CRANK-NICOLSON SCHEME FOR ASIAN OPTION

By

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ABSTRACT

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Finite difference scheme has been widely used in financial mathematics. In particular, the Black-Scholes option pricing model can be transformed into a partial differential equation and numerical solution for option pricing can be approximated using the Crank-Nicolson difference scheme. This approach provides a stable scheme under different volatility condition. Besides, it allows us to acquire the option value at different times, including time zero in a single iteration.

The thesis begins with a brief introduction to option pricing and a review on probability theory in Chapter 1 and 2, followed by a summary of some basic ideas and techniques for option of European style in Chapter 3. Chapter 4 and 5 contain the main results of this thesis and Chapter 6 is the conclusion.

In Chapter 4, we obtain the value of Asian option by solving a two-dimensional Black-Scholes equation using a simple Crank-Nicolson finite difference scheme. If S is the stock price and Z is the average stock price at time t, then the Black-Scholes equation for the Asian option price $\mathrm{F}(\mathrm{Z}, \mathrm{S}, \mathrm{t})$ is given by $\frac{\partial \mathrm{F}}{\partial \mathrm{t}} + \mathrm{rS}\frac{\partial \mathrm{F}}{\partial \mathrm{S}} + \frac{\sigma^2 \mathrm{S}^2}{2}\frac{\partial^2 \mathrm{F}}{\partial \mathrm{S}^2} + \frac{\partial \mathrm{Z_t}}{\partial \mathrm{t}}\frac{\partial \mathrm{F}}{\partial \mathrm{Z}} - \mathrm{rF} = 0$ with terminal value

$\mathrm{F(Z,S,T)} = \Phi(\mathrm{Z,S})$, where Φ is the payoff value at terminal time T. Then, using Crank-Nicolson finite difference scheme, it is approximated by $\frac{F_{i,j}^{h+1}-F_{i,j}^{h}}{\Delta t} + \frac{1}{2}\left[L_{i,j}^{h+1} + L_{i,j}^{h}\right] = 0$, where $L_{i,j}^{h} = \frac{\sigma^2 S_j^2}{2(\Delta S)^2}\left[F_{i,j+1}^{h} - 2F_{i,j}^{h} + F_{i,j-1}^{h}\right] + \frac{rS_j}{2\Delta S}\left[F_{i,j+1}^{h} - F_{i,j-1}^{h}\right] + \frac{S_j}{\Delta Z}\left[F_{i+1,j}^{h} - F_{i,j}^{h}\right] - rF_{i,j}^{h}$ and $F_{i,j}^{h}$ is the option value at time $h\Delta t$, stock price $j\Delta S$ and average stock price $i\Delta Z$. Essentially, the Crank-Nicolson scheme is an average of the forward and backward finite difference scheme. Since a terminal value condition is given, the Black-Scholes equation given above need to be solved backward in time for all values of S and Z. However, in numerical solution, we need to bring it into a finite domain. Thus boundary conditions arising from financial consideration need to be imposed as well. With proper boundary conditions, if the values on top layer (option values at time h) are known, values of the next layer at time $h-1$ can be obtained by solving the linear system arising from Crank-Nicolson scheme. We do this iteratively for $h = T,\ T-1 \dots 1$ to obtain the approximate Asian option values. Finally, these values were compared to those from other methods and found to be favorable.

In chapter 5, we solve the Asian pricing problem again by reducing it to the solution of a one-dimensional equation applying a *Change of Numéraire Argument* due to Jan Večeř [12, 13]. The result obtained is also comparable with option values obtained by solving a two-dimensional equation.

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FACULTY OF ENGINEERING AND SCIENCES

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Date : 08th August 2012

SUBMISSION OF THESIS

It is hereby certified that **LEE TSE YUENG** (ID No: **09UIM02242**) has completed this thesis entitled "CRANK-NICOLSON SCHEME FOR ASIAN OPTION" under the supervision of DR CHIN SEONG TAH (Supervisor) from the Department of Mathematical and Actuarial Sciences, Faculty of Engineering and Science.

I understand that the University will upload softcopy of my thesis in pdf format into UTAR Institutional Repository, which may be made accessible to UTAR community and public.

Yours truly,

(LEE TSE YUENG)

APPROVAL SHEET

This thesis entitled "**CRANK-NICOLSON SCHEME FOR ASIAN OPTION**" was prepared by LEE TSE YUENG and submitted as partial fulfillment of the requirements for the degree of Master of Mathematical Sciences at Universiti Tunku Abdul Rahman.

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DECLARATION

I, Lee Tse Yueng hereby declare that the thesis is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UTAR or other institutions.

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(LEE TSE YUENG)

Date : ________________________

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LIST OF ABBREVIATIONS

KLSE CI	Kuala Lumpur Stock Exchange Composite Index
PDE	Partial Differential Equation
SDE	Stochastic Differential Equation

CHAPTER 1

INTRODUCTION

1.1 Stock Price Model

Stock prices fluctuate widely in reaction to new information. Since market participants compete to be the first to profit from new information, as a result, all these information are immediately reflected in current price of the stock market. Hence, successive price changes are not correlated and the movement is unpredictable, since they depend on as-yet unrevealed information. However, we can obtain the expected size of the prices by using statistical method.

As an example, consider the KLSE CI (Kuala Lumpur Stock Exchange Composite Index) daily closing values from January 2, 2004, to February 15, 2008, for a total of 1020 data.

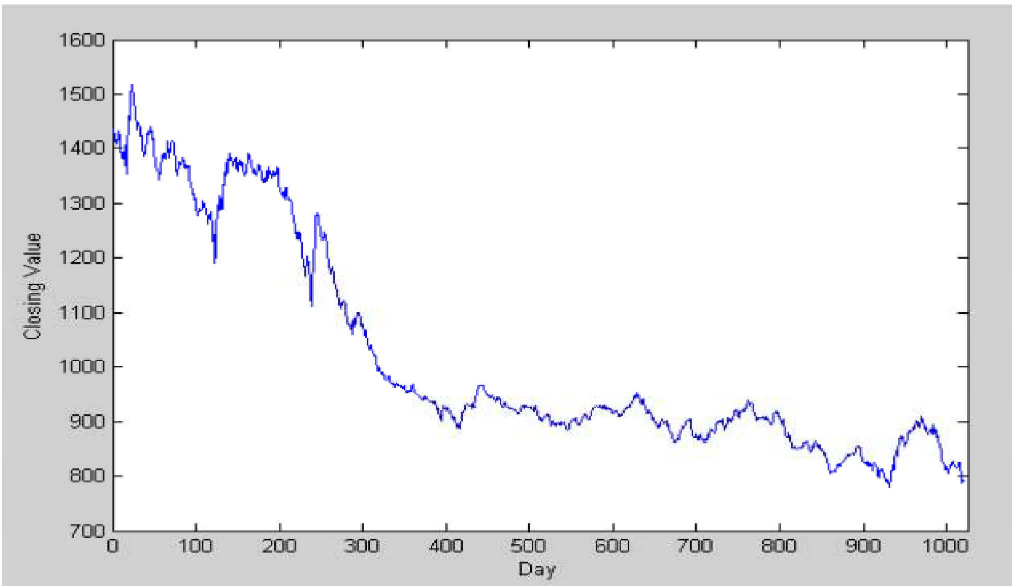

Figure 1.1: KLSE raw data plot

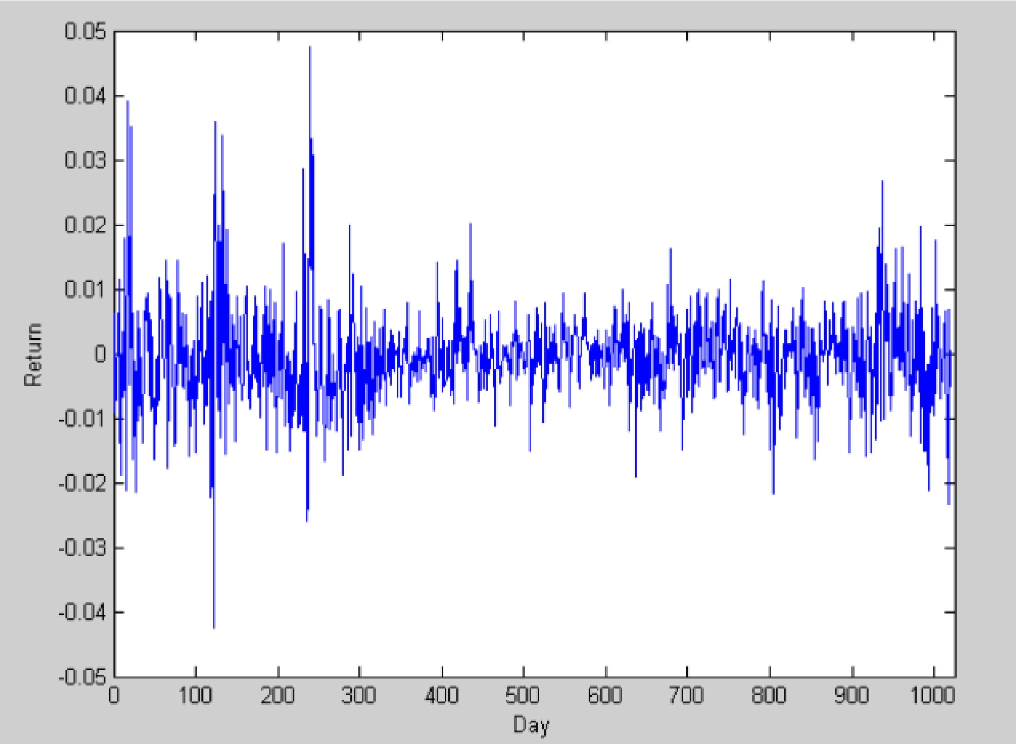

Figure 1.2: Return Series, $y(i) = \log\frac{p(i+1)}{p(i)}$

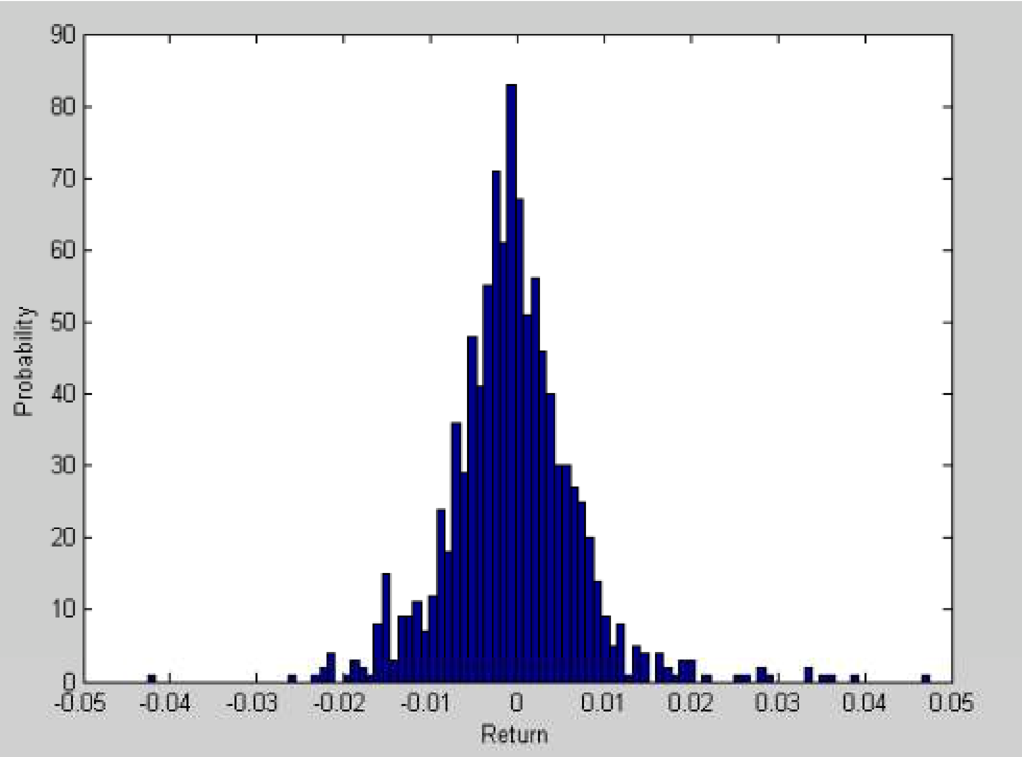

Figure 1.3: Histogram Plot of Return Series

A typical size of the fluctuations, about half of a percent can be identified in this example. The histogram plot above (figure 1.3) indicates that the fluctuations of stock price are uncorrelated and have mean near zero. This typical size is one of the most important statistical quantity that we can extract from the market price history. We may be curious about the form of this distribution, for instance, if it is a normal distribution.

From the shape of the histogram plot in figure 1.3, it is very plausible that stock prices are *lognormally distributed*. This simply means that there are constants ν and σ^2 such that the logarithm of return, $\frac{S_T}{S_0}$ is normally distributed with mean ν and variance σ^2. Symbolically,

$$\mathbb{P}\left[\frac{S_T}{S_0} \in [a, b]\right] = \mathbb{P}\left[\log\left(\frac{S_T}{S_0}\right) \in [\log a\,, \log b]\right]$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{\log a}^{\log b} \exp\left(-\frac{(x-\nu)^2}{2\sigma^2}\right) dx.$$

This is so if we assume stock prices evolve according to

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right]$$

$$= S_0 \exp(\nu + \sigma W_t)$$

where

$$\nu = \left(\mu - \frac{\sigma^2}{2}\right)t$$

and W_t is the standard $\mathbb{P}$ - Brownian motion.

1.2 Mechanics of Option

As stock prices fluctuate widely, market participants need to hedge against their risks. Derivatives provide a rich means for hedging. Derivatives are assets whose values are derived from the value of underlying assets' prices [1]. Option is a type of derivative. It is a contract. An option gives the holder the right, but not the obligation, to choose whether to execute the final transaction or not. There are two basic types of option, the call option and the put option. A call option gives the holder the right, but not the obligation to buy an underlying stock at time T with strike price K, while a put option gives the holder the right (again, not the obligation) to sell an underlying stock at time T with price K. In fact, the terms call and put refer to buying and selling respectively. These are financial terms [2].

A call option will be exercised if the market price of the asset at the expiration time, S_T is greater than the strike price, K that is,

$$S_T > K$$

This kind of option is said to be *in the money* because an asset worth S_T can be purchased for only K.

On the other hand, if the strike price is less than S_T at the expiration time, that is,

$$S_T < K$$

Then, the call option will not be exercised because we can purchase the asset with cheaper price at open market. Thus, the option will be worthless and is said to be *out of the money*.

For put option, all aforesaid conditions are reversed. If the strike price of the asset is less than the market price of the asset at the expiration time, namely,

$$S_T > K$$

Then, the put option will not be exercised and is said to be *out of the money*. The seller can sell the asset to the open market with the market price, which is higher than the price stated in the put option.

The put option will only be exercised when the actual price (market price) of the asset is less than the strike price of the asset at expiration time, that is

$$S_T < K$$

In this situation, the put option is said to be *in the money*.

Regardless of call option or put option, an option is said to be *at the money* (or *on the money*) if and only if the market price of the asset at the expiration time, S_T is equal to the strike price K.

$$S_T = K$$

The tables below summarize all the situations discussed previously:

Table 1.1: Call Option

In the money	$S_T > K$
At the Money	$S_T = K$
Out of the money	$S_T < K$

Table 1.2: Put Option

In the money	$S_T < K$
At the Money	$S_T = K$
Out of the money	$S_T > K$

The payoff of call option and put option at time T may be written respectively as below:

$$C(T) = (S_T - K)_+$$
$$= \max\{(S_T - K), 0\}$$

$$P(T) = (K - S_T)_+$$
$$= \max\{(K - S_T), 0\}$$

These functions can be represented graphically as the following:

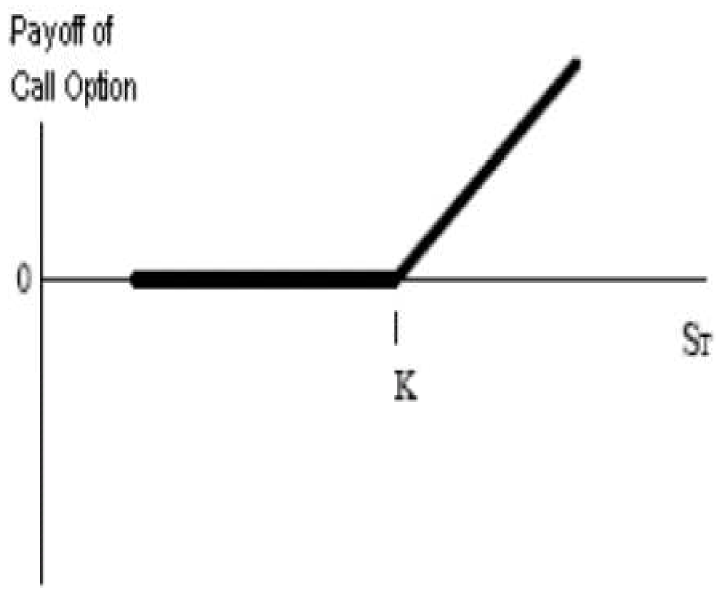

Figure 1.4: Payoff of call option at time *T*.

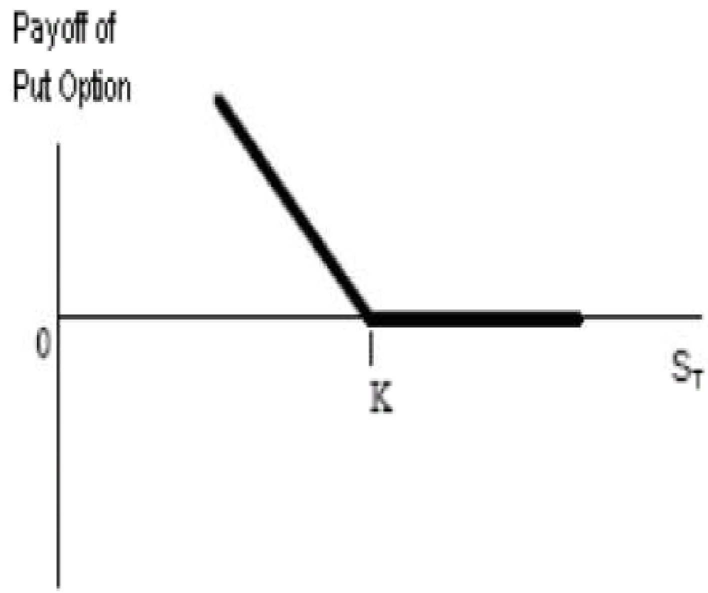

Figure 1.5: Payoff of put option at time *T*.

1.3 Styles of Option

There are three option styles in the market: European style option, American style option and Asian style option. European option is an option that can only be exercised at a specific time T, for a specified price K, while American option allows the holder of the option to exercise it at any time before the expiration date. Asian option, also termed as average option, is an option based on the average price of the underlying stock over the lifetime of the option.

In this thesis, we obtained the value of Asian option by solving a Black-Scholes equation using a Crank-Nicolson finite difference scheme which is stable and easy to program. The Asian option prices so obtained compare favorably with those form simulation method.

In general, the study of Asian option pricing can be divided into three classes: close form solution for the Laplace transform, Monte Carlo simulation and finite difference method for partial differential equation.

Apart from a closed-form formula for a Laplace transform of the Asian option price obtained by H. Geman and M. Yor [4], the price of Asian option is not known in explicit closed form. M. Fu, D. Madan and T. Wang [5] compares Monte Carlo and Laplace transform methods for Asian option pricing. Besides, the theory of Laplace transform is extended by deriving the double Laplace transform of the continuous arithmetic Asian option [4]. V. Linetsky [6] derives a new integral formula for the price of continuously sampled Asian option, but for the cases of low volatility, it converges slowly.

Monte Carlo simulation [7,8,9] and finite difference method for partial differential equation (PDE) [10,11,12,13,14] are the two main numerical method to price the Asian options. However, without the enhancement of variance reduction techniques, Monte Carlo simulation can be computationally expensive and one must also resolves the inherent discretization bias resulting from the approximation of continuous time processes through discrete sampling as shown by Broadie, Glasserman and Kou [15].

In principle, one can find the price of an Asian option by solving a partial differential equation in two space dimensions [16]. Besides, Ingersoll found that the two-dimensional PDE for a floating strike Asian option can be reduced to a one-dimensional PDE[16]. In 1995, Rogers and Shi formulated a one-dimensional PDE which is able to model both floating and fixed strike Asian options [10]. However, since the diffusion term is very small for values of interest on the finite difference grid, it is very hard to solve this PDE numerically. Andreasen applied Rogers and Shi's reduction to discretely sampled Asian option[17]. Večeř J. develops the change of $numéraire$ techniques for pricing Asian options. This technique was extended to jump process by Večeř and Xu [13,14].

In 2001, Kwok, Wong and Lau discussed about the explicit scheme for multivariate option pricing [18]. They found that the correlations among underlying variables deteriorate the accuracy of the computation. Besides, the explicit scheme is very difficult to control the stability in general.

However, these problems can be solved through our works here as PDE governing the value of Asian option with no correlation term. So, the first problem can be eliminated. The von Neumann stability analysis also carries out to ensure our result is stable [19].

Although there are a lot of ways to compute the value of Asian option, the Crank-Nicolson scheme is the only method that can be easily generalized to cope with early exercise decision for an Asian option by comparing the computed option value and immediate exercise value at each node backward in time. Hence, this method can be applied to options without Asian feature, or extended to American style Asian option. Besides, our proposed method is unconditionally stable compared to other methods, for instance, CRR binomial model. The CRR binomial model is only conditional stable of the type $\Delta t \backsim \Delta x^2$. Besides, a forward shooting grid (FSG) approach is required in this CRR model as it cannot record the realized averaged value in almost all Asian options. However, the FSG version of CRR model contains a subtle bias. [20].

CHAPTER 2

REVIEW OF PROBABILITY THEORY

Let us begin by recalling some of the definitions and basic concepts of elementary probability. A *probability space* is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is the set of sample space, $\mathcal{F}$ is a collection of subsets of Ω, events, and $\mathbb{P}$ is the probability measure defined for each event $A \in \mathcal{F}$. The collection $\mathcal{F}$ is a σ-field or σ-algebra, namely, $\Omega \in \mathcal{F}$ and $\mathcal{F}$ is closed under the operations of countable union and taking complements. The probability measure $\mathbb{P}$ must satisfy the usual *axioms of probability* [1,3]:

- $0 \leq \mathbb{P}[A] \leq 1$, for all $A \in \mathcal{F}$,
- $\mathbb{P}[\Omega] = 1$
- $\mathbb{P}[\mathrm{A} \cup \mathrm{B}] = \mathbb{P}[\mathrm{A}] + \mathbb{P}[\mathrm{B}]$ for any disjoint $A, B \in \mathcal{F}$,
- If $A_n \in \mathcal{F}$ for all $n \in \mathbb{N}$ and $A_1 \subseteq A_2 \subseteq \cdots$ then $\mathbb{P}[A_n] \uparrow \mathbb{P}[\cup_n A_n]$ as $n \uparrow \infty$.

Definition 2.1. A real-valued *random variable*, X, is a real-valued function on Ω that is $\mathcal{F}$-measurable. In the case of discrete random variable (that is a random variable that can only take on countable many distinct values) this simply means

$$\{\omega \in \Omega: \mathrm{X}(\omega) = \mathrm{x}\} \in \mathcal{F}$$

so that $\mathbb{P}$ assigns a probability to the event $\{\mathrm{X} = \mathrm{x}\}$. For a general real-valued random variable we require that

$$\{\omega \in \Omega: \mathrm{X}(\omega) \leq \mathrm{x}\} \in \mathcal{F}$$

so that we can define the distribution function, $D(x) = \mathbb{P}[X \leq x]$.

To specify a (discrete time) stochastic process, we require not just a single σ-field $\mathcal{F}$, but an increasing family of them.

Definition 2.2. Let $\mathcal{F}$ be a σ-field. We call $\{\mathcal{F}_t\}_{t\geq 0}$ a *filtration* if

1. $\mathcal{F}_t$ is a sub-σ-algebra of $\mathcal{F}$ for all t.
2. $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s < t$.

The quadruple $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is called a *filtered probability space.*

We are primarily concerned with the natural filtration, $\{\mathcal{F}_t^X\}_{t\geq 0}$, associated with a stochastic process $\{X_t\}_{t\geq 0}$. Let $\mathcal{F}_t^X$ encodes the information generated by the stochastic process X on the interval $[0, t]$. That is $A \in \mathcal{F}_t^X$ if, based upon observations of the trajectory $\{X_t\}_{t\geq 0}$, it is possible to decide whether or not A has occurred.

Definition 2.3. A real-valued *stochastic process* is a family of real-valued function $\{X_t\}_{t\geq 0}$ on Ω . We say that it is *adapted to the filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ if X_t is $\mathcal{F}_t$ measurable for each t.

One can then think of the σ-field $\mathcal{F}_t$ as encoding all the information about the evolution of the stochastic process up until time t, that is, if we know whether each event in $\mathcal{F}_t$ happens or not then we can infer the path followed by the stochastic process up until time t. We shall call the filtration that encodes precisely this information the *natural filtration* associated to the stochastic process $\{X_t\}_{t\geq 0}$.

Notation: If the value of a stochastic variable Z can be completely determined given observations of the trajectory $\{X_t\}_{0\le s\le t}$ then we write $Z \in \mathcal{F}_t^X$. More than one process can be measurable with respect to the same filtration.

Definition 2.4. If $\{\mathrm{Y_t}\}_{\mathrm{t}\ge 0}$ is a stochastic process such that we have $\mathrm{Y} \in \mathcal{F}_{\mathrm{t}}^{\mathrm{X}}$ for all $\mathrm{t} \ge 0$, then we say that $\{\mathrm{Y_t}\}_{\mathrm{t}\ge 0}$ is adapted to the filtration $\{\mathcal{F}_{\mathrm{t}}^{\mathrm{X}}\}_{\mathrm{t}\ge 0}$.

Definition 2.5. Suppose that X is an $\mathcal{F}$-measurable random variable with $\mathbb{E}[|X|] < \infty$. Suppose that $\mathcal{G} \subseteq \mathcal{F}$ is a σ-field; then the *conditional expectation of X given* G, written $\mathbb{E}[X|\mathcal{G}]$, is the $\mathcal{G}$-measurable random variable with the property that for any $A \in \mathcal{G}$

$$\mathbb{E}\big[[X|\mathcal{G}];A\big] \triangleq \int_A \mathbb{E}[X|\mathcal{G}]d\mathbb{P} = \int_A Xd\mathbb{P} \triangleq \mathbb{E}[X;A]$$

The conditional expectation exists, but is only unique up to the addition of a random variable that is zero with probability one.

The tower property of conditional expectations:

Suppose that $\mathcal{F}_i \subseteq \mathcal{F}_j$; then

$$\mathbb{E}\big[\mathbb{E}[X|\mathcal{F}_j]\big|\mathcal{F}_i\big] = \mathbb{E}[X|\mathcal{F}_i]$$

Taking out what is known in conditional expectations:

Suppose that $\mathbb{E}[X]$ and $\mathbb{E}[XY] < \infty$, if Y is $\mathcal{F}_n$-measurable, we have

$$\mathbb{E}[XY|\mathcal{F}_n] = Y\mathbb{E}[X|\mathcal{F}_n].$$

This just says that if Y is known by time n, then if we condition on the information up to time n we can treat Y as a constant.

Definition 2.6. Suppose that $(\Omega, \{\mathcal{F}_n\}_{n\geq 0}, \mathcal{F}, \mathbb{P})$ is a filtered probability space. The sequence of random variables $\{X_n\}_{n\geq 0}$ is a martingale with respect to $\mathbb{P}$ and $\{\mathcal{F}_n\}_{n\geq 0}$ if

$$\mathbb{E}[|X_n|] < \infty, \qquad \forall n,$$

and

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n, \qquad \forall n.$$

Definition 2.7. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let T be a fixed positive number, and let $\mathcal{F}(t)$, $0 \leq t \leq T$, be a filtration of sub-σ-algebras of $\mathcal{F}$. Consider an adapted stochastic process $X(t)$, $0 \leq t \leq T$. Assume that for all $0 \leq s \leq t \leq T$ and for every nonnegative, *Borel-measurable* function f, there is another *Borel-measurable* function g such that

$$\mathbb{E}\big[f\big(X(t)\big)\big|\mathcal{F}(s)\big] = g\big(X(s)\big).$$

Then we say that X is a Markov process.

Theorem 2.1. (*Itô's formula*)

For f such that the partial derivatives $\frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}$ exist and are continuous and $\frac{\partial f}{\partial x} \in \mathcal{H}$, almost surely for each *t* we have

$$\begin{aligned} &f(t, W_t) - f(0, W_0) \\ &\qquad = \int_0^t \frac{\partial f}{\partial x}(s, W_s)dW_s + \int_0^t \frac{\partial f}{\partial s}(s, W_s)dW_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, W_s)ds \end{aligned}$$

Often one writes *Itô* formula in differential notation as:

$$df(t, W_t) = f^{'}(t, W_t)dW_t + \dot{f}(t, W_t)dt + \frac{1}{2}f^{''}(t, W_t)dt$$

Theorem 2.2. (*Girsanov's Theorem*)

Suppose that $\{W_t\}_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion with the natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and that $\{\theta_t\}_{t\geq 0}$ is an $\{\mathcal{F}_t\}_{t\geq 0}$-adapted process such that

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \theta_t^2 dt\right)\right] < \infty$$

Define

$$L_t = \exp\left(-\int_0^t \theta_s d\, W_s - \frac{1}{2}\int_0^T \theta_s^2 ds\right)$$

and let $\mathbb{P}^{(L)}$ be the probability measure defined by

$$\mathbb{P}^{(L)}[A] = \int_A L_t(\omega)\mathbb{P}(d\omega).$$

Then under the probability measure $\mathbb{P}^{(L)}$, the process $\left\{W_t^{(L)}\right\}_{0\leq t\leq T}$, defined by

$$W_t^{(L)} = W_t + \int_0^t \theta_s d\, s,$$

is a standard Brownian motion.

Theorem 2.3. (*Brownian Martingale Representation Theorem*)

Let $\{F_t\}_{t\geq 0}$ denote the natural filtration of the $\mathbb{P}$-Brownian motion $\{W_t\}_{t\geq 0}$. Let $\{M_t\}_{t\geq 0}$ be a square-integrable $(\mathbb{P}, \{W_t\}_{t\geq 0})$-martingale.Then there exists an $\{F_t\}_{t\geq 0}$-predictable process $\{\theta_t\}_{t\geq 0}$ such that with $\mathbb{P}$ -probability one,

$$M_t = M_0 + \int_0^t \theta_S d\, W_s.$$

Theorem 2.4. (*Conditional expectation when measure is changed*)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let Z be an almost surely nonnegative random variable with $\mathbb{E}(Z) = 1$. For $A \in \mathcal{F}$, define

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega) d\,\mathbb{P}(\omega) \;\; for \;\; every\, A \in \mathcal{F}.$$

Then $\widetilde{\mathbb{P}}$ is a probability measure. Furthermore, if X is a nonnegative random variable, then

$$\widetilde{\mathbb{E}}(X) = \mathbb{E}(XZ).$$

If Z is almost surely strictly positive, we also have

$$\mathbb{E}(Y) = \widetilde{\mathbb{E}}\left(\frac{Y}{Z}\right)$$

for every nonnegative random variable Y.

Note: The $\widetilde{\mathbb{E}}$ appearing here is expectation under probability measure $\widetilde{\mathbb{P}}$, that is

$$\widetilde{\mathbb{E}}(X) = \int_\Omega X(\omega) d\,\widetilde{\mathbb{P}}(\omega).$$

Theorem 2.5. (*Radon-Nikodým***)**

Let $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ be equivalent probability measures defined on $(\Omega, \mathcal{F})$. Then there exist an almost surely positive random variable Z such that $\mathbb{E}(Z) = 1$ and

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega) d\,\mathbb{P}(\omega) \;\; for \;\; every\, A \in \mathcal{F}.$$

Note: $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ are equivalent if and only if $\mathbb{P}[A] = 0 \Leftrightarrow \widetilde{\mathbb{P}}(A) = 0$ where $A \in \mathcal{F}$.

CHAPTER 3

EUROPEAN OPTION

3.1 Introduction

European style option (for shortly, European option) is the simplest type of option. As mentioned previously, European option can only be exercised at a specified time T, for a specified price K. Let $\Phi(S)$ be the payoff function at time T and $V(S,t)$ be the option value at time t when $\mathrm{S_t = S}$. Across a time interval δt, we may write the changes δV of option price as

$$\delta V = \frac{\partial V}{\partial t}\delta t + \frac{\partial V}{\partial S}\delta S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\delta S^2 + \cdots\cdots \tag{1}$$

In order to ensure the seller of the option is able to meet the claim, we need a replicating portfolio Π whose value at terminal time T is $V(S,t) = \Phi(S)$. A replicating portfolio Π consists of $D(S,t)$ unit of stock and cash account, C where D and C can be either positive or negative, corresponding to long or short positions. We do not consider $D = 0$ here as we cannot hedge the claim without holding any stocks. The portfolio value $\Pi(S,t)$ is thus

$$\Pi(S,t) = D(S,t)S_t + C(S,t)$$

where S_t denotes stock price at time *t*.

During the short time interval δt, the change of portfolio value becomes

$$\delta\Pi = D\delta S + rC\delta t \tag{2}$$

where r is the interest rate and $rC\delta t$ is the approximate interest paid or earned during time δt. The terms $D\delta S$ is *exact*, there is no other higher order terms like δS^2.

At each time *t*, the expected payoff will change when the stock price changes. Thus, we need to rebalance the portfolio to ensure we are able to meet the claim eventually. So, we have to change the number of units D in response to the new stock price $S(t + \delta t)$ before the beginning of the next time interval. Money that is needed for or generated by this rebalancing is taken out from or deposited into the cash account. We assume that rebalancing is instantaneous so that equation (2) represents the entire change across the short time δt since there is no money to put in or withdrawn from the portfolio, this kind of portfolio is termed as *self-financing* [1].

Therefore, the difference between the two portfolios value (equation (1) and equation (2)) is given as below:

$$\delta(V - \Pi) = \left(\frac{\partial V}{\partial t} - rC\right)\delta t + \left(\frac{\partial V}{\partial S} - D\right)\delta S + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\delta S^2 + \cdots\cdots \qquad (3)$$

Note that the equation above (Equation (3)) depends on the unknown change δS. By choosing $D = \frac{\partial V}{\partial S}$, we are able to eliminate this first order dependence and it becomes

$$\delta(V - \Pi) = \left(\frac{\partial V}{\partial t} - rC\right)\delta t + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\delta S^2 + \cdots\cdots \qquad (4)$$

Since δS^2 is unknown, this changes is still an uncertain quantity. However, it may be effectively deterministic if we average over sufficiently small steps.

Now, let Δt be a time interval. If comparing this time interval with the overall lifetime of the option, it is relatively small. However, it is large if compared with the small time interval δt at which we are able to trade. Define $\Delta t = N\delta t$, and δS_j represents the small price changes for $j = 1, \dots \dots \dots, N$. Since the direction of stock price motion is unpredictable and always changes in an uncertain way over the time, it is said to follow a *stochastic process*. We need a stochastic model for the stock prices.

We assume that in a small time interval δt,

$$\delta S_j = a\delta t + b\sqrt{\delta t}\xi_j \qquad (5)$$

where a refers to the expected rate of change, b is an 'absolute volatility' measuring the motions' expected size and ξ_j is a random variable. At each time-step, ξ_j has a mean of zero and variance equals to one. All these random variables are independent across the successive steps.

The following is the accumulated change of stock price across the time interval Δt

$$\Delta S = \sum_{j=1}^{N} \delta S_j = a\Delta t + b\sqrt{\Delta t}X \qquad (6)$$

where

$$\mathrm{X} = \frac{1}{\sqrt{\mathrm{N}}} \sum_{\mathrm{j}=1}^{\mathrm{N}} \xi_{\mathrm{j}}.$$

Since ξ_j are independent and the random variable X has zero mean and variance is one. By Central Limit Theorem, X follows a normal distribution when N is sufficiently large. Equation (5) and (6) are of the same form, the only difference is the time scale. So, we can argue that the law is precisely the same on all time scales if the ξ_j have a normal distribution.

It has been suggested before that the sum of the squares of price changes is not as random as the changes themselves. In fact, it is much less random than the price change. Indeed,

$$\left(\delta S_j\right)^2 = b^2 \delta t {\xi_j}^2 + 2ab(\delta t)^{3/2} \xi_j + a^2 \delta t^2,$$

which implies

$$\sum_{j=1}^{N} \left(\delta S_j\right)^2 = b^2 \Delta t \frac{1}{N} \sum_{j=1}^{N} {\xi_j}^2 + 2ab(\Delta t)^{3/2} \frac{1}{N^{3/2}} \sum_{j=1}^{N} \xi_j + a^2 (\Delta t)^2 \frac{1}{N^2}$$

$$\longrightarrow b^2 \Delta t$$

as $N \longrightarrow \infty$. Even though the square of the changes in stock price is random on any one step, δt, it will become deterministic if we average across a large number of steps.

Assuming $D = \frac{\partial V}{\partial S}$, the accumulated change from (4) is now becomes:

$$\begin{aligned}
\Delta(V - \Pi) &= \sum_{j=1}^{N} \delta(V - \Pi)_j \\
&= \left(\frac{\partial V}{\partial t} - rC\right) \Delta t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} \sum_{j=1}^{N} \left(\delta S_j\right)^2 \\
&= \left(\frac{\partial V}{\partial t} - rC\right) \Delta t + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} b^2 \Delta t \\
&= \left(\frac{\partial V}{\partial t} - rC + \frac{1}{2} \frac{\partial^2 V}{\partial S^2} b^2\right) \Delta t \qquad (7)
\end{aligned}$$

Since there is no randomness in equation (7), the portfolio $V - \Pi$ is risk-free and it must grow at exactly same rate as any risk-free cash account, namely

$$\Delta(V - \Pi) = r(V - \Pi)\Delta t \tag{8}$$

In finance, the situation above is known as *arbitrage-free*: no party in the market is able to make a riskless profit. An opportunity to lock into risk-free profit is known as arbitrage opportunity.

As $V - \Pi = V - (DS + C)$ and $D = \frac{\partial V}{\partial S}$, from equation (7) and (8), we have

$$\left(\frac{\partial V}{\partial t} - rC + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} b(S,t)^2\right)\Delta t = r(V - \Pi)\,\Delta t$$

$$\frac{\partial V}{\partial t} - rC + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} b(S,t)^2 = r(V - \Pi)$$

$$\frac{\partial V}{\partial t} - rC + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} b(S,t)^2 - r(V - \Pi) = 0$$

$$\frac{\partial V}{\partial t} - rC + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} b(S,t)^2 - r(V - DS - C) = 0$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} b(S,t)^2 - rV + rDS = 0$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} b(S,t)^2 - rV + rS\frac{\partial V}{\partial S} = 0 \tag{9}$$

which is the general version of Black-Scholes equation. The value of any derivative security depending on the stock price S must satisfy the partial difference equation (PDE) (9).

Constructing improved model for the movement of stock price and for pricing option value is still an ongoing research. However, there is a popular model, that is, *lognormal model* $b(S,t) = \sigma S$. Equation (5) now becomes

$$\delta S_j = a(S,t)\delta t + \sigma\sqrt{\delta t}S\xi_j \tag{10}$$

That is, as S varies, the percentage size of the random changes in S is assumed to be constant. We have $\sigma\sqrt{\Delta t}$, the expected size of changes across the time interval Δt where parameter σ is referred as the volatility. For this model, the Black-Scholes equation is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{11}$$

The PDE above contains non-constant coefficients, depending on the independent variable S. If $S = 0$, the coefficients containing terms S^2 and S disappear. However, if we let $x = log\, S$, equation (11) can reduces to the standard heat equation with constant coefficient. It is then easy to construct the exact solution with the help of Green's function of the heat equation. The renowned Black-Scholes formula for the price of European call option is then delivered:

$$V(S,t;K,T) = SN\left(\frac{\log\frac{S}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) - Ke^{-r(T-t)}N\left(\frac{\log\frac{S}{K} + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$

in which N is the cumulative normal distribution

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{\frac{-y^2}{2}}dy.$$

Now, let $\theta = T - t$, the Black-Scholes formula for the prices of a European call option at time t is defined as the following:

$$F(S_t, t) = S_t N(d_1) - Ke^{-r\theta}N(d_2)$$

where

$$d_1 = \frac{\log\frac{S_t}{K} + \left(r + \frac{1}{2}\sigma^2\right)\theta}{\sigma\sqrt{\theta}}$$

$$d_2 = \frac{\log\frac{S_t}{K} + \left(r - \frac{1}{2}\sigma^2\right)\theta}{\sigma\sqrt{\theta}}$$

$$= d_1 - \sigma\sqrt{\theta}$$

and $N(\cdot)$ is the standard normal distribution function, given by

$$N(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy$$

Note that $N(y) \leq 1$ if $y = \infty$.

Using the same way, the price for European put option can also be determined.

We summarize the assumptions that are used in the model [2]:

1. The stock S can be sold and bought.
   - This is essential and important for constructing a hedging portfolio. A portfolio consists of number of stocks holding and a cash account. In order to construct a suitable hedging portfolio, we have to keep on changing the stock holding by selling and buying it.

2. No transaction cost is involved on buying or selling stocks.
   - Here, the transaction cost refers to the charges incurred for the transaction. It is difficult to build the transaction cost in the model. Therefore, for simplicity, we are not considering it in the mathematics model.

3. The market parameters r and b are constant and known.
    - The interest rate, r is differs for different customers or investors. However, it does not have a large effect on the result.
    - As mentioned before, b is the volatility. The option value V is a function of b and is very sensitive to b.

4. No dividend.
    - The underlying stock pays no dividend during the option's life.

5. There is no arbitrage opportunity.
    - No one can make a riskless profit in the market.

6. Stock price follows a Geometric Brownian motion.
    - The motion of stock price cannot be predicted and move in uncertain way. We assume that the motion follows a Geometric Brownian motion.

3.2 Itô's Lemma Approach to Black-Scholes Equation

Geometric Brownian motion, the basic reference model for stock prices is defined by

$$S_t = S_0 \exp(\nu t + \sigma W_t) \tag{12}$$

where

$$\upsilon = \mu - \frac{\sigma^2}{2}$$

and W_t is a $\mathbb{P}$-Brownian motion. By *Itô* formula,

$$dS_t = S_0 \nu \exp(\nu t + \sigma W_t)\, dt + S_0 \sigma \exp(\nu t + \sigma W_t) dW_t$$
$$+ S_0 \frac{1}{2} \sigma^2 \exp(\nu t + \sigma W_t) dt$$
$$= S_t \nu dt + S_t \sigma dW_t + S_t \frac{\sigma^2}{2} dt$$
$$= S_t \left[\mu - \frac{\sigma^2}{2}\right] dt + S_t \sigma dW_t + S_t \frac{\sigma^2}{2} dt$$
$$= S_t \mu dt - S_t \frac{\sigma^2}{2} dt + S_t \sigma dW_t + S_t \frac{\sigma^2}{2} dt$$
$$= S_t(\mu dt + \sigma dW_t) \tag{13}$$

Equation (13) is termed as stochastic differential equation (SDE) for S_t. It can be re-written in the following form:

$$dS_t = S_t \left[r dt + \sigma \left(dW_t + \frac{\mu - r}{\sigma} dt \right) \right]$$
$$= S_t(r dt + \sigma dX_t)$$

where

$$\mathrm{X_t} = W_t + \frac{\mu - r}{\sigma} t$$
$$= W_t + \int_0^t \frac{\mu - r}{\sigma} d\tau.$$

By the Girsanov's theorem, X_t is a standard Brownian motion under the probability measure $\mathbb{P}^{(L)}$ and also a $\mathbb{P}^{(L)}$-martingale.

Let $\tilde{S}_t$ be the discounted stock prices, that is

$$\tilde{S}_t = e^{-rt} S_t$$

It is easy to see that

$$d\tilde{S}_t = \sigma \tilde{S}_t d\mathrm{X_t}$$

Comparing with equation (13), when $\mu = 0$, the discounted stock prices can now be written in the following form

$$\tilde{S}_t = \tilde{S}_0 \exp\left(-\frac{\sigma^2}{2}t + \sigma \mathrm{X_t}\right).$$

and it is a $\mathbb{P}^{(L)}$-martingale.

Let $\Phi_\mathrm{T} = \Phi(\mathrm{S_T})$ be the payoff function at time T. Define

$$\begin{aligned} M_t &= e^{-rT}\mathrm{E}^{\mathbb{P}^{(L)}}[\Phi_\mathrm{T}|\mathrm{S_t} = \mathrm{S}] \\ &= \mathrm{E}^{\mathbb{P}^{(L)}}[e^{-rT}\Phi_\mathrm{T}|\mathrm{S_t} = \mathrm{S}] \\ &= \mathrm{E}^{\mathbb{P}^{(L)}}[e^{-rT}\Phi_\mathrm{T}|\mathcal{F}_\mathrm{t}]. \end{aligned}$$

By the tower property,

$$\begin{aligned} \mathrm{E}^{\mathbb{P}^{(L)}}[M_t|\mathcal{F}_\mathrm{s}] &= \mathrm{E}^{\mathbb{P}^{(L)}}\left[\mathrm{E}^{\mathbb{P}^{(L)}}[e^{-rt}\Phi_\mathrm{T}|\mathcal{F}_\mathrm{t}]\middle|\,\mathcal{F}_\mathrm{s}\right] \\ &= \mathrm{E}^{\mathbb{P}^{(L)}}[e^{-rT}\Phi_\mathrm{T}|\mathcal{F}_\mathrm{s}] \\ &= M_s \end{aligned}$$

for $s < t$. As a consequence, M_t is a $\mathbb{P}^{(L)}$-martingale. Thus, by the Brownian martingale representation theorem, there exists a process θ_τ such that we can write M_t as an *Itô* integral:

$$\begin{aligned} M_t &= M_0 + \int_0^t \theta_s dX_s \\ &= M_0 + \int_0^t \frac{\theta_s}{\sigma\tilde{S}_s}\sigma\tilde{S}_s dX_s \\ &= M_0 + \int_0^t \phi_s d\tilde{S}_s \end{aligned}$$

where $\phi_s = \frac{\theta_s}{\sigma\tilde{S}_s}$ and $d\tilde{S}_s = \sigma\tilde{S}_s dX_s$.

Define $\psi_t = M_t - \phi_t \tilde{S}_t$. Then, the portfolio $\mathrm{e}^{\mathrm{rt}}\psi_t + \phi_t \tilde{S}_t$ replicate $\mathrm{e}^{\mathrm{rt}}M_t$, namely $\mathrm{e}^{\mathrm{rt}}M_t$ has realizable market value. As $\mathrm{e}^{\mathrm{rt}}M_t = \Phi_{\mathrm{T}}$, the option value at time t is then

$$\begin{aligned} \mathrm{e}^{\mathrm{rt}}M_t &= \mathrm{E}^{\mathbb{P}^{(L)}}\left[e^{-r(T-t)}\Phi_{\mathrm{T}} \middle| \mathcal{F}_{\mathrm{t}}\right] \\ &= \mathrm{E}^{\mathbb{P}^{(L)}}\left[e^{-r(T-t)}\Phi_{\mathrm{T}} \middle| \mathrm{S_t} = \mathrm{S}\right] \end{aligned} \tag{14}$$

Now, we introduce a new function, $F(S,t)$. Assume that the function $F(S,t)$ solves the following boundary value problem

$$\frac{\partial}{\partial t}F(S,t) + \frac{\sigma^2 S^2}{2}\frac{\partial^2}{\partial S^2}F(S,t) + rS\frac{\partial}{\partial S}F(S,t) - rF(S,t) = 0, \quad 0 \le t \le T$$

$$F(S,t) = \Phi(S) \tag{15}$$

Define $N_t = \mathrm{e}^{-\mathrm{rt}}F(S,t)$. By *Itô* formula,

$$\begin{aligned} dN_t &= d\left(\mathrm{e}^{-\mathrm{rt}}F(S,t)\right) \\ &= \mathrm{e}^{-\mathrm{rt}}\left(-\mathrm{r}F(S,t) + \frac{\partial F(S,t)}{\partial t}dt + \frac{\partial F(S,t)}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 F(S,t)}{\partial S^2}d^2S_t\right) \\ &= \mathrm{e}^{-\mathrm{rt}}\left(-\mathrm{r}F(S,t) + \frac{\partial F(S,t)}{\partial t}dt + \frac{\partial F(S,t)}{\partial S}(rS_t dt + \sigma S_t dX_t)\right. \\ &\qquad \left. + \frac{1}{2}\frac{\partial^2 F(S,t)}{\partial S^2}(rS_t dt + \sigma S_t dX_t)^2\right) \\ &= \mathrm{e}^{-\mathrm{rt}}\left(-\mathrm{r}F(S,t) + \frac{\partial F(S,t)}{\partial t} + rS_t\frac{\partial F(S,t)}{\partial S} + \frac{\sigma^2 {S_t}^2}{2}\frac{\partial^2 F(S,t)}{\partial S^2}\right)\mathrm{dt} \\ &\qquad + \mathrm{e}^{-\mathrm{rt}}\sigma S_t\frac{\partial F(S,t)}{\partial S}\mathrm{dX_t} \\ &= \mathrm{e}^{-\mathrm{rt}}\sigma S_t\frac{\partial F(S,t)}{\partial S}\mathrm{dX_t} \end{aligned}$$

Then,

$$N_t = N_0 + \int_0^t \mathrm{e}^{-\mathrm{r}\tau}\sigma \mathrm{S}_\tau \frac{\partial F(S,t)}{\partial S}\mathrm{dX}_\tau$$

is a $\mathbb{P}^{(L)}$-martingale.

Since $N_T = \mathrm{e}^{-\mathrm{rT}}\Phi_{\mathrm{T}}$. From the martingale property,

$$\mathrm{E}^{\mathbb{P}^{(L)}}[N_T|\mathcal{F}_t] = \mathrm{N_t} \quad \text{for} \quad t < T$$

$$\Longrightarrow \mathrm{E}^{\mathbb{P}^{(L)}}[\mathrm{e}^{-\mathrm{rT}}\Phi_{\mathrm{T}}|\mathcal{F}_t] = \mathrm{N_t}$$

$$\Longrightarrow \mathrm{E}^{\mathbb{P}^{(L)}}[\mathrm{e}^{-\mathrm{rT}}\Phi_{\mathrm{T}}|\mathcal{F}_t] = \mathrm{e}^{-\mathrm{rt}}F(S,t)$$

$$\therefore \quad F(S,t) = \mathrm{E}^{\mathbb{P}^{(L)}}\left[\mathrm{e}^{-\mathrm{r(T-t)}}\Phi_{\mathrm{T}}\middle|\mathcal{F}_t\right]$$

$$= \mathrm{E}^{\mathbb{P}^{(L)}}\left[\mathrm{e}^{-\mathrm{r(T-t)}}\Phi_{\mathrm{T}}\middle|S_t = S\right]$$

which is actually the option value at time t that we obtained before in expression (14).

3.3 Crank-Nicolson Finite Difference Method

Recall that the Black-Scholes model for European option:

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + rS\frac{\partial F}{\partial S} - rF = 0$$

Consider a function $F(S,t)$ over a two-dimensional grid. Let j and h denote the indices for stock price, S and time t respectively. At a typical point $F(S,t)$, write $F(S,t) = F_j^h$, the expression

$$\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + rS\frac{\partial F}{\partial S} - rF$$

is approximated by the following difference scheme

$$L_j^h = \frac{\sigma^2 S_j^2}{2} D_{ss} + rS_j D_s - rF_j^h$$

where

$S = j\Delta S \quad \text{for} \quad 0 \leq j \leq M$

$t = h\Delta t \quad \text{for} \quad 0 \leq h \leq H$

$$D_{ss} = \frac{F(S_{j+1}, t_h) - 2F(S_j, t_h) + F(S_{j-1}, t_h)}{(\Delta S)^2}$$

$$D_s = \frac{F(S_{j+1}, t_h) - F(S_{j-1}, t_h)}{2\Delta S}$$

After taking the forward time scheme at time h:

$$\frac{F_j^{h+1} - F_j^h}{\Delta t} + L_j^h = 0$$

and backward time scheme at time $h + 1$:

$$\frac{F_j^{h+1} - F_j^h}{\Delta t} + L_j^{h+1} = 0$$

yields the Crank-Nicolson finite difference scheme:

$$\frac{F_j^{h+1} - F_j^h}{\Delta t} + \frac{1}{2}\left(L_j^h + L_j^{h+1}\right) = 0$$

$$F_j^h - \frac{\Delta t}{2} L_j^h = F_j^{h+1} + \frac{\Delta t}{2} L_j^{h+1}$$

where

$$L_j^h = \frac{\sigma^2 S_j^2}{2(\Delta S)^2}\left[F_{j+1}^h - 2F_j^h + F_{j-1}^h\right] + \frac{rS_j}{2\Delta S}\left[F_{j+1}^h - F_{j-1}^h\right] - rF_j^h$$

and

$$L_j^{h+1} = \frac{\sigma^2 S_j^2}{2(\Delta S)^2}\left[F_{j+1}^{h+1} - 2F_j^{h+1} + F_{j-1}^{h+1}\right] + \frac{rS_j}{2\Delta S}\left[F_{j+1}^{h+1} - F_{j-1}^{h+1}\right] - rF_j^{h+1}.$$

Therefore, the Black-Scholes model can be transformed into the following:

$$F_{j-1}^h\left[\frac{r\Delta t S_j}{4\Delta S} - \frac{\Delta t \sigma^2 S_j^2}{4(\Delta S)^2}\right] + F_j^h\left[1 + \frac{\Delta t \sigma^2 S_j^2}{2(\Delta S)^2} + \frac{r\Delta t}{2}\right] - F_{j+1}^h\left[\frac{\Delta t \sigma^2 S_j^2}{4(\Delta S)^2} + \frac{r\Delta t S_j}{4\Delta S}\right]$$

$$= F_{j-1}^{h+1}\left[\frac{\Delta t\sigma^2 S_j^2}{4(\Delta S)^2} - \frac{r\Delta t S_j}{4\Delta S}\right] + F_j^{h+1}\left[1 - \frac{\Delta t\sigma^2 S_j^2}{2(\Delta S)^2} - \frac{r\Delta t}{2}\right]$$

$$+ F_{j+1}^{h+1}\left[\frac{\Delta t\sigma^2 S_j^2}{4(\Delta S)^2} + \frac{r\Delta t S_j}{4\Delta S}\right]$$

Let $\lambda = \dfrac{r\Delta t}{4\Delta S}, \alpha = \dfrac{\Delta t\sigma^2}{4(\Delta S)^2}$ and $\beta = \dfrac{r\Delta t}{2}$

$$\Longrightarrow\ F_{j-1}^{h}\left[\lambda S_j - \alpha S_j^2\right] + F_j^{h}\left[1 + 2\alpha S_j^2 + \beta\right] - F_{j+1}^{h}\left[\alpha S_j^2 + \lambda S_j\right]$$

$$= F_{j-1}^{h+1}\left[\alpha S_j^2 - \lambda S_j\right] + F_j^{h+1}\left[1 - 2\alpha S_j^2 - \beta\right] + F_{j+1}^{h+1}\left[\alpha S_j^2 + \lambda S_j\right] \qquad (16)$$

3.4 Implementation

This program computes the European call option value at time zero. We first set the strike price K, interest rate r, volatility level σ and the terminal time T of the European option value. The time unit is in year. We know the asymptotic value of the option is $S - Ke^{-r(T-t)}$ for large stock price S. However, we do not know how large a value of S is large enough for the asymptotic formula to be correct. Hence, we use a try and error method to determine it. First, we choose a maximum stock price S_{max}. We shall arbitrarily set S_{max} first. Using the chosen S_{max}, we compute the option value at a particular t and s in the interior and denote it by a. Then, we enlarge the chosen S_{max} and compute the option value again at the same t and s, denote this option value by b. If a and b differ by a very small value, the first S_{max} is good enough to be chosen as the maximum stock price. S_{max} is usually some constant multiple of the strike price K.

After that, we set up the number of partition for the time, say H and calculate the time step $dt = \frac{T}{H}$. From practical experience, we found that the accuracy of the calculation has something to do with the ratio $\frac{dS^2}{dt}$. With $\frac{dS^2}{dt} = 50$, both the accuracy and computing time are reasonable. From this ratio, we then find dS. In general, the accuracy of option price depends on the combination of number of steps in stock price dS and time t.

Equation (16) can be expressed in matrix form:

$$AF^j = BF^{j+1} \text{ for } j = 0,1,2,\dots\dots$$

$$\Rightarrow F^j = A^{-1}BF^{j+1} \text{ where } F^j = \left(F_{1,j}, F_{2,j}, F_{3,j}, \dots\dots, F_{m,j}\right)^T ,$$

$A =$

$$\begin{pmatrix} 1+2\alpha S_1^2+\beta & -(\alpha S_2^2+\lambda S_2) & 0 & \cdots & \cdots & 0 \\ \lambda S_1-\alpha S_1^2 & 1+2\alpha S_2^2+\beta & -(\alpha S_3^2+\lambda S_3) & \vdots & \vdots & \vdots \\ 0 & \lambda S_2-\alpha S_2^2 & 1+2\alpha S_3^2+\beta & \vdots & \vdots & 0 \\ \vdots & \cdots & \cdots & \vdots & 0 & -(\alpha S_m^2+\lambda S_m) \\ 0 & \cdots & \cdots & 0 & \lambda S_{m-1}-\alpha S_{m-1}^2 & 1+2\alpha S_{m-1}^2+\beta \end{pmatrix}$$

at time h and

$B =$

$$\begin{pmatrix} 1-2\alpha S_1^2-\beta & \alpha S_2^2+\lambda S_2 & 0 & \cdots & \cdots & 0 \\ \alpha S_1^2-\lambda S_1 & 1-2\alpha S_2^2-\beta & \alpha S_3^2+\lambda S_3 & \vdots & \vdots & \vdots \\ 0 & \alpha S_2^2-\lambda S_2 & 1-2\alpha S_3^2-\beta & \vdots & \vdots & 0 \\ \vdots & \cdots & \cdots & \vdots & 0 & \alpha S_m^2+\lambda S_m \\ 0 & \cdots & \cdots & 0 & \alpha S_{m-1}^2-\lambda S_{m-1} & 1-2\alpha S_{m-1}^2-\beta \end{pmatrix}$$

at time $h+1$.

Note that the matrices A and B are $(m-1)\times(m-1)$ tridiagonal matrix.

Since the boundary values for the option are known at terminal time, we may perform the backward iteration to obtain the option value at time zero.

Remark: In the program code, we denote the option value (as a matrix) by P, i.e.

$$P_{h,j} = \mathrm{F}_{\mathrm{j}}^{\mathrm{h}}.$$

The Matlab function written here is named as *EuropeanOption*. For $S_0 = 20, r = 0.05, \sigma = 0.25$, we may find option value by calling:

[StkPrice Call SpPrice RelErr] = EuropeanOption(20, 0.05, 0.25);

3.5 Stability Analysis

The following is a general form of Black-Scholes equation:

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + rS\frac{\partial F}{\partial S} - rF = 0$$

After applying Crank-Nicolson finite difference scheme, we obtained a approximation linear system:

$$F_{j-1}^{h}\left[\frac{r\Delta t S_j}{4\Delta S} - \frac{\Delta t \sigma^2 S_j^2}{4(\Delta S)^2}\right] + F_j^{h}\left[1 + \frac{\Delta t \sigma^2 S_j^2}{2(\Delta S)^2} + \frac{r\Delta t}{2}\right]$$
$$-F_{j+1}^{h}\left[\frac{\Delta t \sigma^2 S_j^2}{4(\Delta S)^2} + \frac{r\Delta t S_j}{4\Delta S}\right]$$
$$= F_{j-1}^{h+1}\left[\frac{\Delta t \sigma^2 S_j^2}{4(\Delta S)^2} - \frac{r\Delta t S_j}{4\Delta S}\right] + F_j^{h+1}\left[1 - \frac{\Delta t \sigma^2 S_j^2}{2(\Delta S)^2} - \frac{r\Delta t}{2}\right]$$
$$+F_{j+1}^{h+1}\left[\frac{\Delta t \sigma^2 S_j^2}{4(\Delta S)^2} + \frac{r\Delta t S_j}{4\Delta S}\right].$$

Assuming the errors are propagating backward as terminal condition is given.

Let $h+1 = N-k$ and $h = N-(k+1)$.

$$F_{j-1}^{N-(k+1).} \left[\frac{r\Delta t S_j}{4\Delta S} - \frac{\Delta t \sigma^2 S_j^2}{4(\Delta S)^2}\right] + F_j^{N-(k+1).} \left[1 + \frac{\Delta t \sigma^2 S_j^2}{2(\Delta S)^2} + \frac{r\Delta t}{2}\right]$$

$$-F_{j+1}^{N-(k+1).} \left[\frac{\Delta t \sigma^2 S_j^2}{4(\Delta S)^2} + \frac{r\Delta t S_j}{4\Delta S}\right]$$

$$= F_{j-1}^{N-k} \left[\frac{\Delta t \sigma^2 S_j^2}{4(\Delta S)^2} - \frac{r\Delta t S_j}{4\Delta S}\right] + F_j^{N-k} \left[1 - \frac{\Delta t \sigma^2 S_j^2}{2(\Delta S)^2} - \frac{r\Delta t}{2}\right]$$

$$+F_{j+1}^{N-k} \left[\frac{\Delta t \sigma^2 S_j^2}{4(\Delta S)^2} + \frac{r\Delta t S_j}{4\Delta S}\right] \qquad (17)$$

Solutions of equation (17) are assumed to be the following form:

$F_j^{N-(k+1)} = \epsilon^{(k+1)} e^{ij2\pi/\omega}$

$F_{j+1}^{N-(k+1)} = \epsilon^{(k+1)} e^{i(j+1)2\pi/\omega}$

$F_{j-1}^{N-(k+1)} = \epsilon^{(k+1)} e^{i(j-1)2\pi/\omega}$

$F_j^{N-k} = \epsilon^k \, e^{ij2\pi/\omega}$

$F_{j+1}^{N-k} = \epsilon^k \, e^{i(j+1)2\pi/\omega}$

$F_{j-1}^{N-k} = \epsilon^k \, e^{i(j-1)2\pi/\omega}$ (18)

where i is a complex variable, $i = \sqrt{-1}$.

In order to find out how the error changes in time steps, substituting equations (18) into (17), we have

$$\epsilon^{(k+1)} e^{i(j-1)2\pi/\omega} \left[\frac{r\Delta t S_j}{4\Delta S} - \frac{\Delta t \sigma^2 S_j^2}{4(\Delta S)^2}\right] + \epsilon^{(k+1)} e^{ij2\pi/\omega} \left[1 + \frac{\Delta t \sigma^2 S_j^2}{2(\Delta S)^2} + \frac{r\Delta t}{2}\right]$$

$$-\epsilon^{(k+1)} e^{i(j+1)2\pi/\omega} \left[\frac{\Delta t \sigma^2 S_j^2}{4(\Delta S)^2} + \frac{r\Delta t S_j}{4\Delta S}\right]$$

$$= \epsilon^k\, e^{i(j-1)2\pi/\omega}\left[\frac{\Delta t\sigma^2 S_j^2}{4(\Delta S)^2}-\frac{r\Delta t S_j}{4\Delta S}\right]+\epsilon^k\, e^{ij2\pi/\omega}\left[1-\frac{\Delta t\sigma^2 S_j^2}{2(\Delta S)^2}-\frac{r\Delta t}{2}\right]$$
$$+\epsilon^k\, e^{i(j+1)2\pi/\omega}\left[\frac{\Delta t\sigma^2 S_j^2}{4(\Delta S)^2}+\frac{r\Delta t S_j}{4\Delta S}\right]$$

$$\epsilon\left\{e^{-i2\pi/\omega}\left[\frac{r\Delta t S_j}{4\Delta S}-\frac{\Delta t\sigma^2 S_j^2}{4(\Delta S)^2}\right]+\left[1+\frac{\Delta t\sigma^2 S_j^2}{2(\Delta S)^2}+\frac{r\Delta t}{2}\right]\right.$$
$$\left.-\,e^{i2\pi/\omega}\left[\frac{\Delta t\sigma^2 S_j^2}{4(\Delta S)^2}+\frac{r\Delta t S_j}{4\Delta S}\right]\right\}$$
$$= e^{-i2\pi/\omega}\left[\frac{\Delta t\sigma^2 S_j^2}{4(\Delta S)^2}-\frac{r\Delta t S_j}{4\Delta S}\right]+\left[1-\frac{\Delta t\sigma^2 S_j^2}{2(\Delta S)^2}-\frac{r\Delta t}{2}\right]$$
$$+\,e^{i2\pi/\omega}\left[\frac{\Delta t\sigma^2 S_j^2}{4(\Delta S)^2}+\frac{r\Delta t S_j}{4\Delta S}\right]$$

$$\epsilon\left\{\frac{\Delta t\sigma^2 S_j^2}{4(\Delta S)^2}\left[2-e^{-i2\pi/\omega}-e^{i2\pi/\omega}\right]+\frac{r\Delta t S_j}{4\Delta S}\left[e^{-i2\pi/\omega}-e^{i2\pi/\omega}\right]+1+\frac{r\Delta t}{2}\right\}$$
$$=\frac{\Delta t\sigma^2 S_j^2}{4(\Delta S)^2}\left[e^{-i2\pi/\omega}+e^{i2\pi/\omega}-2\right]+\frac{r\Delta t S_j}{4\Delta S}\left[e^{i2\pi/\omega}-e^{-i2\pi/\omega}\right]+1-\frac{r\Delta t}{2}$$

$$\epsilon\left\{\frac{\Delta t\sigma^2 S_j^2}{2(\Delta S)^2}[1-\cos(2\pi/\omega)]-\frac{ir\Delta t S_j}{2\Delta S}[\sin(2\pi/\omega)]+1+\frac{r\Delta t}{2}\right\}$$
$$=\frac{\Delta t\sigma^2 S_j^2}{2(\Delta S)^2}[\cos(2\pi/\omega)-1]+\frac{ir\Delta t S_j}{2\Delta S}[\sin(2\pi/\omega)]+1-\frac{r\Delta t}{2}$$

using identities

$$\cos(2\pi/\omega)=\frac{e^{i2\pi/\omega}+e^{-i2\pi/\omega}}{2}$$
$$\sin(2\pi/\omega)=\frac{e^{i2\pi/\omega}-e^{-i2\pi/\omega}}{2i}$$

By the von Neumann stability analysis (also known as Fourier stability analysis), if $|\epsilon| \leq 1$, the difference equation is stable and vice-versa.

$$|\epsilon| = \frac{\sqrt{\left[\frac{\Delta t\sigma^2 S_j^2}{2(\Delta S)^2}[\cos(2\pi/\omega) - 1] + 1 - \frac{r\Delta t}{2}\right]^2 + \left[\frac{r\Delta t S_j}{2\Delta S}[\sin(2\pi/\omega)]\right]^2}}{\sqrt{\left[\frac{\Delta t\sigma^2 S_j^2}{2(\Delta S)^2}[1 - \cos(2\pi/\omega)] + 1 + \frac{r\Delta t}{2}\right]^2 + \left[-\frac{r\Delta t S_j}{2\Delta S}[\sin(2\pi/\omega)]\right]^2}}$$

$$= \frac{\sqrt{\left[1 - \left(\frac{\Delta t\sigma^2 S_j^2}{2(\Delta S)^2}[1 - \cos(2\pi/\omega)] + \frac{r\Delta t}{2}\right)\right]^2 + \left[\frac{r\Delta t S_j}{2\Delta S}[\sin(2\pi/\omega)]\right]^2}}{\sqrt{\left[1 + \left(\frac{\Delta t\sigma^2 S_j^2}{2(\Delta S)^2}[1 - \cos(2\pi/\omega)] + \frac{r\Delta t}{2}\right)\right]^2 + \left[-\frac{r\Delta t S_j}{2\Delta S}[\sin(2\pi/\omega)]\right]^2}}$$

Since

$$\frac{\Delta t\sigma^2 S_j^2}{2(\Delta S)^2}[1 - \cos(2\pi/\omega)] + \frac{r\Delta t}{2} \geq 0,$$

we have

$$|\epsilon| \leq 1.$$

3.6 Simulation and Analysis

Table below shows the results of different set of parameters with different initial stock price S_0:

Table 3.1: Comparison of Crank-Nicolson scheme and Black-Scholes formula for pricing European call option with K = 20 and T = 1, where T is in year.

r	sigma	S0=35			S0=90		
		Crank-Nicolson	Black-Scholes Formula	Relative Error	Crank-Nicolson	Black-Scholes Formula	Relative Error
0.05	0.25	15.9910	15.9909	3.691E-06	70.9754	70.9754	1.152E-11
	0.35	16.1229	16.1228	6.297E-06	70.9754	70.9754	2.255E-09
	0.4	16.2577	16.2576	6.365E-06	70.9756	70.9756	1.553E-08
	0.5	16.6567	16.6566	5.688E-06	70.9806	70.9806	1.236E-07
0.1	0.25	16.9114	16.9113	1.478E-06	71.9033	71.9033	8.367E-11
	0.35	17.0034	17.0034	3.360E-06	71.9033	71.9033	1.135E-09
	0.4	17.1085	17.1084	3.694E-06	71.9034	71.9034	8.532E-09
	0.5	17.4414	17.4413	3.733E-06	71.9068	71.9068	7.721E-08
0.15	0.25	17.7899	17.7899	4.274E-07	72.7858	72.7858	2.663E-10
	0.35	17.8528	17.8528	1.406E-06	72.7858	72.7858	7.409E-10
	0.4	17.9332	17.9332	1.745E-06	72.7859	72.7859	4.696E-09
	0.5	18.2077	18.2077	2.126E-06	72.7882	72.7882	4.398E-08

All the option values obtained by Crank-Nicolson finite difference scheme and the Black-Scholes formula can be represented graphically as below:

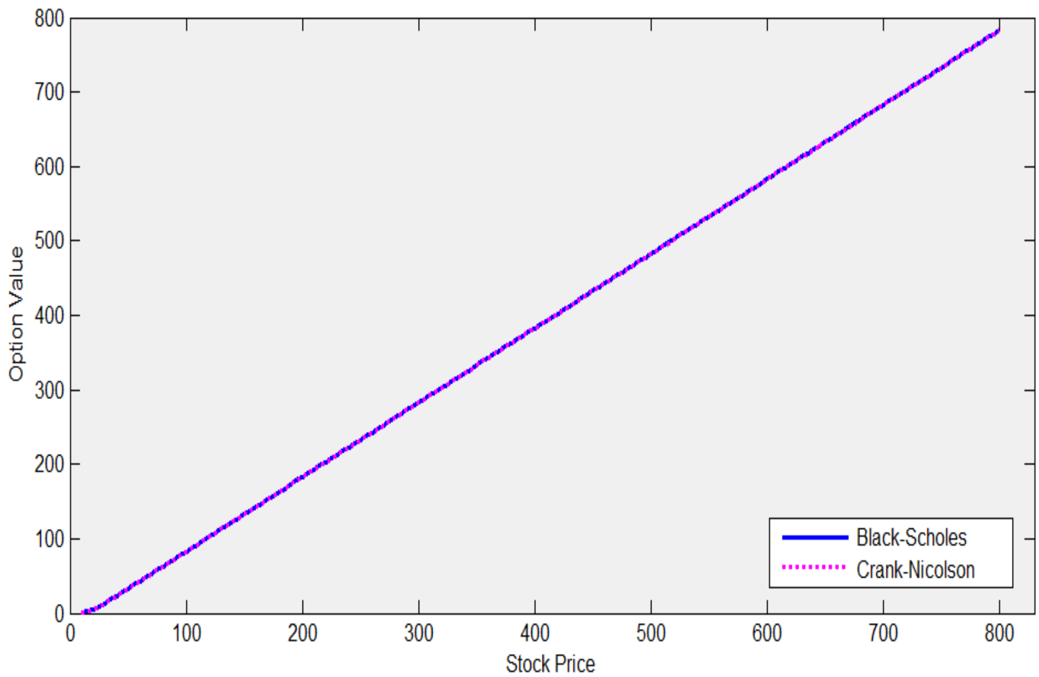

Figure 3.1: Comparison of Crank-Nicolson finite difference scheme and simulation method for pricing the European call option with $K = 20, r = 0.1$, $\sigma = 0.35$ and $T = 1$, where T is in year.

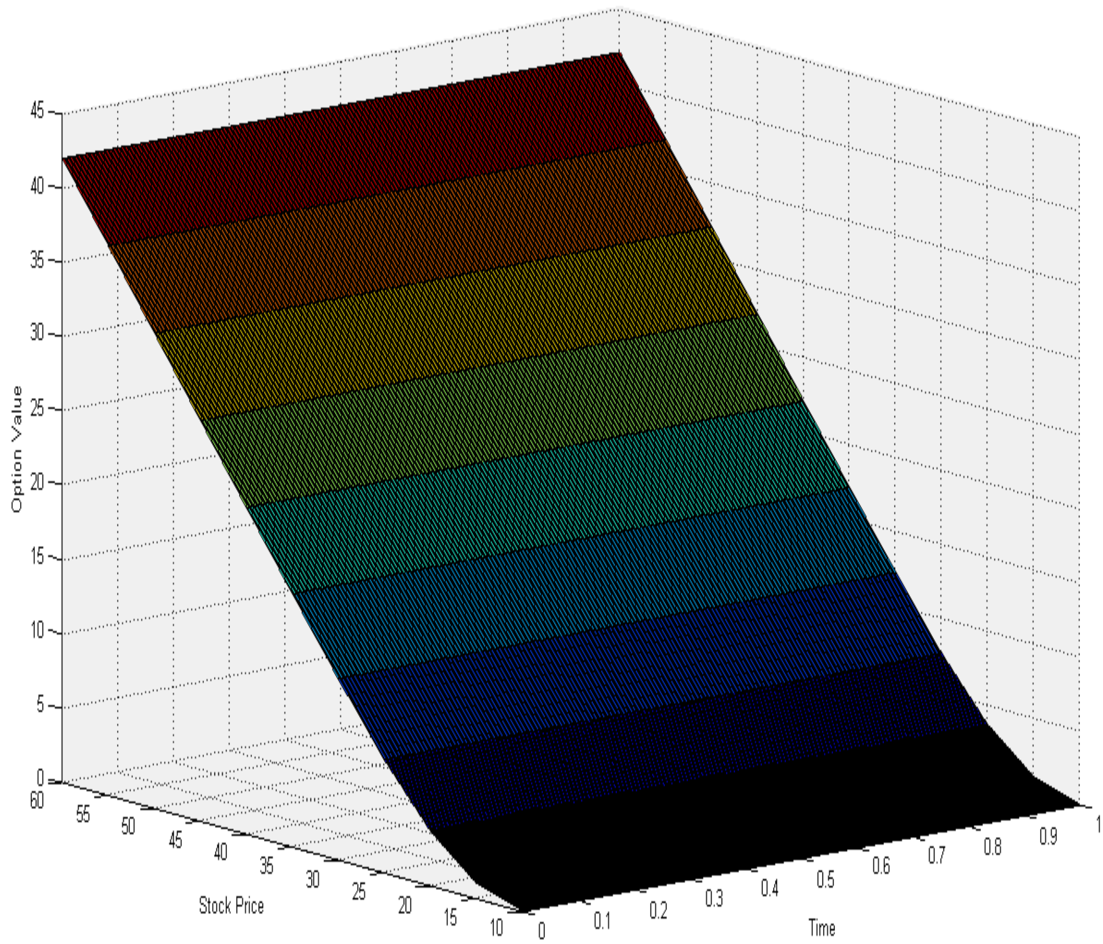

Figure 3.2: A three-dimensional plot of European call option with $K = 20$, $r = 0.1$, $\sigma = 0.35$, and $T = 1$, where T is in year.

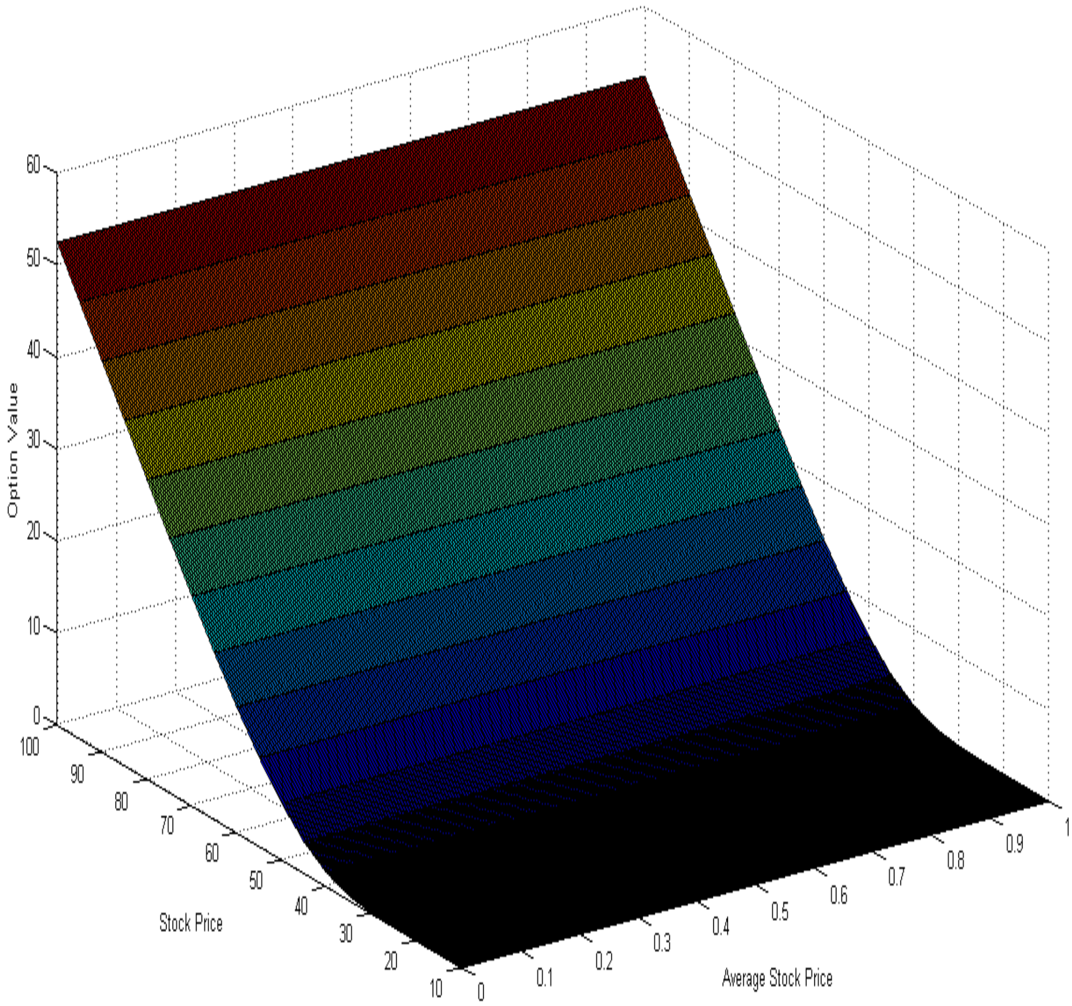

Figure 3.3: A three-dimensional plot of European call option with $K = 50$, $r = 0.1$, $\sigma = 0.35$, and $T = 1$, where T is in year.

3.7 Conclusion

Obviously, the option values obtained by proposed method are quite agreeable with the Matlab build-in function method. It is considered as consistent under different initial stock price and also volatility level.

CHAPTER 4

ASIAN OPTION – A TWO-DIMENSIONAL PDE

4.1 Introduction

Recall that Asian option is an option based on the average price of the underlying stock over the lifetime of the option. The term “Asian” is a reserved word and has no particular significance. Bankers David Spaughton told the story of how both he and Mark Standish were both working for Bankers Trust in 1987. They were in Tokyo, Japan on business when they found this method of pricing option. Hence, they called the option as Asian option.

Asian option is not traded as a standardized contract in any organized exchange. However, it is popular in the over-the-counter (OTC) market. There are several reasons for introducing Asian option. For instance, a corporation expecting to make payment in foreign currency can reduce its average foreign currency exposure by using Asian option. Besides, introducing Asian option can also avoid manipulation of the stock near expiration time. Stock price at time T is subject to manipulation. However, it is not easy to manipulate if we average the stock price.

4.2. Partial Differential Equation for Asian option

Suppose that our market, consisting of a risk-free cash bond, $B_t = e^{rt}$ and a stock with price $\{S_t\}_{t\geq 0}$, is governed by

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where $\{W_t\}_{t\geq 0}$is a $\mathbb{P}$-Brownian motion.

By *Itô's* lemma, we have

$$S_t = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right].$$

The discounted stock price $\tilde{S}_t = e^{-rt}S$ satisfies

$$\begin{aligned} d\tilde{S}_t = d(e^{-rt}S_t) &= -re^{-rt}S_t dt + e^{-rt}dS_t \\ &= -r\tilde{S}_t dt + e^{-rt}\,[\mu dt + \sigma dW_t]\,S_t \\ &= [(\mu - r)dt + \sigma dW_t]\tilde{S}_t \\ &= \sigma\tilde{S}_t d\left[\frac{\mu - r}{\sigma}t + W_t\right] \\ &= \sigma\tilde{S}_t dX_t \end{aligned}$$

where $X_t = \left[\frac{\mu-r}{\sigma}t + W_t\right]$ is a Brownian motion under some risk neutral probability measure $\mathbb{P}^{(\mathrm{L})}$. Again by *Itô's* lemma, we have

$$\tilde{S}_t = S_0 \exp\left[\frac{-\sigma^2 t}{2} + \sigma X_t\right].$$

In terms of X_t, the stock price can be written as

$$S_t = S_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)t + \sigma X_t\right]$$

(see for example, A. Etheridge (2002) for a concise and elegant exposition)[1].

Let $\Phi_{\mathrm{T}} = \Phi(Z_T, S_T) = \max\left(\frac{Z_T}{\mathrm{T}} - \mathrm{K}, 0\right)$ be the payoff function at time T where S_T refers to the stock price at time T and $\frac{Z_T}{\mathrm{T}}$ refers to the average stock price at time T and where $Z_t = \int_0^t S_\tau d\tau$.

From our general theory [1], option value at time t is given by:

$$\begin{aligned}\mathrm{V_t(Z, S)} &= \mathrm{e^{-r(T-t)}E^{\mathbb{P}(L)}}[\Phi(Z_T, S_T)|\mathcal{F}_{\mathrm{t}}] \\ &= \mathrm{e^{-r(T-t)}E^{\mathbb{P}(L)}}[\Phi(Z_T, S_T)|\mathrm{S_t = S, Z_t = Z}]\end{aligned}$$

where $\mathbb{P}^{(L)}$ is the risk neutral probability measure under which the discounted stock price $\tilde{S}_t = e^{-rt}S_t$ is a $\mathbb{P}^{(L)}$-martingale.

Now, we introduce a new function, $F(Z, S, t)$ which solves the terminal value problem

$$\frac{\partial F}{\partial t} + rS\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + \frac{\partial Z_t}{\partial t}\frac{\partial F}{\partial Z} - rF = 0 \tag{19}$$

$$F(Z, S, T) = \Phi(Z, S).$$

Define $N_t = e^{-rt}F(Z, S, t)$. Recall that $dS_t = S_t(rdt + \sigma dX_t)$. By the *Itô's* formula,

$$\begin{aligned}dN_t &= d(e^{-rt}F(Z, S, t)) \\ &= e^{-rt}\left[-rFdt + \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}dS_t + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}d^2S_t + \frac{\partial F}{\partial Z}dZ_t\right] \\ &= e^{-rt}\Bigg[-rFdt + \frac{\partial F}{\partial t}dt + S_t\frac{\partial F}{\partial S}(rdt + \sigma dX_t) \\ &\qquad + \frac{1}{2}S_t^2\frac{\partial^2 F}{\partial S^2}(rdt + \sigma dX_t)^2 + \frac{\partial F}{\partial Z}dZ_t\Bigg]\end{aligned}$$

$$= e^{-rt}\left[-rF + \frac{\partial F}{\partial t} + rS_t\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S_t^2\ \frac{\partial^2 F}{\partial S^2} + \frac{\partial F}{\partial Z}\frac{dZ_t}{\partial t}\right]dt$$

$$+\ e^{-rt}\sigma S_t\frac{\partial F}{\partial S}\ dX_t$$

$$= e^{-rt}\sigma S_t\frac{\partial F}{\partial S}\ dX_t.$$

It follows that

$$\mathrm{N_t} = \mathrm{N_0} + \int_0^{\mathrm{t}} e^{-\mathrm{r\tau}}\mathrm{\sigma S_\tau}\frac{\partial F}{\partial S}\ dX_t$$

is a $\mathbb{P}^{(L)}$-martingale. Since $N_T = e^{-rT}\Phi_{\mathrm{T}}$, by martingale property,

$$E^{\mathbb{P}^{(L)}}[N_T|\mathcal{F}_t] = N_t$$

$$E^{\mathbb{P}^{(L)}}[e^{-rT}\Phi_T|\mathcal{F}_t] = N_t$$

$$E^{\mathbb{P}^{(L)}}[e^{-rT}\Phi_T|\mathcal{F}_t] = e^{-rt}F(S,Z,t)$$

$$\therefore F(Z,S,t) = E^{\mathbb{P}^{(L)}}\left[e^{-r(T-t)}\Phi_T\middle|\mathcal{F}_t\right]$$

$$= E^{\mathbb{P}^{(L)}}\left[e^{-r(T-t)}\Phi_T\middle|S_t = S, Z_t = Z\right]$$

is the option value at time t.

Since the diffusion term $\frac{\partial^2 F}{\partial Z^2}$ is missing, equation (19) is a degenerate diffusion equation. As

$$Z_t = \int_0^t S_\tau d\tau,$$

$$\frac{\partial Z_t}{\partial t} = S_t,$$

equation (19) now assumes the form,

$$\frac{\partial F}{\partial t} + rS\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 F}{\partial S^2} + S\frac{\partial F}{\partial Z} - rF = 0$$

$$F(Z,S,T) = \Phi(Z,S) \qquad (20)$$

4.3 Method of Solution

There are two problems concerning equation (20).

(a). To determine if equation (20) is a well posed problem.

(b). To propose an efficient difference scheme for solving it.

Problem (a) will not be treated here because equation (20) is a degenerate two-dimensional diffusion equation which is known to be a well posed problem under special boundary conditions. The far field boundary conditions are provided by Kangro [21]. The other suitable boundary conditions are derived in the following section.

4.4. Boundary Values

First, we consider the left boundary condition. We found that $S_t = 0$ implies $S_\tau = 0$ for $\tau > t$ and $Z_T = \int_0^T S_\tau d\tau = \int_0^t S_\tau d\tau + \int_t^T S_\tau d\tau = Z_t$. Hence for the Asian call option with payoff $F_C(Z_T, S_T, T) = (\frac{Z_T}{T} - K)_+$, when $S_t = 0$, we obtain

$$F_C(Z, 0, t) = \mathrm{E}^{\mathbb{P}^{(L)}}\left[e^{-r(T-t)}(\frac{Z}{T} - K)_+ | S_t = 0, Z_t = Z\right]$$
$$= e^{-r(T-t)}(\frac{Z}{T} - K)_+$$

as the left boundary condition.

Next we derive the call option price $F_C(Z_t, S_t, t)$ at time t when it is in money, that is, when $\frac{Z_t}{T} > K$.

$$F_C(Z_t, S_t, t) = e^{-r(T-t)}\mathrm{E}^{\mathbb{P}^{(L)}}\left[\left(\frac{Z_T}{T} - K\right)_+ \middle| \mathcal{F}_t\right]$$

$$= e^{-r(T-t)}\mathrm{E}^{\mathbb{P}^{(L)}}\left[\frac{1}{T}\int_t^T S_\tau\, d\tau + \frac{1}{T}\int_0^t S_\tau\, d\tau - K \middle| \mathcal{F}_t\right]$$

$$= e^{-r(T-t)}\mathrm{E}^{\mathbb{P}^{(L)}}\left[\frac{Z_t}{T} - K\right] + e^{-r(T-t)}\frac{1}{T}\mathrm{E}^{\mathbb{P}^{(L)}}\left[\int_t^T S_\tau\, d\tau\right]$$

Integrating $d\left(e^{rt}\tilde{S}_t\right) = re^{rt}\tilde{S}_t dt + e^{rt} d\tilde{S}_t$ and forming conditional expectation, we have

$$\mathrm{E}^{\mathbb{P}^{(L)}}\left[e^{rT}\tilde{S}_T - e^{rt}\tilde{S}_t | \mathcal{F}_t\right] = \mathrm{E}^{\mathbb{P}^{(L)}}\left[\int_t^T rS_\tau d\tau | \mathcal{F}_t\right] + \mathrm{E}^{\mathbb{P}^{(L)}}\left[\int_t^T e^{r\tau}\sigma\tilde{S}_\tau dX_t | \mathcal{F}_t\right].$$

This simplifies to

$$e^{rT}\tilde{S}_t - e^{rt}\tilde{S}_t = \mathrm{E}^{\mathbb{P}^{(L)}}\left[\int_t^T rS_\tau d\tau | \mathcal{F}_t\right],$$

as $\tilde{S}_t$ is a martingale and the second integral on the right hand side is a stochastic integral with mean zero under probability measure $\mathbb{P}^{(L)}$. Thus

$$\mathrm{E}^{\mathbb{P}^{(L)}}\left[\int_t^T S_\tau d\tau | \mathcal{F}_t\right] = \frac{S_t}{r}\left[e^{r(T-t)} - 1\right],$$

and the Asian call option is given by the following when it is in money at time t.

$$F_C(Z_t, S_t, t) = e^{-r(T-t)}\left[\frac{Z_t}{T} - K\right] + e^{-r(T-t)}\frac{S_t}{rT}\left[e^{r(T-t)} - 1\right]$$

$$= e^{-r(T-t)}\left[\frac{Z_t}{T} - K\right] + \frac{S_t}{rT}\left[1 - e^{-r(T-t)}\right] \quad \text{for} \quad Z_t \geq KT \quad . \qquad (21)$$

For large stock price S_t, intuitively the Asian call option must be in money. Hence the same formula

$$F_C(Z_t, S_t, t) = \left(\mathrm{e}^{-r(T-t)}\left[\frac{Z_t}{T} - K\right] + \frac{S_t}{rT}\left[1 - \mathrm{e}^{-r(T-t)}\right]\right) \qquad (22)$$

apply for large S_t.

Next, we consider Asian put option with payoff $F_p(Z_T, S_T, T) = \left(K - \frac{Z_T}{T}\right)_+$. By definition,

$$F_C(Z_t, S_t, t) - F_P(Z_t, S_t, t) = e^{-r(T-t)} \mathrm{E}^{\mathbb{P}^{(L)}} \left[(\frac{Z_T}{T} - K)_+ - (K - \frac{Z_T}{T})_+ | \mathcal{F}_t \right]$$

$$= e^{-r(T-t)} \mathrm{E}^{\mathbb{P}^{(L)}} \left[\frac{Z_T}{T} - K | \mathcal{F}_t \right]$$

$$= e^{-r(T-t)} \frac{1}{T} \mathrm{E}^{\mathbb{P}^{(L)}} [\mathrm{Z_T} | \mathcal{F}_\mathrm{t}] - \ Ke^{-r(T-t)}$$

$$= \frac{e^{-r(T-t)}}{T} \mathrm{E}^{\mathbb{P}^{(L)}} \left[\int_0^{\mathrm{t}} \mathrm{S}_\tau \mathrm{d}\tau \middle| \mathcal{F}_\mathrm{t} + \int_{\mathrm{t}}^{\mathrm{T}} \mathrm{S}_\tau \mathrm{d}\tau \middle| \mathcal{F}_\mathrm{t} \right]$$

$$-Ke^{-r(T-t)}$$

$$= \frac{e^{-r(T-t)}}{T} [Z_t - K] + e^{-r(T-t)} \frac{S_t}{rT} \left[e^{r(T-t)} - 1 \right]$$

In view of (22), we found that for large S_t,

$$F_P(Z_t, S_t, t) = 0$$

4.5. Discretization

Let i, j and h denote the indices for the average stock price Z, stock price S, and time t respectively. Let M, N, H be the number of partitions for Z, S and t respectively. Define

$$\Delta Z = \frac{Z_{max}}{M}, \ \ \Delta S = \frac{S_{max}}{N}, \ \ \Delta t = \frac{T}{H}$$

and let

$$Z_i = i\Delta Z, \ S_j = j\Delta S, \ t_h = h\Delta t$$

for

$$0 \leq i \leq M, 0 \leq j \leq N, 0 \leq h \leq H.$$

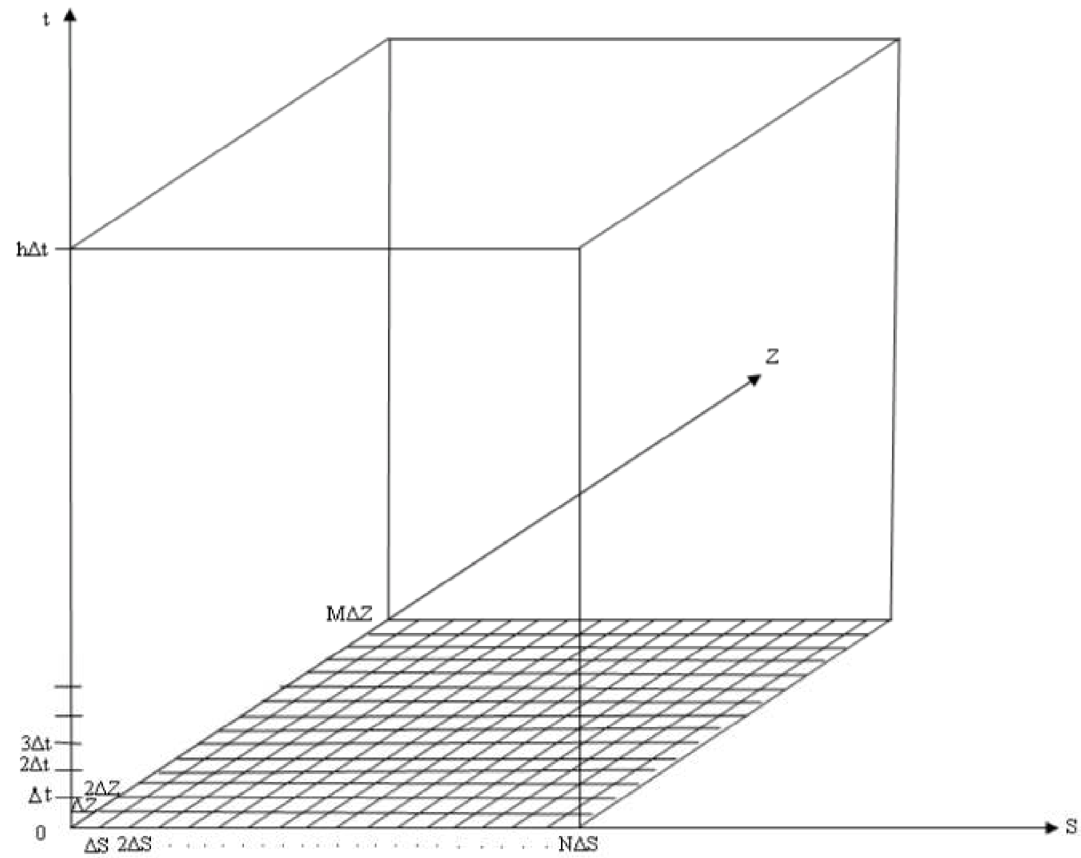

Figure 4.1 : A three-dimensional grid

The nodes $(Z_i,\ S_j, t_h)$ form a uniform grid in $[0, Z_{max}] \times [0, S_{max}] \times [0, T]$. At a node $(i, j, h) \equiv (Z_i, S_j, t_h)$ the expression

$$\frac{\sigma^2 S^2}{2}\frac{\partial^2 F}{\partial S^2} + rS\frac{\partial F}{\partial S} + S\frac{\partial F}{\partial Z} - rF$$

is approximated by the difference scheme

$$L_{i,j}^h = \frac{\sigma^2 S_j^2}{2} D_{SS} + rS_j D_S + S_j D_Z - rF_{i,j}^h$$

where

$Z_i = i\Delta Z$,

$S_j = j\Delta S$,

$t_h = h\Delta t,$

$F_{i,j}^h \cong F(Z_i, S_j, t_h)$

$$D_{SS} = \frac{F(Z_i, S_{j+1}, t_h) - 2F(Z_i, S_j, t_h) + F(Z_i, S_{j-1}, t_h)}{(\Delta S)^2}$$

$$D_S = \frac{F(Z_i, S_{j+1}, t_h) - F(Z_i, S_{j-1}, t_h)}{2\Delta S}$$

$$D_Z = \frac{F(Z_{i+1}, S_j, t_h) - F(Z_i, S_j, t_h)}{\Delta Z}$$

Average the forward time scheme at (i, j, h):

$$\frac{F_{i,j}^{h+1} - F_{i,j}^h}{\Delta t} + L_{i,j}^h = 0$$

with the Backward time scheme at $(i, j, h + 1)$:

$$\frac{F_{i,j}^{h+1} - F_{i,j}^h}{\Delta \mathrm{t}} + L_{i,j}^{h+1} = 0$$

provides the *Crank-Nicolson* scheme:

$$\frac{F_{i,j}^{h+1} - F_{i,j}^h}{\Delta t} + \frac{1}{2}\left(L_{i,j}^h + L_{i,j}^{h+1}\right) = 0$$

$$F_{i,j}^h - \frac{\Delta t}{2} L_{i,j}^h = F_{i,j}^{h+1} + \frac{\Delta t}{2} L_{i,j}^{h+1}$$

where

$$L_{i,j}^h = \frac{\sigma^2 S_j^2}{2(\Delta S)^2}\left[F_{i,j+1}^h - 2F_{i,j}^h + F_{i,j-1}^h\right] + \frac{rS_j}{2\Delta S}\left[F_{i,j+1}^h - F_{i,j-1}^h\right] + \frac{S_j}{\Delta Z}\left[F_{i+1,j}^h - F_{i,j}^h\right] - rF_{i,j}^h$$

and

$$L_{i,j}^{h+1} = \frac{\sigma^2 S_j^2}{2(\Delta S)^2}\left[F_{i,j+1}^{h+1} - 2F_{i,j}^{h+1} + F_{i,j-1}^{h+1}\right] + \frac{rS_j}{2\Delta S}\left[F_{i,j+1}^{h+1} - F_{i,j-1}^{h+1}\right] + \frac{S_j}{\Delta Z}\left[F_{i+1,j}^{h+1} - F_{i,j}^{h+1}\right] - rF_{i,j}^{h+1}$$

Therefore,

$$\left[\frac{r\Delta tS_j}{4\Delta S} - \frac{\sigma^2\Delta t{S_j}^2}{4(\Delta S)^2}\right]F_{i,j-1}^{h} + \left[1 + \frac{\sigma^2\Delta t{S_j}^2}{2(\Delta S)^2} + \frac{\Delta tS_j}{2\Delta Z} + \frac{r\Delta t}{2}\right]F_{i,j}^{h} + \left[-\frac{r\Delta tS_j}{4\Delta S} - \frac{\sigma^2\Delta t{S_j}^2}{4(\Delta S)^2}\right]F_{i,j+1}^{h} - \frac{\Delta tS_j}{2\Delta Z}F_{i+1,j}^{h}$$

$$= \left[\frac{\sigma^2\Delta t{S_j}^2}{4(\Delta S)^2} - \frac{r\Delta tS_j}{4\Delta S}\right]F_{i,j-1}^{h+1} + \left[1 - \frac{\sigma^2\Delta t{S_j}^2}{2(\Delta S)^2} - \frac{\Delta tS_j}{2\Delta Z} - \frac{r\Delta t}{2}\right]F_{i,j}^{h+1} + \left[\frac{\sigma^2\Delta t{S_j}^2}{4(\Delta S)^2} + \frac{r\Delta tS_j}{4\Delta S}\right]F_{i,j+1}^{h+1} + \frac{\Delta tS_j}{2\Delta Z}F_{i+1,j}^{h+1}$$

Let $= \frac{\sigma^2\Delta t}{4(\Delta S)^2}$, $\beta = \frac{r\Delta t}{4\Delta S}$, and $\lambda = \frac{\Delta t}{2\Delta Z}$. The above may be abbreviated to

$$\left(\beta S_j - \alpha {S_j}^2\right)F_{i,j-1}^{h} + \left(1 + 2\alpha{S_j}^2 + \lambda S_j + 2\Delta S\beta\right)F_{i,j}^{h} - \left(\beta S_j + \alpha{S_j}^2\right)F_{i,j+1}^{h} - \lambda S_j F_{i+1,j}^{h}$$

$$= \left(\alpha{S_j}^2 - \beta S_j\right)F_{i,j-1}^{h+1} + \left(1 - 2\alpha{S_j}^2 - \lambda S_j - 2\Delta S\beta\right)F_{i,j}^{h+1} + \left(\alpha{S_j}^2 + \beta S_j\right)F_{i,j+1}^{h+1} + \lambda S_j F_{i+1,j}^{h+1} \qquad (23)$$

The figure below is a visualization of the equation above :

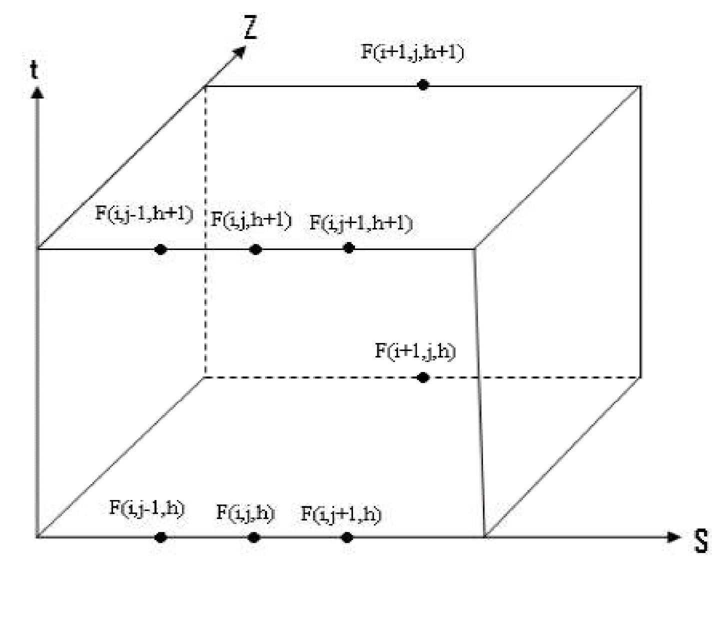

Figure 4.2: Relationship between values of *F* at several points

Note that $F_{i,j}^{h} = F(i,j,h)$ is the option price at time $h\Delta t$, average stock price $i\Delta z$ and stock price $j\Delta s$. As depicted in the Figure 4.2, equation (23) represents a relationship between values of F at the 8 points $(i,j,h), (i,j-1,h), (i,j+1,h), (i+1,j,h), (i,j,h+1), (i,j-1,h+1), (i,j+1,h+1), (i+1,j,h+1)$. At the time of computation, if values of F at 5 points $(i+1,j,h), (i,j,h+1), (i,j-1,h+1), (i,j+1,h+1), (i+1,j,h+1)$ are known, then values of F at $(i,j,h), (i,j-1,h), (i,j+1,h)$, satisfy linear equation (23). This is the case if starting at $t = T$ and $Z = Z_{max}$, iteration is performed backward in time t and in Z. For fixed h and i, corresponding to each of the interior points (i,j,h) where $j = 1,2,..n-1$, there is one and only one linear equation (23) and therefore there are as many equations as unknowns $F(i,j,h)$ for $j = 1,2,..n-1$ and $F(i,j,h)$ may be determined.

4.6. Implementation

The way to calculate the value of Asian option at time zero is similar to the way of finding the value of European option. First of all, we set all the given parameters: terminal time T, maximum stock price S_{max}, strike price K, interest rate r and volatility level σ. We then determine the maximum average stock price as $Z_{max} = KT$ because this would make the payoff function equals to zero. Finally, we determine dS, dZ and dt based on their number of partitions. Note that the time unit is in year.

Let A and B be the matrices for time h and $h+1$ respectively defined as follows:

At time h,

A

$$= \begin{pmatrix} 1+2\alpha S_1^2+\lambda S_1+2\Delta S\beta & -(\beta S_2+\alpha S_2^2) & 0 & \cdots & \cdots & 0 \\ \beta S_1-\alpha S_1^2 & 1+2\alpha S_2^2+\lambda S_2+2\Delta S\beta & -(\beta S_3+\alpha S_3^2) & \cdots & \vdots & \vdots \\ 0 & \beta S_2-\alpha S_2^2 & 1+2\alpha S_3^2+\lambda S_3+2\Delta S\beta & \cdots & \vdots & 0 \\ \vdots & \cdots & \cdots & \cdots & 0 & -(\beta S_n+\alpha S_n^2) \\ 0 & \cdots & \cdots & 0 & \beta S_{n-1}-\alpha S_{n-1}^2 & 1+2\alpha S_n^2+\lambda S_n+2\Delta S\beta \end{pmatrix}$$

At time $h+1$,

B

$$= \begin{pmatrix} 1-2\alpha S_1^2-\lambda S_1-2\Delta S\beta & \alpha S_2^2+\beta S_2 & 0 & \cdots & \cdots & 0 \\ \alpha S_1^2-\beta S_1 & 1-2\alpha S_2^2-\lambda S_2-2\Delta S\beta & \alpha S_3^2+\beta S_3 & \cdots & \vdots & \vdots \\ 0 & \alpha S_2^2-\beta S_2 & 1-2\alpha S_3^2-\lambda S_3-2\Delta S\beta & \cdots & \vdots & 0 \\ \vdots & \cdots & \cdots & \cdots & 0 & \alpha S_n^2+\beta S_n \\ 0 & \cdots & \cdots & 0 & \alpha S_{n-1}^2-\beta S_{n-1} & 1-2\alpha S_n^2-\lambda S_n-2\Delta S\beta \end{pmatrix}$$

Note that the matrices A and B are tridiagonal matrices with size $(m-1)\times(n-1)$ where $m=n$.

Write system (23) in matrix form: $AF^h = BF^{h+1} + b$ where $F^h = (F_{i,1}^h, F_{i,2}^h, \ldots\ldots F_{i,n-1}^h)^T$ and b is the column vector arising from boundary values. Solving, yield $F^h = A^{-1}BF^{h+1} + A^{-1}b$.

When $Z_t > KT$, option price can be calculated according to equation (21). Hence option price is only computed using finite difference scheme when $Z_t \leq KT$. Below are the boundary conditions that we found previously:

1. $F_C(Z_T, S_T, T) = \left(\frac{Z_T}{T} - K\right)_+$
2. When stock price is zero,

   $F_C(Z, 0, t) = \mathrm{e}^{-\mathrm{r(T-t)}}\left(\frac{Z}{T} - K\right)_+$

3. $F_C(Z_t, S_t, T) = \mathrm{e}^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}\left(\frac{Z_t}{T} - K\right) + \frac{\mathrm{S_t}}{\mathrm{rT}}\left[1 - \mathrm{e}^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}\right]$ for $Z_t \geq KT$

4. $F_C(Z_t, S_T, T) = \mathrm{e}^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}\left(\frac{Z_t}{T} - K\right) + \frac{\mathrm{S_t}}{\mathrm{rT}}\left[1 - \mathrm{e}^{-\mathrm{r}(\mathrm{T}-\mathrm{t})}\right]$ $\forall\, t$ if stock price is large.

To compute the value of Asian option, just call our function *AsianOption (Appendix B)* with proper parameters. For instance,

[StkPrice CallPrice SpPrice RelErr] = AsianOption(20, 0.1, 0.5)

4.7. Stability Analysis

Recall that the following is the approximate difference equation after applying *Crank-Nicolson* scheme:

$$F_{i,j-1}^{h}\left[\frac{r\Delta t S_j}{4\Delta S} - \frac{\sigma^2 \Delta t {S_j}^2}{4(\Delta S)^2}\right] + F_{i,j}^{h}\left[1 + \frac{\sigma^2 \Delta t {S_j}^2}{2(\Delta S)^2} + \frac{\Delta t S_j}{2\Delta Z} + \frac{r\Delta t}{2}\right] +$$

$$F_{i,j+1}^{h}\left[-\frac{r\Delta t S_j}{4\Delta S} - \frac{\sigma^2 \Delta t {S_j}^2}{4(\Delta S)^2}\right] - F_{i+1,j}^{h}\frac{\Delta t S_j}{2\Delta Z}$$

$$= F_{i,j-1}^{h+1}\left[\frac{\sigma^2 \Delta t {S_j}^2}{4(\Delta S)^2} - \frac{r\Delta t S_j}{4\Delta S}\right] + F_{i,j}^{h+1}\left[1 - \frac{\sigma^2 \Delta t {S_j}^2}{2(\Delta S)^2} - \frac{\Delta t S_j}{2\Delta Z} - \frac{r\Delta t}{2}\right] +$$

$$F_{i,j+1}^{h+1}\left[\frac{\sigma^2 \Delta t {S_j}^2}{4(\Delta S)^2} + \frac{r\Delta t S_j}{4\Delta S}\right] + F_{i+1,j}^{h+1}\frac{\Delta t S_j}{2\Delta Z}$$

which is derived from the general form of Black-Scholes equation:

$$\frac{\partial F}{\partial t} + rS\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + S\frac{\partial F}{\partial Z} - rF = 0$$

Let $h + 1 = N - k$ and $h = N - (k + 1)$, we have

$$F_{i,j-1}^{N-(k+1)}\left[\frac{r\Delta tS_j}{4\Delta S} - \frac{\sigma^2\Delta tS_j^{\,2}}{4(\Delta S)^2}\right] + F_{i,j}^{N-(k+1)}\left[1 + \frac{\sigma^2\Delta tS_j^{\,2}}{2(\Delta S)^2} + \frac{\Delta tS_j}{2\Delta Z} + \frac{r\Delta t}{2}\right] +$$

$$F_{i,j+1}^{N-(k+1)}\left[-\frac{r\Delta tS_j}{4\Delta S} - \frac{\sigma^2\Delta tS_j^{\,2}}{4(\Delta S)^2}\right] - F_{i+1,j}^{N-(k+1)}\frac{\Delta tS_j}{2\Delta Z}$$

$$= F_{i,j-1}^{N-k}\left[\frac{\sigma^2\Delta tS_j^{\,2}}{4(\Delta S)^2} - \frac{r\Delta tS_j}{4\Delta S}\right] + F_{i,j}^{N-k}\left[1 - \frac{\sigma^2\Delta tS_j^{\,2}}{2(\Delta S)^2} - \frac{\Delta tS_j}{2\Delta Z} - \frac{r\Delta t}{2}\right] +$$

$$F_{i,j+1}^{N-k}\left[\frac{\sigma^2\Delta tS_j^{\,2}}{4(\Delta S)^2} + \frac{r\Delta tS_j}{4\Delta S}\right] + F_{i+1,j}^{N-k}\frac{\Delta tS_j}{2\Delta Z} \tag{24}$$

Solutions of equation (24) are assumed to be the following:

$$F_{i,j}^{N-(k+1)} = \epsilon^{(k+1)}e^{(i+j)2\pi\sqrt{-1}/\omega}$$

$$F_{i,j+1}^{N-(k+1)} = \epsilon^{(k+1)}e^{(i+j+1)2\pi\sqrt{-1}/\omega}$$

$$F_{i,j-1}^{N-(k+1)} = \epsilon^{(k+1)}e^{(i+j-1)2\pi\sqrt{-1}/\omega}$$

$$F_{i+1,j}^{N-(k+1)} = \epsilon^{(k+1)}\, e^{(i+j+1)2\pi\sqrt{-1}/\omega}$$

$$F_{i,j}^{N-k} = \epsilon^{k}\, e^{(i+j)2\pi\sqrt{-1}/\omega}$$

$$F_{i,j+1}^{N-k} = \epsilon^{k}\, e^{(i+j+1)2\pi\sqrt{-1}/\omega}$$

$$F_{i,j-1}^{N-k} = \epsilon^{k}\, e^{(i+j-1)2\pi\sqrt{-1}/\omega}$$

$$F_{i+1,j}^{N-k} = \epsilon^{k}\, e^{(i+j+1)2\pi\sqrt{-1}/\omega} \tag{25}$$

Note that the i here refers to an index. Now, substituting equations (25) into (24), we have

$$\begin{aligned}
&\epsilon^{(k+1)}e^{(i+j-1)2\pi\sqrt{-1}/\omega}\left[\frac{r\Delta tS_j}{4\Delta S} - \frac{\sigma^2\Delta tS_j^{\,2}}{4(\Delta S)^2}\right] \\
&\quad + \epsilon^{(k+1)}e^{(i+j)2\pi\sqrt{-1}/\omega}\left[1 + \frac{\sigma^2\Delta tS_j^{\,2}}{2(\Delta S)^2} + \frac{\Delta tS_j}{2\Delta Z} + \frac{r\Delta t}{2}\right] \\
&\quad + \epsilon^{(k+1)}e^{(i+j+1)2\pi\sqrt{-1}/\omega}\left[-\frac{r\Delta tS_j}{4\Delta S} - \frac{\sigma^2\Delta tS_j^{\,2}}{4(\Delta S)^2}\right] \\
&\quad - \epsilon^{(k+1)}\, e^{(i+j+1)2\pi\sqrt{-1}/\omega}\left[\frac{\Delta tS_j}{2\Delta Z}\right]
\end{aligned}$$

$$= \epsilon^k\, e^{(i+j-1)2\pi\sqrt{-1}/\omega}\left[\frac{\sigma^2\Delta t {S_j}^2}{4(\Delta S)^2} - \frac{r\Delta t S_j}{4\Delta S}\right] + \epsilon^k\, e^{(i+j)2\pi\sqrt{-1}/\omega}\left[1 - \frac{\sigma^2\Delta t {S_j}^2}{2(\Delta S)^2} - \frac{\Delta t S_j}{2\Delta Z} - \frac{r\Delta t}{2}\right] + \epsilon^k\, e^{(i+j+1)2\pi\sqrt{-1}/\omega}\left[\frac{\sigma^2\Delta t {S_j}^2}{4(\Delta S)^2} + \frac{r\Delta t S_j}{4\Delta S}\right] + \epsilon^k\, e^{(i+j+1)2\pi\sqrt{-1}/\omega}\left[\frac{\Delta t S_j}{2\Delta Z}\right]$$

$$\epsilon\left\{e^{-2\pi\sqrt{-1}/\omega}\left[\frac{r\Delta t S_j}{4\Delta S} - \frac{\sigma^2\Delta t {S_j}^2}{4(\Delta S)^2}\right] + \left[1 + \frac{\sigma^2\Delta t {S_j}^2}{2(\Delta S)^2} + \frac{\Delta t S_j}{2\Delta Z} + \frac{r\Delta t}{2}\right] + e^{2\pi\sqrt{-1}/\omega}\left[-\frac{r\Delta t S_j}{4\Delta S} - \frac{\sigma^2\Delta t {S_j}^2}{4(\Delta S)^2} - \frac{\Delta t S_j}{2\Delta Z}\right]\right\}$$

$$= e^{-2\pi\sqrt{-1}/\omega}\left[\frac{\sigma^2\Delta t {S_j}^2}{4(\Delta S)^2} - \frac{r\Delta t S_j}{4\Delta S}\right] + \left[1 - \frac{\sigma^2\Delta t {S_j}^2}{2(\Delta S)^2} - \frac{\Delta t S_j}{2\Delta Z} - \frac{r\Delta t}{2}\right] + e^{2\pi\sqrt{-1}/\omega}\left[\frac{\sigma^2\Delta t {S_j}^2}{4(\Delta S)^2} + \frac{r\Delta t S_j}{4\Delta S} + \frac{\Delta t S_j}{2\Delta Z}\right]$$

$$\epsilon\left\{\frac{\sigma^2\Delta t {S_j}^2}{4(\Delta S)^2}\left[2 - \left(e^{-2\pi\sqrt{-1}/\omega} + e^{2\pi\sqrt{-1}/\omega}\right)\right] + \frac{r\Delta t S_j}{4\Delta S}\left[e^{-2\pi\sqrt{-1}/\omega} - e^{2\pi\sqrt{-1}/\omega}\right] - \frac{\Delta t S_j}{2\Delta Z}\left[e^{2\pi\sqrt{-1}/\omega}\right] + 1 + \frac{\Delta t S_j}{2\Delta Z} + \frac{r\Delta t}{2}\right\}$$

$$= \frac{\sigma^2\Delta t {S_j}^2}{4(\Delta S)^2}\left[e^{-2\pi\sqrt{-1}/\omega} + e^{2\pi\sqrt{-1}/\omega} - 2\right] + \frac{r\Delta t S_j}{4\Delta S}\left[e^{2\pi\sqrt{-1}/\omega} - e^{-2\pi\sqrt{-1}/\omega}\right] + \frac{\Delta t S_j}{2\Delta Z}\left[e^{2\pi\sqrt{-1}/\omega}\right] + 1 - \frac{\Delta t S_j}{2\Delta Z} - \frac{r\Delta t}{2}$$

$$\epsilon\left\{\frac{\sigma^2\Delta t {S_j}^2}{4(\Delta S)^2}\left[2 - 2\cos(2\pi/\omega)\right] + \frac{r\Delta t S_j}{4\Delta S}\left[-2\sqrt{-1}\sin(2\pi/\omega)\right] - \frac{\Delta t S_j}{2\Delta Z}\left[\cos(2\pi/\omega) + \sqrt{-1}\sin(2\pi/\omega)\right] + 1 + \frac{\Delta t S_j}{2\Delta Z} + \frac{r\Delta t}{2}\right\}$$

$$= \frac{\sigma^2 \Delta t {S_j}^2}{4(\Delta S)^2}[2\cos(2\pi/\omega) - 2] + \frac{r\Delta t S_j}{4\Delta S}\left[2\sqrt{-1}\sin(2\pi/\omega)\right] + \frac{\Delta t S_j}{2\Delta Z}\left[\cos(2\pi/\omega) + \sqrt{-1}\sin(2\pi/\omega)\right] + 1 - \frac{\Delta t S_j}{2\Delta Z} - \frac{r\Delta t}{2}$$

$$\epsilon\left\{\frac{\sigma^2 \Delta t {S_j}^2}{2(\Delta S)^2}[1 - \cos(2\pi/\omega)] + \frac{\Delta t S_j}{2\Delta Z}[1 - \cos(2\pi/\omega)] + 1 + \frac{r\Delta t}{2} - \frac{r\Delta t S_j}{2\Delta S}\left[\sqrt{-1}\sin(2\pi/\omega)\right] - \frac{\Delta t S_j}{2\Delta Z}\left[\sqrt{-1}\sin(2\pi/\omega)\right]\right\}$$

$$= \frac{\sigma^2 \Delta t {S_j}^2}{2(\Delta S)^2}[\cos(2\pi/\omega) - 1] + \frac{\Delta t S_j}{2\Delta Z}[\cos(2\pi/\omega) - 1] + 1 - \frac{r\Delta t}{2} + \frac{r\Delta t S_j}{2\Delta S}\left[\sqrt{-1}\sin(2\pi/\omega)\right] + \frac{\Delta t S_j}{2\Delta Z}\left[\sqrt{-1}\sin(2\pi/\omega)\right]$$

using identities

$$e^{2\pi\sqrt{-1}/\omega} = \cos(2\pi/\omega) + \sqrt{-1}\sin(2\pi/\omega)$$

$$\cos(2\pi/\omega) = \frac{e^{2\pi\sqrt{-1}/\omega} + e^{-2\pi\sqrt{-1}/\omega}}{2}$$

$$\sin(2\pi/\omega) = \frac{e^{2\pi\sqrt{-1}/\omega} - e^{-2\pi\sqrt{-1}/\omega}}{2\sqrt{-1}}$$

By the von Neumann stability analysis, the difference equation is say to be stable if and only if $|\epsilon| \leq 1$.

$$|\epsilon| = \frac{\sqrt{\left(\frac{\sigma^2 \Delta t {S_j}^2}{2(\Delta S)^2}[\cos(2\pi/\omega) - 1] + \frac{\Delta t S_j}{2\Delta Z}[\cos(2\pi/\omega) - 1] + 1 - \frac{r\Delta t}{2}\right)^2 + \left(\sin(2\pi/\omega)\left[\frac{r\Delta t S_j}{2\Delta S} + \frac{\Delta t S_j}{2\Delta Z}\right]\right)^2}}{\sqrt{\left(\frac{\sigma^2 \Delta t {S_j}^2}{2(\Delta S)^2}[1 - \cos(2\pi/\omega)] + \frac{\Delta t S_j}{2\Delta Z}[1 - \cos(2\pi/\omega)] + 1 + \frac{r\Delta t}{2}\right)^2 + \left(-\sin(2\pi/\omega)\left[\frac{r\Delta t S_j}{2\Delta S} + \frac{\Delta t S_j}{2\Delta Z}\right]\right)^2}}$$

$$= \frac{\sqrt{\left(1 - \left[\frac{\sigma^2 \Delta t {S_j}^2}{2(\Delta S)^2}[1-\cos(2\pi/\omega)] + \frac{\Delta t S_j}{2\Delta Z}[1-\cos(2\pi/\omega)] + \frac{r\Delta t}{2}\right]\right)^2 + \left(\sin(2\pi/\omega)\left[\frac{r\Delta t S_j}{2\Delta S} + \frac{\Delta t S_j}{2\Delta Z}\right]\right)^2}}{\sqrt{\left(1 + \left[\frac{\sigma^2 \Delta t {S_j}^2}{2(\Delta S)^2}[1-\cos(2\pi/\omega)] + \frac{\Delta t S_j}{2\Delta Z}[1-\cos(2\pi/\omega)] + \frac{r\Delta t}{2}\right]\right)^2 + \left(-\sin(2\pi/\omega)\left[\frac{r\Delta t S_j}{2\Delta S} + \frac{\Delta t S_j}{2\Delta Z}\right]\right)^2}}$$

$$= \frac{\sqrt{\left(1 - \left[[1-\cos(2\pi/\omega)]\left[\frac{\sigma^2 \Delta t {S_j}^2}{2(\Delta S)^2} + \frac{\Delta t S_j}{2\Delta Z}\right] + \frac{r\Delta t}{2}\right]\right) + \left(\sin(2\pi/\omega)\left[\frac{r\Delta t S_j}{2\Delta S} + \frac{\Delta t S_j}{2\Delta Z}\right]\right)^2}}{\sqrt{\left(1 + \left[[1-\cos(2\pi/\omega)]\left[\frac{\sigma^2 \Delta t {S_j}^2}{2(\Delta S)^2} + \frac{\Delta t S_j}{2\Delta Z}\right] + \frac{r\Delta t}{2}\right]\right)^2 + \left(-\sin(2\pi/\omega)\left[\frac{r\Delta t S_j}{2\Delta S} + \frac{\Delta t S_j}{2\Delta Z}\right]\right)^2}}$$

Since

$$[1-\cos(2\pi/\omega)]\left[\frac{\sigma^2 \Delta t {S_j}^2}{2(\Delta S)^2} + \frac{\Delta t S_j}{2\Delta Z}\right] + \frac{r\Delta t}{2} \geq 0,$$

we have

$$|\epsilon| \leq 1.$$

4.8. Simulation and Analysis

Table below compares results obtained by Crank-Nicolson scheme for two-dimensional PDE and by CRR Binomial Tree method (Matlab Build-in function) for pricing the Asian option for a variety of parameters combinations.

Table 4.1: Comparison of Crank-Nicolson finite difference scheme and simulation method for pricing the Asian call option with K = 20 and Tmax = 1, where Tmax is in year.

r	sigma	S0 = K =20			S0 = 35			S0 = 100		
		Crank-Nicolson	CRR Binomial Tree	Relative Error	Crank-Nicolson	CRR Binomial Tree	Relative Error	Crank-Nicolson	CRR Binomial Tree	Relative Error
0.05	0.25	1.3938	1.3702	1.72E-02	15.1148	15.1146	1.50E-05	78.5166	78.5208	5.39E-05
	0.35	1.8253	1.8065	1.04E-02	15.1166	15.1158	4.99E-05	78.5166	78.5208	5.39E-05
	0.4	2.0422	2.0247	8.62E-03	15.1217	15.1200	1.11E-04	78.5166	78.5208	5.39E-05
	0.5	2.4762	2.4603	6.48E-03	15.1556	15.1508	3.16E-04	78.5166	78.5208	5.39E-05
	0.65	3.1241	3.1089	4.90E-03	15.2995	15.2891	6.84E-04	78.5166	78.5208	5.39E-05
	0.8	3.7649	3.7495	4.12E-03	15.5659	15.5503	9.99E-04	78.5166	78.5208	5.36E-05
0.1	0.25	1.6281	1.6061	1.37E-02	15.2102	15.2100	1.33E-05	77.0622	77.0745	1.60E-04
	0.35	2.0329	2.0148	8.99E-03	15.2112	15.2107	3.57E-05	77.0658	77.0745	1.12E-04
	0.4	2.2387	2.2217	7.63E-03	15.2149	15.2137	8.00E-05	77.0658	77.0745	1.12E-04
	0.5	2.6526	2.6370	5.90E-03	15.2413	15.2376	2.46E-04	77.0658	77.0745	1.12E-04
	0.65	3.2737	3.2589	4.55E-03	15.3629	15.3541	5.71E-04	77.0658	77.0745	1.12E-04
	0.8	3.8899	3.8749	3.87E-03	15.5990	15.5855	8.66E-04	77.0659	77.0745	1.12E-04
0.15	0.25	1.8748	1.8548	1.08E-02	15.2873	15.2874	2.36E-06	75.6472	75.6605	1.76E-04
	0.35	2.2469	2.2296	7.76E-03	15.2880	15.2878	1.18E-05	75.6472	75.6605	1.76E-04
	0.4	2.4395	2.4232	6.76E-03	15.2905	15.2899	4.37E-05	75.6472	75.6605	1.76E-04
	0.5	2.8307	2.8155	5.38E-03	15.3109	15.3082	1.77E-04	75.6472	75.6605	1.76E-04
	0.65	3.4227	3.4083	4.23E-03	15.4132	15.4060	4.63E-04	75.6472	75.6605	1.76E-04
	0.8	4.0131	3.9986	3.62E-03	15.6217	15.6102	7.37E-04	75.6472	75.6605	1.76E-04

The following figure shows the option value obtained by two different methods under different stock price:

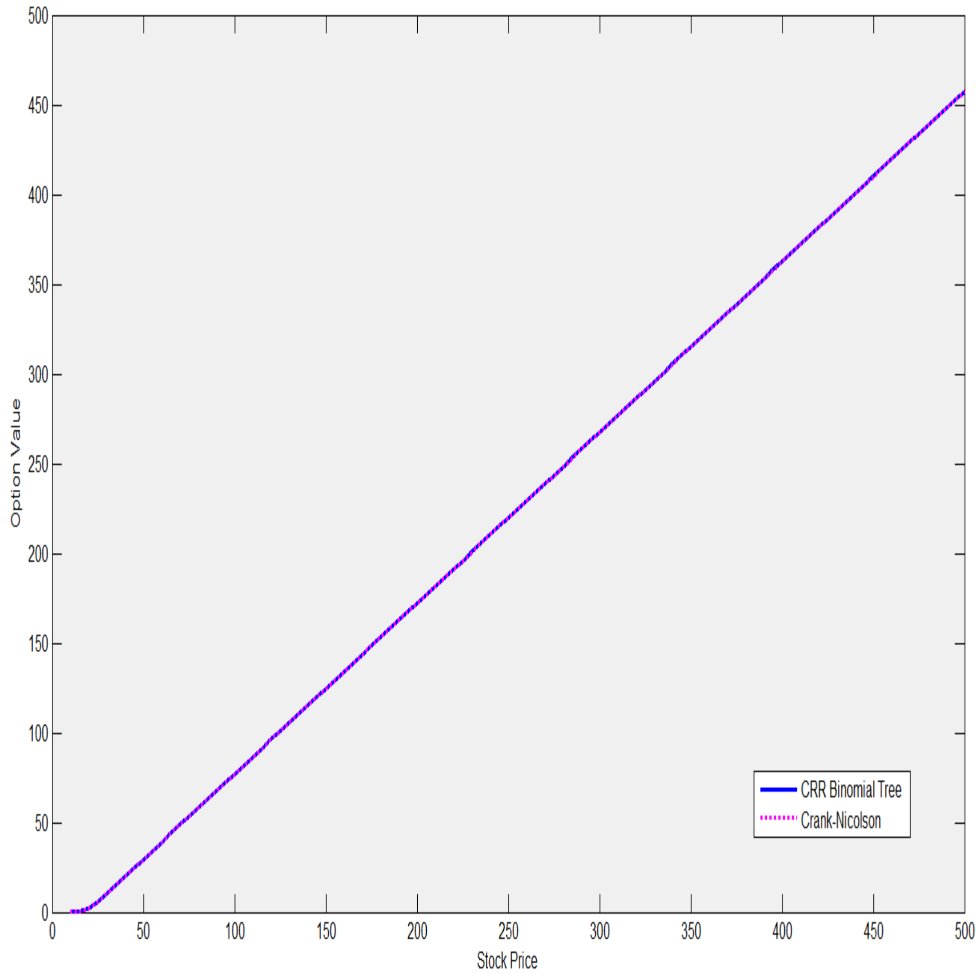

Figure 4.3: Comparison of Crank-Nicolson finite difference scheme and simulation method for pricing the Asian call option under different stock price with $K = 20$, $r = 0.1$, $\sigma = 0.25$, and $T = 1$, where T is in year.

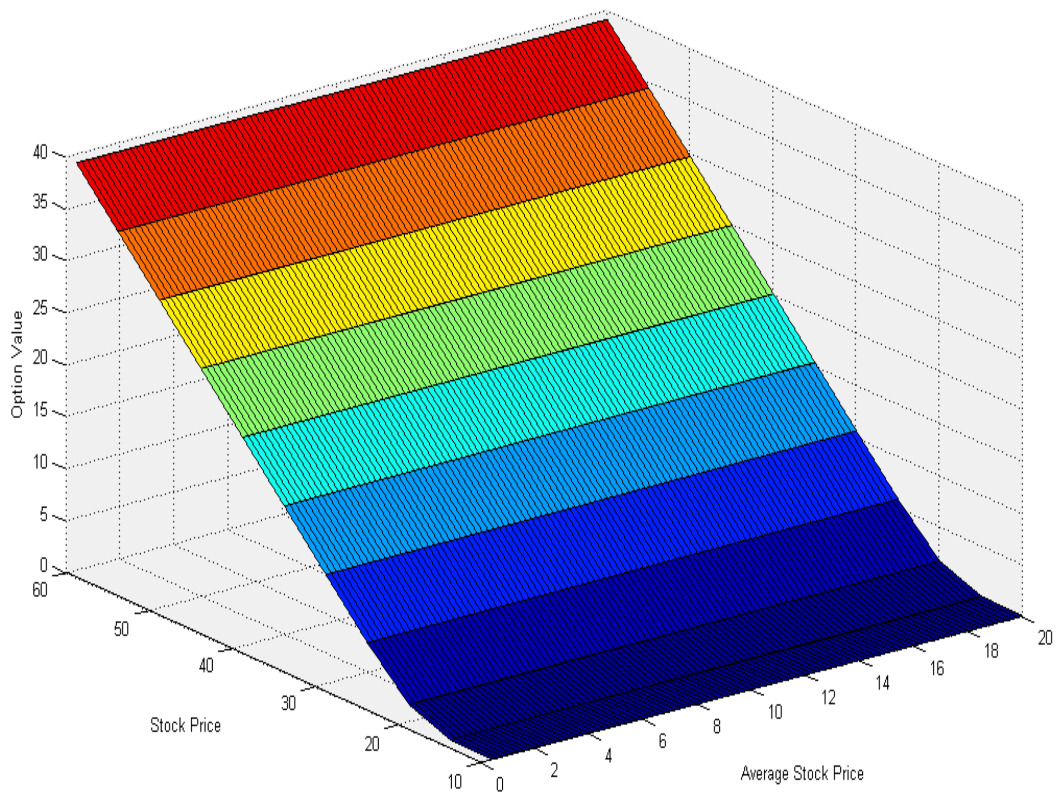

Figure 4.4: A three-dimensional plot of Asian call option with $K = 20$, $r = 0.1$, $\sigma = 0.25$, at time zero.

4.9. Conclusion

From table 4.1, we can see that all the results compute by the Crank-Nicolson scheme is close to the CRR binomial tree method. The method is simple and easy to implement. Moreover, it provides a stable performance at different volatility levels for continuous Asian option.

CHAPTER 5

ASIAN OPTION - A ONE-DIMENSIONAL PDE

5.1. Introduction

As discussed in previous chapter, prices of Asian option can be obtained by solving a two-dimensional PDE using a Crank-Nicolson finite difference scheme. Recently, through a change of *numéraire* argument, Jan Večeř obtained a one -dimensional heat equation whose solution leads to Asian option pricing [13]. This one-dimensional heat equation will be derived here and then solved by a Crank-Nicolson finite difference scheme.

5.2. Change of Numéraire Argument

Assume that

$$dS_t = rS_t dt + \sigma S_t dW_t,$$

where $W_t, 0 \leq t \leq T$, is a Brownian motion under the risk-neutral measure $\mathbb{P}^{(L)}$.

Recall that an Asian call option is an option with payoff

$$F_C(T) = \max\left(\frac{1}{T}\int_0^T S_\tau \, d\tau - K\right)$$

$$= \max\left(\frac{Z_T}{T} - K\right).$$

Let $\gamma(t)$ be a deterministic function of t for $0 \leq t \leq T$. To price this call, we create a portfolio process $X(t)$, consisting of $\gamma(t)$ number of shares of the risky asset and bank borrowing or depositing for $0 \leq t \leq T$.

We select $\gamma(t)$ properly so that

$$X(T) = \frac{1}{T}\int_0^T S_\tau d\tau - K.$$

First, note that

$$e^{r(T-t)}\gamma(t)(dS(t) - rS(t)dt) = d\big(e^{r(T-t)}\gamma(t)S(t)\big) - e^{r(T-t)}S(t)d\gamma(t).$$

At time t, we buy $\gamma(t)$ units of stock and deposit balance $X(t) - \gamma(t)S(t)$ into the bank. Thus

$$dX(t) = \gamma(t)dS(t) + r(X(t) - \gamma(t)S(t)dt), \text{ or}$$

$$dX(t) - rX(t)dt = \gamma(t)(dS(t) - rS(t)dt).$$

Now

$$\begin{aligned} d\left(e^{r(T-t)}X(t)\right) &= e^{r(T-t)}(dX(t) - rX(t)dt) \\ &= e^{r(T-t)}\gamma(t)(dS(t) - rS(t)dt) \\ &= d\big(e^{r(T-t)}\gamma(t)S(t)\big) - e^{r(T-t)}S(t)d\gamma(t). \end{aligned}$$

Integrating yields

$$\begin{aligned} &e^{r(T-t)}X(t) \\ &= e^{rT}X(0) + \int_0^t d\left(e^{r(T-u)}\gamma(u)S(u)\right) - \int_0^t e^{r(T-u)}S(u)d\gamma(u) \\ &= e^{rT}X(0) - e^{rT}\gamma(0)S(0) + e^{r(T-t)}\gamma(t)S(t) - \int_0^t e^{r(T-u)}S(u)d\gamma(u) \end{aligned}$$

which reduce to

$$-K + e^{r(T-t)}\gamma(t)S(t) + \frac{1}{T}\int_0^t S(u)du,$$

if we select

$$X(0) = \frac{1}{rT}(1 - e^{-rT})S(0) - e^{-rT}K$$

$$\gamma(t) = \frac{1}{rT}\left(1 - e^{-r(T-t)}\right) \quad \text{for} \quad 0 \le t \le T.$$

Therefore,

$$X(t) = \frac{1}{rT}\left(1 - e^{-r(T-t)}\right)S(t) - e^{-r(T-t)}K + e^{-r(T-t)}\frac{1}{T}\int_0^t S(u)du \ ,$$

for

$$0 \le t \le T.$$

In particular,

$$X(T) = \frac{1}{T}\int_0^T S(u)du - K \ , \ \ 0 \le t \le T.$$

In terms of $X(T)$, the payoff is

$$F(T) = X^+(T) = \max\{X(T), 0\},$$

and at time $t \le T$, the price of Asian call option is

$$\begin{aligned} F(t) &= E^{\mathbb{P}^{(L)}}\left[e^{-r(T-t)}F(T)\middle|\mathcal{F}_t\right] \\ &= E^{\mathbb{P}^{(L)}}\left[e^{-r(T-t)}X^+(T)\middle|\mathcal{F}_t\right]. \end{aligned}$$

To evaluate this conditional expectation, let

$$Y(t) = \frac{X(t)}{S(t)} = \frac{e^{-rt}X(t)}{e^{-rt}S(t)}$$

be the portfolio value in terms of the number of the stocks. This is a change of *numéraire*. We have changed the unit of account from dollars to assets.

We wish to compute $dY(t)$. Note that:

$$
\begin{aligned}
d\big(e^{-rt}S(t)\big) &= -re^{-rt}S(t)dt + e^{-rt}dS(t) \\
&= -re^{-rt}S(t) + e^{-rt}[rS(t)dt + \sigma S(t)dW(t)] \\
&= \sigma e^{-rt}S(t)dW(t).
\end{aligned}
$$

$$
\begin{aligned}
d([e^{-rt}S(t)]^{-1})) &= -(e^{-rt}S(t))^{-2}d\big(e^{-rt}S(t)\big) \\
&\qquad +(e^{-rt}S(t))^{-3}d\big(e^{-rt}S(t)\big)d(e^{-rt}S(t) \\
&= -(e^{-rt}S(t))^{-2}\sigma(e^{-rt}S(t))dW(t) \\
&\qquad +(e^{-rt}S(t))^{-3}\big(e^{-rt}S(t)\big)^2\sigma^2 dt \\
&= -\sigma(e^{-rt}S(t))^{-1}dW(t) + \sigma^2(e^{-rt}S(t))^{-1}dt\,.
\end{aligned}
$$

$$
\begin{aligned}
d\big(e^{-rt}X(t)\big) &= e^{-rt}(dX(t) - rX(t)dt) \\
&= \gamma(t)e^{-rt}\big(dS(t) - rS(t)\big)dt \\
&= \gamma(t)\sigma e^{-rt}S(t)dW(t).
\end{aligned}
$$

By *Itô's* formula,

$$
\begin{aligned}
dY(t) &= d\big[\big(e^{-rt}X(t)\big)(e^{-rt}S(t))^{-1}\big] \\
&= e^{-rt}X(t)d[(e^{-rt}S(t))^{-1}] + (e^{-rt}S(t))^{-1}\, d[e^{-rt}X(t)] \\
&\qquad + d[(e^{-rt}X(t))]d[(e^{-rt}S(t))^{-1}] \\
&= e^{-rt}X(t)[-\sigma(e^{-rt}S(t))^{-1}dW(t) + \sigma^2(e^{-rt}S(t))^{-1}dt] \\
&\qquad +(e^{-rt}S(t))^{-1}[\sigma\gamma(t)(e^{-rt}S(t)dW(t)] - \sigma^2\gamma(t)dt \\
&= -\sigma Y(t)dW(t) + \sigma^2 Y(t)dt + \sigma\,\gamma(t)dW(t) - \sigma^2\gamma(t)dt \\
&= \sigma[\gamma(t) - Y(t)]dW(t) + \sigma^2[Y(t) - \gamma(t)]dt \\
&= \sigma[\gamma(t) - Y(t)][dW(t) - \sigma dt] \\
&= \sigma[\gamma(t) - Y(t)]d\widetilde{W}(t) \qquad (26)
\end{aligned}
$$

where $\widetilde{W}(t) = W(t) - \sigma t$. By Girsanov's theorem, $\widetilde{W}(t)$ is a Brownian-motion under probability measure $\widetilde{\mathbb{P}}^{(L)}$ defined by

$$\widetilde{\mathbb{P}}^{(L)}(A) = \int_A Z(T) d\mathbb{P}^{(L)},$$

where $Z(t) = \exp\left\{\sigma W(t) - \frac{\sigma^2 t}{2}\right\}$ and $Y(t)$ is a $\widetilde{\mathbb{P}}^{(L)}$-martingale. Being a solution to equation (26), $Y(t)$ is also $\widetilde{\mathbb{P}}^{(L)}$-Markov. As

$$\begin{aligned} S_t &= S_0 \exp\left(\left(r - \frac{\sigma^2}{2}\right) t + \sigma W(t)\right) \\ &= S_0 \exp(rt) Z(t), \end{aligned}$$

where

$$Z(t) = \frac{e^{-rt} S(t)}{S(0)}.$$

Therefore,

$$\begin{aligned} F(t) &= \mathrm{E}^{\mathbb{P}^{(L)}}\left[e^{-r(T-t)} F(T) \middle| \mathcal{F}(t)\right] \\ &= e^{rt} \mathrm{E}^{\mathbb{P}^{(L)}}[e^{-rT} X^+(T) | \mathcal{F}(t)] \\ &= e^{rt} S(0) \mathrm{E}^{\mathbb{P}^{(L)}}\left[e^{-rT} S(T)/S(0) \left(\frac{e^{-rT} X(T)}{e^{-rT} S(T)}\right)^+ \middle| \mathcal{F}(t)\right] \\ &= e^{rt} Z(t) S(0) \mathrm{E}^{\mathbb{P}^{(L)}}[Z(T) Y^+(T)/Z(t) | \mathcal{F}(t)] \\ &= S(t) \mathrm{E}^{\widetilde{\mathbb{P}}^{(L)}}[Y^+(T) | \mathcal{F}(t)] \end{aligned} \quad (27)$$

Because $Y(t)$ is Markov under $\widetilde{\mathbb{P}}^{(L)}$, there must be a function $g(t, y)$ such that

$$g\big(t, Y(t)\big) = E^{\widetilde{\mathbb{P}}^{(L)}}[Y^+(T) | \mathcal{F}(t)] \quad (28)$$

Then at terminal time *T*, we have

$$g\big(T, Y(T)\big) = E^{\widetilde{\mathbb{P}}^{(L)}}[Y^+(T) | \mathcal{F}(T)] = Y^+(T).$$

5.3. Boundary Values

Recall that $Y(t) = \frac{X(t)}{S(t)}$ represents the portfolio value in term of the number of stocks held. As the value for $X(t)$ is positive or negative while $S(t)$ is always positive, $Y(t)$ is either positive or negative. When $Y(t)$ is very negative, the probability that $Y(T)$ is negative or $Y^+(T) = 0$ is near one. This leads to the condition

$$\lim_{y \to -\infty} g(t, y) = 0 \ , \quad 0 \le t \le T.$$

On the other hand, when $Y(t)$ is positive and large, the probability that $Y(T) > 0$ is near one. Therefore, for large $Y(t)$

$$\begin{aligned} g\big(t, Y(t)\big) &= E^{\widetilde{\mathbb{P}}^{(L)}}[Y^+(T)|\mathcal{F}(t)] \\ &= E^{\widetilde{\mathbb{P}}^{(L)}}[Y(T)|\mathcal{F}(t)] \\ &= Y(t). \end{aligned}$$

This gives raise to the boundary condition

$$\lim_{y \to \infty}[g(t, y) - y] = 0 \ , \quad 0 \le t \le T.$$

At the terminal time T, we also have $g(T, y) = y^+$ as the top boundary condition.

Note that the domain for y is unbounded. In numerical calculation, we have to compute in a finite domain. So, we need to truncate the unbounded domain into a bounded domain by setting the maximum value for y.

5.4. Partial Differential Equation for Asian option

In this section, we will derive the one-dimensional heat equation for $g(t, y)$ by obtaining its differential:

$$
\begin{aligned}
dg\big(t, Y(t)\big) &= g_t\big(t, Y(t)\big)dt + g_y\big(t, Y(t)\big)dY(t) \\
&\qquad + \frac{1}{2} g_{yy}\big(t, Y(t)\big)dY(t)dY(t) \\
&= g_t\big(t, Y(t)\big)dt + g_y\big(t, Y(t)\big)\big[\sigma\big(\gamma(t) - Y(t)\big)d\widetilde{W}(t)\big] \\
&\qquad + \frac{1}{2} g_{yy}\big(t, Y(t)\big)\left[\sigma^2(\gamma(t) - Y(t))^2\big(d\widetilde{W}(t)\big)^2\right] \\
&= \left[g_t\big(t, Y(t)\big) + \frac{1}{2}\sigma^2\big(\gamma(t) - Y(t)\big)^2 g_{yy}\big(t, Y(t)\big)\right]dt \\
&\qquad + \sigma\big(\gamma(t) - Y(t)\big)\ g_y\big(t, Y(t)\big)d\widetilde{W}(t)
\end{aligned}
$$

The process $g\big(t, Y(t)\big) = Y^+(t)$ is a martingale under $\widetilde{\mathbb{P}}^{(L)}$, because iterating (28) for $s < t$ yield

$$
\begin{aligned}
E^{\widetilde{\mathbb{P}}^{(L)}}\big[g\big(t, Y(t)\big)\big|\mathcal{F}(s)\big] &= E^{\widetilde{\mathbb{P}}^{(L)}}\left[E^{\widetilde{\mathbb{P}}^{(L)}}[Y^+(T)|\mathcal{F}(t)]\middle|\,\mathcal{F}(s)\right] \\
&= E^{\widetilde{\mathbb{P}}^{(L)}}[Y^+(T)|\mathcal{F}(s)] \\
&= g\big(s, Y(s)\big).
\end{aligned}
$$

The drift term must be zero and we conclude that the function $g(t, y)$ satisfies the PDE

$$
g_t(t, y) + \frac{1}{2}\sigma^2(\gamma(t) - y)^2 g_{yy}(t, y) = 0 \quad , \quad 0 \le t \le T \qquad (29)
$$

5.5. Discretization

Now, consider the function $g(t, y)$ over a two-dimensional grid. As usual, let j and h denote the indices for the y variable and time t respectively. Let N and H be the number of partitions for y and t respectively. Define

$$\Delta y = \frac{y_{max}}{N}, \quad \Delta t = \frac{T}{H}$$

and let

$$y_j = j\Delta y, \; t_h = h\Delta t$$

for

$$0 \leq j \leq N, 0 \leq h \leq H$$

where y_{max} is the maximum value of y for the computation domain.

At point (t, y), the expression $\frac{1}{2}\sigma^2(\gamma(t) - y)^2 g_{yy}(t, y)$ is approximated by the following difference scheme

$$L_j^h = \frac{\sigma^2\left(\gamma(t) - y_j\right)^2}{2} D_{yy}$$

where

$$D_{yy} = \frac{g\left(y_{j+1}, t_h\right) - 2g\left(y_j, t_h\right) + g\left(y_{j-1}, t_h\right)}{(\Delta y)^2}.$$

Thus, we obtain the Crank-Nicolson finite difference scheme:

$$\frac{g_j^{h+1} - g_j^h}{\Delta t} + \frac{1}{2}\left(L_j^h + L_j^{h+1}\right) = 0$$

$$g_j^h - \frac{\Delta t}{2} L_j^h = g_j^{h+1} + \frac{\Delta t}{2} L_j^{h+1}$$

where

$$L_j^h = \frac{\sigma^2\left(\gamma(t) - y_j\right)^2}{2(\Delta y)^2}\left[g_{j+1}^h - 2g_j^h + g_{j-1}^h\right]$$

and

$$L_j^{h+1} = \frac{\sigma^2\big(\gamma(t+\Delta t) - y_j\big)^2}{2(\Delta y)^2}\left[g_{j+1}^{h+1} - 2g_j^{h+1} + g_{j-1}^{h+1}\right].$$

The finite difference scheme can be written as:

$$g_j^h - \frac{\Delta t \sigma^2\big(\gamma(t) - y_j\big)^2}{4(\Delta y)^2}\left[g_{j+1}^h - 2g_j^h + g_{j-1}^h\right]$$

$$= g_j^{h+1} + \frac{\Delta t \sigma^2\big(\gamma(t+\Delta t) - y_j\big)^2}{4(\Delta y)^2}\left[g_{j+1}^{h+1} - 2g_j^{h+1} + g_{j-1}^{h+1}\right],$$

or

$$-\alpha_j^h g_{j-1}^h + \big(1 + 2\alpha_j^h\big)g_j^h - \alpha_j^h g_{j+1}^h$$

$$= \alpha_j^{h+1} g_{j-1}^{h+1} + \big(1 - 2\alpha_j^{h+1}\big)g_j^{h+1} + \alpha_j^{h+1} g_{j+1}^{h+1}$$

After obtaining $g(t,y)$, Asian call option value at time t of the continuously averaged with payoff at time T is

$$V(t) = S(t)g\left(t, \frac{X(t)}{S(t)}\right).$$

5.6. Simulation and Analysis

The following tables present the result of Asian option values. The first table compares the result using suggested method and Matlab build-in CRR binomial tree method under different set of parameters, while the second table shows the option values obtained by solving different dimensional of PDEs using Crank-Nicolson method.

Table 5.1: Comparison of Crank-Nicolson finite difference scheme and CRR Binomial Tree for pricing the Asian call option with K = 20 and Tmax = 1, where Tmax is in year.

r	sigma	S0 = K = 20			S0 = 45			S0 = 100		
		Crank-Nicolson	CRR Binomial Tree	Relative Error	Crank-Nicolson	CRR Binomial Tree	Relative Error	Crank-Nicolson	CRR Binomial Tree	Relative Error
0.05	0.25	1.3684	1.3702	1.27E-03	24.8625	24.8694	2.77E-04	78.5000	78.5208	2.65E-04
	0.35	1.8064	1.8065	6.12E-05	24.8625	24.8694	2.77E-04	78.5000	78.5208	2.65E-04
	0.4	2.0254	2.0247	3.35E-04	24.8626	24.8694	2.76E-04	78.5000	78.5208	2.65E-04
	0.5	2.4625	2.4603	8.91E-04	24.8641	24.8706	2.62E-04	78.5000	78.5208	2.65E-04
	0.65	3.1125	3.1089	1.15E-03	24.8859	24.8901	1.68E-04	78.5000	78.5208	2.65E-04
	0.8	3.7245	3.7495	6.65E-03	24.9671	24.9669	8.31E-06	78.5000	78.5208	2.65E-04
0.1	0.25	1.6098	1.6061	2.28E-03	24.7275	24.7276	3.27E-06	77.0500	77.0745	3.18E-04
	0.35	2.0195	2.0148	2.30E-03	24.7275	24.7276	3.23E-06	77.0500	77.0745	3.18E-04
	0.4	2.2269	2.2217	2.34E-03	24.7275	24.7276	2.69E-06	77.0500	77.0745	3.18E-04
	0.5	2.6433	2.6370	2.39E-03	24.7286	24.7284	7.50E-06	77.0500	77.0745	3.18E-04
	0.65	3.2663	3.2589	2.30E-03	24.7460	24.7440	8.15E-05	77.0500	77.0745	3.18E-04
	0.8	3.8640	3.8749	2.82E-03	24.8153	24.8096	2.28E-04	77.0500	77.0745	3.18E-04
0.15	0.25	1.8582	1.8548	1.82E-03	24.5700	24.5755	2.25E-04	75.6500	75.6605	1.39E-04
	0.35	2.2337	2.2296	1.84E-03	24.5700	24.5755	2.25E-04	75.6500	75.6605	1.39E-04
	0.4	2.4277	2.4232	1.88E-03	24.5700	24.5755	2.25E-04	75.6500	75.6605	1.39E-04
	0.5	2.8211	2.8155	1.96E-03	24.5708	24.5761	2.17E-04	75.6500	75.6605	1.39E-04
	0.65	3.4150	3.4083	1.97E-03	24.5847	24.5885	1.55E-04	75.6500	75.6605	1.39E-04
	0.8	3.9936	3.9986	1.25E-03	24.6438	24.6443	2.24E-05	75.7000	75.6605	5.21E-04

Table 5.2: Comparison of Asian call option value by solving one-dimensional and two-dimensional partial differential equation(PDE) using Crank-Nicolson Scheme with K = 20 and Tmax = 1, where Tmax is in year.

r	sigma	S0 = K = 20		S0 = 45		S0 = 100	
		One - Dimensional PDE	Two - Dimensional PDE	One - Dimensional PDE	Two - Dimensional PDE	One - Dimensional PDE	Two - Dimensional PDE
0.05	0.25	1.3684	1.3938	24.8625	24.8689	78.5000	78.5166
	0.35	1.8064	1.8253	24.8625	24.8689	78.5000	78.5166
	0.4	2.0254	2.0422	24.8626	24.8690	78.5000	78.5166
	0.5	2.4625	2.4762	24.8641	24.8707	78.5000	78.5166
	0.65	3.1125	3.1241	24.8859	24.8933	78.5000	78.5166
	0.8	3.7245	3.7649	24.9671	24.9757	78.5000	78.5166
0.1	0.25	1.6098	1.6281	24.7275	24.7264	77.0500	77.0622
	0.35	2.0195	2.0329	24.7275	24.7264	77.0500	77.0658
	0.4	2.2269	2.2387	24.7275	24.7265	77.0500	77.0658
	0.5	2.6433	2.6526	24.7286	24.7277	77.0500	77.0658
	0.65	3.2663	3.2737	24.7460	24.7458	77.0500	77.0658
	0.8	3.8640	3.8899	24.8153	24.8162	77.0500	77.0659
0.15	0.25	1.8582	1.8748	24.5700	24.5735	75.6500	75.6472
	0.35	2.2337	2.2469	24.5700	24.5735	75.6500	75.6472
	0.4	2.4277	2.4395	24.5700	24.5735	75.6500	75.6472
	0.5	2.8211	2.8307	24.5708	24.5743	75.6500	75.6472
	0.65	3.4150	3.4227	24.5847	24.5888	75.6500	75.6472
	0.8	3.9936	4.0131	24.6438	24.6489	75.7000	75.6472

5.7. Conclusion

The Crank-Nicolson scheme used here has a very simple form and the results obtained are close to the CRR binomial tree method.

CHAPTER 6

CONCLUSION

In this thesis, we apply Crank-Nicolson finite difference scheme to find option value. An abundance literature of numerical option pricing is available in various places, but one with systematic approach is rare to find. In chapter 4, we obtain the value of Asian option by solving an initial value problem of a two-dimensional Black-Scholes equation using a simple Crank-Nicolson finite difference scheme. In chapter 5, we solve the same problem again by reducing it to the solution of a one-dimensional equation applying a *Change of Numéraire Argument* due to Jan Večeř [13]. In these two chapters, we develop a complete and systemic treatment for the solution.

Since we are solving an initial value problem in an unbounded domain, for numerical computation, we have to truncate the unbounded domain into a bounded domain and provide suitable boundary conditions through financial or probabilistic consideration. Currently, we only have asymptotic boundary conditions for large value of stock price. We have difficulty to determine a stock price which is large enough that the boundary conditions are satisfied with high accuracy. Thus our work here is partially based on trial and error. Codes in Matlab are written to test our difference schemes. They are workable and relatively accurate as compared to other methods [Chapter 4, pg 57]. However theoretical works of the effect of boundary conditions on the solution should be studied in the future. Perhaps, we also can try to impose an artificial boundary condition suggested by Han and Wu and Wong and Zhao [22].

As Crank-Nicolson Scheme for the Black-Scholes equation involves a lot of computations, stability analysis were carried out to ensure our result is stable in section 4.7.

The problem of Asian options pricing is closely related to the integral of geometric Brownian motion (called IGBM in the sequel). Indeed, it is essentially a problem about exponential functionals of Brownian motion. In several papers, Marc Yor [23,24] applied the properties of Bessel processes to study the integral of geometric Brownian motion and obtained some of the most important results about pricing of Asian options, in particular the Geman-Yor formula [23,24] for the Laplace transform of Asian option prices and the four known expressions for the probability density function (PDF in the sequel) of IGBM. It will be interesting to know if an exponential functional of Brownian motion will satisfy a simple heat equation through a *Change of Numéraire Argument* as presented here, for then our simple Crank-Nicolson Scheme here is able to solve the highly complicated problem of exponential functional of Brownian motion.

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APPENDIX A

CODE FOR PRICING EUROPEAN OPTION

```
function    [StkPrice    Call    SpPrice    RelErr]    =
EuropeanOption(K, r, sigma)


% Terminal time. Time unit in year
T = 1;

% Maximum stock price
Smax = 800;


% Compute the number of steps in stock price
% number of time step
H = 500;
% increment of time step
dt = 1/H;
% increment of stock price for each step
dS = sqrt(50*dt);
% number of step in Stock price
M = round(Smax/dS);


% The axis of stock price
s = dS : dS : Smax;


% Set the last interior stk price
M1 = M - 1;


% p(i,j) = Option Price at time i*dt(i=1..H); stk
price j*dS(j=1..M-1)
% left bdy value automatically taken care of during
initialization
% right bdy value at Smax = Smax - K*exp(-r(T-t));
% initialize p
p = zeros(H+1,M1);
% p(H+1,j) is the payoff at terminal time T when stk
price is j*dS
% P(1,j) is the payoff at time = 0
```

```
% Top Boundary
for j= 1 : M1
    p(H+1,j) = max(s(j)-K,0);
end


% Define coefficients of difference equation:
% a(j)p(i-1,j-1) + b(j)p(i-1,j) + c(j)p(i-1,j+1) =
a1(j)p(i,j-1)+b1(j)p(i,j)+c1(j)p(i,j+1);
% Here i for time, j for stk price at time i
% b(j) at diagonal,a(j) to the left and c(j) to the
right


for j=1:M1
    a(j)=dt/4*(r*j-sigma^2*j^2);
    b(j)=1+dt*(r/2+1/2*sigma^2*j^2);
    c(j)=-dt/4*(r*j+sigma^2*j^2);
end


for j=1:M1
    a1(j) = -a(j);
    b1(j) = 1 - dt/2*(sigma^2*j^2 + r);
    c1(j) = -c(j);
end


% Construct coefficients matrices A and B
% where Ap(i-1,:) + down bdyValue(i-1) = Bp(i,:) +
up bdyValue(i)
A = zeros(M1,M1);
B = zeros(M1,M1);


% Put b(j) at diagonal, a(j) to left and c(j) to
right
for i = 1 : M1
j = i;
if i>1 A(i,j-1) = a(j); end
    A(i,j) = b(j);
if j<M1 A(i,j+1) = c(j); end;
end


for i = 1 : M1
j = i;
if i>1 B(i,j-1) = a1(j); end
    B(i,j) = b1(j);
if j<M1 B(i,j+1) = c1(j); end;
end
```

```
% vd,vu bdy Value vector of option at smax,time down
and up respectively
vd=zeros(M1,1);
vu=zeros(M1,1);


% Find Option Price
for i = H : -1 : 1
    timeu = (H+1-i)*dt;
    timed = (H+1-i-1)*dt;
    vd(M1) = c(M1)*(Smax -K*exp(-r*timed));
    vu(M1) = c1(M1)*(Smax-K*exp(-r*timeu));
    p(i,:) = inv(A)* (vu-vd+B*p(i+1,:)');
end


% Check
for i = 1 : M1
    idS(i) = i;
end;


format short g
StkPrice = 10 : 5 : Smax;
SpPrice = spline(idS, p(1,:), StkPrice/dS);
[Call, Put] = blsprice(StkPrice, K, r, T, sigma);
RelErr = abs(Call - SpPrice)./Call;
['StkPrice   ' 'Call     '  'SpPrice    ' 'RelErr']
[StkPrice' Call' SpPrice' RelErr'];

figure,plot(StkPrice',Call,'b',StkPrice',SpPrice,'r'
),xlabel('Stock Price'),ylabel('Option Value')
```

APPENDIX B

CODE FOR PRICING ASIAN OPTION BY SOLVING A TWO-DIMENSIONAL PDE

```
function [StkPrice CallPrice SpPrice RelErr] =
AsianOption(K, r, sigma)

    Smax = 500; % maximum stock price
    Tmax = 1;  % terminal time in year
    Zmax = K*Tmax; % maximum accumulation of stock
price

    M = 1000; % number of partition for average
stock price
    N = 1000; % number of partition for stock price
    T = 100;  % number of partition for time

    dS = Smax/N; % increment of stock price for each
step
    dZ = Zmax/M; % increment of average stock price
for each step
    dt = Tmax/T; % increment of time for each step

    ratio = dt/dZ;
    ratio2 = dt/(dS)^2;
    ratio3 = dZ/(dS)^2 ;


    s = dS : dS : Smax; % The axis of stock price
    z = dZ : dZ : Zmax; % The axis of average stock
price
    tLine = 0 : dt : Tmax;
    % p(i,j,k) for Option Price  at time kdt, stk
price jdS, avg stk price idz

    % Top Boundary
    for i = 1 : M
        for j = 1 : N-1
            p(i,j,T+1)=max((i-1)*dZ/Tmax - K,0);
            % Payoff of call Option
        end
    end
```

```
for j = 1 : N-1
    a(j)=-dt/4*((sigma^2*(j)^2)-r*(j));
    b(j)=1 + (dt/2)*(sigma^2*(j)^2+(j)*dS/dZ+r);
    c(j)=-dt/4*(r*(j)+sigma^2*(j)^2);
end

for j = 1 : N-1
    a1(j)=-dt/4*((-sigma^2*(j)^2)+r*(j));
    b1(j)=1-dt/2*(sigma^2*(j)^2+(j)*dS/dZ+r);
    c1(j)=dt/4*(sigma^2*(j)^2+r*(j));
end


A=zeros(N-1,N-1);B=zeros(N-1,N-1);


% Put b(j) at diagonal of A; a(j) to left and
c(j) to right

    for i = 1:N-1
        j=i;
        if i>1 A(i,j-1)=a(j); end
        A(i,j)=b(j);
        if j<N-1 A(i,j+1)=c(j); end;
    end

for i = 1:N-1
    j=i;
    if i>1 B(i,j-1)=a1(j); end
    B(i,j)=b1(j);
    if j<N-1 B(i,j+1)=c1(j); end;
end



vdL = zeros(N-1,1);
vuL = zeros(N-1,1);

vdR = zeros(N-1,1);
vuR = zeros(N-1,1);

vu = zeros(N-1,1);
vd = zeros(N-1,1);

vu1 = zeros(N-1,1);
vd1 = zeros(N-1,1);
```

```
format short
for k = T : -1 : 1

    timeu = (T+1-k-1)*dt;
    timed = (T+1-k)*dt;

    for j = N-1 : -1 : 1
        vd(j) = -(dt*j*dS/(2*dZ))*((exp(-
r*timed))*(Zmax/Tmax - K)+ ((j*dS)/(r*Tmax))*(1-
exp(-r*timed)));
        vu(j) = (dt*j*dS/(2*dZ))*((exp(-
r*timeu))*(Zmax/Tmax - K)+ ((j*dS)/(r*Tmax))*(1-
exp(-r*timeu)));

    end

   for i = M-1

    vdR(N-1)= c(N-1) * max((exp(-
r*timed))*(i*dZ/Tmax - K)+(Smax/r*Tmax)*(1-exp(-
r*timed)),0);
    vuR(N-1)= c1(N-1) * max((exp(-
r*timeu))*(i*dZ/Tmax - K)+(Smax/r*Tmax)*(1-exp(-
r*timeu)),0);

    vdL(N-1)= a(1)*((exp(-
r*timed))*max(i*dZ/Tmax - K,0));
    vuL(N-1)= a1(1)*((exp(-
r*timeu))*max(i*dZ/Tmax - K,0));

    p(M,:,k)=inv(A)*(vuL-vdL + vuR-vdR + vu-vd +
B*p(M,:,k+1)');

   end

   for i = M-2 : -1 : 0

    vdR(N-1)= c(N-1) * max((exp(-
r*timed))*(i*dZ/Tmax - K)+(Smax/r*Tmax)*(1-exp(-
r*timed)),0); % OK
    vuR(N-1)= c1(N-1) * max((exp(-
r*timeu))*(i*dZ/Tmax - K)+(Smax/r*Tmax)*(1-exp(-
r*timeu)),0);% OK

    vdL(N-1)= a(1)*((exp(-
r*timed))*max(i*dZ/Tmax - K,0)); % OK
    vuL(N-1)= a1(1)*((exp(-
r*timeu))*max(i*dZ/Tmax - K,0)); % OK

     for j = N-1 : -1 : 1
      vu1(j) = (dt*j*dS/(2*dZ))*p(i+2,j,k+1)';
```

```
            vd1(j) = -(dt*j*dS/(2*dZ))*p(i+2,j,k)';
        end

         p(i+1,:,k)=inv(A)*(vuL-vdL + vuR-vdR + vu1-
vd1 + B*p(i+1,:,k+1)');

         end
    end


     CN=p(1,:,1);


% Check
for i=1:N-1
    idS(i)=i;
end;

format long g

StkPrice = 10 : 5 : Smax;
size = length(StkPrice);

SpPrice = spline(idS,CN,StkPrice/dS);


for i = 1 : size
    StockSpec(i) = stockspec(sigma, StkPrice(i));

    RateSpec = intenvset('Rates', r,
'StartDates','1-Jan-2004', 'EndDates', '31-Dec-
2004','Compounding',-1);

    ValuationDate = '1-Jan-2004';
    Maturity = '31-Dec-2004';
    TimeSpec = crrtimespec(ValuationDate, Maturity,
100);

    CRRTree(i) = crrtree(StockSpec(i), RateSpec,
TimeSpec);


    OptSpec = 'call';
    Strike = K;
    Settle = '01-Jan-2004';
    ExerciseDates = '31-Dec-2004';

    CallPrice(i) = asianbycrr(CRRTree(i), OptSpec,
Strike, Settle, ExerciseDates);
end
```

```
RelErr = abs(CallPrice-SpPrice)./CallPrice;
['StkPrice   ' 'Simulation     '  'CN    ' 'RelErr']
[StkPrice' CallPrice' SpPrice' RelErr'];

figure,plot(StkPrice,CallPrice,'b',StkPrice,SpPrice,
'r'),xlabel('Stock Price'),ylabel('Option Value')
```

APPENDIX C

CODE FOR PRICING ASIAN OPTION BY SOLVING A ONE – DIMENSIONAL PDE

Code for pricing Asian Option by Solving a One-Dimensional PDE

```
function [y, g, S0] = AsianTransform(S0, K, r, sigma)


Tmax = 1; % Set the terminal time. Time unit in year
Ymax = 1; % set the Max Y value
Ymin = -Ymax;

X0 = (S0/(r*Tmax))*(1-exp(-r*Tmax)) - K*exp(-r*Tmax);
Y0 = X0/S0;

H = 2000; % number of time step
Y = 2000; % number of Y step


dt = Tmax/H; % increment of time step
dY = Ymax/Y; % increment of Y step


t = dt : dt : Tmax;
y = Ymin : dY : Ymax;
N = size(y,2);


% Number of holding shares
for h = 1 : H
        Q(h) = ((1/r*Tmax)*(1-exp(-r*(Tmax - (h-
1)*dt))));
end


% Calculate the coefficient
for h = 1 : H
    for j = 1:N
        alpha(h,j) = (dt*(sigma^2)*(Q(h)-
(y(j)))^2)/(4*((dY)^2));
    end
end
```

```
% Initialize g
g=zeros(H,N);

%Terminal condition % Top Boundary
for j= 1 : N
    g(H,j)=max(y(j),0);
end

%Left Boundary
for h = 1 : H
    g(h,1)= 0;
end

%Right Boundary
for h = 1 : H
    g(h,N)= max(y(j),0);
end

% Set initial Value
for h = H-1 : -1 : 1
    for j = 3 : N-1
        g(h,j)=0;
    end
end

% Find the g value
for h = H-1 : -1 : 1
    for i = 1 : 4000
        for j = 1 : N-2
            g(h,j+1)= (alpha(h+1,j+1)*g(h+1,j) + (1-
2*alpha(h+1,j+1))*g(h+1,j+1) +
alpha(h+1,j+1)*g(h+1,j+2)+ alpha(h,j+1)*g(h,j) +
alpha(h,j+1)*g(h,j+2))/(1+2*alpha(h,j+1));
        end
    end
end
```