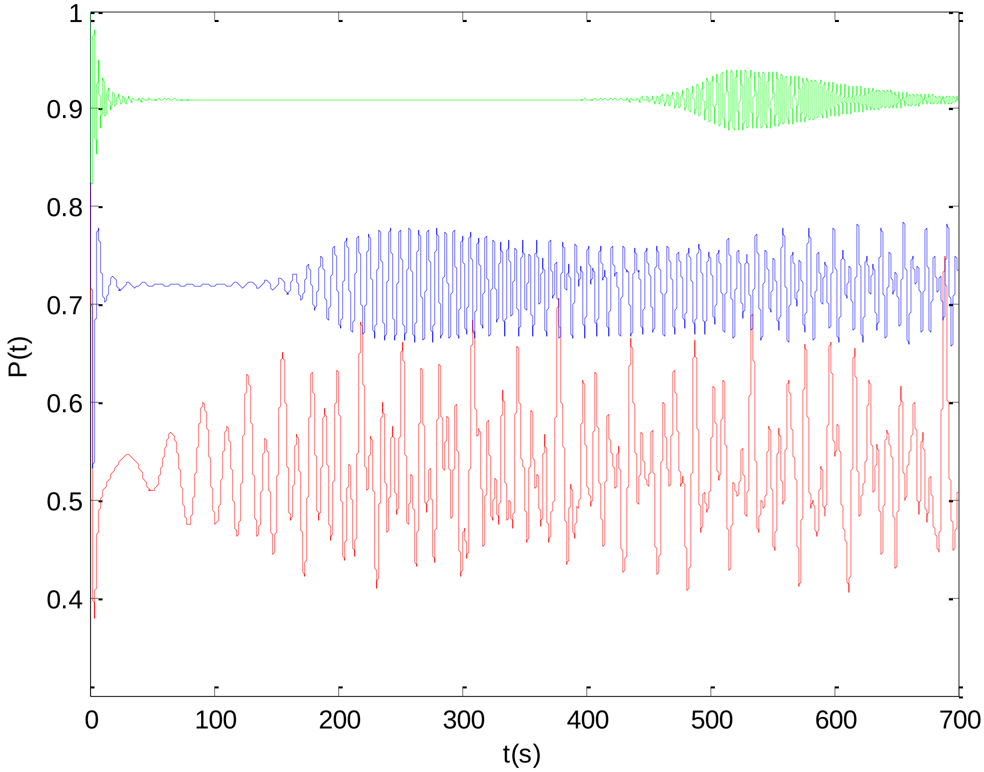


Figure 4.6: Thermal state, $\lambda = 0.1s^{-1}$, $\bar{n} = 15$, Δ= 0 rad/s (red), Δ= 0.5 rad/s (blue), Δ= 1.5 rad/s (green)

From Figure 4.5 and 4.6, it could be seen that when detuning Δ increases from Δ=0 rad/s to Δ=1.5 rad/s, both graphs are shifted upwards. These mean that the atom is more probable to stay in the ground state as detuning increases. For example, from both graphs in Figure 4.5 and 4.6 with Δ=1.5 rad/s, the probability that the atom is in the ground state lies roughly in between 0.8 and 1, which shows very high probability that the atom will remain in the ground state. This is because the increase in detuning means the difference between the monochromatic light frequency and frequency corresponds to the energy gap between the two-level atom becomes larger. Therefore, the electron in the atom is less probable to be excited to upper energy level since the energy E carried by the photon (E=hf), where h is Planck's constant and f is light frequency) is either insufficient or too much for the excitation of the electron to the upper energy level. Hence, it can be deduced that when the monochromatic light frequency is very far different from the frequency corresponds to energy gap of the atom, the photon will not be absorbed by the atom and P(t) will approach 1 very closely. From Figure 4.5, it is interesting to find out that the collapses and revivals features of the Rabi oscillations are maintained even though the detuning increases. From Figure 4.6, long quiescent period starts to form as detuning increases.

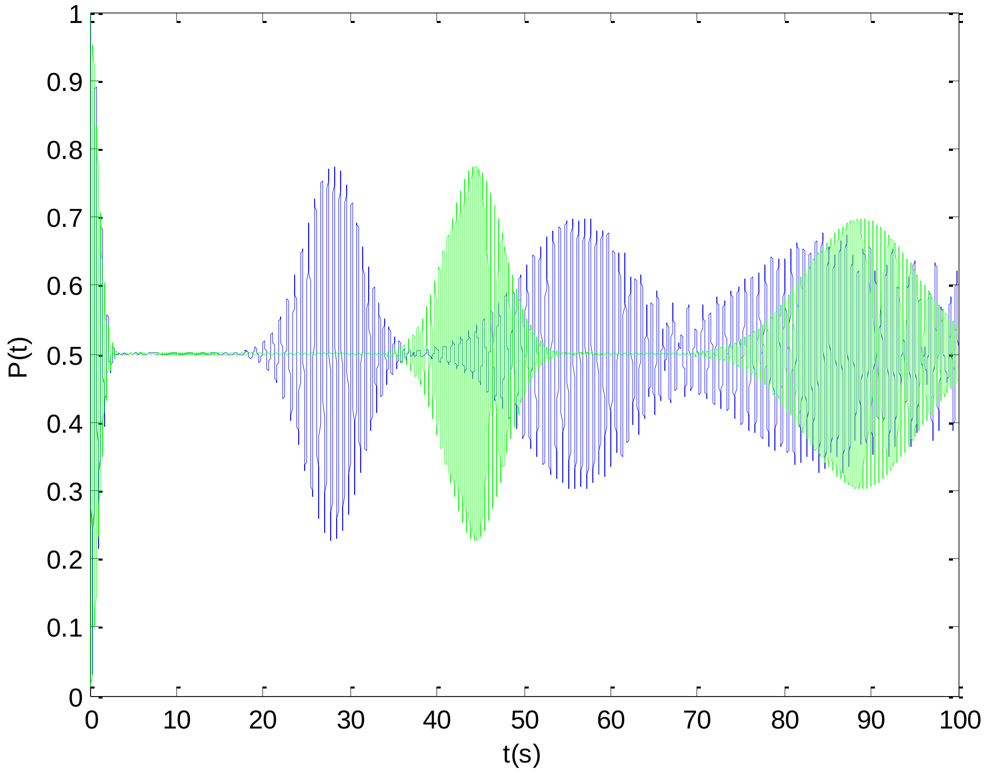


Figure 4.7: Coherent state, $\boldsymbol{\lambda = 1s^{-1}}$, Δ= $\boldsymbol{0rad/s}$, $\boldsymbol{\bar{n} = 20}$ (blue), $\boldsymbol{\bar{n} = 50}$ (green)

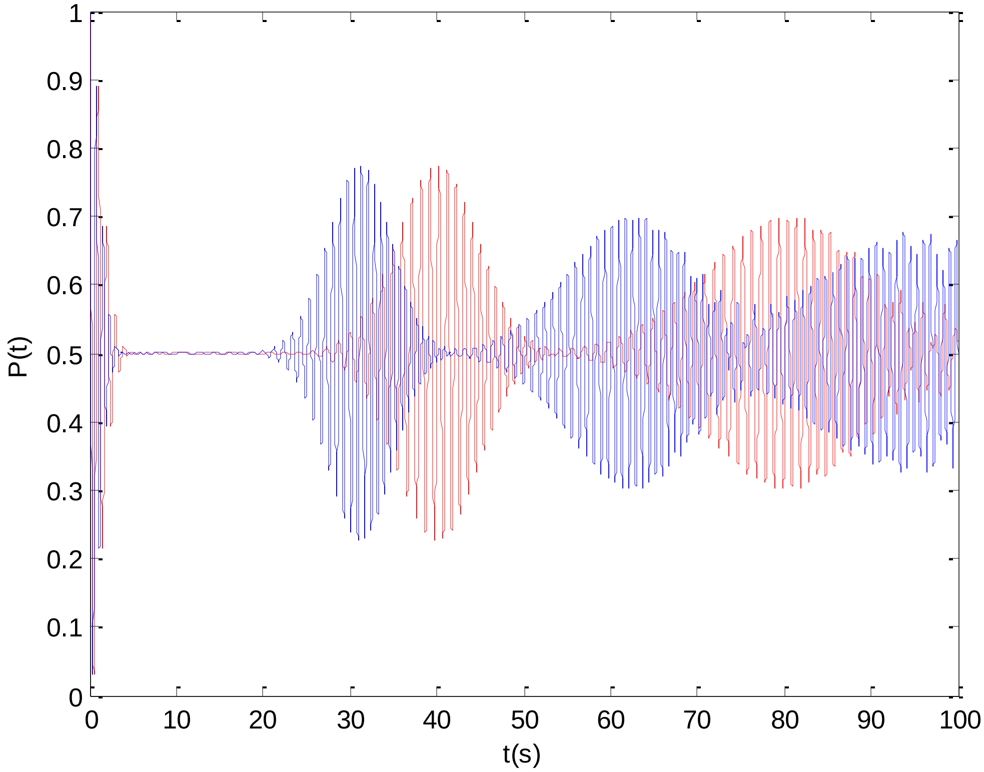


Figure 4.8: Coherent state, $\lambda = 0.7s^{-1}$(red), $\lambda = 0.9s^{-1}$(blue), $\Delta = 0rad/s, \bar{n} = 20$.

Let Ω be the Rabi frequency of the single-photon JCM of resonant case. The Rabi frequency occurs at a frequency $\Omega(\bar{n})$ approximately equals to $2\lambda\bar{n}^{1/2}$, where λ is the interaction strength and $\bar{n}$ is the mean photon number.

Let the time at which the first peak of the revival occurs to be T_{rev}. This peak of revival occurs because of the sum of a number of oscillating terms in sum (4.201) which are in phase. Therefore, T_{rev} can be estimated by making the terms at the peak of the photon number probability distribution of coherent state to be in phase as shown below.

$$\Omega(\bar{n})T_{rev} - \Omega(\bar{n}-1)T_{rev} = 2\pi$$

$$2\lambda\bar{n}^{1/2}T_{rev} - 2\lambda(\bar{n}-1)^{1/2}T_{rev} = 2\pi. \qquad (4.208)$$

Since $(\bar{n}-1)^{1/2} = \bar{n}^{1/2}\left(1 - \frac{1}{\bar{n}}\right)^{1/2}$ and

$$\left(1-\frac{1}{\bar{n}}\right)^{1/2} = 1+\frac{1/2}{1!}\left(-\frac{1}{\bar{n}}\right)+\frac{(1/2)(1/2-1)}{2!}\left(-\frac{1}{\bar{n}}\right)^{2}+\ \dots\dots\dots$$

$$\approx 1-\frac{1}{2}\left(\frac{1}{\bar{n}}\right),$$

$$(\bar{n}-1)^{1/2} \approx \bar{n}^{1/2}\left(1-\frac{1}{2\bar{n}}\right) \text{ when } \bar{n} \gg 1. \tag{4.209}$$

By substituting (4.209) into (4.208), we have

$$2\lambda\bar{n}^{1/2}T_{rev} - 2\lambda\bar{n}^{1/2}\left(1-\frac{1}{2\bar{n}}\right)T_{rev} = 2\pi$$

$$\lambda(\bar{n})^{-1/2}T_{rev} = 2\pi$$

$$T_{rev} = \frac{2\pi}{\lambda}\bar{n}^{1/2}. \tag{4.210}$$

From (4.210), it can be deduced that when mean photon number $\bar{n}$ increases, the time of the first peak of revival, T_{rev} increases. This is evident from Figure 4.7, in which the T_{rev} for $\bar{n}$ = 20 is about 28s. Then, when $\bar{n}$ is increased to $\bar{n}$ = 50, T_{rev} becomes around 45s. From (4.210), it can be computed that $T_{rev} = 28.1s$ at $\bar{n}$ = 20 and $T_{rev} = 44.4s$ at $\bar{n}$ = 50. So, this explains quantitatively the reason why the Rabi Oscillations in Figure 4.7 shifts to the right when $\bar{n}$ rises.

Apart from that, (4.210) also shows that the T_{rev} is inversely proportional to the interaction strength, λ. This gives the reason why the Rabi Oscillations in Figure 4.8 shifts to the left when λ increases. The shift is caused by the drop of T_{rev} from 40.1s to 31.2s as λ increases from 0.7s^{-1} to 0.9s^{-1}, as calculated from (4.210). The calculated values of T_{rev} also agree well with the results given in Figure 4.8, which is T_{rev} ≈ 40s at λ = 0.7s^{-1} and T_{rev} ≈ 30s at λ = 0.9s^{-1}.

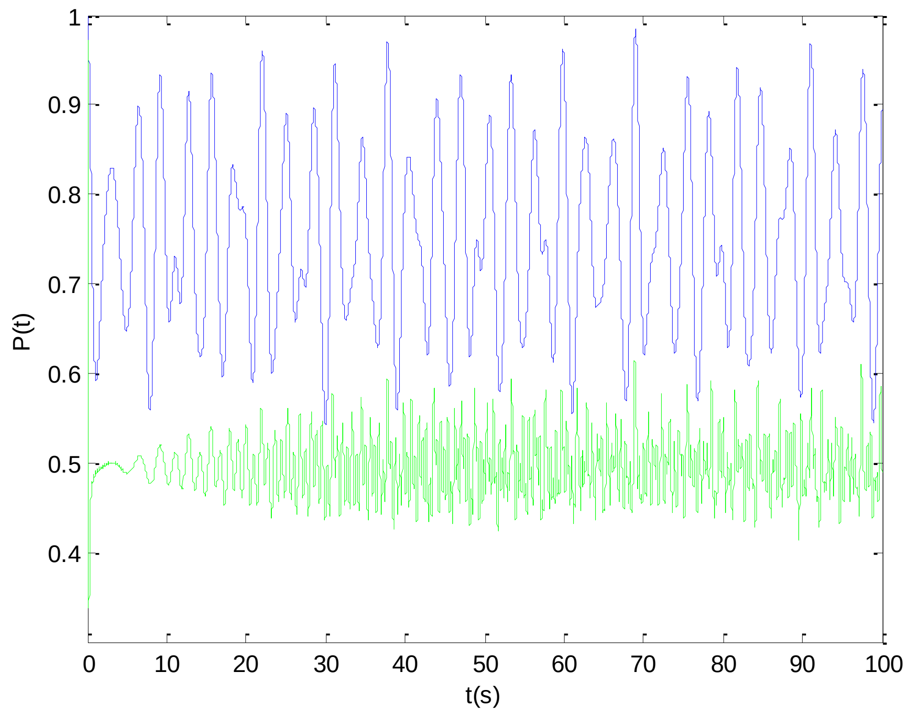


Figure 4.9: Thermal state, $\lambda = 1s^{-1}$, $\Delta = 0 rad/s$, $\bar{n} = 1$ (blue), $\bar{n} = 50$ (green)

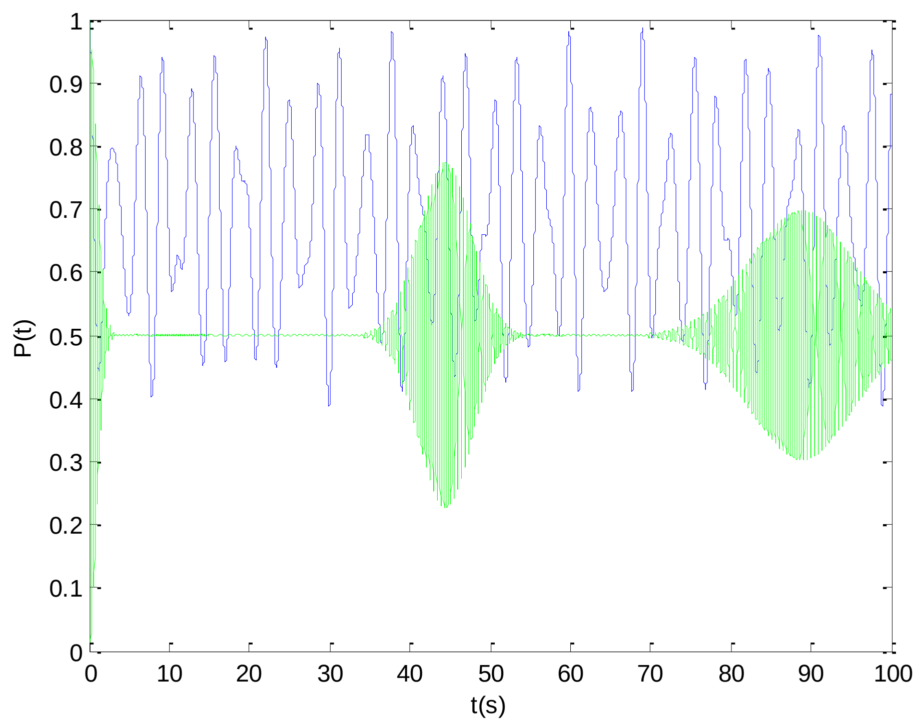


Figure 4.10: Coherent state, $\lambda = 1s^{-1}$, $\Delta = 0 rad/s$, $\bar{n} = 1$ (blue), $\bar{n} = 50$ (green)

In both Figure 4.9 and 4.10, the graph is shifted upwards when the mean photon number $\bar{n}$ decreases from 50 to 1. This indicates that when the mean photon number $\bar{n}$ is very low ($\bar{n}$ = 1), it is more probable that the atom will remain in its ground state. This is because when $\bar{n}$ is very low, it implies that a very low number of photons are present inside the cavity containing the single two-level atom. Hence, the atom is less probable to interact with the photons and get excited to the upper energy level. Therefore, the atom is more probable to remain in the ground state. Besides that, it could also be observed from Figure 4.10 that the Rabi Oscillatory behavior disappears and chaotic results are exhibited when coherent state at low mean photon number is used as the initial field state. For the case in Figure 4.9, the results obtained remains chaotic at low mean photon number. When $\bar{n}$ becomes very large, certain fluctuating regions in Figure 4.10 become quiescent periods. This means for the case of initial coherent state, the atom has equal probability to be excited or de-excited when mean photon number is very large.

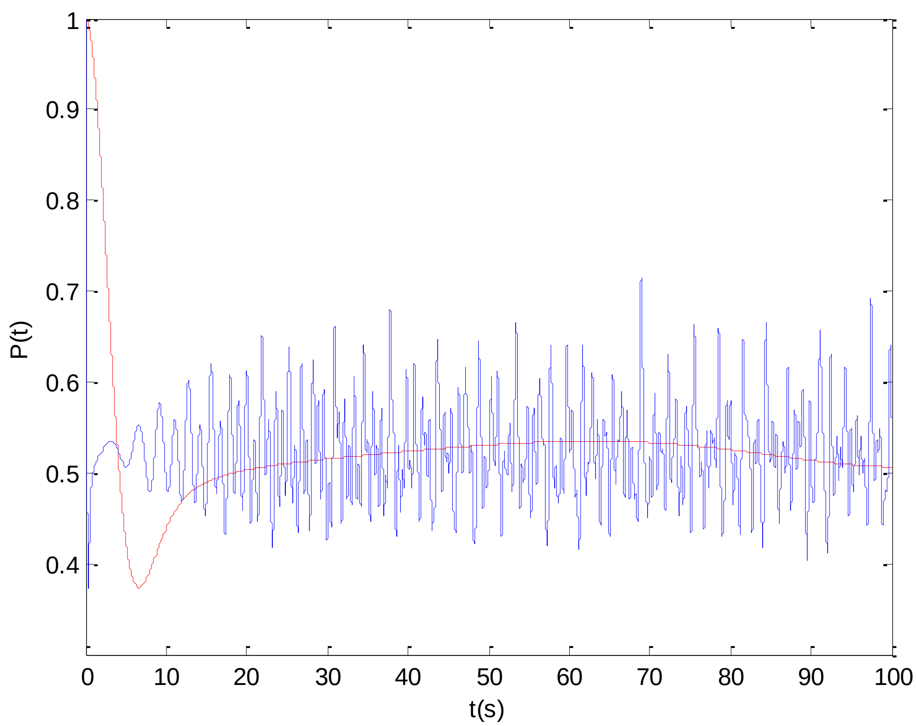


Figure 4.11: Thermal state, $\lambda = 0.05s^{-1}$(red), $\lambda = 1s^{-1}$(blue), $\Delta = 0rad/s, \bar{n} = 20$

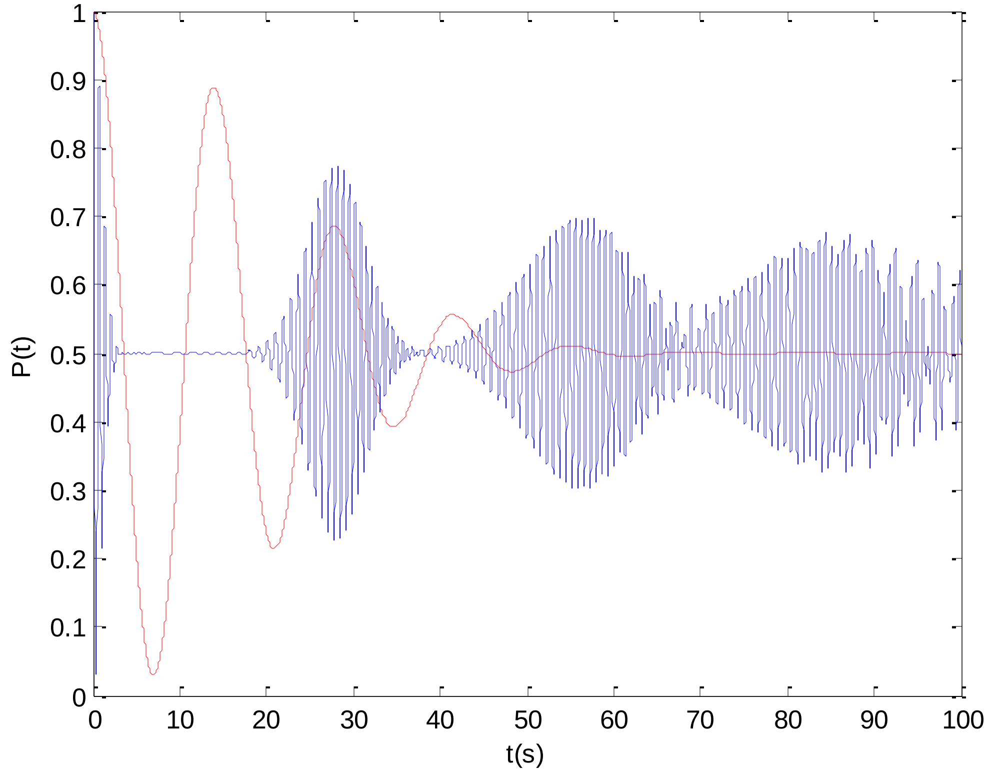


Figure 4.12: Coherent state, $\boldsymbol{\lambda = 0.05s^{-1}}$(red), $\boldsymbol{\lambda = 1s^{-1}}$(blue), Δ= $\boldsymbol{0rad/s, \bar{n} = 20}$

It is also interesting to find out from Figure 4.11 that when the interaction strength λ is decreased to a very low value, the original chaotic behavior disappears. At this stage, P(t) is approximately equal to a constant value of 0.5 and hence there is no information whether the atom is in the excited or ground state.

For the case in Figure 4.12, at low interaction strength $\lambda = 0.05\mathrm{s}^{-1}$, the original collapses and revivals behavior of the Rabi Oscillations are replaced by a small number of oscillations and then followed by a long quiescent period as shown in Figure 4.12. During the quiescent period, the energy state of the atom is also unknown.

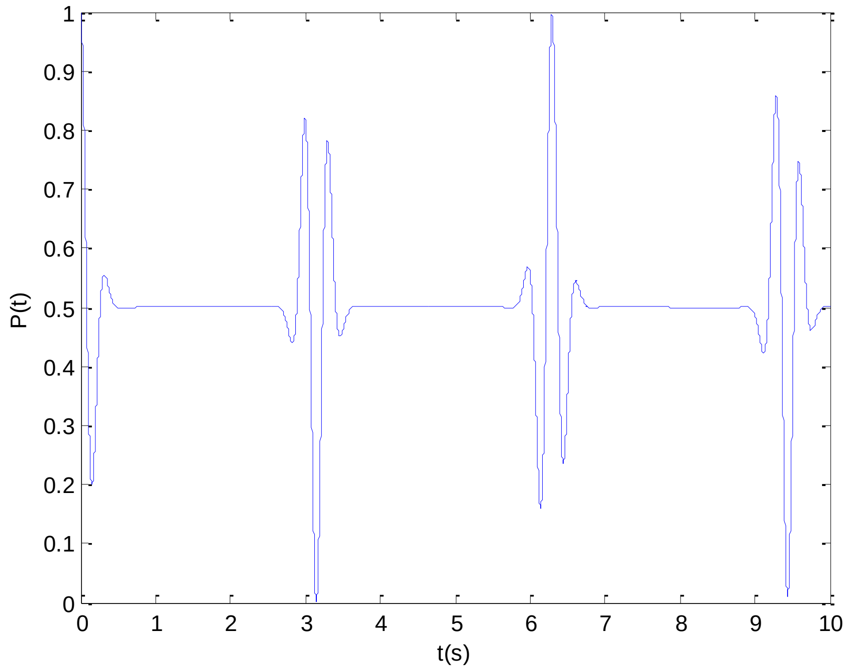


Figure 4.13: Two-photon JCM, coherent state, $\lambda = 1s^{-1}$, Δ= $0rad/s, \bar{n} = 10$.

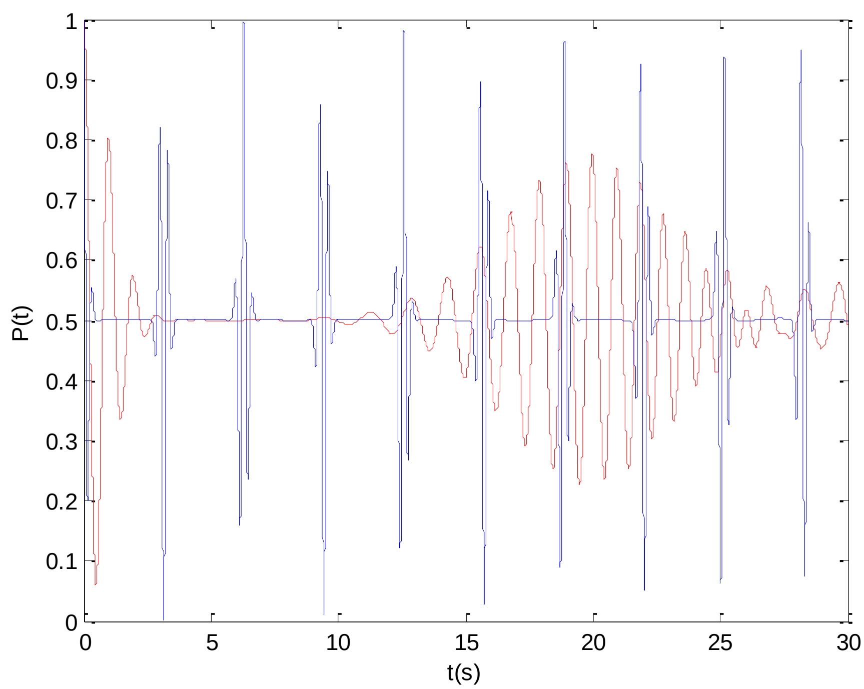


Figure 4.14: Coherent state, $\lambda = 1s^{-1}$, Δ= $0\text{rad}/s, \bar{n} = 10$, single-photon JCM(red), two-photon JCM(blue).

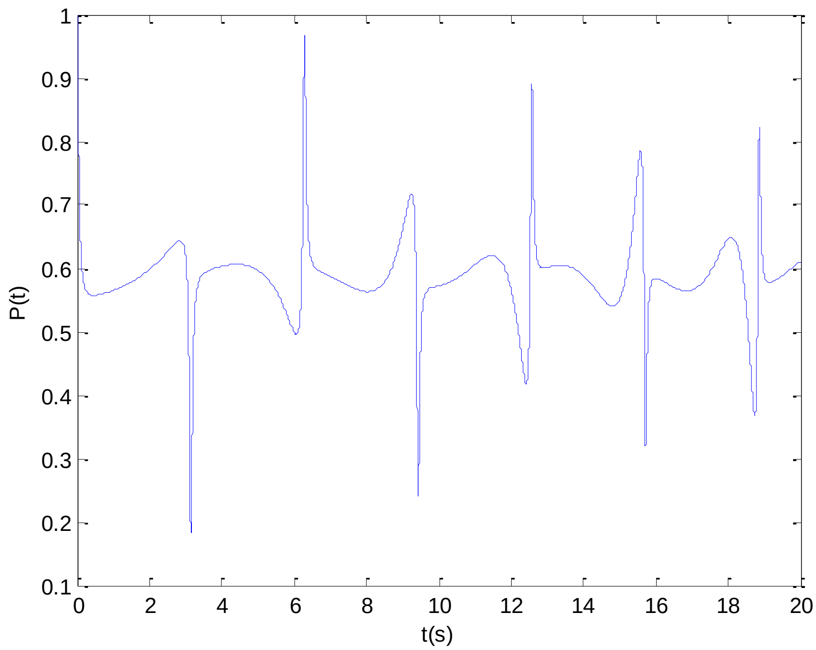


Figure 4.15: Two-photon JCM, thermal state, $\lambda = 1s^{-1}$, $\Delta = 0rad/s, \bar{n} = 10$.

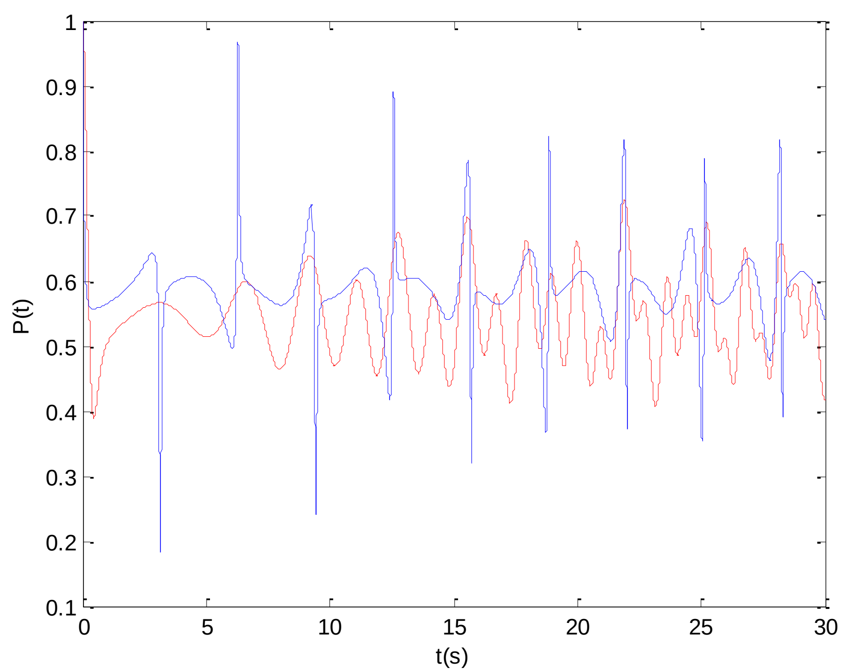


Figure 4.16: Thermal state, $\lambda = 1s^{-1}$, $\Delta = 0rad/s, \bar{n} = 10$, single-photon JCM(red), two-photon JCM(blue).

Figure 4.13 gives the relationship between probability that the atom is in its ground state and time in the two-photon JCM. It could be observed from Figure 4.13 that the atom oscillates between upper and lower energy levels with quiescence periods in between the oscillations. Then, Figure 4.14 compares the single-photon and two-photon JCM results obtained by using coherent initial field states. One of the major differences between them is that the Rabi oscillations of the energy states of the atom in two-photon JCM have sharp peaks or sharp dips compared with the case of single-photon JCM. Besides that, the number of cycles of the Rabi oscillations exhibited in two-photon JCM is also less than the single-photon case. Apart from that, the quiescent period in between the oscillations in two-photon JCM is also shorter than that in single-photon JCM. This also means that the Rabi Oscillations in two-photon JCM collapse and revive faster than that in single-photon JCM.

From Figure 4.15, it could be observed that the results generated by using thermal initial field state in two-photon JCM consist of sharp peaks and dips. By comparing with single-photon JCM case in Figure 4.16, the results shown in two-photon JCM is less chaotic compared with single-photon JCM.

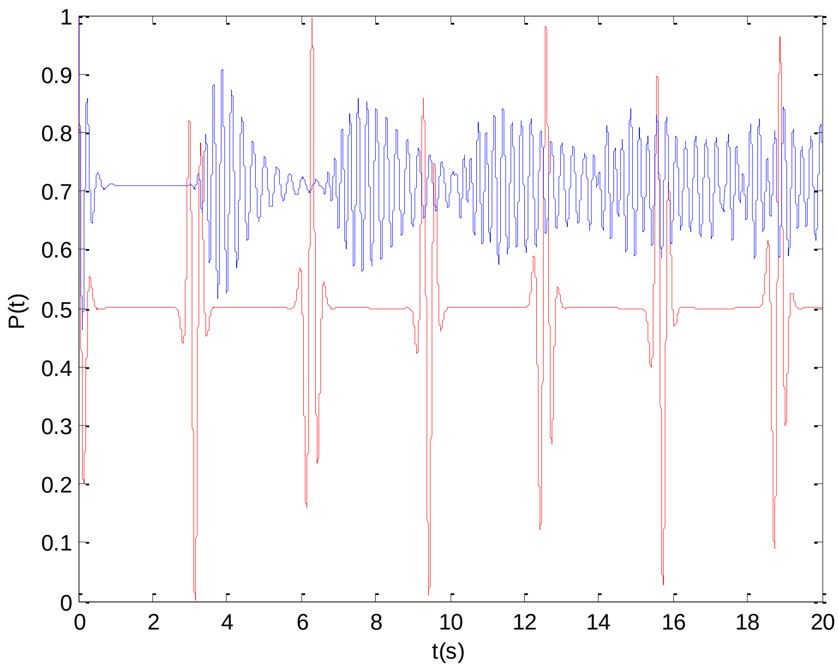


Figure 4.17: Two-photon JCM, coherent state, $\boldsymbol{\lambda} = \mathbf{1s^{-1}}$, $\Delta = \mathbf{0rad/s}$(red), $\Delta = \mathbf{15rad/s}$ (blue), $\overline{\boldsymbol{n}} = \mathbf{10}$.

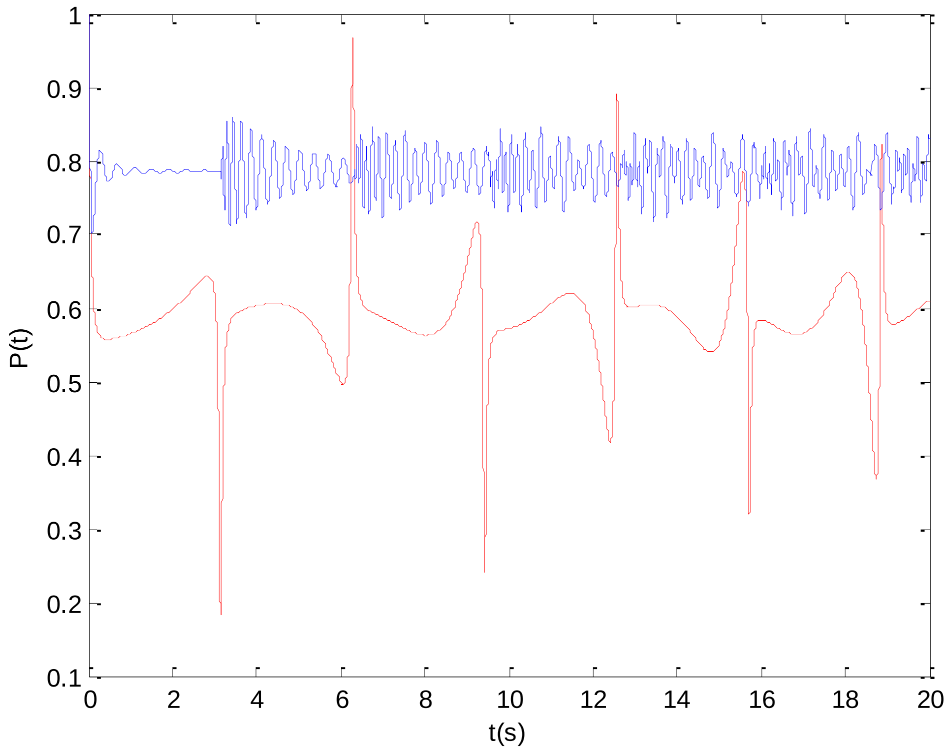


Figure 4.18: Two-photon JCM, thermal state, $\boldsymbol{\lambda = 1s^{-1}}$, $\Delta = \mathbf{0rad/s}$(red), $\Delta = \mathbf{15rad/s}$ (blue), $\boldsymbol{\bar{n} = 10}$.

From Figure 4.17, it could be observed that when detuning Δ increases from 0 rad/s to 15rad/s, the Rabi Oscillations are shifted upwards. Besides that, the sharp peaks and dips observed in the resonant case also disappear. The P(t) also becomes more oscillatory when the detuning becomes very large. The shifting of the Rabi Oscillations in Figure 4.17 shows that the atom is more probable to stay in the ground state when detuning increases.

For the case in Figure 4.18, when the detuning increases, the graph is also shifted upwards and the sharp peaks and dips are replaced by a small number of oscillations at first and then followed by chaotic behavior at later time. This result also shows that the atom is more probable to remain in the ground state as detuning increases. This shares the same conclusion with the case shown in Figure 4.17.

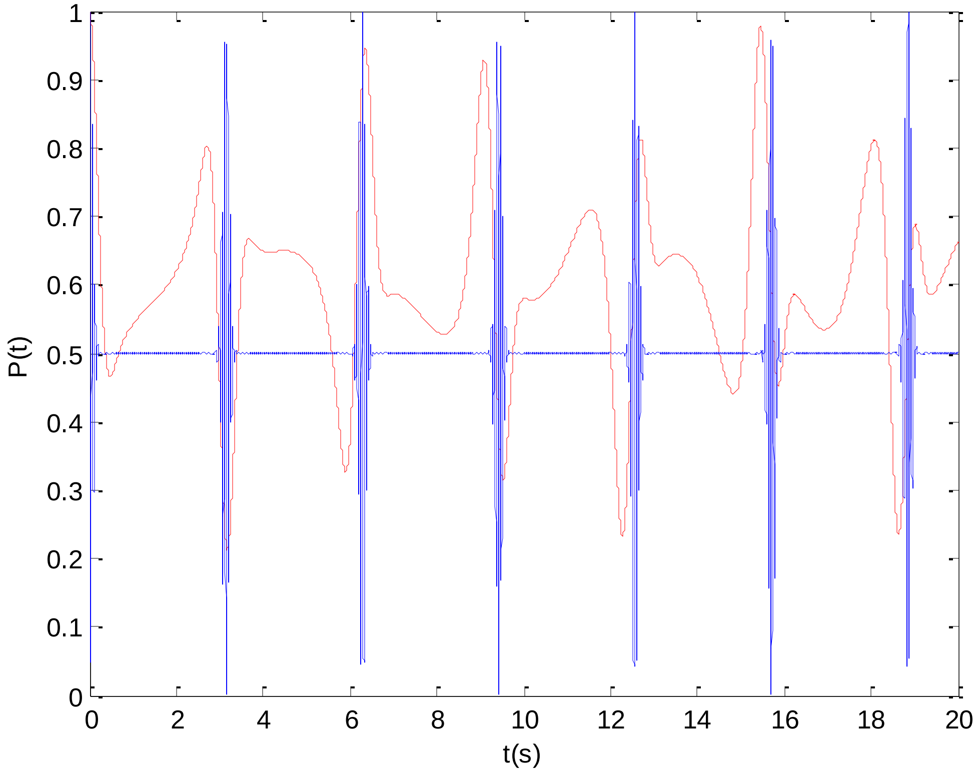


Figure 4.19: Two-photon JCM, coherent state, $\boldsymbol{\lambda = 1s^{-1}}$, Δ= $\mathbf{0rad/s}$, $\boldsymbol{\bar{n} = 3}$(red), $\boldsymbol{\bar{n} = 50}$(blue).

From Figure 4.19, it can be seen that when mean photon number of the coherent state becomes very small, the Rabi oscillations are replaced by fluctuations in the energy state of the atom. The Rabi Oscillations and quiescent periods are only observed when the mean photon number is sufficiently large. By comparing the red colour graph in Figure 4.17 ($\bar{n}$ = 10**)** and blue colour graph in Figure 4.19 ($\bar{n}$ = 50), it could be concluded that the Rabi frequency increases as mean coherent photon number becomes larger.

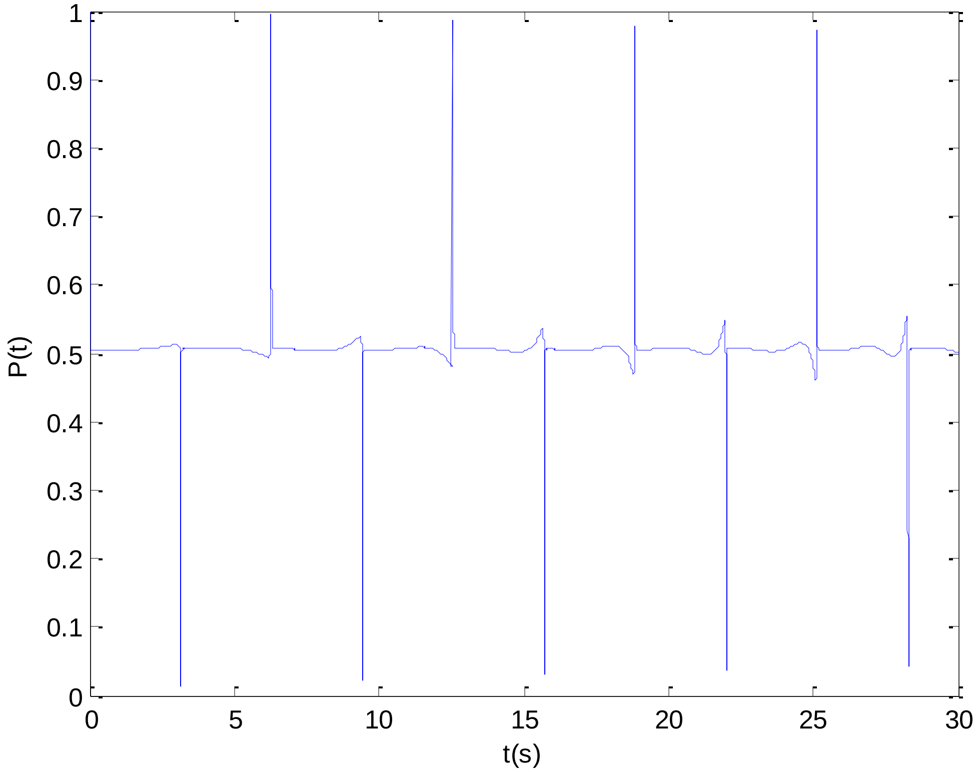


Figure 4.20: Two-photon JCM, thermal state, $\boldsymbol{\lambda = 1s^{-1}}$, Δ= $\boldsymbol{0rad/s, \bar{n} = 150}$.

From Figure 4.20, it is interesting to find that when the mean thermal photon number is very large ($\bar{n} = 150$), the chaotic behavior of the results shown in Figure 4.15 disappears. The sharp peaks and dips are separated clearly by short quiescent period in between them. The peak indicates that the atom is in the ground state while the dip means that the atom is in the excited state. For the quiescent periods in between the peaks and dips, there is no information whether the atom is in the ground or excited state.

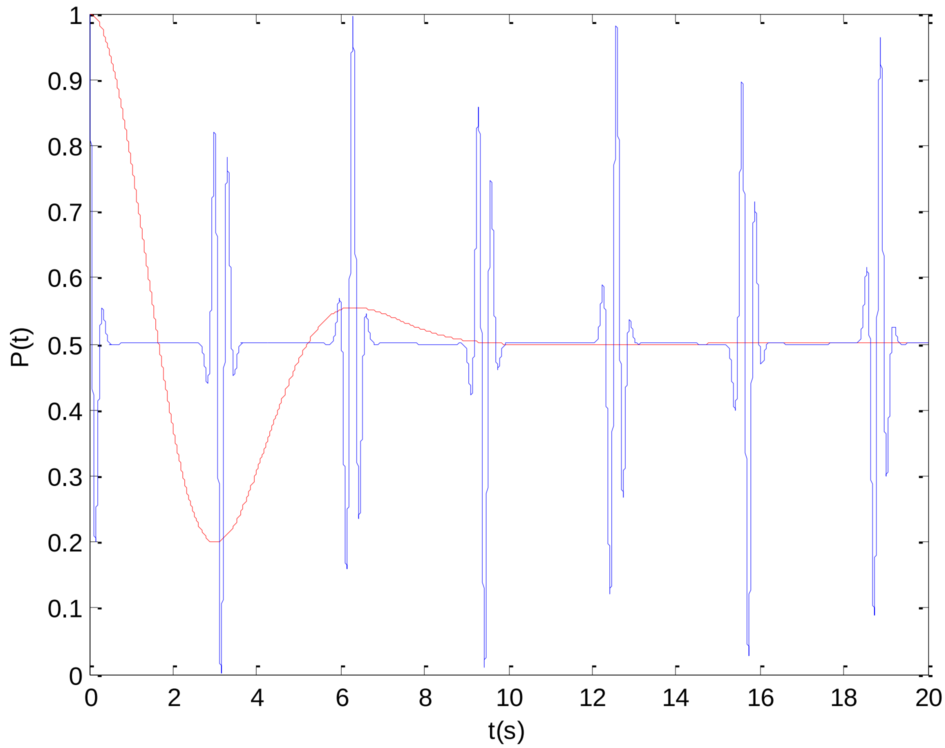


Figure 4.21: Two-photon JCM, coherent state, $\boldsymbol{\lambda = 0.05s^{-1}}$(red), $\boldsymbol{\lambda = 1s^{-1}}$(blue), $\boldsymbol{\Delta = 0rad/s, \bar{n} = 10}$.

From Figure 4.21, the sharp peaks, dips and oscillations disappear when the interaction strength λ becomes very small. At low interaction strength, there is only a small oscillation of the atom's energy state at the beginning. Then, the energy state of the atom enters a quiescent stage in which the energy state of the atom is not known. This quiescent behavior at small λ is also exhibited by the atom in single-photon JCM case shown in Figure 4.12.

5.1 Recommendations

In this theoretical study of Jaynes-Cummings Model (JCM), the Rotating Wave Approximation (RWA) has been made to omit the non-energy conserving terms. However, it is worth to study the JCM without RWA in order to investigate the effect of these non-energy conserving terms on the behavior of the combined atom-field system. Therefore, the JCM-type unitary operator derived without RWA in the methodology could be used as future in-depth study of JCM. Besides that, the combined atom-field system studied in this project is also modeled as ideal and closed system. In fact, the combined system could be generalized to open system for future study. In open system, the energy dissipation and noise will be taken into consideration. Hence, the JCM Hamiltonian for open system has a different form. Apart from that, the JCM could also be generalized to multilevel atom or multi-atom case. Other than that, JCM could also be extended to the case where a two-level atom interacts with multimode electromagnetic field. In this case, the field Hamiltonian will be modified as the summation of the field Hamiltonian of each mode.

5.2 Conclusion

In conclusion, the Jaynes-Cummings Model (JCM) is solved in this project. During the process of solving JCM, the method of quantizing the electromagnetic field has been studied. Besides that, the quantization of the energy of two-level atom and Electric Dipole Approximation are discussed as well in Literature Review. After the JCM Hamiltonian is found in Schrodinger Picture, it is transformed unitarily into the Interaction Picture. The unitary operator which describes the time evolution of the quantum state has been derived from the Interaction Picture JCM Hamiltonian. Finally, with the unitary operator and known initial quantum state, the quantum state of the combined atom-field system is solved. Then, the probability that the atom is in the ground state as function of time has been derived. By using the probability functions, the graphs of probability against time are plotted with different detunings, interaction strengths and mean photon number. The collapse and revival feature of the Rabi Oscillations has been discussed and the effects of detuning, interaction strength and mean photon number on Rabi Oscillations are also investigated. The discussions of results are focused on the single-photon and two-photon JCM. Then, for three-photon and generalized k-photon JCM, only the expressions of their respective probability functions are derived.

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