GENERALISATION OF n-CENTRALISER RINGS AND THEIR GRAPHS

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By

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ABSTRACT

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Let $\text{Cent}(R)$ denote the set of all distinct centralisers in a ring R. A ring R is said to be an n-centraliser ring if $|\text{Cent}(R)| = n$, where $n \in \mathbb{N}$. The question of how the number of distinct centralisers in a ring can influence its structure and commutativity has recently captured the attention of several researchers. Therefore, the study of the n-centraliser rings is a prospective research topic in ring theory. In this dissertation, we first investigate the characterisation for all n-centraliser finite rings for $n \in \{6, 7, 8, 9, 10, 11\}$ and compute their commuting probabilities. Subsequently, we classify the structures for all finite rings in which the cardinality of the maximal non-commuting set is 5.

To extend the study of n-centraliser rings, we introduce the notion of (m, n)-centraliser rings, which is a generalisation of n-centraliser rings. For any m distinct elements $r_1, r_2, \cdots, r_m$ in a ring R, the m-element centraliser of $\{r_1, r_2, \cdots, r_m\}$ in R, denoted by $C_R(\{r_1, r_2, \cdots, r_m\})$, is defined as $C_R(\{r_1, r_2, \cdots, r_m\}) = \{s \in R \mid sr_1 = r_1 s, sr_2 = r_2 s, \cdots, sr_m = r_m s\}$, where $m \in \mathbb{N}$ with $m \geqslant 2$. We denote the set of all distinct m-element centralisers in a ring R by $m - \textit{Cent}(R)$, where $m \in \mathbb{N}$ with $m \geqslant 2$. A ring R is called an (m, n)-centraliser ring if $|m - \textit{Cent}(R)| = n$, where $n \in \mathbb{N}$. Throughout this dissertation, we study the characterisation for some (m, n)-centraliser finite rings for $n \leqslant 10$.

To establish an association between a graph and a ring, we introduce the idea of the *non-centraliser graph* of rings. The non-centraliser graph of a ring R, denoted by Υ_R, is a graph where the vertex set is R, and the edge set consists of $\{x, y\}$, where x, y are two distinct elements in R such that $C_R(x) \neq C_R(y)$. In this dissertation, we discuss various graph theoretical properties of the non-centraliser graph of finite rings.

Keywords: Finite ring, n-centraliser ring, (m, n)-centraliser ring, non-centraliser graph of ring, non-commuting set, commuting probability.

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LEE KONG CHIAN FACULTY OF ENGINEERING AND SCIENCE

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Date: 14 December 2023

SUBMISSION OF DISSERTATION

It is hereby certified that **Chan Tai Chong** (ID No: **21UEM07287**) has completed this dissertation entitled "GENERALISATION OF n-CENTRALISER RINGS AND THEIR GRAPHS" under the supervision of Dr. Qua Kiat Tat (Supervisor) from the Department of Mathematical and Actuarial Sciences, Lee Kong Chian Faculty of Engineering and Science and Dr. Denis Wong Chee Keong (Co-supervisor) from the Department of Mathematical and Actuarial Sciences, Lee Kong Chian Faculty of Engineering and Science.

I understand that University will upload softcopy of my dissertation in pdf format into UTAR Institutional Repository, which may be made accessible to UTAR community and public.

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APPROVAL SHEET

This dissertation entitled "**GENERALISATION OF n-CENTRALISER RINGS AND THEIR GRAPHS**" was prepared by CHAN TAI CHONG and submitted as partial fulfillment of the requirements for the degree of Master of Science at Universiti Tunku Abdul Rahman.

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DECLARATION

I **Chan Tai Chong** hereby declare that the dissertation is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UTAR or other institutions.

(Chan Tai Chong)

Date: 14 December 2023

LIST OF TABLES

LIST OF FIGURES

TABLE OF CONTENTS

CHAPTER 1

BACKGROUNDS AND LITERATURE REVIEWS

1.1 Introduction

Let R be a ring. The centraliser of r in R, denoted by $C_R(r)$, is defined as $C_R(r) = \{s \in R \mid sr = rs\}$. The centre of R, denoted by $Z(R)$, is defined as $Z(R) = \{s \in R \mid sr = rs \text{ for any } r \in R\}$. For any subring S of R, we let R/S to represent the factor group of $(R, +)$ by $(S, +)$ and let $|R : S|$ to represent the index of $(S, +)$ in $(R, +)$.

By determining the commutativity of a ring, we can investigate the complexity of its structures. This is because as the commutativity of a ring decreases, the complexity of its structure increases. The centraliser of a ring and the commutativity of a ring are inextricably linked. If a ring has a smaller number of distinct centralisers, then its commutativity is higher. Hence, its structure is lower in complexity compared to other ring structures. For instance, if a ring has only one centraliser, then it is a commutative ring.

In this dissertation, we primarily focus on topics related to the centraliser of a ring. In the remainder of this chapter, we shall give some background and literature reviews that are relevant to our study.

1.2 n-Centraliser Rings

Let $\mathrm{Cent}(R)$ denote the set of all distinct centralisers in a ring R, and $\mathrm{Cent}(R) = \{C_R(r) \mid r \in R\}$. A ring R is said to be an n-centraliser ring if $|\mathrm{Cent}(R)| = n$, where $n \in \mathbb{N}$. The notion of n-centraliser rings first appeared in Dutta et al. (2015). By the definition of n-centraliser rings, we note that for any ring R, R is a 1-centraliser ring if and only if R is commutative. Nath et al. (2022) have proven that there does not exist any 2-centraliser ring and 3-centraliser ring. Motivated by this result, the following questions naturally arise: *"Does there exist an n-centraliser ring for any positive integer $n \neq 2, 3$? Can we characterise an n-centraliser ring?"*. They have verified the existence of a $(p+2)$-centraliser ring for any prime p. At the same time, they have classified all 4-centraliser and 5-centraliser finite rings. Here, we state the results proven by Nath et al. (2022), as follows:

[A1] For any non-commutative ring R, $|\mathrm{Cent}(R)| \geqslant 4$.

[A2] If R is a ring with $R/Z(R) \cong \mathbb{Z}_p \times \mathbb{Z}_p$ for some prime p, then $|\mathrm{Cent}(R)| = p + 2$.

[A3] For any finite ring R, R is a 4-centraliser finite ring if and only if $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

[A4] For any finite ring R, R is a 5-centraliser finite ring if and only if $R/Z(R) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

Dutta et al. (2018*a*) determined the possible values of $|R : Z(R)|$ for any 6-centraliser and 7-centraliser finite rings. Also, they found the possible values of $|\text{Cent}(R)|$ when $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Later on, Dutta et al. (2023) further studied the characterisation of n-centraliser finite rings for $n \leqslant 7$. In the following, we list the results proven by Dutta et al. (2018*a*), as follows:

[A5] If R is a 6-centraliser finite ring, then $|R : Z(R)| = 8, 12$ or 16.

[A6] If R is a 7-centraliser finite ring, then $|R : Z(R)| = 12, 18, 20, 24$ or 25.

[A7] If R is a ring with $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $|\text{Cent}(R)| = 6$ or 8.

1.3 Non-Commuting Set of Finite Rings

Dutta et al. (2018*a*) introduced the definition of non-commuting set of a finite ring. Let X be a subset of a finite ring R. If $ab \neq ba$ for any two distinct elements $a, b \in X$, then X is called non-commuting set of R. Moreover, X is said to be the maximal non-commuting set of R if its cardinality is the largest one among all such sets. In the same paper, Dutta et al. (2018*a*) obtained several results regarding the relations between the centralisers and non-commuting sets of a finite ring. Besides that, they completely determined the characterisation for all finite rings with cardinality of the maximal non-commuting set is t, where $t \in \{3, 4\}$. In the following, we list the results proven by Dutta et al. (2018*a*). Among these results is one that involves the concept of an irredundant cover. The definition of this concept will be explained in Section 2.2.

Lemma 1.3.1. Let $\{x_1, x_2, \cdots, x_t\}$ be the maximal non-commuting set of a finite ring R. Then the following statements hold.

(a) $R = \bigcup\limits_{i=1}^{t} C_R(x_i)$.

(b) $\bigcap_{i=1}^{t} C_R(x_i) = Z(R)$.

(c) $\{C_R(x_i) \mid i = 1, 2, \cdots, t\}$ is an irredundant cover of R.

(d) $t \geqslant 3$.

(e) $t + 1 \leqslant |\mathrm{Cent}(R)|$.

(f) $t = 3$ if and only if $|\mathrm{Cent}(R)| = 4$.

(g) $t = 4$ if and only if $|\mathrm{Cent}(R)| = 5$.

(h) If $C_R(r)$ is commutative for any $r \in R - Z(R)$, then for any $r_1, r_2 \in R - Z(R)$, either $C_R(r_1) = C_R(r_2)$ or $C_R(r_1) \cap C_R(r_2) = Z(R)$.

Proof. See Proposition 2.4, Proposition 2.5 and Theorem 2.8 in Dutta et al. (2018*a*). □

1.4 (m, n)-Centraliser Rings

In this part, we introduce the notion of (m, n)-centraliser rings, which is a generalisation of n-centraliser rings.

For any m distinct elements $r_1, r_2, \cdots, r_m$ in a ring R, the m-element centraliser of $\{r_1, r_2, \cdots, r_m\}$ in R, denoted by $C_R(\{r_1, r_2, \cdots, r_m\})$, is defined as $C_R(\{r_1, r_2, \cdots, r_m\}) = \{s \in R \mid sr_1 = r_1 s, sr_2 = r_2 s, \cdots, sr_m = r_m s\}$, where $m \in \mathbb{N}$ with $m \geqslant 2$. Note that $C_R(\{r_1, r_2, \cdots, r_m\}) = \bigcap_{i=1}^{m} C_R(r_i)$. We denote the set of all distinct m-element centralisers in a ring R by $m - \textbf{\textit{Cent}}(R)$, where $m \in \mathbb{N}$ with $m \geqslant 2$. A ring R is called an (m, n)-centraliser ring if $|m - \textbf{\textit{Cent}}(R)| = n$, where $n \in \mathbb{N}$. In the following, we give an elementary result regarding the (m, n)-centraliser rings.

Proposition 1.4.1. Let $m \in \mathbb{N}$ with $m \geqslant 2$ and let R be a ring. If R is commutative, then R is an $(m, 1)$-centraliser ring, and the converse holds when $m = 2$.

Proof. If R is commutative, then $C_R(r) = R$ for any $r \in R$. This gives that $\bigcap_{i=1}^{m} C_R(r_i) = R$ for any m distinct elements $r_1, r_2, \cdots, r_m \in R$. Therefore, $m - \textbf{\textit{Cent}}(R) = \{R\}$ and so, $|m - \textbf{\textit{Cent}}(R)| = 1$. Consequently, R is an $(m, 1)$-centraliser ring.

Next, we suppose to the contrary that R is non-commutative. Then, there exist two distinct elements $r_1, r_2 \in R$ such that $r_1 r_2 \neq r_2 r_1$. This gives that $C_R(r_1) \neq C_R(r_2)$. Thus, we have $C_R(r_1) \cap C_R(0) = C_R(r_1) \cap R = C_R(r_1) \in 2 - \textbf{\textit{Cent}}(R)$ and $C_R(r_2) \cap C_R(0) = C_R(r_2) \cap R = C_R(r_2) \in 2 - \textbf{\textit{Cent}}(R)$. This implies that $\{C_R(r_1), C_R(r_2)\} \subseteq 2 - \textbf{\textit{Cent}}(R)$ and hence, $|2 - \textbf{\textit{Cent}}(R)| \geqslant 2$, which contradicts the fact that $|2 - \textbf{\textit{Cent}}(R)| = 1$. So, the given statement is true. $\square$

In general, the converse of Proposition 1.4.1 is not necessarily true for $m \geqslant 3$. For example, $R = \{\left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right] | \, a, b \in \mathbb{Z}_2\}$ is a non-commutative ring and $\mathrm{Cent}(R) = \{R, \{\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]\}, \{\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]\}, \{\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]\}\}$. It follows that $3 -$

$\mathit{Cent}(R) = \{\{\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]\}\}$. So, $|3-\mathit{Cent}(R)| = 1$ and consequently, R is a $(3,1)$-centraliser ring.

1.5 Commuting Probability of Finite Rings

In order to obtain a systematic way to express the commutativity of a finite ring, the idea of the commuting probability of a finite ring is first introduced by MacHale (1976). In this dissertation, we denote the commuting probability of a finite ring R as $\mathrm{Prob}(R)$. $\mathrm{Prob}(R)$ is the probability that a randomly selected two elements (with replacement) from a finite ring R will commute with each other. That is,

$$\mathrm{Prob}(R) = \frac{|\{(r,s) \in R \times R \mid rs = sr\}|}{|R \times R|}. \tag{1.1}$$

By (1.1), we have

$$\mathrm{Prob}(R) = \frac{\sum\limits_{r\in R} |C_R(r)|}{|R|^2} \tag{1.2}$$

and hence

$$\mathrm{Prob}(R) = \frac{|Z(R)|}{|R|} + \frac{\sum\limits_{r\in R-Z(R)} |C_R(r)|}{|R|^2}. \tag{1.3}$$

By the definition of $\mathrm{Prob}(R)$, we note that for any finite ring R, $\mathrm{Prob}(R) = 1$ if and only if R is commutative. MacHale (1976) has shown that for any finite non-commutative ring R, $\mathrm{Prob}(R) \leqslant \frac{5}{8}$. Moreover, the equality attains if and only if $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. In other words, for any finite ring R, either all of the elements commute or at most $\frac{5}{8}$ of the elements commute.

From the above discussion, we observe that there are some interesting relations between $|\mathrm{Cent}(R)|$ and $\mathrm{Prob}(R)$. For instance, for any finite ring R, R is a 1-centraliser finite ring if and only if $\mathrm{Prob}(R) = 1$. Besides that, for any finite ring R, R is a 4-centraliser finite ring if and only if $\mathrm{Prob}(R) = \frac{5}{8}$. Other than that, in view of [A4] and (1.3), we can demonstrate the following result.

Proposition 1.5.1. For any finite ring R, if R is a 5-centraliser finite ring, then $\mathrm{Prob}(R) = \frac{11}{27}$.

Proof. By [A4], we have $|R : Z(R)| = 9$. Since R is a 5-centraliser finite ring, then R is non-commutative. Hence, $Z(R) \subset C_R(r) \subset R$ for any $r \in R - Z(R)$. Thus, we have $|Z(R)| < |C_R(r)| < |R| = 9|Z(R)|$ for any $r \in R - Z(R)$. For any $r \in R - Z(R)$, since $Z(R)$ is an additive subgroup of $C_R(r)$, and $C_R(r)$ is an additive subgroup of R , then $|C_R(r)|$ is divisible by $|Z(R)|$, and $|R|$ is divisible by $|C_R(r)|$. Hence, $|C_R(r)| = 3|Z(R)| = \frac{|R|}{3}$ for any $r \in R - Z(R)$. Consequently, by (1.3), it follows that

$$\begin{aligned}
\mathrm{Prob}(R) &= \frac{|Z(R)|}{|R|} + \frac{\sum\limits_{r \in R - Z(R)} |C_R(r)|}{|R|^2} \\
&= \frac{1}{9} + \frac{\left(|R| - \frac{|R|}{9}\right)\left(\frac{|R|}{3}\right)}{|R|^2} \\
&= \frac{11}{27}.
\end{aligned}$$

□

1.6 Non-Centraliser Graph of Finite Rings

Recently, the study of ring structures by using the properties of graphs has grown in popularity. Graph theory is able to offer visual aids that help us to understand the ring structures more clearly. Moreover, associating a graph to a ring is an interdisciplinary topic that aims to reveal the relations between ring theory and graph theory, and is beneficial for these two branches of study.

There are various types of graphs associated with rings that have appeared in academia. In 2015, Erfanian et al. (2015) introduced the non-commuting graph of a ring. The non-commuting graph of a ring R, denoted by Γ_R, is a simple graph that considers $R - Z(R)$ as the vertices of Γ_R and connects two distinct vertices x and y whenever $xy \neq yx$. The authors discussed various graph theoretical properties of this graph. They affirmed that for any finite non-commutative ring R, the diameter of Γ_R is at most 2, the girth of Γ_R is 3, and Γ_R is Hamiltonian.

Dutta et al. (2018*b*) generalised the notion of the non-commuting graph of a ring to the non-commuting graph of subrings S, K of a ring R, denoted as $\Gamma_{S,K}$, is a simple graph whose vertex set is $(S \cup K) - ((\bigcap_{s \in S} C_K(s)) \cup (\bigcap_{k \in K} C_S(k)))$, and two distinct vertices a, b are adjacent if and only if $a \in S$ or $b \in S$ and $ab \neq ba$. The authors investigated the diameter, girth and some dominating sets of $\Gamma_{S,K}$. Besides that, they confirmed that there does not exist any finite non-commutative ring R with subrings S, K and $S \subseteq K$ such that $\Gamma_{S,K}$ is a star graph or complete bipartite graph.

In 2021, Nath et al. (2021) introduced another generalisation of the non-commuting graph of rings, namely the r-non-commuting graph of a ring, where $r \in R$ is a fixed element. The r-non-commuting graph of R, denoted by Γ_R^r, is a simple graph whose vertex set is R and two vertices x and y are adjacent if and only if $xy - yx \neq r$ and $xy - yx \neq -r$. The authors characterised some finite non-commutative ring R such that Γ_R^r is a tree or star graph. Additionally, Nath et al. (2021) verified that for any finite non-commutative ring R, if $r = ab - ba$ for some $a, b \in R$, then Γ_R^r is not regular. They also demonstrated that there does not exist any finite non-commutative ring R such that Γ_R^r is a lollipop graph.

Inspired by the study of the non-commuting graph of rings, we introduce the idea of the non-centraliser graph of rings in this dissertation. The non-centraliser graph of a ring R, denoted by Υ_R, is a graph with the vertex set is R, and the edge set consists of $\{x, y\}$, where x, y are two distinct elements in R such that $C_R(x) \neq C_R(y)$. Following the definition of Υ_R, we note that Υ_R is a simple graph. By the definition of Υ_R, we can deduce that a ring R is commutative if and only if Υ_R is an empty graph.

1.7 Objectives and Problem Statements

This dissertation embarks on the following objectives, namely:

[O1] To investigate how the number of distinct centralisers in a finite ring can affect its structure and commutativity.

[O2] To obtain more relations between the centralisers and non-commuting sets of a finite ring.

[O3] To generalise the notion of the n-centraliser ring.

[O4] To use the idea of centraliser to associate a graph to a finite ring.

In order to achieve the objective of this dissertation, there have four problem statements that need to be solved, as follows:

[P1] Characterise all n-centraliser finite rings for $n \in \{6, 7, 8, 9, 10, 11\}$ and compute their commuting probability.

[P2] Determine the structures for all finite rings with cardinality of the maximal non-commuting set is 5.

[P3] Characterise some of the (m, n)-centraliser finite rings for $n \leqslant 10$.

[P4] Investigate various graph theoretical properties of the non-centraliser graph of a finite ring.

1.8 Thesis Organization

Here, we give a brief description of the succeeding chapters in this dissertation. In Chapter 2, we investigate some relations between the centralisers and non-commuting sets of a finite ring. We also establish some lemmas that are useful for the construction of our main results. Next, we construct some results to show the existence of n-centraliser rings for some $n \in \mathbb{N}$. We also study the characterisation for all n-centraliser finite rings for $n \in \{6, 7, 8, 9, 10, 11\}$ and compute their probabilities.

In Chapter 3, we provide some results for finite rings with $|R : Z(R)| = 16$. Subsequently, we classify the structures for all finite rings with cardinality of the maximal non-commuting set is 5.

In Chapter 4, we state some requirements that will be applied in the proof of our main theorems. We also compute $|m - \textit{Cent}(R)|$ for some classes of finite rings. Next, we obtain the characterisation for some (m, n)-centraliser finite rings for $n \leqslant 10$.

In Chapter 5, we obtain various graph theoretical properties of the non-centraliser graph of finite rings.

Finally, in the last chapter, we summarize our dissertation and identify some future works on our topics.

CHAPTER 2

n-CENTRALISER FINITE RINGS AND THEIR COMMUTING PROBABILITIES

2.1 Introduction

In this chapter, we attempt to describe the properties of n-centraliser finite rings. In Section 2.2, we first determine some relations between the centralisers and non-commuting sets of a finite ring. We also prove some lemmas which are important in obtaining our main results. Next, we give some results to show the existence of n-centraliser rings for some $n \in \mathbb{N}$. In Sections 2.3-2.7, we study the characterisation for all n-centraliser finite rings for $n \in \{6, 7, 8, 9, 10, 11\}$ and compute their commuting probabilities.

2.2 Preliminary Results

Following Abdollahi et al. (2007), a cover for a group G is a collection of proper subgroups whose union is the whole G. Moreover, a cover is called irredundant if no proper subcollection is also a cover. Let $f(n)$ be the maximum value of $|G : \bigcap_{i=1}^{n} X_i|$, where $\{X_1, X_2, \cdots, X_n\}$ is an irredundant cover of a group G. Tomkinson (1987) has showed that $f(3) = 4$ and $f(4) = 9$. Furthermore, Bryce et al. (1997), Abdollahi et al. (2005) and Abdollahi and Jafarian Amiri (2007) have proved that $f(5) = 16$, $f(6) = 36$ and $f(7) = 81$, respectively. We will use these results in the sequel.

Before we proceed further, we state some useful lemmas regarding the cover of a group. In this chapter, we will frequently use Lemma 2.2.1.

Lemma 2.2.1. Let $H, X_1, X_2, \cdots, X_t$ be the proper subgroups of a finite group G with $|G : X_i| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_t$. If $G = H \cup X_1 \cup X_2 \cup \cdots \cup X_t$, then $\gamma_1 \leqslant t$. Further, if $\gamma_1 = t$, then $\gamma_1 = \gamma_2 = \cdots = \gamma_t = t$ and $X_i \cap X_j$ is a subgroup of H for any two distinct $i, j \in \{1, 2, \cdots, t\}$.

Proof. See Lemma 3.3 in Tomkinson (1987). □

Lemma 2.2.2. Let $\{X_1, X_2, \cdots, X_t\}$ be an irredundant cover of a group G. Then $\bigcap_{j=1, j\neq i}^{t} X_j = \bigcap_{j=1}^{t} X_j$ for any $i \in \{1, 2, \cdots, t\}$.

Proof. See Lemma 2.2(b) in Bryce et al. (1997). □

In the following, we investigate some relations between the centralisers and non-commuting sets of a finite ring, which are needed in the construction of the main results.

Lemma 2.2.3. Let $\{x_1, x_2, \cdots, x_t\}$ be the maximal non-commuting set of a finite ring R. Let $|\mathrm{Cent}(R)| \leqslant t+3$ and let $r \in R - Z(R)$. Then $C_R(r)$ is commutative if and only if $C_R(r) = C_R(x_i)$ for some $i \in \{1, 2, \cdots, t\}$.

Proof. First, we consider the sufficiency part. Suppose to the contrary that $C_R(x_k)$ is non-commutative for some $k \in \{1, 2, \cdots, t\}$. Without loss of generality, we assume that $k = 1$. Let $\{d_1, d_2, \cdots, d_u\}$ be the maximal non-commuting set of $C_R(x_1)$. By Lemma 1.3.1(d), we have $u \geqslant 3$. For any $i \in \{1, 2, 3\}$, since there exists some $j \in \{1, 2, 3\} - \{i\}$ such that $d_j \notin C_R(d_i)$ but $d_j \in R$ and $d_j \in C_R(x_1)$, then $C_R(d_i) \neq R$ and $C_R(d_i) \neq C_R(x_1)$. For any $i \in \{1, 2, 3\}$ and $j \in \{2, \cdots, t\}$, since $x_1 \in C_R(d_i)$ but $x_1 \notin C_R(x_j)$, then $C_R(d_i) \neq C_R(x_j)$. This shows that $\{R, C_R(x_1), C_R(x_2), \cdots, C_R(x_t), C_R(d_1), C_R(d_2), C_R(d_3)\} \subseteq \mathrm{Cent}(R)$. So, we obtain $|\mathrm{Cent}(R)| \geqslant t + 4$, which leads to a contradiction. Conversely, let $C_R(r)$ is commutative, where $r \in R - Z(R)$. By Lemma 1.3.1(a), $r \in C_R(x_i)$ for some $i \in \{1, 2, \cdots, t\}$. By sufficiency part, $C_R(x_i)$ is commutative. It follows that $C_R(x_i) \leqslant C_R(r)$. Since $x_i \in C_R(r)$ and $C_R(r)$ is commutative, then $C_R(r) \leqslant C_R(x_i)$. Hence, $C_R(r) = C_R(x_i)$. □

Lemma 2.2.4. Let t be the cardinality of the maximal non-commuting set of a finite ring R. Then $|\mathrm{Cent}(R)| = t + 1$ if and only if $C_R(r)$ is commutative for any $r \in R - Z(R)$.

Proof. Let $\{x_1, x_2, \cdots, x_t\}$ be the maximal non-commuting set of R. We first prove the necessity part. We note that $\{R, C_R(x_1), C_R(x_2), \cdots, C_R(x_t)\} \subseteq \mathrm{Cent}(R)$. Since $|\mathrm{Cent}(R)| = t + 1$, then $\mathrm{Cent}(R) = \{R, C_R(x_1), C_R(x_2), \cdots, C_R(x_t)\}$. Therefore, for any $r \in R - Z(R)$, $C_R(r) = C_R(x_i)$ for some $i \in \{1, 2, \cdots, t\}$. By Lemma 2.2.3, it follows that $C_R(r)$ is commutative for any $r \in R - Z(R)$. Conversely, let $w \in R - Z(R)$, then by Lemma 1.3.1(a), $w \in C_R(x_i)$ for some $i \in \{1, 2, \cdots, t\}$. By the hypothesis, $C_R(r)$ is commutative for any $r \in R - Z(R)$. Since $C_R(x_i)$ is commutative, then $C_R(x_i) \leqslant C_R(w)$. Since $x_i \in C_R(w)$ and $C_R(w)$ is commutative, then $C_R(w) \leqslant C_R(x_i)$. Therefore, $C_R(w) = C_R(x_i)$. Hence, we obtain $\mathrm{Cent}(R) = \{R, C_R(x_1), C_R(x_2), \cdots, C_R(x_t)\}$ and so, $|\mathrm{Cent}(R)| = t + 1$. □

As a direct consequence of Lemma 2.2.4 and Lemma 1.3.1(h), we have the following result.

Corollary 2.2.5. Let t be the cardinality of the maximal non-commuting set of a finite ring R. If $|\mathrm{Cent}(R)| = t + 1$, then for any $r_1, r_2 \in R - Z(R)$, either $C_R(r_1) = C_R(r_2)$ or $C_R(r_1) \cap C_R(r_2) = Z(R)$.

Lemma 2.2.6. Let $\{x_1, x_2, \cdots, x_t\}$ be the maximal non-commuting set of a finite ring R. If $|\mathrm{Cent}(R)| = t + 2$ and $C_R(r)$ is non-commutative for some $r \in R - Z(R)$, then $C_R(r)$ contains three distinct $C_R(x_i)$'s.

Proof. In view of Lemma 2.2.3, we have $C_R(x_i)$ is commutative for any $i \in \{1, 2, \cdots, t\}$. Since $C_R(r)$ is non-commutative, then $C_R(r) \neq C_R(x_i)$ for any $i \in \{1, 2, \cdots, t\}$. Thus, $\{R, C_R(r), C_R(x_1), C_R(x_2), \cdots, C_R(x_t)\} \subseteq \mathrm{Cent}(R)$. Since $|\mathrm{Cent}(R)| = t + 2$, then $\mathrm{Cent}(R) = \{R, C_R(r), C_R(x_1), C_R(x_2), \cdots, C_R(x_t)\}$. Let $\{d_1, d_2, \cdots, d_u\}$ be the maximal non-commuting set of $C_R(r)$. By Lemma 1.3.1(d), we have $u \geqslant 3$. For any $i \in \{1, 2, 3\}$, since there exists some $j \in \{1, 2, 3\} - \{i\}$ such that $d_j \notin C_R(d_i)$ but $d_j \in R$ and $d_j \in C_R(r)$, then $C_R(d_i) \neq R$ and $C_R(d_i) \neq C_R(r)$. It follows that $\{C_R(d_1), C_R(d_2), C_R(d_3)\} \subseteq \mathrm{Cent}(R) - \{R, C_R(r)\} = \{C_R(x_1), C_R(x_2), \cdots, C_R(x_t)\}$. This gives that $C_R(d_1) = C_R(x_{l_1}), C_R(d_2) = C_R(x_{l_2})$ and $C_R(d_3) = C_R(x_{l_3})$ for three distinct $l_1, l_2, l_3 \in \{1, 2, \cdots, t\}$. For any $i \in \{1, 2, 3\}$, since $r \in C_R(d_i) = C_R(x_{l_i})$ and $C_R(x_{l_i})$ is commutative, then $C_R(x_{l_i}) \leqslant C_R(r)$. Consequently, $C_R(r)$ contains three distinct $C_R(x_i)$'s. □

To obtain the following lemma, we can use similar arguments as in the proof of Lemma 2.2.6.

Lemma 2.2.7. Let $\{x_1, x_2, \cdots, x_t\}$ be the maximal non-commuting set of a finite ring R. If $|\mathrm{Cent}(R)| = t + 3$ and $C_R(r)$ is non-commutative for some $r \in R - Z(R)$, then $C_R(r)$ contains two distinct $C_R(x_i)$'s.

Proof. In view of Lemma 2.2.3, we have $C_R(x_i)$ is commutative for any $i \in \{1, 2, \cdots, t\}$. Since $C_R(r)$ is non-commutative, then $C_R(r) \neq C_R(x_i)$ for any $i \in \{1, 2, \cdots, t\}$. Thus, $\{R, C_R(r), C_R(x_1), C_R(x_2), \cdots, C_R(x_t)\} \subseteq \mathrm{Cent}(R)$. Since $|\mathrm{Cent}(R)| = t + 3$, then $\mathrm{Cent}(R) = \{R, C_R(r), C_R(x_1), C_R(x_2), \cdots, C_R(x_t), C_R(a)\}$ for some $a \in R - Z(R)$. Let $\{d_1, d_2, \cdots, d_u\}$ be the maximal non-commuting set of $C_R(r)$. By Lemma 1.3.1(d), we have $u \geqslant 3$. For any $i \in \{1, 2, 3\}$, since there exists some $j \in \{1, 2, 3\} - \{i\}$ such that $d_j \notin C_R(d_i)$ but $d_j \in R$ and $d_j \in C_R(r)$, then $C_R(d_i) \neq R$ and $C_R(d_i) \neq C_R(r)$. It follows that $\{C_R(d_1), C_R(d_2), C_R(d_3)\} \subseteq \mathrm{Cent}(R) - \{R, C_R(r)\} = \{C_R(x_1), C_R(x_2), \cdots, C_R(x_t), C_R(a)\}$. This implies that $\{C_R(d_{k_1}), C_R(d_{k_2})\} \subseteq \{C_R(x_1), C_R(x_2), \cdots, C_R(x_t)\}$ for two distinct $k_1, k_2 \in \{1, 2, 3\}$. This gives that $C_R(d_{k_1}) = C_R(x_{l_1})$ and $C_R(d_{k_2}) = C_R(x_{l_2})$ for two distinct $l_1, l_2 \in \{1, 2, \cdots, t\}$. For any $i \in \{1, 2\}$, since $r \in C_R(d_{k_i}) = C_R(x_{l_i})$ and $C_R(x_{l_i})$ is commutative, then $C_R(x_{l_i}) \leqslant C_R(r)$. Consequently, $C_R(r)$ contains two distinct $C_R(x_i)$'s. □

Lemma 2.2.8. Let $\{x_1, x_2, \cdots, x_t\}$ be the maximal non-commuting set of a finite ring R. Let $|\mathrm{Cent}(R)| = t + 4$. Let $C_R(a_1), C_R(a_2), C_R(a_3)$ be three distinct proper centralisers of R that are different from $C_R(x_i)$ for any $i \in \{1, 2, \cdots, t\}$. Then the following statements hold.

(a) If $C_R(x_i)$ is non-commutative for some $i \in \{1, 2, \cdots, t\}$, then $a_1, a_2, a_3 \in C_R(x_i)$ and a_1, a_2, a_3 do not commute with each other.

(b) At most one $C_R(x_i)$'s is non-commutative.

(c) Let $D \in \{C_R(x_i) \mid i = 1, 2, \cdots, t\}$ with D is non-commutative, let $Q \in \{C_R(a_1), C_R(a_2), C_R(a_3)\}$ with Q is non-commutative and let $\{q_1, q_2, \cdots, q_v\}$ be the maximal non-commuting set of Q. Then $3 \leqslant v \leqslant t - 2$ and $\{C_R(q_i) \mid i = 1, 2 \cdots, v\} \subseteq \{C_R(x_i) \mid i = 1, 2, \cdots, t\}$. Moreover, if $t = 5$, then $D \in \{C_R(q_1), C_R(q_2), C_R(q_3)\}$.

Proof. (a) Let $\{d_1, d_2, \cdots, d_u\}$ be the maximal non-commuting set of $C_R(x_i)$. By Lemma 1.3.1(d), we have $u \geqslant 3$. For any $j \in \{1, 2, 3\}$, since there exists some $k \in \{1, 2, 3\} - \{j\}$ such that $d_k \notin C_R(d_j)$ but $d_k \in R$ and $d_k \in C_R(x_i)$, then $C_R(d_j) \neq R$ and $C_R(d_j) \neq C_R(x_i)$. For any $j \in \{1, 2, 3\}$ and $k \in \{1, 2, \cdots, t\} - \{i\}$, since $x_i \in C_R(d_j)$ but $x_i \notin C_R(x_k)$, then $C_R(d_j) \neq C_R(x_k)$. Therefore, we have $\{C_R(d_1), C_R(d_2), C_R(d_3)\} \subseteq \mathrm{Cent}(R) - \{R, C_R(x_1), C_R(x_2), \cdots, C_R(x_t)\} = \{C_R(a_1), C_R(a_2), C_R(a_3)\}$, which gives that $\{C_R(d_1), C_R(d_2), C_R(d_3)\} = \{C_R(a_1), C_R(a_2), C_R(a_3)\}$. Without loss of generality, we assume that $C_R(d_j) = C_R(a_j)$ for any $j \in \{1, 2, 3\}$. Since $x_i \in C_R(d_j) = C_R(a_j)$ for any $j \in \{1, 2, 3\}$, then we obtain $a_1, a_2, a_3 \in C_R(x_i)$. Next, we suppose that $a_j a_k = a_k a_j$ for two distinct $j, k \in \{1, 2, 3\}$. Hence, we have $a_j \in C_R(a_k) = C_R(d_k)$ and thus, $d_k \in C_R(a_j) = C_R(d_j)$, which is a contradiction. Consequently, a_1, a_2, a_3 do not commute with each other.

(b) Suppose that there have at least two $C_R(x_i)$'s are non-commutative.

Without any loss, we assume that $C_R(x_1), C_R(x_2)$ are non-commutative. By Lemma 2.2.8(a), we have $a_1, a_2, a_3 \in C_R(x_i)$ for any $i \in \{1, 2\}$ and a_1, a_2, a_3 do not commute with each other. Now, we consider for $C_R(a_1 + x_1)$. Since $a_2 \notin C_R(a_1 + x_1)$ but $a_2 \in R$ and $a_2 \in C_R(x_1)$, then $C_R(a_1 + x_1) \neq R$ and $C_R(a_1 + x_1) \neq C_R(x_1)$. Since $x_1 \in C_R(a_1 + x_1)$ but $x_1 \notin C_R(x_i)$ for any $i \in \{2, \cdots, t\}$, then $C_R(a_1 + x_1) \neq C_R(x_i)$ for any $i \in \{2, \cdots, t\}$. Since $x_2 \notin C_R(a_1 + x_1)$ but $x_2 \in C_R(a_i)$ for any $i \in \{1, 2, 3\}$, then $C_R(a_1 + x_1) \neq C_R(a_i)$ for any $i \in \{1, 2, 3\}$. This gives that $\{R, C_R(x_1), C_R(x_2), \cdots, C_R(x_t), C_R(a_1), C_R(a_2), C_R(a_3), C_R(a_1 + x_1)\} \subseteq \mathrm{Cent}(R)$. Consequently, we obtain $|\mathrm{Cent}(R)| \geqslant t + 5$, which leads to a contradiction.

(c) Without loss of generality, we assume that $Q = C_R(a_1)$. For any $i \in \{1, 2, \cdots, v\}$, since there exists some $j \in \{1, 2, \cdots, v\} - \{i\}$ such that $q_j \notin C_R(q_i)$ but $q_j \in R$ and $q_j \in C_R(a_1)$, then $C_R(q_i) \neq R$ and $C_R(q_i) \neq C_R(a_1)$. By Lemma 2.2.8(a), a_1, a_2, a_3 do not commute with each other. For any $i \in \{1, 2, \cdots, v\}$, since $a_1 \in C_R(q_i)$ but $a_1 \notin C_R(a_2)$ and $a_1 \notin C_R(a_3)$, then $C_R(q_i) \neq C_R(a_2)$ and $C_R(q_i) \neq C_R(a_3)$. Consequently, we obtain $\{C_R(q_i) \mid i = 1, 2, \cdots, v\} \subseteq \mathrm{Cent}(R) - \{R, C_R(a_1), C_R(a_2), C_R(a_3)\} = \{C_R(x_i) \mid i = 1, 2, \cdots, t\}$. By Lemma 1.3.1(d), we have $v \geqslant 3$. Now, we claim that $v \leqslant t - 2$. Suppose to the contrary that $v \geqslant t - 1$, then $a_1 \in \bigcap_{i=1}^{t-1} C_R(q_i) = \bigcap_{i=1, i \neq j}^{t} C_R(x_i)$ for some $j \in \{1, 2, \cdots, t\}$. Hence, by Lemma 1.3.1(b), (c) and Lemma 2.2.2, we obtain $a_1 \in Z(R)$; a contradiction. Thus, we have $v \leqslant t-2$, as claimed. It follows that $3 \leqslant v \leqslant t - 2$ and $\{C_R(q_i) \mid i = 1, 2, \cdots, v\} \subseteq \{C_R(x_i) \mid i = 1, 2, \cdots, t\}$. Next, we consider $t = 5$. Thus, we have $v = 3$ and $\{C_R(q_1), C_R(q_2), C_R(q_3)\} \subseteq$

$\{C_R(x_i) \mid i = 1, 2, \cdots, 5\}$. Assume that $D \notin \{C_R(q_1), C_R(q_2), C_R(q_3)\}$. By Lemma 2.2.8(a), $a_1 \in D$. Hence, $a_1 \in D \cap C_R(q_1) \cap C_R(q_2) \cap C_R(q_3)$. This implies that $a_1 \in \bigcap_{i=1, i\neq j}^{5} C_R(x_i)$ for some $j \in \{1, 2, \cdots, 5\}$. Therefore, it follows from Lemma 1.3.1(b), (c) and Lemma 2.2.2 that $a_1 \in Z(R)$, which is impossible.

□

The following two results can be proved in a manner similar to that used to prove Lemma 2.2.8.

Lemma 2.2.9. Let $\{x_1, x_2, \cdots, x_t\}$ be the maximal non-commuting set of a finite ring R. Let $|\text{Cent}(R)| = t + 5$. Let $C_R(a_1), C_R(a_2), C_R(a_3), C_R(a_4)$ be four distinct proper centralisers of R that are different from $C_R(x_i)$ for any $i \in \{1, 2, \cdots, t\}$. Then the following statements hold.

(a) If $C_R(x_i)$ is non-commutative for some $i \in \{1, 2, \cdots, t\}$, then there exist three distinct $l_1, l_2, l_3 \in \{1, 2, 3, 4\}$ such that $a_{l_1}, a_{l_2}, a_{l_3} \in C_R(x_i)$ and $a_{l_1}, a_{l_2}, a_{l_3}$ do not commute with each other.

(b) At most one $C_R(x_i)$'s is non-commutative.

Proof. (a) Let $\{d_1, d_2, \cdots, d_u\}$ be the maximal non-commuting set of $C_R(x_i)$. By Lemma 1.3.1(d), we have $u \geqslant 3$. For any $j \in \{1, 2, 3\}$, since there exists some $k \in \{1, 2, 3\} - \{j\}$ such that $d_k \notin C_R(d_j)$ but $d_k \in R$ and $d_k \in C_R(x_i)$, then $C_R(d_j) \neq R$ and $C_R(d_j) \neq C_R(x_i)$. For any $j \in \{1, 2, 3\}$ and $k \in \{1, 2, \cdots, t\} - \{i\}$, since $x_i \in C_R(d_j)$ but $x_i \notin C_R(x_k)$, then $C_R(d_j) \neq C_R(x_k)$. Therefore, we have $\{C_R(d_1), C_R(d_2), C_R(d_3)\} \subseteq \text{Cent}(R) - \{R, C_R(x_1), C_R(x_2), \cdots, C_R(x_t)\} = \{C_R(a_1), C_R(a_2), C_R(a_3), C_R(a_4)\}$. This gives that $C_R(d_1) =$

$C_R(a_{l_1}), C_R(d_2) = C_R(a_{l_2})$ and $C_R(d_3) = C_R(a_{l_3})$ for three distinct $l_1, l_2, l_3 \in \{1, 2, 3, 4\}$. Since $x_i \in C_R(d_j) = C_R(a_{l_j})$ for any $j \in \{1, 2, 3\}$, then we obtain $a_{l_1}, a_{l_2}, a_{l_3} \in C_R(x_i)$. Next, we suppose that $a_{l_j} a_{l_k} = a_{l_k} a_{l_j}$ for two distinct $j, k \in \{1, 2, 3\}$. Hence, we have $a_{l_j} \in C_R(a_{l_k}) = C_R(d_k)$ and thus, $d_k \in C_R(a_{l_j}) = C_R(d_j)$, which is a contradiction. Consequently, $a_{l_1}, a_{l_2}, a_{l_3}$ do not commute with each other.

(b) Suppose that there have at least two $C_R(x_i)$'s are non-commutative. Without any loss, we assume that $C_R(x_1), C_R(x_2)$ are non-commutative. By Lemma 2.2.9(a), there exist three distinct $l_1, l_2, l_3 \in \{1, 2, 3, 4\}$ such that a_{l_1}, a_{l_2}, $a_{l_3} \in C_R(x_1)$ and $a_{l_1}, a_{l_2}, a_{l_3}$ do not commute with each other. Also, by Lemma 2.2.9(a), there exist three distinct $k_1, k_2, k_3 \in \{1, 2, 3, 4\}$ such that $a_{k_1}, a_{k_2}, a_{k_3} \in C_R(x_2)$. Since $|\{l_1, l_2, l_3\} \cap \{k_1, k_2, k_3\}| \geqslant 2$, then without any loss, we have $k_1 = l_1$ and $k_2 = l_2$. Now, we consider for $C_R(a_{l_1} + x_1)$ and $C_R(a_{l_2} + x_1)$. Since $a_{l_2} \notin C_R(a_{l_1} + x_1)$ but $a_{l_2} \in R$ and $a_{l_2} \in C_R(x_1)$, then $C_R(a_{l_1} + x_1) \neq R$ and $C_R(a_{l_1} + x_1) \neq C_R(x_1)$. Since $a_{l_1} \notin C_R(a_{l_2} + x_1)$ but $a_{l_1} \in R$ and $a_{l_1} \in C_R(x_1)$, then $C_R(a_{l_2} + x_1) \neq R$ and $C_R(a_{l_2} + x_1) \neq C_R(x_1)$. For any $i \in \{1, 2\}$ and $j \in \{2, \cdots, t\}$, since $x_1 \in C_R(a_{l_i} + x_1)$ but $x_1 \notin C_R(x_j)$, then $C_R(a_{l_i} + x_1) \neq C_R(x_j)$. For any $i \in \{1, 2\}$, since $x_2 \notin C_R(a_{l_i} + x_1)$ but $x_2 \in C_R(a_{l_1})$ and $x_2 \in C_R(a_{l_2})$, then $C_R(a_{l_i} + x_1) \neq C_R(a_{l_1})$ and $C_R(a_{l_i} + x_1) \neq C_R(a_{l_2})$. For any $i \in \{1, 2\}$, since $a_{l_3} \notin C_R(a_{l_i} + x_1)$ but $a_{l_3} \in C_R(a_{l_3})$, then $C_R(a_{l_i} + x_1) \neq C_R(a_{l_3})$. Since $a_{l_1} \in C_R(a_{l_1} + x_1)$ but $a_{l_1} \notin C_R(a_{l_2} + x_1)$, then $C_R(a_{l_1} + x_1) \neq C_R(a_{l_2} + x_1)$. This yields that $\{R, C_R(x_1), C_R(x_2), \cdots, C_R(x_t), C_R(a_{l_1}), C_R(a_{l_2}), C_R(a_{l_3}), C_R(a_{l_1} + x_1),$

$C_R(a_{l_2}+x_1)\} \subseteq \text{Cent}(R)$. Consequently, we obtain $|\text{Cent}(R)| \geqslant t+6$, which leads to a contradiction. □

Lemma 2.2.10. Let $\{x_1, x_2, \cdots, x_t\}$ be the maximal non-commuting set of a finite ring R. Let $|\text{Cent}(R)| = t+6$. Let $C_R(a_1), C_R(a_2), \cdots, C_R(a_5)$ be five distinct proper centralisers of R that are different from $C_R(x_i)$ for any $i \in \{1, 2, \cdots, t\}$. Then the following statements hold.

(a) If $C_R(x_i)$ is non-commutative for some $i \in \{1, 2, \cdots, t\}$, then there exist three distinct $l_1, l_2, l_3 \in \{1, 2, \cdots, 5\}$ such that $a_{l_1}, a_{l_2}, a_{l_3} \in C_R(x_i)$ and $a_{l_1}, a_{l_2}, a_{l_3}$ do not commute with each other.

(b) At most one $C_R(x_i)$'s is non-commutative.

Proof. (a) Let $\{d_1, d_2, \cdots, d_u\}$ be the maximal non-commuting set of $C_R(x_i)$. By Lemma 1.3.1(d), we have $u \geqslant 3$. For any $j \in \{1, 2, 3\}$, since there exists some $k \in \{1, 2, 3\} - \{j\}$ such that $d_k \notin C_R(d_j)$ but $d_k \in R$ and $d_k \in C_R(x_i)$, then $C_R(d_j) \neq R$ and $C_R(d_j) \neq C_R(x_i)$. For any $j \in \{1, 2, 3\}$ and $k \in \{1, 2, \cdots, t\} - \{i\}$, since $x_i \in C_R(d_j)$ but $x_i \notin C_R(x_k)$, then $C_R(d_j) \neq C_R(x_k)$. Therefore, we have $\{C_R(d_1), C_R(d_2), C_R(d_3)\} \subseteq \text{Cent}(R) - \{R, C_R(x_1), C_R(x_2), \cdots, C_R(x_t)\} = \{C_R(a_1), C_R(a_2), \cdots, C_R(a_5)\}$. This gives that $C_R(d_1) = C_R(a_{l_1}), C_R(d_2) = C_R(a_{l_2})$ and $C_R(d_3) = C_R(a_{l_3})$ for three distinct $l_1, l_2, l_3 \in \{1, 2, \cdots, 5\}$. Since $x_i \in C_R(d_j) = C_R(a_{l_j})$ for any $j \in \{1, 2, 3\}$, then we obtain $a_{l_1}, a_{l_2}, a_{l_3} \in C_R(x_i)$. Next, we suppose that $a_{l_j}a_{l_k} = a_{l_k}a_{l_j}$ for two distinct $j, k \in \{1, 2, 3\}$. Hence, we have $a_{l_j} \in C_R(a_{l_k}) = C_R(d_k)$ and thus, $d_k \in C_R(a_{l_j}) = C_R(d_j)$, which is a contradiction. Consequently, $a_{l_1}, a_{l_2}, a_{l_3}$ do not commute with each other.

(b) Suppose that there have at least two $C_R(x_i)$'s are non-commutative. Without any loss, we assume that $C_R(x_1), C_R(x_2)$ are non-commutative. By Lemma 2.2.10(a), there exist three distinct $l_1, l_2, l_3 \in \{1, 2, \cdots, 5\}$ such that $a_{l_1}, a_{l_2}, a_{l_3} \in C_R(x_1)$ and $a_{l_1}, a_{l_2}, a_{l_3}$ do not commute with each other. Also, by Lemma 2.2.10(a), there exist three distinct $k_1, k_2, k_3 \in \{1, 2, \cdots 5\}$ such that $a_{k_1}, a_{k_2}, a_{k_3} \in C_R(x_2)$. Since $|\{l_1, l_2, l_3\} \cap \{k_1, k_2, k_3\}| \geqslant 1$, then without any loss, we have $k_1 = l_1$. Now, we consider for $C_R(a_{l_1} + x_1), C_R(a_{l_2} + x_1)$ and $C_R(a_{l_3} + x_1)$. For any $i \in \{1, 2, 3\}$, since there exists some $j \in \{1, 2, 3\} - \{i\}$ such that $a_{l_j} \notin C_R(a_{l_i} + x_1)$ but $a_{l_j} \in R$ and $a_{l_j} \in C_R(x_1)$, then $C_R(a_{l_i} + x_1) \neq R$ and $C_R(a_{l_i} + x_1) \neq C_R(x_1)$. For any $i \in \{1, 2, 3\}$ and $j \in \{2, \cdots, t\}$, since $x_1 \in C_R(a_{l_i} + x_1)$ but $x_1 \notin C_R(x_j)$, then $C_R(a_{l_i} + x_1) \neq C_R(x_j)$. Since $x_2 \notin C_R(a_{l_1} + x_1)$ but $x_2 \in C_R(a_{l_1})$, then $C_R(a_{l_1} + x_1) \neq C_R(a_{l_1})$. For any two distinct $i, j \in \{1, 2, 3\}$, since $a_{l_j} \notin C_R(a_{l_i} + x_1)$ but $a_{l_j} \in C_R(a_{l_j})$, then $C_R(a_{l_i} + x_1) \neq C_R(a_{l_j})$. Here, we distinguish our proof into the following three cases.

Case 1: $|\{l_2, l_3\} \cap \{k_2, k_3\}| = 0$. Since $x_2 \notin C_R(a_{l_1} + x_1)$ but $x_2 \in C_R(a_{k_2})$ and $x_2 \in C_R(a_{k_3})$, then $C_R(a_{l_1} + x_1) \neq C_R(a_{k_2})$ and $C_R(a_{l_1} + x_1) \neq C_R(a_{k_3})$. This gives that $\{R, C_R(x_1), C_R(x_2), \cdots, C_R(x_t), C_R(a_{l_1}), C_R(a_{l_2}), C_R(a_{l_3}), C_R(a_{k_2}), C_R(a_{k_3}), C_R(a_{l_1} + x_1)\} \subseteq \mathrm{Cent}(R)$. So, we obtain $|\mathrm{Cent}(R)| \geqslant t + 7$, which is a contradiction.

Case 2: $|\{l_2, l_3\} \cap \{k_2, k_3\}| = 1$. Thus, without any loss, we have $k_2 = l_2$. Since $x_2 \notin C_R(a_{l_2} + x_1)$ but $x_2 \in C_R(a_{l_2})$, then $C_R(a_{l_2} + x_1) \neq C_R(a_{l_2})$. For any $i \in \{1,2\}$, since $x_2 \notin C_R(a_{l_i}+x_1)$ but $x_2 \in C_R(a_{k_3})$, then $C_R(a_{l_i}+x_1) \neq C_R(a_{k_3})$. Since $a_{l_2} \notin C_R(a_{l_1}+x_1)$ but $a_{l_2} \in C_R(a_{l_2}+x_1)$, then $C_R(a_{l_1}+x_1) \neq C_R(a_{l_2}+x_1)$. This gives that $\{R, C_R(x_1), C_R(x_2), \cdots, C_R(x_t), C_R(a_{l_1}), C_R(a_{l_2}), C_R(a_{l_3}), C_R(a_{k_3}), C_R(a_{l_1}+x_1), C_R(a_{l_2}+x_1)\} \subseteq \text{Cent}(R)$. So, we obtain $|\text{Cent}(R)| \geqslant t+7$, which is a contradiction.

Case 3: $|\{l_2, l_3\} \cap \{k_2, k_3\}| = 2$. Thus, without any loss, we have $k_2 = l_2$ and $k_3 = l_3$. For any $i \in \{2,3\}$, since $x_2 \notin C_R(a_{l_i}+x_1)$ but $x_2 \in C_R(a_{l_i})$, then $C_R(a_{l_i}+x_1) \neq C_R(a_{l_i})$. For any two distinct $i, j \in \{1,2,3\}$, since $a_{l_j} \notin C_R(a_{l_i}+x_1)$ but $a_{l_j} \in C_R(a_{l_j}+x_1)$, then $C_R(a_{l_i}+x_1) \neq C_R(a_{l_j}+x_1)$. This gives that $\{R, C_R(x_1), C_R(x_2), \cdots, C_R(x_t), C_R(a_{l_1}), C_R(a_{l_2}), C_R(a_{l_3}), C_R(a_{l_1}+x_1), C_R(a_{l_2}+x_1), C_R(a_{l_3}+x_1)\} \subseteq \text{Cent}(R)$. So, we obtain $|\text{Cent}(R)| \geqslant t+7$, which is a contradiction. □

Next, we establish some lemmas that are useful for subsequent results.

Lemma 2.2.11. Let R be a finite ring and let $r_1, r_2 \in R - Z(R)$. Then $|R : Z(R)| \leqslant |R : C_R(r_1)||R : C_R(r_2)||C_R(r_1) \cap C_R(r_2) : Z(R)|$. In particular, if $C_R(r_1) \cap C_R(r_2) = Z(R)$, then $|R : Z(R)| \leqslant |R : C_R(r_1)||R : C_R(r_2)|$.

Proof. Since $C_R(r_1) + C_R(r_2) \subseteq R$, then $|C_R(r_1) + C_R(r_2)| \leqslant |R|$. Hence, we have $\frac{|C_R(r_1)||C_R(r_2)|}{|C_R(r_1) \cap C_R(r_2)|} \leqslant |R|$. So, we obtain $|R : Z(R)| \leqslant |R : C_R(r_1)||R : C_R(r_2)||C_R(r_1) \cap C_R(r_2) : Z(R)|$. Furthermore, if $C_R(r_1) \cap C_R(r_2) = Z(R)$, then we obtain $|R : Z(R)| \leqslant |R : C_R(r_1)||R : C_R(r_2)|$ directly. □

Lemma 2.2.12. Let R be a finite ring and let $r_1, r_2 \in R - Z(R)$ with $r_1 r_2 \neq$

r_2r_1. If $C_R(r_1)$ is commutative with $|R : C_R(r_1)| = p$ for some prime p, then $C_R(r_1) \cap C_R(r_2) = Z(R)$ and $|R : Z(R)| \leqslant p|R : C_R(r_2)|$.

Proof. Let $a \in C_R(r_1) \cap C_R(r_2)$. Since $a \in C_R(r_1)$ and $C_R(r_1)$ is commutative, then $C_R(r_1) \leqslant C_R(a)$. Since $r_2 \notin C_R(r_1)$ but $r_2 \in C_R(a)$, then $C_R(r_1) < C_R(a)$. This gives that $|C_R(r_1)| < |C_R(a)|$. Since $|C_R(r_1)| \mid |C_R(a)|$, then we conclude that $|C_R(a)| = |R|$. This implies that $C_R(a) = R$. This yields that $a \in Z(R)$, which follows that $C_R(r_1) \cap C_R(r_2) \leqslant Z(R)$. On the other hand, it is obvious that $Z(R) \leqslant C_R(r_1) \cap C_R(r_2)$. Consequently, we obtain $C_R(r_1) \cap C_R(r_2) = Z(R)$. Furthermore, by Lemma 2.2.11, we have $|R : Z(R)| \leqslant p|R : C_R(r_2)|$. □

The following result follows immediately from Lemma 2.2.3 and Lemma 2.2.12.

Corollary 2.2.13. Let $\{x_1, x_2, \cdots, x_t\}$ be the maximal non-commuting set of a finite ring R. If $|\mathrm{Cent}(R)| \leqslant t + 3$ and $|R : C_R(x_i)| = p$ for some $i \in \{1, 2, \cdots, t\}$ and prime p, then $C_R(x_i) \cap C_R(x_j) = Z(R)$ and $|R : Z(R)| \leqslant p|R : C_R(x_j)|$ for any $j \in \{1, 2, \cdots, t\} - \{i\}$.

Lemma 2.2.14. Let $\{x_1, x_2, \cdots, x_t\}$ be the maximal non-commuting set of a finite ring R. Let $|\mathrm{Cent}(R)| \leqslant t + 3$ and $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_t$. If $\gamma_2 = 2$, then $|\mathrm{Cent}(R)| = 4$. Furthermore, if $\gamma_2 = 3$, then $|\mathrm{Cent}(R)| = 5$.

Proof. If $R/Z(R)$ is cyclic, then R is commutative; a contradiction. So, we have $R/Z(R)$ is not cyclic. Suppose that $\gamma_2 = 2$. In view of Corollary 2.2.13, we have $|R : Z(R)| \leqslant 2(2) = 4$. Since $\gamma_2 \mid |R : Z(R)|$, then $|R : Z(R)| = 2$ or 4.

If $|R : Z(R)| = 2$, then $R/Z(R) \cong \mathbb{Z}_2$, which is impossible because $R/Z(R)$ is not cyclic. Thus, $|R : Z(R)| = 4$. This gives that $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ as $R/Z(R)$ is not cyclic. By [A2], we obtain $|\mathrm{Cent}(R)| = 4$. Next, we assume that $\gamma_2 = 3$. It follows from Corollary 2.2.13 that $|R : Z(R)| \leqslant 3(3) = 9$. Since $\gamma_2 \mid |R : Z(R)|$, then $|R : Z(R)| = 3, 6$ or 9. If $|R : Z(R)| = 3$ or 6, then $R/Z(R) \cong \mathbb{Z}_3$ or $\mathbb{Z}_6$, which contradicts the fact that $R/Z(R)$ is not cyclic. Therefore, $|R : Z(R)| = 9$. This implies that $R/Z(R) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ as $R/Z(R)$ is not cyclic. So, we have $|\mathrm{Cent}(R)| = 5$ by [A2]. □

Lemma 2.2.15. Let R be a finite ring. Let $r \in R - Z(R)$ with $|C_R(r) : Z(R)| = m$. Let M be the set of all non-trivial positive divisors of m. If m is square-free, or m is not square-free with $\frac{m}{t}$ is square-free for any $t \in M$, then $C_R(r)$ is commutative.

Proof. If m is square-free, then $C_R(r)/Z(R) \cong \mathbb{Z}_m$. Thus, $C_R(r)/Z(R)$ is cyclic and hence, $C_R(r)$ is commutative. Next, we consider for m is not square-free. Suppose to the contrary that $C_R(r)$ is non-commutative, then $C_R(r)$ satisfies $Z(R) < Z(C_R(r)) < C_R(r)$. So, we have $|C_R(r) : Z(C_R(r))| = \frac{m}{t}$ for some $t \in M$. From the given assumption, $\frac{m}{t}$ is square-free. This implies that $C_R(r)/Z(C_R(r)) \cong \mathbb{Z}_{\frac{m}{t}}$ and so, $C_R(r)/Z(C_R(r))$ is cyclic. This yields that $C_R(r)$ is commutative; a contradiction. □

Lemma 2.2.16. If R is a finite ring with $|R : Z(R)| = p^2q$ for some primes p, q, then $C_R(r)$ is commutative for any $r \in R - Z(R)$.

Proof. Since $|R : Z(R)| = p^2q$, then for any $r \in R - Z(R)$, $|C_R(r) : Z(R)| = p, q, p^2$ or pq. Therefore, by Lemma 2.2.15, it follows that $C_R(r)$ is commutative for any $r \in R - Z(R)$. □

Lemma 2.2.17. If R is a finite ring, then $|R : Z(R)| \neq p^2q$ for any two distinct primes p, q.

Proof. Clearly, the result holds when R is commutative. Now, we consider for R is non-commutative. Suppose to the contrary that $|R : Z(R)| = p^2q$ for two distinct primes p, q. Let $\{x_1, x_2, \cdots, x_t\}$ be the maximal non-commuting set of R. Without loss of generality, we assume that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_t$. By Lemma 2.2.16 and Lemma 2.2.4, we have $|\mathrm{Cent}(R)| = t + 1$. Since $R/Z(R)$ is not cyclic, then $R/Z(R) \cong \mathbb{Z}_p \times \mathbb{Z}_{pq}$. Here, we let $|A_1|$ and $|A_2|$ be the total number of elements with order pq in $\mathbb{Z}_{pq}$ and $R/Z(R)$, respectively. It is obvious that $|A_1| < |A_2|$.

If $\gamma_1 = p^2$ or pq, then $\gamma_i \neq p$ for any $i \in \{2, \cdots, t\}$. If $\gamma_1 = p$ or q, then by Corollary 2.2.13, we have $\gamma_i \neq p$ for any $i \in \{2, \cdots, t\}$. Therefore, $|C_R(x_i) : Z(R)| \neq pq$ for any $i \in \{2, \cdots, t\}$. From Lemma 1.3.1(a), $R/Z(R) = \bigcup_{i=1}^{t}[C_R(x_i)/Z(R)]$. This implies that $R/Z(R)$ has at most $|A_1|$ elements of order pq. This gives that $|A_2| \leqslant |A_1|$; a contradiction. So, we can conclude that $|R : Z(R)| \neq p^2q$ for any two distinct primes p, q. □

In the following, we give some results regarding the existence of n-centraliser rings for some $n \in \mathbb{N}$.

Proposition 2.2.18. There exists a $(p^k + 2)$-centraliser ring for any prime p and $k \in \mathbb{N}$.

Proof. Let p be a prime and let $n \in \mathbb{N}$ with $n \geqslant 2$. Let $M(a_1, a_2, \cdots, a_n)$

be defined by $M(a_1, a_2, \cdots, a_n) = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$ for any $a_1, a_2, \cdots, a_n \in \mathbb{Z}_p$, where $M(a_1, a_2, \cdots, a_n)$ is a square matrix of order n. Consider the ring $R = \{M(a_1, a_2, \cdots, a_n) \mid a_1, a_2, \cdots, a_n \in \mathbb{Z}_p\}$. Let $M(a_1, a_2, \cdots, a_n) \in R$ and let $M(b_1, b_2, \cdots, b_n) \in C_R(M(a_1, a_2, \cdots, a_n))$. Therefore, we have $a_1 b_i = b_1 a_i$ for any $i \in \{2, \cdots, n\}$. Here, we consider the following three cases.

Case 1: $a_1 = a_2 = \cdots = a_n = 0$. It is clear that $C_R(M(a_1, a_2, \cdots, a_n)) = R$.

Case 2: $a_1 = 0$ and $a_u \neq 0$ for some $u \in \{2, \cdots, n\}$. Therefore, we have $b_1 = 0$ and $b_2, \cdots, b_n \in \mathbb{Z}_p$. So, we obtain $C_R(M(a_1, a_2, \cdots, a_n)) = \{M(0, b_2, \cdots, b_n) \mid b_2, \cdots, b_n \in \mathbb{Z}_p\}$.

Case 3: $a_1 \neq 0$. Hence, we have $b_i = a_1^{-1} a_i b_1$ for any $i \in \{2, \cdots, n\}$ and $b_1 \in \mathbb{Z}_p$. Let $l_i = a_1^{-1} a_i$ for any $i \in \{2, \cdots, n\}$. So, we obtain $C_R(M(a_1, a_2, \cdots, a_n)) = \{M(b_1, l_2 b_1, \cdots, l_n b_1) \mid b_1 \in \mathbb{Z}_p\}$.

By combining all the cases, we obtain $|\text{Cent}(R)| = 1 + 1 + p^{n-1} = p^{n-1} + 2$. Consequently, by letting $k = n - 1$, we obtain the desired result. □

Proposition 2.2.19. There exists a $(p^2 + p + 2)$-centraliser ring for any prime p.

Proof. Let p be a prime. Let $M(a_1, a_2, a_3)$ be defined by $M(a_1, a_2, a_3) = \begin{bmatrix} a_1 & a_2 \\ a_3 & 0 \end{bmatrix}$ for any $a_1, a_2, a_3 \in \mathbb{Z}_p$. Consider the ring $R = \{M(a_1, a_2, a_3) \mid a_1, a_2, a_3 \in \mathbb{Z}_p\}$, where the multiplication operation of R is defined as $M(a_1, a_2, a_3)M(b_1, b_2, b_3) = M(a_1 b_1 + a_2 b_3, a_1 b_2, a_3 b_1)$ for any $M(a_1, a_2, a_3), M(b_1, b_2, b_3) \in R$. Let $M(a_1,$

$a_2, a_3) \in R$ and let $M(b_1, b_2, b_3) \in C_R(M(a_1, a_2, a_3))$. Therefore, we have $a_1b_2 = b_1a_2$, $a_1b_3 = b_1a_3$ and $a_2b_3 = b_2a_3$. Here, we consider the following four cases.

Case 1: $a_1 = a_2 = a_3 = 0$. It is clear that $C_R(M(a_1, a_2, a_3)) = R$.

Case 2: $a_1 = 0$ and $a_2 \neq 0$. Hence, we have $b_1 = 0$, $b_3 = a_2^{-1}a_3b_2$ and $b_2 \in \mathbb{Z}_p$. Let $l = a_2^{-1}a_3$. Thus, we obtain $C_R(M(a_1, a_2, a_3)) = \{M(0, b_2, lb_2) \mid b_2 \in \mathbb{Z}_p\}$.

Case 3: $a_1 = a_2 = 0$ and $a_3 \neq 0$. So, we have $b_1 = b_2 = 0$ and $b_3 \in \mathbb{Z}_p$. Therefore, we obtain $C_R(M(a_1, a_2, a_3)) = \{M(0, 0, b_3) \mid b_3 \in \mathbb{Z}_p\}$.

Case 4: $a_1 \neq 0$. Hence, we have $b_2 = a_1^{-1}a_2b_1$, $b_3 = a_1^{-1}a_3b_1$ and $b_1 \in \mathbb{Z}_p$. Let $l_2 = a_1^{-1}a_2$ and $l_3 = a_1^{-1}a_3$. Thus, we obtain $C_R(M(a_1, a_2, a_3)) = \{M(b_1, l_2b_1, l_3b_1) \mid b_1 \in \mathbb{Z}_p\}$.

By combining all the cases, we obtain $|\mathrm{Cent}(R)| = 1 + p + 1 + p^2 = p^2 + p + 2$. □

Proposition 2.2.20. There exists a $(p^k + p + 3)$-centraliser ring for any prime p and $k \in \mathbb{N}$ with $k \geqslant 3$.

Proof. Let p be a prime and let $n \in \mathbb{N}$ with $n \geqslant 4$. Let $M(a_1, a_2, a_3, a_4, \cdots, a_n)$ be defined by $M(a_1, a_2, a_3, a_4, \cdots, a_n) = \begin{bmatrix} a_1 & a_2 & \cdots & 0 \\ a_3 & 0 & \cdots & 0 \\ a_4 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \cdots & 0 \end{bmatrix}$ for any $a_1, a_2, a_3, a_4, \cdots, a_n \in \mathbb{Z}_p$, where $M(a_1, a_2, a_3, a_4 \cdots, a_n)$ is a square matrix of order n.

Consider the ring $R = \{M(a_1, a_2, a_3, a_4 \cdots, a_n) \mid a_1, a_2, a_3, a_4 \cdots, a_n \in \mathbb{Z}_p\}$, where the multiplication operation of R is defined as $M(a_1, a_2, a_3, a_4 \cdots, a_n)$ $M(b_1, b_2, b_3, b_4 \cdots, b_n) = M(a_1b_1 + a_2b_3, a_1b_2, a_3b_1, a_4b_1, \cdots, a_nb_1)$ for any $M(a_1, a_2, a_3, a_4 \cdots, a_n), M(b_1, b_2, b_3, b_4 \cdots, b_n) \in R$. Let $M(a_1, a_2, a_3, a_4, \cdots, a_n) \in R$ and let $M(b_1, b_2, b_3, b_4, \cdots, b_n) \in C_R(M(a_1, a_2, a_3, a_4 \cdots, a_n))$. Therefore, we have $a_2b_3 = b_2a_3$ and $a_1b_i = b_1a_i$ for any $i \in \{2, 3, 4, \cdots, n\}$. Here, we consider the following five cases.

Case 1: $a_1 = a_2 = a_3 = a_4 = \cdots = a_n = 0$. It is clear that $C_R(M(a_1, a_2, a_3, a_4, \cdots, a_n)) = R$.

Case 2: $a_1 = 0$, $a_2 \neq 0$. Thus, we have $b_1 = 0$, $b_3 = a_2^{-1}a_3b_2$ and $b_2, b_4, \cdots, b_n \in \mathbb{Z}_p$. Let $l = a_2^{-1}a_3$. So, we obtain $C_R(M(a_1, a_2, a_3, a_4, \cdots, a_n))$ $= \{M(0, b_2, lb_2, b_4, \cdots, b_n) \mid b_2, b_4 \cdots, b_n \in \mathbb{Z}_p\}$.

Case 3: $a_1 = a_2 = 0$ and $a_3 \neq 0$. Hence, we have $b_1 = b_2 = 0$ and $b_3, b_4, \cdots, b_n \in \mathbb{Z}_p$. Therefore, we obtain $C_R(M(a_1, a_2, a_3, a_4, \cdots, a_n)) = \{M(0, 0, b_3, b_4, \cdots, b_n) \mid b_2, b_4 \cdots, b_n \in \mathbb{Z}_p\}$.

Case 4: $a_1 = a_2a_3 = 0$ and $a_u \neq 0$ for some $u \in \{4, \cdots, n\}$. Hence, we have $b_1 = 0$ and $b_2, b_3, b_4, \cdots, b_n \in \mathbb{Z}_p$. So, we obtain $C_R(M(a_1, a_2, a_3, a_4, \cdots, a_n)) = \{M(0, b_2, b_3, b_4, \cdots, b_n) \mid b_2, b_3, b_4 \cdots, b_n \in \mathbb{Z}_p\}$.

Case 5: $a_1 \neq 0$. So, we have $b_i = a_1^{-1}a_ib_1$ for any $i \in \{2, 3, \cdots, n\}$

and $b_1 \in \mathbb{Z}_p$. Let $l_i = a_1^{-1}a_i$ for any $i \in \{2, 3, 4, \cdots, n\}$. Hence, we obtain $C_R(M(a_1, a_2, a_3, a_4, \cdots, a_n)) = \{M(b_1, l_2b_1, l_3b_1, l_4b_1, \cdots, l_nb_1) \mid b_1 \in \mathbb{Z}_p\}$.

By combining all the cases, we obtain $|\text{Cent}(R)| = 1+p+1+1+p^{n-1} = p^{n-1}+p+3$. Consequently, by letting $k = n-1$, we obtain the desired result. □

Proposition 2.2.21. If R is a finite non-commutative ring with $|R : Z(R)| = p^3$, then $|\text{Cent}(R)| = p^2 + p + 2$ or $p^2 + 2$.

Proof. Let $k = |\text{Cent}(R)| - 1$. Let $C_R(r_1), C_R(r_2), \cdots, C_R(r_k)$ be k distinct proper centralisers of R, where $r_1, r_2, \cdots, r_k \in R - Z(R)$. By Lemma 2.2.16, $C_R(r)$ is commutative for any $r \in R - Z(R)$. Therefore, by Lemma 1.3.1(h), we have $C_R(r_i) \cap C_R(r_j) = Z(R)$ for any $i \in \{1, 2, \cdots, k\}$. From Lemma 1.3.1(a), $R = \bigcup_{i=1}^{k} C_R(r_i)$. By using the principle of inclusion-exclusion, we get $|R| = \sum_{i=1}^{k} |C_R(r_i)| + (1-k)|Z(R)|$ and so, $p^3|Z(R)| = \sum_{i=1}^{k} |C_R(r_i)| + (1-k)|Z(R)|$. We consider two cases in this proof.

Case 1: $|R : C_R(r_i)| = p^2$ for any $i \in \{1, 2, \cdots, k\}$. Then, we have $p^3|Z(R)| = k(p|Z(R)) + (1-k)|Z(R)|$, which yields that $k = \frac{p^3-1}{p-1} = p^2+p+1$. Consequently, we obtain $|\text{Cent}(R)| = p^2 + p + 2$.

Case 2: $|R : C_R(r_i)| = p$ for some $i \in \{1, 2, \cdots, k\}$. By Lemma 2.2.11, it follows that $|R : C_R(r_j)| = p^2$ for any $j \in \{1, 2, \cdots, k\} - \{i\}$. Then, we have $p^3|Z(R)| = p^2|Z(R)| + (k-1)(p|Z(R)) + (1-k)|Z(R)|$, which yields that $k = \frac{p^3-p^2}{p-1} + 1 = p^2 + 1$. Consequently, we obtain $|\text{Cent}(R)| = p^2 + 2$. □

The following proposition is a particular case of [A2] and Proposition 2.2.21.

Proposition 2.2.22. If R is a non-commutative ring with order p^3, then $|\mathrm{Cent}(R)| = p+2, p^2+p+2$ or p^2+2.

Proof. Since $Z(R) < R$, then $|Z(R)| = 1, p$ or p^2. If $R/Z(R)$ is cyclic, then R is commutative, which is a contradiction. So, we have $R/Z(R)$ is not cyclic. If $|Z(R)| = p^2$, then $|R : Z(R)| = p$, which follows that $R/Z(R) \cong \mathbb{Z}_p$. This shows that $R/Z(R)$ is cyclic, which is a contradiction. If $|Z(R)| = p$, then $|R : Z(R)| = p^2$, which implies that $R/Z(R) \cong \mathbb{Z}_p \times \mathbb{Z}_p$ as $R/Z(R)$ is not cyclic. Therefore, we obtain $|\mathrm{Cent}(R)| = p+2$ by [A2]. If $|Z(R)| = 1$, then $|R : Z(R)| = p^3$. So, by Proposition 2.2.21, we obtain $|\mathrm{Cent}(R)| = p^2+p+2$ or p^2+2. Consequently, we have $|\mathrm{Cent}(R)| = p+2, p^2+p+2$ or p^2+2. □

We conclude this section by giving a preliminary characterisation for all n-centraliser finite rings with $n \geqslant 6$.

Theorem 2.2.23. *If R is an n-centraliser finite ring with $n \geqslant 6$, then $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, or $|R : Z(R)| \geqslant 16$ with $|R : Z(R)|$ is not square-free and $|R : Z(R)| \neq p^2q$ for any two distinct primes p, q. Furthermore, if $n \neq 6$ and 8, then $R/Z(R) \ncong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Moreover, if $n-2$ is not prime, then $|R : Z(R)| \neq p^2$ for any prime p.*

Proof. If $|R : Z(R)|$ is square-free, then $R/Z(R) \cong \mathbb{Z}_{|R:Z(R)|}$. It follows that $R/Z(R)$ is cyclic, which is impossible because $R/Z(R)$ is not cyclic. If $|R : Z(R)| = 4$ or 9, then $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$ as $R/Z(R)$ is not cyclic. By [A2], we obtain $|\mathrm{Cent}(R)| = 4$ or 5, which leads to a contradiction. Let $\{x_1, x_2, \cdots, x_t\}$ be the maximal non-commuting set of R. Without loss of generality, we suppose that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_t$. If $|R : Z(R)| = 8$, then $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as $R/Z(R)$ is not cyclic. By Lemma 2.2.16 and Lemma 2.2.4, $|\mathrm{Cent}(R)| = t + 1$. Hence, by Lemma 2.2.14, $\gamma_2 = 4$. Therefore, $|C_R(x_1) : Z(R)| \leqslant 4$ and $|C_R(x_i) : Z(R)| = 2$ for any $i \in \{2, \cdots, t\}$. By Lemma 1.3.1(a), $R/Z(R) = \bigcup_{i=1}^{t} [C_R(x_i)/Z(R)]$. Hence, $R/Z(R)$ has at most 2 elements of order 4. Since $\mathbb{Z}_2 \times \mathbb{Z}_4$ has 4 elements of order 4, then $R/Z(R) \ncong \mathbb{Z}_2 \times \mathbb{Z}_4$. Consequently, by Lemma 2.2.17, we have $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, or $|R : Z(R)| \geqslant 16$ with $|R : Z(R)|$ is not square-free and $|R : Z(R)| \neq p^2 q$ for any two distinct primes p, q. By [A7], it follows that $R/Z(R) \ncong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ when $n \neq 6$ and 8. We next assume that $|R : Z(R)| = p^2$ for some prime p. Thus, $R/Z(R) \cong \mathbb{Z}_p \times \mathbb{Z}_p$ as $R/Z(R)$ is not cyclic. Therefore, we obtain $|\mathrm{Cent}(R)| = p + 2$ by [A2] and so, $n - 2 = p$, a contradiction is reached. □

2.3 6-Centraliser Finite Rings

In this section, we classify the structure for all 6-centraliser finite rings and compute their commuting probabilities.

Theorem 2.3.1. *Let R be a 6-centraliser finite ring. Let $X_1, X_2, \cdots, X_5$ be the 5 distinct proper centralisers of R with $|R : X_1| \leqslant |R : X_2| \leqslant \cdots \leqslant |R : X_5|$. Then the cardinality of the maximal non-commuting set of R is 5. Furthermore, R satisfies one of the following structures:*

(a) $|R : X_1| = 2$, $|R : X_i| = 4$ *for any* $i \in \{2, 3, 4, 5\}$, $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *and* $\text{Prob}(R) = \frac{7}{16}$.

(b) $|R : X_i| = 4$ *for any* $i \in \{1, 2, 3, 4, 5\}$, $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *and* $\text{Prob}(R) = \frac{19}{64}$.

Proof. Let $\{x_1, x_2, \cdots, x_t\}$ be the maximal non-commuting set of R. Without loss of generality, we assume that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_t$. By Lemma 1.3.1(a), we have $R = \bigcup_{i=1}^{t} C_R(x_i)$. From Lemma 1.3.1(d)-(g), we obtain $t = 5$. By Corollary 2.2.5, we have $C_R(x_i) \cap C_R(x_j) = Z(R)$ for any two distinct $i, j \in \{1, 2, 3, 4, 5\}$. By Lemma 2.2.14, $\gamma_2 \geqslant 4$. So, we obtain $\gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = 4$ by Lemma 2.2.1. By using the principle of inclusion-exclusion, we get $|R| = \sum_{i=1}^{5} |C_R(x_i)| - 4|Z(R)|$. Thus, we have $\gamma_1 = \frac{|R:Z(R)|}{4}$. By Lemma 2.2.11, it follows that $|R : Z(R)| \leqslant 4(4) = 16$. Therefore, we have $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $|R : Z(R)| = 16$ by Theorem 2.2.23. Suppose that $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\gamma_1 = 2$. By Corollary 2.2.5, it follows that for any $r_1, r_2 \in R - Z(R)$, either $C_R(r_1) = C_R(r_2)$ or $C_R(r_1) \cap C_R(r_2) = Z(R)$. Consequently, by (1.3), we obtain

$$\begin{aligned}
\text{Prob}(R) &= \frac{|Z(R)|}{|R|} + \frac{\sum\limits_{r \in R - Z(R)} |C_R(r)|}{|R|^2} \\
&= \frac{1}{8} + \frac{\left(\frac{|R|}{2} - \frac{|R|}{8}\right)\left(\frac{|R|}{2}\right) + 4\left(\frac{|R|}{4} - \frac{|R|}{8}\right)\left(\frac{|R|}{4}\right)}{|R|^2} \\
&= \frac{7}{16}.
\end{aligned}$$

Next, we assume that $|R : Z(R)| = 16$. Thus, $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_2 \times$

$\mathbb{Z}_2\times\mathbb{Z}_4, \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ or $\mathbb{Z}_4\times\mathbb{Z}_4$ as $R/Z(R)$ is not cyclic. Also, we have $\gamma_1 = 4$. Therefore, $|C_R(x_i) : Z(R)| = 4$ for any $i \in \{1, 2, 3, 4, 5\}$. This shows that $R/Z(R)$ has at least 5 elements of order 2. Since $\mathbb{Z}_2\times\mathbb{Z}_8$ and $\mathbb{Z}_4\times\mathbb{Z}_4$ have exactly 3 elements of order 2, then $R/Z(R) \ncong \mathbb{Z}_2\times\mathbb{Z}_8$ and $\mathbb{Z}_4\times\mathbb{Z}_4$. Here, we suppose that $R/Z(R) \cong \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4$. To simplify writing, we let $\overline{r} = r+Z(R)$ for any $r \in R$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ has 7 elements of order 2 and 8 elements of order 4, then $C_R(x_j)/Z(R) \cong \mathbb{Z}_2\times\mathbb{Z}_2$ for some $j \in \{1, 2, 3, 4, 5\}$ and $C_R(x_i)/Z(R) \cong \mathbb{Z}_4$ for any $i \in \{1, 2, 3, 4, 5\}-\{j\}$. Hence, we have $C_R(x_{k_1})/Z(R) \cong C_R(x_{k_2})/Z(R) \cong \mathbb{Z}_4$ for two distinct $k_1, k_2 \in \{1, 2, 3, 4, 5\} - \{j\}$. Thus, there exist some $a \in C_R(x_{k_1}) - Z(R), b \in C_R(x_{k_2}) - Z(R)$ such that $C_R(x_{k_1})/Z(R) = \{\overline{0}, \overline{a}, \overline{2a}, \overline{3a}\}$ and $C_R(x_{k_2})/Z(R) = \{\overline{0}, \overline{b}, \overline{2b}, \overline{3b}\}$. This implies that $R/Z(R) = \{\overline{ma + nb} \mid m, n \in \mathbb{Z}_4\} \cong \mathbb{Z}_4 \times \mathbb{Z}_4$, which is a contradiction. Consequently, $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Lastly, from (1.3), we obtain

$$\begin{aligned}
\text{Prob}(R) &= \frac{|Z(R)|}{|R|} + \frac{\sum\limits_{r\in R-Z(R)} |C_R(r)|}{|R|^2} \\
&= \frac{1}{16} + \frac{\left(|R| - \frac{|R|}{16}\right)\left(\frac{|R|}{4}\right)}{|R|^2} \\
&= \frac{19}{64}.
\end{aligned}$$

□

From [A7], we know that the converse of structure (a) in Theorem 2.3.1 is not necessarily true. Apart from this, the converse of structure (b) in Theorem 2.3.1 is also not necessarily true. For example, $R = \{\left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right] | \, a, b \in \mathbb{Z}_2\} \times \{\left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right] | \, a, b \in \mathbb{Z}_2\}$ is a 16-centraliser finite ring with $R/Z(R) \cong \mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$. In the following, we provide an example of a 6-centraliser finite ring, which is appeared in the proof of Proposition 2.2.18.

Example 2.3.2. $R = \left\{ \left[\begin{smallmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right] \middle| \, a, b, c \in \mathbb{Z}_2 \right\}$ is a 6-centraliser finite ring with

$$\begin{aligned}
R &= C_R\left(\left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]\right), \\
X_1 &= C_R\left(\left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]\right) = C_R\left(\left[\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]\right) = C_R\left(\left[\begin{smallmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]\right) \\
&= \left\{ \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right] \right\}, \\
X_2 &= C_R\left(\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]\right) = \left\{ \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right] \right\}, \\
X_3 &= C_R\left(\left[\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]\right) = \left\{ \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right] \right\}, \\
X_4 &= C_R\left(\left[\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]\right) = \left\{ \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right] \right\}, \\
X_5 &= C_R\left(\left[\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]\right) = \left\{ \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right] \right\}.
\end{aligned}$$

We note that $\left\{ \left[\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right] \right\}$ is a non-commuting set of R with cardinality 5. Also, we note that there does not exist a non-commuting set of R with cardinality 6. Thus, the cardinality of the maximal non-commuting set of R is 5. Besides that, we have $|R : X_1| = 2$, $|R : X_i| = 4$ for any $i \in \{2, 3, 4, 5\}$. Since $Z(R) = \left\{ \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right] \right\}$, then we have $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Lastly, from (1.3), we obtain

$$\begin{aligned}
\text{Prob}(R) &= \frac{|Z(R)|}{|R|} + \frac{\sum\limits_{r \in R - Z(R)} |C_R(r)|}{|R|^2} \\
&= \frac{1}{8} + \frac{3(4) + 4(2)}{8^2} \\
&= \frac{7}{16}.
\end{aligned}$$

2.4 7-Centraliser Finite Rings

In this section, we characterise all 7-centraliser finite rings and compute their commuting probabilities.

Theorem 2.4.1. *Let R be a 7-centraliser finite ring. Then the cardinality of the maximal non-commuting set of R is 6. Moreover, $R/Z(R) \cong \mathbb{Z}_5 \times \mathbb{Z}_5$ and* $\mathrm{Prob}(R) = \frac{29}{125}$.

Proof. Let $\{x_1, x_2, \cdots, x_t\}$ be the maximal non-commuting set of R. Without loss of generality, we suppose that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_t$. By Lemma 1.3.1(a), we have $R = \bigcup\limits_{i=1}^{t} C_R(x_i)$. In view of Lemma 1.3.1(d)-(g), we have $t = 5$ or 6. By Lemma 2.2.14, we have $\gamma_2 \geqslant 4$. From Theorem 2.2.23, we have $|R : Z(R)| \geqslant 16$ with $|R : Z(R)|$ is not square-free and $|R : Z(R)| \neq p^2 q$ for any two distinct primes p, q.

For $t = 5$, by Lemma 2.2.1, it follows that $\gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = 4$. In view of Lemma 2.2.4, there exists some $r \in R - Z(R)$ such that $C_R(r)$ is non-commutative. By Lemma 2.2.6, $C_R(r)$ contains $C_R(x_{l_1}), C_R(x_{l_2}), C_R(x_{l_3})$ for three distinct $l_1, l_2, l_3 \in \{1, 2, 3, 4, 5\}$. Thus, we have $R = C_R(r) \cup \left(\bigcup\limits_{i \in A} C_R(x_i) \right)$ for some $A \subset \{1, 2, 3, 4, 5\}$ with $|A| \leqslant 2$. Obviously, $|A| \neq 0$. If $|A| = 1$, then by Lemma 2.2.1, it follows that $\gamma_i = 1$ for some $i \in A$, which is impossible. Therefore, $|A| = 2$. From Lemma 2.2.1, it follows that $\gamma_i = 2$ for some $i \in A$. Since $\gamma_2 = 4$, then $i = 1$. By Corollary 2.2.13, it follows that $|R : Z(R)| \leqslant 2(4) = 8$, which leads to a contradiction. Consequently, we obtain $t = 6$.

By Corollary 2.2.5, we have $C_R(x_i) \cap C_R(x_j) = Z(R)$ for any two distinct $i, j \in \{1, 2, \cdots, 6\}$. By Lemma 2.2.1, we obtain $\gamma_2 \leqslant 5$. Suppose that $\gamma_2 = 4$, then $|R : Z(R)| \leqslant 4(4) = 16$ by Lemma 2.2.11. This yields that $|R : Z(R)| = 16$. By using the principle of inclusion-exclusion, we have $|R| = \sum_{i=1}^{6} |C_R(x_i)| - 5|Z(R)|$, which yields that $\sum_{i=1}^{6} \frac{16}{\gamma_i} = 21$. This contradicts with the fact that $\sum_{i=1}^{6} \frac{16}{\gamma_i}$ is even. So, we have $\gamma_2 = 5$. By Lemma 2.2.11, it follows that $|R : Z(R)| \leqslant 5(5) = 25$, which implies that $|R : Z(R)| = 25$. Consequently, $R/Z(R) \cong \mathbb{Z}_5 \times \mathbb{Z}_5$ as $R/Z(R)$ is not cyclic. Lastly, by (1.3), we obtain

$$\begin{aligned}
\text{Prob}(R) &= \frac{|Z(R)|}{|R|} + \frac{\sum\limits_{r \in R - Z(R)} |C_R(r)|}{|R|^2} \\
&= \frac{1}{25} + \frac{\left(|R| - \frac{|R|}{25}\right)\left(\frac{|R|}{5}\right)}{|R|^2} \\
&= \frac{29}{125}.
\end{aligned}$$

□

As an immediate consequence of Theorem 2.4.1 and [A2], we obtain a complete characterisation for all 7-centraliser finite rings.

Theorem 2.4.2. *For any finite ring R, R is a 7-centraliser finite ring if and only if $R/Z(R) \cong \mathbb{Z}_5 \times \mathbb{Z}_5$.*

In the following, we provide an example of a 7-centraliser finite ring, which is appeared in the proof of Proposition 2.2.18.

Example 2.4.3. $R = \left\{ \left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right] \middle| \, a, b \in \mathbb{Z}_5 \right\}$ is a 7-centraliser finite ring with

$$C_R\left(\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]\right) = R,$$

$$C_R\left(\left[\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right]\right) = \left\{\left[\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right] \middle| \, x \in \mathbb{Z}_5\right\},$$

$$C_R\left(\left[\begin{smallmatrix} a & la \\ 0 & 0 \end{smallmatrix}\right]\right) = \left\{\left[\begin{smallmatrix} x & lx \\ 0 & 0 \end{smallmatrix}\right] \middle| \, x \in \mathbb{Z}_5\right\}$$

for any $a \in \mathbb{Z}_5$ with $a \neq 0$ and $l \in \mathbb{Z}_5$. We note that $\{\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 2 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 3 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 4 \\ 0 & 0 \end{smallmatrix}\right]\}$ is a non-commuting set of R with cardinality 6. Also, we note that there does not exist a non-commuting set of R with cardinality 7. Thus, the cardinality of the maximal non-commuting set of R is 6. Since $Z(R) = \{\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]\}$, then we have $R/Z(R) \cong \mathbb{Z}_5 \times \mathbb{Z}_5$. Lastly, from (1.3), we obtain

$$\begin{aligned}
\text{Prob}(R) &= \frac{|Z(R)|}{|R|} + \frac{\sum\limits_{r \in R - Z(R)} |C_R(r)|}{|R|^2} \\
&= \frac{1}{25} + \frac{24(5)}{25^2} \\
&= \frac{29}{125}.
\end{aligned}$$

2.5 8-Centraliser Finite Rings

In this section, we investigate the structure for all 8-centraliser finite rings and compute their commuting probabilities.

Theorem 2.5.1. *Let R be an 8-centraliser finite ring. Then the cardinality of the maximal non-commuting set of R is 7. Further, $|R : C_R(r)| = 4$ for any $r \in R - Z(R)$, $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\text{Prob}(R) = \frac{11}{32}$.*

Proof. Let $\{x_1, x_2, \cdots, x_t\}$ be the maximal non-commuting set of R. Without loss of generality, we assume that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_t$. By Lemma 1.3.1(a), we have $R = \bigcup\limits_{i=1}^{t} C_R(x_i)$. In view of Lemma 1.3.1(d)-(g), we have $t = 5, 6$ or 7. By Lemma 2.2.14, we have $\gamma_2 \geqslant 4$. By Theorem 2.2.23, we have $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, or $|R : Z(R)| \geqslant 16$ with $|R : Z(R)|$ is not square-free, $|R : Z(R)| \neq p^2 q$ for any two distinct primes p, q, and $|R : Z(R)| \neq p^2$ for any prime p.

First, we suppose that $t = 5$ (respectively, $t = 6$). By Lemma 2.2.4, there exists some $r \in R - Z(R)$ such that $C_R(r)$ is non-commutative. By Lemma 2.2.7 (respectively, Lemma 2.2.6), $C_R(r)$ contains $C_R(x_{l_1}), C_R(x_{l_2})$ for two distinct $l_1, l_2 \in \{1, 2, \cdots, 5\}$ (respectively, $C_R(x_{l_1}), C_R(x_{l_2}), C_R(x_{l_3})$ for three distinct $l_1, l_2, l_3 \in \{1, 2, \cdots, 6\}$). Therefore, we have $R = C_R(r) \cup \left(\underset{i \in A}{\cup} C_R(x_i) \right)$ for some $A \subset \{1, 2, \cdots, 5\}$ (respectively, $A \subset \{1, 2, \cdots, 6\}$) with $|A| \leqslant 3$. Clearly, $|A| \neq 0$. If $|A| = 1$, then by Lemma 2.2.1, it follows that $\gamma_i = 1$ for some $i \in A$, which is impossible. So, $|A| = 2$ or 3. In view of Lemma 2.2.1, $\gamma_i = 2$ or 3 for some $i \in A$. Since $\gamma_2 \geqslant 4$, then $i = 1$. Hence, we obtain $|R : Z(R)| \leqslant 3\gamma_2$ by Corollary 2.2.13. From Lemma 2.2.1, $\gamma_2 \leqslant 5$. So, $|R : Z(R)| \leqslant 15$. This yields that $|R : Z(R)| = 8$, which contradicts with Lemma 2.2.16. Consequently, we obtain $t = 7$.

By Corollary 2.2.5, we have $C_R(x_i) \cap C_R(x_j) = Z(R)$ for any two distinct $i, j \in \{1, 2, \cdots, 7\}$. By Lemma 2.2.11, we have $|R : Z(R)| \leqslant \gamma_2^2$ and $|C_R(x_1) : Z(R)| \leqslant \gamma_2$. By Lemma 2.2.1, we obtain $\gamma_2 \leqslant 6$. Now, we assume that $\gamma_2 = 6$. Then, $|R : Z(R)| \leqslant 36$. If $|R : Z(R)| = 24$, then $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_{12}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ as $R/Z(R)$ is not cyclic. Therefore, $|C_R(x_1) : Z(R)| \leqslant 6$ and $|C_R(x_i) : Z(R)| \leqslant 4$ for any $i \in \{2, 3, \cdots, 7\}$. This implies that $R/Z(R)$ has at most 2 elements of order 6. Also, there does not exist any element of order 12 in $R/Z(R)$. We have reached a contradiction as $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ has 14 elements of order 6 and $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ has an element of order 12. If $|R : Z(R)| = 36$, then $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_{18}, \mathbb{Z}_3 \times \mathbb{Z}_{12}$ or $\mathbb{Z}_6 \times \mathbb{Z}_6$ as $R/Z(R)$ is not cyclic. Hence, $|C_R(x_i) : Z(R)| \leqslant 6$ for any $i \in \{1, 2, \cdots, 7\}$. This shows that $R/Z(R)$ has at most 14 elements of order 6. Also, there does not exist any element of order 12 and order 18 in $R/Z(R)$. This contradicts with the fact that $\mathbb{Z}_6 \times \mathbb{Z}_6$ has 24 elements of order 6, $\mathbb{Z}_3 \times \mathbb{Z}_{12}$ has an element of order 12 and $\mathbb{Z}_2 \times \mathbb{Z}_{18}$ has an element of order 18. Next, we suppose that $\gamma_2 = 5$. Thus, $|R : Z(R)| \leqslant 25$, which is a contradiction. Consequently, $\gamma_2 = 4$. Therefore, $|R : Z(R)| \leqslant 16$. By using the principle of inclusion-exclusion, we obtain $|R| = \sum_{i=1}^{7} |C_R(x_i)| - 6|Z(R)|$, which gives that $\sum_{i=1}^{7} \frac{|R:Z(R)|}{\gamma_i} = |R : Z(R)| + 6$. If $|R : Z(R)| = 16$, then $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4 \times \mathbb{Z}_4$ as $R/Z(R)$ is not cyclic. By Lemma 2.2.11, we obtain $\gamma_1 = 4$. Since $\sum_{i=3}^{7} \frac{16}{\gamma_i} = 14$, then it can be easily seen that $\gamma_3 = \gamma_4 = 4$ and $\gamma_5 = \gamma_6 = \gamma_7 = 8$. Therefore, $|C_R(x_i) : Z(R)| = 4$ for any $i \in \{1, 2, 3, 4\}$ and $|C_R(x_i) : Z(R)| = 2$ for any $i \in \{5, 6, 7\}$. This shows that $R/Z(R)$ has at least 7 elements of order 2. Since $\mathbb{Z}_2 \times \mathbb{Z}_8$ and $\mathbb{Z}_4 \times \mathbb{Z}_4$ have exactly 3 elements of order 2, then $R/Z(R) \ncong \mathbb{Z}_2 \times \mathbb{Z}_8$ and $\mathbb{Z}_4 \times \mathbb{Z}_4$. For the sake of simplicity, we let $\overline{r} = r + Z(R)$ for any $r \in R$. Suppose that $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ has 7 elements of order 2 and 8 elements of order 4, then $C_R(x_i)/Z(R) \cong \mathbb{Z}_4$ for any $i \in \{1, 2, 3, 4\}$ and $C_R(x_j)/Z(R) \cong \mathbb{Z}_2$ for any $j \in \{5, 6, 7\}$. Thus, there exist some $a \in C_R(x_1) - Z(R), b \in C_R(x_2) - Z(R)$ such that $C_R(x_1)/Z(R) = \{\overline{0}, \overline{a}, \overline{2a}, \overline{3a}\}$ and $C_R(x_2)/Z(R) = \{\overline{0}, \overline{b}, \overline{2b}, \overline{3b}\}$.

This implies that $R/Z(R) = \{\overline{ma+nb} \mid m, n \in \mathbb{Z}_4\} \cong \mathbb{Z}_4 \times \mathbb{Z}_4$, which is a contradiction. Consequently, $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Here, we let $A = \{C_R(x_i)/Z(R) \mid i = 1, 2, \cdots, 7\}$. Without loss of generality, $C_R(x_1)/Z(R)$ and $C_R(x_2)/Z(R)$ can be written as $B_1 = C_R(x_1)/Z(R) = \{0, \overline{x_1}, \overline{a}, \overline{x_1+a}\}$ and $B_2 = C_R(x_2)/Z(R) = \{0, \overline{x_2}, \overline{b}, \overline{x_2+b}\}$ for some $a \in C_R(x_1) - Z(R), b \in C_R(x_2) - Z(R)$. Hence, we note that there have 6 possibilities for A, that is,

$$A_1 = \{B_1, B_2, \{\overline{0}, \overline{x_1+x_2}, \overline{a+b}, \overline{x_1+x_2+a+b}\}, \{\overline{0}, \overline{x_1+b}, \overline{x_2+a+b}, \overline{x_1+x_2+a}\}, \{\overline{0}, \overline{x_1+x_2+b}\}, \{\overline{0}, \overline{x_2+a}\}, \{\overline{0}, \overline{x_1+a+b}\}\},$$

$$A_2 = \{B_1, B_2, \{\overline{0}, \overline{x_1+x_2+b}, \overline{x_2+a}, \overline{x_1+a+b}\}, \{\overline{0}, \overline{x_1+x_2}, \overline{a+b}, \overline{x_1+x_2+a+b}\}, \{\overline{0}, \overline{x_1+b}\}, \{\overline{0}, \overline{x_2+a+b}\}, \{\overline{0}, \overline{x_1+x_2+a}\}\},$$

$$A_3 = \{B_1, B_2, \{\overline{0}, \overline{x_1+x_2+b}, \overline{x_2+a}, \overline{x_1+a+b}\}, \{\overline{0}, \overline{x_1+b}, \overline{x_2+a+b}, \overline{x_1+x_2+a}\}, \{\overline{0}, \overline{x_1+x_2}\}, \{\overline{0}, \overline{a+b}\}, \{\overline{0}, \overline{x_1+x_2+a+b}\}\},$$

$$A_4 = \{B_1, B_2, \{\overline{0}, \overline{x_1+x_2}, \overline{x_2+a+b}, \overline{x_1+a+b}\}, \{\overline{0}, \overline{x_1+b}, \overline{x_2+a}, \overline{x_1+x_2+a+b}\}, \{\overline{0}, \overline{x_1+x_2+b}\}, \{\overline{0}, \overline{a+b}\}, \{\overline{0}, \overline{x_1+x_2+a}\}\},$$

$$A_5 = \{B_1, B_2, \{\overline{0}, \overline{x_1+x_2+b}, \overline{a+b}, \overline{x_1+x_2+a}\}, \{\overline{0}, \overline{x_1+x_2}, \overline{x_2+a+b}, \overline{x_1+a+b}\}, \{\overline{0}, \overline{x_1+b}\}, \{\overline{0}, \overline{x_2+a}\}, \{\overline{0}, \overline{x_1+x_2+a+b}\}\},$$

$$A_6 = \{B_1, B_2, \{\overline{0}, \overline{x_1+x_2+b}, \overline{a+b}, \overline{x_1+x_2+a}\}, \{\overline{0}, \overline{x_1+b}, \overline{x_2+a}, \overline{x_1+x_2+a+b}\}, \{\overline{0}, \overline{x_1+x_2}\}, \{\overline{0}, \overline{x_2+a+b}\}, \{\overline{0}, \overline{x_1+a+b}\}\}.$$

By Corollary 2.2.5, it follows that for any $r_1, r_2 \in R - Z(R)$, either $C_R(r_1) = C_R(r_2)$ or $C_R(r_1) \cap C_R(r_2) = Z(R)$. Therefore, it can be easily seen that for any $u \in C_R(x_i) - Z(R), v \in C_R(x_j) - Z(R)$, if $i, j \in \{1, 2, \cdots, 7\}$ with $i \neq j$, then $uv \neq vu$. On the other hand, by Lemma 2.2.4, we have $C_R(x_i)$ is commutative for any $i \in \{1, 2, \cdots, 7\}$. If $A = A_1$ or A_3, then

$$(x_1 + b)(x_1 + x_2 + a) = (x_1 + x_2 + a)(x_1 + b)$$
$$\Rightarrow (x_1x_2 - x_2x_1) + (bx_1 - x_1b) + (ba - ab) = 0. \tag{2.1}$$

Simultaneously, when $A = A_1$ (respectively, $A = A_3$), we have

$$(x_1 + x_2)(a + b) = (a + b)(x_1 + x_2)$$
$$\Rightarrow (x_1b - bx_1) + (x_2a - ax_2) = 0 \tag{2.2}$$

and

$$(x_1 + x_2 + b)(x_2 + a) \neq (x_2 + a)(x_1 + x_2 + b)$$
$$\Rightarrow (x_1x_2 - x_2x_1) + (x_2a - ax_2) + (ba - ab) \neq 0 \tag{2.3}$$

(respectively,

$$(x_1 + x_2)(a + b) \neq (a + b)(x_1 + x_2)$$
$$\Rightarrow (x_1b - bx_1) + (x_2a - ax_2) \neq 0 \tag{2.4}$$

and

$$(x_1 + x_2 + b)(x_2 + a) = (x_2 + a)(x_1 + x_2 + b)$$
$$\Rightarrow (x_1x_2 - x_2x_1) + (x_2a - ax_2) + (ba - ab) = 0). \tag{2.5}$$

By taking $(2.1) + (2.2) - (2.3)$ (respectively, $(2.1) + (2.4) - (2.5)$), it shows that

$0 \neq 0$; a contradiction. If $A = A_2$, then

$$(x_2 + a)(x_1 + a + b) = (x_1 + a + b)(x_2 + a)$$
$$\Rightarrow (x_2x_1 - x_1x_2) + (x_2a - ax_2) + (ab - ba) = 0, \tag{2.6}$$

$$(x_1 + x_2)(a + b) = (a + b)(x_1 + x_2)$$
$$\Rightarrow (x_1b - bx_1) + (x_2a - ax_2) = 0 \tag{2.7}$$

and

$$(x_1 + b)(x_2 + a + b) \neq (x_2 + a + b)(x_1 + b)$$
$$\Rightarrow (x_1x_2 - x_2x_1) + (x_1b - bx_1) + (ba - ab) \neq 0. \tag{2.8}$$

By taking $(2.6) - (2.7) + (2.8)$, it shows that $0 \neq 0$; a contradiction. If $A = A_4$ or A_5, then

$$(x_1 + x_2)(x_1 + a + b) = (x_1 + a + b)(x_1 + x_2)$$
$$\Rightarrow (x_1b - bx_1) + (x_2x_1 - x_1x_2) + (x_2a - ax_2) = 0. \tag{2.9}$$

Simultaneously, when $A = A_4$ (respectively, $A = A_5$), we have

$$(x_1 + b)(x_2 + a) = (x_2 + a)(x_1 + b)$$
$$\Rightarrow (x_1x_2 - x_2x_1) + (ba - ab) = 0 \tag{2.10}$$

and

$$(x_1 + x_2 + b)(a + b) \neq (a + b)(x_1 + x_2 + b)$$
$$\Rightarrow (x_1 b - b x_1) + (x_2 a - a x_2) + (ba - ab) \neq 0 \tag{2.11}$$

(respectively,

$$(x_1 + b)(x_2 + a) \neq (x_2 + a)(x_1 + b)$$
$$\Rightarrow (x_1 x_2 - x_2 x_1) + (ba - ab) \neq 0 \tag{2.12}$$

and

$$(x_1 + x_2 + b)(a + b) = (a + b)(x_1 + x_2 + b)$$
$$\Rightarrow (x_1 b - b x_1) + (x_2 a - a x_2) + (ba - ab) = 0). \tag{2.13}$$

By taking $(2.9) + (2.10) - (2.11)$ (respectively, $(2.9) + (2.12) - (2.13)$), it shows that $0 \neq 0$; a contradiction. If $A = A_6$, then

$$(x_1 + x_2 + b)(a + b) = (a + b)(x_1 + x_2 + b)$$
$$\Rightarrow (x_1 b - b x_1) + (x_2 a - a x_2) + (ba - ab) = 0, \tag{2.14}$$

$$(x_1 + b)(x_2 + a) = (x_2 + a)(x_1 + b)$$
$$\Rightarrow (x_1 x_2 - x_2 x_1) + (ba - ab) = 0 \tag{2.15}$$

and

$$(x_1 + x_2)(x_1 + a + b) \neq (x_1 + a + b)(x_1 + x_2)$$

$$\Rightarrow (x_1 b - b x_1) + (x_2 x_1 - x_1 x_2) + (x_2 a - a x_2) \neq 0. \qquad (2.16)$$

By taking $(2.14) - (2.15) - (2.16)$, it shows that $0 \neq 0$; a contradiction. So, we can conclude that $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Thus, we have $\gamma_i = 4$ for any $i \in \{2, 3, \cdots, 7\}$. Also, we have $\gamma_1 = 4$. Lastly, by (1.3), we obtain

$$\begin{aligned}
\text{Prob}(R) &= \frac{|Z(R)|}{|R|} + \frac{\sum\limits_{r \in R - Z(R)} |C_R(r)|}{|R|^2} \\
&= \frac{1}{8} + \frac{\left(|R| - \frac{|R|}{8}\right)\left(\frac{|R|}{4}\right)}{|R|^2} \\
&= \frac{11}{32}.
\end{aligned}$$

This completes the proof. □

Note that [A7] has shown that the converse of Theorem 2.5.1 is not necessarily true. In the following, we provide an example of an 8-centraliser finite ring, which is appeared in the proof of Proposition 2.2.19.

Example 2.5.2. Consider the ring $R = \{\left[\begin{smallmatrix} a & b \\ c & 0 \end{smallmatrix}\right] | \, a, b, c \in \mathbb{Z}_2\}$, where the multiplication operation of R is defined as $\left[\begin{smallmatrix} a & b \\ c & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} x & y \\ z & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} ax+bz & ay \\ cx & 0 \end{smallmatrix}\right]$ for any $\left[\begin{smallmatrix} a & b \\ c & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} x & y \\ z & 0 \end{smallmatrix}\right] \in R$. By the proof of Proposition 2.2.19, R is an 8-centraliser finite ring with

$$C_R\left(\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]\right) = R,$$

$$C_R\left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]\right) = \left\{\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]\right\},$$

$$C_R\left(\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]\right)=\left\{\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right],\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right]\right\},$$

$$C_R\left(\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]\right)=\left\{\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right],\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right]\right\},$$

$$C_R\left(\left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right]\right)=\left\{\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right],\left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right]\right\},$$

$$C_R\left(\left[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right]\right)=\left\{\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right],\left[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right]\right\},$$

$$C_R\left(\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]\right)=\left\{\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right],\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]\right\},$$

$$C_R\left(\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right]\right)=\left\{\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right],\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right]\right\}.$$

We notice that $\left\{\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right],\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right],\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right],\left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right],\left[\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right],\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right],\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right]\right\}$ is a maximal non-commuting set of R. Thus, the cardinality of the maximal non-commuting set of R is 7. Besides that, we have $|R : C_R(r)| = 4$ for any $r \in R - Z(R)$. Since $Z(R) = \{\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]\}$, then we have $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Lastly, from (1.3), we obtain

$$\begin{aligned}\text{Prob}(R) &= \frac{|Z(R)|}{|R|} + \frac{\sum\limits_{r\in R-Z(R)} |C_R(r)|}{|R|^2} \\ &= \frac{1}{8} + \frac{7(2)}{8^2} \\ &= \frac{11}{32}.\end{aligned}$$

2.6 9-Centraliser Finite Rings

In this section, we characterise all 9-centraliser finite rings and compute their commuting probabilities. To this end, we apply similar techniques as in Qu and Chen (2010) to construct the main results in this section.

Lemma 2.6.1. Let $\{x_1, x_2, \cdots, x_5\}$ be the maximal non-commuting set of R.

Let $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_5$. If $|R : Z(R)| = 16$ and $\gamma_1 = 4$, then $|\mathrm{Cent}(R)| \neq 9, 10$ and 11.

Proof. Suppose to the contrary that $|\mathrm{Cent}(R)| = 9, 10$ or 11. By Lemma 1.3.1(a), we have $R = \bigcup_{i=1}^{5} C_R(x_i)$. By Lemma 2.2.1, we obtain $\gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = 4$. Since $R/Z(R) = \bigcup_{i=1}^{5} [C_R(x_i)/Z(R)]$, then it can be easily seen that $C_R(x_i)/Z(R) \cap C_R(x_j)/Z(R) = \{Z(R)\}$ for any two distinct $i, j \in \{1, 2, \cdots, 5\}$. From Lemma 2.2.15, we have $C_R(x_i)$ is commutative for any $i \in \{1, 2, \cdots, 5\}$. For the sake of simplicity, we let $\overline{r} = r + Z(R)$ for any $r \in R$ and let $\overline{S} = S/Z(R)$ for any $S \leqslant R$.

By Lemma 2.2.4, there exists some $a_1 \in R - Z(R)$ such that $C_R(a_1)$ is non-commutative with $|\overline{C_R(a_1)}| = 8$. Without loss of generality, we let $a_1 \in C_R(x_1)$ and let $A = \overline{C_R(a_1)} - \overline{C_R(x_1)} = \{\overline{a_2}, \overline{a_3}, \overline{a_4}, \overline{a_5}\}$, where $a_2, a_3, a_4, a_5 \in R - Z(R)$. Now, we claim that $|\overline{C_R(x_i)} \cap A| = 1$ for any $i \in \{2, 3, 4, 5\}$. Suppose to the contrary that $|\overline{C_R(x_i)} \cap A| \geqslant 2$ for some $i \in \{2, 3, 4, 5\}$. Without loss of generality, we let $\overline{a_2}, \overline{a_3} \in \overline{C_R(x_i)}$. If $|C_R(a_2) \cap C_R(a_3)| = 4|Z(R)|$, then $a_1 \in C_R(x_1) \cap C_R(a_2) \cap C_R(a_3) = C_R(x_1) \cap C_R(x_i) = Z(R)$, which is a contradiction. If $|C_R(a_2) \cap C_R(a_3)| = 8|Z(R)|$, then $C_R(a_2) = C_R(a_3)$ with $|\overline{C_R(a_2)}| = 8$. Since $C_R(a_2)$ is non-commutative and $\overline{0}, \overline{a_2}, \overline{a_3} \in \overline{Z(C_R(a_2))}$, then $|\overline{Z(C_R(a_2))}| = 4$. It follows that $|C_R(a_2) : Z(C_R(a_2))| = 2$, which gives that $C_R(a_2)/Z(C_R(a_2))$ is cyclic. This leads to $C_R(a_2)$ is commutative; a contradiction. So, $|\overline{C_R(x_i)} \cap A| = 1$ for any $i \in \{2, 3, 4, 5\}$, as claimed. Without loss of generality, we let $a_i \in C_R(x_i)$ for any $i \in \{2, 3, 4, 5\}$. Recall that, $C_R(x_1) < C_R(a_1)$. For any $i \in \{2, 3, 4, 5\}$, since $a_1 \notin C_R(x_i)$ but

$a_1 \in C_R(a_i)$, then $C_R(x_i) < C_R(a_i)$. For any two distinct $i, j \in \{1, 2, \cdots, 5\}$, since $x_i \in C_R(a_i)$ but $x_i \notin C_R(a_j)$, then $C_R(a_i) \neq C_R(a_j)$. Consequently, we obtain $|\mathrm{Cent}(R)| \geqslant 1 + 5 + 5 = 11$, which gives that $|\mathrm{Cent}(R)| = 11$. We claim that $C_R(x_i + a_i) = C_R(x_i)$ for any $i \in \{3, 4, 5\}$. Let $i \in \{3, 4, 5\}$. Since $x_i + a_i \in C_R(x_i)$, then $C_R(x_i) \leqslant C_R(x_i + a_i)$. Thus, $C_R(x_i + a_i) \neq C_R(x_j), C_R(a_j)$ for any $j \in \{1, 2, \cdots, 5\} - \{i\}$. Since $a_1 \notin C_R(x_i + a_i)$ but $a_1 \in R, C_R(a_i)$, then $C_R(x_i + a_i) \neq R, C_R(a_i)$. Since $|\mathrm{Cent}(R)| = 11$, then we obtain $C_R(x_i + a_i) = C_R(x_i)$, as claimed. This shows that $a_2 \notin C_R(x_i + a_i)$ for any $i \in \{3, 4, 5\}$. So, we have $\overline{C_R(a_2)} = \{\overline{0}, \overline{x_2}, \overline{a_2}, \overline{x_2 + a_2}, \overline{a_1}, \overline{a_3}, \overline{a_4}, \overline{a_5}\}$. This gives that $|\overline{C_R(a_1)} \cap \overline{C_R(a_2)}| = 6$, which contradicts the fact that $|\overline{C_R(a_1)} \cap \overline{C_R(a_2)}|$ is divide $|\overline{R}|$. Consequently, $|\mathrm{Cent}(R)| \neq 9, 10$ and 11. $\square$

Lemma 2.6.2. Let t be the cardinality of the maximal non-commuting set of a finite ring R. If R is a 9-centraliser finite ring, then $t \neq 5$.

Proof. Assume that $t = 5$. Let $\{x_1, x_2, x_3, x_4, x_5\}$ be the maximal non-commuting set of R. Without loss of generality, we assume that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \gamma_3 \leqslant \gamma_4 \leqslant \gamma_5$. By Lemma 1.3.1(b) and (c), we have $\{C_R(x_i) \mid i = 1, 2, 3, 4, 5\}$ is an irredundant cover of R with intersection $Z(R)$. Thus, $|R : Z(R)| \leqslant f(5) = 16$ and therefore, by Theorem 2.2.23, we obtain $|R : Z(R)| = 16$. By Lemma 2.2.1, we have $\gamma_2 \leqslant 4$. In view of Lemma 2.6.1, it follows that $\gamma_1 = 2$. If $C_R(x_1)$ is commutative, then by Lemma 2.2.12, we obtain $|R : Z(R)| \leqslant 2(4) = 8$, which is impossible. Consequently, $C_R(x_1)$ is non-commutative. Let $C_R(a_1), C_R(a_2), C_R(a_3)$ be three distinct proper centralisers of R that are different from $C_R(x_i)$ for any $i \in \{1, 2, 3, 4, 5\}$. Assume that $\gamma_2 = 2$. From Lemma 2.2.8(b), $C_R(x_2)$ is commutative. Therefore, by Lemma

2.2.12, we obtain $|R : Z(R)| \leqslant 2(2) = 4$; a contradiction. So, $\gamma_2 = 4$. Hence, it follows from Lemma 2.2.1 that $\gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = 4$. From Lemma 2.2.15, we have $C_R(x_i)$ is commutative for any $i \in \{2, 3, 4, 5\}$. We continue the proof by considering three cases.

Case 1: $C_R(a_1), C_R(a_2), C_R(a_3)$ are commutative. Now, we claim that $C_R(x_1) \cap C_R(x_2) = Z(R)$. If $w \in C_R(x_1) \cap C_R(x_2)$, then $x_1, x_2 \in C_R(w)$. Obviously, $C_R(w) \neq C_R(x_i)$ for any $i \in \{1, 2, 3, 4, 5\}$. If $C_R(w) = C_R(a_i)$ for some $i \in \{1, 2, 3\}$, then $x_1x_2 = x_2x_1$ as $C_R(a_i)$ is commutative; a contradiction. Hence, $C_R(w) = R$ and it follows that $w \in Z(R)$. This leads to $C_R(x_1) \cap C_R(x_2) \leqslant Z(R)$. On the other hand, it is clear that $Z(R) \leqslant C_R(x_1) \cap C_R(x_2)$. It follows that $C_R(x_1) \cap C_R(x_2) = Z(R)$. Hence, $|R : Z(R)| \leqslant 2(4) = 8$ by Lemma 2.2.11, which is a contradiction.

Case 2: Some of the $C_R(a_1), C_R(a_2), C_R(a_3)$ are commutative but not all of them. Without loss of generality, we let $P, Q \in \{C_R(a_1), C_R(a_2), C_R(a_3)\}$ with P is commutative and Q is non-commutative. By Lemma 2.2.8(c), $C_R(x_1) \in \{C_R(q_1), C_R(q_2), C_R(q_3)\} \subseteq \{C_R(x_i) \mid i = 1, 2, 3, 4, 5\}$, where $\{q_1, q_2, q_3\}$ is a maximal non-commuting set of Q. Thus, there exists some $j \in \{2, 3, 4, 5\}$ such that $C_R(x_j) \in \{C_R(x_i) \mid i = 1, 2, 3, 4, 5\} - \{C_R(q_1), C_R(q_2), C_R(q_3)\}$. Here, we claim that $C_R(x_1) \cap C_R(x_j) = Z(R)$. If $w \in C_R(x_1) \cap C_R(x_j)$, then $x_1, x_j \in C_R(w)$. Clearly, $C_R(w) \neq C_R(x_i)$ for any $i \in \{1, 2, 3, 4, 5\}$. If $C_R(w) = P$, then $x_1x_j = x_jx_1$ as P is commutative; a contradiction. If $C_R(w) = Q$, then $a_k \in C_R(q_1) \cap C_R(q_2) \cap C_R(q_3) \cap C_R(x_j)$ for some $k \in \{1, 2, 3\}$ and hence,

$a_k \in \bigcap_{i=1, i\neq l}^{5} C_R(x_i)$ for some $l \in \{2, 3, 4, 5\} - \{j\}$. It follows by Lemma 2.2.2 that $a_k \in Z(R)$, which is a contradiction. So, $C_R(w) = R$, which yields that $w \in Z(R)$. This shows that $C_R(x_1) \cap C_R(x_j) \leqslant Z(R)$. On the other hand, it is clear that $Z(R) \leqslant C_R(x_1) \cap C_R(x_j)$. Therefore, $C_R(x_1) \cap C_R(x_j) = Z(R)$. By Lemma 2.2.11, it follows that $|R : Z(R)| \leqslant 2(4) = 8$, which is a contradiction.

Case 3: $C_R(a_1), C_R(a_2), C_R(a_3)$ are non-commutative. By Lemma 2.2.8(c), for any $i \in \{1, 2, 3\}, C_R(x_1) \in A_i = \{C_R(u_{i_1}), C_R(u_{i_2}), C_R(u_{i_3})\} \subseteq \{C_R(x_j) \mid j = 1, 2, 3, 4, 5\}$, where $\{u_{i_1}, u_{i_2}, u_{i_3}\}$ is a maximal non-commuting set of $C_R(a_i)$. Thus, there exists some $k \in \{2, 3, 4, 5\}$ such that $C_R(x_k) \in A_i \cap A_j$ for two distinct $i, j \in \{1, 2, 3\}$. This implies that $a_i, a_j \in C_R(x_k)$. Since $C_R(x_k)$ is commutative, then $a_i a_j = a_j a_i$, which contradicts with Lemma 2.2.8(a). Therefore, $t \neq 5$. This completes the proof. □

Lemma 2.6.3. Let t be the cardinality of the maximal non-commuting set of a finite ring R. If R is a 9-centraliser finite ring, then $t \neq 6$.

Proof. Assume that $t = 6$. Let $\{x_1, x_2, \cdots, x_6\}$ be the maximal non-commuting set of R. Without loss of generality, we assume that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_6$. By Lemma 1.3.1(c), we have $\{C_R(x_i) \mid i = 1, 2, \cdots, 6\}$ is an irredundant cover of R. By Lemma 2.2.3, $C_R(x_i)$ is commutative for any $i \in \{1, 2, \cdots, 6\}$ and there exist some $a, b \in R - Z(R)$ such that $C_R(a)$ and $C_R(b)$ are two distinct non-commutative proper centralisers of R.

By Lemma 2.2.7, $C_R(a)$ contains $C_R(x_{l_1}), C_R(x_{l_2})$ for two distinct $l_1, l_2 \in \{1, 2, \cdots, 6\}$. Therefore, $\{C_R(a)\} \cup \left(\bigcup_{i \in A} \{C_R(x_i)\} \right)$ is an irredundant cover of

R for some $A \subset \{1, 2, \cdots, 6\}$ with $|A| \leqslant 4$. Obviously, $|A| \neq 0$. If $|A| = 1$, then by Lemma 2.2.1, it follows that $\gamma_i = 1$ for some $i \in A$, which is impossible. Thus, $|A| = 2, 3$ or 4. Now, we assume that $|A| = 2$ or 3. This implies that $\gamma_i = 2$ or 3 for some $i \in A$ by Lemma 2.2.1. In view of Lemma 2.2.14, $\gamma_2 \geqslant 4$. Therefore, $i = 1$. By Corollary 2.2.13, it follows that $|R : Z(R)| \leqslant 3\gamma_2$. By Lemma 2.2.1, $\gamma_2 \leqslant 5$. It follows that $|R : Z(R)| \leqslant 15$, which contradicts with Theorem 2.2.23. Hence, $|A| = 4$. Let $\{\alpha_1, \alpha_2, \cdots, \alpha_u\}$ be the maximal non-commuting set of $C_R(a)$. For any $i \in \{1, 2, \cdots, u\}$, since there exists some $j \in \{1, 2, \cdots, u\} - \{i\}$ such that $\alpha_j \notin C_R(\alpha_i)$ but $\alpha_j \in R, C_R(a)$, then $C_R(\alpha_i) \neq R, C_R(a)$. Therefore, we have $\{C_R(\alpha_1), C_R(\alpha_2), \cdots, C_R(\alpha_u)\} \subseteq \text{Cent}(R) - \{R, C_R(a)\} = \{C_R(x_1), C_R(x_2), \cdots, C_R(x_6), C_R(b)\}$. Hence, there have at least $u - 1$ distinct centralisers in $\{C_R(\alpha_1), C_R(\alpha_2), \cdots, C_R(\alpha_u)\}$ are commutative. If $C_R(\alpha_i)$ is commutative for some $i \in \{1, 2, \cdots, u\}$, then $C_R(\alpha_i) < C_R(a)$ as $a \in C_R(\alpha_i)$. This implies that there have at least $u - 1$ distinct commutative proper centralisers of R are contained in $C_R(a)$. Since $|A| = 4$, then there have exactly two distinct centralisers in $\{C_R(x_i) \mid i = 1, 2, \cdots, 6\}$ are contained in $C_R(a)$. Hence, there have exactly two distinct commutative proper centralisers of R are contained in $C_R(a)$. Thus, we have $2 \geqslant u - 1$, and hence, $u \leqslant 3$. So, we get $u = 3$ by Lemma 1.3.1(d). Without loss of generality, we let $C_R(\alpha_1), C_R(\alpha_2)$ be the two distinct commutative proper centralisers of R which contains in $C_R(a)$. Let $\{\beta_1, \beta_2, \cdots, \beta_v\}$ be the maximal non-commuting set of $C_R(b)$. By using similar arguments, we have $v = 3$ and $C_R(\beta_1), C_R(\beta_2)$ are the two distinct commutative proper centralisers of R which contains in $C_R(b)$. Hence, $C_R(\alpha_3) = C_R(b)$ and $C_R(\beta_3) = C_R(a)$. Now, we claim that $C_R(\alpha_i) \neq C_R(\beta_j)$ for any $i, j \in \{1, 2\}$. Assume to the contrary that $C_R(\alpha_i) = C_R(\beta_j)$ for some $i, j \in \{1, 2\}$. Since $\alpha_i \in C_R(a) = C_R(\beta_3)$, then $\beta_3 \in C_R(\alpha_i) = C_R(\beta_j)$. This yields that $\beta_3\beta_j = \beta_j\beta_3$, which is a contradiction. Therefore, our claim is true. So, we have $\{C_R(x_{i_1}), C_R(x_{i_2}), C_R(a), C_R(b)\}$ is an irredundant cover of R for two distinct $i_1, i_2 \in \{1, 2, \cdots, 6\}$. Next, we claim that $C_R(x_{i_1}) \cap C_R(x_{i_2}) = Z(R)$. Let $w \in C_R(x_{i_1}) \cap C_R(x_{i_2})$. Since $C_R(x_{i_1}), C_R(x_{i_2})$ are commutative, then $C_R(x_{i_1}), C_R(x_{i_2}) \leqslant C_R(w)$. Obviously, $C_R(w) \neq C_R(x_i)$ for any $i \in \{1, 2, \cdots, 6\}$. If $C_R(w) = C_R(a)$ or $C_R(b)$, then $C_R(x_{i_1}), C_R(x_{i_2}) \leqslant C_R(a)$ or $C_R(b)$, which contradicts the definition of irredundant cover of R. So, $C_R(w) = R$, which yields that $w \in Z(R)$. Consequently, $C_R(x_{i_1}) \cap C_R(x_{i_2}) \leqslant Z(R)$. On the other hand, it is obvious that $Z(R) \leqslant C_R(x_{i_1}) \cap C_R(x_{i_2})$. Therefore, $C_R(x_{i_1}) \cap C_R(x_{i_2}) = Z(R)$, as desired. So, we have $C_R(x_{i_1}) \cap C_R(x_{i_2}) \cap C_R(a) \cap C_R(b) = Z(R)$. It follows that $|R : Z(R)| \leqslant f(4) = 9$, which contradicts with Theorem 2.2.23. Consequently, $t \neq 6$. □

Lemma 2.6.4. Let t be the cardinality of the maximal non-commuting set of a finite ring R. If R is a 9-centraliser finite ring, then $t \neq 7$.

Proof. Assume that $t = 7$. Let $\{x_1, x_2, \cdots, x_7\}$ be the maximal non-commuting set of R. Without loss of generality, we assume that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_7$. By Lemma 1.3.1(c), we have $\{C_R(x_i) \mid i = 1, 2, \cdots, 7\}$ is an irredundant cover of R. By Lemma 2.2.3, $C_R(x_i)$ is commutative for any $i \in \{1, 2, \cdots, 7\}$ and there exists some $a \in R - Z(R)$ such that $C_R(a)$ is a non-commutative proper centraliser of R.

In view of Lemma 2.2.6, $C_R(a)$ contains $C_R(x_{l_1}), C_R(x_{l_2}), C_R(x_{l_3})$ for three distinct $l_1, l_2, l_3 \in \{1, 2, \cdots, 7\}$. Therefore, $\{C_R(a)\} \cup \left(\underset{i \in A}{\cup} \{C_R(x_i)\} \right)$ is an irredundant cover of R for some $A \subset \{1, 2, \cdots, 7\}$ with $|A| \leqslant 4$. Clearly, $|A| \neq 0$. If $|A| = 1$, then by Lemma 2.2.1, it follows that $\gamma_i = 1$ for some $i \in A$, which is impossible. Hence, $|A| = 2, 3$ or 4. Now, we claim that if $C_R(x_i) \nleqslant C_R(a)$ for some $i \in \{1, 2, \cdots, 7\}$, then $C_R(x_i) \cap C_R(a) = Z(R)$. Let $w \in C_R(x_i) \cap C_R(a)$. Since $C_R(x_i)$ is commutative, then $C_R(x_i) \leqslant C_R(w)$. Clearly, $C_R(w) \neq C_R(a), C_R(x_j)$ for any $j \in \{1, 2 \cdots, 7\} - \{i\}$. If $C_R(w) = C_R(x_i)$, then $C_R(w)$ is commutative. This implies that $C_R(w) \leqslant C_R(a)$. Therefore, $C_R(x_i) \leqslant C_R(a)$, which is a contradiction. So, $C_R(w) = R$ and it follows that $w \in Z(R)$. Hence, we obtain $C_R(x_i) \cap C_R(a) \leqslant Z(R)$. On the other hand, it is clear that $Z(R) \leqslant C_R(x_i) \cap C_R(a)$. Consequently, $C_R(x_i) \cap C_R(a) = Z(R)$, as desired. Since $C_R(x_i) \nleqslant C_R(a)$ for any $i \in A$, then $\underset{i \in A}{\cap} C_R(x_i) \cap C_R(a) = Z(R)$. Thus, we obtain $|R : Z(R)| \leqslant \max\{f(3), f(4), f(5)\} = 16$. Therefore, by Theorem 2.2.23, it follows that $|R : Z(R)| = 16$. Since $C_R(a)$ is non-commutative, then by Lemma 2.2.15, we have $|R : C_R(a)| = 2$. Since $C_R(x_i) \nleqslant C_R(a)$ for any $i \in A$, then $C_R(x_i) \cap C_R(a) = Z(R)$ for any $i \in A$ and hence, by Lemma 2.2.11, we get $|R : C_R(x_i)| = 8$ for any $i \in A$, which contradicts with Lemma 2.2.1. So, $t \neq 7$. □

Theorem 2.6.5. *Let R be a 9-centraliser finite ring. Then the cardinality of the maximal non-commuting set of R is 8. Moreover, $R/Z(R) \cong \mathbb{Z}_7 \times \mathbb{Z}_7$ and* $\mathrm{Prob}(R) = \frac{55}{343}$.

Proof. Let $\{x_1, x_2, \cdots, x_t\}$ be the maximal non-commuting set of R. Without loss of generality, we assume that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant$

γ_t. By Lemma 1.3.1(d)-(g) and Lemmas 2.6.2-2.6.4, we obtain $t = 8$. By Lemma 1.3.1(a), we have $R = \bigcup_{i=1}^{8} C_R(x_i)$. By Corollary 2.2.5, we have $C_R(x_i) \cap C_R(x_j) = Z(R)$ for any two distinct $i, j \in \{1, 2, \cdots, 8\}$. By Theorem 2.2.23, we obtain $|R : Z(R)| \geqslant 16$ with $|R : Z(R)|$ is not square-free, $|R : Z(R)| \neq p^2q$ for any two distinct primes p, q, and $|R : Z(R)| \neq p^2$ for any prime p with $p \neq 7$.

From Lemma 2.2.11, we have $|R : Z(R)| \leqslant \gamma_2^2$ and $|C_R(x_1) : Z(R)| \leqslant \gamma_2$. In view of Lemma 2.2.1 and Lemma 2.2.14, it follows that $4 \leqslant \gamma_2 \leqslant 7$. Now, we suppose that $\gamma_2 = 4$. Then, $|R : Z(R)| \leqslant 16$, which implies that $|R : Z(R)| = 16$. By using the principle of inclusion-exclusion, we obtain $|R| = \sum_{i=1}^{8} |C_R(x_i)| - 7|Z(R)|$. Thus, we get $\sum_{i=1}^{8} \frac{16}{\gamma_i} = 23$, which is impossible because $\sum_{i=1}^{8} \frac{16}{\gamma_i}$ is even. Next, we assume that $\gamma_2 = 5$. Thus, $|R : Z(R)| \leqslant 25$, a contradiction is reached. Here, we assume that $\gamma_2 = 6$. Hence, $|R : Z(R)| \leqslant 36$. If $|R : Z(R)| = 24$, then $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_{12}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ as $R/Z(R)$ is not cyclic. Therefore, $|C_R(x_1) : Z(R)| \leqslant 6$ and $|C_R(x_i) : Z(R)| \leqslant 4$ for any $i \in \{2, 3, \cdots, 8\}$. Consequently, $R/Z(R)$ has at most 2 elements of order 6. Also, there does not exist any element of order 12 in $R/Z(R)$. This contradicts with the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ has 14 elements of order 6 and $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ has an element of order 12. If $|R : Z(R)| = 36$, then $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_{18}, \mathbb{Z}_3 \times \mathbb{Z}_{12}$ or $\mathbb{Z}_6 \times \mathbb{Z}_6$ as $R/Z(R)$ is not cyclic. Thus, $|C_R(x_i) : Z(R)| \leqslant 6$ for any $i \in \{1, 2, \cdots, 8\}$. This implies that $R/Z(R)$ has at most 16 elements of order 6. Also, there does not exist any element of order 12 and order 18 in $R/Z(R)$. We have reached a contradiction as $\mathbb{Z}_6 \times \mathbb{Z}_6$ has 24 elements of order 6, $\mathbb{Z}_3 \times \mathbb{Z}_{12}$ has an element of order 12 and $\mathbb{Z}_2 \times \mathbb{Z}_{18}$ has an element of order 18. So, $\gamma_2 = 7$. Consequently, we have $|R:Z(R)| \leqslant 49$, which follows that $|R:Z(R)| = 49$. Finally, we obtain $R/Z(R) \cong \mathbb{Z}_7 \times \mathbb{Z}_7$ as $R/Z(R)$ is not cyclic. Lastly, by (1.3), we obtain

$$\begin{aligned}
\operatorname{Prob}(R) &= \frac{|Z(R)|}{|R|} + \frac{\sum\limits_{r \in R - Z(R)} |C_R(r)|}{|R|^2} \\
&= \frac{1}{49} + \frac{\left(|R| - \frac{|R|}{49}\right)\left(\frac{|R|}{7}\right)}{|R|^2} \\
&= \frac{55}{343}.
\end{aligned}$$

This completes the proof. □

As an immediate consequence of Theorem 2.6.5 and [A2], we obtain a complete characterisation for all 9-centraliser finite rings.

Theorem 2.6.6. *For any finite ring R, R is a 9-centraliser finite ring if and only if $R/Z(R) \cong \mathbb{Z}_7 \times \mathbb{Z}_7$.*

In the following, we provide an example of a 9-centraliser finite ring, which is appeared in the proof of Proposition 2.2.18.

Example 2.6.7. $R = \{\left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right] |\, a, b \in \mathbb{Z}_7\}$ is a 9-centraliser finite ring with

$$\begin{aligned}
&C_R\left(\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]\right) = R, \\
&C_R\left(\left[\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right]\right) = \left\{\left[\begin{smallmatrix} 0 & x \\ 0 & 0 \end{smallmatrix}\right] \middle|\, x \in \mathbb{Z}_7\right\}, \\
&C_R\left(\left[\begin{smallmatrix} a & la \\ 0 & 0 \end{smallmatrix}\right]\right) = \left\{\left[\begin{smallmatrix} x & lx \\ 0 & 0 \end{smallmatrix}\right] \middle|\, x \in \mathbb{Z}_7\right\}
\end{aligned}$$

for any $a \in \mathbb{Z}_7$ with $a \neq 0$ and $l \in \mathbb{Z}_7$. We note that $\{\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 2 \\ 0 & 0 \end{smallmatrix}\right],$

$\left[\begin{smallmatrix}1&3\\0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}1&4\\0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}1&5\\0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}1&6\\0&0\end{smallmatrix}\right]\}$ is a non-commuting set of R with cardinality 8. Also, we note that there does not exist a non-commuting set of R with cardinality 9. Thus, the cardinality of the maximal non-commuting set of R is 8. Since $Z(R) = \{\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]\}$, then we have $R/Z(R) \cong \mathbb{Z}_7 \times \mathbb{Z}_7$. Lastly, from (1.3), we obtain

$$\begin{aligned}\text{Prob}(R) &= \frac{|Z(R)|}{|R|} + \frac{\sum\limits_{r \in R - Z(R)} |C_R(r)|}{|R|^2} \\ &= \frac{1}{49} + \frac{48(7)}{49^2} \\ &= \frac{55}{343}.\end{aligned}$$

2.7 10-Centraliser Finite Rings

In this section, we determine the structure for all 10-centraliser finite rings and compute their commuting probabilities.

Lemma 2.7.1. Let t be the cardinality of the maximal non-commuting set of a finite ring R. If R is an n-centraliser finite ring with $n \in \{10, 11\}$, then $t \neq 5$.

Proof. Assume that $t = 5$. Let $\{x_1, x_2, \cdots, x_5\}$ be the maximal non-commuting set of R. Without loss of generality, we suppose that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_5$. By Lemma 1.3.1(b) and (c), we have $\{C_R(x_i) \mid i = 1, 2, \cdots, 5\}$ is an irredundant cover of R with intersection $Z(R)$. Hence, we have $|R : Z(R)| \leqslant f(5) = 16$ and so, by Theorem 2.2.23, we obtain $|R : Z(R)| = 16$. By Lemma 2.2.1, we have $\gamma_2 \leqslant 4$. From Lemma 2.6.1, we have $\gamma_1 = 2$. For the sake of simplicity, we write $\overline{r} = r + Z(R)$ for any $r \in R$ and $\overline{S} = S/Z(R)$ for any $S \leqslant R$.

If $C_R(x_1)$ is commutative, then by Lemma 2.2.12, we obtain $|\overline{R}| \leqslant 2(4) = 8$; a contradiction. So, we have $C_R(x_1)$ is non-commutative. By Lemma 2.2.9(b) and Lemma 2.2.10(b), we have $C_R(x_i)$ is commutative for any $i \in \{2, 3, 4, 5\}$. If $\gamma_2 \leqslant 3$, then by Lemma 2.2.12, we obtain $|\overline{R}| \leqslant 3(2) = 6$; a contradiction. Therefore, $\gamma_2 = 4$. In view of Lemma 2.2.1, we obtain $\gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = 4$. Since $\overline{C_R(x_1)} = 8$, then $\overline{R} - \overline{C_R(x_1)} = \{\overline{x_2}, \overline{x_3}, \overline{x_4}, \overline{x_5}, \overline{r_2}, \overline{r_3}, \overline{r_4}, \overline{r_5}\}$ for some $r_2, r_3, r_4, r_5 \in R - C_R(x_1)$. By Lemma 2.2.11, we obtain $|\overline{C_R(x_i)} \cap \overline{C_R(x_1)}| = 2$ for any $i \in \{2, 3, 4, 5\}$. Likewise, we obtain $|\overline{C_R(x_i)} \cap (\overline{R} - \overline{C_R(x_1)})| = 2$ for any $i \in \{2, 3, 4, 5\}$. Without loss of generality, we have

$$\overline{C_R(x_2)} = \{\overline{0}, \overline{d_2}, \overline{x_2}, \overline{r_2}\},$$

$$\overline{C_R(x_3)} = \{\overline{0}, \overline{d_3}, \overline{x_3}, \overline{r_3}\},$$

$$\overline{C_R(x_4)} = \{\overline{0}, \overline{d_4}, \overline{x_4}, \overline{r_4}\},$$

$$\overline{C_R(x_5)} = \{\overline{0}, \overline{d_5}, \overline{x_5}, \overline{r_5}\}$$

for some $\overline{d_2}, \overline{d_3}, \overline{d_4}, \overline{d_5} \in \overline{C_R(x_1)}$.

Assume that $\overline{d_i} \neq \overline{d_j}$ for any two distinct $i, j \in \{2, 3, 4, 5\}$. Then, we have

$$\overline{C_R(d_2)} \supset \{\overline{0}, \overline{d_2}, \overline{x_2}, \overline{r_2}, \overline{x_1}\} \text{ with } |\overline{C_R(d_2)}| = 8,$$

$$\overline{C_R(d_3)} \supset \{\overline{0}, \overline{d_3}, \overline{x_3}, \overline{r_3}, \overline{x_1}\} \text{ with } |\overline{C_R(d_3)}| = 8,$$

$$\overline{C_R(d_4)} \supset \{\overline{0}, \overline{d_4}, \overline{x_4}, \overline{r_4}, \overline{x_1}\} \text{ with } |\overline{C_R(d_4)}| = 8,$$

$$\overline{C_R(d_5)} \supset \{\overline{0}, \overline{d_5}, \overline{x_5}, \overline{r_5}, \overline{x_1}\} \text{ with } |\overline{C_R(d_5)}| = 8.$$

Also, we have $\overline{C_R(x_1)} = \{\overline{0}, \overline{x_1}, \overline{d_2}, \overline{d_3}, \overline{d_4}, \overline{d_5}, \overline{h_1}, \overline{h_2}\}$ for some $h_1, h_2 \in C_R(x_1) - Z(R)$. For any $i \in \{1, 2\}$ and $j \in \{2, 3, 4, 5\}$, since $x_j \notin C_R(h_i)$ but $x_j \in R, C_R(x_j), C_R(d_j)$, then $C_R(h_i) \neq R, C_R(x_j), C_R(d_j)$. We claim that $C_R(h_i) \neq C_R(x_1)$ for any $i \in \{1, 2\}$. Suppose that $C_R(h_i) = C_R(x_1)$ for some $i \in \{1, 2\}$. It follows that $\overline{0}, \overline{x_1}, \overline{h_i} \in \overline{Z(C_R(x_1))}$ and thus, $|\overline{Z(C_R(x_1))}| = 4$. This leads to $|C_R(x_1) : Z(C_R(x_1))| = 2$ and so, $C_R(x_1)/Z(C_R(x_1))$ is cyclic. This yields that $C_R(x_1)$ is commutative, which is a contradiction. So, $C_R(h_i) \neq C_R(x_1)$ for any $i \in \{1, 2\}$. This gives that $|\mathrm{Cent}(R)| \geqslant 11$ and so, $|\mathrm{Cent}(R)| = 11$. Since $|\mathrm{Cent}(R)| = 11$, then $C_R(h_1) = C_R(h_2)$. Since $|\overline{C_R(d_2)} \cap \overline{C_R(x_1)}| \geqslant 3$, then we have $|\overline{C_R(d_2)} \cap \overline{C_R(x_1)}| = 4$. If $\overline{h_i} \in \overline{C_R(d_2)}$ for some $i \in \{1, 2\}$, then $\overline{0}, \overline{x_1}, \overline{h_1}, \overline{h_2}, \overline{d_2} \in \overline{C_R(h_1)}$. This implies $|\overline{C_R(x_1)} \cap \overline{C_R(h_1)}| \geqslant 5$ and it follows that $C_R(x_1) = C_R(h_1)$; a contradiction. So, $\overline{d_k} \in \overline{C_R(d_2)}$ for some $k \in \{3, 4, 5\}$. Without loss of generality, we assume that $k = 3$. By Lemma 2.2.11, we have $|\overline{C_R(d_i)} \cap \overline{C_R(x_4)}| = 2$ for any $i \in \{2, 3\}$. Since $\overline{d_2}, \overline{d_3} \notin \overline{C_R(x_4)}$, then $\overline{x_4} \notin \overline{C_R(d_2)}, \overline{C_R(d_3)}$. If $\overline{d_4} \in \overline{C_R(d_i)}$ for some $i \in \{2, 3\}$, then $|\overline{C_R(d_i)} \cap \overline{C_R(x_1)}| \geqslant 5$ and it follows that $\overline{C_R(d_i)} = \overline{C_R(x_1)}$, which is a contradiction. Thus, $\overline{d_4} \notin \overline{C_R(d_2)}, \overline{C_R(d_3)}$ and therefore, $\overline{r_4} \in \overline{C_R(d_2)}, \overline{C_R(d_3)}$. So, we have

$$\overline{C_R(d_2)} \supset \{\overline{0}, \overline{d_2}, \overline{d_3}, \overline{x_1}, \overline{r_4}, \overline{x_2}, \overline{r_2}\} \text{ with } |\overline{C_R(d_2)}| = 8,$$

$$\overline{C_R(d_3)} \supset \{\overline{0}, \overline{d_2}, \overline{d_3}, \overline{x_1}, \overline{r_4}, \overline{x_3}, \overline{r_3}\} \text{ with } |\overline{C_R(d_3)}| = 8.$$

This shows that $|\overline{C_R(d_2)} \cap \overline{C_R(d_3)}| = 5$ or 6. We have reached a contradiction as $|\overline{C_R(d_2)} \cap \overline{C_R(d_3)}|$ is divide $|R|$.

Consequently, $\overline{d_i} = \overline{d_j}$ for two distinct $i, j \in \{2, 3, 4, 5\}$. Without loss of generality, we assume that $i = 2$ and $j = 3$. Then, we have $\overline{C_R(d_2)} \supset \{\overline{0}, \overline{d_2}, \overline{x_2}, \overline{r_2}, \overline{x_3}, \overline{r_3}, \overline{x_1}\}$ with $|\overline{C_R(d_2)}| = 8$. By Lemma 2.2.11, it follows that $|\overline{C_R(d_2)} \cap \overline{C_R(x_1)}| = 4$ and so, $|\overline{C_R(d_2)} \cap (\overline{R} - \overline{C_R(x_1)})| = 4$. This shows that $\overline{x_k}, \overline{r_k} \notin \overline{C_R(d_2)}$ for any $k \in \{4, 5\}$. This implies that $|\overline{C_R(d_2)} \cap \overline{C_R(x_k)}| \leqslant 2$ for any $k \in \{4, 5\}$. Therefore, by Lemma 2.2.11, we obtain $|\overline{C_R(d_2)} \cap \overline{C_R(x_k)}| = 2$ for any $k \in \{4, 5\}$. So, we have $\overline{d_k} \in \overline{C_R(d_2)}$ for any $k \in \{4, 5\}$. If $\overline{d_k} = \overline{d_2}$ for some $k \in \{4, 5\}$, then $\overline{x_k}, \overline{r_k} \in \overline{C_R(d_2)}$; a contradiction. Hence, $\overline{d_k} \neq \overline{d_2}$ for any $k \in \{4, 5\}$. Since $|\overline{C_R(d_2)}| = 8$, then $\overline{d_4} = \overline{d_5}$. Therefore, we have $\overline{C_R(x_1)} = \{\overline{0}, \overline{x_1}, \overline{d_2}, \overline{d_4}, \overline{h_1}, \overline{h_2}, \overline{h_3}, \overline{h_4}\}$ for some $h_1, h_2, h_3, h_4 \in C_R(x_1) - Z(R)$. Since $\{h_1, h_i, x_2, x_3, x_4, x_5\}$ is not a non-commuting set of R for any $i \in \{2, 3, 4\}$, then $h_1 h_i = h_i h_1$ for any $i \in \{2, 3, 4\}$. It follows that $\overline{C_R(h_1)} \supset \{\overline{0}, \overline{h_1}, \overline{h_2}, \overline{h_3}, \overline{h_4}, \overline{x_1}\}$ with $|\overline{C_R(h_1)}| = 8$. This gives that $|\overline{C_R(x_1)} \cap \overline{C_R(h_1)}| \geqslant 6$. Consequently, we obtain $\overline{C_R(x_1)} = \overline{C_R(h_1)}$. This contradicts with the fact that $\overline{d_2} \notin \overline{C_R(h_1)}$. □

Lemma 2.7.2. Let $\{x_1, x_2, \cdots, x_6\}$ be the maximal non-commuting set of a finite ring R. If R is an n-centraliser finite ring with $n \in \{10, 11\}$, then $|R : Z(R)| = 16, 24, 32$ or 36. Furthermore, if $|R : C_R(x_1)| \leqslant |R : C_R(x_2)| \leqslant \cdots \leqslant |R : C_R(x_6)|$, then $|R : C_R(x_2)| = 4$.

Proof. From Lemma 1.3.1(b) and (c), we have $\{C_R(x_i) \mid i = 1, 2, \cdots, 6\}$ is an irredundant cover of R with intersection $Z(R)$. Thus, we have $|R : Z(R)| \leqslant f(6) = 36$. Therefore, by Theorem 2.2.23, we obtain $|R : Z(R)| = 16, 24, 27, 32$ or 36. If $|R : Z(R)| = 27$, then by Lemma 2.2.16 and Lemma 2.2.4, we obtain $|\mathrm{Cent}(R)| = 7$, which is a contradiction. So, $|R : Z(R)| = 16, 24, 32$ or 36.

By Lemma 2.2.1, it follows that $|R : C_R(x_2)| \leqslant 5$ and hence, $|R : C_R(x_2)| \leqslant 4$. Assume that $|R : C_R(x_2)| \leqslant 3$. If $C_R(x_2)$ is commutative, then by Lemma 2.2.12, we obtain $|R : Z(R)| \leqslant 3(3) = 9$, which is a contradiction. If $C_R(x_2)$ is non-commutative, then by Lemma 2.2.8(b) and Lemma 2.2.9(b), $C_R(x_1)$ is commutative. Thus, it follows from Lemma 2.2.12 that $|R : Z(R)| \leqslant 3(3) = 9$, which is a contradiction again. Consequently, $|R : C_R(x_2)| = 4$, as desired. □

The following lemma is required before we proceed further.

Lemma 2.7.3. Let $H_1, H_2, \cdots, H_t$ be the proper subgroups of a group $(G, +)$ with $|G : H_1| \leqslant |G : H_2| \leqslant \cdots \leqslant |G : H_t|$. If $G = H_1 \cup H_2 \cup \cdots \cup H_t$, then $|G| \leqslant \sum_{i=2}^{t} |H_i|$. Moreover, the equality attains if and only if $H_1 + H_r = G$ for any $r \in \{2, \cdots, t\}$ and $H_r \cap H_s \subset H_1$ for any two distinct $r, s \in \{1, 2, \cdots, t\}$.

Proof. See Theorem 1 in Cohn (1994). □

Lemma 2.7.4. Let $\{x_1, x_2, \cdots, x_6\}$ be the maximal non-commuting set of a finite ring R. If R is a 10-centraliser finite ring with $|R : Z(R)| = 16$, then $|R : C_R(x_i)| = 4$ for any $i \in \{1, 2, \cdots, 6\}$, and $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4 \times \mathbb{Z}_4$.

Proof. Without loss of generality, we suppose that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_6$. From Lemma 1.3.1(a), we have $R = \bigcup_{i=1}^{6} C_R(x_i)$. By Lemma 2.7.2, we have $\gamma_2 = 4$. If $\gamma_4 \neq 4$, then by Lemma 2.7.3, we obtain $|R| \leqslant 2(\frac{|R|}{4}) + 3(\frac{|R|}{8}) = \frac{7}{8}|R|$, which is impossible. So, we have $\gamma_3 = \gamma_4 = 4$.

For the sake of simplicity, we write $\overline{r} = r + Z(R)$ for any $r \in R$ and let $\overline{S} = S/Z(R)$ for any $S \leqslant R$.

We claim that $\gamma_1 = 4$. Suppose to the contrary that $\gamma_1 = 2$. By Lemma 2.2.11, we obtain $|\overline{C_R(x_1)} \cap \overline{C_R(x_2)}| = 2$. Hence, $\overline{C_R(x_1)} \cap \overline{C_R(x_2)} = \{\overline{0}, \overline{a}\}$ for some $\overline{a} \in \overline{R} - \overline{Z(R)}$. So, we have

$$\overline{C_R(x_1)} = \{\overline{0}, \overline{x_1}, \overline{a}, \overline{b}, \overline{a+b}, \overline{x_1+a}, \overline{x_1+b}, \overline{x_1+a+b}\},$$

$$\overline{C_R(x_2)} = \{\overline{0}, \overline{x_2}, \overline{a}, \overline{x_2+a}\}$$

for some $\overline{b} \in \overline{R} - \overline{Z(R)}$. If $ab = ba$, then $C_R(x_1)$ is commutative. Therefore, by Lemma 2.2.12, it follows that $|\overline{R}| \leqslant 2(4) = 8$; a contradiction. So, $ab \neq ba$. Thus, we have

$$\overline{C_R(a)} \supseteq \{\overline{0}, \overline{x_1}, \overline{x_2}, \overline{a}\},$$

$$\overline{C_R(b)} \supseteq \{\overline{0}, \overline{x_1}, \overline{b}, \overline{x_1+b}\},$$

$$\overline{C_R(a+b)} \supseteq \{\overline{0}, \overline{x_1}, \overline{a+b}, \overline{x_1+a+b}\},$$

$$\overline{C_R(x_1+a)} \supseteq \{\overline{0}, \overline{x_1}, \overline{a}, \overline{x_1+a}\}.$$

It can be easily checked that $R, C_R(x_1), C_R(x_2), \cdots, C_R(x_6), C_R(a), C_R(b), C_R(a+b), C_R(x_1+a)$ are 11 distinct centralisers of R. We have reached a contradiction. Consequently, $\gamma_1 = 4$.

By Lemma 2.2.15, we have $C_R(x_i)$ is commutative for any $i \in \{1, 2, \cdots, 6\}$. Next, we claim that $\gamma_5 = \gamma_6 = 4$. If $\gamma_5 = 8$, then $\gamma_6 = 8$ and so,

$|\overline{R}| \leqslant \sum\limits_{i=1}^{6} |\overline{C_R(x_i)}| - 5 = 15$, which is impossible. So, $\gamma_5 = 4$. Assume that $\gamma_6 = 8$. Thus, $|\overline{C_R(x_6)}| = 2$ and hence, $\overline{C_R(x_i)} \cap \overline{C_R(x_6)} = \overline{Z(R)}$ for any $i \in \{1, 2, \cdots, 5\}$. If $\overline{C_R(x_i)} \cap \overline{C_R(x_j)} = \overline{Z(R)}$ for any two distinct $i, j \in \{1, 2, \cdots, 5\}$, then $|\overline{R}| = \sum\limits_{i=1}^{6} |\overline{C_R(x_i)}| - 5 = 17$, which is impossible. Therefore, $\overline{C_R(x_i)} \cap \overline{C_R(x_j)} \neq \overline{Z(R)}$ for two distinct $i, j \in \{1, 2, \cdots, 5\}$. So, there exists some $\overline{w} \in (\overline{C_R(x_i)} \cap \overline{C_R(x_j)}) - \overline{Z(R)}$, which gives that $C_R(x_i) \cup C_R(x_j) \subseteq C_R(w)$. This shows that $R = C_R(w) \cup \left(\bigcup\limits_{k=1, k \neq i,j}^{6} C_R(x_k) \right)$. Therefore, from Lemma 2.2.1, we obtain $\gamma_6 = 4$, which is a contradiction. Consequently, $\gamma_6 = 4$.

Since $|\overline{R}| = 16$, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4 \times \mathbb{Z}_4$ as $\overline{R}$ is not cyclic. Since $|\overline{C_R(x_i)}| = 4$ for any $i \in \{1, 2, \cdots, 6\}$, then there does not exist any element of order 8 in $\overline{R}$ and consequently, $\overline{R} \ncong \mathbb{Z}_2 \times \mathbb{Z}_8$. Assume that $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ has 8 elements of order 4, then there exist four distinct $l_1, l_2, l_3, l_4 \in \{1, 2, \cdots, 6\}$ such that $\overline{C_R(x_{l_i})} \cong \mathbb{Z}_4$ for any $i \in \{1, 2, 3, 4\}$. Without loss of generality, we assume that $l_1 = 1, l_2 = 2, l_3 = 3, l_4 = 4$. If $\overline{C_R(x_1)} \cap \overline{C_R(x_i)} \neq \overline{Z(R)}$ for any $i \in \{2, 3, 4\}$, then there exists some $\overline{w_i} \in (\overline{C_R(x_1)} \cap \overline{C_R(x_i)}) - \overline{Z(R)}$ for any $i \in \{2, 3, 4\}$. This gives that $\overline{0}, \overline{x_1}, \overline{w_2}, \overline{w_3}, \overline{w_4} \in \overline{C_R(x_1)}$. Since $\overline{w_i} \neq \overline{0}, \overline{x_1}$ for any $i \in \{2, 3, 4\}$, then $\overline{w_i} = \overline{w_j}$ for two distinct $i, j \in \{2, 3, 4\}$. So, we obtain $w_i \in C_R(x_1) \cap C_R(x_i) \cap C_R(x_j)$ and therefore, $C_R(x_1) \cup C_R(x_i) \cup C_R(x_j) \subseteq C_R(w_i)$. It follows that $R = C_R(w_i) \cup \left(\bigcup\limits_{k=2, k \neq i,j}^{6} C_R(x_k) \right)$. In view of Lemma 2.2.1, we obtain $\gamma_k \leqslant 3$ for some $k \in \{2, \cdots, 6\} - \{i, j\}$, which leads to a contradiction. Consequently, $\overline{C_R(x_1)} \cap \overline{C_R(x_i)} = \overline{Z(R)}$ for some $i \in \{2, 3, 4\}$. Since $\overline{C_R(x_1)} \cong \mathbb{Z}_4$ and $\overline{C_R(x_i)} \cong \mathbb{Z}_4$, then there exist some $a \in C_R(x_1) - Z(R), b \in C_R(x_i) - Z(R)$

such that $\overline{C_R(x_1)} = \{\overline{0}, \overline{a}, \overline{2a}, \overline{3a}\}$ and $\overline{C_R(x_i)} = \{\overline{0}, \overline{b}, \overline{2b}, \overline{3b}\}$. This implies that $\overline{R} = \{\overline{ma + nb} \mid m, n \in \mathbb{Z}_4\} \cong \mathbb{Z}_4 \times \mathbb{Z}_4$, which is a contradiction. Therefore, $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4 \times \mathbb{Z}_4$. □

Lemma 2.7.5. Let $\{x_1, x_2, \cdots, x_6\}$ be the maximal non-commuting set of a finite ring R. If R is an n-centraliser finite ring with $n \in \{10, 11\}$, then $|R : Z(R)| \neq 24$ and 36.

Proof. Assume that $|R : Z(R)| = 24$ or 36. Without loss of generality, we suppose that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_6$. From Lemma 1.3.1(a), we have $R/Z(R) = \bigcup_{i=1}^{6} [C_R(x_i)/Z(R)]$. By Lemma 2.7.2, we have $\gamma_2 = 4$. Let ${}_m|G|$ denote the total number of elements with order m in an additive group G. For the sake of simplicity, we write $\overline{r} = r + Z(R)$ for any $r \in R$ and $\overline{S} = S/Z(R)$ for any $S \leqslant R$.

If $|\overline{R}| = 24$, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_{12}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ as $\overline{R}$ is not cyclic. Thus, $|\overline{C_R(x_1)}| \leqslant 12$ and $|\overline{C_R(x_i)}| \leqslant 6$ for any $i \in \{2, 3, \cdots, 6\}$. This yields that $\overline{R}$ has at most ${}_{12}|\mathbb{Z}_{12}|$ elements of order 12. Since ${}_{12}|\mathbb{Z}_{12}| < {}_{12}|\mathbb{Z}_2 \times \mathbb{Z}_{12}|$, then $\overline{R} \ncong \mathbb{Z}_2 \times \mathbb{Z}_{12}$ and so, $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$. If $|\overline{C_R(x_1)}| = 8$, then $\overline{R}$ has at most $5({}_6|\mathbb{Z}_6|) = 10$ elements of order 6, which contradicts the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ has 14 elements of order 6. Therefore, $|\overline{C_R(x_1)}| = 6$ or 12. It follows that $\overline{C_R(x_1)} \cong \mathbb{Z}_6$ or $\mathbb{Z}_2 \times \mathbb{Z}_6$. Here, we claim that $\gamma_5 \neq 4$. Assume that $\gamma_5 = 4$, then $\overline{C_R(x_i)} \cong \mathbb{Z}_6$ for any $i \in \{2, 3, 4, 5\}$. This gives that $\overline{C_R(x_i)}$ has exactly 2 elements of order 3 for any $i \in \{1, 2, \cdots, 5\}$. By the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ has exactly 2 elements of order 3, then there exists some $\overline{a} \in \overline{R} - \overline{Z(R)}$ with order 3 such that $\overline{a} \in \bigcap_{i=1}^{5} \overline{C_R(x_i)}$. So, by Lemma 1.3.1(b), (c) and Lemma 2.2.2, we obtain $\overline{a} \in \overline{Z(R)}$, which leads to a contradiction. Consequently, $\gamma_5 \neq 4$ and so, $|\overline{C_R(x_i)}| \leqslant 4$ for any $i \in \{5, 6\}$. This implies that $\overline{R}$ has at most ${}_6|\mathbb{Z}_2 \times \mathbb{Z}_6| + 3({}_6|\mathbb{Z}_6|) = 12$ elements of order 6. We have reached a contradiction as $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ has 14 elements of order 6.

If $|\overline{R}| = 36$, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_{18}, \mathbb{Z}_3 \times \mathbb{Z}_{12}$ or $\mathbb{Z}_6 \times \mathbb{Z}_6$ as $\overline{R}$ is not cyclic. Hence, $|\overline{C_R(x_1)}| \leqslant 18$, $|\overline{C_R(x_2)}| = 9$ and $|\overline{C_R(x_i)}| \leqslant 9$ for any $i \in \{3, 4, 5, 6\}$. This leads to $\overline{R}$ has at most ${}_{12}|\mathbb{Z}_{12}|$ elements of order 12 and ${}_{18}|\mathbb{Z}_{18}|$ elements of order 18. Since ${}_{12}|\mathbb{Z}_{12}| < {}_{12}|\mathbb{Z}_3 \times \mathbb{Z}_{12}|$ and ${}_{18}|\mathbb{Z}_{18}| < {}_{18}|\mathbb{Z}_2 \times \mathbb{Z}_{18}|$, then $R/Z(R) \ncong \mathbb{Z}_2 \times \mathbb{Z}_{18}$ and $\mathbb{Z}_3 \times \mathbb{Z}_{12}$. It follows that $\overline{R} \cong \mathbb{Z}_6 \times \mathbb{Z}_6$. Therefore, we have $|\overline{C_R(x_1)}| = 9, \overline{C_R(x_1)} \cong \mathbb{Z}_2 \times \mathbb{Z}_6$ or $\overline{C_R(x_1)} \cong \mathbb{Z}_3 \times \mathbb{Z}_6$. This implies that $\overline{R}$ has at most ${}_6|\mathbb{Z}_3 \times \mathbb{Z}_6| + 4({}_6|\mathbb{Z}_6|) = 16$ elements of order 6. This contradicts with the fact that $\mathbb{Z}_6 \times \mathbb{Z}_6$ has 24 elements of order 6. □

Lemma 2.7.6. Let $\{x_1, x_2, \cdots, x_6\}$ be the maximal non-commuting set of a finite ring R. If R is a 10-centraliser finite ring, then $|R : Z(R)| \neq 32$.

Proof. Assume that $|R : Z(R)| = 32$. Without loss of generality, we suppose that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_6$. From Lemma 1.3.1(a), we have $R = \bigcup_{i=1}^{6} C_R(x_i)$. By Lemma 2.7.2, we have $\gamma_2 = 4$. If $\gamma_4 \neq 4$, then by Lemma 2.7.3, we obtain $|R| \leqslant 2(\frac{|R|}{4}) + 3(\frac{|R|}{8}) = \frac{7}{8}|R|$, which is impossible. So, we have $\gamma_3 = \gamma_4 = 4$. For the sake of simplicity, we write $\overline{r} = r + Z(R)$ for any $r \in R$ and $\overline{S} = S/Z(R)$ for any $S \leqslant R$. Here, we break the proof into the following two cases.

Case 1: $\gamma_1 = 2$. If $C_R(x_1)$ is commutative, then by Lemma 2.2.12, we obtain $|\overline{R}| \leqslant 2(4) = 8$; a contradiction. Therefore, $C_R(x_1)$ is non-commutative. By Lemma 2.2.8(b), we have $C_R(x_i)$ is commutative for any $i \in \{2, 3, \cdots, 6\}$. Since $|\overline{C_R(x_1)}| = 16$, then $\overline{R} - \overline{C_R(x_1)} = \{\overline{x_2}, \overline{x_3}, \overline{x_4}, \overline{x_5}, \overline{x_6}, \overline{r_1}, \overline{r_2}, \cdots, \overline{r_{11}}\}$ for some $r_1, r_2, \cdots, r_{11} \in R - C_R(x_1)$. We claim that $\overline{r_i} \notin \overline{C_R(x_j)} \cap \overline{C_R(x_k)}$ for any $i \in \{1, 2, \cdots, 11\}$ and $j, k \in \{2, 3, \cdots, 6\}$ with $j \neq k$. If $\overline{r_i} \in \overline{C_R(x_j)} \cap \overline{C_R(x_k)}$ for some $i \in \{1, 2, \cdots, 11\}$ and $j, k \in \{2, 3, \cdots, 6\}$ with $j \neq k$, then $C_R(x_j) \cup C_R(x_k) \subseteq C_R(r_i)$. It is clear that $C_R(r_i) \neq R, C_R(x_l)$ for any $l \in \{1, 2, \cdots, 6\}$. Therefore, by Lemma 2.2.8(a), we obtain $r_i \in C_R(x_1)$; a contradiction. So, our claim is true. By Lemma 2.2.11, we have $|\overline{C_R(x_i)} \cap \overline{C_R(x_1)}| = 4$ for any $i \in \{2, 3, 4\}$. Likewise, we have $|\overline{C_R(x_i)} \cap (\overline{R} - \overline{C_R(x_1)})| = 4$ for any $i \in \{2, 3, 4\}$. By applying Lemma 2.2.11 again, we have $|\overline{C_R(x_i)} \cap \overline{C_R(x_j)}| \geqslant 2$ for any two distinct $i, j \in \{2, 3, 4\}$. If $|\overline{C_R(x_2)} \cap \overline{C_R(x_3)} \cap \overline{C_R(x_4)}| \geqslant 2$, then without loss of generality, we have

$$\overline{C_R(x_2)} \supset \{\overline{0}, \overline{d_1}, \overline{x_2}, \overline{r_1}, \overline{r_2}, \overline{r_3}\},$$

$$\overline{C_R(x_3)} \supset \{\overline{0}, \overline{d_1}, \overline{x_3}, \overline{r_4}, \overline{r_5}, \overline{r_6}\},$$

$$\overline{C_R(x_4)} \supset \{\overline{0}, \overline{d_1}, \overline{x_4}, \overline{r_7}, \overline{r_8}, \overline{r_9}\}$$

for some $\overline{d_1} \in \overline{C_R(x_1)} - \overline{Z(R)}$. It follows that $C_R(x_2) \cup C_R(x_3) \cup C_R(x_4) \subseteq C_R(d_1)$. This shows that $\overline{x_2}, \overline{x_3}, \overline{x_4}, \overline{r_1}, \overline{r_2}, \cdots, \overline{r_9} \in \overline{C_R(d_1)}$ and hence, $|\overline{C_R(d_1)}| = 16$. Therefore, we have $|\overline{C_R(d_1)} \cap \overline{C_R(x_1)}| \leqslant 4$. Hence, by Lemma 2.2.11, we obtain $|\overline{R}| \leqslant 2(2)(4) = 16$, which is a contradiction. Consequently, $|\overline{C_R(x_2)} \cap \overline{C_R(x_3)} \cap \overline{C_R(x_4)}| = 1$. So, without loss of generality, we have

$$\overline{C_R(x_2)} = \{\overline{0}, \overline{d_1}, \overline{d_2}, \overline{d_4}, \overline{x_2}, \overline{r_1}, \overline{r_2}, \overline{r_3}\},$$

$$\overline{C_R(x_3)} = \{\overline{0}, \overline{d_1}, \overline{d_3}, \overline{d_5}, \overline{x_3}, \overline{r_4}, \overline{r_5}, \overline{r_6}\},$$

$$\overline{C_R(x_4)} = \{\overline{0}, \overline{d_2}, \overline{d_3}, \overline{d_6}, \overline{x_4}, \overline{r_7}, \overline{r_8}, \overline{r_9}\}$$

for some $\overline{d_1}, \overline{d_2}, \cdots, \overline{d_6} \in \overline{C_R(x_1)} - \overline{Z(R)}$. It follows that $C_R(x_2) \cup C_R(x_3) \subseteq C_R(d_1), C_R(x_2) \cup C_R(x_4) \subseteq C_R(d_2)$ and $C_R(x_3) \cup C_R(x_4) \subseteq C_R(d_3)$. It is obvious that $C_R(d_i) \neq R, C_R(x_l)$ for any $i \in \{1, 2, 3\}$ and $l \in \{1, 2, \cdots, 6\}$. Since $|\overline{C_R(x_2)} \cap \overline{C_R(x_3)} \cap \overline{C_R(x_4)}| = 1$, then $C_R(d_i) \neq C_R(d_j)$ for any two distinct $i, j \in \{1, 2, 3\}$. Therefore, by Lemma 2.2.8(a), we have d_1, d_2, d_3 do not commute with each other. Now, we consider for $C_R(d_1 + x_1)$. Since $d_2 \notin C_R(d_1 + x_1)$ but $d_2 \in R, C_R(x_1)$, then $C_R(d_1 + x_1) \neq R, C_R(x_1)$. For any $i \in \{2, 3, \cdots, 6\}$, since $x_1 \in C_R(d_1 + x_1)$ but $x_1 \notin C_R(x_i)$, then $C_R(d_1 + x_1) \neq C_R(x_i)$. Since $x_2 \notin C_R(d_1 + x_1)$ but $x_2 \in C_R(d_1)$, then $C_R(d_1 + x_1) \neq C_R(d_1)$. Since $d_2, d_3 \notin C_R(d_1 + x_1)$ but $d_2 \in C_R(d_2)$ and $d_3 \in C_R(d_3)$, then $C_R(d_1 + x_1) \neq C_R(d_2), C_R(d_3)$. Consequently, we obtain $|\mathrm{Cent}(R)| \geqslant 11$, which is a contradiction.

Case 2: $\gamma_1 = 4$. Now, we want to show that $\gamma_5 = \gamma_6 = 4$. By Lemma 2.2.8(b), we have $C_R(x_i), C_R(x_j)$ are commutative for two distinct $i, j \in \{1, 2, 3, 4\}$. By Lemma 2.2.11, it follows that $C_R(x_i) \cap C_R(x_j) \neq Z(R)$ and hence, there exists some $r \in (C_R(x_i) \cap C_R(x_j)) - Z(R)$, which gives that $C_R(x_i) \cup C_R(x_j) \subseteq C_R(r)$. This yields that $R = C_R(r) \cup \left(\bigcup_{k=1, k\neq i,j}^{6} C_R(x_k) \right)$. Therefore, by Lemma 2.2.1, we obtain $\gamma_5 = \gamma_6 = 4$, as desired. By Lemma 2.2.8(b), there exist five distinct $k_1, k_2, \cdots, k_5 \in \{1, 2, \cdots, 6\}$ such that $C_R(x_{k_i})$ is commutative for any $i \in \{1, 2, \cdots, 5\}$. Without loss of generality, we assume that $k_1 = 1, k_2 = 2, \cdots, k_5 = 5$. Let $k \in \{2,3,4,5\}$. By Lemma 2.2.11, we have $C_R(x_1) \cap C_R(x_k) \neq Z(R)$. Thus, there exists some $w_k \in (C_R(x_1) \cap C_R(x_k)) - Z(R)$, which gives that $C_R(x_1) \cup C_R(x_k) \subseteq C_R(w_k)$. Clearly, $C_R(w_k) \neq R, C_R(x_i)$ for any $i \in \{1,2,\cdots,6\}$. Since $|\mathrm{Cent}(R)| = 10$, then we have $C_R(w_u) = C_R(w_v)$ for two distinct $u, v \in \{2,3,4,5\}$. It follows that $C_R(x_1) \cup C_R(x_u) \cup C_R(x_v) \subseteq C_R(w_u)$. This implies that $R = C_R(w_u) \cup \left(\bigcup_{i=2, i\neq u,v}^{6} C_R(x_i) \right)$. Consequently, by Lemma 2.2.1, we obtain $\gamma_i \leqslant 3$ for some $i \in \{2,3,\cdots,6\} - \{u,v\}$, which leads to a contradiction. □

Lemma 2.7.7. Let t be the cardinality of the maximal non-commuting set of a finite ring R. If R is a 10-centraliser finite ring, then $t \neq 7$.

Proof. Assume that $t = 7$. Let $\{x_1, x_2, \cdots, x_7\}$ be the maximal non-commuting set of R. Without loss of generality, we suppose that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_7$. From Lemma 1.3.1(a), we have $R = \bigcup_{i=1}^{7} C_R(x_i)$. By Lemma 2.2.3, we have $C_R(x_i)$ is commutative for any $i \in \{1,2,\cdots,7\}$, and $C_R(a), C_R(b)$ are two distinct non-commutative proper centralisers of R for some $a, b \in R - Z(R)$. By Theorem 2.2.23, we have $|R : Z(R)| \geqslant 16$ with $|R : Z(R)|$ is not square-free, $|R : Z(R)| \neq p^2 q$ for any two distinct primes p, q, and $|R : Z(R)| \neq p^2$ for any prime p.

First, we claim that $\gamma_i \geqslant 4$ for any $i \in \{1,2,\cdots,7\}$. Assume that $\gamma_1 \leqslant 3$, then by Corollary 2.2.13, we obtain $|R : Z(R)| \leqslant 3\gamma_2$. By Lemma 2.2.1, we have $\gamma_2 \leqslant 6$. If $\gamma_2 \leqslant 5$, then $|R : Z(R)| \leqslant 15$, which is a contradiction. If $\gamma_2 = 6$, then $|R : Z(R)| \leqslant 18$, which is a contradiction again. Therefore, $\gamma_1 \geqslant 4$ and so, $\gamma_i \geqslant 4$ for any $i \in \{1,2,\cdots,7\}$, as claimed.

Next, we want to show that $C_R(a)$ contains exactly two distinct $C_R(x_i)$'s. From Lemma 2.2.7, we have $R = C_R(a) \cup \left(\underset{i\in A}{\cup} C_R(x_i)\right)$ for some $A \subset \{1, 2, \cdots, 7\}$ with $|A| \leqslant 5$. Obviously, $|A| \neq 0$. Suppose that $|A| \leqslant 3$, then by Lemma 2.2.1, it follows that $\gamma_i \leqslant |A| \leqslant 3$ for some $i \in A$. This contradicts with the fact that $\gamma_i \geqslant 4$. Assume that $|A| = 4$. Thus, we have $R = C_R(a) \cup \left(\overset{4}{\underset{i=1}{\cup}} C_R(x_{k_i})\right)$ for four distinct $k_1, k_2, k_3, k_4 \in \{1, 2, \cdots, 7\}$. By Lemma 2.2.1, we obtain $\gamma_{k_1} = \gamma_{k_2} = \gamma_{k_3} = \gamma_{k_4} = 4$. Here, we claim that $C_R(x_{k_1}) \cap C_R(x_{k_2}) \neq Z(R)$. Suppose to the contrary that $C_R(x_{k_1}) \cap C_R(x_{k_2}) = Z(R)$. By Lemma 2.2.11, we obtain $|R : Z(R)| \leqslant 16$. Therefore, we have $|R : Z(R)| = 16$. Since $C_R(a)$ is non-commutative, then by Lemma 2.2.15, we obtain $|R : C_R(a)| = 2$. Thus, by Lemma 2.2.11, we have $C_R(a) \cap C_R(x_{k_i}) \neq Z(R)$ for any $i \in \{1, 2, 3, 4\}$. Let $i \in \{1, 2, 3, 4\}$. Then, there exists some $w_i \in (C_R(a) \cap C_R(x_{k_i})) - Z(R)$. Since $C_R(x_{k_i})$ is commutative, then $C_R(x_{k_i}) \leqslant C_R(w_i)$. Obviously, $C_R(w_i) \neq R, C_R(x_j)$ for any $j \in \{1, 2, \cdots, 7\} - \{k_i\}$. If $C_R(w_i) = C_R(a)$, then $C_R(x_{k_i}) \leqslant C_R(a)$. On the other hand, if $C_R(w_i) = C_R(x_{k_i})$, then $C_R(w_i)$ is commutative and hence, $C_R(w_i) \leqslant C_R(a)$ and so, $C_R(x_{k_i}) \leqslant C_R(a)$. In both situations, we obtain a contradiction because $C_R(x_{k_i}) \nleqslant C_R(a)$. So, we obtain $C_R(w_i) = C_R(b)$ and therefore, $C_R(x_{k_i}) \leqslant C_R(b)$. It follows that $R = C_R(a) \cup C_R(b)$. So, by Lemma 2.2.1, we obtain $|R : C_R(b)| = 1$, which is a contradiction. Consequently, $C_R(x_{k_1}) \cap C_R(x_{k_2}) \neq Z(R)$, as claimed. Thus, there exists some $r \in (C_R(x_{k_1}) \cap C_R(x_{k_2})) - Z(R)$, which implies that $C_R(x_{k_1}) \cup C_R(x_{k_2}) \subseteq C_R(r)$. Obviously, $C_R(r) \neq R, C_R(x_i)$ for any $i \in \{1, 2, \cdots, 7\}$. Since $C_R(x_{k_1}), C_R(x_{k_2}) \nleqslant C_R(a)$, then $C_R(r) \neq C_R(a)$. So, we obtain $C_R(r) =$

$C_R(b)$. This gives that $R = C_R(a) \cup C_R(b) \cup C_R(x_{k_3}) \cup C_R(x_{k_4})$. Since $|C_R(b)| > |C_R(x_{k_1})|$, then $|R : C_R(b)| \leqslant 3$. So, by Lemma 2.2.11, we have $C_R(b) \cap C_R(x_{k_i}) \neq Z(R)$ for any $i \in \{3, 4\}$. Let $i \in \{3, 4\}$. Then, there exists some $w_i \in (C_R(b) \cap C_R(x_{k_i})) - Z(R)$. Since $C_R(x_{k_i})$ is commutative, then $C_R(x_{k_i}) \leqslant C_R(w_i)$. Clearly, $C_R(w_i) \neq R, C_R(x_j)$ for any $j \in \{1, 2, \cdots, 7\} - \{k_i\}$. Since $C_R(x_{k_i}) \nleqslant C_R(a)$, then $C_R(w_i) \neq C_R(a)$. So, we can conclude that $C_R(w_i) = C_R(b)$ or $C_R(x_{k_i})$. If $C_R(w_i) = C_R(b)$, then $C_R(x_{k_i}) \leqslant C_R(b)$. On the other hand, if $C_R(w_i) = C_R(x_{k_i})$, then $C_R(w_i)$ is commutative and hence, $C_R(w_i) \leqslant C_R(b)$ and so, $C_R(x_{k_i}) \leqslant C_R(b)$. In both situations, we have $R = C_R(a) \cup C_R(b)$. Therefore, by Lemma 2.2.1, we obtain $|R : C_R(b)| = 1$, which is a contradiction. Consequently, $|A| \neq 4$ and so, $|A| = 5$. It follows that $C_R(a)$ contains exactly two distinct $C_R(x_i)$'s, as claimed. By using a manner entirely similar to that used to prove $C_R(a)$ contains exactly two distinct $C_R(x_i)$'s, we will obtain $C_R(b)$ is also contains exactly two distinct $C_R(x_i)$'s.

In view of Lemma 1.3.1(c), we have $\{C_R(a), C_R(x_{k_1}), C_R(x_{k_2}), \cdots, C_R(x_{k_5})\}$ is an irredundant cover of R for five distinct $k_1, k_2, \cdots, k_5 \in \{1, 2, \cdots, 7\}$ with $k_1 < k_2 < \cdots < k_5$. Here, we claim that $|R : Z(R)| = 16$. We distinguish our proof into the following two cases.

Case 1: $C_R(x_{k_1}) \cap C_R(x_{k_2}) = Z(R)$. By Lemma 2.7.3, we have $|R| \leqslant \frac{|R|}{4} + 4|C_R(x_{k_2})|$, which gives that $\gamma_{k_2} \leqslant 5$. Therefore, it follows from Lemma 2.2.11 that $|R : Z(R)| \leqslant \gamma_{k_2}^2$. If $\gamma_{k_2} = 5$, then $|R : Z(R)| \leqslant 25$, which leads to a contradiction. Therefore, $\gamma_{k_2} = 4$. It follows that $|R : Z(R)| \leqslant 16$ and consequently, $|R : Z(R)| = 16$.

Case 2: $C_R(x_{k_1}) \cap C_R(x_{k_2}) \neq Z(R)$. This implies that there exists some $r \in (C_R(x_{k_1}) \cap C_R(x_{k_2})) - Z(R)$, which follows that $C_R(x_{k_1}) \cup C_R(x_{k_2}) \subseteq C_R(r)$. Clearly, $C_R(r) \neq R, C_R(x_i)$ for any $i \in \{1, 2, \cdots, 7\}$. If $C_R(r) = C_R(a)$, then $C_R(x_{k_1}) \cup C_R(x_{k_2}) \subseteq C_R(a)$, which contradicts the definition of irredundant cover of R. So, we obtain $C_R(r) = C_R(b)$, which gives that $C_R(x_{k_1}) \cup C_R(x_{k_2}) \subseteq C_R(b)$. Since $C_R(b)$ contains exactly two distinct $C_R(x_i)$'s, then we have $\{C_R(a), C_R(b), C_R(x_{k_3}), C_R(x_{k_4}), C_R(x_{k_5})\}$ is an irredundant cover of R. We claim that $C_R(x_{k_3}) \cap C_R(x_{k_4}) = Z(R)$. Let $w \in C_R(x_{k_3}) \cap C_R(x_{k_4})$. Thus, we have $C_R(x_{k_3}) \cup C_R(x_{k_4}) \subseteq C_R(w)$. It is obvious that $C_R(w) \neq C_R(x_i)$ for any $i \in \{1, 2, \cdots, 7\}$. If $C_R(w) = C_R(a)$ or $C_R(b)$, then $C_R(x_{k_1}) \cup C_R(x_{k_2}) \subseteq C_R(a)$ or $C_R(b)$, which contradicts the definition of irredundant cover of R. So, we obtain $C_R(w) = R$, which implies that $w \in Z(R)$. This gives that $C_R(x_{k_3}) \cap C_R(x_{k_4}) \leqslant Z(R)$. On the other hand, it is clear that $Z(R) \leqslant C_R(x_{k_3}) \cap C_R(x_{k_4})$. Hence, $C_R(x_{k_3}) \cap C_R(x_{k_4}) = Z(R)$, as claimed. So, we have $|R : Z(R)| \leqslant f(5) = 16$ and consequently, $|R : Z(R)| = 16$.

By these two cases, we obtain $|R : Z(R)| = 16$, as desired. Since $C_R(a)$ is non-commutative, then by Lemma 2.2.15, we have $|R : C_R(a)| = 2$. If $\gamma_{k_3} \neq 4$, then by Lemma 2.7.3, we have $|R| \leqslant 2(\frac{|R|}{4}) + 3(\frac{|R|}{8}) = \frac{7}{8}|R|$, which is impossible. So, we have $\gamma_{k_1} = \gamma_{k_2} = \gamma_{k_3} = 4$. Let $i \in \{1, 2, 3\}$. By Lemma 2.2.11, it follows that $C_R(a) \cap C_R(x_{k_i}) \neq Z(R)$. So, there exists some $w_i \in (C_R(a) \cap C_R(x_{k_i})) - Z(R)$. Since $C_R(x_{k_i})$ is commutative, then $C_R(x_{k_i}) \leqslant C_R(w_i)$. It is obvious that $C_R(w_i) \neq R, C_R(x_j)$ for any $j \in \{1, 2, \cdots, 7\} - \{k_i\}$. If $C_R(w_i) = C_R(a)$, then $C_R(x_{k_i}) \leqslant C_R(a)$. On the other hand, if $C_R(w_i) = C_R(x_{k_i})$, then $C_R(w_i)$ is commutative and hence, $C_R(w_i) \leqslant C_R(a)$ and therefore, $C_R(x_{k_i}) \leqslant C_R(a)$. In both situations, we have reached a contradiction as $C_R(x_{k_i}) \nleqslant C_R(a)$. So, we obtain $C_R(w_i) = C_R(b)$, which gives that $C_R(x_{k_i}) \leqslant C_R(b)$. This implies that $C_R(b)$ contains three distinct $C_R(x_i)$'s. This contradicts with the fact that $C_R(b)$ contains exactly two distinct $C_R(x_i)$'s. Consequently, $t \neq 7$. □

Lemma 2.7.8. Let t be the cardinality of the maximal non-commuting set of a finite ring R. If R is a 10-centraliser finite ring, then $t \neq 8$.

Proof. Assume that $t = 8$. Let $\{x_1, x_2, \cdots, x_8\}$ be the maximal non-commuting set of R. Without loss of generality, we suppose that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_8$. From Lemma 1.3.1(c), we have $\{C_R(x_i) \mid i = 1, 2, \cdots, 8\}$ is an irredundant cover of R. By Lemma 2.2.3, we have $C_R(x_i)$ is commutative for any $i \in \{1, 2, \cdots, 8\}$, and $C_R(a)$ is non-commutative for some $a \in R - Z(R)$.

By Lemma 2.2.6, we have $\{C_R(a)\} \cup \left(\underset{i \in A}{\cup} \{C_R(x_i)\}\right)$ is an irredundant cover of R for some $A \subset \{1, 2, \cdots, 8\}$ with $|A| \leqslant 5$. Clearly, $|A| \neq 0$. If $|A| = 1$, then by Lemma 2.2.1, it follows that $\gamma_i = 1$ for some $i \in A$, which is a contradiction. Therefore, $|A| = 2, 3, 4$ or 5. Now, we claim that if $i \in A$, then $C_R(x_i) \cap C_R(a) = Z(R)$. This claim can be proved by using a manner entirely similar to that used to prove Lemma 2.6.4. Thus, we have $|R : Z(R)| \leqslant \max\{f(3), f(4), f(5), f(6)\} = 36$. Therefore, by Theorem 2.2.23, we obtain $|R : Z(R)| = 16, 24, 27, 32$ or 36. If $|R : Z(R)| = 27$, then by Lemma 2.2.16 and Lemma 2.2.4, we obtain $|\mathrm{Cent}(R)| = 9$, which is a contradiction. So,

$|R : Z(R)| = 16, 24, 32$ or 36. Since $C_R(a)$ is non-commutative, then by Lemma 2.2.15, we have

$$|R : C_R(a)| \begin{cases} = 2 & \text{if } |R : Z(R)| = 16, \\ \leqslant 3 & \text{if } |R : Z(R)| = 24 \text{ or } 36, \\ \leqslant 4 & \text{if } |R : Z(R)| = 32. \end{cases}$$

Since $C_R(x_i) \cap C_R(a) = Z(R)$ for any $i \in A$, then by Lemma 2.2.11, we obtain $\gamma_i \geqslant 8$ for any $i \in A$. But, by Lemma 2.2.1, we have $\gamma_i \leqslant |A| \leqslant 5$ for some $i \in A$. We have reached a contradiction. Consequently, $t \neq 8$. □

Lemma 2.7.9. Let $\{x_1, x_2, \cdots, x_9\}$ be the maximal non-commuting set of a finite ring R. Let $|R : C_R(x_1)| \leqslant |R : C_R(x_2)| \leqslant \cdots \leqslant |R : C_R(x_9)|$. If R is a 10-centraliser finite ring, then R satisfies one of the following structures:

(a) $|R : C_R(x_i)| = 4$ for any $i \in \{1, 2, 3\}$, $|R : C_R(x_i)| = 8$ for any $i \in \{4, 5, \cdots, 9\}$ and $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

(b) $|R : C_R(x_1)| = 2^{\mu-3}$, $|R : C_R(x_i)| = 8$ for any $i \in \{2, 3, \cdots, 9\}$ and $R/Z(R) \cong \mathbb{Z}_2^{\mu}$ for some $\mu \in \{4, 5, 6\}$.

Proof. From Lemma 1.3.1(a), we have $R = \bigcup_{i=1}^{9} C_R(x_i)$. By Corollary 2.2.5, we have $C_R(x_i) \cap C_R(x_j) = Z(R)$ for any two distinct $i, j \in \{1, 2, \cdots, 9\}$. Let $|R : C_R(x_i)| = \gamma_i$ for any $i \in \{1, 2, \cdots, 9\}$. By Lemma 2.2.11, we have $|R : Z(R)| \leqslant \gamma_2^2$. In view of Lemma 2.2.1 and Lemma 2.2.14, we have $4 \leqslant \gamma_2 \leqslant 8$. By Theorem 2.2.23, we have $|R : Z(R)| \geqslant 16$ with $|R : Z(R)|$ is not square-free, $|R : Z(R)| \neq p^2 q$ for any two distinct primes p, q, and $|R : Z(R)| \neq p^2$ for any prime p. For the sake of simplicity, we write $\overline{r} = r + Z(R)$ for any $r \in R$ and $\overline{S} = S/Z(R)$ for any $S \leqslant R$.

If $\gamma_2 = 5$, then $|\overline{R}| \leqslant 25$, which is a contradiction. If $\gamma_2 = 7$, then $|\overline{R}| \leqslant 49$, which is a contradiction again. Assume that $\gamma_2 = 6$. Therefore, $|\overline{R}| \leqslant 36$. By Lemma 2.2.11, we have $|\overline{C_R(x_1)}| \leqslant 6$. If $|\overline{R}| = 24$, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_{12}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ as $\overline{R}$ is not cyclic. Thus, $|\overline{C_R(x_1)}| \leqslant 6$ and $|\overline{C_R(x_i)}| \leqslant 4$ for any $i \in \{2, 3, \cdots, 9\}$. This leads to $\overline{R}$ has at most 2 elements of order 6. Also, there does not exist any element of order 12 in $\overline{R}$. We have reached a contradiction as $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ has 14 elements of order 6 and $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ has an element of order 12. If $|\overline{R}| = 36$, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_{18}, \mathbb{Z}_3 \times \mathbb{Z}_{12}$ or $\mathbb{Z}_6 \times \mathbb{Z}_6$ as $\overline{R}$ is not cyclic. Thus, $|\overline{C_R(x_i)}| \leqslant 6$ for any $i \in \{1, 2, \cdots, 9\}$. This shows that $\overline{R}$ has at most 18 elements of order 6. Also, there does not exist any element of order 12 and order 18 in $\overline{R}$. This leads to a contradiction as $\mathbb{Z}_6 \times \mathbb{Z}_6$ has 24 elements of order 6, $\mathbb{Z}_3 \times \mathbb{Z}_{12}$ has an element of order 12 and $\mathbb{Z}_2 \times \mathbb{Z}_{18}$ has an element of order 18.

Here, we consider for $|\overline{R}| = 16$. It follows that $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4 \times \mathbb{Z}_4$ as $\overline{R}$ is not cyclic. Since $|\overline{C_R(x_i)}|$ is even for any $i \in \{1, 2, \cdots, 9\}$, then $\overline{R}$ has at least 9 elements of order 2. Since $\mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_4 \times \mathbb{Z}_4$ does not have 9 elements of order 2, then $\overline{R} \ncong \mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ and $\mathbb{Z}_4 \times \mathbb{Z}_4$. Therefore, $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

If $\gamma_2 = 4$, then $|\overline{R}| \leqslant 16$, which gives that $|\overline{R}| = 16$. So, we have

$\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. By Lemma 2.2.11, we have $\gamma_1 = 4$. Since $|\overline{R}| = \sum_{i=1}^{9} |\overline{C_R(x_i)}| - 8$, then we have $\sum_{i=3}^{9} |\overline{C_R(x_i)}| = 16$. Thus, it can be easily seen that $\gamma_3 = 4, \gamma_4 = \gamma_5 = \cdots = \gamma_9 = 8$.

If $\gamma_2 = 8$, then $|\overline{R}| \leqslant 64$. In view of Lemma 2.2.1, we have $\gamma_2 = \gamma_3 = \cdots = \gamma_9 = 8$. Since $|\overline{R}| = \sum_{i=1}^{9} |\overline{C_R(x_i)}| - 8$, then we have $\gamma_1 = \frac{|\overline{R}|}{8}$. If $|\overline{R}| = 16$, then we have $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. If $|\overline{R}| = 24$, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_{12}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ as $\overline{R}$ is not cyclic. It follows that $|\overline{C_R(x_1)}| = 8$ and $|\overline{C_R(x_i)}| = 3$ for any $i \in \{2, 3, \cdots, 9\}$. This shows that there does not exist any element of order 6 and order 12 in $\overline{R}$. This contradicts with the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ has an element of order 6 and $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ has an element of order 12. Here, we let $_m|G|$ denote the total number of elements with order m in an additive group G. If $|\overline{R}| = 32$, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_{16}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4 \times \mathbb{Z}_8$ as $\overline{R}$ is not cyclic. Thus, $|\overline{C_R(x_1)}| = 8$ and $|\overline{C_R(x_i)}| = 4$ for any $i \in \{2, 3, \cdots, 9\}$. Consequently, $\overline{R}$ has at least 9 elements of order 2 and $\overline{R}$ has at most $_8|\mathbb{Z}_8|$ elements of order 8. Also, $\overline{R}$ does not have any element of order 16. Since $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ does not have 9 elements of order 2, then $\overline{R} \ncong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4$. Since $_8|\mathbb{Z}_8| < {}_8|\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8|$ and $_8|\mathbb{Z}_8| < {}_8|\mathbb{Z}_4 \times \mathbb{Z}_8|$, then $\overline{R} \ncong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8$ and $\mathbb{Z}_4 \times \mathbb{Z}_8$. Since $\mathbb{Z}_2 \times \mathbb{Z}_{16}$ has an element of order 16, then $\overline{R} \ncong \mathbb{Z}_2 \times \mathbb{Z}_{16}$. Suppose that $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ has 16 elements of order 4, then there exist two distinct $l_1, l_2 \in \{2, 3, \cdots, 9\}$ such that $\overline{C_R(x_{l_1})} \cong \overline{C_R(x_{l_2})} \cong \mathbb{Z}_4$. It follows that $\overline{C_R(x_{l_1})} = \langle \overline{a} \rangle = \{\overline{0}, \overline{a}, \overline{2a}, \overline{3a}\}$ and $\overline{C_R(x_{l_2})} = \langle \overline{b} \rangle = \{\overline{0}, \overline{b}, \overline{2b}, \overline{3b}\}$ for some $\overline{a} \in \overline{C_R(x_{l_1})} - \overline{Z(R)}$ and $\overline{b} \in \overline{C_R(x_{l_2})} - \overline{Z(R)}$. Thus, we have $\overline{C_R(2a+b)} \supseteq$

$\langle\overline{2a+b}\rangle = \{\overline{0}, \overline{2a+b}, \overline{2b}, \overline{2a+3b}\}$. Since $\overline{C_R(2a+b)} \cap \overline{C_R(x_{l_2})} \neq \overline{Z(R)}$, then by Corollary 2.2.5, we obtain $\overline{C_R(2a+b)} = \overline{C_R(x_{l_2})}$. This gives that $\overline{2a+b} = \overline{b}$ or $\overline{3b}$. So, we obtain $\overline{2a} = \overline{0}$ or $\overline{2b}$, which is a contradiction. Consequently, we have $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. If $|\overline{R}| = 40$, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_{20}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{10}$ as $\overline{R}$ is not cyclic. Therefore, $|\overline{C_R(x_1)}| = 8$ and $|\overline{C_R(x_i)}| = 5$ for any $i \in \{2, 3, \cdots, 9\}$. This leads to there does not exist any element of order 10 and order 20 in $\overline{R}$, which contradicts the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{10}$ has an element of order 10 and $\mathbb{Z}_2 \times \mathbb{Z}_{20}$ has an element of order 20. If $|\overline{R}| = 48$, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_{24}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{12}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ or $\mathbb{Z}_4 \times \mathbb{Z}_{12}$ as $\overline{R}$ is not cyclic. Thus, $|\overline{C_R(x_1)}| = 8$ and $|\overline{C_R(x_i)}| = 6$ for any $i \in \{2, 3, \cdots, 9\}$. It follows that $\overline{R}$ has at most 16 elements of order 6. Also, there does not exist any element of order 12 and order 24 in $\overline{R}$. We have reached a contradiction as $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ has 30 elements of order 6, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{12}, \mathbb{Z}_4 \times \mathbb{Z}_{12}$ have an element of order 12 and $\mathbb{Z}_2 \times \mathbb{Z}_{24}$ has an element of order 24. If $|\overline{R}| = 56$, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_{28}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{14}$ as $\overline{R}$ is not cyclic. Hence, $|\overline{C_R(x_1)}| = 8$ and $|\overline{C_R(x_i)}| = 7$ for any $i \in \{2, 3, \cdots, 9\}$. It follows that there does not exist any element of order 14 and order 28 in $\overline{R}$, which leads to a contradiction as $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{14}$ has an element of order 14 and $\mathbb{Z}_2 \times \mathbb{Z}_{28}$ has an element of order 28. If $|\overline{R}| = 64$, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_{32}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{16}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_{16}, \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ or $\mathbb{Z}_8 \times \mathbb{Z}_8$ as $\overline{R}$ is not cyclic. Thus, $|\overline{C_R(x_i)}| = 8$ for any $i \in \{1, 2, \cdots, 9\}$. This implies that $\overline{R}$ has at least 9 elements of order 2 and $\overline{R}$ has at most $9({}_4|\mathbb{Z}_2 \times \mathbb{Z}_4|) = 36$ elements of order 4. Also, there does not exist any element of order 16 and order 32 in $\overline{R}$. Since $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ and $\mathbb{Z}_8 \times \mathbb{Z}_8$ does not have 9 elements of order

2, then $\overline{R} \ncong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ and $\mathbb{Z}_8 \times \mathbb{Z}_8$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ has 48 elements of order 4, then $\overline{R} \ncong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{16}$ and $\mathbb{Z}_4 \times \mathbb{Z}_{16}$ have an element of order 16, then $\overline{R} \ncong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{16}$ and $\mathbb{Z}_4 \times \mathbb{Z}_{16}$. Since $\mathbb{Z}_2 \times \mathbb{Z}_{32}$ has an element of order 32, then $\overline{R} \ncong \mathbb{Z}_2 \times \mathbb{Z}_{32}$. Suppose that $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8$ has 32 elements of order 8, then there exist two distinct $l_1, l_2 \in \{1, 2, \cdots, 9\}$ such that $\overline{C_R(x_{l_1})} \cong \mathbb{Z}_8$ and $\overline{C_R(x_{l_2})} \cong \mathbb{Z}_8$. Thus, there exist some $\overline{a} \in \overline{C_R(x_{l_1})} - \overline{Z(R)}, \overline{b} \in \overline{C_R(x_{l_2})} - \overline{Z(R)}$ such that $\overline{C_R(x_{l_1})} = \{\overline{0}, \overline{a}, \overline{2a}, \cdots, \overline{7a}\}$ and $\overline{C_R(x_{l_2})} = \{\overline{0}, \overline{b}, \overline{2b}, \cdots, \overline{7b}\}$. This implies that $\overline{R} = \{\overline{ma + nb} \mid m, n \in \mathbb{Z}_8\} \cong \mathbb{Z}_8 \times \mathbb{Z}_8$, which is a contradiction. Assume that $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ has 32 elements of order 4, then there exist two distinct $l_1, l_2 \in \{1, 2, \cdots, 9\}$ such that $\overline{C_R(x_{l_1})} \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ and $\overline{C_R(x_{l_2})} \cong \mathbb{Z}_2 \times \mathbb{Z}_4$. Since $|\overline{C_R(x_{l_1})} + \overline{C_R(x_{l_2})}| = \frac{|\overline{C_R(x_{l_1})}||\overline{C_R(x_{l_2})}|}{|\overline{C_R(x_{l_1})} \cap \overline{C_R(x_{l_2})}|} = 64$, then we have $\overline{R} = \overline{C_R(x_{l_1})} + \overline{C_R(x_{l_2})}$. It can be easily checked that if $\overline{a} \in \overline{C_R(x_{l_1})}, \overline{b} \in \overline{C_R(x_{l_2})}$ with order of $\overline{a}$ is 4 or order of $\overline{b}$ is 4 but not both, then the order of $\overline{a + b}$ is 4. This implies that $\overline{R}$ has at least 32 elements of order 4. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ has exactly 32 elements of order 4, then there exist some $\overline{u} \in \overline{C_R(x_{l_1})}, \overline{v} \in \overline{C_R(x_{l_2})}$ with order of $\overline{u}$ and $\overline{v}$ are 4 such that the order of $\overline{u + v}$ is not 4. Thus, the order of $\overline{u + v}$ is 1 or 2. It follows that $\overline{mu + mv} = \overline{0}$ for some $m \in \{1, 2\}$, which gives that $\overline{mu} = \overline{-mv}$. Since $\overline{mu} \in \overline{C_R(x_{l_1})}, \overline{-mv} \in \overline{C_R(x_{l_2})}$, then $\overline{mu} \in \overline{C_R(x_{l_1})} \cap \overline{C_R(x_{l_2})} = \overline{Z(R)}$. This yields that the order of $\overline{u}$ is 1 or 2, which is a contradiction. Consequently, we have $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. □

Theorem 2.7.10. *Let R be a 10-centraliser finite ring. Let $X_1, X_2, \cdots, X_9$ be the 9 distinct proper centralisers of R with $|R : X_1| \leqslant |R : X_2| \leqslant \cdots \leqslant |R : X_9|$.*

Let t *be the cardinality of the maximal non-commuting set of* R*. Then* R *satisfies one of the following structures:*

(a) $t = 6$, $|R : X_i| = 2$ *for any* $i \in \{1, 2, 3\}$, $|R : X_i| = 4$ *for any* $i \in \{4, 5, \cdots, 9\}$, $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *or* $\mathbb{Z}_4 \times \mathbb{Z}_4$*, and* $\mathrm{Prob}(R) = \frac{11}{32}$.

(b) $t = 9$, $|R : X_i| = 4$ *for any* $i \in \{1, 2, 3\}$, $|R : X_i| = 8$ *for any* $i \in \{4, 5, \cdots, 9\}$, $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *and* $\mathrm{Prob}(R) = \frac{1}{4}$.

(c) $t = 9$, $|R : X_1| = 2^{\mu-3}$, $|R : X_i| = 8$ *for any* $i \in \{2, 3, \cdots, 9\}$, $R/Z(R) \cong \mathbb{Z}_2^{\mu}$ *and* $\mathrm{Prob}(R) = \frac{1}{8} + \frac{7}{2^{2\mu-3}}$ *for some* $\mu \in \{4, 5, 6\}$.

Proof. In view of Lemma 1.3.1(d)-(g), Lemma 2.7.1, Lemma 2.7.7 and Lemma 2.7.8, we have $t = 6$ or 9. First, we consider for $t = 6$. Let $\{x_1, x_2, \cdots, x_6\}$ be the maximal non-commuting set of R. By Lemma 2.7.2 and Lemmas 2.7.4-2.7.6, we have $|R : C_R(x_i)| = 4$ for any $i \in \{1, 2, \cdots, 6\}$, and $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4 \times \mathbb{Z}_4$. From Lemma 1.3.1(a), we have $R = \bigcup_{i=1}^{6} C_R(x_i)$. By Lemma 2.2.15, we have $C_R(x_i)$ is commutative for any $i \in \{1, 2, \cdots, 6\}$. We claim that $C_R(x_i) \cap C_R(x_j) \cap C_R(x_k) = Z(R)$ for any three distinct $i, j, k \in \{1, 2, \cdots, 6\}$. If not, then there exists some $r \in (C_R(x_i) \cap C_R(x_j) \cap C_R(x_k)) - Z(R)$ for three distinct $i, j, k \in \{1, 2, \cdots, 6\}$, which follows that $C_R(x_i) \cup C_R(x_j) \cup C_R(x_k) \subseteq C_R(r)$. Therefore, $R = C_R(r) \cup \left(\bigcup_{l=1, l \neq i,j,k}^{6} C_R(x_l) \right)$. By Lemma 2.2.1, it follows that $\gamma_l \leqslant 3$ for some $l \in \{1, 2, \cdots, 6\} - \{i, j, k\}$, which leads to a contradiction. So, our claim is true. For the sake of simplicity, we write $\overline{r} = r + Z(R)$ for any $r \in R$ and $\overline{S} = S/Z(R)$ for any $S \leqslant R$. If $\overline{C_R(x_i)} \cap \overline{C_R(x_j)} = \overline{Z(R)}$ for any two distinct $i, j \in \{1, 2, \cdots, 6\}$, then $|\overline{R}| = \sum_{i=1}^{6} |\overline{C_R(x_i)}| - 5 = 19$, which is impossible. So, $\overline{C_R(x_{k_1})} \cap \overline{C_R(x_{k_2})} \neq \overline{Z(R)}$ for two distinct $k_1, k_2 \in \{1, 2, \cdots, 6\}$. Without loss of generality, we assume that $k_1 = 1, k_2 = 4$. It follows that there exists some $\overline{b_1} \in (\overline{C_R(x_1)} \cap \overline{C_R(x_4)}) - \overline{Z(R)}$, which implies that $C_R(x_1) \cup C_R(x_4) \subseteq C_R(b_1)$. Since $|C_R(b_1)| > |C_R(x_1)|$, then $|R : C_R(b_1)| = 2$. Therefore, by Lemma 2.2.11, we have $C_R(b_1) \cap C_R(x_u) \neq Z(R)$ for any $u \in \{2, 3, 5, 6\}$. Let $u \in \{2, 3, 5, 6\}$. Thus, there exists some $\overline{b_u} \in (\overline{C_R(b_1)} \cap \overline{C_R(x_u)}) - \overline{Z(R)}$. Since $C_R(x_u)$ is commutative, then $C_R(x_u) \leqslant C_R(b_u)$. Clearly, $C_R(b_u) \neq R, C_R(x_i)$ for any $i \in \{1, 2, \cdots, 6\} - \{u\}$. If $C_R(b_u) = C_R(b_1)$, then $C_R(x_u) \leqslant C_R(b_1)$. On the other hand, if $C_R(b_u) = C_R(x_u)$, then $C_R(b_u)$ is commutative and hence, $C_R(b_u) \leqslant C_R(b_1)$ and so, $C_R(x_u) \leqslant C_R(b_1)$. In both situations, we obtain $b_1 \in C_R(x_1) \cap C_R(x_4) \cap C_R(x_u) = Z(R)$, which is a contradiction. Since $|\mathrm{Cent}(R)| = 10$, then $C_R(b_{l_1}) = C_R(b_{l_2})$ for two distinct $l_1, l_2 \in \{2, 3, 5, 6\}$. If $C_R(b_{l_1}) = C_R(b_{l_3})$ for some $l_3 \in \{2, 3, 5, 6\} - \{l_1, l_2\}$, then we obtain $b_{l_1} \in C_R(x_{l_1}) \cap C_R(x_{l_2}) \cap C_R(x_{l_3}) = Z(R)$, which is a contradiction. Thus, we have $C_R(b_{l_1}) = C_R(b_{l_2}) \neq C_R(b_{l_3}), C_R(b_{l_4})$, where $l_3, l_4 \in \{2, 3, 5, 6\} - \{l_1, l_2\}$ with $l_3 \neq l_4$. Since $|\mathrm{Cent}(R)| = 10$, then $C_R(b_{l_3}) = C_R(b_{l_4})$. Without loss of generality, we assume that $l_1 = 2, l_2 = 5, l_3 = 3, l_4 = 6$. Thus, we have $\overline{b_2} \in \overline{C_R(x_2)} \cap \overline{C_R(x_5)}$ and $\overline{b_3} \in \overline{C_R(x_3)} \cap \overline{C_R(x_6)}$. This gives that

$$\overline{C_R(x_1)} = \{\overline{0}, \overline{x_1}, \overline{b_1}, \overline{x_1 + b_1}\},$$

$$\overline{C_R(x_2)} = \{\overline{0}, \overline{x_2}, \overline{b_2}, \overline{x_2 + b_2}\},$$

$$\overline{C_R(x_3)} = \{\overline{0}, \overline{x_3}, \overline{b_3}, \overline{x_3 + b_3}\},$$

$$\overline{C_R(x_4)} = \{\overline{0}, \overline{x_4}, \overline{b_1}, \overline{x_4 + b_1}\},$$

$$\overline{C_R(x_5)} = \{\overline{0}, \overline{x_5}, \overline{b_2}, \overline{x_5 + b_2}\},$$

$$\overline{C_R(x_6)} = \{\overline{0}, \overline{x_6}, \overline{b_3}, \overline{x_6 + b_3}\},$$

$$\overline{C_R(b_1)} \supset \{\overline{0}, \overline{x_1}, \overline{b_1}, \overline{x_1+b_1}, \overline{x_4}, \overline{x_4+b_1}\} \text{ with } |\overline{C_R(b_1)}| = 8,$$

$$\overline{C_R(b_2)} \supset \{\overline{0}, \overline{x_2}, \overline{b_2}, \overline{x_2+b_2}, \overline{x_5}, \overline{x_5+b_2}\} \text{ with } |\overline{C_R(b_2)}| = 8,$$

$$\overline{C_R(b_3)} \supset \{\overline{0}, \overline{x_3}, \overline{b_3}, \overline{x_3+b_3}, \overline{x_6}, \overline{x_6+b_3}\} \text{ with } |\overline{C_R(b_3)}| = 8.$$

Here, we claim that $C_R(x_i+b_j) = C_R(x_i)$ for any $i \in \{j, j+3\}$ and $j \in \{1,2,3\}$. Let $i \in \{j, j+3\}$ and $j \in \{1,2,3\}$. Since $C_R(x_i)$ is commutative, then $C_R(x_i) \leqslant C_R(x_i+b_j)$. Obviously, $C_R(x_i+b_j) \neq R, C_R(x_k)$ for any $k \in \{1,2,\cdots,6\}-\{i\}$. Since $x_j, x_{j+3} \in C_R(b_j)$ but $x_j \notin C_R(x_{j+3}+b_j)$ and $x_{j+3} \notin C_R(x_j+b_j)$, then $C_R(x_i+b_j) \neq C_R(b_j)$. If $C_R(x_i+b_j) = C_R(b_u)$ for some $u \in \{1,2,3\}-\{j\}$, then $x_i+b_j \in C_R(x_i) \cap C_R(x_u) \cap C_R(x_{u+3}) = Z(R)$, which is a contradiction. This implies that $C_R(x_i+b_j) = C_R(x_i)$. For any $r \in R - Z(R)$, since $r = a+z$ for some $a \in \{x_1, x_2, x_3, x_4, x_5, x_6, b_1, b_2, b_3, x_1+b_1, x_2+b_2, x_3+b_3, x_4+b_1, x_5+b_2, x_6+b_3\}$ and $z \in Z(R)$, then we have $C_R(r) = C_R(a)$. Consequently, by (1.3), we obtain

$$\begin{aligned}
\text{Prob}(R) &= \frac{|Z(R)|}{|R|} + \frac{\sum\limits_{r \in R-Z(R)} |C_R(r)|}{|R|^2} \\
&= \frac{1}{16} + \frac{(3|Z(R)|)\left(\frac{|R|}{2}\right) + (12|Z(R)|)\left(\frac{|R|}{4}\right)}{|R|^2} \\
&= \frac{1}{16} + \frac{\left(\frac{3|R|}{16}\right)\left(\frac{|R|}{2}\right) + \left(\frac{12|R|}{16}\right)\left(\frac{|R|}{4}\right)}{|R|^2} \\
&= \frac{11}{32}.
\end{aligned}$$

Next, for $t = 9$, by Lemma 2.7.9, it follows that R satisfies one of the following structures:

(i) $|R : X_i| = 4$ for any $i \in \{1, 2, 3\}$, $|R : X_i| = 8$ for any $i \in \{4, 5, \cdots, 9\}$ and $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

(ii) $|R : X_1| = 2^{\mu-3}$, $|R : X_i| = 8$ for any $i \in \{2, 3, \cdots, 9\}$ and $R/Z(R) \cong \mathbb{Z}_2^{\mu}$ for some $\mu \in \{4, 5, 6\}$.

By Corollary 2.2.5, it follows that for any $r_1, r_2 \in R - Z(R)$, either $C_R(r_1) = C_R(r_2)$ or $C_R(r_1) \cap C_R(r_2) = Z(R)$. Consequently, by (1.3), the $\text{Prob}(R)$ of structures (i) and (ii) are

$$\begin{aligned}\text{Prob}(R) &= \frac{|Z(R)|}{|R|} + \frac{\sum\limits_{r \in R - Z(R)} |C_R(r)|}{|R|^2} \\ &= \frac{1}{16} + \frac{3\left(\frac{|R|}{4} - \frac{|R|}{16}\right)\left(\frac{|R|}{4}\right) + 6\left(\frac{|R|}{8} - \frac{|R|}{16}\right)\left(\frac{|R|}{8}\right)}{|R|^2} \\ &= \frac{1}{4}\end{aligned}$$

and

$$\begin{aligned}\text{Prob}(R) &= \frac{|Z(R)|}{|R|} + \frac{\sum\limits_{r \in R - Z(R)} |C_R(r)|}{|R|^2} \\ &= \frac{1}{2^{\mu}} + \frac{\left(\frac{|R|}{2^{\mu-3}} - \frac{|R|}{2^{\mu}}\right)\left(\frac{|R|}{2^{\mu-3}}\right) + 8\left(\frac{|R|}{8} - \frac{|R|}{2^{\mu}}\right)\left(\frac{|R|}{8}\right)}{|R|^2} \\ &= \frac{1}{8} + \frac{7}{2^{2\mu-3}},\end{aligned}$$

respectively. This completes the proof. □

We obtain a partial converse of Theorem 2.7.10, as follows:

Theorem 2.7.11. *If R is a finite ring with $R/Z(R) \cong \mathbb{Z}_4 \times \mathbb{Z}_4$, then* $|\mathrm{Cent}(R)| = 10$.

Proof. Let $\{x_1, x_2, \cdots, x_t\}$ be the maximal non-commuting set of R. Without loss of generality, we suppose that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_t$. By Lemma 1.3.1(a), we have $R = \bigcup_{i=1}^{t} C_R(x_i)$. For the sake of simplicity, we write $\overline{r} = r + Z(R)$ for any $r \in R$ and $\overline{S} = S/Z(R)$ for any $S \leqslant R$.

By the fact that $\mathbb{Z}_4 \times \mathbb{Z}_4$ has exactly 12 elements of order 4 and 3 elements of order 2, then $\overline{R}$ has exactly 6 cyclic subgroups of order 4 and 3 cyclic subgroups of order 2. Since every cyclic subgroup of order 4 contains an element of order 2, then for any cyclic subgroup of order 2, there exists some cyclic subgroup of order 4 such that the cyclic subgroup of order 2 is contained in the cyclic subgroup of order 4. Since every group is a union of cyclic subgroups, then $\overline{R}$ is a union of exactly 6 cyclic subgroups of order 4. This shows that there does not exist any 7 distinct elements in $\overline{R}$ such that they do not commute with each other. Therefore, we have $t \leqslant 6$. In view of Lemma 1.3.1(d), (f), (g), [A3] and [A4], we obtain $t = 5$ or 6.

First, we claim that if $\gamma_i = 2$ for some $i \in \{1, 2, \cdots, t\}$, then $\overline{C_R(x_i)} \cong \mathbb{Z}_2 \times \mathbb{Z}_4$. If $\gamma_i = 2$ for some $i \in \{1, 2, \cdots, t\}$, then $|\overline{C_R(x_i)}| = 8$. Therefore, $\overline{C_R(x_i)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_8$. If $\overline{C_R(x_i)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_8$, then $\overline{R}$ has at least 7 elements of order 2 or $\overline{R}$ has an element of order 8, which contradicts the fact that $\mathbb{Z}_4 \times \mathbb{Z}_4$ has exactly 3 elements of order 2 and $\mathbb{Z}_4 \times \mathbb{Z}_4$ does not exist any element of order 8. We now claim that $\gamma_i \leqslant 4$ for any $i \in \{1, 2, \cdots, t\}$. Assume that $\gamma_i = 8$ for some $i \in \{1, 2, \cdots, t\}$. Thus, $|\overline{C_R(x_i)}| = 2$ and so, the order of $\overline{x_i}$ is 2. If $\gamma_j = 2$ for some $j \in \{1, 2, \cdots, t\} - \{i\}$, then $\overline{C_R(x_j)} \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ and it follows that $\overline{C_R(x_j)}$ has exactly 3 elements of order 2. Since $\mathbb{Z}_4 \times \mathbb{Z}_4$ has exactly 3 elements of order 2, then $\overline{x_i} \in \overline{C_R(x_j)}$, which is impossible. Therefore, $\gamma_j \geqslant 4$ for any $j \in \{1, 2, \cdots, t\} - \{i\}$. Hence, we have $|\overline{C_R(x_j)}| \leqslant 4$ for any $j \in \{1, 2, \cdots, t\} - \{i\}$. This shows that $\overline{R}$ has at most $(t-1)(2) = 2t - 2 \leqslant 10$ elements of order 4, which leads to a contradiction as $\mathbb{Z}_4 \times \mathbb{Z}_4$ has 12 elements of order 4. Next, we claim that if $\gamma_i = 2$ for some $i \in \{1, 2, \cdots, t\}$, then the order of $\overline{x_i}$ is 2. Assume that the order of $\overline{x_i}$ is 4 or 8. Since $\overline{C_R(x_i)} \cong \mathbb{Z}_2 \times \mathbb{Z}_4$, then the order of $\overline{x_i}$ is 4. It follows that $\overline{0}, \overline{x_i}, \overline{2x_i}, \overline{3x_i} \in \overline{Z(C_R(x_i))}$ and hence, $|\overline{Z(C_R(x_i))}| \geqslant 4$. This implies that $|C_R(x_i) : Z(C_R(x_i))| \leqslant 2$. If $|C_R(x_i) : Z(C_R(x_i))| = 1$, then $C_R(x_i)$ is commutative. On the other hand, if $|C_R(x_i) : Z(C_R(x_i))| = 2$, then $C_R(x_i)/Z(C_R(x_i))$ is cyclic, which follows that $C_R(x_i)$ is commutative. In both situations, we obtain $C_R(x_i)$ is commutative. Therefore, by Lemma 2.2.12, we obtain $|\overline{R}| \leqslant 2(4) = 8$, which is a contradiction. Here, we claim that $\gamma_i = 4$ for any $i \in \{1, 2, \cdots, t\}$. We first assume that $\gamma_i = \gamma_j = 2$ for two distinct $i, j \in \{1, 2, \cdots, t\}$. Thus, we have $\overline{C_R(x_i)} \cong \mathbb{Z}_2 \times \mathbb{Z}_4$. This shows that $\overline{C_R(x_i)}$ has exactly 3 elements of order 2. Note that, the order of $\overline{x_j}$ is 2. Since $\mathbb{Z}_4 \times \mathbb{Z}_4$ has exactly 3 elements of order 2, it follows that $\overline{x_j} \in \overline{C_R(x_i)}$, which is impossible. Next, we assume that $\gamma_1 = 2$ and $\gamma_i = 4$ for any $i \in \{2, 3, \cdots, t\}$. Hence, we have $\overline{C_R(x_1)} \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ and $|\overline{C_R(x_i)}| = 4$ for any $i \in \{2, 3, \cdots, t\}$. Let w_1, w_2, w_3 be three distinct elements of order 4 in $\overline{C_R(x_1)}$. If for any $j \in \{1, 2, 3\}$, $\overline{w_j} \in \overline{C_R(x_i)}$ for some $i \in \{2, 3, \cdots, t\}$, then $\overline{R}$ has at most $[4 + (t-1)(2)] - 3 = 2t - 1 \leqslant 11$ elements of order 4, which contradicts the fact that $\mathbb{Z}_4 \times \mathbb{Z}_4$ has 12 elements of order 4. Therefore, there exists some $j \in \{1, 2, 3\}$ such that $\overline{w_j} \notin \overline{C_R(x_i)}$ for any $i \in \{2, 3, \cdots, t\}$. Note that, the orders of $\overline{x_1}$ and $\overline{w_j}$ are 2 and 4, respectively. Thus, $\overline{C_R(x_1)}$ can be written as $\overline{C_R(x_1)} = \{\overline{0}, \overline{x_1}, \overline{w_j}, \overline{2w_j}, \overline{3w_j}, \overline{x_1 + w_j}, \overline{x_1 + 2w_j}, \overline{x_1 + 3w_j}\}$. If $x_1 w_j = w_j x_1$, then $C_R(x_1)$ is commutative. Therefore, in view of Lemma 2.2.12, we obtain $|\overline{R}| \leqslant 2(4) = 8$, which leads to a contradiction. Therefore, $x_1 w_j \neq w_j x_1$. This gives that $\{w_j, x_1, x_2, \cdots, x_t\}$ is a non-commuting set of R with cardinality $t+1$, which contradicts the fact that the cardinality of the maximal non-commuting set of R is t. Consequently, $\gamma_i = 4$ for any $i \in \{1, 2, \cdots, t\}$, as claimed.

If $t = 5$, then $|\overline{C_R(x_i)}| = 4$ for any $i \in \{1, 2, \cdots, 5\}$. This leads to $\overline{R}$ has at most 10 elements of order 4. This contradicts with the fact that $\mathbb{Z}_4 \times \mathbb{Z}_4$ has 12 elements of order 4. So, $t = 6$. Since $\overline{R}$ has 12 elements of order 4, then $\overline{C_R(x_i)} \cong \mathbb{Z}_4$ for any $i \in \{1, 2, \cdots, 6\}$. Since $\overline{C_R(x_i)}$ is cyclic for any $i \in \{1, 2, \cdots, 6\}$, then $C_R(x_i)$ is commutative for any $i \in \{1, 2, \cdots, 6\}$. Let $\overline{b_1}, \overline{b_2}, \overline{b_3}$ be three distinct elements of order 2 in $\overline{R}$. We claim that $C_R(x_i) \cap C_R(x_j) \cap C_R(x_k) = Z(R)$ for any three distinct $i, j, k \in \{1, 2, \cdots, 6\}$. This claim can be proved in a manner entirely similar to that used to prove Theorem 2.7.10. Thus, without loss of generality, we have $\overline{b_1} \in \overline{C_R(x_1)} \cap \overline{C_R(x_4)}$, $\overline{b_2} \in \overline{C_R(x_2)} \cap \overline{C_R(x_5)}$, $\overline{b_3} \in \overline{C_R(x_3)} \cap \overline{C_R(x_6)}$. Hence, we have

$$\overline{C_R(x_1)} = \{\overline{0}, \overline{x_1}, \overline{b_1}, \overline{x_1 + b_1}\},$$

$$\overline{C_R(x_2)} = \{\overline{0}, \overline{x_2}, \overline{b_2}, \overline{x_2 + b_2}\},$$

$$\overline{C_R(x_3)} = \{\overline{0}, \overline{x_3}, \overline{b_3}, \overline{x_3 + b_3}\},$$

$$\overline{C_R(x_4)} = \{\overline{0}, \overline{x_4}, \overline{b_1}, \overline{x_4 + b_1}\},$$

$$\overline{C_R(x_5)} = \{\overline{0}, \overline{x_5}, \overline{b_2}, \overline{x_5 + b_2}\},$$

$$\overline{C_R(x_6)} = \{\overline{0}, \overline{x_6}, \overline{b_3}, \overline{x_6 + b_3}\},$$

$$\overline{C_R(b_1)} \supset \{\overline{0}, \overline{x_1}, \overline{b_1}, \overline{x_1 + b_1}, \overline{x_4}, \overline{x_4 + b_1}\} \text{ with } |\overline{C_R(b_1)}| = 8,$$

$$\overline{C_R(b_2)} \supset \{\overline{0}, \overline{x_2}, \overline{b_2}, \overline{x_2 + b_2}, \overline{x_5}, \overline{x_5 + b_2}\} \text{ with } |\overline{C_R(b_2)}| = 8,$$

$$\overline{C_R(b_3)} \supset \{\overline{0}, \overline{x_3}, \overline{b_3}, \overline{x_3 + b_3}, \overline{x_6}, \overline{x_6 + b_3}\} \text{ with } |\overline{C_R(b_3)}| = 8.$$

By using a manner entirely similar to that used to prove Theorem 2.7.10, we will obtain $C_R(x_i + b_j) = C_R(x_i)$ for any $i \in \{j, j+3\}$ and $j \in \{1, 2, 3\}$. For any $r \in R - Z(R)$, since $r = a + z$ for some $a \in \{x_1, x_2, x_3, x_4, x_5, x_6, b_1, b_2, b_3, x_1 + b_1, x_2 + b_2, x_3 + b_3, x_4 + b_1, x_5 + b_2, x_6 + b_3\}$ and $z \in Z(R)$, then we have $C_R(r) = C_R(a)$. Consequently, we obtain $|\mathrm{Cent}(R)| = 10$, as required. □

In general, the converse of Theorem 2.7.10 is not necessarily true. For example, $R_1 = \{\left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right] | \, a, b \in \mathbb{Z}_2\} \times \{\left[\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right] | \, a, b \in \mathbb{Z}_2\}$ is a 16-centraliser finite ring with $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $R_2 = \left\{ \left[\begin{smallmatrix} a & b & c & d & e \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \end{smallmatrix}\right] \middle| \, a, b, c, d, e \in \mathbb{Z}_2 \right\}$ is a 18-centraliser finite ring with $R_2/Z(R_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $R_3 = \left\{ \left[\begin{smallmatrix} a & b & c & d & e & f \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{smallmatrix}\right] \middle| \, a, b, c, d, e, f \in \mathbb{Z}_2 \right\}$ is a 34-centraliser finite ring with $R_3/Z(R_3) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. In the following, we provide an example of a 10-centraliser finite ring, which is appeared in the proof of Proposition 2.2.18.

Example 2.7.12. Let $M(a, b, c, d)$ be defined by $M(a, b, c, d) = \left[\begin{smallmatrix} a & b & c & d \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{smallmatrix}\right]$ for any $a, b, c, d \in \mathbb{Z}_2$. The ring $R = \{M(a, b, c, d) \mid a, b, c, d \in \mathbb{Z}_2\}$ is a 10-centraliser finite ring with

$$
\begin{aligned}
R &= C_R(M(0,0,0,0)),\\
X_1 &= C_R(M(0,1,0,0)) = C_R(M(0,0,1,0)) = C_R(M(0,0,0,1))\\
&= C_R(M(0,1,1,0)) = C_R(M(0,1,0,1)) = C_R(M(0,0,1,1))\\
&= C_R(M(0,1,1,1))\\
&= \{M(0,0,0,0), M(0,1,0,0)), M(0,0,1,0), M(0,0,0,1),\\
&\quad M(0,1,1,0), M(0,1,0,1), M(0,0,1,1), M(0,1,1,1)\},\\
X_2 &= C_R(M(1,0,0,0)) = \{M(0,0,0,0), M(1,0,0,0)\},\\
X_3 &= C_R(M(1,1,0,0)) = \{M(0,0,0,0), M(1,1,0,0)\},\\
X_4 &= C_R(M(1,0,1,0)) = \{M(0,0,0,0), M(1,0,1,0)\},\\
X_5 &= C_R(M(1,0,0,1)) = \{M(0,0,0,0), M(1,0,0,1)\},\\
X_6 &= C_R(M(1,1,1,0)) = \{M(0,0,0,0), M(1,1,1,0)\},\\
X_7 &= C_R(M(1,1,0,1)) = \{M(0,0,0,0), M(1,1,0,1)\},\\
X_8 &= C_R(M(1,0,1,1)) = \{M(0,0,0,0), M(1,0,1,1)\},\\
X_9 &= C_R(M(1,1,1,1)) = \{M(0,0,0,0), M(1,1,1,1)\}.
\end{aligned}
$$

We note that $\{M(0,1,0,0), M(1,0,0,0), M(1,1,0,0), M(1,0,1,0), M(1,0,0,1), M(1,1,1,0), M(1,1,0,1), M(1,0,1,1), M(1,1,1,1)\}$ is a non-commuting set of R with cardinality 9. Also, we note that there does not exist a non-commuting set of R with cardinality 10. Thus, the cardinality of the maximal non-commuting set of R is 9. Besides that, we have $|R : X_1| = 2$, $|R : X_i| = 8$ for any $i \in \{2,3,\cdots,9\}$. Since $Z(R) = \{M(0,0,0,0)\}$, then we have $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Lastly, from (1.3), we obtain

$$\begin{aligned}\text{Prob}(R) &= \frac{|Z(R)|}{|R|} + \frac{\sum\limits_{r \in R - Z(R)} |C_R(r)|}{|R|^2} \\ &= \frac{1}{16} + \frac{7(8) + 8(2)}{16^2} \\ &= \frac{11}{32}.\end{aligned}$$

2.8 11-Centraliser Finite Rings

In this section, we investigate the structure for all 11-centraliser finite rings and compute their commuting probabilities.

Lemma 2.8.1. Let $\{x_1, x_2, \cdots, x_6\}$ be the maximal non-commuting set of a finite ring R. If R is an 11-centraliser finite ring, then $|R : Z(R)| \neq 16$.

Proof. Without loss of generality, we suppose that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_6$. From Lemma 1.3.1(a), we have $R = \bigcup\limits_{i=1}^{6} C_R(x_i)$. By Lemma 2.7.2, we have $\gamma_2 = 4$. If $\gamma_4 \neq 4$, then by Lemma 2.7.3, we obtain $|R| \leqslant 2(\frac{|R|}{4}) + 3(\frac{|R|}{8}) = \frac{7|R|}{8}$, which is impossible. So, we have $\gamma_3 = \gamma_4 = 4$. For the sake of simplicity, we write $\overline{r} = r + Z(R)$ and $\overline{S} = S/Z(R)$ for any $S \leqslant R$.

We claim that $\gamma_1 = 4$. Suppose to the contrary that $\gamma_1 = 2$. By Lemma 2.2.11, we obtain $|\overline{C_R(x_1)} \cap \overline{C_R(x_2)}| = 2$. Hence, $\overline{C_R(x_1)} \cap \overline{C_R(x_2)} = \{\overline{0}, \overline{a}\}$ for some $\overline{a} \in \overline{R} - \overline{Z(R)}$. So, we have

$$\overline{C_R(x_1)} = \{\overline{0}, \overline{x_1}, \overline{a}, \overline{b}, \overline{a+b}, \overline{x_1+a}, \overline{x_1+b}, \overline{x_1+a+b}\},$$

$$\overline{C_R(x_2)} = \{\overline{0}, \overline{x_2}, \overline{a}, \overline{x_2+a}\}$$

for some $\overline{b} \in \overline{R} - \overline{Z(R)}$. If $ab = ba$, then $C_R(x_1)$ is commutative. Therefore, by Lemma 2.2.12, it follows that $|\overline{R}| \leqslant 2(4) = 8$; a contradiction. So, $ab \neq ba$. Thus, we have

$$\begin{aligned}
\overline{C_R(a)} &\supseteq \{\overline{0}, \overline{x_1}, \overline{x_2}, \overline{a}\}, \\
\overline{C_R(x_1+a)} &\supseteq \{\overline{0}, \overline{x_1}, \overline{a}, \overline{x_1+a}\}, \\
\overline{C_R(b)} &\supseteq \{\overline{0}, \overline{x_1}, \overline{b}, \overline{x_1+b}\}, \\
\overline{C_R(x_1+b)} &\supseteq \{\overline{0}, \overline{x_1}, \overline{b}, \overline{x_1+b}\}, \\
\overline{C_R(a+b)} &\supseteq \{\overline{0}, \overline{x_1}, \overline{a+b}, \overline{x_1+a+b}\}, \\
\overline{C_R(x_1+a+b)} &\supseteq \{\overline{0}, \overline{x_1}, \overline{a+b}, \overline{x_1+a+b}\}.
\end{aligned}$$

It can be easily checked that $R, C_R(x_1), C_R(x_2), \cdots, C_R(x_6), C_R(a), C_R(x_1 + a), A, B$ are 11 distinct centralisers of R for any $A \in \{C_R(b), C_R(x_1+b)\}$ and $B \in \{C_R(a+b), C_R(x_1+a+b)\}$. Since $|\mathrm{Cent}(R)| = 11$, then we obtain $C_R(b) = C_R(x_1+b)$ and $C_R(a+b) = C_R(x_1+a+b)$. Assume that $C_R(u)$ is non-commutative for some $u \in \{b, a+b\}$. By Lemma 2.2.15, $|R : C_R(u)| = 2$. Therefore, by Lemma 2.2.11, it follows that $|\overline{C_R(u)} \cap \overline{C_R(x_2)}| = 2$. Since $\overline{u} \notin \overline{C_R(x_2)}$, then $\overline{x_2} \notin \overline{C_R(u)}$. Since $\overline{a} \notin \overline{C_R(u)}$, then we have $\overline{x_2+a} \in \overline{C_R(u)}$. This gives that

$$\overline{C_R(x_2+a)} \supseteq \{\overline{0}, \overline{x_2}, \overline{a}, \overline{x_2+a}, \overline{u}\}.$$

It can be easily checked that $R, C_R(x_1), C_R(x_2), \cdots, C_R(x_6), C_R(a), C_R(x_1 + a), C_R(b), C_R(a+b), C_R(x_2+a)$ are 12 distinct centralisers of R. We have reached a contradiction. Consequently, $C_R(b), C_R(a+b)$ are commutative. In view of Lemma 2.2.15, we have $C_R(x_i)$ is commutative for any $i \in \{2, 3, \cdots, 6\}$. Suppose that $ux_i = x_i u$ for some $u \in \{b, a+b\}$ and $i \in \{2, 3, \cdots, 6\}$. Since $C_R(u)$ and $C_R(x_i)$ are commutative, then $C_R(u) \leqslant C_R(x_i)$ and $C_R(x_i) \leqslant C_R(u)$. This yields that $C_R(u) = C_R(x_i)$, which is a contradiction. Therefore, $ux_i \neq x_i u$ for any $u \in \{b, a+b\}$ and $i \in \{2, 3, \cdots, 6\}$. This implies that $\{x_2, x_3, \cdots, x_6, b, a+b\}$ is a non-commuting set of R with cardinality 7, which leads to a contradiction. Consequently, we obtain $\gamma_1 = 4$, as claimed.

By Lemma 2.2.15, we have $C_R(x_i)$ is commutative for any $i \in \{1, 2, \cdots, 6\}$. We claim that $\gamma_5 = \gamma_6 = 4$. This claim can be proved by using a manner entirely similar to that used to prove Lemma 2.7.4. Now, we claim that $C_R(x_i) \cap C_R(x_j) \cap C_R(x_k) = Z(R)$ for any three distinct $i, j, k \in \{1, 2, \cdots, 6\}$. If not, then there exists some $r \in (C_R(x_i) \cap C_R(x_j) \cap C_R(x_k)) - Z(R)$ for three distinct $i, j, k \in \{1, 2, \cdots, 6\}$ such that $C_R(x_i) \cup C_R(x_j) \cup C_R(x_k) \subseteq C_R(r)$. Therefore, $R = C_R(r) \cup \left(\underset{l=1, l\neq i,j,k}{\overset{6}{\cup}} C_R(x_l) \right)$. By Lemma 2.2.1, it follows that $\gamma_l \leqslant 3$ for some $l \in \{1, 2, \cdots, 6\} - \{i, j, k\}$, which leads to a contradiction. So, our claim is true. Let $C_R(a_1), C_R(a_2), C_R(a_3), C_R(a_4)$ be four distinct proper centralisers of R that are different from $C_R(x_i)$ for any $i \in \{1, 2, \cdots, 6\}$. We next claim that there exists some $u \in \{1, 2, 3, 4\}$ such that $\overline{a_u} \notin \overline{C_R(x_j)} \cap \overline{C_R(x_k)}$ for any two distinct $j, k \in \{1, 2, \cdots, 6\}$. Suppose to the contrary that for any $u \in \{1, 2, 3, 4\}, \overline{a_u} \in \overline{C_R(x_{k_u})} \cap \overline{C_R(x_{l_u})}$ for two distinct $k_u, l_u \in \{1, 2, \cdots, 6\}$, then $|\overline{R}| \leqslant \sum\limits_{i=1}^{6} |\overline{C_R(x_i)}| - 5 - 4 = 15$, which is impossible. Consequently, there exists some $u \in \{1, 2, 3, 4\}$ such that $\overline{a_u} \notin \overline{C_R(x_j)} \cap \overline{C_R(x_k)}$ for any two distinct $j, k \in \{1, 2, \cdots, 6\}$. Without loss of generality, we assume that $u = 1$ and let $\overline{a_1} \in \overline{C_R(x_1)}$. Let $i \in \{2, 3, \cdots, 6\}$. In view of Lemma 2.2.11, we obtain $|\overline{C_R(a_1)} \cap \overline{C_R(x_i)}| = 2$. Thus, there exists exactly one $\overline{w_i} \in (\overline{C_R(a_1)} \cap \overline{C_R(x_i)}) - \overline{Z(R)}$. Since $C_R(x_i)$ is commutative, then $C_R(x_i) \leqslant C_R(w_i)$. Clearly, $C_R(w_i) \neq R, C_R(x_j)$ for any $j \in \{1, 2, \cdots, 6\} - \{i\}$. If $C_R(w_i) = C_R(a_1)$, then $C_R(x_i) \leqslant C_R(a_1)$. On the other hand, if $C_R(w_i) = C_R(x_i)$, then $C_R(w_i)$ is commutative and hence, $C_R(w_i) \leqslant C_R(a_1)$ and so, $C_R(x_i) \leqslant C_R(a_1)$. In both situations, we obtain $a_1 \in C_R(x_i)$, which is a contradiction. If $C_R(w_{l_1}) = C_R(w_{l_2}) = C_R(w_{l_3})$ for three distinct $l_1, l_2, l_3 \in \{2, 3, \cdots, 6\}$, then we obtain $w_{l_1} \in C_R(x_{l_1}) \cap C_R(x_{l_2}) \cap C_R(x_{l_3}) = Z(R)$, which is a contradiction. Therefore, $C_R(w_{l_1}) = C_R(w_{l_2}) = C_R(w_{l_3})$ does not exist for any $l_1, l_2, l_3 \in \{2, 3, \cdots, 6\}$. Since $|\mathrm{Cent}(R)| = 11$, then without any loss, we have $C_R(w_2) = C_R(w_3)$ and $C_R(w_4) = C_R(w_5)$ with $C_R(w_2) \neq C_R(w_4) \neq C_R(w_6)$. Thus, we have $\overline{w_2} \in \overline{C_R(x_2)} \cap \overline{C_R(x_3)}$ and $\overline{w_4} \in \overline{C_R(x_4)} \cap \overline{C_R(x_5)}$. So, we obtain $\overline{C_R(a_1)} = \overline{C_R(a_1)} \cap \overline{R} = \overline{C_R(a_1)} \cap (\bigcup\limits_{i=1}^{6} \overline{C_R(x_i)}) = \bigcup\limits_{i=1}^{6} (\overline{C_R(a_1)} \cap \overline{C_R(x_i)}) = \{\overline{0}, \overline{x_1}, \overline{a_1}, \overline{x_1 + a_1}, \overline{w_2}, \overline{w_4}, \overline{w_6}\}$, which contradicts the fact that $|\overline{C_R(a_1)}|$ is divide $|R|$. $\square$

Lemma 2.8.2. Let $\{x_1, x_2, \cdots, x_6\}$ be the maximal non-commuting set of a finite ring R. If R is an 11-centraliser finite ring, then $|R : Z(R)| \neq 32$.

Proof. Assume that $|R : Z(R)| = 32$. Without loss of generality, we suppose that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_6$. From Lemma 1.3.1(a), we have $R = \bigcup\limits_{i=1}^{6} C_R(x_i)$. By Lemma 2.7.2, we have $\gamma_2 = 4$. If $\gamma_4 \neq 4$, then by Lemma 2.7.3, we obtain $|R| \leqslant 2(\frac{|R|}{4}) + 3(\frac{|R|}{8}) = \frac{7|R|}{8}$, which is impossible. So, we have

$\gamma_3 = \gamma_4 = 4$. For the sake of simplicity, we write $\overline{r} = r + Z(R)$ for any $r \in R$ and $\overline{S} = S/Z(R)$ for any $S \leqslant R$. We consider two cases in this proof.

Case 1: $\gamma_1 = 2$. If $C_R(x_1)$ is commutative, then by Lemma 2.2.12, we obtain $|\overline{R}| \leqslant 2(4) = 8$; a contradiction. Therefore, $C_R(x_1)$ is non-commutative. By Lemma 2.2.9(b), we have $C_R(x_i)$ is commutative for any $i \in \{2, 3, \cdots, 6\}$. We claim that $C_R(x_2) \cap C_R(x_3) \cap C_R(x_4) \neq Z(R)$. Suppose that $C_R(x_2) \cap C_R(x_3) \cap C_R(x_4) = Z(R)$. By Lemma 2.2.11, we have $|\overline{C_R(x_i)} \cap \overline{C_R(x_j)}| \geqslant 2$ for any two distinct $i, j \in \{2, 3, 4\}$. So, we have

$$\overline{C_R(x_2)} \supset \{\overline{0}, \overline{w_1}, \overline{w_2}\},$$

$$\overline{C_R(x_3)} \supset \{\overline{0}, \overline{w_1}, \overline{w_3}\},$$

$$\overline{C_R(x_4)} \supset \{\overline{0}, \overline{w_2}, \overline{w_3}\}$$

for some $\overline{w_1}, \overline{w_2}, \overline{w_3} \in \overline{R} - \overline{Z(R)}$. It follows that $C_R(x_2) \cup C_R(x_3) \subseteq C_R(w_1)$, $C_R(x_2) \cup C_R(x_4) \subseteq C_R(w_2)$ and $C_R(x_3) \cup C_R(x_4) \subseteq C_R(w_3)$. Obviously, $C_R(w_i) \neq R, C_R(x_j)$ for any $i \in \{1, 2, 3\}$ and $j \in \{1, 2, \cdots, 6\}$. Since $\overline{C_R(x_1)} \cap \overline{C_R(x_2)} \cap \overline{C_R(x_3)} = \overline{Z(R)}$, then $C_R(w_i) \neq C_R(w_j)$ for any two distinct $i, j \in \{1, 2, 3\}$. In view of Lemma 2.2.9(a), it follows that $w_i w_j \neq w_j w_i$ for two distinct $i, j \in \{1, 2, 3\}$. This contradicts with the fact that $C_R(x_k)$ is commutative for any $k \in \{2, 3, 4\}$. So, $C_R(x_2) \cap C_R(x_3) \cap C_R(x_4) \neq Z(R)$. Therefore, there exists some $r \in (C_R(x_2) \cap C_R(x_3) \cap C_R(x_4)) - Z(R)$ such that $C_R(x_2) \cup C_R(x_3) \cup C_R(x_4) \subseteq C_R(r)$. Thus, we have $R = C_R(r) \cup C_R(x_1) \cup C_R(x_5) \cup C_R(x_6)$. Since $|C_R(r)| > |C_R(x_2)|$, then $|R : C_R(r)| = 2$. If $\gamma_6 \neq 4$, then by Lemma 2.7.3, we obtain $|R| \leqslant |C_R(x_1)| + |C_R(x_5)| + |C_R(x_6)| \leqslant \frac{|R|}{2} + \frac{|R|}{4} + \frac{|R|}{8} = \frac{7|R|}{8}$, which is impossible. So, we have $\gamma_5 = \gamma_6 = 4$. Since $|\overline{C_R(x_1)}| = 16$, then $\overline{R} - \overline{C_R(x_1)} = \{\overline{x_2}, \overline{x_3}, \overline{x_4}, \overline{x_5}, \overline{x_6}, \overline{r_1}, \overline{r_2}, \cdots, \overline{r_{11}}\}$ for some $r_1, r_2, \cdots, r_{11} \in R - C_R(x_1)$. By Lemma 2.2.11, we have $|\overline{C_R(r)} \cap \overline{C_R(x_1)}| = 8$ and $|\overline{C_R(x_i)} \cap \overline{C_R(x_1)}| = 4$ for any $i \in \{4, 5, 6\}$. We claim that $\overline{r_i} \notin \overline{C_R(x_j)} \cap \overline{C_R(x_k)}$ for any $i \in \{1, 2, \cdots, 11\}$ and $j, k \in \{4, 5, 6\}$ with $j \neq k$. If $\overline{r_i} \in \overline{C_R(x_j)} \cap \overline{C_R(x_k)}$ for some $i \in \{1, 2, \cdots, 11\}$ and $j, k \in \{4, 5, 6\}$ with $j \neq k$, then we obtain $|\overline{R} - \overline{C_R(x_1)}| \leqslant |\overline{C_R(r)} - \overline{C_R(x_1)}| + |\overline{C_R(x_5)} - \overline{C_R(x_1)}| + |\overline{C_R(x_6)} - \overline{C_R(x_1)}| - 1 = 15$, which is a contradiction. So, our claim is true. If $|\overline{C_R(x_4)} \cap \overline{C_R(x_5)} \cap \overline{C_R(x_6)}| \geqslant 2$, then without loss of generality, we have

$$\overline{C_R(x_4)} \supset \{\overline{0}, \overline{d_1}, \overline{x_4}, \overline{r_1}, \overline{r_2}, \overline{r_3}\},$$

$$\overline{C_R(x_5)} \supset \{\overline{0}, \overline{d_1}, \overline{x_5}, \overline{r_4}, \overline{r_5}, \overline{r_6}\},$$

$$\overline{C_R(x_6)} \supset \{\overline{0}, \overline{d_1}, \overline{x_6}, \overline{r_7}, \overline{r_8}, \overline{r_9}\}$$

for some $\overline{d_1} \in \overline{C_R(x_1)} - \overline{Z(R)}$. It follows that $C_R(x_4) \cup C_R(x_5) \cup C_R(x_6) \subseteq C_R(d_1)$. This shows that $\overline{x_4}, \overline{x_5}, \overline{x_6}, \overline{r_1}, \overline{r_2}, \cdots, \overline{r_9} \in \overline{C_R(d_1)}$ and hence, $|\overline{C_R(d_1)}| = 16$. Therefore, we have $|\overline{C_R(d_1)} \cap \overline{C_R(x_1)}| \leqslant 4$. Hence, by Lemma 2.2.11, we obtain $|R : Z(R)| \leqslant 2(2)(4) = 16$, which is a contradiction. Consequently, $|\overline{C_R(x_4)} \cap \overline{C_R(x_5)} \cap \overline{C_R(x_6)}| = 1$. By Lemma 2.2.11, it follows that $|\overline{C_R(x_i)} \cap \overline{C_R(x_j)}| \geqslant 2$ for any two distinct $i, j \in \{4, 5, 6\}$. So, we have

$$\overline{C_R(x_4)} \supset \{\overline{0}, \overline{w_1}, \overline{w_2}\},$$

$$\overline{C_R(x_5)} \supset \{\overline{0}, \overline{w_1}, \overline{w_3}\},$$

$$\overline{C_R(x_6)} \supset \{\overline{0}, \overline{w_2}, \overline{w_3}\}$$

for some $\overline{w_1}, \overline{w_2}, \overline{w_3} \in \overline{R} - \overline{Z(R)}$. Hence, by using similar arguments as in above, we will obtain $w_i w_j \neq w_j w_i$ for two distinct $i, j \in \{1, 2, 3\}$. This contradicts with the fact that $C_R(x_k)$ is commutative for any $k \in \{4, 5, 6\}$.

Case 2: $\gamma_1 = 4$. Now, we want to show that $\gamma_5 = \gamma_6 = 4$. By Lemma 2.2.9(b), we have $C_R(x_i), C_R(x_j)$ are commutative for two distinct $i, j \in \{1, 2, 3, 4\}$. By Lemma 2.2.11, it follows that $C_R(x_i) \cap C_R(x_j) \neq Z(R)$ and hence, there exists some $r \in (C_R(x_i) \cap C_R(x_j)) - Z(R)$ such that $C_R(x_i) \cup C_R(x_j) \subseteq C_R(r)$. This yields that $R = C_R(r) \cup \left(\bigcup_{k=1, k \neq i,j}^{6} C_R(x_k) \right)$. Therefore, by Lemma 2.2.1, we obtain $\gamma_5 = \gamma_6 = 4$, as desired. By Lemma 2.2.9(b), there have at least five $C_R(x_i)$'s are commutative. Without loss of generality, we assume that $C_R(x_i)$ is commutative for any $i \in \{1, 2, \cdots, 5\}$. Let $k \in \{2, 3, \cdots, 6\}$ with $C_R(x_k)$ is commutative. By Lemma 2.2.11, we have $C_R(x_1) \cap C_R(x_k) \neq Z(R)$. Thus, there exists some $w_k \in (C_R(x_1) \cap C_R(x_k)) - Z(R)$ such that $C_R(x_1) \cup C_R(x_k) \subseteq C_R(w_k)$. Clearly, $C_R(w_k) \neq R, C_R(x_i)$ for any $i \in \{1, 2, \cdots, 6\}$. We claim that if $C_R(x_u), C_R(x_v)$ are commutative for two distinct $u, v \in \{2, 3, \cdots, 6\}$, then $C_R(w_u) \neq C_R(w_v)$. Suppose that $C_R(w_u) = C_R(w_v)$. It follows that $C_R(x_1) \cup C_R(x_u) \cup C_R(x_v) \subseteq C_R(w_u)$. This implies that $R = C_R(w_u) \cup \left(\bigcup_{i=2, i \neq u,v}^{6} C_R(x_i) \right)$. Consequently, by Lemma 2.2.1, we obtain $\gamma_i \leqslant 3$ for some $i \in \{2, 3, \cdots, 6\} - \{u, v\}$, which leads to a contradiction. Hence, our claim is proved. If $C_R(x_6)$ is commutative, then since $|\mathrm{Cent}(R)| = 11$, it follows that $C_R(w_u) = C_R(w_v)$ for two distinct $u, v \in \{2, 3, \cdots, 6\}$, which leads to a contradiction. So, $C_R(x_6)$ is non-commutative. In view of Lemma 2.2.9(a), we have $x_6 \in C_R(w_{l_1}), C_R(w_{l_2}), C_R(w_{l_3})$ and $w_{l_1}, w_{l_2}, w_{l_3}$ do not commute with each other for three distinct $l_1, l_2, l_3 \in \{2,3,4,5\}$. Now, we consider for $C_R(w_{l_1} + x_6)$ and $C_R(w_{l_2} + x_6)$. For any $i \in \{1,2\}$, since $w_{l_3} \notin C_R(w_{l_i} + x_6)$ but $w_{l_3} \in R, C_R(x_6), C_R(w_{l_3})$, then $C_R(w_{l_i} + x_6) \neq R, C_R(x_6), C_R(w_{l_3})$. For any $i \in \{1,2\}$ and $j \in \{1,2,\cdots,5\}$, since $x_6 \in C_R(w_{l_i} + x_6)$ but $x_6 \notin C_R(x_j)$, then $C_R(w_{l_i} + x_6) \neq C_R(x_j)$. For any two distinct $i, j \in \{1,2\}$, since $w_{l_j} \notin C_R(w_{l_i} + x_6)$ but $w_{l_j} \in C_R(w_{l_j})$, then $C_R(w_{l_i} + x_6) \neq C_R(w_{l_j})$. For any $i \in \{1,2\}$, since $x_{l_i} \notin C_R(w_{l_i} + x_6)$ but $x_{l_i} \in C_R(w_{l_i})$, then $C_R(w_{l_i} + x_6) \neq C_R(w_{l_i})$. Since $w_{l_1} \in C_R(w_{l_1} + x_6)$ but $w_{l_1} \notin C_R(w_{l_2} + x_6)$, then $C_R(w_{l_1} + x_6) \neq C_R(w_{l_2} + x_6)$. This gives that $\{R, C_R(x_1), C_R(x_2), \cdots, C_R(x_6), C_R(w_{l_1}), C_R(w_{l_2}), C_R(w_{l_3}), C_R(w_{l_1}+x_6), C_R(w_{l_2} + x_6)\} \subseteq \mathrm{Cent}(R)$. Consequently, we obtain $|\mathrm{Cent}(R)| \geqslant 12$, which leads to a contradiction. $\square$

Lemma 2.8.3. Let $\{x_1, x_2, \cdots, x_7\}$ be the maximal non-commuting set of a finite ring R. If R is an 11-centraliser finite ring, then $|R : Z(R)| = 16, 24, 32, 36, 40, 48, 54, 56, 60, 64, 72$ or 80. Furthermore, if $|R : C_R(x_1)| \leqslant |R : C_R(x_2)| \leqslant \cdots \leqslant |R : C_R(x_7)|$, then $4 \leqslant |R : C_R(x_2)| \leqslant 6$.

Proof. From Lemma 1.3.1(b) and (c), we have $\{C_R(x_i) \mid i = 1,2,\cdots,7\}$ is an irredundant cover of R with intersection $Z(R)$. Thus, we have $|R : Z(R)| \leqslant f(7) = 81$. Therefore, by Theorem 2.2.23, we obtain $|R : Z(R)| = 16, 24, 27, 32, 36, 40, 48, 54, 56, 60, 64, 72, 80$ or 81. If $|R : Z(R)| = 27$, then it follows from Lemma 2.2.16 and Lemma 2.2.4 that $|\mathrm{Cent}(R)| = 8$, which is a contradiction. Hence, $|R : Z(R)| = 16, 24, 32, 36, 40, 48, 54, 56, 60, 64, 72, 80$ or

81.

By Lemma 2.2.1, we have $|R : C_R(x_2)| \leqslant 6$. Assume that $|R : C_R(x_2)| \leqslant 3$. If $C_R(x_2)$ is commutative, then by Lemma 2.2.12, we obtain $|R : Z(R)| \leqslant 3(3) = 9$, which is a contradiction. If $C_R(x_2)$ is non-commutative, then by Lemma 2.2.8(b), $C_R(x_1)$ is commutative. It follows from Lemma 2.2.12 that $|R : Z(R)| \leqslant 3(3) = 9$, which is a contradiction again. Consequently, $4 \leqslant |R : C_R(x_2)| \leqslant 6$. Since $|R : C_R(x_2)|$ is not divide 81, then $|R : Z(R)| \neq 81$. $\square$

Lemma 2.8.4. Let $\{x_1, x_2, \cdots, x_7\}$ be the maximal non-commuting set of a finite ring R. If R is an 11-centraliser finite ring, then $|R : Z(R)| \neq 16$.

Proof. Assume that $|R : Z(R)| = 16$. Without loss of generality, we suppose that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_7$. From Lemma 1.3.1(a), we have $R/Z(R) = \bigcup_{i=1}^{7} [C_R(x_i)/Z(R)]$. By Lemma 2.8.3, we have $\gamma_2 = 4$. If $\gamma_3 \neq 4$, then by Lemma 2.7.3, we obtain $|R| \leqslant \frac{|R|}{4} + 5(\frac{|R|}{8}) = \frac{7|R|}{8}$, which is impossible. So, we have $\gamma_3 = 4$. For the sake of simplicity, we write $\overline{r} = r + Z(R)$ for any $r \in R$ and $\overline{S} = S/Z(R)$ for any $S \leqslant R$.

We claim that $\gamma_1 = 4$. Suppose to the contrary that $\gamma_1 = 2$. By Lemma 2.2.11, we obtain $|\overline{C_R(x_1)} \cap \overline{C_R(x_2)}| = 2$. Hence, $\overline{C_R(x_1)} \cap \overline{C_R(x_2)} = \{\overline{0}, \overline{a}\}$ for some $\overline{a} \in \overline{R} - \overline{Z(R)}$. So, we have

$$\overline{C_R(x_1)} = \{\overline{0}, \overline{x_1}, \overline{a}, \overline{b}, \overline{a+b}, \overline{x_1+a}, \overline{x_1+b}, \overline{x_1+a+b}\},$$

$$\overline{C_R(x_2)} = \{\overline{0}, \overline{x_2}, \overline{a}, \overline{x_2+a}\}$$

for some $\overline{b} \in \overline{R} - \overline{Z(R)}$. If $ab = ba$, then $C_R(x_1)$ is commutative. Therefore, by Lemma 2.2.12, it follows that $|\overline{R}| \leqslant 2(4) = 8$; a contradiction. So, $ab \neq ba$. Thus, we have

$$\overline{C_R(a)} \supseteq \{\overline{0}, \overline{x_1}, \overline{x_2}, \overline{a}\},$$

$$\overline{C_R(x_1 + a)} \supseteq \{\overline{0}, \overline{x_1}, \overline{a}, \overline{x_1 + a}\},$$

$$\overline{C_R(b)} \supseteq \{\overline{0}, \overline{x_1}, \overline{b}, \overline{x_1 + b}\},$$

$$\overline{C_R(a + b)} \supseteq \{\overline{0}, \overline{x_1}, \overline{a + b}, \overline{x_1 + a + b}\}.$$

It can be easily checked that $R, C_R(x_1), C_R(x_2), \cdots, C_R(x_7), C_R(a)$, $C_R(x_1 + a), C_R(b), C_R(a + b)$ are 12 distinct centralisers of R. We have reached a contradiction. Thus, $\gamma_1 = 4$. By Lemma 2.2.15, we have $C_R(x_i)$ is commutative for any $i \in \{1, 2, \cdots, 7\}$. If $\gamma_4 = 8$, then by Lemma 2.7.3, it follows that $\gamma_5 = \gamma_6 = \gamma_7 = 8$. Hence, we obtain $|\overline{R}| \leqslant \sum_{i=1}^{7} |\overline{C_R(x_i)}| - 6 = 14$, which is impossible. So, $\gamma_4 = 4$.

Next, we want to show that $\gamma_6 = 8$. Suppose to the contrary that $\gamma_6 = 4$. Thus, $\sum_{i=1}^{7} |\overline{C_R(x_i)}| - 6 \geqslant 20$. Therefore, there exist 4 distinct $\overline{r_1}, \overline{r_2}, \overline{r_3}, \overline{r_4} \in \overline{R} - \overline{Z(R)}$ such that for any $i \in \{1, 2, 3, 4\}$, $\overline{r_i} \in \overline{C_R(x_{k_i})} \cap \overline{C_R(x_{l_i})}$ for two distinct $k_i, l_i \in \{1, 2, \cdots, 7\}$. It is clear that for any $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2, \cdots, 7\}$, $C_R(r_i) \neq R, C_R(x_j)$. Since $|\text{Cent}(R)| = 11$, then $C_R(r_i) = C_R(r_j)$ for two distinct $i, j \in \{1, 2, 3, 4\}$. Thus, we have $\overline{0}, \overline{r_i}, \overline{r_j} \in \overline{C_R(x_{k_i})} \cap \overline{C_R(x_{l_i})}$. This shows that $|\overline{C_R(x_{k_i})} \cap \overline{C_R(x_{l_i})}| \geqslant 3$, which follows that $|\overline{C_R(x_{k_i})} \cap \overline{C_R(x_{l_i})}| = 4$. Hence, we obtain $\overline{C_R(x_{k_i})} = \overline{C_R(x_{l_i})}$, which is a contradiction. So, $\gamma_6 = 8$.

Let $C_R(a_1), C_R(a_2), C_R(a_3)$ be three distinct proper centralisers of R that are different from $C_R(x_i)$ for any $i \in \{1, 2, \cdots, 7\}$. We claim that there exists some $u \in \{1, 2, 3\}$ such that $\overline{a_u} \notin \overline{C_R(x_j)} \cap \overline{C_R(x_k)}$ for any two distinct $j, k \in \{1, 2, \cdots, 7\}$. Suppose to the contrary that for any $u \in \{1, 2, 3\}, \overline{a_u} \in \overline{C_R(x_{k_u})} \cap \overline{C_R(x_{l_u})}$ for two distinct $k_u, l_u \in \{1, 2, \cdots, 7\}$, then $|\overline{R}| \leqslant \sum_{i=1}^{7} |\overline{C_R(x_i)}| - 6 - 3 = 15$, which is impossible. Consequently, there exists some $u \in \{1, 2, 3\}$ such that $\overline{a_u} \notin \overline{C_R(x_j)} \cap \overline{C_R(x_k)}$ for any two distinct $j, k \in \{1, 2, \cdots, 7\}$. Without loss of generality, we assume that $u = 1$. Hence, $\overline{a_1} \in \overline{C_R(x_i)}$ for some $i \in \{1, 2, \cdots, 7\}$ with $|\overline{C_R(x_i)}| = 4$. Without any loss, we assume that $\overline{a_1} \in \overline{C_R(x_1)}$. If $C_R(a_1)$ is commutative, then $C_R(a_1) = C_R(x_1)$, which is a contradiction. Thus, $C_R(a_1)$ is non-commutative. Since $C_R(x_1) < C_R(a_1)$, we have $|R : C_R(a_1)| = 2$. Let $i \in \{2, 3, 4\}$. In view of Lemma 2.2.11, we obtain $|\overline{C_R(a_1)} \cap \overline{C_R(x_i)}| = 2$. Thus, there exists exactly one $\overline{w_i} \in (\overline{C_R(a_1)} \cap \overline{C_R(x_i)}) - \overline{Z(R)}$. Since $C_R(x_i)$ is commutative, then $C_R(x_i) \leqslant C_R(w_i)$. Clearly, $C_R(w_i) \neq R, C_R(x_j)$ for any $j \in \{1, 2, \cdots, 7\} - \{i\}$. If $C_R(w_i) = C_R(a_1)$, then $C_R(x_i) \leqslant C_R(a_1)$. On the other hand, if $C_R(w_i) = C_R(x_i)$, then $C_R(w_i)$ is commutative and hence, $C_R(w_i) \leqslant C_R(a_1)$ and so, $C_R(x_i) \leqslant C_R(a_1)$. In both situations, we obtain $a_1 \in C_R(x_i)$, which is a contradiction. Since $|\mathrm{Cent}(R)| = 11$, then without any loss, we have $C_R(w_2) = C_R(w_3)$. Thus, we have $\overline{w_2} \in \overline{C_R(x_2)} \cap \overline{C_R(x_3)}$. So, we obtain $\overline{C_R(a_1)} = \overline{C_R(a_1)} \cap \overline{R} = \overline{C_R(a_1)} \cap (\bigcup_{i=1}^{7} \overline{C_R(x_i)}) = \bigcup_{i=1}^{7} (\overline{C_R(a_1)} \cap \overline{C_R(x_i)})$. Since $\overline{C_R(a_1)} \cap \overline{C_R(x_i)} = \{\overline{0}\}$ for any $i \in \{6, 7\}$, we have $\overline{C_R(a_1)} = \bigcup_{i=1}^{5} (\overline{C_R(a_1)} \cap \overline{C_R(x_i)}) \supseteq \{\overline{0}, \overline{x_1}, \overline{a_1}, \overline{x_1 + a_1}, \overline{w_2}, \overline{w_4}\}$. This shows that $|\overline{C_R(a_1)}| = 6$ or 7, which contradicts the fact that $|\overline{C_R(a_1)}|$ is divide $|R|$. $\square$

Lemma 2.8.5. Let $\{x_1, x_2, \cdots, x_7\}$ be the maximal non-commuting set of a finite ring R. If R is an 11-centraliser finite ring, then $|R : Z(R)| \neq 24, 40$ and 56 .

Proof. Assume that $|R : Z(R)| = 24, 40$ or 56. Thus, we have $|R : Z(R)| = 8p$ for some prime $p \in \{3, 5, 7\}$. Without loss of generality, we suppose that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_7$. From Lemma 1.3.1(a), we have $R/Z(R) = \bigcup_{i=1}^{7}[C_R(x_i)/Z(R)]$. By Lemma 2.8.3, we have $4 \leqslant \gamma_2 \leqslant 6$. Let $_m|G|$ denote the total number of elements with order m in an additive group G. For the sake of simplicity, we write $\overline{r} = r + Z(R)$ for any $r \in R$ and $\overline{S} = S/Z(R)$ for any $S \leqslant R$.

Since $\overline{R}$ is not cyclic, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_{4p}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2p}$. Thus, $|\overline{C_R(x_1)}| \leqslant 4p$ and $|\overline{C_R(x_i)}| \leqslant 2p$ for any $i \in \{2, 3, \cdots, 7\}$. This yields that $\overline{R}$ has at most $_{4p}|\mathbb{Z}_{4p}|$ elements of order $4p$. Since $_{4p}|\mathbb{Z}_{4p}| < {}_{4p}|\mathbb{Z}_2 \times \mathbb{Z}_{4p}|$, then $\overline{R} \ncong \mathbb{Z}_2 \times \mathbb{Z}_{4p}$ and so, $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2p}$. We now claim that $|\overline{C_R(x_1)}| = 2p$ or $4p$. If $|\overline{R}| = 24$ with $|\overline{C_R(x_1)}| = 4$, then $|\overline{C_R(x_i)}| \leqslant 4$ for any $i \in \{2, 3, \cdots, 7\}$. This implies that there does not exist any element of order 6 in $\overline{R}$. We have reached a contradiction as $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ has an element of order 6. If $|\overline{R}| = 24$ with $|\overline{C_R(x_1)}| = 8$, then $|\overline{C_R(x_i)}| \leqslant 8$ for any $i \in \{2, 3, \cdots, 7\}$. Therefore, $\overline{R}$ has at most $6({}_6|\mathbb{Z}_6|) = 12$ elements of order 6, which contradicts the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ has 14 elements of order 6. If $|\overline{R}| = 40$ with $|\overline{C_R(x_1)}| = 8$, then $|\overline{C_R(x_i)}| \leqslant 8$ for any $i \in \{2, 3, \cdots, 7\}$. This leads to there does not exist any element of order 10 in $\overline{R}$, which contradicts the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{10}$ has an element of order 10. Therefore, $|\overline{C_R(x_1)}| = 2p$ or $4p$. It follows that $\overline{C_R(x_1)} \cong \mathbb{Z}_{2p}$ or $\mathbb{Z}_2 \times \mathbb{Z}_{2p}$. Here, we claim that $\gamma_6 \neq 4$. Assume that $\gamma_6 = 4$, then $\overline{C_R(x_i)} \cong \mathbb{Z}_{2p}$ for any $i \in \{2, 3, \cdots, 6\}$. This gives that $\overline{C_R(x_i)}$ has exactly $p-1$ elements of order p for any $i \in \{1, 2, \cdots, 6\}$. By the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2p}$ has exactly $p-1$ elements of order p, then there exists some $\overline{a} \in \overline{R} - \overline{Z(R)}$ with order p such that $\overline{a} \in \bigcap_{i=1}^{6} \overline{C_R(x_i)}$. So, by Lemma 1.3.1(b), (c) and Lemma 2.2.2, we obtain $\overline{a} \in \overline{Z(R)}$, which leads to a contradiction. Consequently, $\gamma_6 \neq 4$ and so, $|\overline{C_R(x_i)}| < 2p$ for any $i \in \{6, 7\}$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ has 14 elements of order 6, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{10}$ has 28 elements of order 10, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{14}$ has 42 elements of order 14, then it follows that $\overline{C_R(x_1)} \cong \mathbb{Z}_2 \times \mathbb{Z}_{2p}$ and $\overline{C_R(x_i)} \cong \mathbb{Z}_{2p}$ for any $i \in \{2, 3, 4, 5\}$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2p}$ has exactly $p-1$ elements of order p, then there exists some $\overline{a} \in \overline{R} - \overline{Z(R)}$ with order p such that $\overline{a} \in \bigcap_{i=2}^{5} \overline{C_R(x_i)}$. For any $i \in \{2, 3, 4, 5\}$, since $\overline{C_R(x_i)}$ is cyclic, then $C_R(x_i)$ is commutative. Therefore, we have $\bigcup_{i=2}^{5} C_R(x_i) \subseteq C_R(a)$. This gives that $R = C_R(a) \cup C_R(x_1) \cup C_R(x_6) \cup C_R(x_7)$. By Lemma 2.7.3, we have $|R| \leqslant |C_R(x_1)| + |C_R(x_6)| + |C_R(x_7)|$. So, we obtain $|R| < \frac{|R|}{2} + 2(\frac{|R|}{4}) = |R|$, which is a contradiction. □

Lemma 2.8.6. Let $\{x_1, x_2, \cdots, x_7\}$ be the maximal non-commuting set of a finite ring R. Let $|\mathrm{Cent}(R)| = 11$. Let $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_7$. If $|R : Z(R)| = 32$ or 64, then $\gamma_1 = 2$.

Proof. Assume that $\gamma_1 \neq 2$. By Lemma 2.8.3, we have $\gamma_2 = 4$. Hence, $\gamma_1 = 4$. From Lemma 1.3.1(a), we have $R/Z(R) = \bigcup_{i=1}^{7} [C_R(x_i)/Z(R)]$. If $\gamma_3 \neq 4$, then by Lemma 2.7.3, we obtain $|R| \leqslant \frac{|R|}{4} + 5(\frac{|R|}{8}) = \frac{7|R|}{8}$, which is impossible. So, we have $\gamma_3 = 4$. Now, we want to show that $\gamma_5 = 4$. Suppose that $\gamma_5 \geqslant 8$. By Lemma 2.2.8(b), we have $C_R(x_i), C_R(x_j)$ are commutative for two distinct $i, j \in \{1, 2, 3\}$. By Lemma 2.2.11, it follows that $C_R(x_i) \cap C_R(x_j) \neq Z(R)$ and hence, there exists some $r \in (C_R(x_i) \cap C_R(x_j)) - Z(R)$, which gives that

$C_R(x_i) \cup C_R(x_j) \subseteq C_R(r)$. This yields that $R = C_R(r) \cup \left(\bigcup_{k=1, k\neq i,j}^{7} C_R(x_k) \right)$. Therefore, by Lemma 2.7.3, we obtain $|R| \leqslant 2(\frac{|R|}{4}) + 3(\frac{|R|}{8}) = \frac{7|R|}{8}$, which is a contradiction. Thus, $\gamma_5 = 4$. Next, we want to prove that $\gamma_7 = 4$. By Lemma 2.2.8(b), we have $C_R(x_{l_1}), C_R(x_{l_2}), C_R(x_{l_3}), C_R(x_{l_4})$ are commutative for four distinct $l_1, l_2, l_3, l_4 \in \{1, 2, \cdots, 5\}$. Without loss of generality, we assume that $l_1 = 1, l_2 = 2, l_3 = 3, l_4 = 4$. Let $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$. By Lemma 2.2.11, it follows that $C_R(x_i) \cap C_R(x_j) \neq Z(R)$ and hence, there exists some $w_{i,j} \in (C_R(x_i) \cap C_R(x_j)) - Z(R)$, which gives that $C_R(x_i) \cup C_R(x_j) \subseteq C_R(w_{i,j})$. Clearly, $C_R(w_{i,j}) \neq R, C_R(x_k)$ for any $k \in \{1, 2, \cdots, 7\}$. Now, we choosing $w_{1,2}, w_{1,3}, w_{1,4}, w_{2,3}$. Since $|\mathrm{Cent}(R)| = 11$, then there exist two distinct $b_1, b_2 \in \{w_{1,2}, w_{1,3}, w_{1,4}, w_{2,3}\}$ such that $C_R(b_1) = C_R(b_2)$. Thus, $R = C_R(b_1) \cup C_R(x_k) \cup C_R(x_5) \cup C_R(x_6) \cup C_R(x_7)$ for some $k \in \{1, 2, 3, 4\}$. If $\gamma_7 \geqslant 8$, then by Lemma 2.7.3, we have $|R| \leqslant 3(\frac{R}{4}) + \frac{|R|}{8} = \frac{7|R|}{8}$, which is impossible. So, $\gamma_7 = 4$.

Here, we claim that $C_R(x_{l_1}) \cap C_R(x_{l_2}) \cap C_R(x_{l_3}) \cap C_R(x_{l_4}) = Z(R)$ for any four distinct $l_1, l_2, l_3, l_4 \in \{1, 2, \cdots, 7\}$. If not, then there exists some $r \in (C_R(x_{l_1}) \cap C_R(x_{l_2}) \cap C_R(x_{l_3}) \cap C_R(x_{l_4})) - Z(R)$, which gives that $C_R(x_{l_1}) \cup C_R(x_{l_2}) \cup C_R(x_{l_3}) \cup C_R(x_{l_4}) \subseteq C_R(r)$. Therefore, we have $R = C_R(r) \cup C_R(x_{l_5}) \cup C_R(x_{l_6}) \cup C_R(x_{l_7})$ for three distinct $l_5, l_6, l_7 \in \{1, 2, \cdots, 7\} - \{l_1, l_2, l_3, l_4\}$. So, we obtain $|R| \leqslant \frac{3|R|}{4}$ by Lemma 2.7.3, which is impossible. Therefore, our claim is true. In view of Lemma 2.2.8(b), there have at least six $C_R(x_i)$'s are commutative. Without loss of generality, we assume that $C_R(x_i)$ is commutative for any $i \in \{1, 2, \cdots, 6\}$. Let $i, j \in \{1, 2, \cdots, 6\}$ with $i \neq j$. By Lemma 2.2.11, it follows that $C_R(x_i) \cap C_R(x_j) \neq Z(R)$ and hence, there exists some $w_{i,j} \in (C_R(x_i) \cap C_R(x_j)) - Z(R)$, which gives that $C_R(x_i) \cup C_R(x_j) \subseteq C_R(w_{i,j})$. It is obvious that $C_R(w_{i,j}) \neq R, C_R(x_k)$ for any $k \in \{1, 2, \cdots, 7\}$. Now, we choosing $w_{1,2}, w_{1,3}, w_{4,5}, w_{4,6}, w_{5,6}$. Since $|\mathrm{Cent}(R)| = 11$ and $C_R(x_{l_1}) \cap C_R(x_{l_2}) \cap C_R(x_{l_3}) \cap C_R(x_{l_4}) = Z(R)$ for any four distinct $l_1, l_2, l_3, l_4 \in \{1, 2, \cdots, 7\}$, then we have $C_R(w_{1,2}) = C_R(w_{1,3})$, $C_R(b_1) = C_R(b_2)$ and $C_R(b_3) \neq C_R(w_{1,2}) \neq C_R(b_1)$ for three distinct $b_1, b_2, b_3 \in \{w_{4,5}, w_{4,6}, w_{5,6}\}$. Therefore, $R = C_R(w_{1,2}) \cup C_R(b_1) \cup C_R(x_7)$. In view of Lemma 2.7.3, we obtain $|R| \leqslant \frac{|R|}{2} + \frac{|R|}{4} = \frac{3|R|}{4}$. We have reached a contradiction. □

Lemma 2.8.7. Let $\{x_1, x_2, \cdots, x_7\}$ be the maximal non-commuting set of a finite ring R. If R is an 11-centraliser finite ring, then $|R : Z(R)| \neq 32$.

Proof. Assume that $|R : Z(R)| = 32$. Without loss of generality, we suppose that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_7$. From Lemma 1.3.1(a), we have $R/Z(R) = \bigcup_{i=1}^{7} [C_R(x_i)/Z(R)]$. By Lemma 2.8.3 and Lemma 2.8.6, we have $\gamma_1 = 2$ and $\gamma_2 = 4$. If $\gamma_3 \neq 4$, then by Lemma 2.7.3, we obtain $|R| \leqslant \frac{|R|}{4} + 5(\frac{|R|}{8}) = \frac{7|R|}{8}$, which is impossible. So, we have $\gamma_3 = 4$. If $\gamma_4 \geqslant 16$, then by Lemma 2.7.3, we obtain $|R| \leqslant 2(\frac{|R|}{4}) + 4(\frac{|R|}{16}) = \frac{3|R|}{4}$, which is impossible. So, we have $\gamma_4 \leqslant 8$. For the sake of simplicity, we write $\overline{r} = r + Z(R)$ for any $r \in R$ and $\overline{S} = S/Z(R)$ for any $S \leqslant R$.

If $C_R(x_1)$ is commutative, then by Lemma 2.2.12, we obtain $|\overline{R}| \leqslant 2(4) = 8$; a contradiction. Therefore, $C_R(x_1)$ is non-commutative. By Lemma 2.2.8(b), we have $C_R(x_i)$ is commutative for any $i \in \{2, 3, \cdots, 7\}$. Since $|\overline{C_R(x_1)}| = 16$, then $\overline{R} - \overline{C_R(x_1)} = \{\overline{x_2}, \overline{x_3}, \overline{x_4}, \overline{x_5}, \overline{x_6}, \overline{x_7}, \overline{r_1}, \overline{r_2}, \cdots, \overline{r_{10}}\}$ for some $r_1, r_2, \cdots, r_{10} \in R - C_R(x_1)$. We claim that $\overline{r_i} \notin \overline{C_R(x_j)} \cap \overline{C_R(x_k)}$ for any $i \in \{1, 2, \cdots, 10\}$ and $j, k \in \{2, 3, \cdots, 7\}$ with $j \neq k$. If $\overline{r_i} \in \overline{C_R(x_j)} \cap \overline{C_R(x_k)}$ for some $i \in \{1, 2, \cdots, 10\}$ and $j, k \in \{2, 3, \cdots, 7\}$ with $j \neq k$, then $C_R(x_j) \cup C_R(x_k) \subseteq C_R(r_i)$. It is clear that $C_R(r_i) \neq R, C_R(x_l)$ for any $l \in \{1, 2, \cdots, 7\}$. Therefore, by Lemma 2.2.8(a), we obtain $r_i \in C_R(x_1)$; a contradiction. So, our claim is true. By Lemma 2.2.11, we have

$$|\overline{C_R(x_i)} \cap \overline{C_R(x_1)}| = \begin{cases} 4 & \text{if } \gamma_i = 4, \\ 2 & \text{if } \gamma_i = 8, \end{cases}$$

where $i \in \{1, 2, \cdots, 7\}$. Likewise, we have

$$|\overline{C_R(x_i)} \cap (\overline{R} - \overline{C_R(x_1)})| = \begin{cases} 4 & \text{if } \gamma_i = 4, \\ 2 & \text{if } \gamma_i = 8, \end{cases}$$

where $i \in \{1, 2, \cdots, 7\}$. If $|\overline{C_R(x_2)} \cap \overline{C_R(x_3)} \cap \overline{C_R(x_4)}| \geqslant 2$, then without loss of generality, we have

$$\begin{aligned} \overline{C_R(x_2)} &\supset \{\overline{0}, \overline{d_1}, \overline{x_2}, \overline{r_1}, \overline{r_2}, \overline{r_3}\}, \\ \overline{C_R(x_3)} &\supset \{\overline{0}, \overline{d_1}, \overline{x_3}, \overline{r_4}, \overline{r_5}, \overline{r_6}\}, \\ \overline{C_R(x_4)} &\supseteq \{\overline{0}, \overline{d_1}, \overline{x_4}, \overline{r_7}\} \end{aligned}$$

for some $\overline{d_1} \in \overline{C_R(x_1)} - \overline{Z(R)}$. It follows that $C_R(x_2) \cup C_R(x_3) \cup C_R(x_4) \subseteq C_R(d_1)$. This shows that $\overline{x_2}, \overline{x_3}, \overline{x_4}, \overline{r_1}, \overline{r_2}, \cdots, \overline{r_7} \in \overline{C_R(d_1)}$ and hence, $|\overline{C_R(d_1)}| = 16$. Therefore, we have $|\overline{C_R(d_1)} \cap \overline{C_R(x_1)}| \leqslant 4$. Hence, by Lemma 2.2.11, we obtain $|\overline{R}| \leqslant 2(2)(4) = 16$, which is a contradiction. Consequently, $|\overline{C_R(x_2)} \cap \overline{C_R(x_3)} \cap \overline{C_R(x_4)}| = 1$. If $\gamma_4 = 4$, then by using a manner entirely similar to that used to prove Lemma 2.7.6, we will obtain $|\mathrm{Cent}(R)| \geqslant 12$, which leads to a contradiction. Therefore, $\gamma_4 = 8$. In view of Lemma 2.7.3, it follows $\gamma_5 = \gamma_6 = \gamma_7 = 8$. By Lemma 2.2.11, we have $|\overline{C_R(x_2)} \cap \overline{C_R(x_3)}| = 2$ or 4. Thus, without any loss, we have

$$\overline{C_R(x_2)} = \{\overline{0}, \overline{d_1}, \overline{x_2}, \overline{r_1}, \overline{r_2}, \overline{r_3}, \overline{d_2}, \overline{d_3}\},$$

$$\overline{C_R(x_3)} = \{\overline{0}, \overline{d_1}, \overline{x_3}, \overline{r_4}, \overline{r_5}, \overline{r_6}, \overline{d_4}, \overline{d_5}\}.$$

for some $\overline{d_1}, \overline{d_2}, \overline{d_3}, \overline{d_4}, \overline{d_5} \in \overline{C_R(x_1)} - \overline{Z(R)}$. We first consider $|\{\overline{d_2}, \overline{d_3}\} \cap \{\overline{d_4}, \overline{d_5}\}| = 0$. Hence, we have

$$\overline{C_R(d_1)} \supset \overline{C_R(x_2)} \cup \overline{C_R(x_3)} \cup \{\overline{x_1}\},$$

$$\overline{C_R(d_2)} \supset \overline{C_R(x_2)} \cup \{\overline{x_1}\},$$

$$\overline{C_R(d_3)} \supset \overline{C_R(x_2)} \cup \{\overline{x_1}\},$$

$$\overline{C_R(d_4)} \supset \overline{C_R(x_3)} \cup \{\overline{x_1}\},$$

$$\overline{C_R(d_5)} \supset \overline{C_R(x_3)} \cup \{\overline{x_1}\}.$$

Since $|\overline{C_R(d_1)} \cap \overline{C_R(d_i)}| \geqslant 9$ for any $i \in \{2, 3, 4, 5\}$, then $|\overline{C_R(d_1)} \cap \overline{C_R(d_i)}| = 16$ for any $i \in \{2, 3, 4, 5\}$. This yields that $\overline{C_R(d_1)} = \overline{C_R(d_2)} = \overline{C_R(d_3)} = \overline{C_R(d_4)} = \overline{C_R(d_5)}$. So, we have $\overline{0}, \overline{d_1}, \overline{d_2}, \overline{d_3}, \overline{d_4}, \overline{d_5} \in \overline{Z(C_R(d_1))}$, which gives that $|\overline{Z(C_R(d_1))}| \geqslant 8$. If $|C_R(d_1) : Z(C_R(d_1))| = 1$, then $C_R(d_1)$ is commutative. On the other hand, if $|C_R(d_1) : Z(C_R(d_1))| = 2$, then $C_R(d_1)/Z(C_R(d_1))$ is cyclic, which follows that $C_R(d_1)$ is commutative. In both situations, we obtain $C_R(d_1)$ is commutative. This leads to $x_1x_2 = x_2x_1$, which is a contradiction. So, we have

$$\overline{C_R(x_2)} = \{\overline{0}, \overline{d_1}, \overline{x_2}, \overline{r_1}, \overline{r_2}, \overline{r_3}, \overline{d_2}, \overline{d_3}\},$$

$$\overline{C_R(x_3)} = \{\overline{0}, \overline{d_1}, \overline{x_3}, \overline{r_4}, \overline{r_5}, \overline{r_6}, \overline{d_2}, \overline{d_3}\},$$

$$\overline{C_R(d_1)} \supset \{\overline{0}, \overline{x_2}, \overline{x_3}, \overline{r_1}, \overline{r_2}, \overline{r_3}, \overline{r_4}, \overline{r_5}, \overline{r_6}, \overline{x_1}, \overline{d_1}, \overline{d_2}, \overline{d_3}\}.$$

Next, without loss of generality, we have

$$\overline{C_R(x_4)} = \{\overline{0}, \overline{x_4}, \overline{r_7}, \overline{w_4}\},$$

$$\overline{C_R(x_5)} = \{\overline{0}, \overline{x_5}, \overline{r_8}, \overline{w_5}\},$$

$$\overline{C_R(x_6)} = \{\overline{0}, \overline{x_6}, \overline{r_9}, \overline{w_6}\},$$

$$\overline{C_R(x_7)} = \{\overline{0}, \overline{x_7}, \overline{r_{10}}, \overline{w_7}\}$$

for some $\overline{w_4}, \overline{w_5}, \overline{w_6}, \overline{w_7} \in \overline{C_R(x_1)} - \overline{Z(R)}$. If $C_R(d_1) = C_R(w_i)$ for some $i \in \{4, 5, 6, 7\}$, then $\overline{x_i} \in \overline{C_R(d_1)}$. This leads to $|\overline{C_R(d_1)} \cap \overline{C_R(x_1)}| < 8$. Hence, it follows from Lemma 2.2.11 that $|\overline{R}| < 2(2)(8) = 32$; a contradiction. So, $C_R(d_1) \neq C_R(w_i)$ for any $i \in \{4, 5, 6, 7\}$. We claim that $\overline{w_u} \neq \overline{w_v}$ for two distinct $u, v \in \{4, 5, 6, 7\}$. Suppose that $\overline{w_u} = \overline{w_v}$ for any two distinct $u, v \in \{4, 5, 6, 7\}$, then we have

$$\overline{C_R(w_4)} \supset \{\overline{0}, \overline{x_4}, \overline{x_5}, \overline{x_6}, \overline{x_7}, \overline{r_7}, \overline{r_8}, \overline{r_9}, \overline{r_{10}}, \overline{w_4}, \overline{x_1}\}.$$

Since $|R : C_R(d_1)| = |R : C_R(w_4)| = 2$, then by Lemma 2.2.11, it follows that

$|\overline{C_R(d_1)} \cap \overline{C_R(w_4)}| = 8$. Thus, we have $\overline{d_1} \in \overline{C_R(w_4)}$. It is obvious that $C_R(d_1)$ and $C_R(w_4)$ are not equal to $R, C_R(x_i)$ for any $i \in \{1, 2, \cdots, 7\}$. Therefore, from Lemma 2.2.8(a), we obtain $d_1 w_4 \neq w_4 d_1$, which is a contradiction. So, we have $\overline{w_u} \neq \overline{w_v}$ for two distinct $u, v \in \{4, 5, 6, 7\}$. Without loss of generality, we assume that $u = 4$ and $v = 5$. Thus, we have

$$\overline{C_R(w_4)} \supset \{\overline{0}, \overline{x_4}, \overline{r_7}, \overline{w_4}, \overline{x_1}\},$$

$$\overline{C_R(w_5)} \supset \{\overline{0}, \overline{x_5}, \overline{r_8}, \overline{w_5}, \overline{x_1}\}.$$

Since $x_4 \in C_R(w_4)$ but $x_4 \notin C_R(w_5)$, then $C_R(w_4) \neq C_R(w_5)$. Clearly, $C_R(d_1), C_R(w_4)$ and $C_R(w_5)$ are not equal to $R, C_R(x_i)$ for any $i \in \{1, 2, \cdots, 7\}$. Hence, we have $\mathrm{Cent}(R) = \{R, C_R(x_1), C_R(x_2), \cdots, C_R(x_7), C_R(d_1), C_R(w_4), C_R(w_5)\}$. Note that, there exist at least 8 distinct $\overline{h_1}, \overline{h_2}, \cdots, \overline{h_8} \in \overline{C_R(x_1)} - \overline{Z(R)}$ such that for any $i \in \{1, 2, \cdots, 8\}$, $\overline{h_i} \notin \overline{C_R(x_j)}$ for any $j \in \{2, 3, \cdots, 7\}$. So, we are forced to conclude that $C_R(h_1) = C_R(h_2) = \cdots = C_R(h_8) = C_R(x_1)$. This implies that $\overline{0}, \overline{x_1}, \overline{h_1}, \overline{h_2}, \cdots, \overline{h_8} \in \overline{Z(C_R(x_1))}$, which gives that $|\overline{Z(C_R(x_1))}| \geqslant 10$ and it follows that $|\overline{Z(C_R(x_1))}| = 16$. So, we obtain $|C_R(x_1) : Z(C_R(x_1))| = 1$, which yields that $C_R(x_1)$ is commutative. We have reached a contradiction. $\square$

Lemma 2.8.8. Let $\{x_1, x_2, \cdots, x_7\}$ be the maximal non-commuting set of a finite ring R. If R is an 11-centraliser finite ring, then $|R : Z(R)| \neq 36, 54$ and 60.

Proof. Assume that $|R : Z(R)| = 36, 54$ or 60. Without loss of generality, we suppose that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_7$. From Lemma

1.3.1(a), we have $R/Z(R) = \bigcup_{i=1}^{7}[C_R(x_i)/Z(R)]$. By Lemma 2.8.3, we have $4 \leqslant \gamma_2 \leqslant 6$. Let ${}_m|G|$ denote the total number of elements with order m in an additive group G. For the sake of simplicity, we write $\overline{r} = r + Z(R)$ for any $r \in R$ and $\overline{S} = S/Z(R)$ for any $S \leqslant R$.

If $|\overline{R}| = 36$, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_{18}, \mathbb{Z}_3 \times \mathbb{Z}_{12}$ or $\mathbb{Z}_6 \times \mathbb{Z}_6$ as $\overline{R}$ is not cyclic. Hence, $|\overline{C_R(x_1)}| \leqslant 18$ and $|\overline{C_R(x_i)}| \leqslant 9$ for any $i \in \{2, 3, \cdots, 7\}$. This leads to $\overline{R}$ has at most ${}_{12}|\mathbb{Z}_{12}|$ elements of order 12 and ${}_{18}|\mathbb{Z}_{18}|$ elements of order 18. Since ${}_{12}|\mathbb{Z}_{12}| < {}_{12}|\mathbb{Z}_3 \times \mathbb{Z}_{12}|$ and ${}_{18}|\mathbb{Z}_{18}| < {}_{18}|\mathbb{Z}_2 \times \mathbb{Z}_{18}|$, then $R/Z(R) \ncong \mathbb{Z}_2 \times \mathbb{Z}_{18}$ and $\mathbb{Z}_3 \times \mathbb{Z}_{12}$. It follows that $\overline{R} \cong \mathbb{Z}_6 \times \mathbb{Z}_6$. Therefore, we have $\overline{C_R(x_1)} \cong \mathbb{Z}_6, |\overline{C_R(x_1)}| = 9, \overline{C_R(x_1)} \cong \mathbb{Z}_2 \times \mathbb{Z}_6$ or $\overline{C_R(x_1)} \cong \mathbb{Z}_3 \times \mathbb{Z}_6$. This implies that $\overline{R}$ has at most ${}_6|\mathbb{Z}_3 \times \mathbb{Z}_6| + 6({}_6|\mathbb{Z}_6|) = 20$ elements of order 6. This contradicts with the fact that $\mathbb{Z}_6 \times \mathbb{Z}_6$ has 24 elements of order 6.

If $|\overline{R}| = 54$, then $\overline{R} \cong \mathbb{Z}_3 \times \mathbb{Z}_{18}, \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_6$ as $\overline{R}$ is not cyclic. Since $\gamma_2 = 6$, then by Lemma 2.2.1, we obtain $\gamma_2 = \gamma_3 = \cdots = \gamma_7 = 6$. Therefore, $|\overline{C_R(x_1)}| \leqslant 27$ and $|\overline{C_R(x_i)}| = 9$ for any $i \in \{2, 3, \cdots, 7\}$. This shows that $\overline{R}$ has at most ${}_{18}|\mathbb{Z}_{18}|$ elements of order 18. Since ${}_{18}|\mathbb{Z}_{18}| < {}_{18}|\mathbb{Z}_2 \times \mathbb{Z}_{18}|$, then $R/Z(R) \ncong \mathbb{Z}_3 \times \mathbb{Z}_{18}$. So, $\overline{R} \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_6$. If $|\overline{C_R(x_1)}| = 9$ or 27, then $\overline{R}$ does not exist any element of order 2. We have reached a contradiction as $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_6$ has an elements of order 2. Hence, $|\overline{C_R(x_1)}| = 18$ and it follows that $\overline{C_R(x_1)} \cong \mathbb{Z}_3 \times \mathbb{Z}_6$. This implies that $\overline{R}$ has at most ${}_6|\mathbb{Z}_3 \times \mathbb{Z}_6|$ elements of order 6. Consequently, we obtain ${}_6|\overline{R}| = {}_6|\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_6| \leqslant {}_6|\mathbb{Z}_3 \times \mathbb{Z}_6|$, which is a contradiction.

If $|\overline{R}| = 60$, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_{30}$ as $\overline{R}$ is not cyclic. Thus, $|\overline{C_R(x_1)}| \leqslant 30$ and $|\overline{C_R(x_i)}| \leqslant 15$ for any $i \in \{2, 3, \cdots, 7\}$. It follows that $\overline{R}$ has at most ${}_{30}|\mathbb{Z}_{30}|$ elements of order 30. This leads to ${}_{30}|\overline{R}| = {}_{30}|\mathbb{Z}_2 \times \mathbb{Z}_{30}| \leqslant {}_{30}|\mathbb{Z}_{30}|$, a contradiction is reached. □

Lemma 2.8.9. Let $\{x_1, x_2, \cdots, x_7\}$ be the maximal non-commuting set of a finite ring R. If R is an 11-centraliser finite ring, then $|R : Z(R)| \neq 48$.

Proof. Assume that $|R : Z(R)| = 48$. Without loss of generality, we suppose that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_7$. From Lemma 1.3.1(a), we have $R/Z(R) = \bigcup_{i=1}^{7} [C_R(x_i)/Z(R)]$. By Lemma 2.8.3, we have $\gamma_2 = 4$ or 6. Let ${}_m|G|$ denote the total number of elements with order m in an additive group G. For the sake of simplicity, we write $\overline{r} = r + Z(R)$ for any $r \in R$ and $\overline{S} = S/Z(R)$ for any $S \leqslant R$.

Since $\overline{R}$ is not cyclic, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_{24}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{12}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ or $\mathbb{Z}_4 \times \mathbb{Z}_{12}$. Hence, $|\overline{C_R(x_1)}| \leqslant 24$ and $|\overline{C_R(x_i)}| \leqslant 12$ for any $i \in \{2, 3, \cdots, 7\}$. This shows that $\overline{R}$ has at most ${}_{24}|\mathbb{Z}_{24}|$ elements of order 24. Since ${}_{24}|\mathbb{Z}_{24}| < {}_{24}|\mathbb{Z}_2 \times \mathbb{Z}_{24}|$, then $\overline{R} \ncong \mathbb{Z}_2 \times \mathbb{Z}_{24}$. We first claim that $\gamma_1 \neq 6$. If $\gamma_1 = 6$, then $\gamma_2 = 6$ and hence, we obtain $\gamma_2 = \gamma_3 = \cdots = \gamma_7 = 6$ by Lemma 2.2.1. It follows that $|\overline{C_R(x_i)}| = 8$ for any $i \in \{1, 2, \cdots, 7\}$. This leads to there does not exist any element of order 6 and order 12 in $\overline{R}$. This contradicts with the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ has an element of order 6, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{12}$ and $\mathbb{Z}_4 \times \mathbb{Z}_{12}$ have an element of order 12. Next, we claim that $\gamma_6 \neq 4$. Suppose that $\gamma_6 = 4$, then $\overline{C_R(x_i)} \cong \mathbb{Z}_2 \times \mathbb{Z}_6$ or $\mathbb{Z}_{12}$ for any $i \in \{2, 3, \cdots, 6\}$. This gives that $\overline{C_R(x_i)}$ has exactly 2 elements of order 3 for any $i \in \{2, 3, \cdots, 6\}$. Since 3 is divide

$|\overline{C_R(x_1)}|$, then there exists an element of order 3 in $\overline{C_R(x_1)}$. By the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{12}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ and $\mathbb{Z}_4 \times \mathbb{Z}_{12}$ have exactly 2 elements of order 3, then there exists some $\overline{a} \in \overline{R} - \overline{Z(R)}$ with order 3 such that $\overline{a} \in \bigcap_{i=1}^{6} \overline{C_R(x_i)}$. So, by Lemma 1.3.1(b), (c) and Lemma 2.2.2, we obtain $\overline{a} \in \overline{Z(R)}$, which leads to a contradiction. Consequently, $\gamma_6 \neq 4$.

We claim that $\gamma_5 \neq 4$. Suppose that $\gamma_5 = 4$, then $\overline{C_R(x_i)} \cong \mathbb{Z}_2 \times \mathbb{Z}_6$ or $\mathbb{Z}_{12}$ for any $i \in \{2, 3, 4, 5\}$. This gives that $\overline{C_R(x_i)}$ has exactly 2 elements of order 3 for any $i \in \{2, 3, 4, 5\}$. Since 3 is divide $|\overline{C_R(x_1)}|$, then there exists an element of order 3 in $\overline{C_R(x_1)}$. By the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{12}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ and $\mathbb{Z}_4 \times \mathbb{Z}_{12}$ have exactly 2 elements of order 3, then there exists some $\overline{a} \in \overline{R} - \overline{Z(R)}$ with order 3 such that $\overline{a} \in \bigcap_{i=1}^{5} \overline{C_R(x_i)}$. In view of Lemma 2.2.8(b), there exist four distinct $l_1, l_2, l_3, l_4 \in \{1, 2, 3, 4, 5\}$ such that $C_R(x_{l_1}), C_R(x_{l_2}), C_R(x_{l_3}), C_R(x_{l_4})$ are commutative. Therefore, we have $\bigcup_{i=1}^{4} C_R(x_{l_i}) \subseteq C_R(a)$. It follows that $R = C_R(a) \cup C_R(x_{l_5}) \cup C_R(x_6) \cup C_R(x_7)$, where $l_5 \in \{1, 2, 3, 4, 5\} - \{l_1, l_2, l_3, l_4\}$. So, by Lemma 2.7.3, we obtain $|R| \leqslant \frac{|R|}{2} + 2(\frac{|R|}{6}) = \frac{5|R|}{6}$, which is impossible. So, $\gamma_5 \neq 4$. Therefore, we have $|\overline{C_R(x_1)}| \leqslant 24$, $|\overline{C_R(x_i)}| \leqslant 12$ for any $i \in \{2, 3, 4\}$ and $|\overline{C_R(x_j)}| \leqslant 8$ for any $j \in \{5, 6, 7\}$. This gives that $\overline{R}$ has at most ${}_{12}|\mathbb{Z}_2 \times \mathbb{Z}_{12}| + 3({}_{12}|\mathbb{Z}_{12}|) = 8 + 3(4) = 20$ elements of order 12. Since $\mathbb{Z}_4 \times \mathbb{Z}_{12}$ has 24 elements of order 12, then $\overline{R} \ncong \mathbb{Z}_4 \times \mathbb{Z}_{12}$.

Next, we want to show that $\gamma_4 \neq 4$. Suppose that $\gamma_4 = 4$, then $\overline{C_R(x_i)} \cong \mathbb{Z}_2 \times \mathbb{Z}_6$ or $\mathbb{Z}_{12}$ for any $i \in \{2, 3, 4\}$. This gives that $\overline{C_R(x_i)}$ has exactly 2 elements of order 3 for any $i \in \{2, 3, 4\}$. Since 3 is divide $|\overline{C_R(x_1)}|$, then there exists an element of order 3 in $\overline{C_R(x_1)}$. By the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{12}$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ have exactly 2 elements of order 3, then there exists some $\overline{a} \in \overline{R} - \overline{Z(R)}$ with order 3 such that $\overline{a} \in \bigcap_{i=1}^{4} \overline{C_R(x_i)}$. In view of Lemma 2.2.8(b), there exist three distinct $l_1, l_2, l_3 \in \{1, 2, 3, 4\}$ such that $C_R(x_{l_1}), C_R(x_{l_2}), C_R(x_{l_3})$ are commutative. Therefore, we have $\bigcup_{i=1}^{3} C_R(x_{l_i}) \subseteq C_R(a)$. It follows that $R = C_R(a) \cup C_R(x_{l_4}) \cup C_R(x_5) \cup C_R(x_6) \cup C_R(x_7)$, where $l_4 \in \{1, 2, 3, 4\} - \{l_1, l_2, l_3\}$. If $\gamma_7 \neq 6$, then by Lemma 2.7.3, we obtain $|R| \leqslant \frac{|R|}{2} + 2(\frac{|R|}{6}) + \frac{|R|}{8} = \frac{23|R|}{24}$; a contradiction. So, $\gamma_7 = 6$ and hence, $\gamma_5 = \gamma_6 = 6$. From Lemma 2.7.3 again, we have $|R : C_R(a)| = 2$ and $|R : C_R(x_{l_4})| = 2$. Since $\gamma_2 \geqslant 4$, then $l_4 = 1$. Hence, we have $|\overline{C_R(a)}| = |\overline{C_R(x_1)}| = 24$ and $|\overline{C_R(x_5)}| = |\overline{C_R(x_6)}| = |\overline{C_R(x_7)}| = 8$. Note that, $\overline{R} = \overline{C_R(a)} \cup \overline{C_R(x_1)} \cup \overline{C_R(x_5)} \cup \overline{C_R(x_6)} \cup \overline{C_R(x_7)}$. Assume that $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{12}$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{12}$ has 16 elements of order 12, then it follows that $\overline{C_R(a)}, \overline{C_R(x_1)} \cong \mathbb{Z}_2 \times \mathbb{Z}_{12}$. This implies that $\overline{R}$ has at most $2({}_6|\mathbb{Z}_2 \times \mathbb{Z}_{12}|) = 2(6) = 12$ elements of order 6. We have reached a contradiction as $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{12}$ has 14 elements of order 6. Next, we suppose that $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$. Hence, $\overline{C_R(a)}, \overline{C_R(x_1)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$. This yields that $\overline{R}$ has at most $2({}_6|\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6|) = 2(14) = 28$ elements of order 6, which leads to a contradiction as $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ has 30 elements of order 6.

Therefore, we have $|\overline{C_R(x_1)}| \leqslant 24$, $|\overline{C_R(x_i)}| \leqslant 12$ for any $i \in \{2, 3\}$ and $|\overline{C_R(x_j)}| \leqslant 8$ for any $j \in \{4, 5, 6, 7\}$. Assume that $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ has 30 elements of order 6, then it follows that $\overline{C_R(x_1)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$, $\overline{C_R(x_2)}, \overline{C_R(x_3)} \cong \mathbb{Z}_2 \times \mathbb{Z}_6$ and $\overline{C_R(x_u)}, \overline{C_R(x_v)} \cong \mathbb{Z}_6$ for two distinct $u, v \in \{4, 5, 6, 7\}$. This gives that $\overline{C_R(x_i)}$ has exactly 2 elements of order 3 for any $i \in \{1, 2, 3, u, v\}$. By the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ has exactly 2 elements of order 3, then there exists some $\overline{a} \in \overline{R} - \overline{Z(R)}$ with order 3 such that $\overline{a} \in \bigcap_{i\in\{1,2,3,u,v\}} \overline{C_R(x_i)}$. In view of Lemma 2.2.8(b), there exist four distinct $l_1, l_2, l_3, l_4 \in \{1, 2, 3, u, v\}$ such that $C_R(x_{l_1}), C_R(x_{l_2}), C_R(x_{l_3}), C_R(x_{l_4})$ are commutative. Therefore, we have $\bigcup_{i=1}^{4} C_R(x_{l_i}) \subseteq C_R(a)$. It follows that $R = C_R(a) \cup C_R(x_{l_5}) \cup C_R(x_{l_6}) \cup C_R(x_{l_7})$, where $l_5 \in \{1, 2, 3, u, v\} - \{l_1, l_2, l_3, l_4\}$ and $l_6, l_7 \in \{4, 5, 6, 7\} - \{u, v\}$ with $l_6 \neq l_7$. Hence, we obtain $|R| \leqslant \frac{|R|}{2} + 2(\frac{|R|}{6}) = \frac{5|R|}{6}$ by Lemma 2.7.3, a contradiction is reached. Consequently, we have $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{12}$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{12}$ has 16 elements of order 12, then it follows that $\overline{C_R(x_1)} \cong \mathbb{Z}_2 \times \mathbb{Z}_{12}$ and $\overline{C_R(x_2)}, \overline{C_R(x_3)} \cong \mathbb{Z}_{12}$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{12}$ has 14 elements of order 6, then we have $\overline{C_R(x_u)}, \overline{C_R(x_v)} \cong \mathbb{Z}_6$ for two distinct $u, v \in \{4, 5, 6, 7\}$. This gives that $\overline{C_R(x_i)}$ has exactly 2 elements of order 3 for any $i \in \{1, 2, 3, u, v\}$. By using similar arguments as in above, we will obtain $|R| \leqslant \frac{|R|}{2} + 2(\frac{|R|}{6}) = \frac{5|R|}{6}$, which leads to a contradiction. □

Lemma 2.8.10. Let $\{x_1, x_2, \cdots, x_7\}$ be the maximal non-commuting set of a finite ring R. If R is an 11-centraliser finite ring, then $|R : Z(R)| \neq 64$.

Proof. Assume that $|R : Z(R)| = 64$. Without loss of generality, we suppose that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_7$. From Lemma 1.3.1(a), we have $R/Z(R) = \bigcup_{i=1}^{7} [C_R(x_i)/Z(R)]$. By Lemma 2.8.3 and Lemma 2.8.6, we have $\gamma_1 = 2$ and $\gamma_2 = 4$. If $\gamma_3 \neq 4$, then by Lemma 2.7.3, we obtain $|R| \leqslant \frac{|R|}{4} + 5(\frac{|R|}{8}) = \frac{7|R|}{8}$, which is impossible. So, we have $\gamma_3 = 4$. If $\gamma_4 \geqslant 16$, then by Lemma 2.7.3, we obtain $|R| \leqslant 2(\frac{|R|}{4}) + 4(\frac{|R|}{16}) = \frac{3|R|}{4}$, which is impossible. So, we have $\gamma_4 \leqslant 8$. For the sake of simplicity, we write $\overline{r} = r + Z(R)$ for any $r \in R$ and $\overline{S} = S/Z(R)$ for any $S \leqslant R$.

If $C_R(x_1)$ is commutative, then by Lemma 2.2.12, we obtain $|\overline{R}| \leqslant 2(4) = 8$; a contradiction. Therefore, $C_R(x_1)$ is non-commutative. By Lemma 2.2.8(b), we have $C_R(x_i)$ is commutative for any $i \in \{2, 3, \cdots, 7\}$. Since $|\overline{C_R(x_1)}| = 32$, then $\overline{R} - \overline{C_R(x_1)} = \{\overline{x_2}, \overline{x_3}, \overline{x_4}, \overline{x_5}, \overline{x_6}, \overline{x_7}, \overline{r_1}, \overline{r_2}, \cdots, \overline{r_{26}}\}$ for some $r_1, r_2, \cdots, r_{26} \in R - C_R(x_1)$. We claim that $\overline{r_i} \notin \overline{C_R(x_j)} \cap \overline{C_R(x_k)}$ for any $i \in \{1, 2, \cdots, 26\}$ and $j, k \in \{2, 3, \cdots, 7\}$ with $j \neq k$. If $\overline{r_i} \in \overline{C_R(x_j)} \cap \overline{C_R(x_k)}$ for some $i \in \{1, 2, \cdots, 26\}$ and $j, k \in \{2, 3, \cdots, 7\}$ with $j \neq k$, then $C_R(x_j) \cup C_R(x_k) \subseteq C_R(r_i)$. It is clear that $C_R(r_i) \neq R, C_R(x_l)$ for any $l \in \{1, 2, \cdots, 7\}$. Therefore, by Lemma 2.2.8(a), we obtain $r_i \in C_R(x_1)$; a contradiction. So, our claim is true. By Lemma 2.2.11, we have

$$|\overline{C_R(x_i)} \cap \overline{C_R(x_1)}| = \begin{cases} 8 & \text{if } \gamma_i = 4, \\ 4 & \text{if } \gamma_i = 8, \end{cases}$$

where $i \in \{1, 2, \cdots, 7\}$. Likewise, we have

$$|\overline{C_R(x_i)} \cap (\overline{R} - \overline{C_R(x_1)})| = \begin{cases} 8 & \text{if } \gamma_i = 4, \\ 4 & \text{if } \gamma_i = 8, \end{cases}$$

where $i \in \{1, 2, \cdots, 7\}$. If $|\overline{C_R(x_2)} \cap \overline{C_R(x_3)} \cap \overline{C_R(x_4)}| \geqslant 2$, then without loss of generality, we have

$$\overline{C_R(x_2)} \supset \{\overline{0}, \overline{d_1}, \overline{x_2}, \overline{r_1}, \overline{r_2}, \cdots, \overline{r_7}\},$$

$$\overline{C_R(x_3)} \supset \{\overline{0}, \overline{d_1}, \overline{x_3}, \overline{r_8}, \overline{r_9}, \cdots, \overline{r_{14}}\},$$

$$\overline{C_R(x_4)} \supset \{\overline{0}, \overline{d_1}, \overline{x_4}, \overline{r_{15}}, \overline{r_{16}}, \overline{r_{17}}\}$$

for some $\overline{d_1} \in \overline{C_R(x_1)} - \overline{Z(R)}$. It follows that $C_R(x_2) \cup C_R(x_3) \cup C_R(x_4) \subseteq C_R(d_1)$. This shows that $\overline{x_2}, \overline{x_3}, \overline{x_4}, \overline{r_1}, \overline{r_2}, \cdots, \overline{r_{17}} \in \overline{C_R(d_1)}$ and thus, $|\overline{C_R(d_1)}| = 32$. Therefore, we have $|\overline{C_R(d_1)} \cap \overline{C_R(x_1)}| \leqslant 8$. Hence, by Lemma 2.2.11, we obtain $|\overline{R}| \leqslant 2(2)(8) = 32$, which is a contradiction. Consequently, $|\overline{C_R(x_2)} \cap \overline{C_R(x_3)} \cap \overline{C_R(x_4)}| = 1$. We continue the proof by considering two cases.

Case 1: $\gamma_4 = 4$. By Lemma 2.2.11, we have $|\overline{C_R(x_i)} \cap \overline{C_R(x_j)}| \geqslant 4$ for any two distinct $i, j \in \{2, 3, 4\}$. Thus, without loss of generality, we have

$$\overline{C_R(x_2)} = \{\overline{0}, \overline{d_1}, \overline{d_2}, \overline{d_3}, \overline{d_4}, \overline{d_5}, \overline{d_6}, \overline{d_{10}}, \overline{x_2}, \overline{r_1}, \overline{r_2}, \cdots, \overline{r_7}\},$$

$$\overline{C_R(x_3)} = \{\overline{0}, \overline{d_1}, \overline{d_2}, \overline{d_3}, \overline{d_7}, \overline{d_8}, \overline{d_9}, \overline{d_{11}}, \overline{x_3}, \overline{r_8}, \overline{r_9}, \cdots, \overline{r_{14}}\},$$

$$\overline{C_R(x_4)} = \{\overline{0}, \overline{d_4}, \overline{d_5}, \overline{d_6}, \overline{d_7}, \overline{d_8}, \overline{d_9}, \overline{d_{12}}, \overline{x_4}, \overline{r_{15}}, \overline{r_{16}}, \cdots, \overline{r_{21}}\}$$

for some $\overline{d_1}, \overline{d_2}, \cdots, \overline{d_{12}} \in \overline{C_R(x_1)} - \overline{Z(R)}$. It follows that $C_R(x_2) \cup C_R(x_3) \subseteq C_R(d_1), C_R(x_2) \cup C_R(x_4) \subseteq C_R(d_4)$ and $C_R(x_3) \cup C_R(x_4) \subseteq C_R(d_7)$. It is obvious that $C_R(d_i) \neq R, C_R(x_l)$ for any $i \in \{1, 4, 7\}$ and $l \in \{1, 2, \cdots, 7\}$. Since $|\overline{C_R(x_2)} \cap \overline{C_R(x_3)} \cap \overline{C_R(x_4)}| = 1$, then $C_R(d_i) \neq C_R(d_j)$ for any two distinct $i, j \in \{1, 4, 7\}$. Therefore, by Lemma 2.2.8(a), we have d_1, d_4, d_7 do not commute with each other. Now, we consider for $C_R(d_1 + x_1)$. Since $d_4 \notin C_R(d_1 + x_1)$ but $d_4 \in R, C_R(x_1)$, then $C_R(d_1 + x_1) \neq R, C_R(x_1)$. For any $i \in \{2, 3, \cdots, 7\}$, since $x_1 \in C_R(d_1 + x_1)$ but $x_1 \notin C_R(x_i)$, then $C_R(d_1 + x_1) \neq C_R(x_i)$. Since $x_2 \notin C_R(d_1 + x_1)$ but $x_2 \in C_R(d_1)$, then $C_R(d_1 + x_1) \neq C_R(d_1)$. Since $d_4, d_7 \notin C_R(d_1 + x_1)$ but $d_4 \in C_R(d_4)$ and $d_7 \in C_R(d_7)$, then $C_R(d_1 + x_1) \neq C_R(d_4), C_R(d_7)$. Consequently, we obtain $|\text{Cent}(R)| \geqslant 12$, which is a contradiction.

Case 2: $\gamma_4 = 8$. In view of Lemma 2.7.3, it follows $\gamma_5 = \gamma_6 = \gamma_7 = 8$. By Lemma 2.2.11, we have $|\overline{C_R(x_2)} \cap \overline{C_R(x_3)}| = 4$ or 8. Thus, without any loss, we have

$$\overline{C_R(x_2)} = \{\overline{0}, \overline{d_1}, \overline{d_2}, \overline{d_3}, \overline{x_2}, \overline{r_1}, \overline{r_2}, \cdots, \overline{r_7}, \overline{d_4}, \overline{d_5}, \overline{d_6}, \overline{d_7}\},$$

$$\overline{C_R(x_3)} = \{\overline{0}, \overline{d_1}, \overline{d_2}, \overline{d_3}, \overline{x_3}, \overline{r_8}, \overline{r_9}, \cdots, \overline{r_{14}}, \overline{d_8}, \overline{d_9}, \overline{d_{10}}, \overline{d_{11}}\}.$$

for some $\overline{d_1}, \overline{d_2}, \cdots, \overline{d_{11}} \in \overline{C_R(x_1)} - \overline{Z(R)}$. We first consider $|\{\overline{d_4}, \overline{d_5}, \overline{d_6}, \overline{d_7}\} \cap \{\overline{d_8}, \overline{d_9}, \overline{d_{10}}, \overline{d_{11}}\}| = 0$. Hence, we have

$$\overline{C_R(d_i)} \supset \overline{C_R(x_2)} \cup \overline{C_R(x_3)} \cup \{\overline{x_1}\} \text{ for any } i \in \{1, 2, 3\},$$

$$\overline{C_R(d_i)} \supset \overline{C_R(x_2)} \cup \{\overline{x_1}\} \text{ for any } i \in \{4, 5, 6, 7\},$$

$$\overline{C_R(d_j)} \supset \overline{C_R(x_3)} \cup \{\overline{x_1}\} \text{ for any } i \in \{8, 9, 10, 11\}.$$

Since $|\overline{C_R(d_1)} \cap \overline{C_R(d_i)}| \geqslant 17$ for any $i \in \{2, 3, \cdots, 11\}$, then $|\overline{C_R(d_1)} \cap \overline{C_R(d_i)}| = 32$ for any $i \in \{2, 3, \cdots, 11\}$. This yields that $\overline{C_R(d_1)} = \overline{C_R(d_2)} = \cdots = \overline{C_R(d_{11})}$. So, we have $\overline{0}, \overline{d_1}, \overline{d_2}, \cdots, \overline{d_{11}} \in \overline{Z(C_R(d_1))}$, which gives that $|\overline{Z(C_R(d_1))}| \geqslant 16$. If $|C_R(d_1) : Z(C_R(d_1))| = 1$, then $C_R(d_1)$ is commutative. On the other hand, if $|C_R(d_1) : Z(C_R(d_1))| = 2$, then $C_R(d_1)/Z(C_R(d_1))$ is cyclic, which follows that $C_R(d_1)$ is commutative. In both situations, we obtain $C_R(d_1)$ is commutative. This leads to $x_1x_2 = x_2x_1$, which is a contradiction. So, we have

$$\overline{C_R(x_2)} = \{\overline{0}, \overline{d_1}, \overline{d_2}, \overline{d_3}, \overline{x_2}, \overline{r_1}, \overline{r_2}, \cdots, \overline{r_7}, \overline{d_4}, \overline{d_5}, \overline{d_6}, \overline{d_7}\},$$

$$\overline{C_R(x_3)} = \{\overline{0}, \overline{d_1}, \overline{d_2}, \overline{d_3}, \overline{x_3}, \overline{r_8}, \overline{r_9}, \cdots, \overline{r_{14}}, \overline{d_4}, \overline{d_5}, \overline{d_6}, \overline{d_7}\},$$

$$\overline{C_R(d_1)} \supset \{\overline{0}, \overline{x_2}, \overline{x_3}, \overline{r_1}, \overline{r_2}, \cdots, \overline{r_{14}}, \overline{d_1}, \overline{d_2}, \cdots, \overline{d_7}\}.$$

Next, without loss of generality, we have

$$\overline{C_R(x_4)} \supset \{\overline{0}, \overline{x_4}, \overline{r_{15}}, \overline{r_{16}}, \overline{r_{17}}, \overline{w_4}\},$$

$$\overline{C_R(x_5)} \supset \{\overline{0}, \overline{x_5}, \overline{r_{18}}, \overline{r_{19}}, \overline{r_{20}}, \overline{w_5}\},$$

$$\overline{C_R(x_6)} \supset \{\overline{0}, \overline{x_6}, \overline{r_{21}}, \overline{r_{22}}, \overline{r_{23}}, \overline{w_6}\},$$

$$\overline{C_R(x_7)} \supset \{\overline{0}, \overline{x_7}, \overline{r_{24}}, \overline{r_{25}}, \overline{r_{26}}, \overline{w_7}\}$$

for some $\overline{w_4}, \overline{w_5}, \overline{w_6}, \overline{w_7} \in \overline{C_R(x_1)} - \overline{Z(R)}$. If $C_R(d_1) = C_R(w_i)$ for some $i \in \{4, 5, 6, 7\}$, then $\overline{x_i} \in \overline{C_R(d_1)}$. This leads to $|\overline{C_R(d_1)} \cap \overline{C_R(x_1)}| < 16$. Hence, it follows from Lemma 2.2.11 that $|\overline{R}| < 2(2)(16) = 64$; a contradiction. So, $C_R(d_1) \neq C_R(w_i)$ for any $i \in \{4, 5, 6, 7\}$. We claim that $\overline{w_u} \neq \overline{w_v}$ for two distinct $u, v \in \{4, 5, 6, 7\}$. Suppose that $\overline{w_u} = \overline{w_v}$ for any two distinct $u, v \in \{4, 5, 6, 7\}$, then we have

$$\overline{C_R(w_4)} \supset \{\overline{0}, \overline{x_4}, \overline{x_5}, \overline{x_6}, \overline{x_7}, \overline{r_{15}}, \overline{r_{16}}, \cdots, \overline{r_{26}}, \overline{w_4}, \overline{x_1}\}.$$

Since $|R : C_R(d_1)| = |R : C_R(w_4)| = 2$, then by Lemma 2.2.11, it follows that $|\overline{C_R(d_1)} \cap \overline{C_R(w_4)}| = 16$. Thus, we have $\overline{d_1} \in \overline{C_R(w_4)}$. It is obvious that $C_R(d_1)$ and $C_R(w_4)$ are not equal to $R, C_R(x_i)$ for any $i \in \{1, 2, \cdots, 7\}$. Therefore, from Lemma 2.2.8(a), we obtain $d_1w_4 \neq w_4d_1$, which is a contradiction. So, we have $\overline{w_u} \neq \overline{w_v}$ for two distinct $u, v \in \{4,5,6,7\}$. Without loss of generality, we assume that $u = 4$ and $v = 5$. Thus, we have

$$\overline{C_R(x_4)} \supset \{\overline{0}, \overline{x_4}, \overline{r_{15}}, \overline{r_{16}}, \overline{r_{17}}, \overline{w_4}, \overline{x_1}\},$$

$$\overline{C_R(x_5)} \supset \{\overline{0}, \overline{x_5}, \overline{r_{18}}, \overline{r_{19}}, \overline{r_{20}}, \overline{w_5}, \overline{x_1}\}.$$

Since $x_4 \in C_R(w_4)$ but $x_4 \notin C_R(w_5)$, then $C_R(w_4) \neq C_R(w_5)$. Clearly, $C_R(d_1), C_R(w_4)$ and $C_R(w_5)$ are not equal to $R, C_R(x_i)$ for any $i \in \{1, 2, \cdots, 7\}$. Hence, we have $\text{Cent}(R) = \{R, C_R(x_1), C_R(x_2), \cdots, C_R(x_7), C_R(d_1), C_R(w_4), C_R(w_5)\}$. Note that, there exist at least 12 distinct $\overline{h_1}, \overline{h_2}, \cdots, \overline{h_{12}} \in \overline{C_R(x_1)} - \overline{Z(R)}$ such that for any $i \in \{1, 2, \cdots, 12\}$, $\overline{h_i} \notin \overline{C_R(x_j)}$ for any $j \in \{2, 3, \cdots, 7\}$. So, we are forced to conclude that $C_R(h_1) = C_R(h_2) = \cdots = C_R(h_{12}) = C_R(x_1)$. This implies that $\overline{0}, \overline{x_1}, \overline{h_1}, \overline{h_2}, \cdots, \overline{h_{12}} \in \overline{Z(C_R(x_1))}$, which gives that $|\overline{Z(C_R(x_1))}| \geqslant 14$ and it follows that $|\overline{Z(C_R(x_1))}| \geqslant 16$. If $|C_R(x_1) : Z(C_R(x_1))| = 1$, then $C_R(x_1)$ is commutative. On the other hand, if $|C_R(x_1) : Z(C_R(x_1))| = 2$, then $C_R(x_1)/Z(C_R(x_1))$ is cyclic, which follows that $C_R(x_1)$ is commutative. In both situations, we obtain $C_R(x_1)$ is commutative. We have reached a contradiction. □

Lemma 2.8.11. Let $\{x_1, x_2, \cdots, x_7\}$ be the maximal non-commuting set of a finite ring R. If R is an 11-centraliser finite ring, then $|R : Z(R)| \neq 72$.

Proof. Assume that $|R : Z(R)| = 72$. Without loss of generality, we suppose that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_7$. From Lemma 1.3.1(a), we have $R/Z(R) = \bigcup_{i=1}^{7}[C_R(x_i)/Z(R)]$. By Lemma 2.8.3, we have $\gamma_2 = 4$ or 6. Let ${}_m|G|$

denote the total number of elements with order m in an additive group G. For the sake of simplicity, we write $\overline{r} = r + Z(R)$ for any $r \in R$ and $\overline{S} = S/Z(R)$ for any $S \leqslant R$. Since $\overline{R}$ is not cyclic, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_{36}, \mathbb{Z}_3 \times \mathbb{Z}_{24}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{18}, \mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_6$ or $\mathbb{Z}_6 \times \mathbb{Z}_{12}$. Hence, $|\overline{C_R(x_1)}| \leqslant 36$ and $|\overline{C_R(x_i)}| \leqslant 18$ for any $i \in \{2, 3, \cdots, 7\}$. This shows that $\overline{R}$ has at most ${}_{36}|\mathbb{Z}_{36}|$ elements of order 36 and $\overline{R}$ has at most ${}_{24}|\mathbb{Z}_{24}|$ elements of order 24. Since ${}_{36}|\mathbb{Z}_{36}| < {}_{36}|\mathbb{Z}_2 \times \mathbb{Z}_{36}|$ and ${}_{24}|\mathbb{Z}_{24}| < {}_{24}|\mathbb{Z}_3 \times \mathbb{Z}_{24}|$, then $\overline{R} \ncong \mathbb{Z}_2 \times \mathbb{Z}_{36}$ and $\mathbb{Z}_3 \times \mathbb{Z}_{24}$.

Suppose that $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{18}$. We first claim that if $|\overline{C_R(x_i)}| = 18$ for some $i \in \{2, 3, \cdots, 7\}$, then $\overline{C_R(x_i)} \ncong \mathbb{Z}_3 \times \mathbb{Z}_6$. If not, then $\overline{R}$ has ${}_3|\mathbb{Z}_3 \times \mathbb{Z}_6| = 8$ elements of order 3, which contradicts the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{18}$ has only 2 elements of order 3. Next, we want to show that $\gamma_6 \neq 4$. Assume that $\gamma_6 = 4$, then $\overline{C_R(x_i)} \cong \mathbb{Z}_{18}$ for any $i \in \{2, 3, \cdots, 6\}$. This gives that $\overline{C_R(x_i)}$ has exactly 2 elements of order 3 for any $i \in \{2, 3, \cdots, 6\}$. Since 3 is divide $|\overline{C_R(x_1)}|$, then there exists an element of order 3 in $\overline{C_R(x_1)}$. By the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{18}$ has exactly 2 elements of order 3, then there exists some $\overline{a} \in \overline{R} - \overline{Z(R)}$ with order 3 such that $\overline{a} \in \bigcap\limits_{i=1}^{6} \overline{C_R(x_i)}$. So, by Lemma 1.3.1(b), (c) and Lemma 2.2.2, we obtain $\overline{a} \in \overline{Z(R)}$, which leads to a contradiction. Consequently, $\gamma_6 \neq 4$ and so, $|\overline{C_R(x_i)}| \leqslant 12$ for any $i \in \{6, 7\}$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{18}$ has 42 elements of order 18, then it follows that $\overline{C_R(x_1)} \cong \mathbb{Z}_2 \times \mathbb{Z}_{18}$ and $\overline{C_R(x_i)} \cong \mathbb{Z}_{18}$ for any $i \in \{2, 3, 4, 5\}$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{18}$ has exactly 2 elements of order 3, then there exists some $\overline{a} \in \overline{R} - \overline{Z(R)}$ with order 3 such that $\overline{a} \in \bigcap\limits_{i=2}^{5} \overline{C_R(x_i)}$. For any $i \in \{2, 3, 4, 5\}$, since $\overline{C_R(x_i)}$ is cyclic, then $C_R(x_i)$ is commutative. Therefore, we have $\bigcup\limits_{i=2}^{5} C_R(x_i) \subseteq C_R(a)$. This gives that $R = C_R(a) \cup C_R(x_1) \cup C_R(x_6) \cup$

$C_R(x_7)$. So, by Lemma 2.7.3, we obtain $|R| \leqslant \frac{|R|}{2} + 2(\frac{|R|}{6}) = \frac{5|R|}{6}$, which is a contradiction.

Next, we suppose that $\overline{R} \cong \mathbb{Z}_6 \times \mathbb{Z}_{12}$. First, we consider $|\overline{C_R(x_1)}| \leqslant 24$. Since $\mathbb{Z}_6 \times \mathbb{Z}_{12}$ has 32 elements of order 12, then it follows that $\overline{C_R(x_1)} \cong \mathbb{Z}_2 \times \mathbb{Z}_{12}$ and $\overline{C_R(x_i)} \cong \mathbb{Z}_{12}$ for any $i \in \{2, 3, \cdots, 7\}$. This yields that $\overline{R}$ has at most ${}_6|\mathbb{Z}_2 \times \mathbb{Z}_{12}| + 6({}_6|\mathbb{Z}_{12}|) = 6 + 6(2) = 18$ elements of order 6, which contradicts the fact that $\mathbb{Z}_6 \times \mathbb{Z}_{12}$ has 24 elements of order 6. Therefore, $|\overline{C_R(x_1)}| = 36$. If $C_R(x_1)$ is commutative, then by Lemma 2.2.12, we obtain $|\overline{R}| \leqslant 2(6) = 12$, which is a contradiction. Therefore, $C_R(x_1)$ is non-commutative. By Lemma 2.2.8(b), we have $C_R(x_i)$ is commutative for any $i \in \{2, 3, \cdots, 7\}$. We claim that if $\gamma_2 = 4$, then there exist four distinct $l_1, l_2, l_3, l_4 \in \{3, 4, 5, 6, 7\}$ such that $\gamma_{l_1} = \gamma_{l_2} = \gamma_{l_3} = \gamma_{l_4} = 6$. If not, then $\overline{R}$ has at most ${}_{12}|\mathbb{Z}_3 \times \mathbb{Z}_{12}| + 3({}_{12}|\mathbb{Z}_{12}|) = 16 + 3(4) = 28$ elements of order 12, which contradicts the fact that $\mathbb{Z}_6 \times \mathbb{Z}_{12}$ has 32 elements of order 12. Therefore, our claim is true. On the other hand, if $\gamma_2 = 6$, then by Lemma 2.2.1, we obtain $\gamma_2 = \gamma_3 = \cdots = \gamma_7 = 6$. In both situations, we have $\gamma_{l_1} = \gamma_{l_2} = \gamma_{l_3} = \gamma_{l_4} = 6$ for four distinct $l_1, l_2, l_3, l_4 \in \{3, 4, 5, 6, 7\}$. Let $i \in \{l_1, l_2, l_3, l_4\}$. In view of Lemma 2.2.11, we have $C_R(x_2) \cap C_R(x_i) \neq Z(R)$, which implies that there exists some $w_i \in (C_R(x_2) \cap C_R(x_i)) - Z(R)$ and it follows that $C_R(x_2) \cup C_R(x_i) \subseteq C_R(w_i)$. It is obvious that $C_R(w_i) \neq R, C_R(x_j)$ for any $j \in \{1, 2, \cdots, 7\}$. Assume that $C_R(w_j) = C_R(w_k) = C_R(w_l)$ for three distinct $j, k, l \in \{l_1, l_2, l_3, l_4\}$. Hence, we have $C_R(x_2) \cup C_R(x_j) \cup C_R(x_k) \cup C_R(x_l) \subseteq C_R(w_j)$. It follows that $R = C_R(w_j) \cup C_R(x_1) \cup C_R(x_{l_5}) \cup C_R(x_{l_6})$, where $l_5 \in \{3, 4, 5, 6, 7\} - \{l_1, l_2, l_3, l_4\}$ and $l_6 \in \{l_1, l_2, l_3, l_4\} - \{j, k, l\}$. So, by

Lemma 2.7.3, we obtain $|R| \leqslant \frac{|R|}{2} + \frac{|R|}{4} + \frac{|R|}{6} = \frac{11|R|}{12}$, a contradiction is reached. Consequently, there does not exist any three distinct $j, k, l \in \{l_1, l_2, l_3, l_4\}$ such that $C_R(w_j) = C_R(w_k) = C_R(w_l)$. Since $|\text{Cent}(R)| = 11$, then it follows that $C_R(w_j) = C_R(w_k)$ for two distinct $j, k \in \{l_1, l_2, l_3, l_4\}$. Therefore, we have $R = C_R(w_j) \cup C_R(x_1) \cup C_R(x_u) \cup C_R(x_v) \cup C_R(x_l)$, where $u, v \in \{l_1, l_2, l_3, l_4\} - \{j, k\}$ with $u \neq v$ and $l \in \{3, 4, 5, 6, 7\} - \{l_1, l_2, l_3, l_4\}$. Thus, by Lemma 2.7.3, we obtain $|R| \leqslant \frac{|R|}{2} + 2(\frac{|R|}{6}) + |C_R(x_l)|$, which yields that $\gamma_l \leqslant 6$. By Lemma 2.2.11, we have $C_R(x_2) \cap C_R(x_l) \neq Z(R)$. So, there exists some $w_l \in (C_R(x_2) \cap C_R(x_l)) - Z(R)$ and hence, $C_R(x_2) \cup C_R(x_l) \subseteq C_R(w_l)$. Clearly, $C_R(w_l) \neq R, C_R(x_i)$ for any $i \in \{1, 2, \cdots, 7\}$. If $C_R(w_l) = C_R(w_j)$, then $R = C_R(w_l) \cup C_R(x_1) \cup C_R(x_u) \cup C_R(x_v)$. So, we obtain $|R| \leqslant \frac{|R|}{2} + 2(\frac{|R|}{6}) = \frac{5|R|}{6}$ by Lemma 2.7.3, which is a contradiction. If $C_R(w_l) = C_R(w_u) = C_R(w_v)$, then $R = C_R(w_l) \cup C_R(x_1) \cup C_R(x_j) \cup C_R(x_k)$. So, we obtain $|R| \leqslant \frac{|R|}{2} + 2(\frac{|R|}{6}) = \frac{5|R|}{6}$ by Lemma 2.7.3, which is a contradiction again. Since $|\text{Cent}(R)| = 11$, then we have $C_R(w_m) \neq C_R(w_n)$ for two distinct $m, n \in \{l, u, v\}$. Therefore, $C_R(w_j), C_R(w_m), C_R(w_n)$ are three distinct proper centralisers of R that are different from $C_R(x_i)$ for any $i \in \{1, 2, \cdots, t\}$. From Lemma 2.2.8(a), we have $w_j, w_m, w_n \in C_R(x_1)$ and w_j, w_m, w_n do not commute with each other. We now consider for $C_R(w_j + x_1)$. Since $w_m \notin C_R(w_j + x_1)$ but $w_m \in R, C_R(x_1)$, then $C_R(w_j + x_1) \neq R, C_R(x_1)$. For any $i \in \{2, 3, \cdots, 7\}$, since $x_1 \in C_R(w_j + x_1)$ but $x_1 \notin C_R(x_i)$, then $C_R(w_j + x_1) \neq C_R(x_i)$. Since $x_j \notin C_R(w_j + x_1)$ but $x_j \in C_R(w_j)$, then $C_R(w_j + x_1) \neq C_R(w_j)$. Since $w_m, w_n \notin C_R(w_j + x_1)$ but $w_m \in C_R(w_m)$ and $w_n \in C_R(w_n)$, then $C_R(w_j + x_1) \neq C_R(w_m), C_R(w_n)$. Consequently, we obtain $|\text{Cent}(R)| \geqslant 12$, a contradiction is reached.

Hence, we have $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_6$. We first claim that $\gamma_6 \neq 4$. Suppose that $\gamma_6 = 4$, then $\overline{C_R(x_i)} \cong \mathbb{Z}_3 \times \mathbb{Z}_6$ for any $i \in \{2, 3, \cdots, 6\}$. This gives that $\overline{C_R(x_i)}$ has exactly 8 elements of order 3 for any $i \in \{2, 3, \cdots, 6\}$. Since 3 is divide $|\overline{C_R(x_1)}|$, then there exists an element of order 3 in $\overline{C_R(x_1)}$. By the fact that $\mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_6$ has exactly 8 elements of order 3, then there exists some $\overline{a} \in \overline{R} - \overline{Z(R)}$ with order 3 such that $\overline{a} \in \bigcap_{i=1}^{6} \overline{C_R(x_i)}$. So, by Lemma 1.3.1(b), (c) and Lemma 2.2.2, we obtain $\overline{a} \in \overline{Z(R)}$, which leads to a contradiction. Consequently, $\gamma_6 \neq 4$. We next claim that $\gamma_5 \neq 4$. Assume that $\gamma_5 = 4$, then $\overline{C_R(x_i)} \cong \mathbb{Z}_3 \times \mathbb{Z}_6$ for any $i \in \{2, 3, 4, 5\}$. This gives that $\overline{C_R(x_i)}$ has exactly 8 elements of order 3 for any $i \in \{2, 3, 4, 5\}$. Since 3 is divide $|\overline{C_R(x_1)}|$, then there exists an element of order 3 in $\overline{C_R(x_1)}$. By the fact that $\mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_6$ has exactly 8 elements of order 3, then there exists some $\overline{a} \in \overline{R} - \overline{Z(R)}$ with order 3 such that $\overline{a} \in \bigcap_{i=1}^{5} \overline{C_R(x_i)}$. In view of Lemma 2.2.8(b), there exist four distinct $l_1, l_2, l_3, l_4 \in \{1, 2, 3, 4, 5\}$ such that $C_R(x_{l_1}), C_R(x_{l_2}), C_R(x_{l_3}), C_R(x_{l_4})$ are commutative. Therefore, we have $\bigcup_{i=1}^{4} C_R(x_{l_i}) \subseteq C_R(a)$. It follows that $R = C_R(a) \cup C_R(x_{l_5}) \cup C_R(x_6) \cup C_R(x_7)$, where $l_5 \in \{1, 2, 3, 4, 5\} - \{l_1, l_2, l_3, l_4\}$. So, by Lemma 2.7.3, we obtain $|R| \leqslant \frac{|R|}{2} + 2(\frac{|R|}{6}) = \frac{5|R|}{6}$, which is impossible. So, $\gamma_5 \neq 4$. If $\gamma_1 \geqslant 4$, then $\overline{R}$ has at most $4({}_6|\mathbb{Z}_3 \times \mathbb{Z}_6|) + 3({}_6|\mathbb{Z}_2 \times \mathbb{Z}_6|) = 4(8) + 3(6) = 50$ elements of order 6, which leads to a contradiction as $\mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_6$ has 56 elements of order 6. Therefore, $\gamma_1 \leqslant 3$. If $C_R(x_1)$ is commutative, then it follows from Lemma 2.2.11 that $|\overline{R}| \leqslant 3(6) = 18$, which is a contradiction. Thus, $C_R(x_1)$ is non-commutative. From Lemma 2.2.8(b), we have $C_R(x_i)$ is commutative for any $i \in \{2, 3, \cdots, 7\}$. Next, we want to show that $\gamma_4 \neq 4$. Suppose that $\gamma_4 = 4$, then $\overline{C_R(x_i)} \cong \mathbb{Z}_3 \times \mathbb{Z}_6$ for any $i \in \{2, 3, 4\}$. This gives that $\overline{C_R(x_i)}$ has exactly 8 elements of order 3 for any

$i \in \{2,3,4\}$. If 3 is not divide $|\overline{C_R(x_5)}|$, then $|\overline{C_R(x_5)}| = 2, 4$ or 8. This implies that $\overline{R}$ has at most ${}_6|\mathbb{Z}_6 \times \mathbb{Z}_6| + 3({}_6|\mathbb{Z}_3 \times \mathbb{Z}_6|) + 2({}_6|\mathbb{Z}_6|) = 24 + 3(8) + 2(2) = 52$ elements of order 6. This contradicts with the fact that $\mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_6$ has 56 elements of order 6. Therefore, 3 is divide $|\overline{C_R(x_5)}|$. Since 3 is divide $|\overline{C_R(x_5)}|$, then there exists an element of order 3 in $\overline{C_R(x_5)}$. By the fact that $\mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_6$ has exactly 8 elements of order 3, then there exists some $\overline{a} \in \overline{R} - \overline{Z(R)}$ with order 3 such that $\overline{a} \in \bigcap_{i=2}^{5} \overline{C_R(x_i)}$. Thus, we have $\bigcup_{i=2}^{5} C_R(x_i) \subseteq C_R(a)$. This implies that $R = C_R(a) \cup C_R(x_1) \cup C_R(x_6) \cup C_R(x_7)$. By Lemma 2.7.3, it follows that $|R| \leqslant \frac{|R|}{2} + 2(\frac{|R|}{6}) = \frac{5|R|}{6}$, which is impossible. So, $\gamma_4 \neq 4$. We now claim that $C_R(x_2) \cap C_R(x_i) \neq Z(R)$ for any $i \in \{3,4,5,6\}$. We first consider the case where $\gamma_2 = 4$. If $\gamma_6 \geqslant 18$, then $\overline{R}$ has at most ${}_6|\mathbb{Z}_6 \times \mathbb{Z}_6| + 2({}_6|\mathbb{Z}_3 \times \mathbb{Z}_6|) + 2({}_6|\mathbb{Z}_2 \times \mathbb{Z}_6|) = 24 + 2(8) + 2(6) = 52$ elements of order 6, which contradicts the fact that $\mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_6$ has 56 elements of order 6. Hence, $\gamma_6 \leqslant 12$. Therefore, by Lemma 2.2.11, it follows that $C_R(x_2) \cap C_R(x_i) \neq Z(R)$ for any $i \in \{3,4,5,6\}$. We next consider the case where $\gamma_2 = 6$. From Lemma 2.2.1, we have $\gamma_6 = 6$. Therefore, by Lemma 2.2.11, it follows that $C_R(x_2) \cap C_R(x_i) \neq Z(R)$ for any $i \in \{3,4,5,6\}$. By combining these two cases, our claim is proved. Let $i \in \{3,4,5,6\}$. Hence, there exists some $w_i \in (C_R(x_2) \cap C_R(x_i)) - Z(R)$, which gives that $C_R(x_2) \cup C_R(x_i) \subseteq C_R(w_i)$. It is obvious that $C_R(w_i) \neq R, C_R(x_j)$ for any $j \in \{1, 2, \cdots, 7\}$. Assume that $C_R(w_j) = C_R(w_k) = C_R(w_l)$ for three distinct $j, k, l \in \{3,4,5,6\}$. Hence, we have $C_R(x_2) \cup C_R(x_j) \cup C_R(x_k) \cup C_R(x_l) \subseteq C_R(w_j)$. It follows that $R = C_R(w_j) \cup C_R(x_1) \cup C_R(x_h) \cup C_R(x_7)$, where $h \in \{3,4,5,6\} - \{j,k,l\}$. So, by Lemma 2.7.3, we obtain $|R| \leqslant \frac{|R|}{2} + \frac{|R|}{4} + \frac{|R|}{6} = \frac{11|R|}{12}$, a contradiction is reached. Consequently, there does not exist any three distinct $j, k, l \in \{3, 4, 5, 6\}$ such that $C_R(w_j) = C_R(w_k) = C_R(w_l)$. Since $|\mathrm{Cent}(R)| = 11$, then it follows that $C_R(w_j) = C_R(w_k)$ for two distinct $j, k \in \{3, 4, 5, 6\}$. Therefore, we have $R = C_R(w_j) \cup C_R(x_1) \cup C_R(x_u) \cup C_R(x_v) \cup C_R(x_7)$, where $u, v \in \{3, 4, 5, 6\} - \{j, k\}$ with $u \neq v$. Thus, by Lemma 2.7.3, we obtain $|R| \leqslant \frac{|R|}{2} + \frac{|R|}{4} + \frac{|R|}{6} + |C_R(x_l)|$, which yields that $\gamma_7 \leqslant 12$. If $\gamma_2 = 4$, then it follows from Lemma 2.2.11 that $C_R(x_2) \cap C_R(x_7) \neq Z(R)$. On the other hand, if $\gamma_2 = 6$, then by Lemma 2.2.1, $\gamma_7 = 6$. Therefore, by Lemma 2.2.11, it follows that $C_R(x_2) \cap C_R(x_7) \neq Z(R)$. So, there exists some $w_7 \in (C_R(x_2) \cap C_R(x_7)) - Z(R)$ and hence, $C_R(x_2) \cup C_R(x_7) \subseteq C_R(w_7)$. Clearly, $C_R(w_7) \neq R, C_R(x_i)$ for any $i \in \{1, 2, \cdots, 7\}$. If $C_R(w_7) = C_R(w_j)$, then $R = C_R(w_l) \cup C_R(x_1) \cup C_R(x_u) \cup C_R(x_v)$. So, we obtain $|R| \leqslant \frac{|R|}{2} + \frac{|R|}{4} + \frac{|R|}{6} = \frac{11|R|}{12}$ by Lemma 2.7.3, which is a contradiction. If $C_R(w_7) = C_R(w_u) = C_R(w_v)$, then $R = C_R(w_7) \cup C_R(x_1) \cup C_R(x_j) \cup C_R(x_k)$. So, we obtain $|R| \leqslant \frac{|R|}{2} + \frac{|R|}{4} + \frac{|R|}{6} = \frac{11|R|}{12}$ by Lemma 2.7.3, which is a contradiction again. Since $|\mathrm{Cent}(R)| = 11$, then we have $C_R(w_m) \neq C_R(w_n)$ for two distinct $m, n \in \{7, u, v\}$. Therefore, $C_R(w_j), C_R(w_m), C_R(w_n)$ are three distinct proper centralisers of R that are different from $C_R(x_i)$ for any $i \in \{1, 2, \cdots, t\}$. From Lemma 2.2.8(a), we have $w_j, w_m, w_n \in C_R(x_1)$ and w_j, w_m, w_n do not commute with each other. We now consider for $C_R(w_j + x_1)$. Since $w_m \notin C_R(w_j + x_1)$ but $w_m \in R, C_R(x_1)$, then $C_R(w_j + x_1) \neq R, C_R(x_1)$. For any $i \in \{2, 3, \cdots, 7\}$, since $x_1 \in C_R(w_j + x_1)$ but $x_1 \notin C_R(x_i)$, then $C_R(w_j + x_1) \neq C_R(x_i)$. Since $x_j \notin C_R(w_j + x_1)$ but $x_j \in C_R(w_j)$, then $C_R(w_j + x_1) \neq C_R(w_j)$. Since $w_m, w_n \notin C_R(w_j + x_1)$ but $w_m \in C_R(w_m)$ and $w_n \in C_R(w_n)$, then $C_R(w_j + x_1) \neq C_R(w_m), C_R(w_n)$. Consequently, we obtain $|\mathrm{Cent}(R)| \geqslant 12$, a contradiction is reached. □

Lemma 2.8.12. Let $\{x_1, x_2, \cdots, x_7\}$ be the maximal non-commuting set of a finite ring R. If R is an 11-centraliser finite ring, then $|R : Z(R)| \neq 80$.

Proof. Assume that $|R : Z(R)| = 80$. Without loss of generality, we suppose that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_7$. From Lemma 1.3.1(a), we have $R/Z(R) = \bigcup_{i=1}^{7}[C_R(x_i)/Z(R)]$. By Lemma 2.8.3, we have $\gamma_2 = 4$ or 5. Let $_m|G|$ denote the total number of elements with order m in an additive group G. For the sake of simplicity, we write $\overline{r} = r + Z(R)$ for any $r \in R$ and $\overline{S} = S/Z(R)$ for any $S \leqslant R$.

Since $\overline{R}$ is not cyclic, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_{40}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{20}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{10}$ or $\mathbb{Z}_4 \times \mathbb{Z}_{20}$. Hence, $|\overline{C_R(x_1)}| \leqslant 40$ and $|\overline{C_R(x_i)}| \leqslant 20$ for any $i \in \{2, 3, \cdots, 7\}$. This shows that $\overline{R}$ has at most $_{40}|\mathbb{Z}_{40}|$ elements of order 40. Since $_{40}|\mathbb{Z}_{40}| < {}_{40}|\mathbb{Z}_2 \times \mathbb{Z}_{40}|$, then $\overline{R} \ncong \mathbb{Z}_2 \times \mathbb{Z}_{40}$. We first claim that $\gamma_1 \neq 5$. Assume that $\gamma_1 = 5$, then $|\overline{C_R(x_1)}| = 16$ and $|\overline{C_R(x_i)}| \leqslant 16$ for any $i \in \{2, 3, \cdots, 7\}$. This shows that $|\overline{R}|$ has at most $6({}_{10}|\mathbb{Z}_{10}|) = 6(4) = 24$ elements of order 10. Also, there does not exist any element of order 20. This contradicts with the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{10}$ has 60 elements of order 10 and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{20}, \mathbb{Z}_4 \times \mathbb{Z}_{20}$ have an element of order 20. Thus, $\gamma_1 \neq 5$. We next claim that $\gamma_6 \neq 4$. Suppose that $\gamma_6 = 4$, then $\overline{C_R(x_i)} \cong \mathbb{Z}_2 \times \mathbb{Z}_{10}$ or $\mathbb{Z}_{20}$ for any $i \in \{2, 3, \cdots, 6\}$. This gives that $\overline{C_R(x_i)}$ has exactly 4 elements of order 5 for any $i \in \{2, 3, \cdots, 6\}$. Since 5 is divide $|\overline{C_R(x_1)}|$, then there exists an element of order 5 in $\overline{C_R(x_1)}$. By the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{20}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{10}$ and $\mathbb{Z}_4 \times \mathbb{Z}_{20}$ have exactly 4 elements of order 5, then there exists some $\overline{a} \in \overline{R} - \overline{Z(R)}$ with order 5 such that $\overline{a} \in \bigcap_{i=1}^{6} \overline{C_R(x_i)}$. So, by Lemma 1.3.1(b), (c) and Lemma 2.2.2, we obtain

$\overline{a} \in \overline{Z(R)}$, which leads to a contradiction. Consequently, $\gamma_6 \neq 4$.

We claim that $\gamma_5 \neq 4$. Suppose that $\gamma_5 = 4$, then $\overline{C_R(x_i)} \cong \mathbb{Z}_2 \times \mathbb{Z}_{10}$ or $\mathbb{Z}_{20}$ for any $i \in \{2,3,4,5\}$. This gives that $\overline{C_R(x_i)}$ has exactly 4 elements of order 5 for any $i \in \{2,3,4,5\}$. Since 5 is divide $|\overline{C_R(x_1)}|$, then there exists an element of order 5 in $\overline{C_R(x_1)}$. By the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{20}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{10}$ and $\mathbb{Z}_4 \times \mathbb{Z}_{20}$ have exactly 4 elements of order 5, then there exists some $\overline{a} \in \overline{R} - \overline{Z(R)}$ with order 5 such that $\overline{a} \in \bigcap_{i=1}^{5} \overline{C_R(x_i)}$. In view of Lemma 2.2.8(b), there exist four distinct $l_1, l_2, l_3, l_4 \in \{1,2,3,4,5\}$ such that $C_R(x_{l_1}), C_R(x_{l_2}), C_R(x_{l_3}), C_R(x_{l_4})$ are commutative. Therefore, we have $\bigcup_{i=1}^{4} C_R(x_{l_i}) \subseteq C_R(a)$. It follows that $R = C_R(a) \cup C_R(x_{l_5}) \cup C_R(x_6) \cup C_R(x_7)$, where $l_5 \in \{1,2,3,4,5\} - \{l_1, l_2, l_3, l_4\}$. So, by Lemma 2.7.3, we obtain $|R| \leqslant \frac{|R|}{2} + 2(\frac{|R|}{5}) = \frac{9|R|}{10}$, which is impossible. So, $\gamma_5 \neq 4$. Therefore, we have $|\overline{C_R(x_1)}| \leqslant 40$, $|\overline{C_R(x_i)}| \leqslant 20$ for any $i \in \{2,3,4\}$ and $|\overline{C_R(x_j)}| \leqslant 16$ for any $j \in \{5,6,7\}$. This gives that $\overline{R}$ has at most ${}_{20}|\mathbb{Z}_2 \times \mathbb{Z}_{20}| + 3({}_{20}|\mathbb{Z}_{20}|) = 16 + 3(8) = 40$ elements of order 20. Since $\mathbb{Z}_4 \times \mathbb{Z}_{20}$ has 48 elements of order 20, then $\overline{R} \ncong \mathbb{Z}_4 \times \mathbb{Z}_{20}$.

Next, we want to show that $\gamma_4 \neq 4$. Suppose that $\gamma_4 = 4$, then $\overline{C_R(x_i)} \cong \mathbb{Z}_2 \times \mathbb{Z}_{10}$ or $\mathbb{Z}_{20}$ for any $i \in \{2,3,4\}$. This gives that $\overline{C_R(x_i)}$ has exactly 4 elements of order 5 for any $i \in \{2,3,4\}$. Since 5 is divide $|\overline{C_R(x_1)}|$, then there exists an element of order 5 in $\overline{C_R(x_1)}$. By the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{20}$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{10}$ have exactly 4 elements of order 5, then there exists some $\overline{a} \in \overline{R} - \overline{Z(R)}$ with order 5 such that $\overline{a} \in \bigcap_{i=1}^{4} \overline{C_R(x_i)}$. In view of Lemma 2.2.8(b), there exist three distinct $l_1, l_2, l_3 \in \{1,2,3,4\}$ such that $C_R(x_{l_1}), C_R(x_{l_2}), C_R(x_{l_3})$ are commutative. Therefore, we have $\bigcup_{i=1}^{3} C_R(x_{l_i}) \subseteq C_R(a)$. It follows that $R = C_R(a) \cup C_R(x_{l_4}) \cup C_R(x_5) \cup C_R(x_6) \cup C_R(x_7)$, where $l_4 \in \{1,2,3,4\} - \{l_1, l_2, l_3\}$. We claim that $\gamma_7 \neq 8$. If $\gamma_7 = 8$, then $|\overline{C_R(x_7)}| = 10$ and so, $\overline{C_R(x_7)} \cong \mathbb{Z}_{10}$. This shows that $\overline{C_R(x_7)}$ has exactly 4 elements of order 5. By the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{20}$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{10}$ have exactly 4 elements of order 5, then $\overline{a} \in \overline{C_R(x_7)}$. Since $\overline{C_R(x_7)}$ is cyclic, then $C_R(x_7)$ is commutative, which follows that $C_R(x_7) \leqslant C_R(a)$. Therefore, we have $R = C_R(a) \cup C_R(x_{l_4}) \cup C_R(x_5) \cup C_R(x_6)$. So, from Lemma 2.2.1, we obtain $|R| \leqslant \frac{|R|}{2} + 2(\frac{|R|}{5}) = \frac{9|R|}{10}$; a contradiction. So, $\gamma_7 \neq 8$. If $\gamma_6 \neq 5$, then by Lemma 2.7.3, we obtain $|R| \leqslant \frac{|R|}{2} + \frac{|R|}{5} + 2(\frac{|R|}{8}) = \frac{19|R|}{20}$; a contradiction. So, $\gamma_6 = 5$ and hence, $\gamma_5 = 5$. From Lemma 2.7.3 again, we have $|R : C_R(a)| = 2$ and $|R : C_R(x_{l_4})| = 2$. Since $\gamma_2 \geqslant 4$, then $l_4 = 1$. Hence, we have $|\overline{C_R(a)}| = |\overline{C_R(x_1)}| = 40$, $|\overline{C_R(x_5)}| = |\overline{C_R(x_6)}| = 16$ and $|\overline{C_R(x_7)}| \leqslant 16$ with $|\overline{C_R(x_7)}| \neq 10$. Note that, $\overline{R} = \overline{C_R(a)} \cup \overline{C_R(x_1)} \cup \overline{C_R(x_5)} \cup \overline{C_R(x_6)} \cup \overline{C_R(x_7)}$. Assume that $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{20}$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{20}$ has 32 elements of order 20, then it follows that $\overline{C_R(a)}, \overline{C_R(x_1)} \cong \mathbb{Z}_2 \times \mathbb{Z}_{20}$. This implies that $\overline{R}$ has at most $2({}_{10}|\mathbb{Z}_2 \times \mathbb{Z}_{20}|) = 2(12) = 24$ elements of order 10. We have reached a contradiction as $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{20}$ has 28 elements of order 10. Next, we suppose that $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{10}$. Hence, $\overline{C_R(a)}, \overline{C_R(x_1)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{10}$. This yields that $\overline{R}$ has at most $2({}_{10}|\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{10}|) = 2(28) = 56$ elements of order 10, which leads to a contradiction as $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{10}$ has 60 elements of order 10. So, $\gamma_4 \neq 4$.

Therefore, we have $|\overline{C_R(x_1)}| \leqslant 40$, $|\overline{C_R(x_i)}| \leqslant 20$ for any $i \in \{2,3\}$ and $|\overline{C_R(x_j)}| \leqslant 16$ for any $j \in \{4,5,6,7\}$. Assume that $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{10}$.

Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{10}$ has 60 elements of order 10, then it follows that $\overline{C_R(x_1)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{10}$, $\overline{C_R(x_2)}, \overline{C_R(x_3)} \cong \mathbb{Z}_2 \times \mathbb{Z}_{10}$ and $\overline{C_R(x_u)}, \overline{C_R(x_v)} \cong \mathbb{Z}_{10}$ for two distinct $u, v \in \{4, 5, 6, 7\}$. This gives that $\overline{C_R(x_i)}$ has exactly 4 elements of order 5 for any $i \in \{1, 2, 3, u, v\}$. By the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{10}$ has exactly 4 elements of order 5, then there exists some $\overline{a} \in \overline{R} - \overline{Z(R)}$ with order 5 such that $\overline{a} \in \bigcap_{i=1,2,3,u,v} \overline{C_R(x_i)}$. In view of Lemma 2.2.8(b), there exist four distinct $l_1, l_2, l_3, l_4 \in \{1, 2, 3, u, v\}$ such that $C_R(x_{l_1}), C_R(x_{l_2}), C_R(x_{l_3}), C_R(x_{l_4})$ are commutative. Therefore, we have $\bigcup_{i=1}^{4} C_R(x_{l_i}) \subseteq C_R(a)$. It follows that $R = C_R(a) \cup C_R(x_{l_5}) \cup C_R(x_{l_6}) \cup C_R(x_{l_7})$, where $l_5 \in \{1, 2, 3, u, v\} - \{l_1, l_2, l_3, l_4\}$ and $l_6, l_7 \in \{4, 5, 6, 7\} - \{u, v\}$ with $l_6 \neq l_7$. Hence, we obtain $|R| \leqslant \frac{|R|}{2} + 2(\frac{|R|}{5}) = \frac{9|R|}{10}$ by Lemma 2.7.3, a contradiction is reached. Consequently, we have $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{20}$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{20}$ has 32 elements of order 20, then it follows that $\overline{C_R(x_1)} \cong \mathbb{Z}_2 \times \mathbb{Z}_{20}$ and $\overline{C_R(x_2)}, \overline{C_R(x_3)} \cong \mathbb{Z}_{20}$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{20}$ has 28 elements of order 10, then we have $\overline{C_R(x_u)}, \overline{C_R(x_v)} \cong \mathbb{Z}_{10}$ for two distinct $u, v \in \{4, 5, 6, 7\}$. This gives that $\overline{C_R(x_i)}$ has exactly 4 elements of order 5 for any $i \in \{1, 2, 3, u, v\}$. By using similar arguments as in above, we will obtain $|R| \leqslant \frac{|R|}{2} + 2(\frac{|R|}{5}) = \frac{9|R|}{10}$, which leads to a contradiction. □

Lemma 2.8.13. Let t be the cardinality of the maximal non-commuting set of a finite ring R. If R is an 11-centraliser finite ring, then $t \neq 8$.

Proof. Assume that $t = 8$. Let $\{x_1, x_2, \cdots, x_8\}$ be the maximal non-commuting set of R. Without loss of generality, we suppose that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_8$. From Lemma 1.3.1(a), we have $R = \bigcup_{i=1}^{8} C_R(x_i)$. By Lemma 2.2.3, we have $C_R(x_i)$ is commutative for any $i \in \{1, 2, \cdots, 8\}$ and $C_R(a), C_R(b)$ are two distinct non-commutative proper centralisers of R for some

$a, b \in R - Z(R)$. By Theorem 2.2.23, we have $|R : Z(R)| \geqslant 16$ with $|R : Z(R)|$ is not square-free, $|R : Z(R)| \neq p^2 q$ for any two distinct primes p, q, and $|R : Z(R)| \neq p^2$ for any prime p.

First, we claim that $\gamma_i \geqslant 4$ for any $i \in \{1, 2, \cdots, 8\}$. Assume that $\gamma_1 \leqslant 3$, then by Corollary 2.2.13, we obtain $|R : Z(R)| \leqslant 3\gamma_2$. By Lemma 2.2.1, we have $\gamma_2 \leqslant 7$. If $\gamma_2 \leqslant 5$, then $|R : Z(R)| \leqslant 15$; a contradiction. If $\gamma_2 = 6$, then $|R : Z(R)| \leqslant 18$; a contradiction. If $\gamma_2 = 7$, then $|R : Z(R)| \leqslant 21$; a contradiction. Therefore, $\gamma_1 \geqslant 4$ and so, $\gamma_i \geqslant 4$ for any $i \in \{1, 2, \cdots, 8\}$, as claimed

Next, we want to show that $C_R(a)$ contains exactly two distinct $C_R(x_i)$'s. From Lemma 2.2.7, we have $R = C_R(a) \cup \left(\underset{i \in A}{\cup} C_R(x_i) \right)$ for some $A \subset \{1, 2, \cdots, 8\}$ with $|A| \leqslant 6$. Obviously, $|A| \neq 0$. Suppose that $|A| \leqslant 3$, then by Lemma 2.2.1, it follows that $\gamma_i \leqslant |A| \leqslant 3$ for some $i \in A$. This contradicts with the fact that $\gamma_i \geqslant 4$. Assume that $|A| = 4$ or 5. Thus, we have $R = C_R(a) \cup \left(\bigcup_{i=1}^{|A|} C_R(x_{k_i}) \right)$ for $|A|$ distinct $k_1, k_2, \cdots, k_{|A|} \in \{1, 2, \cdots, 8\}$. Without loss of generality, we assume that $\gamma_{k_1} \leqslant \gamma_{k_2} \leqslant \cdots \leqslant \gamma_{|A|}$. We claim that $\gamma_{k_1} = 4$. By Lemma 2.2.1, $\gamma_{k_1} \leqslant |A|$. For $|A| = 4$, we have $\gamma_{k_1} = 4$. For $|A| = 5$, we have $\gamma_{k_1} = 4$ or 5. If $\gamma_{k_1} = 5$, then by Lemma 2.2.1, we obtain $\gamma_{k_2} = \gamma_{k_3} = \gamma_{k_4} = \gamma_{k_5} = 5$. Therefore, by Corollary 2.2.13, we obtain $|R : Z(R)| \leqslant 5(5) = 25$, which is a contradiction. So, $\gamma_{k_1} = 4$. We next claim that $\gamma_{k_i} \neq 5$ and 7 for any $i \in \{2, \cdots, |A|\}$. Suppose that $\gamma_{k_i} = 5$ or 7 for some $i \in \{2, \cdots, |A|\}$, then by Corollary 2.2.13, we obtain $|R : Z(R)| \leqslant 4\gamma_2$. If $\gamma_2 = 5$, then $|R : Z(R)| \leqslant 20$, which is a contradiction. If $\gamma_2 = 7$, then $|R : Z(R)| \leqslant 28$, which is a contradiction again. Therefore, $\gamma_{k_i} \neq 5$ and 7 for any $i \in \{2, \cdots, |A|\}$. In view of Lemma 2.7.3, we have $|R| \leqslant \sum_{i=1}^{|A|} |C_R(x_{k_i})| \leqslant \frac{|R|}{4} + (|A| - 1)|C_R(x_{k_2})|$, which gives that $\gamma_{k_2} \leqslant \frac{4(|A|-1)}{3} \leqslant 5$. Therefore, $\gamma_{k_2} = 4$. In view of Lemma 2.7.3 again, we have $|R| \leqslant \sum_{i=1}^{|A|} |C_R(x_{k_i})| \leqslant 2(\frac{|R|}{4}) + (|A| - 2)|C_R(x_{k_3})|$, which gives that $\gamma_{k_3} \leqslant 2(|A| - 2) \leqslant 6$. Therefore, $\gamma_{k_3} = 4$ or 6. Here, we consider $C_R(x_{k_1}) \cap C_R(x_{k_2}) = Z(R)$. By Lemma 2.2.11, we obtain $|R : Z(R)| \leqslant 16$ and hence, $|R : Z(R)| = 16$. Since $C_R(a)$ is non-commutative, then by Lemma 2.2.15, we obtain $|R : C_R(a)| = 2$. We claim that if $\gamma_{k_i} < 8$ for some $i \in \{1, 2, \cdots, |A|\}$, then $C_R(x_{k_i}) \leqslant C_R(b)$. By Lemma 2.2.11, we have $C_R(a) \cap C_R(x_{k_i}) \neq Z(R)$. It follows that there exists some $w \in (C_R(a) \cap C_R(x_{k_i})) - Z(R)$. Since $C_R(x_{k_i})$ is commutative, then $C_R(x_{k_i}) \leqslant C_R(w)$. It is clear that $C_R(w) \neq R, C_R(x_j)$ for any $j \in \{1, 2, \cdots, 8\} - \{k_i\}$. If $C_R(w) = C_R(a)$, then $C_R(x_{k_i}) \leqslant C_R(a)$. On the other hand, if $C_R(w) = C_R(x_{k_i})$, then $C_R(w)$ is commutative and hence, $C_R(w) \leqslant C_R(a)$ and so, $C_R(x_{k_i}) \leqslant C_R(a)$. In both situations, we obtain a contradiction because $C_R(x_{k_i}) \nleqslant C_R(a)$. So, we obtain $C_R(w) = C_R(b)$ and therefore, $C_R(x_{k_i}) \leqslant C_R(b)$. If $|A| = 4$, then $R = C_R(a) \cup C_R(b) \cup C_R(x_{k_4})$. Thus, it follows from Lemma 2.2.1 that $\gamma_{k_4} = 2$, which is a contradiction. If $|A| = 5$, then $R = C_R(a) \cup C_R(b) \cup C_R(x_{k_4}) \cup C_R(x_{k_5})$. If $\gamma_{k_5} \neq 4$, then by Lemma 2.7.3, we obtain $|R| \leqslant |C_R(a)| + |C_R(x_{k_4})| + |C_R(x_{k_5})| \leqslant \frac{|R|}{2} + \frac{|R|}{4} + \frac{|R|}{8} = \frac{7|R|}{8}$, which is impossible. So, we have $\gamma_{k_4} = \gamma_{k_5} = 4$ and it follows that $R = C_R(a) \cup C_R(b)$. Thus, by Lemma 2.2.1, we obtain $|R : C_R(b)| = 1$, which is a contradiction. Consequently, $C_R(x_{k_1}) \cap C_R(x_{k_2}) \neq Z(R)$. Thus, there exists some $r \in$

$(C_R(x_{k_1}) \cap C_R(x_{k_2})) - Z(R)$ such that $C_R(x_{k_1}) \cup C_R(x_{k_2}) \subseteq C_R(r)$. It is obvious that $C_R(r) \neq R, C_R(x_i)$ for any $i \in \{1, 2, \cdots, 8\}$. Since $C_R(x_{k_1}), C_R(x_{k_2}) \nleqslant C_R(a)$, then $C_R(r) \neq C_R(a)$. So, we obtain $C_R(r) = C_R(b)$. This gives that $R = C_R(a) \cup C_R(b) \cup \left(\overset{|A|}{\underset{i=3}{\cup}} C_R(x_{k_i}) \right)$. Since $|C_R(b)| > |C_R(x_{k_1})|$, then $|R : C_R(b)| \leqslant 3$. We claim that if $\gamma_{k_i} \leqslant 6$ for some $i \in \{3, \cdots, |A|\}$, then $C_R(x_{k_i}) \leqslant C_R(b)$. Assume that $C_R(b) \cap C_R(x_{k_i}) = Z(R)$, then by Lemma 2.2.11, we obtain $|R : Z(R)| \leqslant 3\gamma_{k_i}$. If $\gamma_{k_i} \leqslant 5$, then $|R : Z(R)| \leqslant 15$; a contradiction. If $\gamma_{k_i} = 6$, then $|R : Z(R)| \leqslant 18$; a contradiction. So, $C_R(b) \cap C_R(x_{k_i}) \neq Z(R)$. Thus, there exists some $w \in (C_R(b) \cap C_R(x_{k_i})) - Z(R)$. Since $C_R(x_{k_i})$ is commutative, then $C_R(x_{k_i}) \leqslant C_R(w)$. It is clear that $C_R(w) \neq R, C_R(x_j)$ for any $j \in \{1, 2, \cdots, 8\} - \{k_i\}$. Since $C_R(x_{k_i}) \nleqslant C_R(a)$, then $C_R(w) \neq C_R(a)$. So, we conclude that $C_R(w) = C_R(b)$ or $C_R(x_{k_i})$. If $C_R(w) = C_R(b)$, then $C_R(x_{k_i}) \leqslant C_R(b)$. On the other hand, if $C_R(w) = C_R(x_{k_i})$, then $C_R(w)$ is commutative and hence, $C_R(w) \leqslant C_R(b)$ and so, $C_R(x_{k_i}) \leqslant C_R(b)$. In both situations, we obtain $C_R(x_{k_i}) \leqslant C_R(b)$, as claimed. If $|A| = 4$, then $R = C_R(a) \cup C_R(b) \cup C_R(x_{k_4})$. Therefore, by Lemma 2.2.1, we obtain $\gamma_{k_4} = 2$, a contradiction is reached. If $|A| = 5$, then $R = C_R(a) \cup C_R(b) \cup C_R(x_{k_4}) \cup C_R(x_{k_5})$. If $\gamma_{k_5} \neq 4$, then by Lemma 2.7.3, we obtain $|R| \leqslant |C_R(a)| + |C_R(x_{k_4})| + |C_R(x_{k_5})| \leqslant \frac{|R|}{2} + \frac{|R|}{4} + \frac{|R|}{6} = \frac{11|R|}{12}$, which is impossible. So, we have $\gamma_{k_4} = \gamma_{k_5} = 4$ and it follows that $R = C_R(a) \cup C_R(b)$. Thus, by Lemma 2.2.1, we obtain $|R : C_R(b)| = 1$, which is a contradiction. Consequently, $|A| \neq 4$ and 5. So, $|A| = 6$. It follows that $C_R(a)$ contains exactly two distinct $C_R(x_i)$'s, as desired. By using a manner entirely similar to that used to prove $C_R(a)$ contains exactly two distinct $C_R(x_i)$'s, we will obtain $C_R(b)$ is also contains exactly two distinct $C_R(x_i)$'s.

Since $C_R(a), C_R(b)$ contains exactly two distinct $C_R(x_i)$'s, then we have $C_R(x_{u_1}), C_R(x_{u_2}) \leqslant C_R(a)$ and $C_R(x_{v_1}), C_R(x_{v_2}) \leqslant C_R(b)$ for some $u_1, u_2, v_1, v_2 \in \{1, 2, \cdots, 8\}$ with $u_1 \neq u_2$ and $v_1 \neq v_2$. We claim that $|\{u_1, u_2\} \cap \{v_1, v_2\}| = 0$. Suppose to the contrary that $|\{u_1, u_2\} \cap \{v_1, v_2\}| \geqslant 1$. Without any loss, we assume that $u_1 = v_1$. Now, we consider for $C_R(x_{u_1} + x_{v_2})$. Since $x_i \in R, C_R(x_i)$ but $x_i \notin C_R(x_{u_1}+x_{v_2})$ for any $i \in \{u_1, v_2\}$, then $C_R(x_{u_1}+x_{v_2}) \neq R, C_R(x_{u_1}), C_R(x_{v_2})$. Since $x_{u_1} \in C_R(a), C_R(b)$ but $x_{u_1} \notin C_R(x_{u_1} + x_{v_2})$, then $C_R(x_{u_1} + x_{v_2}) \neq C_R(a), C_R(b)$. So, $C_R(x_{u_1} + x_{v_2}) = C_R(x_i)$ for some $i \in \{1, 2, \cdots, 8\} - \{u_1, v_2\}$. Since $b \in C_R(x_{u_1} + x_{v_2})$, then $b \in C_R(x_i)$. Since $C_R(x_i)$ is commutative, then we have $C_R(x_i) \leqslant C_R(b)$. This contradicts with the fact that $C_R(b)$ contains exactly two distinct $C_R(x_i)$'s. Consequently, $|\{u_1, u_2\} \cap \{v_1, v_2\}| = 0$, as claimed. Therefore, in view of Lemma 1.3.1(c), we have $\{C_R(a), C_R(b), C_R(x_{k_1}), C_R(x_{k_2}), C_R(x_{k_3}), C_R(x_{k_4})\}$ is an irredundant cover of R for four distinct $k_1, k_2, k_3, k_4 \in \{1, 2, \cdots, 8\}$ with $k_1 < k_2 < k_3 < k_4$. Next, we claim that $C_R(b) \cap C_R(x_{k_1}) = Z(R)$. Let $w \in C_R(b) \cap C_R(x_{k_1})$. Since $C_R(x_{k_1})$ is commutative, then $C_R(x_{k_1}) \leqslant C_R(w)$. Clearly, $C_R(w) \neq C_R(x_i)$ for any $i \in \{1, 2, \cdots, 8\} - \{k_1\}$. If $C_R(w) = C_R(a)$ or $C_R(b)$, then $C_R(x_{k_1}) \leqslant C_R(a)$ or $C_R(b)$, which is a contradiction. If $C_R(w) = C_R(x_{k_1})$, then $C_R(w)$ is commutative and hence, $C_R(w) \leqslant C_R(b)$ and therefore, $C_R(x_{k_1}) \leqslant C_R(b)$, which is a contradiction again. So, we obtain $C_R(w) = R$, which implies that $w \in Z(R)$. This gives that $C_R(b) \cap C_R(x_{k_1}) \leqslant Z(R)$. On the other hand, it is obvious that $Z(R) \leqslant C_R(b) \cap C_R(x_{k_1})$. Hence, $C_R(b) \cap C_R(x_{k_1}) = Z(R)$, as claimed. So, we have $|R : Z(R)| \leqslant f(6) = 36$ and consequently, $|R : Z(R)| = 16, 24, 27, 32$ or 36. If $|R : Z(R)| = 27$, then by Lemma 2.2.16 and Lemma 2.2.4, we have

$|\text{Cent}(R)| = 9$, a contradiction is reached. Therefore, $|R : Z(R)| = 16, 24, 32$ or 36. Since $C_R(b)$ is non-commutative, then by Lemma 2.2.15, we have

$$|R : C_R(b)| \begin{cases} = 2 & \text{if } |R : Z(R)| = 16, \\ \leqslant 3 & \text{if } |R : Z(R)| = 24 \text{ or } 36, \\ \leqslant 4 & \text{if } |R : Z(R)| = 32. \end{cases}$$

By Lemma 2.2.11, we have $|R : Z(R)| \leqslant \gamma_{k_1}|R : C_R(b)|$. By Lemma 2.7.3, $|R| \leqslant |C_R(b)| + \sum_{i=1}^{4} |C_R(x_{k_i})| \leqslant |C_R(b)| + 4|C_R(x_{k_1})|$, which gives that $\gamma_{k_1} \leqslant \frac{4|R|}{|R|-|C_R(b)|}$. Assume that $|R : C_R(b)| = 2$, then $\gamma_{k_1} \leqslant 8$ and thus, $|R : Z(R)| \leqslant 8(2) = 16$, which implies that $|R : Z(R)| = 16$. If $\gamma_{k_1} \neq 8$, then $|R : Z(R)| \leqslant 2(4) = 8$; a contradiction. Therefore, $\gamma_{k_1} = 8$ and so, $\gamma_{k_2} = \gamma_{k_3} = \gamma_{k_4} = 8$. We claim that $\gamma_{u_1} = \gamma_{u_2} = \gamma_{v_1} = \gamma_{v_2} = 4$. If $\gamma_i = 8$ for some $i \in \{u_1, u_2, v_1, v_2\}$, then $|R : Z(R)| \leqslant 4 + (4-1) + (4-1) + 5(2-1) = 15$, which is impossible. So, $\gamma_{u_1} = \gamma_{u_2} = \gamma_{v_1} = \gamma_{v_2} = 4$. If $a + Z(R) = b + Z(R)$, then $C_R(a) = C_R(b)$, which is impossible. Thus, $a + Z(R) \neq b + Z(R)$. Since $a + Z(R) \in (C_R(x_{u_1}) \cap C_R(x_{u_2}))/Z(R)$ and $b + Z(R) \in (C_R(x_{v_1}) \cap C_R(x_{v_2}))/Z(R)$, then we obtain $|R : Z(R)| \leqslant 4 + 3(4-1) + 4(2-1) - 2 = 15$, which is impossible. Therefore, $|R : C_R(b)| \neq 2$. If $|R : C_R(b)| = 3$, then $\gamma_{k_1} \leqslant 6$ and thus, $|R : Z(R)| \leqslant 6(3) = 18$; a contradiction. If $|R : C_R(b)| = 4$, then $\gamma_{k_1} \leqslant 5$ and thus, $|R : Z(R)| \leqslant 5(4) = 20$; a contradiction. Consequently, $t \neq 8$. □

Lemma 2.8.14. Let t be the cardinality of the maximal non-commuting set of a finite ring R. If R is an 11-centraliser finite ring, then $t \neq 9$.

Proof. Assume that $t = 9$. Let $\{x_1, x_2, \cdots, x_9\}$ be the maximal non-commuting set of R. Without loss of generality, we suppose that $|R : C_R(x_i)| = \gamma_i$, where $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_9$. From Lemma 1.3.1(c), we have $\{C_R(x_i) \mid i = 1, 2, \cdots, 9\}$ is an irredundant cover of R. By Lemma 2.2.3, we have $C_R(x_i)$ is commutative for any $i \in \{1, 2, \cdots, 9\}$ and $C_R(a)$ is non-commutative for some $a \in R - Z(R)$.

By Lemma 2.2.6, we have $\{C_R(a)\} \cup \left(\underset{i\in A}{\cup}\{C_R(x_i)\}\right)$ is an irredundant cover of R for some $A \subset \{1, 2, \cdots, 9\}$ with $|A| \leqslant 6$. Clearly, $|A| \neq 0$. If $|A| = 1$, then by Lemma 2.2.1, it follows that $\gamma_i = 1$ for some $i \in A$, which is a contradiction. We claim that if $i \in A$, then $C_R(x_i) \cap C_R(a) = Z(R)$. This claim can be proved by using a manner entirely similar to that used to prove Lemma 2.6.4. Thus, we have $|R : Z(R)| \leqslant \max\{f(3), f(4), f(5), f(6), f(7)\} = 81$. Therefore, by Theorem 2.2.23, we obtain $|R : Z(R)| = 16, 24, 27, 32, 36, 40, 48, 54, 56, 60, 64, 72, 80$ or 81. If $|R : Z(R)| = 27$, then by Lemma 2.2.16 and Lemma 2.2.4, we obtain $|\mathrm{Cent}(R)| = 10$, which is a contradiction. So, $|R : Z(R)| = 16, 24, 32, 36, 40, 48, 54, 56, 60, 64, 72, 80$ or 81. Since $C_R(a)$ is non-commutative, then by Lemma 2.2.15, we have

$$|R : C_R(a)| \begin{cases} = 2 & \text{if } |R : Z(R)| = 16, \\ \leqslant 3 & \text{if } |R : Z(R)| = 24, 36, 54 \text{ or } 81, \\ \leqslant 4 & \text{if } |R : Z(R)| = 32, \\ \leqslant 5 & \text{if } |R : Z(R)| = 40 \text{ or } 60, \\ \leqslant 6 & \text{if } |R : Z(R)| = 48, \\ \leqslant 7 & \text{if } |R : Z(R)| = 56, \\ \leqslant 8 & \text{if } |R : Z(R)| = 64, \\ \leqslant 9 & \text{if } |R : Z(R)| = 72, \\ \leqslant 10 & \text{if } |R : Z(R)| = 80. \end{cases}$$

Since $C_R(x_i) \cap C_R(a) = Z(R)$ for any $i \in A$, then by Lemma 2.2.11, we obtain $\gamma_i \geqslant 8$ for any $i \in A$. But, by Lemma 2.2.1, we have $\gamma_i \leqslant |A| \leqslant 6$ for some $i \in A$. We have reached a contradiction. Consequently, $t \neq 9$. □

Lemma 2.8.15. Let $\{x_1, x_2, \cdots, x_{10}\}$ be the maximal non-commuting set of a finite ring R. Let $|R : C_R(x_1)| \leqslant |R : C_R(x_2)| \leqslant \cdots \leqslant |R : C_R(x_{10})|$. If R is an 11-centraliser finite ring, then $|R : C_R(x_1)| = 3^{\mu-2}$, $|R : C_R(x_i)| = 9$ for any $i \in \{2, 3, \cdots, 10\}$ and $R/Z(R) \cong \mathbb{Z}_3^{\mu}$ for some $\mu \in \{3, 4\}$.

Proof. From Lemma 1.3.1(a), we have $R = \bigcup_{i=1}^{10} C_R(x_i)$. By Corollary 2.2.5, we have $C_R(x_i) \cap C_R(x_j) = Z(R)$ for any two distinct $i, j \in \{1, 2, \cdots, 10\}$. Let $|R : C_R(x_i)| = \gamma_i$ for any $i \in \{1, 2, \cdots, 10\}$. By Lemma 2.2.11, we have $|R : Z(R)| \leqslant \gamma_2^2$. In view of Lemma 2.2.1 and Lemma 2.2.14, we have $4 \leqslant \gamma_2 \leqslant 9$.

By Theorem 2.2.23, we have $|R : Z(R)| \geqslant 16$ with $|R : Z(R)|$ is not square-free, $|R : Z(R)| \neq p^2 q$ for any two distinct primes p, q, and $|R : Z(R)| \neq p^2$ for any prime p. For the sake of simplicity, we write $\overline{S} = S/Z(R)$ for any $S \leqslant R$.

If $\gamma_2 = 4$, then $|\overline{R}| \leqslant 16$, which gives that $|\overline{R}| = 16$. Since $|\overline{R}| = \sum_{i=1}^{10} |\overline{C_R(x_i)}| - 9$, then we obtain $|\overline{R}|$ is odd, which is a contradiction. If $\gamma_2 = 5$, then $|\overline{R}| \leqslant 25$, which is a contradiction. Assume that $\gamma_2 = 6$. Therefore, $|\overline{R}| \leqslant 36$. By Lemma 2.2.11, we have $|\overline{C_R(x_1)}| \leqslant 6$. If $|\overline{R}| = 24$, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_{12}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ as $\overline{R}$ is not cyclic. Thus, $|\overline{C_R(x_1)}| \leqslant 6$ and $|\overline{C_R(x_i)}| \leqslant 4$ for any $i \in \{2, 3, \cdots, 10\}$. This leads to $\overline{R}$ has at most 2 elements of order 6. Also, there does not exist any element of order 12 in $\overline{R}$. We have reached a contradiction as $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ has 14 elements of order 6 and $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ has an element of order 12. If $|\overline{R}| = 36$, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_{18}, \mathbb{Z}_3 \times \mathbb{Z}_{12}$ or $\mathbb{Z}_6 \times \mathbb{Z}_6$ as $\overline{R}$ is not cyclic. Thus, $|\overline{C_R(x_i)}| \leqslant 6$ for any $i \in \{1, 2, \cdots, 10\}$. This shows that $\overline{R}$ has at most 20 elements of order 6. Also, there does not exist any element of order 12 and order 18 in $\overline{R}$. This leads to a contradiction as $\mathbb{Z}_6 \times \mathbb{Z}_6$ has 24 elements of order 6, $\mathbb{Z}_3 \times \mathbb{Z}_{12}$ has an element of order 12 and $\mathbb{Z}_2 \times \mathbb{Z}_{18}$ has an element of order 18. If $\gamma_2 = 7$, then $|\overline{R}| \leqslant 49$, which is a contradiction. If $\gamma_2 = 8$, then $|\overline{R}| \leqslant 64$. From Lemma 2.2.11, we have $|\overline{C_R(x_1)}| \leqslant 8$. If $|\overline{R}| = 16, 32$ or 64, then since $|\overline{R}| = \sum_{i=1}^{10} |\overline{C_R(x_i)}| - 9$, it follows that $|\overline{R}|$ is odd, a contradiction is reached. If $|\overline{R}| = 24$, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_{12}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ as $\overline{R}$ is not cyclic. It follows that $|\overline{C_R(x_1)}| \leqslant 8$ and $|\overline{C_R(x_i)}| \leqslant 3$ for any $i \in \{2, 3, \cdots, 10\}$. This shows that $\overline{R}$ has at most 2 elements of order 6. Also, there does not exist any element of order 12 in $\overline{R}$. This contradicts with the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ has 14 elements of order

6 and $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ has an element of order 12. If $|\overline{R}| = 40$, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_{20}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{10}$ as $\overline{R}$ is not cyclic. Therefore, $|\overline{C_R(x_1)}| \leqslant 8$ and $|\overline{C_R(x_i)}| \leqslant 5$ for any $i \in \{2, 3, \cdots, 10\}$. This leads to there does not exist any element of order 10 and order 20 in $\overline{R}$, which contradicts the fact that $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{10}$ has an element of order 10 and $\mathbb{Z}_2 \times \mathbb{Z}_{20}$ has an element of order 20. If $|\overline{R}| = 48$, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_{24}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{12}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ or $\mathbb{Z}_4 \times \mathbb{Z}_{12}$ as $\overline{R}$ is not cyclic. Thus, $|\overline{C_R(x_1)}| \leqslant 8$ and $|\overline{C_R(x_i)}| \leqslant 6$ for any $i \in \{2, 3, \cdots, 10\}$. It follows that $\overline{R}$ has at most 20 elements of order 6. Also, there does not exist any element of order 12 and order 24 in $\overline{R}$. We have reached a contradiction as $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ has 30 elements of order 6, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{12}, \mathbb{Z}_4 \times \mathbb{Z}_{12}$ have an element of order 12 and $\mathbb{Z}_2 \times \mathbb{Z}_{24}$ has an element of order 24. If $|\overline{R}| = 56$, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_{28}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{14}$ as $\overline{R}$ is not cyclic. Hence, $|\overline{C_R(x_1)}| \leqslant 8$ and $|\overline{C_R(x_i)}| \leqslant 7$ for any $i \in \{2, 3, \cdots, 10\}$. It follows that there does not exist any element of order 14 and order 28 in $\overline{R}$, which leads to a contradiction as $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{14}$ has an element of order 14 and $\mathbb{Z}_2 \times \mathbb{Z}_{28}$ has an element of order 28. Consequently, $\gamma_2 = 9$. Therefore, by Lemma 2.2.1, we obtain $\gamma_2 = \gamma_3 = \cdots = \gamma_{10} = 9$. Since $|\overline{R}| = \sum_{i=1}^{10} |\overline{C_R(x_i)}| - 9$, then we have $\gamma_1 = \frac{|\overline{R}|}{9}$. Let ${}_m|G|$ denote the total number of elements with order m in an additive group G. If $|\overline{R}| = 27$, then $R/Z(R) \cong \mathbb{Z}_3 \times \mathbb{Z}_9$ or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ as $\overline{R}$ is not cyclic. Hence, $|\overline{C_R(x_1)}| = 9$ and $|\overline{C_R(x_i)}| = 3$ for any $i \in \{2, 3, \cdots, 10\}$. This shows that $\overline{R}$ has at most ${}_9|\mathbb{Z}_9|$ elements of order 9. Since ${}_9|\mathbb{Z}_9| < {}_9|\mathbb{Z}_3 \times \mathbb{Z}_9|$, then $\overline{R} \ncong \mathbb{Z}_3 \times \mathbb{Z}_9$. If $|\overline{R}| = 36$, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_{18}, \mathbb{Z}_3 \times \mathbb{Z}_{12}$ or $\mathbb{Z}_6 \times \mathbb{Z}_6$ as $\overline{R}$ is not cyclic. Therefore, $|\overline{C_R(x_1)}| = 9$ and $|\overline{C_R(x_i)}| = 4$ for any $i \in \{2, 3, \cdots, 10\}$. This shows that there does not exist any element of order 6, order 12 and order 18 in $\overline{R}$. This leads to a

contradiction as $\mathbb{Z}_6 \times \mathbb{Z}_6$ has an elements of order 6, $\mathbb{Z}_3 \times \mathbb{Z}_{12}$ has an element of order 12 and $\mathbb{Z}_2 \times \mathbb{Z}_{18}$ has an element of order 18. If $|\overline{R}| = 54$, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_{27}$ or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_6$ as $\overline{R}$ is not cyclic. Thus, $|\overline{C_R(x_1)}| = 9$ and $|\overline{C_R(x_i)}| = 6$ for any $i \in \{2, 3, \cdots, 10\}$. This shows that there does not exist any element of order 27. Since 2 is divide $|\overline{C_R(x_i)}|$ for any $i \in \{2, 3, \cdots, 10\}$, then $\overline{R}$ has at least 9 elements of order 2. We have reached a contradiction as $\mathbb{Z}_2 \times \mathbb{Z}_{27}$ has an element of order 27 and $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_6$ has only 1 element of order 2. If $|\overline{R}| = 72$, then $\overline{R} \cong \mathbb{Z}_2 \times \mathbb{Z}_{36}, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{18}, \mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_6, \mathbb{Z}_3 \times \mathbb{Z}_{24}$ or $\mathbb{Z}_6 \times \mathbb{Z}_{12}$ as $\overline{R}$ is not cyclic. Therefore, $|\overline{C_R(x_1)}| = 9$ and $|\overline{C_R(x_i)}| = 8$ for any $i \in \{2, 3, \cdots, 10\}$. This implies that there does not exist any element of order 6, order 12, order 18, order 24 and order 36 in $\overline{R}$. This leads to a contradiction as $\mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_6$ has an element of order 6, $\mathbb{Z}_6 \times \mathbb{Z}_{12}$ has an element of order 12, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{18}$ has an element of order 18, $\mathbb{Z}_3 \times \mathbb{Z}_{24}$ has an element of order 24 and $\mathbb{Z}_2 \times \mathbb{Z}_{36}$ has an element of order 36. If $|\overline{R}| = 81$, then $\overline{R} \cong \mathbb{Z}_3 \times \mathbb{Z}_{27}, \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_9, \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_9 \times \mathbb{Z}_9$ as $\overline{R}$ is not cyclic. Thus, $|\overline{C_R(x_i)}| = 9$ for any $i \in \{1, 2 \cdots, 10\}$. It follows that $\overline{R}$ has at most $10({}_9|\mathbb{Z}_9|) = 60$ elements of order 9. Also, there does not exist any element of order 27 in $\overline{R}$. Since $\mathbb{Z}_9 \times \mathbb{Z}_9$ has 72 elements of order 9 and $\mathbb{Z}_3 \times \mathbb{Z}_{27}$ has an element of order 27, then $\overline{R} \ncong \mathbb{Z}_9 \times \mathbb{Z}_9$ and $\mathbb{Z}_3 \times \mathbb{Z}_{27}$. Assume that $\overline{R} \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_9$. To simplify writing, we let $\overline{r} = r + Z(R)$ for any $r \in R$. Since $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_9$ has 54 elements of order 8, then there exist two distinct $l_1, l_2 \in \{1, 2, \cdots, 10\}$ such that $\overline{C_R(x_{l_1})}, \overline{C_R(x_{l_2})} \cong \mathbb{Z}_9$. Hence, there exist some $\overline{a} \in \overline{C_R(x_{l_1})} - \overline{Z(R)}, \overline{b} \in \overline{C_R(x_{l_2})} - \overline{Z(R)}$ such that $\overline{C_R(x_{l_1})} = \{\overline{0}, \overline{a}, \overline{2a}, \cdots, \overline{8a}\}$ and $\overline{C_R(x_{l_2})} = \{\overline{0}, \overline{b}, \overline{2b}, \cdots, \overline{8b}\}$. This implies that $\overline{R} = \{\overline{ma + nb} \mid m, n \in \mathbb{Z}_9\} \cong \mathbb{Z}_9 \times \mathbb{Z}_9$, which leads to a contradiction.

Consequently, $\overline{R} \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. □

Theorem 2.8.16. *Let R be an 11-centraliser finite ring. Let $X_1, X_2, \cdots, X_{10}$ be the 10 distinct proper centralisers of R with $|R : X_1| \leqslant |R : X_2| \leqslant \cdots \leqslant |R : X_{10}|$. Let t be the cardinality of the maximal non-commuting set of R. Then $|R : X_1| = 3^{\mu-2}$, $|R : X_i| = 9$ for any $i \in \{2, 3, \cdots, 10\}$, $R/Z(R) \cong \mathbb{Z}_3^{\mu}$ and $\mathrm{Prob}(R) = \frac{1}{9} + \frac{8}{3^{2\mu-2}}$ for some $\mu \in \{3, 4\}$.*

Proof. By Lemma 1.3.1(d)-(g), Lemma 2.7.1, Lemma 2.7.2, Lemma 2.7.5, Lemmas 2.8.1-2.8.5, Lemmas 2.8.7-2.8.14, we obtain $t = 10$. Thus, it follows from Lemma 2.8.15 that $|R : X_i| = 3^{\mu-2}$, $|R : X_i| = 9$ for any $i \in \{2, 3, \cdots, 10\}$ and $R/Z(R) \cong \mathbb{Z}_3^{\mu}$ for some $\mu \in \{3, 4\}$. In view of Corollary 2.2.5, it follows that for any $r_1, r_2 \in R - Z(R)$, either $C_R(r_1) = C_R(r_2)$ or $C_R(r_1) \cap C_R(r_2) = Z(R)$. Consequently, by (1.3), we obtain

$$\begin{aligned}\mathrm{Prob}(R) &= \frac{|Z(R)|}{|R|} + \frac{\sum\limits_{r \in R - Z(R)} |C_R(r)|}{|R|^2} \\ &= \frac{1}{3^{\mu}} + \frac{\left(\frac{|R|}{3^{\mu-2}} - \frac{|R|}{3^{\mu}}\right)\left(\frac{|R|}{3^{\mu-2}}\right) + 9\left(\frac{|R|}{9} - \frac{|R|}{3^{\mu}}\right)\left(\frac{|R|}{9}\right)}{|R|^2} \\ &= \frac{1}{9} + \frac{8}{3^{2\mu-2}}.\end{aligned}$$

We are done. □

In general, the converse of the above theorem is not necessarily true. For example, $R_1 = \{\left[\begin{smallmatrix} a & b \\ c & 0 \end{smallmatrix}\right] \mid a, b, c \in \mathbb{Z}_3\}$ is a 14-centraliser finite ring with $R_1/Z(R_1) \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, where the multiplication operation of R is defined as $\left[\begin{smallmatrix} a & b \\ c & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} x & y \\ z & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} ax+bz & ay \\ cx & 0 \end{smallmatrix}\right]$ for any $\left[\begin{smallmatrix} a & b \\ c & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} x & y \\ z & 0 \end{smallmatrix}\right] \in R$. Besides that, $R_2 =$

$\left\{ \left[\begin{smallmatrix} a & b & c & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix} \right] \middle| a, b, c, d \in \mathbb{Z}_3 \right\}$ is a 29-centraliser finite ring with $R_2/Z(R_2) \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. In the following, we provide an example of an 11-centraliser finite ring, which is appeared in the proof of Proposition 2.2.18.

Example 2.8.17. Let $M(a, b, c)$ be defined by $M(a, b, c) = \left[\begin{smallmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right]$ for any $a, b, c \in \mathbb{Z}_3$. The ring $R = \{M(a, b, c) \mid a, b, c \in \mathbb{Z}_3\}$ is an 11-centraliser finite ring with

$$\begin{aligned}
R &= C_R(M(0,0,0)), \\
X_1 &= C_R(M(0,0,1)) = C_R(M(0,0,2)) = C_R(M(0,1,0)) \\
&= C_R(M(0,1,1)) = C_R(M(0,1,2)) = C_R(M(0,2,0)) \\
&= C_R(M(0,2,1)) = C_R(M(0,2,2)) \\
&= \{M(0,0,0), M(0,0,1), M(0,0,2), M(0,1,0), M(0,1,1) \\
&\quad M(0,1,2), M(0,2,0), M(0,2,1), M(0,2,2)\}, \\
X_2 &= C_R(M(1,0,0)) = C_R(M(2,0,0)) \\
&= \{M(0,0,0), M(1,0,0), M(2,0,0)\}, \\
X_3 &= C_R(M(1,0,1)) = C_R(M(2,0,2)) \\
&= \{M(0,0,0), M(1,0,1), M(2,0,2)\}, \\
X_4 &= C_R(M(1,0,2)) = C_R(M(2,0,1)) \\
&= \{M(0,0,0), M(1,0,2), M(2,0,1)\}, \\
X_5 &= C_R(M(1,1,0)) = C_R(M(2,2,0)) \\
&= \{M(0,0,0), M(1,1,0), M(2,2,0)\}, \\
X_6 &= C_R(M(1,1,1)) = C_R(M(2,2,2)) \\
&= \{M(0,0,0), M(1,1,1), M(2,2,2)\},
\end{aligned}$$

$$
\begin{aligned}
X_7 &= C_R(M(1,1,2)) = C_R(M(2,2,1)) \\
&= \{M(0,0,0), M(1,1,2), M(2,2,1)\}, \\
X_8 &= C_R(M(1,2,0)) = C_R(M(2,1,0)) \\
&= \{M(0,0,0), M(1,2,0), M(2,1,0)\}, \\
X_9 &= C_R(M(1,2,1)) = C_R(M(2,1,2)) \\
&= \{M(0,0,0), M(1,2,1), M(2,1,2)\}, \\
X_{10} &= C_R(M(1,2,2)) = C_R(M(2,1,1)) \\
&= \{M(0,0,0), M(1,2,2), M(2,1,1)\}.
\end{aligned}
$$

Note that, $\{M(0,0,1), M(1,0,0), M(1,0,1), M(1,0,2), M(1,1,0), M(1,1,1), M(1,1,2), M(1,2,0), M(1,2,1), M(1,2,2)\}$ is a non-commuting set of R with cardinality 10. Also, we note that there does not exist a non-commuting set of R with cardinality 11. Thus, the cardinality of the maximal non-commuting set of R is 10. Besides that, we have $|R : X_1| = 3$, $|R : X_i| = 9$ for any $i \in \{2, 3, \cdots, 10\}$. Since $Z(R) = \{M(0,0,0)\}$, then we have $R/Z(R) \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Lastly, from (1.3), we obtain

$$
\begin{aligned}
\text{Prob}(R) &= \frac{|Z(R)|}{|R|} + \frac{\sum\limits_{r \in R - Z(R)} |C_R(r)|}{|R|^2} \\
&= \frac{1}{27} + \frac{8(9) + 18(3)}{27^2} \\
&= \frac{17}{81}.
\end{aligned}
$$

We conclude this chapter by summarizing the main theorems regarding the n-centraliser ring. Here, we list the results obtained by Dutta et al. (2018a) and Nath et al. (2022).

characterisation	Obtained by
R is a 1-centraliser ring if and only if R is commutative.	By definition
There does not exist any 2-centraliser and 3-centraliser ring.	Nath et al. (2022)
R is a 4-centraliser finite ring if and only if $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ if and only if the cardinality of the maximal non-commuting set of R is 3.	Dutta et al. (2018a), Nath et al. (2022)
R is a 5-centraliser finite ring if and only if $R/Z(R) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ if and only if the cardinality of the maximal non-commuting set of R is 4.	Dutta et al. (2018a), Nath et al. (2022)
If R is a 6-centraliser finite ring, then $\lvert R : Z(R)\rvert = 8, 12$ or 16.	Dutta et al. (2018a)
If R is a 7-centraliser finite ring, then $\lvert R : Z(R)\rvert = 12, 18, 20, 24$ or 25.	Dutta et al. (2018a)

Table 2.1: Characterisation for all n-centraliser finite rings with $n \leqslant 7$

In this chapter, we have obtained the characterisation for all n-centraliser finite rings for $n \in \{6, 7, 8, 9, 10, 11\}$ in terms of $R/Z(R)$. Also, we have determined their cardinality of the maximal non-commuting set, index of proper centralisers and $\mathrm{Prob}(R)$. In the following, we present the results that we have obtained. Let $n \in \{6,7,8,9,10,11\}$. Let R be an n-centraliser finite ring. Let $X_1, X_2, \cdots, X_{n-1}$ be the $n-1$ distinct proper centralisers of R with $|R : X_1| \leqslant |R : X_2| \leqslant \cdots \leqslant |R : X_{n-1}|$. Let t be the cardinality of the maximal non-commuting set of R.

n	characterisation			
	t	Index of proper centralisers of R	Isomorphism of $R/Z(R)$	$\mathrm{Prob}(R)$
6	5	$\|R : X_1\| = 2$, $\|R : X_i\| = 4$ for any $i \in \{2,3,4,5\}$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\frac{7}{16}$
	5	$\|R : X_i\| = 4$ for any $i \in \{1,2,3,4,5\}$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\frac{19}{64}$
7	6	$\|R : X_i\| = 5$ for any $i \in \{1,2,\cdots,6\}$	$\mathbb{Z}_5 \times \mathbb{Z}_5$	$\frac{29}{125}$
8	7	$\|R : X_i\| = 4$ for any $i \in \{1,2,\cdots,7\}$	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\frac{11}{32}$
9	8	$\|R : X_i\| = 7$ for any $i \in \{1,2,\cdots,8\}$	$\mathbb{Z}_7 \times \mathbb{Z}_7$	$\frac{55}{343}$
10	6	$\|R : X_i\| = 2$ for any $i \in \{1,2,3\}$, $\|R : X_i\| = 4$ for any $i \in \{4,5,\cdots,9\}$,	$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$\frac{11}{32}$
	6	$\|R : X_i\| = 2$ for any $i \in \{1,2,3\}$, $\|R : X_i\| = 4$ for any $i \in \{4,5,\cdots,9\}$,	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$\frac{11}{32}$

	9	$\lvert R : X_i\rvert = 4$ for any $i \in \{1,2,3\}$, $\lvert R : X_i\rvert = 8$ for any $i \in \{4,5,\cdots,9\}$	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$	$\frac{1}{4}$
	9	$\lvert R : X_1\rvert = 2$, $\lvert R : X_i\rvert = 8$ for any $i \in \{2,3,\cdots,9\}$	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$	$\frac{11}{32}$
	9	$\lvert R : X_1\rvert = 4$, $\lvert R : X_i\rvert = 8$ for any $i \in \{2,3,\cdots,9\}$	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$	$\frac{23}{128}$
	9	$\lvert R : X_i\rvert = 8$ for any $i \in \{1,2,\cdots,9\}$	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$	$\frac{71}{512}$
11	10	$\lvert R : X_1\rvert = 3$, $\lvert R : X_i\rvert = 9$ for any $i \in \{2,3,\cdots,10\}$	$\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3$	$\frac{17}{81}$
	10	$\lvert R : X_i\rvert = 9$ for any $i \in \{1,2,\cdots,10\}$	$\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3$	$\frac{89}{729}$

Table 2.2: Characterisation for all n-centraliser finite rings with $6 \leqslant n \leqslant 11$

CHAPTER 3

FINITE RINGS WITH CARDINALITY OF THE MAXIMAL NON-COMMUTING SET IS 5

3.1 Introduction

In this chapter, we study the structures for all finite rings with cardinality of the maximal non-commuting set is 5. To achieve it, we have applied some of the similar techniques which have been used in Amiri and Madadi (2016) to prove our main result. Our main result is:

Main Result. *If R is a finite ring with cardinality of the maximal non-commuting set is 5, then R satisfies one of the following structures:*

(a) $|\mathrm{Cent}(R)| = 6$*, and* $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

(b) $|\mathrm{Cent}(R)| = 16$ *and* $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

For the purpose of convenience, for any finite ring R, we denote $\overline{r} = r + Z(R)$ for any $r \in R$ and denote $\overline{S} = S/Z(R)$ for any $S \leqslant R$ in this chapter. In Section 3.2, we obtain some results for finite rings with $|\overline{R}| = 16$. Finally, in Section 3.3, we give the proof of our main result.

3.2 Preliminary Results

In this section, we establish some results that are helpful for the proof of the main result.

Lemma 3.2.1. Let R be a finite ring. Let $r \in R - Z(R)$ with $|\overline{C_R(r)}| = pm$ for some prime p and $m \in \mathbb{N}$. If $C_R(r)$ is non-commutative, then the order of $\overline{r}$ is not m.

Proof. Suppose that the order of $\overline{r}$ is m. Since $C_R(r)$ is non-commutative, then $\overline{C_R(r)}$ satisfies $\overline{Z(R)} < \overline{Z(C_R(r))} < \overline{C_R(r)}$. Since $\overline{r} \in \overline{Z(C_R(r))}$, then $|\overline{Z(C_R(r))}|$ is divisible by m. Hence, $|C_R(r) : Z(C_R(r))| = p$. This leads to $C_R(r)/Z(C_R(r))$ is cyclic. It yields that $C_R(r)$ is commutative, which leads to a contradiction. $\square$

Lemma 3.2.2. Let R be a finite ring. Let $r_1 \in R - Z(R)$ with $|\overline{C_R(r_1)}| = p_1p_2p_3$ for some primes p_1, p_2, p_3. If $C_R(r_1)$ is non-commutative, then $C_R(r_1) \neq C_R(r_2)$ for any $r_2 \in R - Z(R)$ with $\overline{r_1} \notin < \overline{r_2} >$.

Proof. Assume that $C_R(r_1) = C_R(r_2)$ for some $r_2 \in R - Z(R)$ with $\overline{r_1} \notin < \overline{r_2} >$. Since $C_R(r_1)$ is non-commutative, then $\overline{C_R(r_1)}$ satisfies $\overline{Z(R)} < \overline{Z(C_R(r_1))} < \overline{C_R(r_1)}$. Suppose that $|\overline{Z(C_R(r_1))}| = p_ip_j$ for two distinct $i, j \in \{1, 2, 3\}$, then $|C_R(r_1) : Z(C_R(r_1))| = p_k$, where $k \in \{1, 2, 3\} - \{i, j\}$. This yields that $C_R(r_1)/Z(C_R(r_1))$ is cyclic. Consequently, $C_R(r_1)$ is commutative, which is a contradiction. Thus, we have $|\overline{Z(C_R(r_1))}| = p_i$ for some $i \in \{1, 2, 3\}$. This gives that $\overline{Z(C_R(r_1))}$ is cyclic with order p_i. So, we obtain $\overline{r_1} \in \overline{Z(C_R(r_1))} = \overline{Z(C_R(r_2))} =< \overline{r_2} >$, which leads to a contradiction. $\square$

In the following, we provide some results for finite rings with $|\overline{R}| = 16$.

Lemma 3.2.3. Let R be a finite ring with $|\mathrm{Cent}(R)| > 6$ and $|\overline{R}| = 16$. Let $\{x_1, x_2, x_3, x_4, x_5\}$ be the maximal non-commuting set of R. If $|R : C_R(x_i)| \neq 2$ for any $i \in \{1, 2, 3, 4, 5\}$, then $|\mathrm{Cent}(R)| = 16$.

Proof. From Lemma 1.3.1(a), we have $\overline{R} = \bigcup_{i=1}^{5} \overline{C_R(x_i)}$. Given that $|R : C_R(x_i)| \neq 2$ for any $i \in \{1, 2, 3, 4, 5\}$ and $|\overline{R}| = 16$, thus it can be easily seen that $|\overline{C_R(x_i)}| = 4$ for any $i \in \{1, 2, 3, 4, 5\}$ and $\overline{C_R(x_i)} \cap \overline{C_R(x_j)} = \overline{Z(R)}$ for any two distinct $i, j \in \{1, 2, 3, 4, 5\}$. By Lemma 2.2.15, $C_R(x_i)$ is commutative for any $i \in \{1, 2, 3, 4, 5\}$. Thus, it can be easily checked that for any $r \in R - Z(R)$, $C_R(r)$ is non-commutative if and only if $|\overline{C_R(r)}| = 8$. Since $|\mathrm{Cent}(R)| > 6$, then there exists some $a_1 \in R - Z(R)$ such that $C_R(a_1)$ is non-commutative with $|\overline{C_R(a_1)}| = 8$. Without loss of generality, we assume that $a_1 \in C_R(x_1)$. Therefore, we have

$$\overline{C_R(x_1)} = \{\overline{0}, \overline{x_1}, \overline{a_1}, \overline{x_1 + a_1}\}$$

and

$$\overline{C_R(a_1)} = \{\overline{0}, \overline{x_1}, \overline{a_1}, \overline{x_1 + a_1}, \overline{a_2}, \overline{a_3}, \overline{a_4}, \overline{a_5}\}$$

for some $a_2, a_3, a_4, a_5 \in R - Z(R)$. Now, we claim that $|\overline{C_R(x_i)} \cap A| = 1$ for any $i \in \{2, 3, 4, 5\}$, where $A = \{\overline{a_2}, \overline{a_3}, \overline{a_4}, \overline{a_5}\}$. Suppose that $|\overline{C_R(x_i)} \cap A| \geqslant 2$ for some $i \in \{2, 3, 4, 5\}$. So, we have $a_{k_1}, a_{k_2} \in C_R(x_i)$ for two distinct $k_1, k_2 \in \{2, 3, 4, 5\}$. If $|C_R(a_{k_1}) \cap C_R(a_{k_2})| = 4|Z(R)|$, then $a_1 \in C_R(x_1) \cap C_R(a_{k_1}) \cap C_R(a_{k_2}) = C_R(x_1) \cap C_R(x_i) = Z(R)$, which is a contradiction. If

$|C_R(a_{k_1}) \cap C_R(a_{k_2})| = 8|Z(R)|$, then $C_R(a_{k_1}) = C_R(a_{k_2})$ with $|\overline{C_R(a_{k_1})}| = 8$, which contradicts with Lemma 3.2.1 and Lemma 3.2.2. So, our claim is true. Without loss of generality, we let $a_i \in C_R(x_i)$ for any $i \in \{2, 3, 4, 5\}$. Hence, we have

$$\overline{C_R(x_i)} = \{\overline{0}, \overline{x_i}, \overline{a_i}, \overline{x_i + a_i}\}$$

for any $i \in \{2, 3, 4, 5\}$. For any $i \in \{2, 3, 4, 5\}$, since $a_1 \in C_R(a_i)$ but $a_1 \notin C_R(x_i)$, then $C_R(x_i) < C_R(a_i)$. So, $|\overline{C_R(a_i)}| = 8$ for any $i \in \{2, 3, 4, 5\}$. Next, we claim that $|\overline{C_R(x_j + a_j)}| = 8$ for some $j \in \{1, 2, 3, 4, 5\}$. Assume that $|\overline{C_R(x_i + a_i)}| = 4$ for any $i \in \{1, 2, 3, 4, 5\}$. Thus, $C_R(x_i + a_i) = C_R(x_i)$ for any $i \in \{1, 2, 3, 4, 5\}$. This implies that

$$\overline{C_R(a_2)} = \{\overline{0}, \overline{x_2}, \overline{a_2}, \overline{x_2 + a_2}, \overline{a_1}, \overline{a_3}, \overline{a_4}, \overline{a_5}\}.$$

This shows that $|\overline{C_R(a_1)} \cap \overline{C_R(a_2)}| = 6$, which contradicts the fact that $|\overline{C_R(a_1)} \cap \overline{C_R(a_2)}|$ is divide $|\overline{R}|$. Consequently, we have $|\overline{C_R(x_j + a_j)}| = 8$ for some $j \in \{1, 2, 3, 4, 5\}$, as claimed. We claim that $|\overline{C_R(a_j)} \cap \{\overline{a_i}, \overline{x_i + a_i}\}| = 1$ for any $i \in \{1, 2, 3, 4, 5\} - \{j\}$. If $|\overline{C_R(a_j)} \cap \{\overline{a_i}, \overline{x_i + a_i}\}| = 2$ for some $i \in \{1, 2, 3, 4, 5\} - \{j\}$, then $\overline{a_j} \in \overline{C_R(x_i)}$, which is a contradiction. We next claim that $|\overline{C_R(x_j + a_j)} \cap \{\overline{a_i}, \overline{x_i + a_i}\}| = 1$ for any $i \in \{1, 2, 3, 4, 5\} - \{j\}$. If $|\overline{C_R(x_j + a_j)} \cap \{\overline{a_i}, \overline{x_i + a_i}\}| = 2$ for some $i \in \{1, 2, 3, 4, 5\} - \{j\}$, then $\overline{x_j + a_j} \in \overline{C_R(x_i)}$, which is a contradiction. By Lemma 3.2.1 and Lemma 3.2.2, we obtain $C_R(a_j) \neq C_R(x_j + a_j)$. It follows that $\overline{C_R(a_j)} \cap \overline{C_R(x_j + a_j)} = \overline{C_R(x_j)}$. Thus, we have $\overline{x_i + a_i} \in \overline{C_R(a_j)}$ or $\overline{x_i + a_i} \in \overline{C_R(x_j + a_j)}$ but not both. This implies that $|\overline{C_R(x_i+a_i)}| \geqslant 5$ for any $i \in \{1,2,3,4,5\} - \{j\}$. Hence, we have $|\overline{C_R(x_i+a_i)}| = 8$ for any $i \in \{1,2,3,4,5\} - \{j\}$. Consequently, we obtain $|\mathrm{Cent}(R)| = 1+5+10 = 16$ by Lemma 3.2.1 and Lemma 3.2.2, as desired. □

Lemma 3.2.4. Let R be a finite ring with $|\mathrm{Cent}(R)| > 6$ and $|\overline{R}| = 16$. Let $\{x_1, x_2, x_3, x_4, x_5\}$ be the maximal non-commuting set of R. If $|R : C_R(x_k)| = 2$ for some $k \in \{1,2,3,4,5\}$, then $|R : C_R(x_l)| = 2$ for some $l \in \{1,2,3,4,5\} - \{k\}$.

Proof. Suppose that $|R : C_R(x_l)| \geqslant 4$ for any $l \in \{1,2,3,4,5\} - \{k\}$. Without loss of generality, we assume that $k = 1$. Thus, $\overline{C_R(x_1)}$ can be written as

$$\overline{C_R(x_1)} = \{\overline{0}, \overline{x_1}, \overline{a}, \overline{b}, \overline{a+b}, \overline{x_1+a}, \overline{x_1+b}, \overline{x_1+a+b}\}$$

for some $a, b \in R - Z(R)$. By Lemma 1.3.1(a), we have $\overline{R} = \bigcup_{i=1}^{5} \overline{C_R(x_i)}$. Hence, by Lemma 2.7.3, we obtain $|\overline{C_R(x_i)}| = 4$ for any $i \in \{2,3,4,5\}$. Assume that $ab = ba$, then $C_R(x_1)$ is commutative. Thus, it follows from Lemma 2.2.12 that $|\overline{R}| \leqslant 2(4) = 8$, which is a contradiction. So, $ab \neq ba$. By Lemma 2.2.11, we obtain $|\overline{C_R(x_1)} \cap \overline{C_R(x_i)}| = 2$ for any $i \in \{2,3,4,5\}$. Then, there exist four elements $w_2, w_3, w_4, w_5 \in \{a, b, a+b, x_1+a, x_1+b, x_1+a+b\}$ such that $\overline{w_i} \in \overline{C_R(x_i)}$ for any $i \in \{2,3,4,5\}$. Let $A = \{a, b, a+b, x_1+a, x_1+b, x_1+a+b\} - \{w_2, w_3, w_4, w_5\}$. Now, we claim that $A = \{u_3, x_1+u_3\}$ for some $u_3 \in \{a, b, a+b\}$. Suppose to the contrary that $A \neq \{w, x_1+w\}$ for any $w \in \{a, b, a+b\}$. Hence, we note that there exist two distinct $u, v \in \{a, b, a+b\}$ such that $u, v \in A$, $u, x_1+v \in A$ or $x_1+u, x_1+v \in A$. Since all the elements in the set A are non-commute with x_2, x_3, x_4, x_5, then we have $\{\alpha, \beta, x_2, x_3, x_4, x_5\}$

is a non-commuting set of R with cardinality 6, where $\alpha \in \{u, x_1 + u\}$ and $\beta \in \{v, x_1 + v\}$. This contradicts with the fact that the cardinality of the maximal non-commuting set of R is 5. Therefore, $A = \{u_3, x_1 + u_3\}$ for some $u_3 \in \{a, b, a + b\}$, as claimed. Without loss of generality, we have

$$\overline{C_R(x_2)} = \{\overline{0}, \overline{x_2}, \overline{u_1}, \overline{x_2 + u_1}\},$$
$$\overline{C_R(x_3)} = \{\overline{0}, \overline{x_3}, \overline{x_1 + u_1}, \overline{x_1 + x_3 + u_1}\},$$
$$\overline{C_R(x_4)} = \{\overline{0}, \overline{x_4}, \overline{u_2}, \overline{x_4 + u_2}\},$$
$$\overline{C_R(x_5)} = \{\overline{0}, \overline{x_5}, \overline{x_1 + u_2}, \overline{x_1 + x_5 + u_2}\},$$

where $u_1, u_2 \in \{a, b, a + b\} - \{u_3\}$ with $u_1 \neq u_2$. By Lemma 2.2.15, $C_R(x_i)$ is commutative for any $i \in \{2, 3, 4, 5\}$. Consequently, we obtain

$$\overline{C_R(u_1)} = \{\overline{0}, \overline{x_1}, \overline{u_1}, \overline{x_1 + u_1}, \overline{x_2}, \overline{x_2 + u_1}, \overline{x_4 + u_2}, \overline{x_1 + x_5 + u_2}\}$$

and

$$\overline{C_R(x_1 + u_1)} = \{\overline{0}, \overline{x_1}, \overline{u_1}, \overline{x_1 + u_1}, \overline{x_3}, \overline{x_1 + x_3 + u_1}, \overline{x_4 + u_2}, \overline{x_1 + x_5 + u_2}\}.$$

This shows that $|\overline{C_R(u_1)} \cap \overline{C_R(x_1 + u_1)}| = 6$. We have reached a contradiction as $|\overline{C_R(u_1)} \cap \overline{C_R(x_1 + u_1)}|$ is divide $|\overline{R}|$. □

Lemma 3.2.5. Let R be a finite ring with $|\mathrm{Cent}(R)| > 6$ and $|\overline{R}| = 16$. Let $\{x_1, x_2, x_3, x_4, x_5\}$ be the maximal non-commuting set of R. If $|R : C_R(x_k)| = |R : C_R(x_l)| = 2$ for two distinct $k, l \in \{1, 2, 3, 4, 5\}$, then $|\mathrm{Cent}(R)| = 16$.

Proof. Without loss of generality, we assume that $k = 1$ and $l = 2$. By Lemma

2.2.11, we obtain $|\overline{C_R(x_1)} \cap \overline{C_R(x_2)}| = 4$. Therefore, $\overline{C_R(x_1)} \cap \overline{C_R(x_2)}$ can be written as $\overline{C_R(x_1)} \cap \overline{C_R(x_2)} = \{\overline{0}, \overline{a}, \overline{b}, \overline{a+b}\}$ for some $a, b \in R - Z(R)$. So, we have

$$\overline{C_R(x_1)} = \{\overline{0}, \overline{x_1}, \overline{a}, \overline{b}, \overline{a+b}, \overline{x_1+a}, \overline{x_1+b}, \overline{x_1+a+b}\},$$

$$\overline{C_R(x_2)} = \{\overline{0}, \overline{x_2}, \overline{a}, \overline{b}, \overline{a+b}, \overline{x_2+a}, \overline{x_2+b}, \overline{x_2+a+b}\}.$$

Suppose that $ab = ba$, then $C_R(x_1)$ and $C_R(x_2)$ are commutative. Therefore, by Lemma 2.2.12, we obtain $|\overline{R}| \leqslant 2(2) = 4$, which is a contradiction. So, $ab \neq ba$. Thus, we have

$$\begin{aligned}
\overline{C_R(a)} &= \{\overline{0}, \overline{a}, \overline{x_1}, \overline{x_2}, \overline{x_1+x_2}, \overline{x_1+a}, \overline{x_2+a}, \overline{x_1+x_2+a}\},\\
\overline{C_R(b)} &= \{\overline{0}, \overline{b}, \overline{x_1}, \overline{x_2}, \overline{x_1+x_2}, \overline{x_1+b}, \overline{x_2+b}, \overline{x_1+x_2+b}\},\\
\overline{C_R(a+b)} &= \{\overline{0}, \overline{a+b}, \overline{x_1}, \overline{x_2}, \overline{x_1+x_2}, \overline{x_1+a+b}, \overline{x_2+a+b},\\
&\quad \overline{x_1+x_2+a+b}\},\\
\overline{C_R(x_1+x_2)} &= \{\overline{0}, \overline{x_1+x_2}, \overline{a}, \overline{b}, \overline{a+b}, \overline{x_1+x_2+a}, \overline{x_1+x_2+b},\\
&\quad \overline{x_1+x_2+a+b}\}.
\end{aligned}$$

Apart from this, we have

$$\overline{C_R(u+v)} = \{\overline{0}, \overline{u}, \overline{v}, \overline{u+v}\}$$

for any $u \in \{x_1, x_2, x_1+x_2\}$ and $v \in \{a, b, a+b\}$. Consequently, we have $|\mathrm{Cent}(R)| = 1 + 2 + 4 + 9 = 16$. This completes the proof. $\square$

3.3 Proof of Main Result

In this section, we give the proof of our main result.

Theorem 3.3.1. *If R is a finite ring with cardinality of the maximal non-commuting set is 5, then R satisfies one of the following structures:*

(a) $|\mathrm{Cent}(R)| = 6$, *and* $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

(b) $|\mathrm{Cent}(R)| = 16$ *and* $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Let $\{x_1, x_2, x_3, x_4, x_5\}$ be the maximal non-commuting set of R. By Lemma 1.3.1(d), we have $|\mathrm{Cent}(R)| \geqslant 6$. If $|\mathrm{Cent}(R)| = 6$, then by Theorem 2.3.1, we get $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Next, we consider for $|\mathrm{Cent}(R)| > 6$. By Lemma 1.3.1(b), (c), it follows that $\{C_R(x_i) \mid i = 1, 2, 3, 4, 5\}$ is an irredundant cover of R with intersection $Z(R)$. Therefore, we obtain $|R : Z(R)| \leqslant f(5) = 16$. Thus, by Theorem 2.2.23, we have $|R : Z(R)| = 8$ or 16. If $|R : Z(R)| = 8$, then by Lemma 2.2.16 and Lemma 2.2.4, we get $|\mathrm{Cent}(R)| = 6$, which is a contradiction. So, we have $|R : Z(R)| = 16$. In view of Lemma 3.2.3, Lemma 3.2.4 and Lemma 3.2.5, we obtain $|\mathrm{Cent}(R)| = 16$. Suppose to the contrary that $R/Z(R) \ncong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Then, there exists some $a \in R - Z(R)$ such that the order of $\overline{a}$ is 4 or 8. Since $\gcd(3, \text{order of } \overline{a}) = 1$, then $\overline{C_R(a)} = \overline{C_R(3a)}$. Since $\overline{a} \neq \overline{3a}$, then we obtain $|\mathrm{Cent}(R)| < |R : Z(R)|$, which leads to a contradiction. So, we can conclude that $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. We are done. □

CHAPTER 4

(m, n)-CENTRALISER FINITE RINGS

4.1 Introduction

Inspired by the works of Ashrafi et al. (2020), we intend to determine the structures of (m, n)-centraliser finite rings. In Section 4.2, we first give some requirements which will be used in the construction of our main results. We also compute $|m - \textbf{\textit{Cent}}(R)|$ for certain classes of finite rings. Finally, in Section 4.3, we describe the characterisation for some (m, n)-centraliser finite rings for $n \leqslant 10$.

4.2 Some Requirements and Some Computations of $|m - \textbf{\textit{Cent}}(R)|$

In this section, we establish some useful lemmas that will be employed in the construction of our main results. Besides that, we compute $|m - \textbf{\textit{Cent}}(R)|$ by imposing some assumptions on the finite ring R.

Lemma 4.2.1. Let $m \in \mathbb{N}$ with $m \geqslant 2$ and let R be a non-commutative ring. Then the following statements hold.

(a) If $|Z(R)| = m - 1$, then $|\mathrm{Cent}(R)| \leqslant |m - \textbf{\textit{Cent}}(R)|$.

(b) If $|Z(R)| \geqslant m$, then $|\mathrm{Cent}(R)| \leqslant |m - \textbf{\textit{Cent}}(R)| - 1$.

Proof. We claim that $\mathrm{Cent}(R) - \{R\} \subset (m - \textbf{\textit{Cent}}(R)) - \{R\}$. Let $C_R(r_1) \in$

$\text{Cent}(R) - \{R\}$, where $r_1 \in R - Z(R)$. Hence, we have $C_R(r_1) = \bigcap_{i=1}^{m} C_R(r_i) \in (m - \textbf{\textit{Cent}}(R)) - \{R\}$ for $m-1$ distinct $r_2, r_3, \cdots, r_m \in Z(R)$. This shows that $\text{Cent}(R) - \{R\} \subseteq (m - \textbf{\textit{Cent}}(R)) - \{R\}$. Suppose that $\text{Cent}(R) - \{R\} = (m - \textbf{\textit{Cent}}(R)) - \{R\}$. Since R is non-commutative, then there exist two distinct $r_1, r_2 \in R - Z(R)$ such that $r_1 r_2 \neq r_2 r_1$. Thus, there exists some $r \in R - Z(R)$ such that

$$C_R(r) = \begin{cases} C_R(r_1) \cap C_R(r_2) & \text{if } m = 2, \\ \bigcap_{i=1}^{m} C_R(r_i) \text{ for } m-2 \text{ distinct } r_3, r_4, \cdots, r_m \in Z(R) & \text{if } m \geqslant 3. \end{cases}$$

This implies that $r \in \bigcap_{i=1}^{m} C_R(r_i)$, which gives that $r_1, r_2 \in C_R(r)$. This contradicts with the fact that $r_1, r_2 \notin \bigcap_{i=1}^{m} C_R(r_i)$. Consequently, $\text{Cent}(R) - \{R\} \subset (m - \textbf{\textit{Cent}}(R)) - \{R\}$, as claimed.

(a) Since $|Z(R)| = m - 1$, we have $R \notin m - \textbf{\textit{Cent}}(R)$. It follows that $\text{Cent}(R) - \{R\} \subset m - \textbf{\textit{Cent}}(R)$. Therefore, $|\text{Cent}(R)| \leqslant |m - \textbf{\textit{Cent}}(R)|$.

(b) Since $|Z(R)| \geqslant m$, we have $R \in m - \textbf{\textit{Cent}}(R)$. It follows that $\text{Cent}(R) \subset m - \textbf{\textit{Cent}}(R)$. Therefore, $|\text{Cent}(R)| \leqslant |m - \textbf{\textit{Cent}}(R)| - 1$. □

Lemma 4.2.2. If R is an n-centraliser finite ring with $4 \leqslant n \leqslant 9$, then every proper centraliser of R is commutative.

Proof. By Lemma 1.3.1(f), (g), Theorem 2.3.1, Theorem 2.4.1, Theorem 2.5.1 and Theorem 2.6.5, it follows that the cardinality of the maximal non-commuting set of R is $n - 1$. So, we obtain every proper centraliser of R is commutative by

Lemma 2.2.4. $\square$

Theorem 4.2.3. *Let $m \in \mathbb{N}$ with $m \geqslant 2$ and let R be a finite non-commutative ring with every proper centraliser of R is commutative. Then the following statements hold.*

(a) If $|Z(R)| = m - 1$, then $|m - \textbf{\textit{Cent}}(R)| = |\mathrm{Cent}(R)|$.

(b) If $|Z(R)| \geqslant m$, then $|m - \textbf{\textit{Cent}}(R)| = |\mathrm{Cent}(R)| + 1$.

Proof. Here, we let $\bigcap\limits_{i=1}^{m} C_R(r_i) \in m - \textbf{\textit{Cent}}(R)$, where all r_i's are in R and distinct from each other.

(a) Since $|Z(R)| = m-1$, then $r_i \in R - Z(R)$ for some $i \in \{1, 2, \cdots, m\}$. If $r_j \in Z(R)$ for any $j \in \{1, 2, \cdots, m\} - \{i\}$, then $\bigcap\limits_{i=1}^{m} C_R(r_i) = C_R(r)$ for some $r \in R - Z(R)$. If $r_j \in R - Z(R)$ for some $j \in \{1, 2, \cdots, m\} - \{i\}$, then by Lemma 1.3.1(h), we obtain $\bigcap\limits_{i=1}^{m} C_R(r_i) = Z(R)$ or $C_R(r)$ for some $r \in R - Z(R)$. Therefore, $m - \textbf{\textit{Cent}}(R) = \{Z(R)\} \cup (\mathrm{Cent}(R) - \{R\})$. Consequently, $|m - \textbf{\textit{Cent}}(R)| = |\mathrm{Cent}(R)|$.

(b) Given that $|Z(R)| \geqslant m$. If $r_i \in Z(R)$ for any $i \in \{1, 2, \cdots, m\}$, then $\bigcap\limits_{i=1}^{m} C_R(r_i) = R$. If $r_i \in R - Z(R)$ for some $i \in \{1, 2, \cdots, m\}$, then by using similar arguments as in part (a), we obtain $\bigcap\limits_{i=1}^{m} C_R(r_i) = Z(R)$ or $C_R(r)$ for some $r \in R - Z(R)$. Therefore, $m - \textbf{\textit{Cent}}(R) = \{Z(R)\} \cup \mathrm{Cent}(R)$. Consequently, $|m - \textbf{\textit{Cent}}(R)| = |\mathrm{Cent}(R)| + 1$. $\square$

Theorem 4.2.4. *Let $m \in \mathbb{N}$ with $m \geqslant 2$ and let R be a finite non-commutative ring with $|Z(R)| \geqslant m-1$. Let t be the cardinality of the maximal non-commuting* *set of R. Then every proper centraliser of R is commutative if and only if*

$$|m-\textit{Cent}(R)| = \begin{cases} t+1 & \textit{if } |Z(R)| = m-1, \\ t+2 & \textit{if } |Z(R)| \geqslant m. \end{cases}$$

Proof. The necessity part follows readily by Lemma 2.2.4 and Theorem 4.2.3. Next, we consider the sufficiency part. In view of Lemma 4.2.1, we obtain $|\mathrm{Cent}(R)| \leqslant t+1$. So, we have $|\mathrm{Cent}(R)| = t+1$ by Lemma 1.3.1(e). Consequently, by Lemma 2.2.4, it follows that every proper centraliser of R is commutative. □

Theorem 4.2.5. *Let $m, n \in \mathbb{N}$ with $m, n \geqslant 2$ and let R be a finite ring with $|R : Z(R)| = p^n$ for some prime p. Let $|C_R(r)| = p|Z(R)|$ or $p^2|Z(R)|$ for any $r \in R-Z(R)$. Let s_1 and s_2 be the total numbers of distinct proper centralisers in R with order $p|Z(R)|$ and $p^2|Z(R)|$, respectively. Then the following statements hold.*

(a) $s_1 + s_2(p+1) = \frac{p^n-1}{p-1}$.

(b) If $|Z(R)| = m-1$, then $|m-\textit{Cent}(R)| = s_1+s_2+1$.

(c) If $|Z(R)| \geqslant m$, then $|m-\textit{Cent}(R)| = s_1+s_2+2$.

Proof. In view of Lemma 2.2.15, we obtain every proper centraliser of R is commutative.

(a) By Lemma 1.3.1(h), it follows that for any $r_1, r_2 \in R-Z(R)$, either $C_R(r_1) = C_R(r_2)$ or $C_R(r_1) \cap C_R(r_2) = Z(R)$. So, we have $|R| - |Z(R)| =$

$s_1(|C_R(a_1)| - |Z(R)|) + s_2(|C_R(a_2)| - |Z(R)|)$, where $a_1, a_2 \in R - Z(R)$ with $|C_R(a_1)| = p|Z(R)|$ and $|C_R(a_2)| = p^2|Z(R)|$. This gives that $s_1 + s_2(p+1) = \frac{p^n-1}{p-1}$.

(b) It follows from Theorem 4.2.3(a) that $|m - \textit{Cent}(R)| = s_1 + s_2 + 1$.

(c) It follows from Theorem 4.2.3(b) that $|m - \textit{Cent}(R)| = s_1 + s_2 + 2$. $\square$

To simplify the writing in Theorem 4.2.6 and Theorem 4.2.7, we denote $\delta(R) = \begin{cases} 1 & \text{if } |Z(R)| = m-1 \\ 0 & \text{if } |Z(R)| \geqslant m \end{cases}$, for any ring R and $m \in \mathbb{N}$ with $m \geqslant 2$.

Theorem 4.2.6. *Let $m \in \mathbb{N}$ with $m \geqslant 2$. Let R_1, R_2 be rings with $|Z(R_1)|$, $|Z(R_2)| \geqslant m-1$. Then*

$$|m - \textit{Cent}(R_1 \times R_2)|$$
$$= \begin{cases} |m - \textit{Cent}(R_1)||m - \textit{Cent}(R_2)| \\ + \delta(R_2)|m - \textit{Cent}(R_1)| + \delta(R_1)|m - \textit{Cent}(R_2)| & \textit{if } m = 2, \\ |m - \textit{Cent}(R_1)||m - \textit{Cent}(R_2)| \\ + \delta(R_2)|m - \textit{Cent}(R_1)| + \delta(R_1)|m - \textit{Cent}(R_2)| & \textit{if } m \geqslant 3. \\ + \delta(R_1)\delta(R_2) \end{cases}$$

Proof. It can be easily seen that

$$\bigcap_{i=1}^{m} C_{R_1 \times R_2}((a_i, b_i))$$
$$= \bigcap_{i=1}^{m} (C_{R_1}(a_i) \times C_{R_2}(b_i))$$

$$= (\bigcap_{i=1}^{m} C_{R_1}(a_i)) \times (\bigcap_{i=1}^{m} C_{R_2}(b_i)) \quad (4.1)$$

for any $(a_1, b_1), (a_2, b_2), \cdots, (a_m, b_m) \in R_1 \times R_2$. Let $(a_1, b_1), (a_2, b_2), \cdots,$ (a_m, b_m) be m distinct elements in $R_1 \times R_2$. Before we continue the proof, without loss of generality, we consider the following two situations.

Situation 1: $|Z(R_1)| \geqslant m$. Since $|Z(R_1)| \geqslant m$, $\bigcap_{i=1}^{m} C_{R_1}(a_i)$ can be written as $\bigcap_{i=1}^{m} C_{R_1}(a_i) = \bigcap_{i=1}^{m} C_{R_1}(x_i)$ for m distinct $x_1, x_2, \cdots, x_m \in R_1$.

Situation 2: $|Z(R_1)| = m - 1$. If not all a_i's are in $Z(R_1)$, then since $|Z(R_1)| = m - 1$, $\bigcap_{i=1}^{m} C_{R_1}(a_i)$ can be written as $\bigcap_{i=1}^{m} C_{R_1}(a_i) = \bigcap_{i=1}^{m} C_{R_1}(x_i)$ for m distinct $x_1, x_2, \cdots, x_m \in R_1$. On the other hand, if all a_i's are in $Z(R_1)$, then $\bigcap_{i=1}^{m} C_{R_1}(a_i) = R_1$.

Now, we break the proof into the following four cases.

Case 1: $|Z(R_1)|, |Z(R_2)| \geqslant m$. By using (4.1) and similar arguments as in Situation 1, we have

$$\bigcap_{i=1}^{m} C_{R_1 \times R_2}((a_i, b_i)) = (\bigcap_{i=1}^{m} C_{R_1}(x_i)) \times (\bigcap_{i=1}^{m} C_{R_2}(y_i))$$

for m distinct $x_1, x_2, \cdots, x_m \in R_1$ and m distinct $y_1, y_2, \cdots, y_m \in R_2$. Consequently, we obtain

$$|m - \textit{Cent}(R_1 \times R_2)| = |m - \textit{Cent}(R_1)||m - \textit{Cent}(R_2)|.$$

Case 2: $|Z(R_1)| \geqslant m, |Z(R_2)| = m-1$. By using (4.1) and similar arguments as in Situation 1 and 2, we have

$$\bigcap_{i=1}^{m} C_{R_1\times R_2}((a_i,b_i))$$
$$= \begin{cases} (\bigcap_{i=1}^{m} C_{R_1}(x_i)) \times (\bigcap_{i=1}^{m} C_{R_2}(y_i)) & \text{if not all } b_i\text{'s are in } Z(R_2), \\ (\bigcap_{i=1}^{m} C_{R_1}(x_i)) \times R_2 & \text{if all } b_i\text{'s are in } Z(R_2) \end{cases}$$

for m distinct $x_1, x_2, \cdots, x_m \in R_1$ and m distinct $y_1, y_2, \cdots, y_m \in R_2$. Consequently, we obtain

$$|m-\textit{Cent}(R_1\times R_2)| = |m-\textit{Cent}(R_1)||m-\textit{Cent}(R_2)| + |m-\textit{Cent}(R_1)|.$$

Case 3: $|Z(R_1)| = m-1, |Z(R_2)| \geqslant m$. By using similar arguments as in Case 2, we obtain

$$|m-\textit{Cent}(R_1\times R_2)| = |m-\textit{Cent}(R_1)||m-\textit{Cent}(R_2)| + |m-\textit{Cent}(R_2)|.$$

Case 4: $|Z(R_1)| = |Z(R_2)| = m-1$. By using (4.1) and similar arguments as in Situation 2, we have

$$\bigcap_{i=1}^{m} C_{R_1\times R_2}((a_i,b_i))$$

$$
= \begin{cases} (\bigcap\limits_{i=1}^{m} C_{R_1}(x_i)) \times (\bigcap\limits_{i=1}^{m} C_{R_2}(y_i)) & \text{if not all } a_i\text{'s are in } Z(R_1) \text{ and not all } b_i\text{'s are in } Z(R_2), \\ R_1 \times (\bigcap\limits_{i=1}^{m} C_{R_2}(y_i)) & \text{if all } a_i\text{'s are in } Z(R_1) \text{ and not all } b_i\text{'s are in } Z(R_2), \\ (\bigcap\limits_{i=1}^{m} C_{R_1}(x_i)) \times R_2 & \text{if not all } a_i\text{'s are in } Z(R_1) \text{ and all } b_i\text{'s are in } Z(R_2), \\ R_1 \times R_2 & \text{if all } a_i\text{'s are in } Z(R_1), \text{ all } b_i\text{'s are in } Z(R_2) \text{ and } m \geqslant 3 \end{cases}
$$

for m distinct $x_1, x_2, \cdots, x_m \in R_1$ and m distinct $y_1, y_2, \cdots, y_m \in R_2$. Consequently, we obtain

$$
\begin{aligned}
&|m - \textbf{\textit{Cent}}(R_1 \times R_2)| \\
&= \begin{cases} |m - \textbf{\textit{Cent}}(R_1)||m - \textbf{\textit{Cent}}(R_2)| \\ \quad + |m - \textbf{\textit{Cent}}(R_1)| + |m - \textbf{\textit{Cent}}(R_2)| & \text{if } m = 2, \\ |m - \textbf{\textit{Cent}}(R_1)||m - \textbf{\textit{Cent}}(R_2)| \\ \quad + |m - \textbf{\textit{Cent}}(R_1)| + |m - \textbf{\textit{Cent}}(R_2)| + 1 & \text{if } m \geqslant 3. \end{cases}
\end{aligned}
$$

By combining all the cases, we obtain the desired result. □

To conclude this section, we give a generalisation of the above theorem.

Theorem 4.2.7. *Let* $m, n \in \mathbb{N}$ *with* $m, n \geqslant 2$. *Let* $R_1, R_2, \cdots, R_n$ *be rings with*

$|Z(R_1)|, |Z(R_2)|, \cdots, |Z(R_n)| \geqslant m-1$. *Let* $\Lambda_n = \{1, 2, \cdots, n\}$. *Then*

$$\left|m-\textit{Cent}\left(\prod_{i=1}^{n} R_i\right)\right| = \begin{cases} \prod_{i=1}^{n} |m-\textit{Cent}(R_i)| + M_{n-1} & \textit{if}\ m=2 \\ \prod_{i=1}^{n} |m-\textit{Cent}(R_i)| + M_{n-1} & \\ + \prod_{i=1}^{n} \delta(R_i) & \textit{if}\ m \geqslant 3 \end{cases},$$

where $M_{n-1} = \sum_{i=1}^{n-1} \left(\sum_{\substack{B_i \subset \Lambda_n \\ \textit{with} \\ |B_i|=i}} \left(\left(\prod_{x\in B_i} \delta(R_x) \right) \left(\prod_{y \in \Lambda_n - B_i} |m-\textit{Cent}(R_y)| \right) \right) \right).$

Proof. To prove this result, we will use mathematical induction. By Theorem 4.2.6, the result holds for $n = 2$. Suppose that there exists some $k \in \mathbb{N}$ with $k \geqslant 3$ such that

$$\left|m-\textit{Cent}\left(\prod_{i=1}^{k} R_i\right)\right| = \begin{cases} \prod_{i=1}^{k} |m-\textit{Cent}(R_i)| + M_{k-1} & \text{if}\ m=2 \\ \prod_{i=1}^{k} |m-\textit{Cent}(R_i)| + M_{k-1} & \\ + \prod_{i=1}^{k} \delta(R_i) & \text{if}\ m \geqslant 3 \end{cases}.$$

Now, we want to show that the result also holds for $n = k+1$. By Theorem 4.2.6, we have

$$\begin{aligned} & \left|m-\textit{Cent}\left(\prod_{i=1}^{k+1} R_i\right)\right| \\ = & \left|m-\textit{Cent}\left(\prod_{i=1}^{k} R_i \times R_{k+1}\right)\right| \end{aligned}$$

$$
= \begin{cases}
\left| m - \textit{Cent}\left(\prod_{i=1}^{k} R_i\right)\right| \left|m - \textit{Cent}\left(R_{k+1}\right)\right| \\
+ \delta(R_{k+1}) \left| m - \textit{Cent}\left(\prod_{i=1}^{k} R_i\right)\right| & \text{if } m = 2, \\
+ \delta\left(\prod_{i=1}^{k} R_i\right) \left|m - \textit{Cent}\left(R_{k+1}\right)\right| \\
\left| m - \textit{Cent}\left(\prod_{i=1}^{k} R_i\right)\right| \left|m - \textit{Cent}\left(R_{k+1}\right)\right| \\
+ \delta(R_{k+1}) \left| m - \textit{Cent}\left(\prod_{i=1}^{k} R_i\right)\right| & \text{if } m \geqslant 3 \\
+ \delta\left(\prod_{i=1}^{k} R_i\right) \left|m - \textit{Cent}\left(R_{k+1}\right)\right| + \delta\left(\prod_{i=1}^{k} R_i\right) \delta(R_{k+1})
\end{cases}
$$

$$
= \begin{cases}
\left(\prod_{i=1}^{k} |m - \textit{Cent}(R_i)| + M_{k-1}\right) \left|m - \textit{Cent}\left(R_{k+1}\right)\right| \\
+ \delta(R_{k+1}) \left(\prod_{i=1}^{k} |m - \textit{Cent}(R_i)| + M_{k-1}\right) & \text{if } m = 2, \\
+ \left(\prod_{i=1}^{k} \delta\left(R_i\right)\right) \left|m - \textit{Cent}\left(R_{k+1}\right)\right| \\
\left(\prod_{i=1}^{k} |m - \textit{Cent}(R_i)| + M_{k-1} + \prod_{i=1}^{k} \delta(R_i)\right) \left|m - \textit{Cent}\left(R_{k+1}\right)\right| & \text{if } m \geqslant 3. \\
+ \delta(R_{k+1}) \left(\prod_{i=1}^{k} |m - \textit{Cent}(R_i)| + M_{k-1} + \prod_{i=1}^{k} \delta(R_i)\right)
\end{cases}
$$

By expanding and simplifying, we obtain

$$
\left| m - \textit{Cent}\left(\prod_{i=1}^{k+1} R_i\right)\right| = \begin{cases}
\prod_{i=1}^{k+1} |m - \textit{Cent}(R_i)| + M_k & \text{if } m = 2 \\
\prod_{i=1}^{k+1} |m - \textit{Cent}(R_i)| + M_k + \prod_{i=1}^{k+1} \delta(R_i) & \text{if } m \geqslant 3,
\end{cases},
$$

as desired. □

4.3 Some (m, n)-Centraliser Finite Rings

In this section, we give the characterisation for some (m, n)-centraliser finite rings with $n \leqslant 10$. We begin with the following result.

Theorem 4.3.1. *Let $m \in \mathbb{N}$ with $m \geqslant 2$ and let R be a non-commutative ring with $|Z(R)| \geqslant m - 1$. Then R is not $(m, 2)$-centraliser ring, $(m, 3)$-centraliser ring, and $(m, 4)$-centraliser ring with $|Z(R)| \geqslant m$.*

Proof. Suppose that R is an $(m, 2)$-centraliser ring, $(m, 3)$-centraliser ring, or $(m, 4)$-centraliser ring with $|Z(R)| \geqslant m$. Then, by Lemma 4.2.1, we obtain $|\mathrm{Cent}(R)| \leqslant 3$, which contradicts the fact that $|\mathrm{Cent}(R)| \geqslant 4$. □

Theorem 4.3.2. *Let $m \in \mathbb{N}$ with $m \geqslant 2$ and let R be a finite ring with $|Z(R)| = m-1$. Then R is an $(m, 4)$-centraliser finite ring if and only if $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. $(\Rightarrow)$: By Lemma 4.2.1(a), we obtain $|\mathrm{Cent}(R)| \leqslant 4$. Since $|\mathrm{Cent}(R)| \geqslant 4$, we have $|\mathrm{Cent}(R)| = 4$. Hence, by [A3], we have $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

$(\Leftarrow)$: In view of [A2], we have $|\mathrm{Cent}(R)| = 4$. Consequently, R is an $(m, 4)$-centraliser finite ring by Lemma 4.2.2 and Theorem 4.2.3(a). □

Theorem 4.3.3. *Let $m \in \mathbb{N}$ with $m \geqslant 2$ and let R be a finite ring with $|Z(R)| \geqslant m-1$. Then R is an $(m, 5)$-centraliser finite ring if and only if $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ with $|Z(R)| \geqslant m$, or $R/Z(R) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ with $|Z(R)| = m - 1$.*

Proof. $(\Rightarrow)$: By Lemma 4.2.1, we obtain $|\mathrm{Cent}(R)| \leqslant 5$. Since $|\mathrm{Cent}(R)| \geqslant 4$, we have $4 \leqslant |\mathrm{Cent}(R)| \leqslant 5$. Hence, by Lemma 4.2.2, Theorem 4.2.3, [A3] and

[A4], we have $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ with $|Z(R)| \geqslant m$, or $R/Z(R) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$ with $|Z(R)| = m - 1$.

($\Leftarrow$): In view of [A2], we have $|\mathrm{Cent}(R)| = 4$ with $|Z(R)| \geqslant m$, or $|\mathrm{Cent}(R)| = 5$ with $|Z(R)| = m - 1$. Consequently, R is an $(m, 5)$-centraliser finite ring by Lemma 4.2.2 and Theorem 4.2.3. □

Theorem 4.3.4. *Let $m \in \mathbb{N}$ with $m \geqslant 2$ and let R be a finite ring with $|Z(R)| = m - 1$. If R is an $(m, 6)$-centraliser finite ring, then $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. By Lemma 4.2.1(a), we obtain $|\mathrm{Cent}(R)| \leqslant 6$. Since $|\mathrm{Cent}(R)| \geqslant 4$, we have $4 \leqslant |\mathrm{Cent}(R)| \leqslant 6$. Hence, by Lemma 4.2.2, Theorem 4.2.3(a) and Theorem 2.3.1, we have $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. □

Theorem 4.3.5. *Let $m \in \mathbb{N}$ with $m \geqslant 2$ and let R be a finite ring with $|Z(R)| \geqslant m$. Then R is an $(m, 6)$-centraliser finite ring if and only if $R/Z(R) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.*

Proof. ($\Rightarrow$): By Lemma 4.2.1(b), we obtain $|\mathrm{Cent}(R)| \leqslant 5$. Since $|\mathrm{Cent}(R)| \geqslant 4$, we have $4 \leqslant |\mathrm{Cent}(R)| \leqslant 5$. Hence, by Lemma 4.2.2, Theorem 4.2.3(b) and [A4], we have $R/Z(R) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

($\Leftarrow$): In view of [A2], we have $|\mathrm{Cent}(R)| = 5$. Consequently, R is an $(m, 6)$-centraliser finite ring by Lemma 4.2.2 and Theorem 4.2.3(b). □

Theorem 4.3.6. *Let $m \in \mathbb{N}$ with $m \geqslant 2$ and let R be a finite ring with $|Z(R)| = m-1$. Then R is an $(m, 7)$-centraliser finite ring if and only if $R/Z(R) \cong \mathbb{Z}_5 \times \mathbb{Z}_5$.*

Proof. (⇒): By Lemma 4.2.1(a), we obtain $|\text{Cent}(R)| \leqslant 7$. Since $|\text{Cent}(R)| \geqslant 4$, we have $4 \leqslant |\text{Cent}(R)| \leqslant 7$. Hence, by Lemma 4.2.2, Theorem 4.2.3(a) and Theorem 2.4.1, we have $R/Z(R) \cong \mathbb{Z}_5 \times \mathbb{Z}_5$.

(⇐): In view of [A2], we have $|\text{Cent}(R)| = 7$. Consequently, R is an $(m, 7)$-centraliser finite ring by Lemma 4.2.2 and Theorem 4.2.3(a). □

Theorem 4.3.7. *Let $m \in \mathbb{N}$ with $m \geqslant 2$ and let R be a finite ring with $|Z(R)| \geqslant m$. If R is an $(m, 7)$-centraliser finite ring, then $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. By Lemma 4.2.1(b), we obtain $|\text{Cent}(R)| \leqslant 6$. Since $|\text{Cent}(R)| \geqslant 4$, we have $4 \leqslant |\text{Cent}(R)| \leqslant 6$. Hence, by Lemma 4.2.2, Theorem 4.2.3(b) and Theorem 2.3.1, we have $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. □

Theorem 4.3.8. *Let $m \in \mathbb{N}$ with $m \geqslant 2$ and let R be a finite ring with $|Z(R)| = m - 1$. If R is an $(m, 8)$-centraliser finite ring, then $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. By Lemma 4.2.1(a), we obtain $|\text{Cent}(R)| \leqslant 8$. Since $|\text{Cent}(R)| \geqslant 4$, we have $4 \leqslant |\text{Cent}(R)| \leqslant 8$. Hence, by Lemma 4.2.2, Theorem 4.2.3(a) and Theorem 2.5.1, we have $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. □

Theorem 4.3.9. *Let $m \in \mathbb{N}$ with $m \geqslant 2$ and let R be a finite ring with $|Z(R)| \geqslant m$. Then R is an $(m, 8)$-centraliser finite ring if and only if $R/Z(R) \cong \mathbb{Z}_5 \times \mathbb{Z}_5$.*

Proof. (⇒): By Lemma 4.2.1(b), we obtain $|\text{Cent}(R)| \leqslant 7$. Since $|\text{Cent}(R)| \geqslant 4$, we have $4 \leqslant |\text{Cent}(R)| \leqslant 7$. Hence, by Lemma 4.2.2, Theorem 4.2.3(b) and Theorem 2.4.1, we have $R/Z(R) \cong \mathbb{Z}_5 \times \mathbb{Z}_5$.

($\Leftarrow$): In view of [A2], we have $|\mathrm{Cent}(R)| = 7$. Consequently, R is an $(m, 8)$-centraliser finite ring by Lemma 4.2.2 and Theorem 4.2.3(b). □

Theorem 4.3.10. *Let $m \in \mathbb{N}$ with $m \geqslant 2$ and let R be a finite ring with $|Z(R)| = m-1$. Then R is an $(m, 9)$-centraliser finite ring if and only if $R/Z(R) \cong \mathbb{Z}_7 \times \mathbb{Z}_7$.*

Proof. ($\Rightarrow$): By Lemma 4.2.1(a), we obtain $|\mathrm{Cent}(R)| \leqslant 9$. Since $|\mathrm{Cent}(R)| \geqslant 4$, we have $4 \leqslant |\mathrm{Cent}(R)| \leqslant 9$. Hence, by Lemma 4.2.2, Theorem 4.2.3(a) and Theorem 2.6.5, we have $R/Z(R) \cong \mathbb{Z}_7 \times \mathbb{Z}_7$.

($\Leftarrow$): In view of [A2], we have $|\mathrm{Cent}(R)| = 9$. Consequently, R is an $(m, 9)$-centraliser finite ring by Lemma 4.2.2 and Theorem 4.2.3(a). □

Theorem 4.3.11. *Let $m \in \mathbb{N}$ with $m \geqslant 2$ and let R be a finite ring with $|Z(R)| \geqslant m$. If R is an $(m, 9)$-centraliser finite ring, then $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. By Lemma 4.2.1(b), we obtain $|\mathrm{Cent}(R)| \leqslant 8$. Since $|\mathrm{Cent}(R)| \geqslant 4$, we have $4 \leqslant |\mathrm{Cent}(R)| \leqslant 8$. Hence, by Lemma 4.2.2, Theorem 4.2.3(b) and Theorem 2.5.1, we have $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. □

We conclude this chapter by giving the characterisation for some $(m, 10)$-centraliser finite rings.

Theorem 4.3.12. *Let $m \in \mathbb{N}$ with $m \geqslant 2$ and let R be a finite ring with $|Z(R)| \geqslant m$. Then R is an $(m, 10)$-centraliser finite ring if and only if $R/Z(R) \cong \mathbb{Z}_7 \times \mathbb{Z}_7$.*

Proof. ($\Rightarrow$): By Lemma 4.2.1(b), we obtain $|\mathrm{Cent}(R)| \leqslant 9$. Since $|\mathrm{Cent}(R)| \geqslant 4$, we have $4 \leqslant |\mathrm{Cent}(R)| \leqslant 9$. Hence, by Lemma 4.2.2, Theorem 4.2.3(b) and Theorem 2.6.5, we have $R/Z(R) \cong \mathbb{Z}_7 \times \mathbb{Z}_7$.

($\Leftarrow$): In view of [A2], we have $|\mathrm{Cent}(R)| = 9$. Consequently, R is an $(m, 10)$-centraliser finite ring by Lemma 4.2.2 and Theorem 4.2.3(b). $\square$

CHAPTER 5

NON-CENTRALISER GRAPHS OF FINITE RINGS

5.1 Introduction

In this chapter, we attempt to discuss various graph theoretical properties of the non-centraliser graphs of finite rings. Let G be a graph. We denote by $V(G)$ the vertex set of G, $E(G)$ the edge set of G. An edge $\{x, y\}$ in $E(G)$ is said to join the vertices x and y. For any two vertices x, y in $V(G)$, we write $x \sim y$ (respectively, $x \not\sim y$) to means that x is adjacent to y (respectively, x is non-adjacent to y). We use the notation $d(x, y)$ to denote the distance between two vertices x and y in $V(G)$. We let $\mathfrak{C}_R(x)$ be defined by $\mathfrak{C}_R(x) = \{y \in R \mid C_R(x) = C_R(y)\}$.

5.2 Some Properties of Υ_R

In this section, we investigate some properties of Υ_R. We first give some examples of Υ_R.

Example 5.2.1. Consider the ring $R_1 = \langle a, b \mid 2a = 2b = 0, a^2 = a, b^2 = b, ab = a, ba = b\rangle$. Note that, R_1 can be simplified as $R_1 = \{0, a, b, a+b\}$. Since $a \neq b$, then $ab \neq ba$. Hence, we have

$$C_{R_1}(0) = R_1,$$

$$C_{R_1}(a) = \{0, a\},$$

$$C_{R_1}(b) = \{0, b\},$$

$$C_{R_1}(a + b) = \{0, a + b\}.$$

This gives that $V(\Upsilon_{R_1}) = \{0, a, b, a + b\}$ and $E(\Upsilon_{R_1}) = \{\{0, a\}, \{0, b\}, \{0, a + b\}, \{a, b\}, \{a, a + b\}, \{b, a + b\}\}$. Therefore, Υ_{R_1} is as follows:

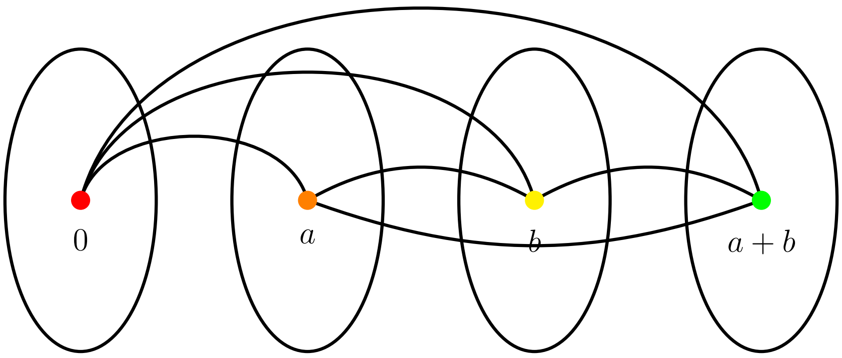

Figure 5.1: Non-centraliser graph Υ_{R_1}

This shows us that Υ_{R_1} is a complete 4-partite graph.

Example 5.2.2. Consider the ring $R_2 = \langle a, b \mid 3a = 3b = 0, a^2 = a, b^2 = b, ab = a, ba = b\rangle$. Note that, R_2 can be simplified as $R_2 = \{0, a, 2a, b, 2b, a + b, a + 2b, 2a + b, 2a + 2b\}$. Since $a \neq b$, then $ab \neq ba$. Hence, we have

$$C_{R_2}(0) = R_2,$$

$$C_{R_2}(a) = C_{R_2}(2a) = \{0, a, 2a\},$$

$$C_{R_2}(b) = C_{R_2}(2b) = \{0, b, 2b\},$$

$$C_{R_2}(a+b) = C_{R_2}(2a+2b) = \{0, a+b, 2a+2b\},$$

$$C_{R_2}(a+2b) = C_{R_2}(2a+b) = \{0, a+2b, 2a+b\}.$$

This gives that $V(\Upsilon_{R_2}) = \{0, a, 2a, b, 2b, a+b, a+2b, 2a+b, 2a+2b\}$ and $E(\Upsilon_{R_2}) = \{\{0,a\}, \{0,2a\}, \{0,b\}, \{0,2b\}, \{0,a+b\}, \{0,2a+2b\}, \{0,a+2b\}, \{0,2a+b\}, \{a,b\}, \{a,2b\}, \{a,a+b\}, \{a,2a+2b\}, \{a,a+2b\}, \{a,2a+b\}, \{2a,b\}, \{2a,2b\}, \{2a,a+b\}, \{2a,2a+2b\}, \{2a,a+2b\}, \{2a,2a+b\}, \{b,a+b\}, \{b,2a+2b\}, \{b,a+2b\}, \{b,2a+b\}, \{2b,a+b\}, \{2b,2a+2b\}, \{2b,a+2b\}, \{2b,2a+b\}, \{a+b,a+2b\}, \{a+b,2a+b\}, \{2a+2b,a+2b\}, \{2a+2b,2a+b\}\}$. Therefore, Υ_{R_2} is as follows:

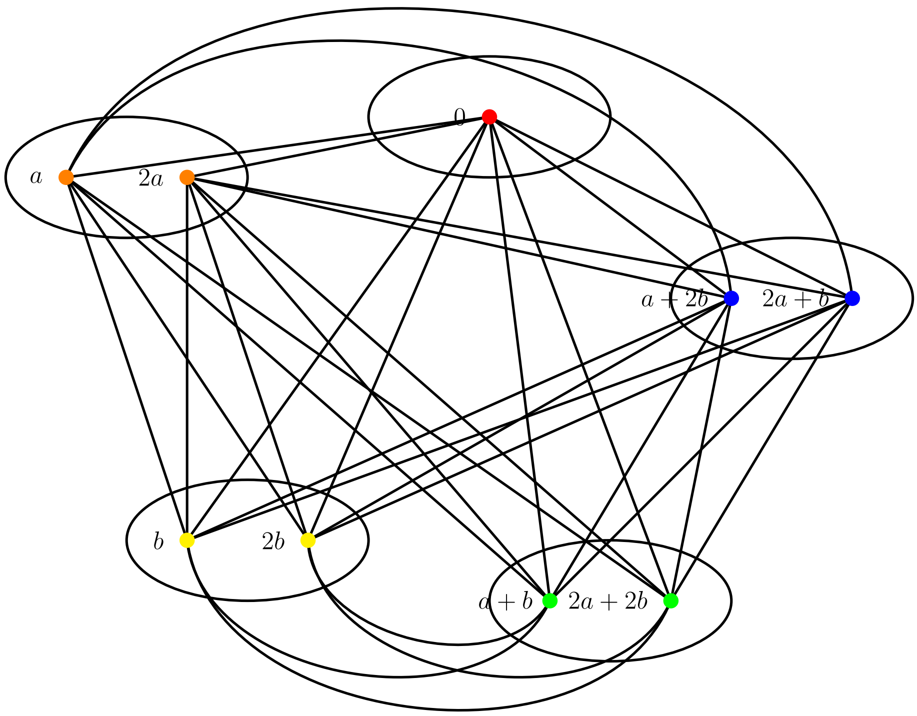

Figure 5.2: Non-centraliser graph Υ_{R_2}

This shows us that Υ_{R_2} is a complete 5-partite graph.

From the above examples, we see that $|\mathrm{Cent}(R_1)| = 4$ and Υ_{R_1} is a complete 4-partite graph. Also, we see that $|\mathrm{Cent}(R_2)| = 5$ and Υ_{R_2} is a complete 5-partite graph. The following result shows that the structure of Υ_R can be affected by $|\mathrm{Cent}(R)|$.

Proposition 5.2.3. Let R be a finite non-commutative ring. Then $|\mathrm{Cent}(R)| = n$ if and only if Υ_R is a complete n-partite graph.

Proof. If $|\mathrm{Cent}(R)| = n$, then $\mathrm{Cent}(R) = \{C_R(r_1), C_R(r_2), \cdots, C_R(r_n)\}$ for n distinct elements $r_1, r_2, \cdots, r_n \in R$. Since $V(\Upsilon_R)$ can be partitioned into n partite sets $\mathfrak{C}_R(r_1), \mathfrak{C}_R(r_2), \cdots, \mathfrak{C}_R(r_n)$ such that for any two vertices $x \in \mathfrak{C}_R(r_i)$ and $y \in \mathfrak{C}_R(r_j)$, $x \sim y$ if and only if $i \neq j$, then Υ_R is a complete n-partite graph. Conversely, assume that Υ_R is a complete n-partite graph but $|\mathrm{Cent}(R)| = m$, where $m \neq n$. Hence, by the necessity part, we obtain Υ_R is a complete m-partite graph. We have reached a contradiction. □

As an immediate consequence of Proposition 5.2.3, [A1], [A3], [A4], Theorem 2.3.1, Theorem 2.4.2, Theorem 2.5.1, Theorem 2.6.6, Theorem 2.7.10 and Theorem 2.8.16, we have the following result.

Proposition 5.2.4. Let R be a finite ring. Then the following statements hold.

(a) Υ_R is an empty graph if and only if R is commutative.

(b) Υ_R is neither a complete bipartite graph nor a complete tripartite graph.

(c) Υ_R is a complete 4-partite graph if and only if $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(d) Υ_R is a complete 5-partite graph if and only if $R/Z(R) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

(e) If Υ_R is a complete 6-partite graph, then $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

(f) Υ_R is a complete 7-partite graph if and only if $R/Z(R) \cong \mathbb{Z}_5 \times \mathbb{Z}_5$.

(g) If Υ_R is a complete 8-partite graph, then $R/Z(R) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

(h) Υ_R is a complete 9-partite graph if and only if $R/Z(R) \cong \mathbb{Z}_7 \times \mathbb{Z}_7$.

(i) If Υ_R is a complete 10-partite graph, then $R/Z(R) \cong \mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

(j) If Υ_R is a complete 11-partite graph, then $R/Z(R) \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

5.2.1 Clique and Colouring

The clique number of a graph G, denoted by $\omega(G)$, is the order of a largest complete subgraph contained in G. The chromatic number of a graph G, denoted by $\chi(G)$, is the smallest number of colours required to colour the vertices of G so that every adjacent vertices in G are assigned different colours. The following result is a direct consequence of Proposition 5.2.3.

Proposition 5.2.1.1. Let R be a finite non-commutative ring. Then $\omega(\Upsilon_R) = \chi(\Upsilon_R) = |\mathrm{Cent}(R)|$.

5.2.2 Domination Number and Independent Set

A subset D of the vertex set of a graph G is called a dominating set of G if every vertex not in D is adjacent to at least one vertex in D. Furthermore, the domination number of G, denoted by $\gamma(G)$, is the cardinality of a smallest dominating set of G. In this part, we first find an upper bound of $\gamma(\Upsilon_R)$.

Proposition 5.2.2.1. Let R be a finite non-commutative ring. Then $\gamma(\Upsilon_R) \leqslant 2$.

Proof. For any $r_1 \in R - Z(R)$, since $C_R(r_1) \neq C_R(0)$, then we have $r_1 \sim 0$. On the other hand, for any $r_2 \in Z(R)$, since $C_R(r_2) \neq C_R(x)$ for some $x \in R - Z(R)$, then we have $r_2 \sim x$. This shows that $\{0, x\}$ is a dominating set of Υ_R. Therefore, we obtain $\gamma(\Upsilon_R) \leqslant 2$. □

We now obtain some sufficient conditions for $\gamma(\Upsilon_R) = 2$.

Proposition 5.2.2.2. Let R be a finite non-commutative ring. If $|Z(R)| \neq 1$ or $2r \neq 0$ for any $r \in R$, then $\gamma(\Upsilon_R) = 2$.

Proof. By Proposition 5.2.2.1, $\gamma(\Upsilon_R) \leqslant 2$. Suppose that $\gamma(\Upsilon_R) = 1$. Let $\{x\}$ be a dominating set of Υ_R. Assume that $|Z(R)| \neq 1$. Let $w \in Z(R) - \{0\}$. Since $C_R(x) = C_R(x + w)$, then $x \not\sim (x + w)$, which leads to a contradiction. So, we have $2r \neq 0$ for any $r \in R$. This gives that $2x \neq 0$ and thus, $x \neq -x$. Since $C_R(x) = C_R(-x)$, then $x \not\sim -x$, which leads to a contradiction again. Hence, $\gamma(\Upsilon_R) = 2$. □

A subset D of the vertex set of a graph G is called an independent set of

G if every two distinct vertices in D are non-adjacent with each other. Moreover, D is said to be a maximal independent set of G if its cardinality is the largest one among all such sets. We next give a necessary condition for maximal independent set of Υ_R.

Proposition 5.2.2.3. Let R be a finite non-commutative ring. If S is a maximal independent set of Υ_R, then $S = \bigcap\limits_{s\in S} \mathfrak{C}_R(s)$.

Proof. Let $w \in S$. Since S is a maximal independent set of Υ_R, then $w \not\sim s$ for any $s \in S - \{w\}$. Hence, $C_R(w) = C_R(s)$ for any $s \in (S - \{w\}) \cup \{w\} = S$. It follows that $w \in \mathfrak{C}_R(s)$ for any $s \in S$ and hence, $w \in \bigcap\limits_{s\in S} \mathfrak{C}_R(s)$. This gives that $S \subseteq \bigcap\limits_{s\in S} \mathfrak{C}_R(s)$. On the other hand, let $w \in \bigcap\limits_{s\in S} \mathfrak{C}_R(s)$. Thus, $w \in \mathfrak{C}_R(s)$ for any $s \in S$ and therefore, we have $C_R(w) = C_R(s)$ for any $s \in S$. If $w \notin S$, then $w \not\sim s$ for any $s \in S - \{w\} = S$. This shows that $S \cup \{w\}$ can be formed an independent set of Υ_R, which contradicts the fact that S is a maximal independent set of Υ_R. Therefore, $w \in S$ and so, $\bigcap\limits_{s\in S} \mathfrak{C}_R(s) \subseteq S$. Consequently, we obtain $S = \bigcap\limits_{s\in S} \mathfrak{C}_R(s)$. □

5.2.3 Diameter and Girth

The diameter of a graph G, denoted by $diam(G)$, is the largest distance between any two vertices of G. We begin this subsection by determine a lower bound of $diam(\Upsilon_R)$.

Proposition 5.2.3.1. Let R be a finite non-commutative ring. Then Υ_R is connected and $diam(\Upsilon_R) \leqslant 2$.

Proof. Let x, y be two distinct elements in R. Here, we consider the following three cases.

Case 1: $x, y \in Z(R)$. Thus, we have $C_R(x) \neq C_R(w)$ and $C_R(w) \neq C_R(y)$ for some $w \in R - Z(R)$. It follows that $x \sim w \sim y$. Therefore, we have $d(x, y) \leqslant 2$.

Case 2: Either $x \in Z(R)$ or $y \in Z(R)$ but not both. Thus, we have $C_R(x) \neq C_R(y)$. It follows that $x \sim y$. Therefore, we have $d(x, y) = 1$.

Case 3: $x, y \notin Z(R)$. Thus, we have $C_R(x) \neq C_R(0)$ and $C_R(0) \neq C_R(y)$. It follows that $x \sim 0 \sim y$. Therefore, we have $d(x, y) \leqslant 2$.

Consequently, by all the cases above, we obtain Υ_R is connected and $diam(\Upsilon_R) \leqslant 2$. $\square$

The following proposition presents some necessary and sufficient conditions for $diam(\Upsilon_R) = 1$.

Proposition 5.2.3.2. Let R be a finite non-commutative ring. Then the following statements are equivalent.

(a) $diam(\Upsilon_R) = 1$.

(b) Υ_R is complete.

(c) $|\mathrm{Cent}(R)| = |R|$.

Proof. (a)⇒(b): Since $diam(\Upsilon_R) = 1$, then $d(x, y) \leqslant 1$ for any two distinct $x, y \in V(\Upsilon_R)$. In view of Proposition 5.2.3.1, Υ_R is connected. So, we have $d(x, y) = 1$ for any two distinct $x, y \in V(\Upsilon_R)$. Consequently, Υ_R is complete.

(b)⇒(c): Assume to the contrary that $|\mathrm{Cent}(R)| < |R|$. Let $|\mathrm{Cent}(R)| = n$. Thus, we have $\mathrm{Cent}(R) = \{C_R(r_1), C_R(r_2), \cdots, C_R(r_n)\}$ for n distinct elements $r_1, r_2, \cdots, r_n \in R$. Let $r \in R - \{r_1, r_2, \cdots, r_n\}$. Since $C_R(r) \in \mathrm{Cent}(R)$, then $C_R(r) = C_R(r_i)$ for some $i \in \{1, 2, \cdots, n\}$. Hence, $r \nsim r_i$, which contradicts the fact that Υ_R is complete.

(c)⇒(a): Since $|\mathrm{Cent}(R)| = |R|$, then $C_R(x) \neq C_R(y)$ for any two distinct $x, y \in R$. This implies that $d(x, y) = 1$ for any $x, y \in R$. Consequently, we obtain $diam(\Upsilon_R) = 1$. □

The girth of a graph G, denoted by $gr(G)$, is the length of the shortest cycle contained in G. In the following, we obtain an exact value of $gr(\Upsilon_R)$.

Proposition 5.2.3.3. Let R be a finite non-commutative ring. Then $gr(\Upsilon_R) = 3$.

Proof. Since R is non-commutative, then there exist two distinct elements $r_1, r_2 \in R$ such that $r_1 r_2 \neq r_2 r_1$. This gives that $C_R(r_1) \neq C_R(r_2) \neq C_R(0)$. It follows that $r_1 \sim r_2 \sim 0 \sim r_1$ is a triangle contained in Υ_R, which implies that $gr(\Upsilon_R) \leqslant 3$. Since Υ_R is a simple graph, then $gr(\Upsilon_R) \geqslant 3$. Consequently, $gr(\Upsilon_R) = 3$. □

5.2.4 Degree

The degree of a vertex x in a graph G is the number of edges incident with x, and is written $deg(x)$. We begin with the following result which gives the degree of each vertex in Υ_R.

Proposition 5.2.4.1. Let R be a finite non-commutative ring. Then for any $x \in V(\Upsilon_R)$,

$$deg(x) = |R| - |\mathfrak{C}_R(x)| \begin{cases} = |R| - |Z(R)| & \text{if } x \in Z(R), \\ \geqslant |R| - |C_R(x)| & \text{if } x \notin Z(R). \end{cases}$$

Proof. Let $x \in V(\Upsilon_R)$. Since $C_R(x) \neq C_R(u)$ for any $u \notin \mathfrak{C}_R(x)$, then $x \sim u$ for any $u \notin \mathfrak{C}_R(x)$. It follows that $deg(x) \geqslant |R| - |\mathfrak{C}_R(x)|$. Since $C_R(x) = C_R(v)$ for any $v \in \mathfrak{C}_R(x)$, then $x \not\sim v$ for any $v \in \mathfrak{C}_R(x)$. It follows that $deg(x) \leqslant |R| - |\mathfrak{C}_R(x)|$. Therefore, we obtain $deg(x) = |R| - |\mathfrak{C}_R(x)|$. If $x \in Z(R)$, then $\mathfrak{C}_R(x) = Z(R)$ and so, we have $deg(x) = |R| - |Z(R)|$. Assume that $x \notin Z(R)$. Since $\mathfrak{C}_R(x) \subseteq C_R(x)$, then we have $deg(x) \geqslant |R| - |C_R(x)|$. □

We use the notation $\delta(G)$ to represent the minimum vertex degree in a graph G. In the following proposition, we give a lower bound of $\delta(\Upsilon_R)$.

Proposition 5.2.4.2. Let R be a finite non-commutative ring. Then $\delta(\Upsilon_R) \geqslant 3$.

Proof. Suppose to the contrary that $\delta(\Upsilon_R) \leqslant 2$ for some non-commutative ring R. Thus, $deg(x) \leqslant 2$ for some $x \in V(\Upsilon_R)$. Hence, we note that x is non-adjacent to at least $|R| - 3$ distinct vertices in $V(\Upsilon_R)$. Therefore, we have $C_R(x) = C_R(r_1) = C_R(r_2) = \cdots = C_R(r_{|R|-3})$ for $|R| - 3$ distinct elements $r_1, r_2, \cdots, r_{|R|-3} \in R$. This yields that $|\mathrm{Cent}(R)| \leqslant 3$, which contradicts with [A1]. □

By applying Proposition 5.2.4.2, we can confirm that Υ_R is not isomorphic to certain graph. For example, Υ_R is not isomorphic to tree. Next, we improve a lower bound of $\delta(\Upsilon_R)$ and develop an upper bound of $\delta(\Upsilon_R)$.

Proposition 5.2.4.3. Let R be a finite non-commutative ring. Then $\frac{|R|}{2} \leqslant \delta(\Upsilon_R) \leqslant |R| - 1$.

Proof. Since $x \in \mathfrak{C}_R(x)$ for any $x \in R$, then $|\mathfrak{C}_R(x)| \geqslant 1$ for any $x \in R$. Since $Z(R) < R$, then $|Z(R)| \leqslant \frac{|R|}{2}$. Since $C_R(x) < R$ for any $x \notin Z(R)$, then $|C_R(x)| \leqslant \frac{|R|}{2}$ for any $x \notin Z(R)$. Thus, it follows from Proposition 5.2.4.1 that $\frac{|R|}{2} \leqslant deg(x) \leqslant |R| - 1$ for any $x \in V(\Upsilon_R)$. Consequently, we obtain $\frac{|R|}{2} \leqslant \delta(\Upsilon_R) \leqslant |R| - 1$. □

By combining the Proposition 5.2.4.2 and Proposition 5.2.4.3, we have the following result.

Proposition 5.2.4.4. Let R be a finite non-commutative ring. Then $\max\{3, \frac{|R|}{2}\} \leqslant \delta(\Upsilon_R) \leqslant |R| - 1$.

Dirac's theorem (see Dirac (1952)) is a well-known theorem in the Hamiltonian problem. Dirac's theorem states that a simple graph G is hamiltonian if

$\delta(G) \geqslant 2$ and $\delta(G) \geqslant \frac{|V(G)|}{2}$. As a direct consequence of Proposition 5.2.4.4 and Dirac's theorem, we have the following result.

Proposition 5.2.4.5. Let R be a finite non-commutative ring. Then Υ_R is Hamiltonian.

5.2.5 Connectivity and Planarity

The edge connectivity (respectively, vertex connectivity) of a graph G, denoted by $\lambda(G)$ (respectively, $\kappa(G)$), is the minimum number of edges (respectively, vertices) required to be eliminated from G so that G is disconnected. Chartrand (1966) has verified that for any simple graph G, if $\delta(G) \geqslant \frac{|V(G)|-1}{2}$, then $\lambda(G) = \delta(G)$. Thus, by combining this result and Proposition 5.2.4.4, we obtain the following proposition immediately.

Proposition 5.2.5.1. Let R be a finite non-commutative ring. Then $\max\{3, \frac{|R|}{2}\} \leqslant \lambda(\Upsilon_R) = \delta(\Upsilon_R) \leqslant |R| - 1$.

Next, we give the relation between the $\kappa(\Upsilon_R)$ and $\lambda(\Upsilon_R)$.

Proposition 5.2.5.2. Let R be a finite non-commutative ring. Then $\kappa(\Upsilon_R) = \lambda(\Upsilon_G) = \delta(\Upsilon_G) = |R| - \max_{r \in R}\{|\mathfrak{C}_R(r)|\}$.

Proof. Let $|\mathrm{Cent}(R)| = n$. Thus, we have $\mathrm{Cent}(R) = \{C_R(r_1), C_R(r_2), \cdots, C_R(r_n)\}$ for n distinct elements $r_1, r_2, \cdots, r_n \in R$. By Proposition 5.2.3, Υ_R is a complete n-partite graph. Since Υ_R is a complete n-partite graph, then $\kappa(\Upsilon_R)$

(respectively, $\lambda(\Upsilon_R)$) is equal to the minimum number of vertices (respectively, edges) required to be eliminated from Υ_R so that Υ_R left only a single partite set (respectively, Υ_R exists an isolated vertex). Thus, we have $\kappa(\Upsilon_R) = \lambda(\Upsilon_R) = |R| - \max_{r\in\{r_1,r_2,\cdots,r_n\}}\{|\mathfrak{C}_R(r)|\} = |R| - \max_{r\in R}\{|\mathfrak{C}_R(r)|\}$. From Proposition 5.2.5.1, $\lambda(\Upsilon_R) = \delta(\Upsilon_R)$. Consequently, we get the desired result. □

Kuratowski (1930) published a famous theorem regarding the planarity of a graph, which is Kuratowski's theorem. Kuratowski's theorem states that a graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph, where K_5 is a complete graph of order 5 and $K_{3,3}$ is a complete bipartite graph with both partite sets having 3 vertices. In the following, we obtain a complete characterisation of the planarity of Υ_R.

Proposition 5.2.5.3. Let R be a finite non-commutative ring. Then Υ_R is planar if and only if $|R| = 4$.

Proof. We first consider the necessity part. Assume that $|Z(R)| \geqslant 3$. Let $\{0, z_1, z_2\} \subseteq Z(R)$ and let $r \in R - Z(R)$. It follows that $C_R(0) = C_R(z_1) = C_R(z_2)$ and $C_R(r) = C_R(r + z_1) = C_R(r + z_2)$ and $C_R(0) \neq C_R(r)$. Since $C_R(a) \neq C_R(b)$ for any $a \in \{0, z_1, z_2\}, b \in \{r, r+z_1, r+z_2\}$, then $a \sim b$ for any $a \in \{0, z_1, z_2\}, b \in \{r, r+z_1, r+z_2\}$. This gives that $K_{3,3}$ is a subgraph contained in Υ_R, which contradicts with the Kuratowski's Theorem. So, $|Z(R)| \leqslant 2$. By [A1], $|\mathrm{Cent}(R)| \geqslant 4$. If $|\mathrm{Cent}(R)| \geqslant 6$, then $\{C_R(r_1), C_R(r_2), \cdots, C_R(r_6)\} \subseteq \mathrm{Cent}(R)$ for six distinct elements $r_1, r_2, \cdots, r_6 \in R$. Since $C_R(a) \neq C_R(b)$ for any $a \in \{r_1, r_2, r_3\}, b \in \{r_4, r_5, r_6\}$, then $a \sim b$ for any $a \in \{r_1, r_2, r_3\}, b \in$

$\{r_4, r_5, r_6\}$. This gives that $K_{3,3}$ is a subgraph contained in Υ_R, which contradicts with the Kuratowski's Theorem. If $|\mathrm{Cent}(R)| = 5$, then $\mathrm{Cent}(R) = \{C_R(r_1), C_R(r_2), \cdots, C_R(r_5)\}$ for five distinct elements $r_1, r_2, \cdots, r_5 \in R$. By [A4], $|R| \geqslant 9$. Let $r \in R - \{r_1, r_2, \cdots, r_5\}$. Since $C_R(r) \in \mathrm{Cent}(R)$, then $C_R(r) = C_R(r_i)$ for some $i \in \{1, 2, \cdots, 5\}$. Without loss of generality, we assume that $i = 1$. Since $C_R(a) \neq C_R(b)$ for any $a \in \{r, r_1, r_2\}, b \in \{r_3, r_4, r_5\}$, then $a \sim b$ for any $a \in \{r, r_1, r_2\}, b \in \{r_3, r_4, r_5\}$. This gives that $K_{3,3}$ is a subgraph contained in Υ_R, which contradicts with the Kuratowski's Theorem. So, $|\mathrm{Cent}(R)| = 4$. Thus, we have $\mathrm{Cent}(R) = \{C_R(r_1), C_R(r_2), C_R(r_3), C_R(r_4)\}$ for four distinct elements $r_1, r_2, r_3, r_4 \in R$. Suppose that $|Z(R)| = 2$ and let $Z(R) = \{0, z\}$. Note that, $C_R(r_1) = C_R(r_1 + z)$ and $C_R(r_3) = C_R(r_3 + z)$. Since $C_R(a) \neq C_R(b)$ for any $a \in \{r_1, r_1 + z, r_2\}, b \in \{r_3, r_3 + z, r_4\}$, then $a \sim b$ for any $a \in \{r_1, r_1 + z, r_2\}, b \in \{r_3, r_3 + z, r_4\}$. This gives that $K_{3,3}$ is a subgraph contained in Υ_R, which contradicts with the Kuratowski's Theorem. Consequently, $|Z(R)| = 1$. So, by [A3], we obtain $|R| = 4$, as required. Conversely, let $R = \{0, r_1, r_2, r_3\}$. Note that, $|\mathrm{Cent}(R)| \leqslant |R| = 4$. By [A1], $|\mathrm{Cent}(R)| \geqslant 4$ and thus, $|\mathrm{Cent}(R)| = 4$. It follows that $C_R(0), C_R(r_1), C_R(r_2), C_R(r_3)$ are distinct from each other, which yields that $a \sim b$ for any two distinct elements $a, b \in R$. Consequently, Υ_R can be drawn in the plane without edges crossing and so, Υ_R is planar. □

5.2.6 Regularity

In this part, we study the regularity of Υ_R. We first prove the following equivalence.

Proposition 5.2.6.1. Let R be a finite non-commutative ring. Then the following statements are equivalent.

(a) Υ_R is regular.

(b) $x + Z(R) = \mathfrak{C}_R(x)$ for any $x \in R$.

(c) $|\mathrm{Cent}(R)| = |R : Z(R)|$.

Proof. Note that, $x + Z(R) \subseteq \mathfrak{C}_R(x)$ for any $x \in R$. By Proposition 5.2.4.1, $deg(x) = |R| - |\mathfrak{C}_R(x)|$ for any $x \in V(\Upsilon_R)$ and $deg(0) = |R| - |Z(R)|$. Hence, we have

$$\begin{aligned}
\Upsilon_R \text{ is regular} &\Leftrightarrow deg(0) = deg(x) \text{ for any } x \in V(\Upsilon_R) \\
&\Leftrightarrow |R| - |Z(R)| = |R| - |\mathfrak{C}_R(x)| \text{ for any } x \in R \\
&\Leftrightarrow |Z(R)| = |\mathfrak{C}_R(x)| \text{ for any } x \in R \\
&\Leftrightarrow |x + Z(R)| = |\mathfrak{C}_R(x)| \text{ for any } x \in R \\
&\Leftrightarrow x + Z(R) = \mathfrak{C}_R(x) \text{ for any } x \in R.
\end{aligned}$$

So, the implication (a)⇔(b) holds.

We now show (b)⇔(c). For necessity part, suppose to the contrary that $|\mathrm{Cent}(R)| < |R : Z(R)|$. It follows that there exist two distinct elements

$r_1 + Z(R), r_2 + Z(R) \in R/Z(R)$ such that $C_R(r_1) = C_R(r_2)$. Therefore, we obtain $r_1 \in \mathfrak{C}_R(r_2) = r_2 + Z(R)$, which implies that $r_1 + Z(R) = r_2 + Z(R)$, a contradiction is reached. Conversely, we assume that $x + Z(R) \neq \mathfrak{C}_R(x)$ for some $x \in R$. Note that, $x + Z(R) \subseteq \mathfrak{C}_R(x)$. Hence, $\mathfrak{C}_R(x) \nsubseteq x + Z(R)$. It follows that there exists some $a \in \mathfrak{C}_R(x)$ but $a \notin x + Z(R)$. Thus, we have $C_R(a) = C_R(x)$ and $a + Z(R) \neq x + Z(R)$. This gives that $|\mathrm{Cent}(R)| < |R : Z(R)|$, which leads to a contradiction. $\square$

Next, we indicate the structure of $R/Z(R)$ when Υ_R is regular.

Proposition 5.2.6.2. Let R be a finite non-commutative ring. If Υ_R is regular, then $R/Z(R) \cong \mathbb{Z}_2^n$ for some $n \geqslant 2$.

Proof. If $R/Z(R) \cong \mathbb{Z}_2$, then $R/Z(R)$ is cyclic. This yields that R is commutative, which is impossible. So, $R/Z(R) \ncong \mathbb{Z}_2$. We want to show that all non-identity elements in $R/Z(R)$ have order 2. Let $x \in R - Z(R)$. Since $C_R(x) = C_R(-x)$, then $x \in \mathfrak{C}_R(-x)$. In view of Proposition 5.2.6.1, $x \in -x + Z(R)$. It follows that $x = -x + z$ for some $z \in Z(R)$, which gives that $2x = z$. This implies that the order of $x + Z(R)$ is 2. So, we can conclude that $R/Z(R) \cong \mathbb{Z}_2^n$ for some $n \geqslant 2$. $\square$

In general, the converse of Proposition 5.2.6.2 is not necessarily true. For example, $R = \left\{ \left[\begin{smallmatrix} a & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right] \middle| \, a, b, c \in \mathbb{Z}_2 \right\}$ is a non-commutative ring with $R/Z(R) \cong \mathbb{Z}_2^3$ but Υ_R is not regular. In the following, we demonstrate that the converse of Proposition 5.2.6.2 is holds for some circumstances.

Proposition 5.2.6.3. Let R be a finite non-commutative ring. If $R/Z(R) \cong \mathbb{Z}_2^2$, then Υ_R is regular.

Proof. In view of [A2], $|\mathrm{Cent}(R)| = 4$. So, by Proposition 5.2.6.1, we have Υ_R is regular. □

Proposition 5.2.6.4. Let R be a finite non-commutative ring. If $R/Z(R) \cong \mathbb{Z}_2^3$, then Υ_R is regular if and only if $|R : C_R(r)| = 4$ for any $r \in R - Z(R)$.

Proof. If Υ_R is regular, then by Proposition 5.2.6.1, we have $|\mathrm{Cent}(R)| = 8$. So, we obtain $|R : C_R(r)| = 4$ for any $r \in R - Z(R)$ by Theorem 2.5.1. Conversely, we assume that Υ_R is not regular. Thus, it follows from Proposition 5.2.6.1 that $|\mathrm{Cent}(R)| \neq 8$. Therefore, by [A7], we have $|\mathrm{Cent}(R)| = 6$. But, by Theorem 2.3.1, we obtain $|R : C_R(r)| \neq 4$ for some $r \in R - Z(R)$. We have reached a contradiction. So, Υ_R is regular. □

Next, we show that Υ_R is not an n-regular graph for some positive integer n.

Proposition 5.2.6.5. Let R be a finite non-commutative ring. Let $n \in \mathbb{N}$ with $n \geqslant 4$ and let D be the set $D = \left\{d \in \mathbb{N} \mid d \leqslant \frac{n}{3}, d \mid n\right\}$. If for any $d \in D$, $1 + \frac{n}{d}$ is square free or $1 + \frac{n}{d} = p^2 q$ for two distinct primes p, q, then Υ_R is not an n-regular graph.

Proof. Assume that Υ_R is an n-regular graph. By Proposition 5.2.4.1, $deg(0) = |R| - |Z(R)|$ and thus, we have $|R| - |Z(R)| = n$. It follows that $|R : Z(R)| = \frac{n+|Z(R)|}{|Z(R)|} = 1 + \frac{n}{|Z(R)|}$. Since $Z(R) < C_R(x) < R$ for any $x \in R - Z(R)$, then $|R| = k_1|C_R(x)| = k_1k_2|Z(R)|$ for some $k_1, k_2 \geqslant 2$, which implies that $|R| = k|Z(R)|$ for some $k \geqslant 4$. Hence, we have $(k-1)|Z(R)| = n$. This shows that $|Z(R)| \mid n$ and $|Z(R)| = \frac{n}{k-1} \leqslant \frac{n}{3}$, which gives that $|Z(R)| \in D$. From the given assumptions, we have $1 + \frac{n}{|Z(R)|}$ is square-free or $1 + \frac{n}{|Z(R)|} = p^2q$ for two distinct primes p, q. If $1 + \frac{n}{|Z(R)|} = p^2q$ for two distinct primes p, q, then by Lemma 2.2.17, we obtain a contradiction. So, $1 + \frac{n}{|Z(R)|}$ is square-free. This implies that $R/Z(R) \cong \mathbb{Z}_{1+\frac{n}{|Z(R)|}}$ and hence, $R/Z(R)$ is cyclic. Consequently, R is commutative. We have reached a contradiction. $\square$

5.2.7 Rings with the Same Non-Centraliser Graph

In this subsection, we determine some sufficient conditions to guarantee the isomorphism between two non-centraliser graphs.

Buckley et al. (2014) have introduced the notion of $\mathbb{Z}$-isoclinic between two rings. Following Buckley et al. (2014), two rings R_1 and R_2 are said to be $\mathbb{Z}$-isoclinic if there exist additive group isomorphisms $\phi : R_1/Z(R_1) \rightarrow R_2/Z(R_2)$ and $\psi : [R_1, R_1] \rightarrow [R_2, R_2]$ such that $\psi([u, v]) = [u', v']$ whenever $\phi(u + Z(R_1)) = u' + Z(R_2)$ and $\phi(v + Z(R_1)) = v' + Z(R_2)$. Recall that, for a ring R, $[x, y] = xy - yx$ is the additive commutator of R and $[R, R] = \{[x_i, y_i] + \cdots + [x_n, y_n] \mid x_1, y_1, \cdots, x_n, y_n \in R, n \in \mathbb{N}\}$ is the commutator subgroup of $(R, +)$. We conclude this chapter by showing that two non-centraliser graphs are isomorphic when two rings are $\mathbb{Z}$-isoclinic and the cardinalities of their centres are equal.

Proposition 5.2.7.1. Let R_1, R_2 be two finite non-commutative rings with $|Z(R_1)| = |Z(R_2)|$. If R_1 and R_2 are $\mathbb{Z}$-isoclinic, then $\Upsilon_{R_1} \cong \Upsilon_{R_2}$.

Proof. Since R_1 and R_2 are $\mathbb{Z}$-isoclinic, then there exists an additive group isomorphism $\phi : R_1/Z(R_1) \to R_2/Z(R_2)$. Thus, $|R_1/Z(R_1)| = |R_2/Z(R_2)|$. Let $|R_1/Z(R_1)| = n$. Hence, we have $R_1/Z(R_1) = \{x_1 + Z(R_1), \cdots, x_n + Z(R_1)\}$ for n distinct elements $x_1, \cdots, x_n \in R_1$, and $R_2/Z(R_2) = \{y_1+Z(R_2), \cdots, y_n+Z(R_2)\}$ for n distinct elements $y_1, \cdots, y_n \in R_2$. Without loss of generality, we assume that $\phi(x_i + Z(R_1)) = y_i + Z(R_2)$ for any $i \in \{1, \cdots, n\}$. Since $|Z(R_1)| = |Z(R_2)|$, then we are able to construct a map $g : Z(R_1) \to Z(R_2)$ such that g is bijective. Note that, for any $u \in R_1$ (respectively, $v \in R_2$), u (respectively, v) can be written in the form $u = x_i + z$ for some $i \in \{1, \cdots, n\}$ and $z \in Z(R_1)$ (respectively, $v = y_i + g(z)$ for some $i \in \{1, \cdots, n\}$ and $g(z) \in Z(R_2)$) and this representation is unique. Here, we construct a map $f : R_1 \to R_2$ such that $f(x_i + z) = y_i + g(z)$ for any $i \in \{1, \cdots, n\}$ and $z \in Z(R_1)$. Assume that f is not injective, then $f(x_i + z_1) = f(x_j + z_2)$ for two distinct $x_i + z_1, x_j + z_2 \in R_1$. Thus, we have $y_i + g(z_1) = y_j + g(z_2)$. If $i = j$, then $g(z_1) = g(z_2)$, which gives that $z_1 = z_2$, which is a contradiction. Therefore, $i \neq j$. It follows that $y_i + Z(R_2) = y_j + Z(R_2)$, which implies that $\phi(x_i + Z(R_1)) = \phi(x_j + Z(R_1))$. This gives that $x_i + Z(R_1) = x_j + Z(R_1)$. So, we obtain $i = j$, which is a contradiction again. Therefore, f is injective. Since $|Z(R_1)| = |Z(R_2)|$, then $|R_1| = |R_2|$ and consequently, f is bijective.

Let $\{u, v\}$ be an arbitrary edge in $E(\Upsilon_{R_1})$. Thus, $C_{R_1}(u) \neq C_{R_1}(v)$. Hence, we know that either $|C_{R_1}(u) - C_{R_1}(v)| \geqslant 1$ or $|C_{R_1}(v) - C_{R_1}(u)| \geqslant 1$

but not both. Without loss of generality, we suppose that $|C_{R_1}(u) - C_{R_1}(v)| \geqslant 1$. Hence, there exists an element $w \in C_{R_1}(u)$ but $w \notin C_{R_1}(v)$. Note that, u, v, w can be written as $u = x_i + z_1, v = x_j + z_2, w = x_k + z_3$ for some $i, j, k \in \{1, \cdots, n\}$ and $z_1, z_2, z_3 \in Z(R_1)$. Since $w \in C_{R_1}(u)$ but $w \notin C_{R_1}(v)$, then $x_k x_i - x_i x_k = 0$ and $x_k x_j - x_j x_k \neq 0$, which gives that $[x_k, x_i] = 0$ and $[x_k, x_j] \neq 0$. Since R_1 and R_2 are $\mathbb{Z}$-isoclinic, then there exists another additive group isomorphism $\psi : [R_1, R_1] \to [R_2, R_2]$ such that for any $s, t \in \{1, \cdots, n\}$, if $\phi(x_s + Z(R_1)) = y_s + Z(R_2)$ and $\phi(x_t + Z(R_1)) = y_t + Z(R_2)$, then $\psi([x_s, x_t]) = [y_s, y_t]$. Since the kernel of f is $\{0\}$, then we note that for any $s, t \in \{1, \cdots, n\}$, $[x_s, x_t] = 0$ if and only if $[y_s, y_t] = 0$. So, we obtain $[y_k, y_i] = 0$ and $[y_k, y_j] \neq 0$, which yields that $y_k \in C_{R_2}(y_i) = C_{R_2}(y_i + g(z_1)) = C_{R_2}(f(x_i + z_1)) = C_{R_2}(f(u))$ but $y_k \notin C_{R_2}(y_j) = C_{R_2}(y_j + g(z_2)) = C_{R_2}(f(x_j + z_2)) = C_{R_2}(f(v))$. It follows that $C_{R_2}(f(u)) \neq C_{R_2}(f(v))$ and so, $\{f(u), f(v)\}$ is an edge in $E(\Upsilon_2)$. This shows that there exists a bijective map $f : V(R_1) \to V(R_2)$ such that for any $u, v \in V(R_1)$, $\{u, v\} \in E(R_1)$ if and only if $\{f(u), f(v)\} \in E(R_2)$. So, we can conclude that $\Upsilon_{R_1} \cong \Upsilon_{R_2}$. $\square$

CHAPTER 6

CONCLUSION

In conclusion, the study of n-centraliser rings has allowed us to delve into understanding the impact of the number of distinct centralisers in a finite ring on its structure and commutativity. Throughout this study, we have meticulously investigated various topics related to the centraliser of a ring. This investigation involved characterising all n-centraliser finite rings for $n \in \{6, 7, 8, 9, 10, 11\}$ and computing their commuting probabilities. Additionally, we have classified the structures for all finite rings with cardinality of the maximal non-commuting set is 5. Moreover, we have generalised the notion of n-centraliser rings and yielded various results regarding this generalisation. To achieve this, we introduced the notion of (m, n)-centraliser rings and determined the characterisation for some (m, n)-centraliser finite rings for $n \leqslant 10$. Finally, we have employed the concept of centralisers to establish a connection between a graph and a ring. This was accomplished by introducing the concept of the non-centraliser graph of rings and discussing various graph-theoretic properties of the non-centraliser graph of finite rings.

On the other hand, we have also identified some open problems for future work. The investigation in Chapter 2 could be extended by considering the existence and characterisation of n-centraliser finite rings for $n \geqslant 12$. Furthermore, the study in Chapter 3 might be continued by considering the cardinality of the maximal non-commuting set to be t, where $t \geqslant 6$. Moreover, in Chapter 4, we imposed the assumption that $|Z(R)| \geqslant m - 1$. The results in Chapter 4 could be further improved by encompassing the cases where $|Z(R)| \leqslant m - 2$. Lastly, in Chapter 5, the applications of the non-centraliser graph of finite rings in various fields remain largely unexplored. The exploration of applications for the non-centraliser graph of finite rings opens up new avenues for research. Furthermore, the discussion in Chapter 5 could be continued by considering the complement of the non-centraliser graph of rings.

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