PERFORMANCE OF THE MAHALANOBIS AND OTHER PARAMETRIC FAMILIES OF UNIVERSAL PORTFOLIOS

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PERFORMANCE OF THE MAHALANOBIS AND OTHER PARAMETRIC FAMILIES OF UNIVERSAL PORTFOLIOS

By

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ABSTRACT

PERFORMANCE OF THE MAHALANOBIS AND OTHER PARAMETRIC FAMILIES OF UNIVERSAL PORTFOLIOS

Lim Wei Xiang

Cover and Ordentlich [9] has shown that the Dirichlet-weighted universal portfolios exhibit some long-range optimal properties. However, the implementation of the portfolio requires large computer memory requirements and long computation time. The wealth achieved by the Dirichlet-weighted universal portfolio cannot exceed that of the best constant rebalanced portfolio. A multiplicative-update universal portfolio, introduced by Helmbold, Schapire, Singer and Warmuth [12], has its limitation when the learning parameter η is restricted to small positive values. We show that the bound on the parameter η is unnecessarily restrictive, and demonstrate that higher investment returns can be achieved by allowing η to take larger positive or negative values. A class of additive-update universal portfolios generated by the Mahalanobis squared divergence is derived, and practical bounds for the valid parametric values of the Mahalanobis universal portfolio are obtained. Any real number can be used as a parameter of the Mahalanobis universal portfolio provided modifications are made when a portfolio component becomes negative. A sufficient condition for the Mahalanobis and Helmbold universal portfolios to achieve wealths exceeding that of the best constant rebalanced portfolio is derived. The performance of the Mahalanobis universal portfolios is demonstrated by running the portfolios on some large stock-data sets covering a period of 1975 trading days. The Dirichlet universal portfolio of order one is a memory-saving universal portfolio that overcomes the shortcomings of the Dirichlet-weighted universal portfolio in large computational memory and time. The mixture-current-run universal portfolio is a mixture of different universal portfolios and follows the current run of the portfolio that achieves the best single-day investment return. This portfolio is shown to be able to perform better than the individual portfolios in the mixture. We show empirically that there are mixture-current-run universal portfolios that can achieve higher wealths than that of the best constant rebalanced portfolio.

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LIST OF ABBREVIATIONS

CSD	Chi-square divergence
BCRP	Best constant rebalanced portfolio
MCR	Mixture-current-run

CHAPTER ONE

INTRODUCTION

Investment decision making using universal portfolios adopts the approach where the investors need not depend on the stochastic model underlying the true distribution of the stock prices. The uniform universal portfolio introduced by Cover [7] has been shown empirically that it can achieve a wealth growth rate close to that of the optimal wealth in an empirical study which includes selected stocks from the New York Stock Exchange for a period of 22 years. Subsequently, it is generalized to the class of Dirichlet-weighted universal portfolios by Cover and Ordentlich [9]. Since the implementation of these universal portfolios requires a large amount of computer memory, it is not practical to run such an algorithm. It is known that the Dirichlet-weighted universal portfolio cannot achieve a higher wealth than that of the best constant rebalanced portfolio (BCRP). It is important to have an effective universal portfolio for trust fund managers to manage the clients' wealths. The universal portfolio should perform well in the long run riding out financial crises or economic downturns in the investment period. This research mainly focusses on the multiplicative-update and additive-update universal portfolios which require much lesser memory requirements in their implementation. The thesis concludes with a study on a mixture of different universal portfolios. The performance of the universal portfolios is studied by running these universal portfolios on some selected stock-price data sets from the Kuala Lumpur Stock Exchange. Four of the data sets are from the period 2 January 2003 until 30 December 2010 which covers the global financial crisis of 2008.

This thesis consists of six chapters. An introduction is given in the first chapter which states the objectives of the research. Then a literature review on the area of universal portfolios followed by the definitions used in the thesis are given. In Chapter Two, a multiplicative-update universal portfolio namely, the Helmbold universal portfolio where the multiplicative scalar in the power of the update-exponential function serves as a parameter is studied. We show that it is unnecessary to restrict the values of this multiplicative scalar or learning parameter to small positive values. In fact higher investment returns can be obtained by using large positive or negative values of this learning parameter. The initial starting portfolio may also be regarded as a parameter affecting the performance of the Helmbold universal portfolio. We present a detailed study of the dependence of the wealth achieved on the initial starting portfolio. We derive an expression for the portfolio on any day depending on the initial starting portfolio. By changing the initial starting portfolio, it may be possible to achieve higher investment wealths. We obtain the Type II Helmbold universal portfolio by using a second-order logarithmic approximation in the objective functions to be maximized and minimized. An algorithm to solve the set of non-linear equations associated with a Type II Helmbold universal portfolio is presented. The performance of the Helmbold and Type II Helmbold universal portfolios are compared by running both universal portfolios on some data sets selected from the local stock exchange.

We note that the Helmbold universal portfolio is obtained by maximizing and minimizing a certain objective functions involving the Kullback-Leibler information measure. In Chapter Three, we propose to generate a family of universal portfolios by maximizing and minimizing the same objective functions using the chi-square divergence (CSD) distance measure. A range of valid parametric values of the additive-update CSD universal portfolio is derived for the selection of a valid parameter. The CSD universal portfolio is run on some real stock data taken from the local stock exchange. The performance of this family of universal portfolios is compared with that of the Helmbold universal portfolios. We derive a larger family of additive-update universal portfolios generated by the Mahalanobis squared divergence in Chapter Four. This family of universal portfolios includes the CSD universal portfolios as a subclass which is studied in Chapter Three. The family of universal portfolios generated by the Mahalanobis squared divergence are associated with symmetric, positive definite matrices. The explicit formulae for the Mahalanobis universal portfolios associated with some special symmetric matrices are derived. A sufficient bound for valid parametric values of the Mahalanobis universal portfolio is obtained. The sufficient bound is practical if the generating matrix is chosen to be a special diagonal matrix. A sufficient condition for the Mahalanobis universal portfolio to achieve a wealth higher than that of the BCRP is derived. An analogous result holds for the Helmbold parametric family of universal portfolios. In order to keep the generated portfolio vectors within the valid range, we modify the portfolio components using translation and normalization whenever a component becomes negative. The modified Mahalanobis universal portfolio ensures that the generated portfolio vectors are genuine portfolio vectors for any real number parameter. These modified portfolios based on any scalar parameter are run on some selected stock-price data sets from the local stock exchange to evaluate their performance.

In Chapter Five, we propose to consider a “Markovian” type Dirichlet universal portfolio. We note that the Cover-Ordentlich Dirichlet-weighted universal portfolio is obtained by weighting the current portfolio components by the accumulated constant rebalanced portfolio wealth return with respect to the Dirichlet probability measure. A family of Dirichlet universal portfolios of order one is derived using a similar weighting procedure where the accumulated constant rebalanced portfolio wealth return is replaced by the latest one-day wealth return. The Dirichlet universal portfolio of order one is run on some data sets selected from the local stock exchange and the dependence of the wealth return on the initial starting portfolio is studied. We identify the relationship between the Dirichlet universal portfolio of order one and the CSD universal portfolio in the last section of Chapter Five. In the last chapter, the problem of mixing two or more universal portfolios with the aim of achieving a higher wealth return is studied. We introduce the mixture-current-run (MCR) universal portfolio which follows the current run of the portfolio that achieves the best single-day wealth return. When the current run changes to a different run, the investment portfolio changes accordingly. An upper bound on the wealth achievable in the MCR universal portfolio is derived and we also estimate the probability of achieving this upper bound. An application of MCR universal portfolio is discussed in the last section in Chapter Six.

1.1 Literature Review

A portfolio is an investment strategy that can reduce the risk of investment by using diversification of assets. It refers to investing in any combination of financial assets which has a lower risk than investing in an individual asset. Besides, it can be shown portfolio investment may give a higher wealth return. In this research, a portfolio is a vector of the proportions of the investment wealth distributed among the stocks invested in a market. Portfolio theory was first developed mathematically by Markowitz [17]. Markowitz treated the portfolio problem as a choice of the mean and variance of a portfolio, that is holding constant variance and maximizing the mean as well as holding constant mean and minimizing the variance. This led to the efficient frontier where the investor could choose his preferred portfolio depending on his risk preference. Sharpe [20] extends Markowitz's work on the portfolio analysis. A simplified model of the relationships among securities for practical applications of the Markowitz portfolio analysis technique is provided by Sharpe.

The theory of rebalanced portfolios for known underlying distributions was introduced by Kelly [15]. Kelly showed that the growth rate of wealth can be maximized by the log-optimal investment where the gambler reinvests his cumulative wealth based on the knowledge given by the received symbols.

This theory was extended to investment in independent and identically distributed markets by Breiman [5]. Mossin [19] extended the one-period portfolio analysis to a optimal portfolio management over several periods. Thorp [34] studied the uses of logarithmic utility over the portfolio selection. A study of "maximum-expected-log" rule against the efficient frontier is given by Markowitz [18]. Bell and Cover [4] showed that the Kelly criterion has good short run, a Kelly investor has at least half a chance of outperforming any other gambler after just one trial. Finkelstein and Whitley [10] showed that the Kelly investor is always ahead of any other gambler on average after any fixed number of bets. An algorithm for maximizing the expected log investment return is presented by Cover [6]. Barron and Cover [3] showed that the increase in exponential growth of wealth is achieved for special extreme case with side information. Algoet and Cover [2] proved that maximizing conditionally expected log return is asymptotically optimal for the market with no restrictions on the distribution. A constant rebalanced portfolio allocates the same proportions of wealth among the available stocks on every day. It is known that the optimal growth rate of wealth is achieved by a constant rebalanced portfolio if the price-relatives are independent and identically distributed. The wealth achieved by the best constant rebalanced portfolio (BCRP) is expected to grow exponentially with a rate determined by stock's volatility.

An investment portfolio is universal if it can be used in a market where no probabilistic model is assumed for the stock prices. It is useful for the investor who only has limited knowledge of the true distribution underlying the market. Cover and Gluss [8] restricted the price-relatives to a finite set and used the Blackwell's approach-exclusion theorem and compound sequential Bayes decision rules to define an investment scheme with universal properties. Subsequently, Cover [7] introduced the uniform universal portfolio and used the Laplace's method of integration to show that the uniform universal portfolio performs asymptotically as well as the BCRP. Cover and his research associates tested the uniform universal portfolio experimentally on some stock data sets from the New York Stock Exchange covering a period of 22 years trading and it is possible to increase the wealth by a large margin. Jamshidian [14] extended the Cover's work to the continuous time framework. The uniform universal portfolio is generalized to the class of Dirichlet-weighted universal portfolios by Cover and Ordentlich [9]. In the same paper, Cover and Ordentlich [9] introduced the notion of side information and focussed the studies on the wealth achievable by the uniform and Dirichlet-weighted $(1/2, 1/2, \dots, 1/2)$ universal portfolios. The authors also derived the theoretical performance bounds of the two special Dirichlet-weighted universal portfolios. Ishijima [13] showed that the Dirichlet-weighted universal portfolios coincide with the optimal Bayes portfolio under the continuous time framework without hindsight. The performance bounds are extended to the general class of Dirichlet-weighted universal portfolios by Tan [21, 22] for any parametric vector $(\alpha_1, \alpha_2, \dots, \alpha_m)$. Gaivoronski and Stella [11] used the nonstationary optimization to construct the Dirichlet-weighted universal portfolios, that it maximizes the expected log cumulative wealth estimated using all historical price relative relatives. Agarwal, Hazan, Kale and Schapire [1] extended the Gaivoronski and Stella's idea by appending a regularization term to minimize the variation of next portfolio.

A universal portfolio requiring a much lesser computation time and memory requirement for its implementation was introduced by Helmbold, Schapire, Singer and Warmuth [12]. The authors used the exponentiated gradient update algorithm that was developed by Kivinen and Warmuth [16] to generate the multiplicative-update universal portfolio. The Helmbold universal portfolio is shown to be outperforming the uniform universal portfolio based on the same stock data from the New York Stock Exchange in [7]. Helmbold, Schapire, Singer and Warmuth also extended the study on Helmbold universal portfolios to include the presence of additional side information. Tan and Tang [32] showed that the Helmbold universal portfolio is sensitive to the initial starting portfolio and it behaves like a constant rebalanced portfolio if the parameter is restricted to a small positive value. They also showed that there are Dirichlet-weighted universal portfolios that can perform better than the Helmbold universal portfolio empirically.

1.2 Definitions

We discuss some basic definitions in this section by considering investment in a market of m stocks. An m-dimensional vector $\mathbf{b}_n = (b_{ni})$ is said to be a *portfolio vector* if $b_{ni} \geq 0$ for $i = 1,2,\ldots,m$ and $\sum_{i=1}^{m} b_{ni} = 1$. The integer n in the context of this thesis refers to the nth trading day. The component b_{ni} is the proportion of the current wealth of the investor which is invested in the ith stock. We denote the *simplex* of portfolio vectors $\mathbf{b} = (b_i)$ by

$$B = \left\{(b_1, b_2, \dots, b_m): b_i \geq 0 \text{ for } i = 1,2, \dots, m, \sum_{i=1}^{m} b_i = 1\right\}. \tag{1.1}$$

The point $\mathbf{b} = (b_1, b_2, \dots, b_m)$ is a boundary point of B if there exists an index j such that $b_j = 0$, where $1 \leq j \leq m$. Let $\mathbf{x}_n = (x_{ni})$ denotes the *price-relative vector* of the market on the nth trading day, where x_{ni} is the ratio of the closing price of the ith stock to its opening price, for $i = 1,2, \dots, m$. The price-relative x_{ni} describes the performance of the ith stock on the nth trading day where the ith stock increases or decreases by a factor of x_{ni} times its previous value.

The wealth achieved in a single day j is

$$\mathbf{b}_j^t \mathbf{x}_j = \sum_{i=1}^{m} b_{ji} x_{ji} \tag{1.2}$$

for $i = 1,2, \dots, m$. Assuming an initial wealth S_0 of 1 unit and given the sequence of price-relative vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$, the wealth achieved at the end of the nth trading day is given by

$$S_n = \prod_{j=1}^{n} \mathbf{b}_j^t \mathbf{x}_j, \tag{1.3}$$

where $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$ is the sequence of portfolio strategies used by the investor.

A constant rebalanced portfolio investment strategy uses the same portfolio vector $\mathbf{b}$ for each trading day. The buy-and-hold strategy for a single stock is a special case of the constant rebalanced portfolio. The optimal wealth achieved by the BCRP is defined as

$$S_n^* = \max_{\mathbf{b}} S_n(\mathbf{b}) = \max_{\mathbf{b}} \prod_{j=1}^{n} \mathbf{b}^t \mathbf{x}_j, \tag{1.4}$$

We denote the BCRP by $\mathbf{b}_n^*$ where

$$S_n(\mathbf{b}_n^*) = \max_{\mathbf{b}} S_n(\mathbf{b}). \tag{1.5}$$

The goal of our research is to find the universal portfolios that can achieve wealths close to that of the BCRP. In fact, we show empirically in this thesis (for example, Chapter Six) that there are universal portfolios that can achieve wealths exceeding that of the BCRP. This demonstrates the importance of the additive-update Mahalanobis universal portfolio which can achieve a wealth exceeding that of the BCRP. In contrast, the Dirichlet-weighted universal portfolios cannot achieve wealths exceeding that of the BCRP. The Mahalanobis universal portfolios introduced in this thesis provides an alternative class of investment portfolios available to the trust fund managers for investment. Empirically the performance potential of these universal portfolios is demonstrated in this research.

CHAPTER TWO

HELMBOLD UNIVERSAL PORTFOLIO

Helmbold *et al.* [12] proposed a universal portfolio that can be implemented by day-to-day multiplicative-update of the current portfolio which requires very much lesser computer memory requirements growing linearly with the number of stocks invested. It was shown that the Helmbold universal portfolio can perform better than the uniform universal portfolio on some stock-price data sets used in [7].

2.1 Two Parameters of the Helmbold Universal Portfolio

The work reported in this section is published in Tan and Lim [24, 25, 29]. In [12], a multiplicative-update universal portfolio where the multiplicative scalar in the power of the update-exponential function serves as a parameter was introduced. Tan and Tang [32] observed that the initial starting portfolio $\mathbf{b}_1$ can be a factor influencing the performance of the Helmbold universal portfolio. By restricting the parameter η of the Helmbold universal portfolio to the narrow range of $0 < \eta \leq 2\sqrt{\frac{2\log m}{N}} < 1$, the Helmbold universal portfolio behaves like a constant rebalanced portfolio. In this section, we propose to remove the unnecessary restriction on η and demonstrate that higher investment wealths can be obtained by large positive values of η or negative η. A consequence of moving further away from $\eta = 0$

is that the resulting Helmbold universal portfolio no longer behaves like a constant rebalanced portfolio. The emphasis of this section is on the dependence of the Helmbold universal portfolio on the parameter η and to study the dependence on the initial starting portfolio $\mathbf{b}_1$.

The *Kullback-Leibler* distance measure is

$$\mathrm{D}(\mathbf{b}_k \parallel \mathbf{b}_n) = \sum_{i=1}^{m} b_{ki} \log\left(\frac{b_{ki}}{b_{ni}}\right), \tag{2.1}$$

where $\mathbf{b}_k = (b_{ki})$ and $\mathbf{b}_n = (b_{ni})$ are any two portfolio vectors.

The *Helmbold universal portfolio* is a sequence of portfolio vectors $\{\mathbf{b}_{n+1}\}$ generated by the following update of b_{ni}:

$$b_{n+1,i} = \frac{b_{ni} \exp\left(\eta \frac{x_{ni}}{\mathbf{b}_n^t \mathbf{x}_n}\right)}{\sum_{j=1}^{m} b_{nj} \exp\left(\eta \frac{x_{nj}}{\mathbf{b}_n^t \mathbf{x}_n}\right)}, \tag{2.2}$$

for $i = 1,2, \dots, m$, where the constant η (any real number) and the initial starting portfolio $\mathbf{b}_1 = (b_{11}, b_{12}, \dots, b_{1m})$ are given. The Helmbold universal portfolio (2.2) is said to be generated by the parameter η and the initial starting portfolio $\mathbf{b}_1$.

First, we derive an expression for the dependence of $\mathbf{b}_{n+1}$ on $\mathbf{b}_1$ for the Helmbold universal portfolio.

Proposition 2.1 For the Helmbold universal portfolio $\mathbf{b}_{n+1} = \left(b_{n+1,i}\right)$ given by (2.2), we have

$$b_{n+1,i} = \frac{b_{1i} \exp\left(\eta \sum_{k=1}^{n} \frac{x_{ki}}{\mathbf{b}_k^t \mathbf{x}_k}\right)}{\sum_{j=1}^{m} b_{1j} \exp\left(\eta \sum_{k=1}^{n} \frac{x_{kj}}{\mathbf{b}_k^t \mathbf{x}_k}\right)} \tag{2.3}$$

where $\mathbf{b}_1 = (b_{1i})$ is the initial starting portfolio.

Proof. From (2.2), expressing b_{ni} as a function of $b_{n-1,i}$, we have

$$b_{n+1,i} = \frac{b_{n-1,i} \exp\left(\eta \frac{x_{n-1,i}}{\mathbf{b}_{n-1}^t \mathbf{x}_{n-1}}\right) \exp\left(\eta \frac{x_{ni}}{\mathbf{b}_n^t \mathbf{x}_n}\right)}{\left[\sum_{j=1}^{m} b_{n-1,j} \exp\left(\eta \frac{x_{n-1,j}}{\mathbf{b}_{n-1}^t \mathbf{x}_{n-1}}\right)\right]\left[\sum_{j=1}^{m} b_{n,j} \exp\left(\eta \frac{x_{n,j}}{\mathbf{b}_n^t \mathbf{x}_n}\right)\right]}.$$

Now expressing $b_{n-1,i}$ as a function of $b_{n-2,i}$ and continuing in this way until b_{2i} is expressed as a function of b_{1i}, we obtain

$$b_{n+1,i} = \frac{b_{1i} \exp\left(\eta \sum_{k=1}^{n} \frac{x_{ki}}{\mathbf{b}_k^t \mathbf{x}_k}\right)}{\prod_{l=1}^{n}\left[\sum_{j=1}^{m} b_{lj} \exp\left(\eta \frac{x_{lj}}{\mathbf{b}_l^t \mathbf{x}_l}\right)\right]}. \tag{2.4}$$

Summing over i in (2.4) where $\sum_{i=1}^{m} b_{n+1,i} = 1$ and noting that the denominator in (2.4) does not depend on i, we conclude that the denominators in (2.3) and (2.4) are equal. □

We remark that the function $\mathbf{b}_{n+1}$ may not be continuous at a boundary point $\mathbf{b}_1$ of the simplex B since $b_{1i} = 0$ implies that $b_{n+,1i} = 0$ for all $n = 1,2,3, \ldots.$

We introduce the *eta-parametric family of Helmbold universal portfolios* which is defined by (2.2) for any real number η.

Proposition 2.2 Consider the objective functions

$$F(\mathbf{b}_{n+1}) = \eta \log(\mathbf{b}_{n+1}^t \mathbf{x}_n) - \mathrm{D}(\mathbf{b}_{n+1} \parallel \mathbf{b}_n)$$

and

$$G(\mathbf{b}_{n+1}) = \eta \log(\mathbf{b}_{n+1}^t \mathbf{x}_n) + \mathrm{D}(\mathbf{b}_{n+1} \parallel \mathbf{b}_n)$$

where $\mathrm{D}(\mathbf{b}_{n+1} \parallel \mathbf{b}_n)$ is the Kullback-Leibler distance measure or relative entropy given by (2.1) and η is positive. By approximating $\log(\mathbf{b}_{n+1}^t \mathbf{x}_n)$ using $\left[\log(\mathbf{b}_n^t \mathbf{x}_n) + \frac{\mathbf{b}_{n+1}^t \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n} - 1\right]$, the maximum of the objective function $F(\mathbf{b}_{n+1})$ is achieved at $\mathbf{b}_{n+1}$ given by (2.2) and the minimum of $G(\mathbf{b}_{n+1})$ is also achieved at $\mathbf{b}_{n+1}$ given by (2.2) where η is replaced by $-\eta$.

Proof. Since $\mathbf{b}_n$ is a portfolio vector for $n = 1,2,3,\ldots$, we need to intoduce the Lagrange multiplier γ in maximizing the objective function

$$\hat{F}(\mathbf{b}_{n+1}, \gamma) = \eta \left[\log(\mathbf{b}_n^t \mathbf{x}_n) + \frac{\mathbf{b}_{n+1}^t \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n} - 1\right] - \sum_{i=1}^{m} b_{n+1,i} \log\left(\frac{b_{n+1,i}}{b_{ni}}\right)$$

$$+ \gamma \left(\sum_{i=1}^{m} b_{n+1,i} - 1\right)$$

and minimizing the objective function

$$\hat{G}(\mathbf{b}_{n+1}, \gamma) = \eta \left[\log(\mathbf{b}_n^t \mathbf{x}_n) + \frac{\mathbf{b}_{n+1}^t \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n} - 1\right] + \sum_{i=1}^{m} b_{n+1,i} \log\left(\frac{b_{n+1,i}}{b_{ni}}\right)$$

$$+ \gamma \left(\sum_{i=1}^{m} b_{n+1,i} - 1\right).$$

Helmbold *et al.* [12] have shown that the maximum of $\hat{F}(\mathbf{b}_{n+1}, \gamma)$ is achieved at $\mathbf{b}_{n+1}$ given by (2.2). The minimum of $\hat{G}(\mathbf{b}_{n+1}, \gamma)$ is achieved when the following m partial derivatives are zero,

$$\frac{\partial \hat{G}(\mathbf{b}_{n+1}, \gamma)}{\partial b_{n+1,i}} = \eta \frac{x_{ni}}{\mathbf{b}_n^t \mathbf{x}_n} + \left[\log\left(\frac{b_{n+1,i}}{b_{ni}}\right) + 1\right] + \gamma = 0$$

for $i = 1,2, \ldots, m$. We obtain

$$b_{n+1,i} = b_{ni} \exp\left(-\eta \frac{x_{ni}}{\mathbf{b}_n^t \mathbf{x}_n}\right) \exp(-\gamma - 1)$$

for $i = 1,2, \ldots, m$. Summing up the components $b_{n+1,i}$ over i, we have

$$1 = exp(-\gamma - 1) \sum_{i=1}^{m} b_{ni}\, exp\left(-\eta \frac{x_{ni}}{\mathbf{b}_n^t \mathbf{x}_n}\right)$$

leading to

$$b_{n+1,i} = \frac{b_{ni} \exp(-\eta \frac{x_{ni}}{\mathbf{b}_n^t \mathbf{x}_n})}{\sum_{j=1}^{m} b_{nj} \exp(-\eta \frac{x_{nj}}{\mathbf{b}_n^t \mathbf{x}_n})}$$

for $i = 1,2, \ldots, m$. Let

$$\begin{aligned} &\tilde{G}\left(b_{n+1,1}, b_{n+1,2}, \ldots, b_{n+1,m-1}\right) \\ &\qquad = \eta \left[\log(\mathbf{b}_n^t \mathbf{x}_n) + \frac{\mathbf{b}_{n+1}^t \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n} - 1\right] \\ &\qquad\qquad + \sum_{i=1}^{m} b_{n+1,i} \log\left(\frac{b_{n+1,i}}{b_{ni}}\right) \end{aligned} \tag{2.5}$$

where $b_{n+1,m} = 1 - \sum_{i=1}^{m-1} b_{n+1,i}$. Then the first partial derivatives of (2.5) are

$$\frac{\partial \tilde{G}\left(b_{n+1,1}, b_{n+1,2}, \ldots, b_{n+1,m-1}\right)}{\partial b_{n+1,i}} = \eta \left[\frac{x_{ni} - x_{nm}}{\mathbf{b}_n^t \mathbf{x}_n}\right] + \log\left(\frac{b_{nm} b_{n+1,i}}{b_{ni} b_{n+1,m}}\right)$$

for $i = 1,2, \ldots, m-1$ and the second partial derivatives of (2.5) are

$$\frac{\partial^2 \tilde{G}\left(b_{n+1,1}, b_{n+1,2}, \ldots, b_{n+1,m-1}\right)}{\partial b_{n+1,i} \partial b_{n+1,j}} = \begin{cases} \dfrac{1}{b_{n+1,i}} + \dfrac{1}{b_{n+1,m}} & \text{for } i = j \\ \dfrac{1}{b_{n+1,m}} & \text{for } i \neq j \end{cases}$$

for $i, j = 1,2, \ldots, m-1$. The Hessian matrix of $\tilde{G}\left(b_{n+1,1}, b_{n+1,2}, \ldots, b_{n+1,m-1}\right)$ is

$$H = \frac{\partial^2 \tilde{G}\left(b_{n+1,1}, b_{n+1,2}, \ldots, b_{n+1,m-1}\right)}{\partial b_{n+1,i} \partial b_{n+1,j}}$$

where

$$h_{ij} = \begin{cases} \phi_i + \phi_m & \text{for } i = j \\ \phi_m & \text{for } i \neq j \end{cases}$$

for $i, j = 1,2, \dots, m-1$ and

$$\phi_k = \frac{1}{b_{n+1,k}}$$

for $k = 1,2, \dots, m$. Let $J_k = \left(J_{ij}\right)$ be the $k \times k$ sub-matrix of H where

$$J_{ij} = \begin{cases} \phi_i + \phi_m & \text{for } i = j \\ \phi_m & \text{for } i \neq j \end{cases}$$

for $k = 1,2, \dots, m-1$. If $b_{n+1,i} > 0$ for $i = 1,2, \dots, m$, then $\phi_i > 0$ for $i = 1,2, \dots, m$. A simple evaluation of the determinant of $J_k = \left(J_{ij}\right)$ shows that

$$|J_k| = \sum_{i=1}^{k-1} (\phi_1 \phi_2 \dots \phi_i \phi_{i+2} \phi_{i+3} \dots \phi_k \phi_m) + \phi_1 \phi_2 \dots \phi_k + \phi_2 \phi_3 \dots \phi_k \phi_m$$

for $k = 1,2, \dots, m-1$ and ϕ_{k+1} is defined to be ϕ_m for a fixed k. Now $\phi_i > 0$ for $i = 1,2, \dots, k$ and $\phi_m > 0$ implies that $|J_k| > 0$ for $k = 1,2, \dots, m-1$. In other words, the principal minors $|J_k|$ of H are all positive. Hence H, the Hessian matrix of $\tilde{G}\left(b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m-1}\right)$ is positive definite. Similarly, if

$$\tilde{F}\left(b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m-1}\right)$$
$$= \eta \left[\log(\mathbf{b}_n^t \mathbf{x}_n) + \frac{\mathbf{b}_{n+1}^t \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n} - 1\right] - \sum_{i=1}^{m} b_{n+1,i} \log\left(\frac{b_{n+1,i}}{b_{ni}}\right)$$

where $b_{n+1,m} = 1 - \sum_{i=1}^{m-1} b_{n+1,i}$, then the Hessian matrix of $\tilde{F}\left(b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m-1}\right)$ is $-H$ which is negative definite. Hence $\tilde{G}\left(b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m-1}\right)$ has a minimum point and $\tilde{F}\left(b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m-1}\right)$ has a maximum point. Furthermore,

$\tilde{F}(b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m-1})$ is concave and $\tilde{G}(b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m-1})$ is convex in the simplex B defined in (1.1). □

We have shown that the eta-parametric family of Helmbold universal portfolios is generated by maximizing and minimizing two different objective functions $F(\mathbf{b}_{n+1})$ and $G(\mathbf{b}_{n+1})$ for $\eta > 0$ and $\eta < 0$ respectively. The objective functions want the current portfolio $\mathbf{b}_{n+1}$ to be close to the previous portfolio $\mathbf{b}_n$ in terms of Kullback-Leibler distance measure. Tan and Tang [32] have shown that the initial starting portfolio $\mathbf{b}_1$ is a parameter that can affect the final wealth achievable by the Helmbold universal portfolio. If the initial starting portfolio is a good one, we require that the subsequent portfolios are close to each other. On the other hand, if the initial starting portfolio is not a good one, we hope to move away from the current portfolio to the right one with the highest investment wealth.

We have run the eta-parametric family of Helmbold universal portfolio on three stock data sets chosen from the Kuala Lumpur Stock Exchange. The period of trading of the stocks selected is from 1 January 2003 until 30 November 2004, consisting of 500 trading days. Each data set consists of three company stocks. Set A consists of the stocks of Malayan Banking, Genting and Amway (M) Holdings. Set B consists of the stocks of Public Bank, Sunrise and YTL Corporation. Finally, set C consists of the stocks of Hong Leong Bank, RHB Capital, and YTL Corporation.

We begin with the initial starting portfolio $\mathbf{b}_1 = (0.3333, 0.3333, 0.3334)$ for all the three data sets. For each data set, the portfolios $\mathbf{b}_{501}$ and the wealths S_{500} achieved after 500 trading days are calculated for selected values of η and are listed in Tables 2.1, 2.2 and 2.3.

Table 2.1: The portfolios $\mathbf{b}_{501}$ and the wealths S_{500} achieved by the Helmbold universal portfolio for selected values of η for data set A, where $\mathbf{b}_1 = (0.3333, 0.3333, 0.3334)$

η	$\mathbf{b}_{501}$	S_{500}
-10.00	(0.0053, 0.9789, 0.0158)	1.4310
-5.00	(0.0678, 0.8140, 0.1182)	1.5449
-3.00	(0.1509, 0.6379, 0.2111)	1.5725
-1.00	(0.2697, 0.4283, 0.3019)	1.5722
-0.75	(0.2857, 0.4035, 0.3109)	1.5707
-0.50	(0.3016, 0.3793, 0.3191)	1.5689
-0.30	(0.3143, 0.3605, 0.3252)	1.5674
-0.20	(0.3207, 0.3513, 0.3281)	1.5666
-0.10	(0.3270, 0.3422, 0.3308)	1.5658
0	(0.3333, 0.3333, 0.3334)	1.5650
0.10	(0.3396, 0.3245, 0.3359)	1.5642
0.20	(0.3458, 0.3159, 0.3382)	1.5633
0.30	(0.3521, 0.3074, 0.3405)	1.5624
0.50	(0.3645, 0.2910, 0.3446)	1.5607
0.75	(0.3797, 0.2712, 0.3490)	1.5585
1.00	(0.3948, 0.2525, 0.3527)	1.5563
3.00	(0.5043, 0.1366, 0.3591)	1.5399
5.00	(0.5935, 0.0700, 0.3365)	1.5266
10.00	(0.7485, 0.0115, 0.2401)	1.4996

Table 2.2: The portfolios $\mathbf{b}_{501}$ and the wealths S_{500} achieved by the Helmbold universal portfolio for selected values of η for data set B, where $\mathbf{b}_1 = (0.3333, 0.3333, 0.3334)$

η	$\mathbf{b}_{501}$	S_{500}
-10.00	(0.8507, 0.1493, 0.0000)	1.8141
-5.00	(0.6872, 0.3109, 0.0018)	1.8110
-3.00	(0.6011, 0.3813, 0.0176)	1.8399
-1.00	(0.4598, 0.3965, 0.1437)	1.9866
-0.75	(0.4323, 0.3868, 0.1810)	2.0223
-0.50	(0.4018, 0.3730, 0.2252)	2.0630
-0.30	(0.3755, 0.3591, 0.2655)	2.0994
-0.20	(0.3617, 0.3511, 0.2872)	2.1188
-0.10	(0.3477, 0.3425, 0.3098)	2.1390
0	(0.3333, 0.3333, 0.3334)	2.1600
0.10	(0.3187, 0.3235, 0.3578)	2.1818

Table 2.2 continued

η	$\mathbf{b}_{501}$	S_{500}
0.20	(0.3040, 0.3132, 0.3828)	2.2042
0.30	(0.2891, 0.3025, 0.4084)	2.2274
0.50	(0.2595, 0.2798, 0.4607)	2.2754
0.75	(0.2232, 0.2501, 0.5267)	2.3382
1.00	(0.1888, 0.2200, 0.5912)	2.4031
3.00	(0.0317, 0.0517, 0.9165)	2.8851
5.00	(0.0039, 0.0091, 0.9871)	3.1899
10.00	(0.0000, 0.0001, 0.9999)	3.5140

Table 2.3: The portfolios $\mathbf{b}_{501}$ and the wealths S_{500} achieved by the Helmbold universal portfolio for selected values of η for data set C, where $\mathbf{b}_1 = (0.3333{,}0.3333{,}0.3334)$

η	$\mathbf{b}_{501}$	S_{500}
-10.00	(0.0232, 0.9768, 0.0000)	1.3126
-5.00	(0.1379, 0.8617, 0.0004)	1.3350
-3.00	(0.2517, 0.7406, 0.0078)	1.3898
-1.00	(0.3657, 0.5190, 0.1153)	1.5868
-0.75	(0.3676, 0.4775, 0.1549)	1.6359
-0.50	(0.3633, 0.4322, 0.2044)	1.6933
-0.30	(0.3548, 0.3937, 0.2515)	1.7454
-0.20	(0.3488, 0.3738, 0.2774)	1.7736
-0.10	(0.3416, 0.3536, 0.3047)	1.8033
0	(0.3333, 0.3333, 0.3334)	1.8343
0.10	(0.3239, 0.3129, 0.3632)	1.8666
0.20	(0.3134, 0.2925, 0.3940)	1.9002
0.30	(0.3020, 0.2724, 0.4256)	1.9350
0.50	(0.2769, 0.2333, 0.4898)	2.0077
0.75	(0.2428, 0.1877, 0.5695)	2.1032
1.00	(0.2076, 0.1475, 0.6449)	2.2017
3.00	(0.0338, 0.0122, 0.9540)	2.8736
5.00	(0.0040, 0.0007, 0.9953)	3.2021
10.00	(0.0000, 0.0000, 1.0000)	3.4416

Let $\|\mathbf{x}\|_1 = \sum_{i=1}^{m}|x_i|$ and $\|\mathbf{x}\|_2 = (\sum_{i=1}^{m} x_i^2)^{1/2}$ denote the ℓ_1 norm and ℓ_2 norm of the vector $\mathbf{x}$ respectively. It is clear from Tables 2.1, 2.2 and 2.3 that $\|\mathbf{b}_{501} - \mathbf{b}_1\|_1$ and $\|\mathbf{b}_{501} - \mathbf{b}_1\|_2$ as functions of η are growing with η as $|\eta|$ gets larger. If η is restricted between 0 and $2\sqrt{\frac{2\log m}{N}}$ (or 0.1326 for $m = 3$, $N = 500$) as recommended in [12], $\|\mathbf{b}_{501} - \mathbf{b}_1\|_1$ and $\|\mathbf{b}_{501} - \mathbf{b}_1\|_2$ are close to 0. If a larger variation of $\mathbf{b}_{501}$ from $\mathbf{b}_1$ is required, it can be achieved by using a larger $|\eta|$ universal portfolio.

In Figures 2.1, 2.2 and 2.3, the graphs of the wealth S_{500} against η are plotted for the data sets A, B and C respectively, where the local maxima are shown. We strongly believe that the local maxima are also the global maxima over all η. For data set A, the maximum wealth achievable is $S_{500}(max) = 1.5755$ at $\eta = -2.0714$. Here is an example of a Helmbold universal portfolio with a negative-valued parameter achieving the maximum wealth. For data sets B and C, the maximum wealth achievable are $S_{500}(max) = 3.6467$ and $S_{500}(max) = 3.5066$ at $\eta = 30.1449$ and $\eta = 115.7115$ respectively. Again, this demonstrates that if η is restricted between 0 and 0.1326 as recommended in [12], it is not possible to achieve the maximum wealth.

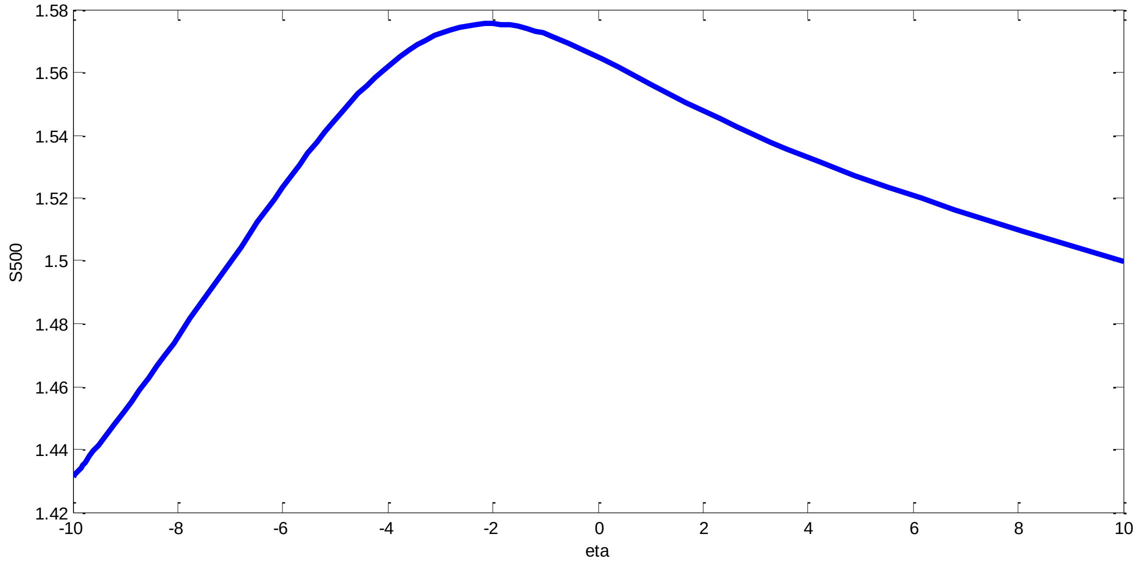

Figure 2.1: Graph of S_{500} against η displaying the local maximum at $\eta = -2.0714$ for data set A, where $\mathbf{b}_1 = (0.3333, 0.3333, 0.3334)$ (Helmbold universal portfolio)

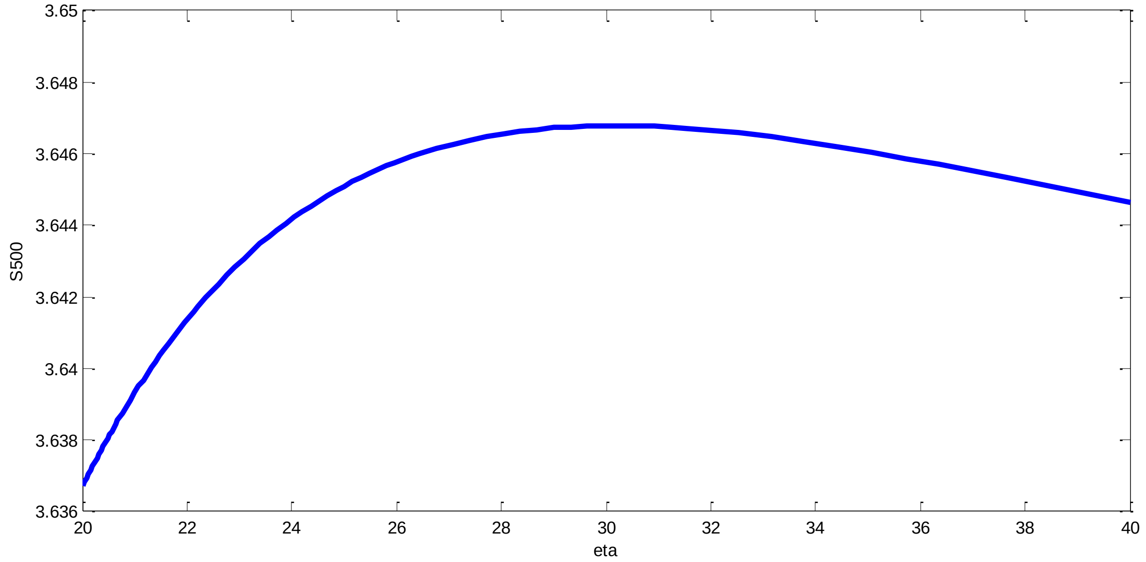

Figure 2.2: Graph of S_{500} against η displaying the local maximum at $\eta = \mathbf{30.1449}$ for data set B, where $\mathbf{b}_1 = (0.3333, 0.3333, 0.3334)$ (Helmbold universal portfolio)

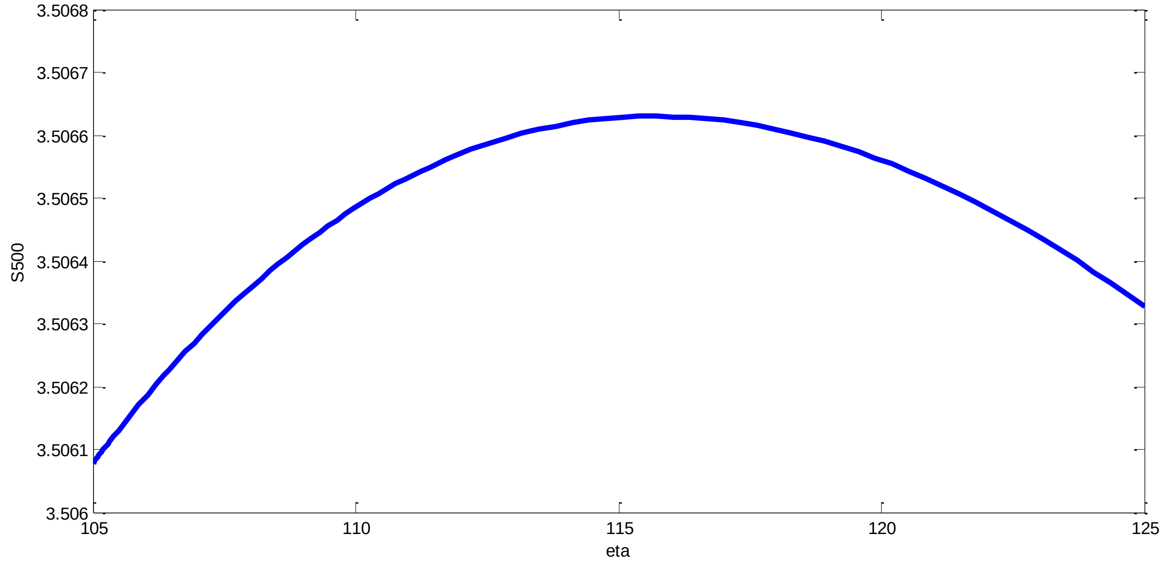

Figure 2.3: Graph of S_{500} against η displaying the local maximum at $\eta = \mathbf{115.7115}$ for data set C, where $\mathbf{b}_1 = (0.3333, 0.3333, 0.3334)$ (Helmbold universal portfolio)

Hence, it is necessary to remove the unnecessary restriction $0 < \eta \leq 2\sqrt{\frac{2\log m}{N}}$ imposed on η in order to achieve a higher investment wealth. Empirical evidence is provided that the maximum investment wealth can be achieved at a negative learning parameter η and at large positive learning parameters. The best η achieving the maximum wealth can be determined from hindsight given the past stock data.

We now study the dependence of the Helmbold universal portfolio on the initial starting portfolio $\mathbf{b}_1$. For data sets A, B and C, the portfolios $\mathbf{b}_{501}$ and the wealths S_{500} achieved after 500 trading days for selected values of $\mathbf{b}_1$ are calculated and displayed in Tables 2.4, 2.5 and 2.6. If $\mathbf{b}_1 = (0.3333, 0.3333, 0.3334)$ is used, then the maximum wealths $S_{500}(max)$ obtained for data sets A, B and C are $1.5755, 3.6467$ and 3.5066 respectively corresponding to the respective $\eta = -2.0714, 30.1449$ and 115.7115 from Figures 2.1, 2,2 and 2.3. From Table 2.4, we observe that by changing $\mathbf{b}_1$ to $(1.0000, 0.0000, 0.0000)$, we can obtain a higher wealth $S_{500} = 1.8534$ compared to 1.5755. Even using $\mathbf{b}_1 = (0.7000, 0.1500, 0.1500)$, we obtain a better $S_{500} = 1.7447$ compared to 1.5755. From Table 2.5, changing $\mathbf{b}_1$ to $(0.2000, 0.2000, 0.6000)$ and $(0.0000, 0.0000, 1.0000)$, we can obtain higher wealths of $S_{500} = 3.8775$ and $S_{500} = 4.2970$ respectively compared to $S_{500} = 3.6467$ for $\mathbf{b}_1 = (0.3333, 0.3333, 0.3334)$. Again, changing $\mathbf{b}_1$ to $(0.2500, 0.2500, 0.5000)$ and $(0.0000, 0.0000, 1.0000)$ for data set C in Table 2.6, higher wealths of $S_{500} = 3.5296$ and $S_{500} = 4.2970$ respectively are obtained which are better than $S_{500} = 3.5066$ for $\mathbf{b}_1 = (0.3333, 0.3333, 0.3334)$.

Table 2.4: The portfolios $\mathbf{b}_{501}$ and the wealths S_{500} achieved by the Helmbold universal portfolio for selected values of $\mathbf{b}_1$ for data set A, where $\eta = -2.0714$

$\mathbf{b}_1$	$\mathbf{b}_{501}$	S_{500}
(0.0000, 0.5000, 0.5000)	(0.0000, 0.6849, 0.3151)	1.4254
(0.1000, 0.4500, 0.4500)	(0.0516, 0.6478, 0.3006)	1.4689
(0.2000, 0.4000, 0.4000)	(0.1107, 0.6057, 0.2836)	1.5138
(0.3000, 0.3500, 0.3500)	(0.1785, 0.5578, 0.2637)	1.5599
(0.4000, 0.3000, 0.3000)	(0.2563, 0.5033, 0.2404)	1.6068
(0.5000, 0.2500, 0.2500)	(0.3454, 0.4414, 0.2132)	1.6538
(0.6000, 0.2000, 0.2000)	(0.4472, 0.3714, 0.1814)	1.7001
(0.7000, 0.1500, 0.1500)	(0.5629, 0.2926, 0.1446)	1.7447
(0.8000, 0.1000, 0.1000)	(0.6934, 0.2044, 0.1023)	1.7862
(0.9000, 0.0500, 0.0500)	(0.8392, 0.1067, 0.0541)	1.8230
(1.0000, 0.0000, 0.0000)	(1.0000, 0.0000, 0.0000)	1.8534
(0.5000, 0.0000, 0.5000)	(0.4415, 0.0000, 0.5585)	1.8142
(0.4500, 0.1000, 0.4500)	(0.3565, 0.1930, 0.4505)	1.7501
(0.4000, 0.2000, 0.4000)	(0.2832, 0.3592, 0.3576)	1.6770
(0.3500, 0.3000, 0.3500)	(0.2215, 0.4991, 0.2794)	1.6008
(0.3000, 0.4000, 0.3000)	(0.1701, 0.6155, 0.2144)	1.5256
(0.2500, 0.5000, 0.2500)	(0.1274, 0.7120, 0.1606)	1.4537
(0.2000, 0.6000, 0.2000)	(0.0921, 0.7919, 0.1160)	1.3862
(0.1500, 0.7000, 0.1500)	(0.0627, 0.8583, 0.0790)	1.3235
(0.1000, 0.8000, 0.1000)	(0.0382, 0.9138, 0.0480)	1.2657
(0.0500, 0.9000, 0.0500)	(0.0175, 0.9605, 0.0220)	1.2125
(0.0000, 1.0000, 0.0000)	(0.0000, 1.0000, 0.0000)	1.1637
(0.5000, 0.5000, 0.0000)	(0.2673, 0.7327, 0.0000)	1.4702
(0.4500, 0.4500, 0.1000)	(0.2506, 0.6811, 0.0683)	1.5030
(0.4000, 0.4000, 0.2000)	(0.2318, 0.6244, 0.1438)	1.5351
(0.3500, 0.3500, 0.3000)	(0.2108, 0.5623, 0.2269)	1.5657
(0.3000, 0.3000, 0.4000)	(0.1874, 0.4947, 0.3179)	1.5941
(0.2500, 0.2500, 0.5000)	(0.1615, 0.4217, 0.4169)	1.6193
(0.2000, 0.2000, 0.6000)	(0.1331, 0.3436, 0.5233)	1.6400
(0.1500, 0.1500, 0.7000)	(0.1024, 0.2611, 0.6365)	1.6551
(0.1000, 0.1000, 0.8000)	(0.0696, 0.1753, 0.7551)	1.6632
(0.0500, 0.0500, 0.9000)	(0.0353, 0.0877, 0.8770)	1.6632
(0.0000, 0.0000, 1.0000)	(0.0000, 0.0000, 1.0000)	1.6539

Table 2.5: The portfolios $\mathbf{b}_{501}$ and the wealths S_{500} achieved by the Helmbold universal portfolio for selected values of $\mathbf{b}_1$ for data set B, where $\eta = 30.1449$

$\mathbf{b}_1$	$\mathbf{b}_{501}$	S_{500}
(0.0000, 0.5000, 0.5000)	(0.0000, 0.0000, 1.0000)	3.7560
(0.1000, 0.4500, 0.4500)	(0.0000, 0.0000, 1.0000)	3.7153
(0.2000, 0.4000, 0.4000)	(0.0000, 0.0000, 1.0000)	3.6825
(0.3000, 0.3500, 0.3500)	(0.0000, 0.0000, 1.0000)	3.6550
(0.4000, 0.3000, 0.3000)	(0.0000, 0.0000, 1.0000)	3.6311
(0.5000, 0.2500, 0.2500)	(0.0000, 0.0000, 1.0000)	3.6095
(0.6000, 0.2000, 0.2000)	(0.0000, 0.0000, 1.0000)	3.5888
(0.7000, 0.1500, 0.1500)	(0.0000, 0.0000, 1.0000)	3.5672
(0.8000, 0.1000, 0.1000)	(0.0000, 0.0000, 1.0000)	3.5408
(0.9000, 0.0500, 0.0500)	(0.0000, 0.0000, 1.0000)	3.4932
(1.0000, 0.0000, 0.0000)	(1.0000, 0.0000, 0.0000)	1.3677
(0.5000, 0.0000, 0.5000)	(0.0000, 0.0000, 1.0000)	4.0243
(0.4500, 0.1000, 0.4500)	(0.0000, 0.0000, 1.0000)	3.8556
(0.4000, 0.2000, 0.4000)	(0.0000, 0.0000, 1.0000)	3.7510
(0.3500, 0.3000, 0.3500)	(0.0000, 0.0000, 1.0000)	3.6704
(0.3000, 0.4000, 0.3000)	(0.0000, 0.0000, 1.0000)	3.6026
(0.2500, 0.5000, 0.2500)	(0.0000, 0.0000, 1.0000)	3.5428
(0.2000, 0.6000, 0.2000)	(0.0000, 0.0000, 1.0000)	3.4886
(0.1500, 0.7000, 0.1500)	(0.0000, 0.0000, 1.0000)	3.4385
(0.1000, 0.8000, 0.1000)	(0.0000, 0.0000, 1.0000)	3.3918
(0.0500, 0.9000, 0.0500)	(0.0000, 0.0000, 1.0000)	3.3472
(0.0000, 1.0000, 0.0000)	(0.0000, 1.0000, 0.0000)	1.5570
(0.5000, 0.5000, 0.0000)	(0.0817, 0.9183, 0.0000)	1.1536
(0.4500, 0.4500, 0.1000)	(0.0000, 0.0000, 1.0000)	3.3677
(0.4000, 0.4000, 0.2000)	(0.0000, 0.0000, 1.0000)	3.5083
(0.3500, 0.3500, 0.3000)	(0.0000, 0.0000, 1.0000)	3.6149
(0.3000, 0.3000, 0.4000)	(0.0000, 0.0000, 1.0000)	3.7072
(0.2500, 0.2500, 0.5000)	(0.0000, 0.0000, 1.0000)	3.7932
(0.2000, 0.2000, 0.6000)	(0.0000, 0.0000, 1.0000)	3.8775
(0.1500, 0.1500, 0.7000)	(0.0000, 0.0000, 1.0000)	3.9639
(0.1000, 0.1000, 0.8000)	(0.0000, 0.0000, 1.0000)	4.0569
(0.0500, 0.0500, 0.9000)	(0.0000, 0.0000, 1.0000)	4.1633
(0.0000, 0.0000, 1.0000)	(0.0000, 0.0000, 1.0000)	4.2970

Table 2.6: The portfolios $\mathbf{b}_{501}$ and the wealths S_{500} achieved by the Helmbold universal portfolio for selected values of $\mathbf{b}_1$ for data set C, where $\eta = 115.7115$

$\mathbf{b}_1$	$\mathbf{b}_{501}$	S_{500}
(0.0000, 0.5000, 0.5000)	(0.0000, 0.0000, 1.0000)	3.6639
(0.1000, 0.4500, 0.4500)	(0.0000, 0.0000, 1.0000)	3.5546
(0.2000, 0.4000, 0.4000)	(0.0000, 0.0000, 1.0000)	3.5260
(0.3000, 0.3500, 0.3500)	(0.0000, 0.0000, 1.0000)	3.5107
(0.4000, 0.3000, 0.3000)	(0.0000, 0.0000, 1.0000)	3.4989
(0.5000, 0.2500, 0.2500)	(0.0000, 0.0000, 1.0000)	3.4869
(0.6000, 0.2000, 0.2000)	(0.0000, 0.0000, 1.0000)	3.4724
(0.7000, 0.1500, 0.1500)	(0.0000, 0.0000, 1.0000)	3.4523

Table 2.6 continued

$\mathbf{b}_1$	$\mathbf{b}_{501}$	S_{500}
$(0.8000, 0.1000, 0.1000)$	$(0.0000, 0.0000, 1.0000)$	3.4197
$(0.9000, 0.0500, 0.0500)$	$(0.0000, 0.0000, 1.0000)$	3.3529
$(1.0000, 0.0000, 0.0000)$	$(1.0000, 0.0000, 0.0000)$	1.3664
$(0.5000, 0.0000, 0.5000)$	$(0.0000, 0.0000, 1.0000)$	3.7505
$(0.4500, 0.1000, 0.4500)$	$(0.0000, 0.0000, 1.0000)$	3.4988
$(0.4000, 0.2000, 0.4000)$	$(0.0000, 0.0000, 1.0000)$	3.5069
$(0.3500, 0.3000, 0.3500)$	$(0.0000, 0.0000, 1.0000)$	3.5069
$(0.3000, 0.4000, 0.3000)$	$(0.0000, 0.0000, 1.0000)$	3.5060
$(0.2500, 0.5000, 0.2500)$	$(0.0000, 0.0000, 1.0000)$	3.5056
$(0.2000, 0.6000, 0.2000)$	$(0.0000, 0.0000, 1.0000)$	3.5066
$(0.1500, 0.7000, 0.1500)$	$(0.0000, 0.0000, 1.0000)$	3.5101
$(0.1000, 0.8000, 0.1000)$	$(0.0000, 0.0000, 1.0000)$	3.5180
$(0.0500, 0.9000, 0.0500)$	$(0.0000, 0.0000, 1.0000)$	3.5358
$(0.0000, 1.0000, 0.0000)$	$(0.0000, 1.0000, 0.0000)$	0.9595
$(0.5000, 0.5000, 0.0000)$	$(1.0000, 0.0000, 0.0000)$	1.0740
$(0.4500, 0.4500, 0.1000)$	$(0.0000, 0.0000, 1.0000)$	3.4651
$(0.4000, 0.4000, 0.2000)$	$(0.0000, 0.0000, 1.0000)$	3.4882
$(0.3500, 0.3500, 0.3000)$	$(0.0000, 0.0000, 1.0000)$	3.5023
$(0.3000, 0.3000, 0.4000)$	$(0.0000, 0.0000, 1.0000)$	3.5154
$(0.2500, 0.2500, 0.5000)$	$(0.0000, 0.0000, 1.0000)$	3.5296
$(0.2000, 0.2000, 0.6000)$	$(0.0000, 0.0000, 1.0000)$	3.5465
$(0.1500, 0.1500, 0.7000)$	$(0.0000, 0.0000, 1.0000)$	3.5680
$(0.1000, 0.1000, 0.8000)$	$(0.0000, 0.0000, 1.0000)$	3.5975
$(0.0500, 0.0500, 0.9000)$	$(0.0000, 0.0000, 1.0000)$	3.6401
$(0.0000, 0.0000, 1.0000)$	$(0.0000, 0.0000, 1.0000)$	4.2970

We may also consider the wealth function S_{500} as a function of one component of $\mathbf{b}_1$ with another component fixed at a certain value, say 0.1000. Table 2.7 tabulates the values of the function S_{500} against $\mathbf{b}_{11} = 0.0000, 0.1000, \dots, 0.9000$, $\mathbf{b}_{12} = 0.0000, 0.1000, \dots, 0.9000$ and $\mathbf{b}_{13} = 0.0000, 0.1000, \dots, 0.9000$ for the data set B. In Figures 2.4, 2.5 and 2.6, the corresponding graphs of S_{500} against b_{11}, b_{12}, and b_{13} are plotted. In Figure 2.5, we observe that S_{500} is discontinuous at the boundary point $\mathbf{b}_1 = (0.1000,0.9000,0.0000)$. Similarly, in Figure 2.6, S_{500} is discontinuous at $\mathbf{b}_1 = (0.9000,0.1000,0.0000)$. In contrast, S_{500} is continuous at all points $\mathbf{b}_1$ in Figure 2.4. Again, in Figures 2.5 and 2.6, we obtain higher wealths of $S_{500} = 4.2352$ and $S_{500} = 4.1194$ at $\mathbf{b}_1 = (0.1000,0.0000,0.9000)$ and

$\mathbf{b}_1 = (0.0000,0.1000,0.9000)$ respectively for data set B compared with $S_{500} = 3.6467$ at $\mathbf{b}_1 = (0.3333,0.3333,0.3334)$.

Table 2.7: The portfolios $\mathbf{b}_{501}$ as a function of one component of $\mathbf{b}_1$ with another component fixed at 0.1000 and the wealths S_{500} achieved by the Helmbold universal portfolio for data set B, where $\eta = 30.1449$

$\mathbf{b}_1$	$\mathbf{b}_{501}$	S_{500}
(0.0000, 0.9000, 0.1000)	(0.0000, 0.0000, 1.0000)	3.4460
(0.1000, 0.8000, 0.1000)	(0.0000, 0.0000, 1.0000)	3.3918
(0.2000, 0.7000, 0.1000)	(0.0000, 0.0000, 1.0000)	3.3682
(0.3000, 0.6000, 0.1000)	(0.0000, 0.0000, 1.0000)	3.3599
(0.4000, 0.5000, 0.1000)	(0.0000, 0.0000, 1.0000)	3.3626
(0.5000, 0.4000, 0.1000)	(0.0000, 0.0000, 1.0000)	3.3757
(0.6000, 0.3000, 0.1000)	(0.0000, 0.0000, 1.0000)	3.4020
(0.7000, 0.2000, 0.1000)	(0.0000, 0.0000, 1.0000)	3.4492
(0.8000, 0.1000, 0.1000)	(0.0000, 0.0000, 1.0000)	3.5408
(0.9000, 0.0000, 0.1000)	(0.0000, 0.0000, 1.0000)	3.7876
(0.1000, 0.0000, 0.9000)	(0.0000, 0.0000, 1.0000)	4.2352
(0.1000, 0.1000, 0.8000)	(0.0000, 0.0000, 1.0000)	4.0569
(0.1000, 0.2000, 0.7000)	(0.0000, 0.0000, 1.0000)	3.9398
(0.1000, 0.3000, 0.6000)	(0.0000, 0.0000, 1.0000)	3.8440
(0.1000, 0.4000, 0.5000)	(0.0000, 0.0000, 1.0000)	3.7572
(0.1000, 0.5000, 0.4000)	(0.0000, 0.0000, 1.0000)	3.6737
(0.1000, 0.6000, 0.3000)	(0.0000, 0.0000, 1.0000)	3.5893
(0.1000, 0.7000, 0.2000)	(0.0000, 0.0000, 1.0000)	3.4991
(0.1000, 0.8000, 0.1000)	(0.0000, 0.0000, 1.0000)	3.3918
(0.1000, 0.9000, 0.0000)	(0.0233, 0.9767, 0.0000)	1.2838
(0.9000, 0.1000, 0.0000)	(0.2361, 0.7639, 0.0000)	1.1622
(0.8000, 0.1000, 0.1000)	(0.0000, 0.0000, 1.0000)	3.5408
(0.7000, 0.1000, 0.2000)	(0.0000, 0.0000, 1.0000)	3.6768
(0.6000, 0.1000, 0.3000)	(0.0000, 0.0000, 1.0000)	3.7599
(0.5000, 0.1000, 0.4000)	(0.0000, 0.0000, 1.0000)	3.8256
(0.4000, 0.1000, 0.5000)	(0.0000, 0.0000, 1.0000)	3.8845
(0.3000, 0.1000, 0.6000)	(0.0000, 0.0000, 1.0000)	3.9411
(0.2000, 0.1000, 0.7000)	(0.0000, 0.0000, 1.0000)	3.9980
(0.1000, 0.1000, 0.8000)	(0.0000, 0.0000, 1.0000)	4.0569
(0.0000, 0.1000, 0.9000)	(0.0000, 0.0000, 1.0000)	4.1194

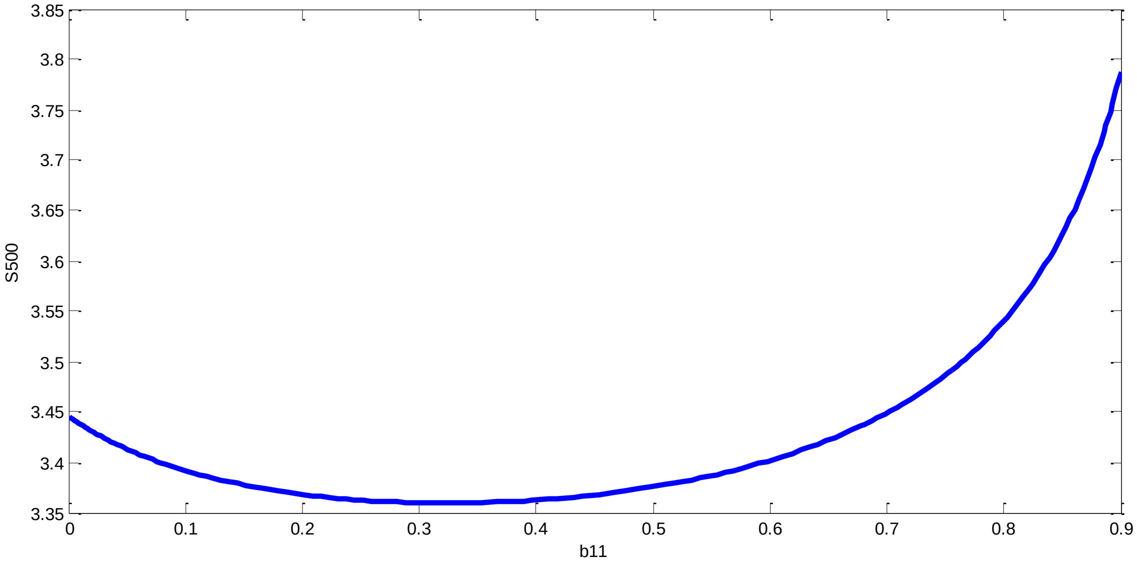

Figure 2.4: Graph of S_{500} against b_{11} for data set B, where $\eta = \mathbf{30.1449}$ (Helmbold universal portfolio)

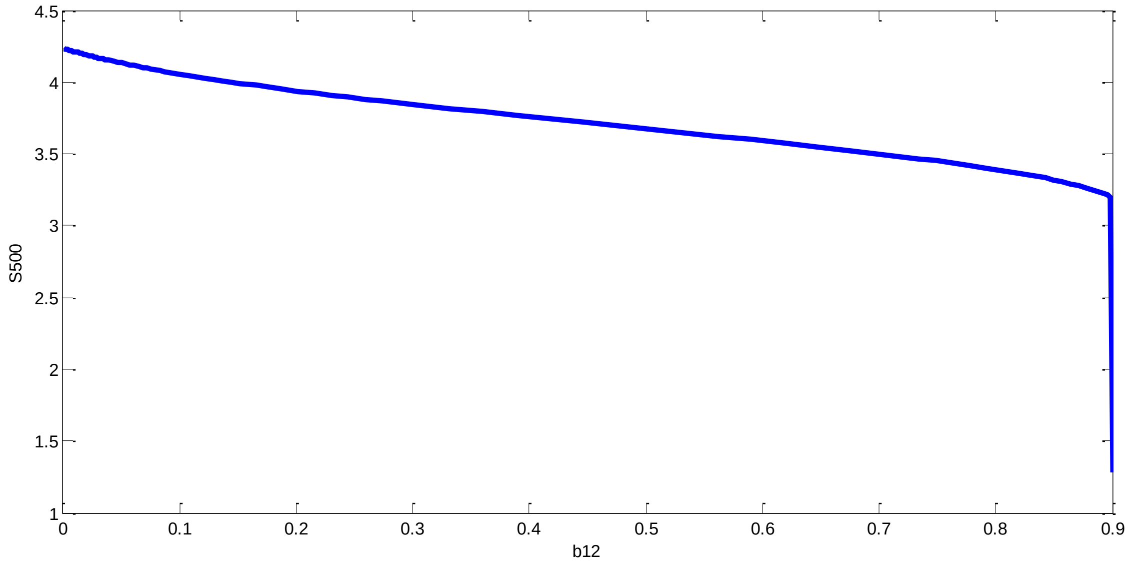

Figure 2.5: Graph of S_{500} against b_{12} for data set B, where $\eta = \mathbf{30.1449}$ (Helmbold universal portfolio)

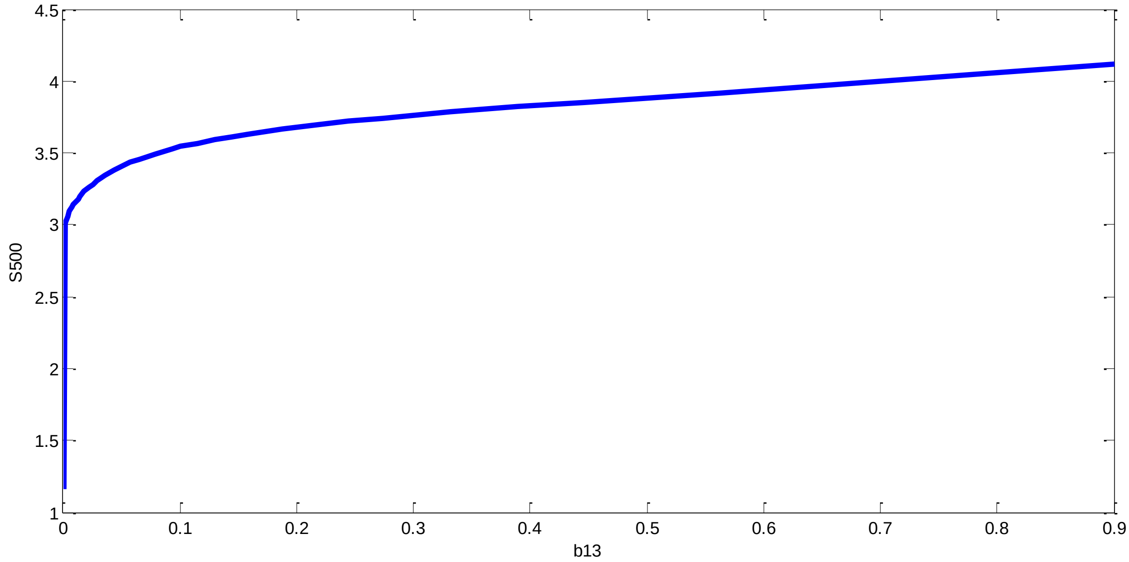

Figure 2.6: Graph of S_{500} against b_{13} for data set B, where $\eta = 30.1449$ (Helmbold universal portfolio)

In conclusion, the achievable universal wealth depends on the initial starting portfolio $\mathbf{b}_1$. An improper choice of $\mathbf{b}_1$ may lead to a lower investment wealth. We have also provided empirical evidence that a choice of same proportions in $\mathbf{b}_1$ may not necessarily lead to the highest wealth return.

2.2 Type II Helmbold Universal Portfolio

Helmbold *et al.* [12] approximated the function $\log\left(\frac{\mathbf{b}_{n+1}^t \mathbf{x}_{n+1}}{\mathbf{b}_n^t \mathbf{x}_n}\right)$ with a first-order Taylor polynomial to derive the portfolio. A second-order logarithmic approximation is used instead in this section to derive the Type II portfolio. By maximizing and minimizing the objective functions, we obtain a set of non-linear equations in the m unknown portfolio variables. The solution of this set of non-linear equations leads to a Type II Helmbold universal portfolio.

The *Type II Helmbold universal portfolio* is a sequence of portfolio vectors $\{\mathbf{b}_{n+1}\}$ generated by the following update of b_{ni}:

$$b_{n+1,i} = \frac{C_{(i)}\exp\left(D_{(i)}(\mathbf{b}_{n+1}^{t}\mathbf{x}_n)\right)}{\sum_{j=1}^{m} C_{(j)}\exp\left(D_{(j)}(\mathbf{b}_{n+1}^{t}\mathbf{x}_n)\right)} \tag{2.6}$$

for $i = 1,2,\dots,m$, where $C_{(i)} = b_{ni}\exp\left(\frac{2\eta x_{ni}}{\mathbf{b}_n^t\mathbf{x}_n}\right)$, $D_{(i)} = \frac{-\eta x_{ni}}{(\mathbf{b}_n^t\mathbf{x}_n)^2}$ and η is any given real number. Note that $\mathbf{b}_{n+1}$ is defined to be the solution to a set of non-linear equations given by (2.6).

The *eta-parametric family of Type II Helmbold universal portfolios* is derived as follow:

Proposition 2.3 Consider the objective functions

$$F(\mathbf{b}_{n+1}) = \eta\log(\mathbf{b}_{n+1}^{t}\mathbf{x}_n) - \mathrm{D}(\mathbf{b}_{n+1} \parallel \mathbf{b}_n)$$

and

$$G(\mathbf{b}_{n+1}) = \eta\log(\mathbf{b}_{n+1}^{t}\mathbf{x}_n) + \mathrm{D}(\mathbf{b}_{n+1} \parallel \mathbf{b}_n)$$

where $\mathrm{D}(\mathbf{b}_{n+1} \parallel \mathbf{b}_n)$ is the *Kullback-Leibler* distance measure or relative entropy given by (2.1) and η is positive. By approximating $\log(\mathbf{b}_{n+1}^{t}\mathbf{x}_n)$ using $\left[\log(\mathbf{b}_n^t\mathbf{x}_n) + \left(\frac{\mathbf{b}_{n+1}^t\mathbf{x}_n}{\mathbf{b}_n^t\mathbf{x}_n} - 1\right) - \frac{1}{2}\left(\frac{\mathbf{b}_{n+1}^t\mathbf{x}_n}{\mathbf{b}_n^t\mathbf{x}_n} - 1\right)^2\right]$, the maximum of the objective function $F(\mathbf{b}_{n+1})$ is achieved at $\mathbf{b}_{n+1}$ given by (2.6) and the minimum of the objective function $G(\mathbf{b}_{n+1})$ is achieved at $\mathbf{b}_{n+1}$ given by (2.6) where η is replaced by $-\eta$.

Proof. The Lagrange multiplier γ is introduced in maximizing and minimizing the objective functions because $\mathbf{b}_n$ is a portfolio vector for $n = 1,2,3, \ldots,$

$$\hat{F}(\mathbf{b}_{n+1},\gamma) = \eta\left[\log(\mathbf{b}_n^t\mathbf{x}_n) + \left(\frac{\mathbf{b}_{n+1}^t\mathbf{x}_n}{\mathbf{b}_n^t\mathbf{x}_n} - 1\right) - \frac{1}{2}\left(\frac{\mathbf{b}_{n+1}^t\mathbf{x}_n}{\mathbf{b}_n^t\mathbf{x}_n} - 1\right)^2\right]$$

$$-\sum_{i=1}^{m} b_{n+1,i}\log\left(\frac{b_{n+1,i}}{b_{ni}}\right) + \gamma\left(\sum_{i=1}^{m} b_{n+1,i} - 1\right)$$

and

$$\hat{G}(\mathbf{b}_{n+1},\gamma) = \eta\left[\log(\mathbf{b}_n^t\mathbf{x}_n) + \left(\frac{\mathbf{b}_{n+1}^t\mathbf{x}_n}{\mathbf{b}_n^t\mathbf{x}_n} - 1\right) - \frac{1}{2}\left(\frac{\mathbf{b}_{n+1}^t\mathbf{x}_n}{\mathbf{b}_n^t\mathbf{x}_n} - 1\right)^2\right]$$

$$+\sum_{i=1}^{m} b_{n+1,i}\log\left(\frac{b_{n+1,i}}{b_{ni}}\right) + \gamma\left(\sum_{i=1}^{m} b_{n+1,i} - 1\right).$$

The maximum of $\hat{F}(\mathbf{b}_{n+1},\gamma)$ is achieved when the following m partial derivatives are zero,

$$\frac{\partial\hat{F}(\mathbf{b}_{n+1},\gamma)}{\partial b_{n+1,i}} = \frac{2\eta x_{ni}}{\mathbf{b}_n^t\mathbf{x}_n} - \frac{\eta x_{ni}(\mathbf{b}_{n+1}^t\mathbf{x}_n)}{(\mathbf{b}_n^t\mathbf{x}_n)^2} - \left[\log\left(\frac{b_{n+1,i}}{b_{ni}}\right) + 1\right] + \gamma = 0$$

for $i = 1,2, \ldots, m$. We obtain

$$b_{n+1,i} = b_{ni}\exp\left(\frac{2\eta x_{ni}}{\mathbf{b}_n^t\mathbf{x}_n}\right)\exp\left(\frac{-\eta x_{ni}(\mathbf{b}_{n+1}^t\mathbf{x}_n)}{(\mathbf{b}_n^t\mathbf{x}_n)^2}\right)\exp(-\gamma - 1)$$

for $i = 1,2, \ldots, m$. Summing up the components $b_{n+1,i}$ over i, we have

$$1 = exp(-\gamma - 1)\sum_{i=1}^{m}\exp\left(\frac{2\eta x_{ni}}{\mathbf{b}_n^t\mathbf{x}_n}\right)\exp\left(\frac{-\eta x_{ni}(\mathbf{b}_{n+1}^t\mathbf{x}_n)}{(\mathbf{b}_n^t\mathbf{x}_n)^2}\right)$$

leading to (2.6) for $i = 1,2, \ldots, m$. It is straight forward to show that the minimum of $\hat{G}(\mathbf{b}_{n+1},\gamma)$ is achieved at $b_{n+1,i}(-\eta)$ in (2.6). □

We introduce the numerical algorithm of solving the set of non-linear equations associated with the Type II Helmbold universal portfolio.

Rewrite (2.6) to be the following equations

$$\sum_{\substack{j=1\\ j\neq i}}^{m} b_{n+1,i}C_{(j)}\exp\left(D_{(j)}(\mathbf{b}_{n+1}^t\mathbf{x}_n)\right)+\left(b_{n+1,i}C_{(i)}-C_{(i)}\right)\exp(D_{(i)}(\mathbf{b}_{n+1}^t\mathbf{x}_n))=0$$

for $i=1,2,\dots,m$ and let the left hand side of the above equation be $f_i^{n+1}(b_{n+1,i})$ where

$$f_i^{n+1}(b_{n+1,i})=\sum_{\substack{j=1\\ j\neq i}}^{m} b_{n+1,i}C_{(j)}\exp\left(D_{(j)}(\mathbf{b}_{n+1}^t\mathbf{x}_n)\right) \qquad (2.7)$$

$$+\left(b_{n+1,i}C_{(i)}-C_{(i)}\right)\exp(D_{(i)}(\mathbf{b}_{n+1}^t\mathbf{x}_n))$$

where $C_{(i)}=b_{ni}\exp\left(\frac{2\eta x_{ni}}{\mathbf{b}_n^t\mathbf{x}_n}\right)$, $D_{(i)}=\frac{-\eta x_{ni}}{(\mathbf{b}_n^t\mathbf{x}_n)^2}$ and η is any real number. When $b_{n+1,j}$ are fixed for all $j\neq i$, the function $f_i^{n+1}(b_{n+1,i})$ has a root $0\leq b_{n+1,i*}\leq 1$, that is $f_i^{n+1}(b_{n+1,i*})=0$. This is due to $f_i^{n+1}(1)>0$ and $f_i^{n+1}(0)<0$. We use Newton's Method to find the root $b_{n+1,i*}$. The algorithm works as follows:

(1) Fix $b_{n+1,j}$ for j $=2,3,\dots,m-1$ and find $0\leq b_{n+1,1*}\leq 1$ such that $f_1^{n+1}(b_{n+1,1*})=0$.

(2) Fix $b_{n+1,j}$ for $j=1,3,\dots,m-1$ and find $0\leq b_{n+1,2*}\leq 1$ such that $f_2^{n+1}(b_{n+1,2*})=0$.

⋮

(m - 1) Fix $b_{n+1,j}$ for $j=1,2,\dots,m-2$ and find $0\leq b_{n+1,m-1*}\leq 1$ such that $f_{m-1}^{n+1}(b_{n+1,m-1*})=0$.

If $b_{n+1,m-1} = b_{n+1,m-1*}$, then the solution of (2.6) is $b_{n+1,1*}$, $b_{n+1,2*}, \ldots$, $b_{n+1,m*}$. Otherwise, repeat (1) to (m - 1).

We apply Newton's Method in numerical analysis to find the root of $f_i^{n+1}(b_{n+1,i}) = 0$. We find a sequence of iterates $b_{n+1,i}$ that converges to the root $b_{n+1,i*}$. Begin by guessing on an initial estimate $b_{n+1,i}$. We can assume the solution $\mathbf{b}_{n+1}$ is close to the given $\mathbf{b}_n$. The initial iterate $b_{n+1,i} = b_{ni}$ is a good start. For subsequent iterates, we apply the Newton formula

$$b_{n+1,i(new)} = b_{n+1,i(old)} - \frac{f_i^{n+1}(b_{n+1,i(old)})}{f'^{n+1}_i(b_{n+1,i(old)})}. \qquad (2.8)$$

In summary, given $f_i^{n+1}(\cdot)$ as a function of $b_{n+1,i}$, where $b_{n+1,j}$ is fixed for all $j \neq i$, we use Newton's Method to find $0 \leq b_{n+1,i*} \leq 1$ such that $f_i^{n+1}(b_{n+1,i*}) = 0$. The iterations are repeated for $i = 1,2, \ldots, m-1$ until the solution $\mathbf{b}_{n+1}$ to (2.6) is obtained.

The eta-parametric family of Type II Helmbold universal portfolio are run on the same three stock data sets that are used in the previous section. We compare the performance of the two types of Helmbold universal portfolios using the same initial starting portfolio $\mathbf{b}_1 = (0.3333, 0.3333, 0.3334)$ on each data set. The portfolios $\mathbf{b}_{501}$ and the maximum wealths $S_{500}(max)$ achieved by respective η's on each data set after 500 trading days are shown in Table 2.8. The both types of Helmbold universal portfolios achieve the same maximum wealths $S_{500}(max)$ for data set A whereas the Helmbold universal portfolio performs slightly better than the Type II Helmbold universal portfolio for data sets B and C.

Table 2.8: The portfolios $\mathbf{b}_{501}$ and the maximum wealths $S_{500}(max)$ achieved by respective η's by the two types of Helmbold universal portfolios for data sets A, B and C, where $\mathbf{b}_1 = (0.3333, 0.3333, 0.3334)$

Data Set	Helmbold universal portfolio	Type II Helmbold universal portfolio
Set A	$S_{500}(max) = 1.5755$ at $\eta = -2.0714$ $\mathbf{b}_{501} = (0.2032, 0.5403, 0.2564)$	$S_{500}(max) = 1.5755$ at $\eta = -2.0714$ $\mathbf{b}_{501} = (0.2032, 0.5405, 0.2563)$
Set B	$S_{500}(max) = 3.6467$ at $\eta = 30.1449$ $\mathbf{b}_{501} = (0.0000, 0.0000, 1.0000)$	$S_{500}(max) = 3.6416$ at $\eta = 25.3080$ $\mathbf{b}_{501} = (0.0000, 0.0000, 1.0000)$
Set C	$S_{500}(max) = 3.5066$ at $\eta = 115.7115$ $\mathbf{b}_{501} = (0.0000, 0.0000, 1.0000)$	$S_{500}(max) = 3.4902$ at $\eta = 17.2895$ $\mathbf{b}_{501} = (0.0000, 0.0000, 1.0000)$

From the empirical results, we observe that the Helmbold universal portfolio performs better than the Type II Helmbold universal portfolio in terms of the final wealth achievement. There is no advantage in using the Type II Helmbold universal portfolio instead of the Helmbold universal portfolio. Furthermore, the implementation of the Type II Helmbold universal portfolio is more complicated and the computation requires more time. The results in this section are presented in Tan and Lim [23].

2.3 Running the Helmbold Universal Portfolios on 10-stock Data Sets

The implementation of the Dirichlet-weighted universal portfolio requires a large computer memory requirements for processing the stock data during the computation. The Helmbold universal portfolio which requires much lesser computer memory requirements can be implemented on any number of stocks. We run the Helmbold universal portfolio on some 10-stock data sets in this section.

We have selected four stock-price data sets D, E, F and G covering the period of 2 January 2003 until 30 December 2010. There is a total of 1975 trading days and the companies in the four data sets are listed in Table 2.9. The selected companies must be active and liquid enough to be traded. These two factors are applied to the market capitalisation. The companies in data sets D, E, F and G are selected from the FTSE Bursa Malaysia Kuala Lumpur Composite Index which comprises the largest 30 companies listed on the Kuala Lumpur Stock Exchange Main Market by full market capitalisation. Different sectors of company are selected in each data set to reduce the risk of investment. Each data set consists of ten companies and there is overlapping of companies in the data sets.

Table 2.9: List of companies in the data sets D, E, F and G

Set D	Set E	Set F	Set G
YTL Corporation	YTL Corporation	YTL Power International	Malaysian Airline System
UMW Holdings	UMW Holdings	PPB Group	Hong Leong Financial Group
MMC Corporation	MMC Corporation	Petronas Dagangan	IOI Corporation
YTL Power International	YTL Power International	Digi.com	YTL Power International
PPB Group	PPB Group	Hong Leong Financial Group	Kuala Lumpur Kepong
Petronas Dagangan	Petronas Dagangan	Malaysian Airline System	Petronas Dagangan
Digi.com	Digi.com	Kuala Lumpur Kepong	MMC Corporation
Malayan Banking	Hong Leong Financial Group	PLUS Expressways	PPB Group
Malaysian Airline System	Malaysian Airline System	IOI Corporation	Digi.com
Kuala Lumpur Kepong	IOI Corporation	Sime Darby	Sime Darby

We start with the same initial starting portfolio $\mathbf{b}_1 = (0.1000, 0.1000, \ldots, 0.1000)$ for all the four 10-stock data sets. Table 2.10 shows the portfolios $\mathbf{b}_{1976}$ and the maximum wealths $S_{1975}(max)$

achieved by respective η's on each data set after 1975 trading days. The maximum wealths $S_{1975}(max)$ achieved for data set D is 18.2486 at $\eta = 0.4138$. Again, data sets E, F and G are examples of the Helmbold universal portfolios with large negative-valued parameters achieving the maximum wealths $S_{1975}(max) = 22.9859, 15.7558$ and 19.9357 at $\eta = -2.3639$, -9.4444 and -83.1143 respectively.

Table 2.10: The portfolios $\mathbf{b}_{1976}$ and the maximum wealths $S_{1975}(max)$ achieved by respective η's by the Helmbold universal portfolios for data sets D, E, F and G, where $\mathbf{b}_1 = (0.1000, 0.1000, \ldots, 0.1000)$

Data set	Best η	$\mathbf{b}_{1976}$	$S_{1975}(max)$
Set D	0.4138	(0.1356, 0.1319, 0.1223, 0.1042, 0.1048, 0.1012, 0.0914, 0.0558, 0.0825, 0.0702)	18.2486
Set E	-2.3639	(0.0085, 0.0107, 0.0161, 0.0391, 0.0409, 0.0469, 0.0797, 0.1374, 0.1487, 0.4719)	22.9859
Set F	-9.4444	(0.0000, 0.0000, 0.0000, 0.0000, 0.0004, 0.0003, 0.0371, 0.8604, 0.0272, 0.0745)	15.7558
Set G	-83.1143	(0.0000, 0.0000, 0.0002, 0.0000, 0.0460, 0.0000, 0.0000, 0.0000, 0.0000, 0.9537)	19.9357

The best constant rebalanced portfolios (BCRP) $\mathbf{b}^*_{1975}$ and the respective wealths S^*_{1975} achieved after 1975 trading days for the four 10-stock data sets are calculated and listed in Table 2.11. From Tables 2.11 and 2.10, the wealths S^*_{1975} achieved by the BCRP's are much higher than the maximum wealths $S_{1975}(max)$ achieved by the Helmbold universal portfolios for all the four 10-stock data sets with the same initial starting portfolios.

Table 2.11: The best constant rebalanced portfolios $\mathbf{b}^*_{1975}$ and the wealths S^*_{1975} achieved for data sets D, E, F and G

Data set	$\mathbf{b}^*_{1975}$	S^*_{1975}
Set D	(0.5981, 0.4019, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000)	37.5867
Set E	(0.5981, 0.4019, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000)	37.5867

Table 2.11 continued

Data set	$\mathbf{b}^*_{1975}$	S^*_{1975}
Set F	(0.4836, 0.3869, 0.1295, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000)	20.7169
Set G	(0.0000, 0.0000, 0.0000, 0.1965, 0.0000, 0.0000, 0.5926, 0.2109, 0.0000, 0.0000)	24.6381

The Helmbold universal portfolios are run on the four 10-stock data sets again but with the initial starting portfolios $\mathbf{b}_1$ are replaced by the BCRP's instead of the same initial starting portfolios. The resulting portfolios $\mathbf{b}_{1976}$ and the maximum wealths $S_{1975}(max)$ achieved after 1975 trading days where $\mathbf{b}_1 = \mathbf{b}^*_{1975}$ are recorded in Table 2.12. Data sets D and E have the same maximum wealths $S_{1975}(max)$ which are 43.2025 achieved at $\eta = -2.8843$. The maximum wealths $S_{1975}(max)$ obtained for data sets F and G are 27.2148 and 39.9419 respectively corresponding to the respective $\eta = -9.5684$ and -5.7511. From Tables 2.10 and 2.12, the maximum wealths $S_{1975}(max)$ achieved by the Helmbold universal portfolio are significantly higher if the initial starting portfolios $\mathbf{b}_1$ are replaced by the BCRP's for all the four 10-stock data sets. It is noteworthy that the maximum wealths $S_{1975}(max)$ achieved by the Helmbold universal portfolio where $\mathbf{b}_1 = \mathbf{b}^*_{1975}$ exceed the wealths S^*_{1975} achieved by the BCRP's from Tables 2.12 and 2.11.

Table 2.12: The portfolios $\mathbf{b}_{1976}$ and the maximum wealths $S_{1975}(max)$ achieved by respective η's by the Helmbold universal portfolios for data sets D, E, F and G, where $\mathbf{b}_1 = \mathbf{b}^*_{1975}$

Data set	Best η	$\mathbf{b}_{1976}$	$S_{1975}(max)$
Set D	-2.8843	(0.4177, 0.5823, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000)	43.2025
Set E	-2.8843	(0.4177, 0.5823, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000)	43.2025
Set F	-9.5684	(0.0934, 0.8502, 0.0563, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000)	27.2148
Set G	-5.7511	(0.0000, 0.0000, 0.0000, 0.1967, 0.0000, 0.0000, 0.2956, 0.5077, 0.0000, 0.0000)	39.9419

CHAPTER THREE

CHI-SQUARE DIVERGENCE UNIVERSAL PORTFOLIO

The multiplicative-update universal portfolio was proposed by Helmbold *et al.* in [12]. In this chapter, we propose to generate a family of universal portfolios by the same method using the chi-square divergence (CSD) distance measure. This leads to a family of additive-updates universal portfolios. The families of universal portfolios generated by this method can be implemented online involving day-to-day updates of the current portfolio.

3.1 The Xi-Parametric Family of Chi-Square Divergence Universal Portfolio

The work reported in this section is published in Tan and Lim [26, 27]. An additive-update universal portfolio is obtained by maximizing and minimizing a certain objective functions involving the CSD distance measure in this section. We compare the performance of the CSD additive-update universal portfolios with that of the Helmbold multiplicative-update universal portfolios by running the portfolios on some selected data sets from the local stock exchange. It is shown that for some parametric values of the CSD universal portfolio, better wealths can be generated from daily investment. Practical bounds for the parametric values of the CSD universal portfolios are obtained for their investment implementation.

The *chi-square divergence* distance measure is

$$\mathrm{D}(\mathbf{b}_k \parallel \mathbf{b}_n) = \sum_{i=1}^{m} \frac{(b_{ki} - b_{ni})^2}{b_{ni}}, \tag{3.1}$$

where $\mathbf{b}_k = (b_{ki})$ and $\mathbf{b}_n = (b_{ni})$ are any two portfolio vectors.

The *CSD universal portfolio* is a sequence of portfolio vectors $\{\mathbf{b}_{n+1}\}$ generated by the following update of b_{ni}:

$$b_{n+1,i} = b_{ni}\left[\frac{\xi(x_{ni} - \mathbf{b}_n^t\mathbf{x}_n)}{(\mathbf{b}_n^t\mathbf{x}_n)} + 1\right] \tag{3.2}$$

for $i = 1,2,\dots,m$, where initial starting portfolio $\mathbf{b}_1$ is given and ξ is any chosen real number such that $b_{ni} \geq 0$ for all $i = 1,2,\dots,m$ and $n = 1,2,3,\dots$. Equation (3.2) defines the *xi-parametric family of CSD universal portfolios* for any appropriate real number ξ such that $b_{ni} \geq 0$ for all $i = 1,2,\dots,m$ and $n = 1,2,3,\dots$. It is clear that this parametric family is defined for certain bounded values of ξ and not for all real ξ, in contrast with the eta-parametric family which is defined for all real η.

First, we show that the xi-parametric family of CSD universal portfolios is obtained by maximizing and minimizing certain objective functions of the doubling rate of the capital function and the CSD distance measure.

Proposition 3.1 Consider the objective functions

$$F(\mathbf{b}_{n+1}) = 2\xi \log(\mathbf{b}_{n+1}^t \mathbf{x}_n) - \mathrm{D}(\mathbf{b}_{n+1} \parallel \mathbf{b}_n)$$

and

$$G(\mathbf{b}_{n+1}) = 2\xi \log(\mathbf{b}_{n+1}^t \mathbf{x}_n) + \mathrm{D}(\mathbf{b}_{n+1} \parallel \mathbf{b}_n)$$

where $\mathrm{D}(\mathbf{b}_{n+1} \parallel \mathbf{b}_n)$ is the CSD distance measure given by (3.1) and $\xi > 0$. By approximating $\log(\mathbf{b}_{n+1}^t \mathbf{x}_n)$ using $\left[\log(\mathbf{b}_n^t \mathbf{x}_n) + \frac{\mathbf{b}_{n+1}^t \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n} - 1\right]$, the maximum of the objective function $F(\mathbf{b}_{n+1})$ is achieved at $\mathbf{b}_{n+1}$ given by (3.2) and the minimum of $G(\mathbf{b}_{n+1})$ is also achieved at $\mathbf{b}_{n+1}$ given by (3.2) where ξ is replaced by $-\xi$.

Proof. Since $\mathbf{b}_n$ is a portfolio vector for $n = 1,2,3,\ldots$, we need to introduce the Lagrange multiplier γ in maximizing the objective function

$$\hat{F}(\mathbf{b}_{n+1}, \gamma) = 2\xi \left[\log(\mathbf{b}_n^t \mathbf{x}_n) + \frac{\mathbf{b}_{n+1}^t \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n} - 1\right] - \sum_{i=1}^{m} \frac{\left(b_{n+1,i} - b_{ni}\right)^2}{b_{ni}} + \gamma\left(\sum_{i=1}^{m} b_{n+1,i} - 1\right)$$

and minimizing the objective function

$$\hat{G}(\mathbf{b}_{n+1}, \gamma) = 2\xi \left[\log(\mathbf{b}_n^t \mathbf{x}_n) + \frac{\mathbf{b}_{n+1}^t \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n} - 1\right] + \sum_{i=1}^{m} \frac{\left(b_{n+1,i} - b_{ni}\right)^2}{b_{ni}} + \gamma\left(\sum_{i=1}^{m} b_{n+1,i} - 1\right).$$

First, the maximum of $\hat{F}(\mathbf{b}_{n+1}, \gamma)$ is achieved when the following m partial derivatives are zero, that is

$$\frac{\partial \hat{F}(\mathbf{b}_{n+1}, \gamma)}{\partial b_{n+1,i}} = 2\xi \frac{x_{ni}}{\mathbf{b}_n^t \mathbf{x}_n} - \frac{2\left(b_{n+1,i} - b_{ni}\right)}{b_{ni}} + \gamma = 0$$

for $i = 1,2,\ldots,m$. We obtain

$$b_{n+1,i} = b_{ni}\left[\frac{\xi x_{ni}}{\mathbf{b}_n^t \mathbf{x}_n} + 1 + \frac{\gamma}{2}\right]$$

for $i = 1,2,\dots,m$. Summing up the components $b_{n+1,i}$ over i, we have

$$\frac{\gamma}{2} = -\sum_{i=1}^{m} \frac{\xi b_{ni} x_{ni}}{\mathbf{b}_n^t \mathbf{x}_n} = -\xi$$

which results in (3.2) for $i = 1,2,\dots,m$. Finally, it is straight forward to show the minimum of $\hat{G}(\mathbf{b}_{n+1},\gamma)$ is achieved at $b_{n+1,i}(-\xi)$ in (3.2) and we omit the proof. Let

$$\begin{aligned}\tilde{F}&\left(b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m-1}\right)\\ &= 2\xi\left[\log(\mathbf{b}_n^t \mathbf{x}_n) + \frac{\mathbf{b}_{n+1}^t \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n} - 1\right]\\ &\quad - \sum_{i=1}^{m} \frac{\left(b_{n+1,i} - b_{ni}\right)^2}{b_{ni}}\end{aligned} \tag{3.3}$$

where $b_{n+1,m} = 1 - \sum_{i=1}^{m-1} b_{n+1,i}$. Then the first partial derivatives of (3.3) are

$$\frac{\partial \tilde{F}\left(b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m-1}\right)}{\partial b_{n+1,i}} = 2\xi\left[\frac{x_{ni} - x_{nm}}{\mathbf{b}_n^t \mathbf{x}_n}\right] - \frac{2\left(b_{n+1,i} - b_{ni}\right)}{b_{ni}} + \frac{2\left(b_{n+1,m} - b_{nm}\right)}{b_{nm}}$$

for $i = 1,2,\dots,m-1$ and the second partial derivatives of (3.3) are

$$\frac{\partial^2 \tilde{F}\left(b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m-1}\right)}{\partial b_{n+1,i}\partial b_{n+1,j}} = \begin{cases} -\dfrac{2}{b_{ni}} - \dfrac{2}{b_{nm}} & \text{for } i = j \\ -\dfrac{2}{b_{nm}} & \text{for } i \neq j \end{cases}$$

for $i, j = 1,2,\dots,m-1$. Define the matrix H where

$$h_{ij} = \begin{cases} \phi_i + \phi_m & \text{for } i = j \\ \phi_m & \text{for } i \neq j \end{cases}$$

for $i, j = 1,2,\dots,m-1$ and

$$\phi_k = \frac{1}{b_{nk}}$$

for $k = 1,2, \dots, m$. From the previous result in Proposition 2.2, the matrix H is positive definite if $b_{ni} > 0$ for $i = 1,2, \dots, m$. Hence the Hessian matrix of $\tilde{F}(b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m-1})$ is $-2H$ which is negative definite. Similarly, if

$$\tilde{G}(b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m-1})$$

$$= 2\xi \left[\log(\mathbf{b}_n^t \mathbf{x}_n) + \frac{\mathbf{b}_{n+1}^t \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n} - 1\right] + \sum_{i=1}^{m} \frac{(b_{n+1,i} - b_{ni})^2}{b_{ni}}$$

where $b_{n+1,m} = 1 - \sum_{i=1}^{m-1} b_{n+1,i}$, the Hessian matrix of $\tilde{G}(b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m-1})$ is $2H$ which is positive definite. Thus $\tilde{F}(b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m-1})$ and $\tilde{G}(b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m-1})$ are concave and convex respectively. The function $\tilde{F}(b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m-1})$ has a maximum point and $\tilde{G}(b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m-1})$ has a minimum point. □

Proposition 3.2 A sufficient condition for $b_{ni} \geq 0$ for all $i = 1,2, \dots, m$ and all positive integers n where $b_{n+1,i}$ is defined in (3.2) is that

$$|\xi| \leq \frac{\min_{n,i}\{x_{ni}\}}{\max_n \left\{\max_i(x_{ni}) - \min_i(x_{ni})\right\}}, \tag{3.4}$$

where $\mathbf{b}_1 \geq 0$ is given.

Proof. Given $\mathbf{b}_n$ as a portfolio vector, then for $b_{n+1,i} \geq 0$ in (3.2), for $i = 1,2, \dots, m$ and $n = 1,2,3, \dots$, we must have

$$\xi(x_{ni} - \mathbf{b}_n^t \mathbf{x}_n) \geq -(\mathbf{b}_n^t \mathbf{x}_n),$$

or equivalently, one of the following inequalities is satisfied:

$$-\frac{(\mathbf{b}_n^t \mathbf{x}_n)}{|x_{ni} - \mathbf{b}_n^t \mathbf{x}_n|} \leq \xi \text{ or } \xi \leq \frac{(\mathbf{b}_n^t \mathbf{x}_n)}{|x_{ni} - \mathbf{b}_n^t \mathbf{x}_n|}. \tag{3.5}$$

Noting that

$$\frac{\min_{n,i}\{x_{ni}\}}{\max_{n}\left\{\max_{i}(x_{ni}) - \min_{i}(x_{ni})\right\}} \leq \frac{(\mathbf{b}_n^t \mathbf{x}_n)}{|x_{ni} - \mathbf{b}_n^t \mathbf{x}_n|}$$

is true for all $i = 1,2, \ldots, m$ and $n = 1,2,3, \ldots$, any ξ satisfying (3.4) will imply that (3.5) satisfied. □

Any ξ satisfying (3.4) will generate a xi-parametric family of CSD universal portfolios. In practice, if the minimum and maximum price-relatives are 0.95 and 1.05 respectively, then the condition (3.4) says that $|\xi| \leq \frac{0.95}{(1.05-0.95)} = 9.5$. In other words, a parametric family of CSD universal portfolios is generated for $-9.5 \leq \xi \leq 9.5$.

For comparison, we run the CSD universal portfolios on the three same data sets designated as A, B and C introduced in Chapter Two with the initial starting portfolio $\mathbf{b}_1 = (0.3333, 0.3333, 0.3334)$. For each data set, the maximum wealths $S_{500}(max)$ achieved within the range of the parameter ξ given by (3.4) and an extended range of ξ are shown in Table 3.1 together with the portfolios $\mathbf{b}_{501}$ and the best ξ values where S_{500} is maximum. The extended range of ξ is the largest interval of ξ that satisfies (3.5) over all n within the whole investment period.

Table 3.1: The portfolios $\mathbf{b}_{501}$ and the maximum wealths $S_{500}(max)$ achieved by respective ξ's within the range of ξ in (3.4) and an extended range of ξ by the CSD universal portfolio for data sets A, B and C, where $\mathbf{b}_1 = (0.3333, 0.3333, 0.3334)$

Data set	Normal range of ξ determined by (3.4)	Extended range of ξ
Set A	$-12.6811 \leq \xi \leq 12.6811$ $S_{500}(max) = 1.5758$ at $\xi = -1.9174$ $\mathbf{b}_{501} = (0.2106, 0.5298, 0.2596)$	$-14.6638 \leq \xi \leq 16.1608$ $S_{500}(max) = 1.5758$ at $\xi = -1.9174$ $\mathbf{b}_{501} = (0.2106, 0.5298, 0.2596)$

Table 3.1 continued

Data set	Normal range of ξ determined by (3.4)	Extended range of ξ
Set B	$-11.2959 \leq \xi \leq 11.2959$ $S_{500}(max) = 3.7860$ at $\xi = 11.2959$ $\mathbf{b}_{501} = (0.0000, 0.0000, 1.0000)$	$-12.1887 \leq \xi \leq 13.0114$ $S_{500}(max) = 3.8394$ at $\xi = 13.0114$ $\mathbf{b}_{501} = (0.0000, 0.0000, 1.0000)$
Set C	$-10.1386 \leq \xi \leq 10.1386$ $S_{500}(max) = 3.6310$ at $\xi = 10.1386$ $\mathbf{b}_{501} = (0.0000, 0.0000, 1.0000)$	$-15.2103 \leq \xi \leq 16.7997$ $S_{500}(max) = 3.6480$ at $\xi = 11.8504$ $\mathbf{b}_{501} = (0.0000, 0.0000, 1.0000)$

In Figures 3.1, 3.2 and 3.3, the superimposed graphs of S_{500} against ξ (CSD universal portfolio) and S_{500} against η (Helmbold universal portfolio) are shown for data sets A, B and C respectively and a limited range of the parametric values, where $\mathbf{b}_1 = (0.3333,0,3333,0.3334)$. There are more fluctuations in the CSD graphs compared with the Helmbold graphs. We can say that the Helmbold wealth function S_n is more stable with respect to changes in its parameter.

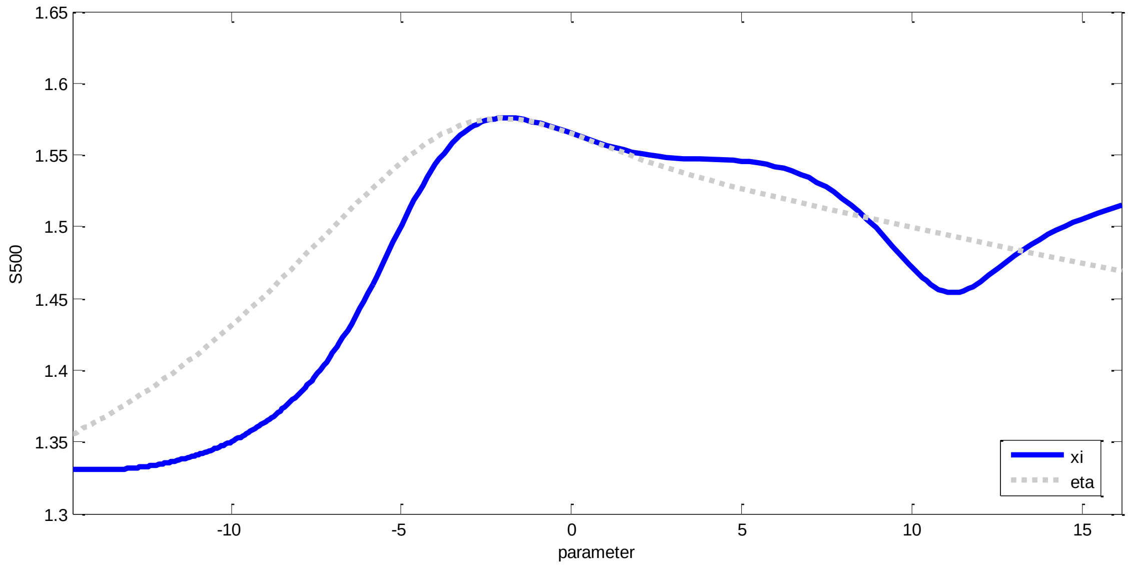

Figure 3.1: Two superimposed graphs of S_{500} against ξ (CSD universal portfolio) and S_{500} against η (Helmbold universal portfolio) for data set A, where $\mathbf{b}_1 = (0.3333,0.3333,0.3334)$

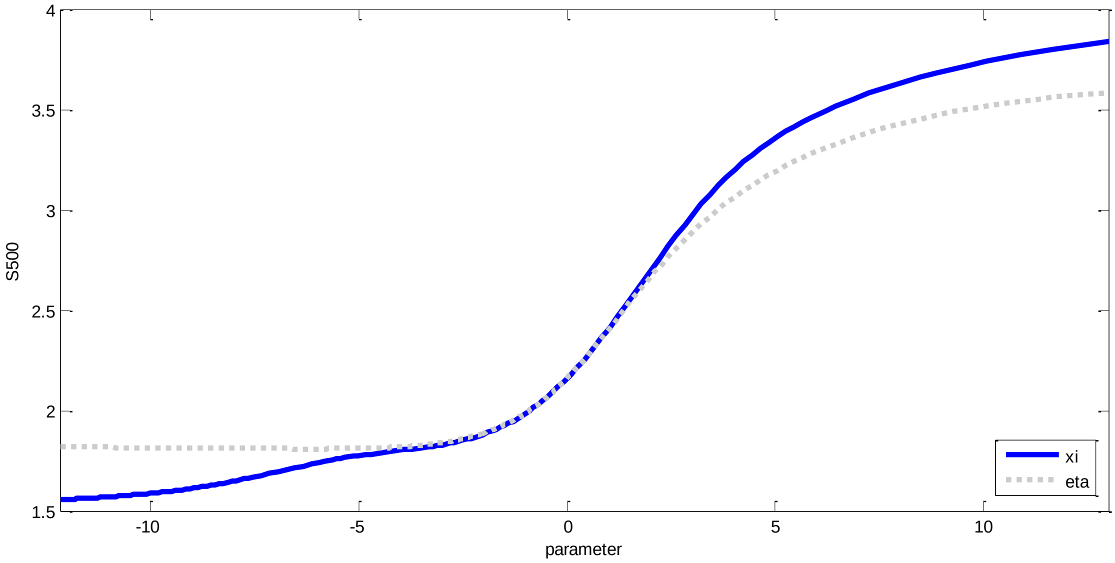

Figure 3.2: Two superimposed graphs of S_{500} against ξ (CSD universal portfolio) and S_{500} against η (Helmbold universal portfolio) for data set B, where $\mathbf{b}_1 = (0.3333, 0.3333, 0.3334)$

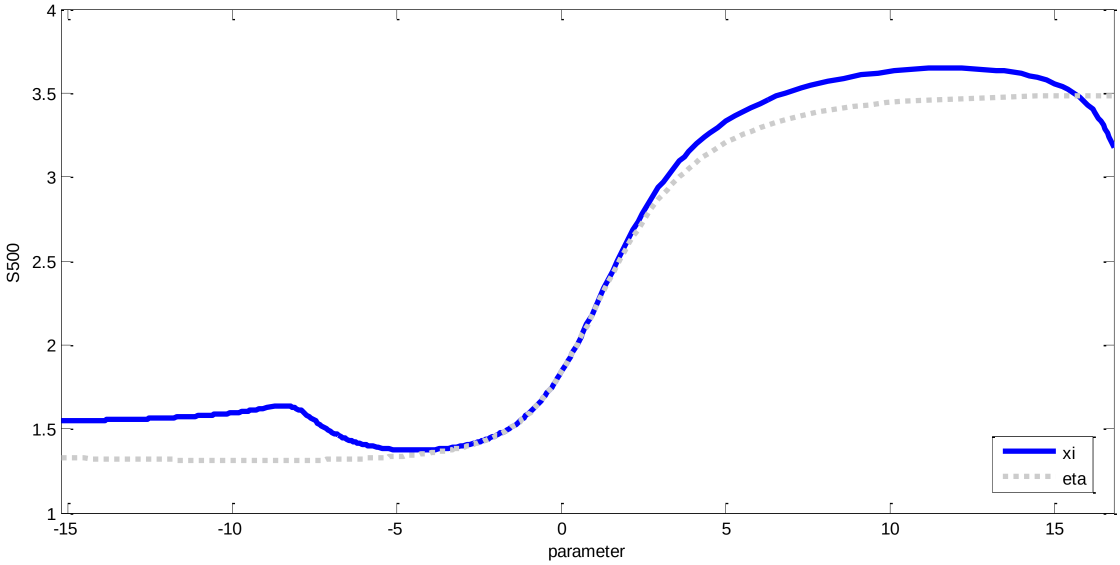

Figure 3.3: Two superimposed graphs of S_{500} against ξ (CSD universal portfolio) and S_{500} against η (Helmbold universal portfolio) for data set C, where $\mathbf{b}_1 = (0.3333, 0.3333, 0.3334)$

It is clear from Tables 3.1 and 2.8 that the CSD universal portfolio achieves a slightly higher wealth of $S_{500} = 1.5758$ at $\xi = -1.9174$ compared with the Helmbold universal portfolio for data set A. For data sets B and C, the CSD universal portfolios achieve significantly higher wealths of $S_{500} = 3.8394$ and $S_{500} = 3.6480$ at $\xi = 13.0114$ and $\xi = 11.8504$ respectively. Although the good-performance results are data-dependent, we can conclude that there are CSD universal portfolios that can outperform the Helmbold universal portfolios. In fact, we can achieve higher wealths from the CSD universal portfolios by changing the initial starting portfolio $\mathbf{b}_1$. Table 3.2 shows that by using $\mathbf{b}_1 = (0.0100{,}0.0100{,}0.9800)$ for the data sets B and C, we are able to achieve higher wealths of $S_{500} = 4.2903$ (compared to $S_{500} = 3.8394$) and $S_{500} = 4.3024$ (compared to $S_{500} = 3.6480$), respectively. Hence the initial starting portfolio $\mathbf{b}_1$ can be regarded as a factor or parameter influencing the performance of the CSD universal portfolio.

Table 3.2: The portfolios $\mathbf{b}_{501}$ and the maximum wealths $S_{500}(max)$ achieved by respective ξ's within an extended range of ξ by the CSD universal portfolio for data sets A, B and C, where $\mathbf{b}_1$ are set as stated

Data set	Extended range of ξ
Set A	$-14.6625 \leq \xi \leq 16.0680$ $\mathbf{b}_1 = (0.2100, 0.5300, 0.2600)$ $S_{500}(max) = 1.4691$ at $\xi = 16.0680$ $\mathbf{b}_{501} = (0.0001, 0.0000, 0.9999)$
Set B	$-13.3201 \leq \xi \leq 12.9848$ $\mathbf{b}_1 = (0.0100, 0.0100, 0.9800)$ $S_{500}(max) = 4.2903$ at $\xi = 12.9848$ $\mathbf{b}_{501} = (0.0000{,}0.0000{,}1.0000)$
Set C	$-15.3125 \leq \xi \leq 12.9973$ $\mathbf{b}_1 = (0.0100, 0.0100, 0.9800)$ $S_{500}(max) = 4.3024$ at $\xi = -15.3125$ $\mathbf{b}_{501} = (0.0012, 0.0000, 0.9988)$

3.2 Running the Chi-Square Divergence Universal Portfolios on 10-stock Data Sets

The CSD universal portfolio can be implemented online using day-to-day updates and it requires much lesser computer memory requirements compared to the Dirichlet-weighted universal portfolio. The computer memory requirements are growing linearly with the number of stocks, so the CSD universal portfolio can be implemented on any number of stocks. In this section, we run the CSD universal portfolios on the same four 10-stock data sets in Section 2.3.

First, we run the CSD universal portfolios on data sets D, E, F and G using $\mathbf{b}_1 = (0.1000, 0.1000, \ldots, 0.1000)$. The portfolios $\mathbf{b}_{1976}$ and the maximum wealths $S_{500}(max)$ achieved by respective ξ's over the range of values of ξ considered on each data set after 1975 trading days are listed in Table 3.3. From Tables 2.10 and 3.3, the maximum wealth $S_{1975}(max) = 18.2486$ achieved by the Helmbold universal portfolio is slightly higher than the maximum wealth $S_{1975}(max) = 18.2431$ achieved by the CSD universal portfolio for data set D. For data sets E, F and G, the maximum wealths achieved by the CSD universal portfolios are $S_{1975}(max) = 29.1040, 22.3262$ and 25.5834 respectively, which are much higher than the maximum wealths achieved by the Helmbold universal portfolios. For data sets F and G, the maximum wealths $S_{1975}(max)$ achieved by the CSD universal portfolios are even higher than the wealth S^*_{1975}

achieved by the best constant rebalanced portfolios (BCRP) from Tables 3.3 and 2.11.

Table 3.3: The portfolios $\mathbf{b}_{1976}$ and the maximum wealths $S_{1975}(max)$ achieved by respective ξ's within an extended range of ξ by the CSD universal portfolio for data sets D, E, F and G, where $\mathbf{b}_1 = (0.1000, 0.1000, \ldots, 0.1000)$

Data set	Smallest ξ	Largest ξ	Best ξ	$\mathbf{b}_{1976}$	$S_{1975}(max)$
Set D	-5.1308	4.6077	0.3769	(0.1333, 0.1300, 0.1178, 0.1053, 0.1051, 0.1018, 0.0921, 0.0593, 0.0831, 0.0722)	18.2431
Set E	-5.1203	4.6078	-2.8760	(0.0035, 0.0057, 0.0014, 0.0310, 0.0250, 0.0224, 0.0261, 0.6996, 0.0260, 0.1593)	29.1040
Set F	-4.9553	5.3162	-4.9553	(0.0003, 0.0001, 0.0000, 0.0000, 0.6796, 0.0000, 0.0008, 0.3062, 0.0003, 0.0127)	22.3262
Set G	-5.1030	4.7028	-3.7942	(0.0013, 0.7729, 0.0184, 0.0044, 0.0626, 0.0017, 0.0000, 0.0026, 0.0021, 0.1340)	25.5834

Next, the CSD universal portfolios are run on the four 10-stock data sets using $\mathbf{b}_1 = \mathbf{b}^*_{1975}$ that is computed in Table 2.11. Table 3.4 shows the resulting portfolios $\mathbf{b}_{1976}$ and the maximum wealths $S_{1975}(max)$ achieved by respective ξ's over the range of values of ξ considered on each data set after 1975 trading days where the initial starting portfolios $\mathbf{b}_1$ are the BCRP's. Data sets D and E achieve the same maximum wealths $S_{1975}(max) = 48.3525$ by the CSD universal portfolios, and it is higher than the maximum wealths $S_{1975}(max)$ achieved by the Helmbold universal portfolios when $\mathbf{b}_1 = \mathbf{b}^*_{1975}$ from Tables 3.4 and 2.12. Whereas for data sets F and G, the maximum wealths achieved by the CSD universal portfolios, $S_{1975}(max) = 26.5128$ and 32.8167 respectively, are lower than the maximum wealths achieved by the Helmbold universal portfolios when $\mathbf{b}_1 = \mathbf{b}^*_{1975}$. From the results, we can conclude that there are CSD universal portfolios that can perform better than the Helmbold universal portfolios and there are CSD universal portfolios that can outperform the BCRP's.

Table 3.4: The portfolios $\mathbf{b}_{1976}$ and the maximum wealths $S_{1975}(max)$ achieved by respective ξ's within an extended range of ξ by the CSD universal portfolio for data sets D, E, F and G, where $\mathbf{b}_1 = \mathbf{b}^*_{1975}$

Data set	Smallest ξ	Largest ξ	Best ξ	$\mathbf{b}_{1976}$	$S_{1975}(max)$
Set D	-9.3310	10.3310	-2.5743	(0.1572, 0.8428, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000)	48.3525
Set E	-9.3310	10.3310	-2.5743	(0.1572, 0.8428, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000)	48.3525
Set F	-4.9959	8.7514	-2.6671	(0.1448, 0.8279, 0.0274, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000)	26.5128
Set G	-3.8807	4.8240	-2.2298	(0.0000, 0.0000, 0.0000, 0.0427, 0.0000, 0.0000, 0.9063, 0.0511, 0.0000, 0.0000)	32.8167

CHAPTER FOUR

MAHALANOBIS UNIVERSAL PORTFOLIO

In Chapter Three, we introduce an additive-update universal portfolio which is generated by the chi-square divergence (CSD) distance measure. It is the objective of this chapter to show that the CSD universal portfolio belongs to a general class of universal portfolios generated by the Mahalanobis squared divergence (alternatively known as the quadratic divergence).

4.1 The Mahalanobis Parametric Family of Additive-Update Universal Portfolio

The results in this section are presented in Tan and Lim [30]. The Mahalanobis squared divergence generates a large family of additive-update universal portfolios containing the subclass of CSD universal portfolios. The Mahalanobis universal portfolio is characterised by three parameters, namely, the positive definite, symmetric matrix generating the divergence, the initial starting portfolio and a scalar parameter.

The *Mahalanobis squared divergence* distance measure with respect to a symmetric, positive definite matrix $A = \left(a_{ij}\right)$ is

$$\mathrm{D_A}(\mathbf{b}_k \parallel \mathbf{b}_n) = [\mathbf{b}_k - \mathbf{b}_n]^t A[\mathbf{b}_k - \mathbf{b}_n], \tag{4.1}$$

where $\mathbf{b}_k = (b_{ki})$ and $\mathbf{b}_n = (b_{ni})$ are any two portfolio vectors. Alternatively, (4.1) can be written as

$$\mathrm{D_A}(\mathbf{b}_k \parallel \mathbf{b}_n) = \sum_{i=1}^{m} a_{ii}[b_{ki} - b_{ni}]^2 + \sum_{i<j} 2a_{ij}[b_{ki} - b_{ni}][b_{kj} - b_{nj}].$$

The *Mahalanobis universal portfolio* is the sequence of universal portfolios $\{\mathbf{b}_{n+1}\}$ given by

$$\mathbf{b}_{n+1} = \mathbf{b}_n + \frac{\xi}{(\mathbf{b}_n^t \mathbf{x}_n)}\left[A^{-1}\mathbf{x}_n - \left(\frac{\mathbf{1}^t A^{-1}\mathbf{x}_n}{\mathbf{1}^t A^{-1}\mathbf{1}}\right)A^{-1}\mathbf{1}\right], \tag{4.2}$$

where initial starting portfolio $\mathbf{b}_1$ is given and ξ is any real number such that $\mathbf{b}_{n+1} \geq 0$ for $n = 1,2,3,\ldots$. The matrix A is assumed to be symmetric and positive definite and $\mathbf{1} = (1,1,\ldots,1)$ denotes a vector consisting of all 1's.

First, we show that the Mahalanobis parametric family universal portfolios maximize and minimize a certain objective functions which is a linear combination of the growth rate of the wealth and the Mahalanobis squared divergence distance measure.

Proposition 4.1 Consider the objective functions

$$F(\mathbf{b}_{n+1}) = 2\xi \log(\mathbf{b}_{n+1}^t \mathbf{x}_n) - \mathrm{D_A}(\mathbf{b}_{n+1} \parallel \mathbf{b}_n)$$

and

$$G(\mathbf{b}_{n+1}) = 2\xi \log(\mathbf{b}_{n+1}^t \mathbf{x}_n) + \mathrm{D_A}(\mathbf{b}_{n+1} \parallel \mathbf{b}_n)$$

where $\mathrm{D_A}(\mathbf{b}_{n+1} \parallel \mathbf{b}_n)$ is the Mahalanobis squared divergence distance measure given by (4.1) and $\xi > 0$. By approximating $\log(\mathbf{b}_{n+1}^t \mathbf{x}_n)$ using $\left[\log(\mathbf{b}_n^t \mathbf{x}_n) + \frac{\mathbf{b}_{n+1}^t \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n} - 1\right]$, the maximum of the objective function $F(\mathbf{b}_{n+1})$ is achieved at $\mathbf{b}_{n+1}$ given by (4.2). Similarly, the minimum of $G(\mathbf{b}_{n+1})$ is also achieved at $\mathbf{b}_{n+1}$ given by (4.2) where ξ is replaced by $-\xi$.

Proof. We introduce the Lagrange multiplier γ for the constraint $\sum_{i=1}^{m} b_{n+1,i} = 1$. Consider maximizing the objective function

$$\begin{aligned}\hat{F}(\mathbf{b}_{n+1},\gamma) = 2\xi\left[\log(\mathbf{b}_n^t\mathbf{x}_n) + \frac{\mathbf{b}_{n+1}^t\mathbf{x}_n}{\mathbf{b}_n^t\mathbf{x}_n} - 1\right] \\ -\sum_{i=1}^{m} a_{ii}\left[b_{n+1,i} - b_{ni}\right]^2 \\ - 2\sum_{i<j} a_{ij}\left[b_{n+1,i} - b_{ni}\right]\left[b_{n+1,j} - b_{nj}\right] \\ + \gamma\left(\sum_{i=1}^{m} b_{n+1,i} - 1\right)\end{aligned} \tag{4.3}$$

and minimizing the objective function

$$\begin{aligned}\hat{G}(\mathbf{b}_{n+1},\gamma) = 2\xi\left[\log(\mathbf{b}_n^t\mathbf{x}_n) + \frac{\mathbf{b}_{n+1}^t\mathbf{x}_n}{\mathbf{b}_n^t\mathbf{x}_n} - 1\right] \\ +\sum_{i=1}^{m} a_{ii}\left[b_{n+1,i} - b_{ni}\right]^2 + 2\sum_{i<j} a_{ij}\left[b_{n+1,i} - b_{ni}\right]\left[b_{n+1,j} - b_{nj}\right] \\ + \gamma\left(\sum_{i=1}^{m} b_{n+1,i} - 1\right).\end{aligned}$$

Differentiating (4.3), we have

$$\begin{aligned}\frac{\partial\hat{F}(\mathbf{b}_{n+1},\gamma)}{\partial b_{n+1,i}} &= 2\xi\frac{x_{ni}}{\mathbf{b}_n^t\mathbf{x}_n} - 2a_{ii}\left[b_{n+1,i} - b_{ni}\right] - \sum_{k=1}^{m-i} 2\,a_{i,i+k}\left[b_{n+1,i+k} - b_{n,i+k}\right] \\ &\quad -\sum_{k=1}^{i-1} 2\,a_{i-k,i}\left[b_{n+1,i-k} - b_{n,i-k}\right] + \gamma \\ &= 2\xi\frac{x_{ni}}{\mathbf{b}_n^t\mathbf{x}_n} - 2\sum_{j=1}^{m} a_{ij}\left[b_{n+1,j} - b_{nj}\right] + \gamma.\end{aligned}$$

Setting $\frac{\partial \hat{F}(\mathbf{b}_{n+1},\gamma)}{\partial b_{n+1,i}} = 0$ for $i = 1,2, \ldots, m$, we obtain

$$2\xi \frac{\mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n} + \gamma \mathbf{1} = 2A[\mathbf{b}_{n+1} - \mathbf{b}_n].$$

Hence

$$\mathbf{b}_{n+1} = \mathbf{b}_n + \frac{\xi}{(\mathbf{b}_n^t \mathbf{x}_n)} A^{-1}\mathbf{x}_n + \frac{\gamma}{2} A^{-1}\mathbf{1}. \tag{4.4}$$

To evaluate γ, pre-multiply (4.4) by $\mathbf{1}^t$ to obtain

$$1 = 1 + \frac{\xi}{(\mathbf{b}_n^t \mathbf{x}_n)} \mathbf{1}^t A^{-1}\mathbf{x}_n + \frac{\gamma}{2} \mathbf{1}^t A^{-1}\mathbf{1}$$

and consequently,

$$\frac{\gamma}{2} = \frac{-\xi}{\mathbf{b}_n^t \mathbf{x}_n} \frac{\mathbf{1}^t A^{-1}\mathbf{x}_n}{\mathbf{1}^t A^{-1}\mathbf{1}}. \tag{4.5}$$

Combining (4.4) and (4.5), we have (4.2). In a similar manner, it can be shown that the minimum of $\hat{G}(\mathbf{b}_{n+1}, \gamma)$ is achieved at $\mathbf{b}_{n+1}$ given by (4.2) where ξ is replaced by $-\xi$. Let

$$\begin{aligned}
\tilde{F}\left(b_{n+1,1}, b_{n+1,2}, \ldots, b_{n+1,m-1}\right) \\
&= 2\xi \left[\log(\mathbf{b}_n^t \mathbf{x}_n) + \frac{\mathbf{b}_{n+1}^t \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n} - 1\right] \\
&- \sum_{i=1}^{m-1} a_{ii}\left[b_{n+1,i} - b_{ni}\right]^2 - a_{mm}\left[b_{n+1,m} - b_{nm}\right]^2 \\
&- 2 \sum_{k=1}^{m-1-i} a_{i,i+k}\left[b_{n+1,i} - b_{ni}\right]\left[b_{n+1,i+k} - b_{n,i+k}\right] \\
&- 2 \sum_{k=1}^{i-1} a_{i-k,i}\left[b_{n+1,i-k} - b_{n,i-k}\right]\left[b_{n+1,i} - b_{ni}\right] \\
&- 2 \sum_{k=1}^{m-1} a_{km}\left[b_{n+1,k} - b_{nk}\right]\left[b_{n+1,m} - b_{nm}\right] \\
&- 2 \sum_{\substack{k<j \\ k,j\neq i \\ j\neq m}} a_{kj}\left[b_{n+1,k} - b_{nk}\right]\left[b_{n+1,j} - b_{nj}\right]
\end{aligned} \tag{4.6}$$

where $b_{n+1,m} = 1 - \sum_{i=1}^{m-1} b_{n+1,i}$. Then the first partial derivatives of (4.6) are

$$\frac{\partial \tilde{F}\left(b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m-1}\right)}{\partial b_{n+1,i}}$$

$$= 2\xi \left[\frac{x_{ni} - x_{nm}}{\mathbf{b}_n^t \mathbf{x}_n}\right] - 2\sum_{k=1}^{m-1} a_{ik}\left[b_{n+1,k} - b_{nk}\right]$$

$$+ 2a_{mm}\left[b_{n+1,m} - b_{nm}\right] + 2\sum_{k=1}^{m-1} a_{km}\left[b_{n+1,k} - b_{nk}\right]$$

$$- 2a_{im}\left[b_{n+1,m} - b_{nm}\right]$$

for $i = 1,2,\dots,m-1$ and the second partial derivatives of (4.6) are

$$\frac{\partial^2 \tilde{F}\left(b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m-1}\right)}{\partial b_{n+1,i}\partial b_{n+1,j}} = \begin{cases} -2a_{ii} + 4a_{im} - 2a_{mm} & \text{for } i = j \\ -2a_{ij} + 2a_{im} + 2a_{jm} - 2a_{mm} & \text{for } i \neq j \end{cases}$$

for $i = 1,2,\dots,m-1$. Given $A = \left(a_{ij}\right)_{i,j=1,2,\dots,m}$ which is positive definite and this implies that the submatrix $\tilde{A} = \left(a_{ij}\right)_{i,j=1,2,\dots,m-1}$ is positive definite. If $A = diag(a_{11}, a_{22}, \dots, a_{mm})$ is a positive diagonal matrix, where $a_{kk} > 0$ for $k = 1,2,\dots,m$, then $a_{ij} = a_{im} = a_{jm} = 0$ for $i \neq j$, $i,j = 1,2,\dots,m-1$, and the Hessian matrix of $\tilde{F}\left(b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m-1}\right)$ is $-2H$ where

$$h_{ij} = \begin{cases} \phi_i + \phi_m & \text{for } i = j \\ \phi_m & \text{for } i \neq j \end{cases}$$

for $i,j = 1,2,\dots,m-1$ and $\phi_k = a_{kk} > 0$ for $k = 1,2,\dots,m$. From the previous result in Proposition 2.2, H is positive definite. Hence the Hessian matrix of $\tilde{F}\left(b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m-1}\right)$ is negative definite. Similarly if

$$\tilde{G}\left(b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m-1}\right)$$

$$= 2\xi \left[\log(\mathbf{b}_n^t \mathbf{x}_n) + \frac{\mathbf{b}_{n+1}^t \mathbf{x}_n}{\mathbf{b}_n^t \mathbf{x}_n} - 1\right]$$

$$+ [\mathbf{b}_{n+1} - \mathbf{b}_n]^t A [\mathbf{b}_{n+1} - \mathbf{b}_n]$$

where $b_{n+1,m} = 1 - \sum_{i=1}^{m-1} b_{n+1,i}$, its Hessian matrix is $2H$ which is positive definite. Hence $\tilde{F}\left(b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m-1}\right)$ and $\tilde{G}\left(b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m-1}\right)$ are concave and convex respectively if A is a positive diagonal matrix, having maximum and minimum points respectively. In general, $\hat{F}(\mathbf{b}_{n+1}, 0)$ and $\hat{G}(\mathbf{b}_{n+1}, 0)$ have the Hessian matrices $-2A$ and $+2A$ respectively where $b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m}$ are m free variables. If A is positive definite and does not depend on $\mathbf{b}_{n+1}$, then $\hat{F}(\mathbf{b}_{n+1}, 0)$ and $\hat{G}(\mathbf{b}_{n+1}, 0)$ are concave and convex respectively over

$$\mathbb{R}^m = \left\{\left(b_{n+1,1}, b_{n+1,2}, \dots, b_{n+1,m}\right): -\infty < b_{n+1,k} < \infty \text{ for } k = 1,2, \dots, m\right\}.$$

If A depends on $\mathbf{b}_{n+1}$ and is positive definite over a sub-region $S \subseteq \mathbb{R}^m$, then $\hat{F}(\mathbf{b}_{n+1}, 0)$ and $\hat{G}(\mathbf{b}_{n+1}, 0)$ are concave and convex respectively over S. Furthermore, $\hat{F}(\mathbf{b}_{n+1}, 0)$ and $\hat{G}(\mathbf{b}_{n+1}, 0)$ have maximum and minimum points respectively. □

The Mahalanobis universal portfolio is an additive-update universal portfolio and hence it is important to derive the sufficient conditions for the "portfolios" generated to be genuine portfolio vectors.

Proposition 4.2 (i) The Mahalanobis universal portfolio (4.2) can be expressed as

$$\mathbf{b}_{n+1} = \mathbf{b}_n + \frac{\xi}{(\mathbf{b}_n^t \mathbf{x}_n)} C(n)\mathbf{x}_n \tag{4.7}$$

where the matrix $C(n) = \left(c_{ij}\right)$ is given by

$$c_{ij} = e_{ij} - \frac{\left(\sum_k e_{ik}\right)\left(\sum_k e_{kj}\right)}{\left(\sum_{k,l} e_{kl}\right)} \tag{4.8}$$

and $E = A^{-1} = \left(e_{ij}\right)$. Furthermore, $C(n)$ is symmetric with zero row sums.

(ii) Given that $\mathbf{b}_n = (b_{ni})$ is a portfolio vector, then for $\mathbf{b}_{n+1}$ defined by (4.7) to be a portfolio, it is necessary and sufficient that

$$-\frac{(\mathbf{b}_n^t \mathbf{x}_n) b_{ni}}{|(C(n)\mathbf{x}_n)_i|} \le \xi \ \text{ or } \ \xi \le \frac{(\mathbf{b}_n^t \mathbf{x}_n) b_{ni}}{|(C(n)\mathbf{x}_n)_i|} \tag{4.9}$$

for $i = 1,2,\dots,m$, where $(C(n)\mathbf{x}_n)_i$ denotes the ith element of the vector $C(n)\mathbf{x}_n$.

(iii) For the sequence $\{\mathbf{b}_{n+1}\}$ defined by (4.7) to be a valid sequence of portfolio vectors, it is sufficient that

$$|\xi| \le \frac{\inf\limits_{n,i}\{x_{ni}\}}{\sup\limits_{n,i}\{x_{ni}\}} \frac{\inf\limits_{n,i}\{b_{ni}\}}{\sup\limits_{n}\{\|C(n)\|_1\}} \tag{4.10}$$

where $\|C(n)\|_1 = \max_i\left\{\sum_{j=1}^{m} \left|c_{ij}\right|\right\}$.

Proof. (i) Comparing (4.2) and (4.7), we must have

$$\begin{aligned} C(n)\mathbf{x}_n &= A^{-1}\mathbf{x}_n - \left(\frac{\mathbf{1}^t A^{-1}\mathbf{x}_n}{\mathbf{1}^t A^{-1}\mathbf{1}}\right) A^{-1}\mathbf{1} \\ &= E\mathbf{x}_n - \left(\frac{\mathbf{1}^t E\mathbf{x}_n}{\mathbf{1}^t E\mathbf{1}}\right) E\mathbf{1}. \end{aligned} \tag{4.11}$$

Identifying the ith element of the vector in (4.11), we have the identity

$$\sum_j c_{ij} x_{nj} = \sum_j e_{ij} x_{nj} - \frac{\left[\sum_{k_2}\left(\sum_{k_3} e_{k_2 k_3}\right) x_{nk_2}\right]}{\left(\sum_{k,l} e_{kl}\right)} \left(\sum_{k_1} e_{ik_1}\right).$$

Comparing coefficients of x_{nj}, we have

$$c_{ij} = e_{ij} - \frac{\left(\sum_{k_3} e_{jk_3}\right)\left(\sum_{k_1} e_{ik_1}\right)}{\left(\sum_{k,l} e_{kl}\right)}.$$

Noting that $\sum_{k_3} e_{jk_3} = \sum_{k_3} e_{k_3 j}$ because E is symmetric, (4.8) follows. Equivalently,

$$C(n) = A^{-1} - \frac{(A^{-1}\mathbf{1})(\mathbf{1}^t A^{-1})}{(\mathbf{1}^t A^{-1}\mathbf{1})}.$$

Consequently, $C(n)$ is symmetric. From (4.8),

$$\sum_j c_{ij} = \sum_j e_{ij} - \sum_k e_{ik} = 0.$$

Thus the row sums of $C(n)$ are zero. Similarly, the column sums of $C(n)$ are also zero.

(ii) Given that $\mathbf{b}_n$ is a portfolio vector, it follows from (4.7) that for $b_{n+1,i} \geq 0$, it is necessary and sufficient that

$$\xi(C(n)\mathbf{x}_n)_i \geq -(\mathbf{b}_n^t \mathbf{x}_n) b_{ni}$$

for $i = 1,2,\dots,m$ and (4.9) is obtained.

(iii) For any vector $\mathbf{y} = (y_i)$, we define

$$\|\mathbf{y}\|_\infty = \max_i\{|y_i|\}.$$

Then, it is easy to deduce that for any matrix C,

$$\|C\mathbf{y}\|_\infty \leq \|\mathbf{y}\|_\infty \|C\|_1$$

where

$$\|C\|_1 = \max_i \left\{ \sum_{j=1}^m |c_{ij}| \right\}.$$

Thus,

$$|(C(n)\mathbf{x}_n)_i| \leq \|C(n)\mathbf{x}_n\|_\infty \leq \|\mathbf{x}_n\|_\infty \|C(n)\|_1$$

for $i = 1,2,\dots,m$. Noting that $(\mathbf{b}_n^t \mathbf{x}_n) \geq \min_i\{x_{ni}\}$, the following inequality holds

$$\frac{\inf\limits_{n,i}\{x_{ni}\}}{\sup\limits_{n,i}\{x_{ni}\}} \frac{\inf\limits_{n,i}\{b_{ni}\}}{\sup\limits_{n}\{\|C(n)\|_1\}} \leq \frac{(\mathbf{b}_n^t \mathbf{x}_n) b_{ni}}{|(C(n)\mathbf{x}_n)_i|}$$

for $i = 1,2, \dots, m$. Thus the condition (4.10) is sufficient for $\{\mathbf{b}_{n+1}\}$ to be a valid sequence of portfolio vectors. □

4.1.1 Mahalanobis Universal Portfolios Generated by Special Symmetric Matrices

We study the Mahalanobis universal portfolios generated by the Mahalanobis squared divergence associated with special symmetric matrices in this subsection. The explicit formulae for the additive-update are derived according to the respective matrices.

We consider investment in a three-stock market using universal portfolios generated by special symmetric, positive definite matrices $A_1(r)$ and $A_2(r,t)$ given below

$$A_1(r) = \frac{1}{(1-r^2)} \begin{bmatrix} 1 & -r & 0 \\ -r & 1+\mathrm{r}^2 & -r \\ 0 & -r & 1 \end{bmatrix} \tag{4.12}$$

where $0 < r < 1$, and

$$A_2(r,t) = \frac{1}{r(r^2-t^2)} \begin{bmatrix} r^2 & 0 & -rt \\ 0 & r^2-t^2 & 0 \\ -rt & 0 & r^2 \end{bmatrix} \tag{4.13}$$

where $0 < t < r$. The corresponding inverse matrices of (4.12) and (4.13) are given by

$$A_1^{-1}(r) = \begin{bmatrix} 1 & r & r^2 \\ r & 1 & r \\ r^2 & r & 1 \end{bmatrix} \tag{4.14}$$

$$A_2^{-1}(r,t) = \begin{bmatrix} r & 0 & t \\ 0 & r & 0 \\ t & 0 & r \end{bmatrix}. \tag{4.15}$$

We now present the xi-parametric families of $A_1(r)$ and $A_2(r,t)$ universal portfolios, omitting the details of the derivation of the formulae which are given in Section 4.1. The xi-parametric family of $A_1(r)$ universal portfolios $\mathbf{b}_{n+1}$ is given by

$$\begin{aligned} b_{n+1,1} &= b_{n1} + \frac{\xi}{(\mathbf{b}_n^t \mathbf{x}_n)}[c_1 x_{n1} + c_2 x_{n2} + c_3 x_{n3}], \\ b_{n+1,2} &= b_{n2} + \frac{\xi}{(\mathbf{b}_n^t \mathbf{x}_n)}[c_2 x_{n1} - 2c_2 x_{n2} + c_2 x_{n3}], \\ b_{n+1,3} &= b_{n3} + \frac{\xi}{(\mathbf{b}_n^t \mathbf{x}_n)}[c_3 x_{n1} + c_2 x_{n2} + c_1 x_{n3}], \end{aligned} \tag{4.16}$$

where

$$\begin{aligned} c_1 &= \frac{(2 + 2r - r^2 - 2r^3 - r^4)}{(3 + 4r + 2r^2)}, \\ c_2 &= \frac{(-1 + r^2)}{(3 + 4r + 2r^2)}, \\ c_3 &= -c_1 - c_2 = \frac{(-1 - 2r + 2r^3 + r^4)}{(3 + 4r + 2r^2)}, \end{aligned} \tag{4.17}$$

for $0 < r < 1$. Note that ξ is a valid parameter only if $b_{n+1,i} \geq 0$ for all $n = 1,2,3,\ldots$ and $i = 1,2,3$, given the initial starting portfolio $\mathbf{b}_1$. The xi-parametric family of $A_2(r,t)$ universal portfolios $\mathbf{b}_{n+1}$ is given by

$$\begin{aligned} b_{n+1,1} &= b_{n1} + \frac{\xi}{(\mathbf{b}_n^t \mathbf{x}_n)}[u_1 x_{n1} + u_2 x_{n2} + u_3 x_{n3}], \\ b_{n+1,2} &= b_{n2} + \frac{\xi}{(\mathbf{b}_n^t \mathbf{x}_n)}[u_2 x_{n1} - 2u_2 x_{n2} + u_2 x_{n3}], \\ b_{n+1,3} &= b_{n3} + \frac{\xi}{(\mathbf{b}_n^t \mathbf{x}_n)}[u_3 x_{n1} + u_2 x_{n2} + u_1 x_{n3}], \end{aligned} \tag{4.18}$$

where

$$u_1 = \frac{2r^2 - t^2}{3r + 2t},$$

$$u_2 = -\left(\frac{r^2 + rt}{3r + 2t}\right), \quad (4.19)$$

$$u_3 = -(u_1 + u_2) = \frac{t^2 + rt - r^2}{3r + 2t},$$

for $0 < t < r$. Again ξ is a valid parameter only if $b_{n+1,i} \geq 0$ for all $n = 1,2,3, \ldots$ and $i = 1,2,3$, given the initial starting portfolio $\mathbf{b}_1$.

In order to compare the performance of the $A_1(r)$ and $A_2(r,t)$ universal portfolios with the Helmbold and CSD universal portfolios, we choose the three same data sets that used in Sections 2.1 and 3.1 with the initial starting portfolio $\mathbf{b}_1 = (0.3333, 0.3333, 0.3334)$.

For data sets A, B and C, the CSD universal portfolios perform better than the Helmbold universal portfolios in Section 3.1. From Table 3.1, the maximum wealths $S_{500}(max)$ achieved by the CSD universal portfolios are $1.5758, 3.8394$ and 3.6480 for data sets A, B and C respectively. We run the $A_1(r)$ and $A_2(r,t)$ universal portfolios on data sets A, B and C to compare their performance with that of the CSD universal portfolios. The portfolios $\mathbf{b}_{501}$ and the maximum wealths $S_{500}(max)$ achieved by respective ξ's over the range of values of ξ considered are displayed in Tables 4.1, 4.2 and 4.3 for nine selected values of r in Table 4.1 and 11 selected pairs of (r,t) in Tables 4.2 and 4.3. In Table 4.1, the $A_1(r)$ universal portfolio achieves a higher wealth of $S_{500}(max) = 1.6435$ for data set A, $r = 0.9$ and $-13.7626 \leq \xi \leq 14.7978$ compared with the value of $S_{500}(max) = 1.5758$ for the CSD universal portfolio. This indicates an increase in wealth of 0.0677 units or 6.77%. In Table 4.2, there is a higher wealth of $S_{500}(max) = 1.6885$ achieved for the $A_2(r,t)$ universal portfolio where $r = 2b_{n2}$ and $t = b_{n2}$ for data set A. This is an increase of 0.1127 units of wealth or 11.27% over the maximum wealth of the CSD universal portfolio. A higher wealth of $S_{500}(max) = 1.5950$ is also observed for data set A in Table 4.3 as the $A_2(r,t)$ universal portfolio where $r = b_{n2} + 1$ and $t = b_{n2}$. This corresponds to a smaller increase of 0.0192 units of wealth or 1.92% over that of the maximum wealth of the CSD universal portfolio. However, it is observed in Tables 4.1, 4.2 and 4.3, that the use of the $A_1(r)$ and $A_2(r,t)$ universal portfolios does not lead to a better performance over the CSD universal portfolios for data sets B and C.

Table 4.1: The portfolios $\mathbf{b}_{501}$ and the maximum wealths $S_{500}(max)$ achieved by respective ξ's within an extended range of ξ by the $A_1(r)$ universal portfolio for selected values of r for data sets A, B and C, where $\mathbf{b}_1 = (0.3333, 0.3333, 0.3334)$

Data set	r	Smallest ξ	Largest ξ	Best ξ	$\mathbf{b}_{501}$	$S_{500}(max)$
Set A	0.1	-1.5551	1.1825	-1.5551	(0.0532, 0.7219, 0.2249)	1.5940
	0.2	-1.7506	1.3834	-1.7506	(0.0533, 0.7060, 0.2407)	1.6005
	0.3	-2.0021	1.6473	-2.0021	(0.0534, 0.6901, 0.2565)	1.6070
	0.4	-2.3360	2.0045	-2.3360	(0.0535, 0.6743, 0.2721)	1.6135
	0.5	-2.8005	2.5094	-2.8005	(0.0536, 0.6588, 0.2875)	1.6199
	0.6	-3.4928	3.2719	-3.4928	(0.0537, 0.6437, 0.3026)	1.6261
	0.7	-4.6402	4.5482	-4.6402	(0.0538, 0.6290, 0.3172)	1.6321
	0.8	-6.9254	7.1078	-6.9254	(0.0539, 0.6149, 0.3313)	1.6379
	0.9	-13.7626	14.7978	-13.7626	(0.0539, 0.6013, 0.3448)	1.6435
Set B	0.1	-0.4724	0.6973	0.6973	(0.0264, 0.1669, 0.8067)	2.6304
	0.2	-0.4978	0.7072	0.7072	(0.0219, 0.1886, 0.7896)	2.6237
	0.3	-0.5350	0.7356	0.7356	(0.0181, 0.2065, 0.7754)	2.6181
	0.4	-0.5890	0.7853	0.7853	(0.0159, 0.2218, 0.7623)	2.6120
	0.5	-0.6690	0.8677	0.8677	(0.0143, 0.2347, 0.7510)	2.6063
	0.6	-0.7936	1.0052	1.0052	(0.0130, 0.2455, 0.7414)	2.6015
	0.7	-1.0064	1.2491	1.2491	(0.0119, 0.2547, 0.7334)	2.5974
	0.8	-1.4391	1.7545	1.7545	(0.0110, 0.2626, 0.7264)	2.5939
	0.9	-2.7490	3.2998	3.2998	(0.0101, 0.2694, 0.7205)	2.5909

Table 4.1 continued

Data set	r	Smallest ξ	Largest ξ	Best ξ	$\mathbf{b}_{501}$	$S_{500}(max)$
Set C	0.1	-0.3859	0.6049	0.6049	(0.1542, 0.0171, 0.8286)	2.4417
	0.2	-0.4137	0.7077	0.7077	(0.1098, 0.0171, 0.8731)	2.4964
	0.3	-0.4515	0.8428	0.8428	(0.0617, 0.0171, 0.9212)	2.5568
	0.4	-0.5040	1.0074	1.0074	(0.0156, 0.0226, 0.9617)	2.6070
	0.5	-0.5795	1.0748	1.0748	(0.0149, 0.0679, 0.9171)	2.5366
	0.6	-0.6950	1.2101	1.2101	(0.0143, 0.1037, 0.8819)	2.4821
	0.7	-0.8900	1.4687	1.4687	(0.0138, 0.1326, 0.8536)	2.4391
	0.8	-1.2830	2.0228	2.0228	(0.0135, 0.1562, 0.8303)	2.4043
	0.9	-2.4687	3.7420	3.7420	(0.0131, 0.1758, 0.8110)	2.3758

Table 4.2: The portfolios $\mathbf{b}_{501}$ and the maximum wealths $S_{500}(max)$ achieved by respective ξ's within an extended range of ξ by the $A_2(r,t)$ universal portfolio for selected values of (r,t) where $r=2t$ for data sets A, B and C, where $\mathbf{b}_1 = (0.3333, 0.3333, 0.3334)$

Data set	r	t	Smallest ξ	Largest ξ	Best ξ	$\mathbf{b}_{501}$	$S_{500}(max)$
Set A	0.10	0.05	-14.4946	9.1316	-3.9838	(0.2610, 0.4554, 0.2835)	1.5663
	0.20	0.10	-7.2473	4.5658	-1.9919	(0.2610, 0.4554, 0.2835)	1.5663
	0.30	0.15	-4.8315	3.0438	-1.3280	(0.2610, 0.4554, 0.2835)	1.5663
	0.50	0.25	-2.8989	1.8263	-0.7968	(0.2610, 0.4554, 0.2835)	1.5663
	1.00	0.50	-1.4494	0.9131	-0.3984	(0.2610, 0.4554, 0.2835)	1.5663
	5.00	2.50	-0.2898	0.1826	-0.0797	(0.2610, 0.4554, 0.2835)	1.5663
	10.00	5.00	-0.1449	0.0913	-0.0398	(0.2611, 0.4553, 0.2836)	1.5663
	20.00	10.00	-0.0724	0.0456	-0.0199	(0.2611, 0.4553, 0.2836)	1.5663
	$2b_{n1}$	b_{n1}	-4.4141	1.2736	-0.1228	(0.3188, 0.3578, 0.3234)	1.5652
	$2b_{n2}$	b_{n2}	-1.6896	11.7141	9.3917	(0.3785, 0.0000, 0.6215)	1.6885
	$2b_{n3}$	b_{n3}	-18.9702	1.2650	-0.1506	(0.3156, 0.3634, 0.3210)	1.5653
Set B	0.10	0.05	-7.2686	11.0391	11.0391	(0.1871, 0.0000, 0.8129)	2.5124
	0.20	0.10	-3.6343	5.5195	5.5194	(0.1871, 0.0000, 0.8129)	2.5124
	0.30	0.15	-2.4228	3.6797	3.6796	(0.1871, 0.0000, 0.8129)	2.5124
	0.50	0.25	-1.4537	2.2078	2.2077	(0.1871, 0.0000, 0.8129)	2.5124
	1.00	0.50	-0.7268	1.1039	1.1039	(0.1871, 0.0000, 0.8129)	2.5124
	5.00	2.50	-0.1453	0.2207	0.2207	(0.1872, 0.0001, 0.8127)	2.5123
	10.00	5.00	-0.0726	0.1103	0.1103	(0.1873, 0.0003, 0.8125)	2.5121
	20.00	10.00	-0.0363	0.0551	0.0551	(0.1874, 0.0005, 0.8120)	2.5118
	$2b_{n1}$	b_{n1}	-0.8953	8.9998	6.2539	(0.0060, 0.3632, 0.6308)	2.5845
	$2b_{n2}$	b_{n2}	-1.1252	11.7313	5.0148	(0.3506, 0.0001, 0.6493)	2.7053
	$2b_{n3}$	b_{n3}	-13.7856	1.0953	1.0953	(0.2076, 0.0000, 0.7924)	2.4134
Set C	0.10	0.05	-5.0734	4.6697	4.6697	(0.3585, 0.0171, 0.6244)	2.2038
	0.20	0.10	-2.5367	2.3348	2.3348	(0.3585, 0.0171, 0.6244)	2.2038
	0.30	0.15	-1.6911	1.5565	1.5565	(0.3585, 0.0171, 0.6244)	2.2038
	0.50	0.25	-1.0146	0.9339	0.9339	(0.3585, 0.0171, 0.6244)	2.2038
	1.00	0.50	-0.5073	0.4669	0.4669	(0.3585, 0.0171, 0.6244)	2.2038
	5.00	2.50	-0.1014	0.0933	0.0933	(0.3585, 0.0174, 0.6241)	2.2034
	10.00	5.00	-0.0507	0.0466	0.0466	(0.3585, 0.0177, 0.6238)	2.2030
	20.00	10.00	-0.0253	0.0233	0.0233	(0.3585, 0.0177, 0.6238)	2.2030
	$2b_{n1}$	b_{n1}	-0.7956	0.7183	0.7183	(0.3487, 0.0215, 0.6297)	2.1924

Table 4.2 continued

Data set	r	t	Smallest ξ	Largest ξ	Best ξ	$\mathbf{b}_{501}$	$S_{500}(max)$
	$2b_{n2}$	b_{n2}	-0.5436	10.3591	3.7002	(0.4177, 0.0002, 0.5821)	2.4972
	$2b_{n3}$	b_{n3}	-5.2933	0.5095	0.5095	(0.3575, 0.0232, 0.6193)	2.1519

Table 4.3: The portfolios $\mathbf{b}_{501}$ and the maximum wealths $S_{500}(max)$ achieved by respective ξ's within an extended range of ξ by the $A_2(r,t)$ universal portfolio for selected values of (r,t) where $r = t + 1$ for data sets A, B and C, where $\mathbf{b}_1 = (0.3333, 0.3333, 0.3334)$

Data set	r	t	Smallest ξ	Largest ξ	Best ξ	$\mathbf{b}_{501}$	$S_{500}(max)$
Set A	1.05	0.05	-1.3357	0.9633	-1.3356	(0.0530, 0.7449, 0.2020)	1.5845
	1.10	0.10	-1.2772	0.9077	-1.2771	(0.0530, 0.7515, 0.1955)	1.5818
	1.15	0.15	-1.2241	0.8587	-1.1981	(0.0592, 0.7478, 0.1930)	1.5794
	1.25	0.25	-1.1313	0.7760	-0.9377	(0.1035, 0.6881, 0.2084)	1.5756
	1.50	0.50	-0.9536	0.6277	-0.5403	(0.1793, 0.5807, 0.2400)	1.5702
	3.50	2.50	-0.4222	0.2527	0.0273	(0.3495, 0.3041, 0.3465)	1.5651
	6	5	-0.2495	0.1452	0.0603	(0.3918, 0.2230, 0.3851)	1.5660
	11	10	-0.1373	0.0785	0.0483	(0.4167, 0.1720, 0.4113)	1.5672
	$b_{n1}+1$	b_{n1}	-1.1212	0.6986	-0.4210	(0.2279, 0.4968, 0.2753)	1.5694
	$b_{n2}+1$	b_{n2}	-0.9859	0.7877	-0.9859	(0.0708, 0.7482, 0.1810)	1.5950
	$b_{n3}+1$	b_{n3}	-1.1283	0.6936	-0.4859	(0.2106, 0.5242, 0.2652)	1.5700
Set B	1.05	0.05	-0.4501	0.7116	0.7116	(0.0351, 0.1262, 0.8387)	2.6424
	1.10	0.10	-0.4443	0.7192	0.7192	(0.0382, 0.1117, 0.8501)	2.6466
	1.15	0.15	-0.4387	0.7268	0.7268	(0.0413, 0.0971, 0.8616)	2.6507
	1.25	0.25	-0.4281	0.7420	0.7420	(0.0476, 0.0678, 0.8846)	2.6588
	1.50	0.50	-0.4039	0.7615	0.7615	(0.0709, 0.0000, 0.9291)	2.6648
	3.50	2.50	-0.2793	0.3043	0.3043	(0.3263, 0.0000, 0.6737)	2.3403
	6	5	-0.2012	0.1745	0.1745	(0.4000, 0.0001, 0.6000)	2.2536
	11	10	-0.1287	0.0942	0.0942	(0.4458, 0.0002, 0.5540)	2.2010
	$b_{n1}+1$	b_{n1}	-0.4051	0.7122	0.7122	(0.0334, 0.1290, 0.8376)	2.6222
	$b_{n2}+1$	b_{n2}	-0.4207	0.7833	0.7833	(0.0413, 0.0354, 0.9234)	2.7187
	$b_{n3}+1$	b_{n3}	-0.4374	0.7079	0.7079	(0.1004, 0.0000, 0.8996)	2.6167
Set C	1.05	0.05	-0.3573	0.4927	0.4927	(0.2131, 0.0171, 0.7697)	2.3709
	1.10	0.10	-0.3494	0.4643	0.4643	(0.2290, 0.0171, 0.7539)	2.3522
	1.15	0.15	-0.3420	0.4392	0.4392	(0.2430, 0.0171, 0.7399)	2.3357
	1.25	0.25	-0.3282	0.3969	0.3969	(0.2666, 0.0171, 0.7163)	2.3082
	1.50	0.50	-0.2986	0.3210	0.3210	(0.3092, 0.0171, 0.6737)	2.2593
	3.50	2.50	-0.1752	0.1292	0.1292	(0.4176, 0.0172, 0.5652)	2.1389
	6	5	-0.1157	0.0742	0.0742	(0.4488, 0.0175, 0.5337)	2.1050
	11	10	-0.0689	0.0401	0.0401	(0.4682, 0.0177, 0.5142)	2.0841
	$b_{n1}+1$	b_{n1}	-0.3133	0.3840	0.3840	(0.2734, 0.0180, 0.7086)	2.2968
	$b_{n2}+1$	b_{n2}	-0.3038	0.4250	0.4250	(0.2528, 0.0133, 0.7339)	2.3504
	$b_{n3}+1$	b_{n3}	-0.3393	0.3213	0.3213	(0.3080, 0.0193, 0.6728)	2.2463

In Table 4.2, we observed that the portfolios $\mathbf{b}_{501}$ and maximum wealths S_{500} achieved by the $A_2(r,t)$ universal portfolios, where r and t are constants independent of n, are approximately equal for data sets A, B and C. Whereas the values of smallest ξ, largest ξ and best ξ vary with respective pairs of (r,t) in some fashion. That is, if $(0.30,0.15)$ is three times the pair $(0.10,0.05)$ in r and t, then the values of ξ for $(0.10,0.05)$ are approximately three times the values of ξ for $(0.30,0.15)$. We strongly believe that the $A_2(r,t)$ universal portfolios have the same behaviour whenever $r = gt$ (where r, g and t are constants) holds for a particular g.

The performance of the $A_1(r)$ and $A_2(r,t)$ universal portfolios depends on the price-relative data set. We have shown that for some data sets, it may be possible to achieve higher investment wealths by using the $A_1(r)$ and $A_2(r,t)$ universal portfolios. The results in this section are reported in Tan and Lim [31].

4.1.2 Mahalanobis Universal Portfolios Generated by Special Diagonal Matrices

The sufficient condition (4.10) for valid parametric values ξ is only useful provided $\inf_{n,i}\{b_{ni}\}$ is bounded away from zero, that is $\inf_{n,i}\{b_{ni}\} > 0$. In practice, this is not easy to verify. Another sufficient condition for valid values of ξ which is more practical is available for the Mahalanobis universal portfolios generated by special diagonal matrices given in the next proposition.

Proposition 4.3 Consider a Mahalanobis universal portfolio generated by a diagonal matrix $A = D = (d_{ii})$ where $d_{ii} = d_i^{-1} > 0$ for $i = 1,2,\dots,m$.

(i) The universal portfolio $\{\mathbf{b}_{n+1}\}$ is given by

$$b_{n+1,i} = b_{ni} + \frac{\xi d_i}{(\mathbf{b}_n^t \mathbf{x}_n)}[x_{ni} - \bar{x}_n(\mathbf{d})] \tag{4.20}$$

for $i = 1,2,\dots,m$ where

$$\bar{x}_n(\mathbf{d}) = \sum_{j=1}^{m} \left(\frac{d_j}{d_1 + d_2 + \cdots + d_m}\right) x_{nj}.$$

(ii) Given that $\mathbf{b}_n = (b_{ni})$ is a portfolio vector, then for $\mathbf{b}_{n+1}$ to be a portfolio vector, it is necessary and sufficient that

$$-\frac{(\mathbf{b}_n^t \mathbf{x}_n) b_{ni}}{d_i |x_{ni} - \bar{x}_n(\mathbf{d})|} \leq \xi \text{ or } \xi \leq \frac{(\mathbf{b}_n^t \mathbf{x}_n) b_{ni}}{d_i |x_{ni} - \bar{x}_n(\mathbf{d})|} \tag{4.21}$$

for $i = 1,2,\dots,m$.

(iii) A sufficient condition for the sequence $\{\mathbf{b}_{n+1}\}$ to be portfolio vectors is that

$$|\xi| \leq \frac{\inf_{n,i}\{x_{ni}\} \inf_{n,i}\{b_{ni}/d_i\}}{\sup_n \left\{\max_i\{x_{ni}\} - \min_i\{x_{ni}\}\right\}}. \tag{4.22}$$

If $d_i = c_i b_{ni}$ for $i = 1,2,\dots,m$, the sufficient condition (4.22) reduces to the following condition independent of $\mathbf{b}_n$:

$$|\xi| \leq \frac{\inf_{n,i}\{x_{ni}\} \inf_{n,i}\{1/c_i\}}{\sup_n \left\{\max_i\{x_{ni}\} - \min_i\{x_{ni}\}\right\}}. \tag{4.23}$$

Proof. (i) Now $A^{-1} = E = D^{-1} = (e_{ii})$ where $e_{ii} = d_i > 0$ for $i = 1,2,\dots,m$. We observe that

$$\frac{\mathbf{1}^t A^{-1} \mathbf{x}_n}{\mathbf{1}^t A^{-1} \mathbf{1}} = \bar{x}_n(\mathbf{d}) = \sum_{j=1}^{m} \left(\frac{d_j}{d_1 + d_2 + \cdots + d_m}\right) x_{nj},$$

$A^{-1}\mathbf{x}_n = (d_i x_{ni})$ and $A^{-1}\mathbf{1} = (d_i)$. From (4.2),

$$b_{n+1,i} = b_{ni} + \frac{\xi}{(\mathbf{b}_n^t \mathbf{x}_n)}[d_i x_{ni} - d_i \bar{x}_n(\mathbf{d})]$$

where (4.20) follows.

(ii) Given that $\mathbf{b}_n$ is a portfolio vecor, it follows from (4.20) that for $b_{n+1,i} \geq 0$, if and only if

$$\xi\big(x_{ni} - \bar{x}_n(\mathbf{d})\big) \geq \frac{-(\mathbf{b}_n^t \mathbf{x}_n) b_{ni}}{d_i}$$

for $i = 1,2, \dots, m$ and the condition (4.21) follows.

(iii) We observe that

$$|x_{ni} - \bar{x}_n(\mathbf{d})| \leq \max_i\{x_{ni}\} - \min_i\{x_{ni}\}$$

and hence

$$\frac{\inf_{n,i}\{x_{ni}\} \inf_{n,i}\{b_{ni}/d_i\}}{\sup_n \left\{\max_i\{x_{ni}\} - \min_i\{x_{ni}\}\right\}} \leq \frac{(\mathbf{b}_n^t \mathbf{x}_n) b_{ni}}{d_i |x_{ni} - \bar{x}_n(\mathbf{d})|}$$

for $i = 1,2, \dots, m$. It is clear that the condition (4.22) is sufficient for $\{\mathbf{b}_{n+1}\}$ to be a valid sequence of portfolio vectors. When $d_i = c_i b_{ni}$ for $i = 1,2, \dots, m$, it is again evident (4.22) becomes (4.23). □

When $c_i = 1$ for $i = 1,2, \dots, m$ in Proposition 4.3(iii), the Mahalanobis universal portfolio generated by $A = D = (d_i^{-1}) = (b_{ni}^{-1})$ is known as the CSD universal portfolio which has been studied in Chapter Three.

We focus on the Mahalanobis universal portfolios generated by special diagonal matrices $D = (d_i^{-1})$ where $d_i = c_i b_{ni}$ for $i = 1,2, \dots, m$, and their performance on the three data sets designated as A, B and C in this thesis. We have run the Helmbold and CSD universal portfolios on the data sets A, B and C using $\mathbf{b}_1 = (0.3333{,}0.3333{,}0.3334)$ in Sections 2.1 and 3.1. Since the maximum wealths achieved by the Helmbold universal portfolios are dominated by the maximum wealths achieved by the CSD universal portfolios for data sets A, B and C, we shall only compare with the CSD universal portfolios.

The Mahalanobis universal portfolios generated by $D = (d_i^{-1})$ for selected values of $d_i = c_i b_{ni}$ for $i = 1,2,\ldots,m$ are run on the data sets A, B and C using $\mathbf{b}_1 = (0.3333{,}0.3333{,}0.3334)$. The portfolios $\mathbf{b}_{501}$ and the maximum wealths S_{500} achieved by each universal portfolio generated by (d_1, d_2, d_3) over the range of values of ξ considered are listed in Tables 4.4, 4.5 and 4.6. The first row of each table lists down the maximum wealth S_{500} achieved by the CSD universal portfolio for comparison. It can be seen from the three tables that the $(d_1, d_2, d_3) = (2b_{n1}, 2b_{n2}, 6b_{n3})$ Mahalanobis universal portfolio always outperform the CSD universal portfolio in terms of maximum wealth S_{500} achieved for data sets A, B and C. In Table 4.4, the $(3b_{n1}, 6b_{n2}, b_{n3})$ Mahalanobis universal portfolio achieves a higher maximum wealth $S_{500} = 1.9108$ than the maximum wealth $S_{500} = 1.5758$ achieved by the CSD universal portfolio for data set A. For data set C, the $(3b_{n1}, b_{n2}, 6b_{n3})$ Mahalanobis universal portfolio achieves the maximum wealth $S_{500} = 3.7799$ which is higher than the maximum wealth $S_{500} = 3.6480$ achieved by the CSD universal portfolio in Table 4.6.

Table 4.4: The portfolios $\mathbf{b}_{501}$ and the maximum wealths $S_{500}(max)$ achieved by respective ξ's within an extended range of ξ by the selected (d_1, d_2, d_3) Mahalanobis universal portfolios for data set A, where $\mathbf{b}_1 = (0.3333,0.3333,0.3334)$

(d_1, d_2, d_3)	Smallest ξ	Largest ξ	Best ξ	$\mathbf{b}_{501}$	$S_{500}(max)$
(b_{n1}, b_{n2}, b_{n3})	-14.6638	16.1608	-1.9174	(0.2106, 0.5298, 0.2596)	1.5758
$(6b_{n1}, b_{n2}, 3b_{n3})$	-3.1077	3.8873	-2.0078	(0.0533, 0.7057, 0.2410)	1.6242
$(b_{n1}, 6b_{n2}, 3b_{n3})$	-4.9417	2.7498	2.1733	(0.4603, 0.0000, 0.5397)	1.6110
$(b_{n1}, 3b_{n2}, 6b_{n3})$	-2.6348	5.3052	2.6447	(0.4000, 0.0003, 0.5997)	1.5835
$(6b_{n1}, 3b_{n2}, b_{n3})$	-2.9890	5.2269	2.7036	(0.7653, 0.0002, 0.2345)	1.6346
$(3b_{n1}, b_{n2}, 6b_{n3})$	-3.3646	7.7393	-1.8684	(0.1848, 0.6923, 0.1230)	1.6375
$(3b_{n1}, 6b_{n2}, b_{n3})$	-5.2591	3.0243	5.2591	(0.0000, 0.9058, 0.0942)	1.9108
$(6b_{n1}, 2b_{n2}, 2b_{n3})$	-2.9395	4.1060	-0.7843	(0.1484, 0.5387, 0.3130)	1.5766
$(2b_{n1}, 6b_{n2}, 2b_{n3})$	-6.9911	2.9861	6.9911	(0.0000, 1.0000, 0.0000)	1.7543
$(2b_{n1}, 2b_{n2}, 6b_{n3})$	-2.8029	7.8468	-1.0109	(0.2342, 0.6063, 0.1596)	1.5980

Table 4.5: The portfolios $\mathbf{b}_{501}$ and the maximum wealths $S_{500}(max)$ achieved by respective ξ's within an extended range of ξ by the selected (d_1, d_2, d_3) Mahalanobis universal portfolios for data set B, where $\mathbf{b}_1 = (0.3333,0.3333,0.3334)$

(d_1, d_2, d_3)	Smallest ξ	Largest ξ	Best ξ	$\mathbf{b}_{501}$	$S_{500}(max)$
(b_{n1}, b_{n2}, b_{n3})	-12.1887	13.0114	13.0114	(0.0000, 0.0000, 1.0000)	3.8394
$(6b_{n1}, b_{n2}, 3b_{n3})$	-4.1560	2.4293	2.4293	(0.0000, 0.0279, 0.9721)	3.4195
$(b_{n1}, 6b_{n2}, 3b_{n3})$	-4.0106	2.3587	2.3587	(0.0291, 0.0000, 0.9709)	3.3248
$(b_{n1}, 3b_{n2}, 6b_{n3})$	-2.0563	4.3663	4.3663	(0.0011, 0.0000, 0.9989)	3.7693
$(6b_{n1}, 3b_{n2}, b_{n3})$	-4.0459	2.9111	2.9111	(0.0000, 0.0000, 1.0000)	3.3396
$(3b_{n1}, b_{n2}, 6b_{n3})$	-2.0858	4.5329	4.5329	(0.0000, 0.0006, 0.9994)	3.7953
$(3b_{n1}, 6b_{n2}, b_{n3})$	-2.2510	2.6682	2.6682	(0.0001, 0.0000, 0.9999)	3.2429
$(6b_{n1}, 2b_{n2}, 2b_{n3})$	-6.0547	2.6942	2.6941	(0.0000, 0.0004, 0.9996)	3.5292
$(2b_{n1}, 6b_{n2}, 2b_{n3})$	-2.7724	2.6674	2.6674	(0.0009, 0.0000, 0.9991)	3.4399
$(2b_{n1}, 2b_{n2}, 6b_{n3})$	-2.0892	6.4927	6.4926	(0.0000, 0.0000, 1.0000)	3.9974

Table 4.6: The portfolios $\mathbf{b}_{501}$ and the maximum wealths $S_{500}(max)$ achieved by respective ξ's within an extended range of ξ by the selected (d_1, d_2, d_3) Mahalanobis universal portfolios for data set C, where $\mathbf{b}_1 = (0.3333,0.3333,0.3334)$

(d_1, d_2, d_3)	Smallest ξ	Largest ξ	Best ξ	$\mathbf{b}_{501}$	$S_{500}(max)$
(b_{n1}, b_{n2}, b_{n3})	-15.2103	16.7997	11.8504	(0.0000, 0.0000, 1.0000)	3.6480
$(6b_{n1}, b_{n2}, 3b_{n3})$	-4.2718	3.0235	3.0234	(0.0000, 0.0032, 0.9968)	3.4880
$(b_{n1}, 6b_{n2}, 3b_{n3})$	-2.8263	2.8594	2.5320	(0.0259, 0.0000, 0.9741)	3.1309
$(b_{n1}, 3b_{n2}, 6b_{n3})$	-2.8606	3.8431	2.8236	(0.0129, 0.0000, 0.9871)	3.3513
$(6b_{n1}, 3b_{n2}, b_{n3})$	-4.1566	3.1274	3.1273	(0.0000, 0.0000, 1.0000)	3.3555
$(3b_{n1}, b_{n2}, 6b_{n3})$	-2.9289	4.7738	4.7738	(0.0000, 0.0001, 0.9999)	3.7799
$(3b_{n1}, 6b_{n2}, b_{n3})$	-2.5521	2.9303	2.9303	(0.0000, 0.0000, 1.0000)	3.2294
$(6b_{n1}, 2b_{n2}, 2b_{n3})$	-6.2426	3.0355	3.0355	(0.0000, 0.0000, 1.0000)	3.5758

Table 4.6 continued

(d_1, d_2, d_3)	Smallest ξ	Largest ξ	Best ξ	$\mathbf{b}_{501}$	$S_{500}(max)$
$(2b_{n1}, 6b_{n2}, 2b_{n3})$	-2.5884	2.8559	2.8559	(0.0004, 0.0000, 0.9996)	3.3278
$(2b_{n1}, 2b_{n2}, 6b_{n3})$	-2.8886	5.8405	4.1410	(0.0000, 0.0000, 1.0000)	3.7027

We have shown that there are Mahalanobis universal portfolios generated by $(c_1 b_{n1}, c_2 b_{n2}, c_3 b_{n3})$ that can outperform the CSD and Helmbold universal portfolios. To select an appropriate parametric value ξ, we can use the sufficient condition (4.23) given by Proposition 4.3(iii).

4.2 Running the Mahalanobis Universal Portfolios on 10-stock Data Sets

The implementation of the Dirichlet-weighted universal portfolio needs computer memory requirements that are growing exponentially with the number of stocks. The disadvantage of the Dirichlet-weighted universal portfolio is the large computer memory requirements required to implement the algorithm if the number of stocks in the portfolio exceeds nine. We run the Mahalanobis universal portfolio which needs much lesser computer memory requirements on the 10-stock data sets D, E, F and G. In the meantime, a sufficient condition for the Mahalanobis or Helmbold universal portfolios to achieve a wealth exceeding that of the best constant rebalanced portfolio (BCRP) is derived in this section.

Proposition 4.4 Given the price-relative vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, suppose $\mathbf{b}_n^*$ is the BCRP.

(i) Given A and $\mathbf{b}_1 = \mathbf{b}_n^*$, suppose the Mahalanobis parametric family is defined for some $-\xi_1 \leq \xi \leq \xi_2$ where $\xi_1, \xi_2 > 0$. If $\xi = 0$ is not a local maximum point of $S_n(\xi)$, the wealth achieved by the Mahalanobis universal portfolio (4.2), then there exists some ξ_0 in the interval $[-\xi_1, \xi_2]$ such that $S_n(\xi_0) > S_n(0)$.

(ii) Consider the Helmbold parametric family where $\mathbf{b}_1 = \mathbf{b}_n^*$. If $\eta = 0$ is not a local maximum point of $S_n(\eta)$, the wealth achieved by the Helmbold universal portfolio (2.2), then there exists some η_0 such that $S_n(\eta_0) > S_n(0)$.

Proof. (i) We observe that when $\xi = 0$ and $\mathbf{b}_1 = \mathbf{b}_n^*$, the portfolio given by (4.2) is a constant rebalanced portfolio $\mathbf{b}_j = \mathbf{b}_n^*$ for $j = 1,2,3,\dots$. Thus $S_n(0)$ at $\xi = 0$ is the wealth achieved by the BCRP $\mathbf{b}_n^*$. The function $S_n(\xi)$ is continuous in ξ for $-\xi_1 \leq \xi \leq \xi_2$. If $\xi = 0$ is not a local maximum point, then there exists some ξ_0 in the interval $[-\xi_1, \xi_2]$ such that $S_n(\xi_0) > S_n(0)$.

(ii) The proof is analogous to (i). □

For data sets D, E, F and G in Tables 2.12 and 2.11, the maximum wealths achieved by the Helmbold universal portfolios, where the initial starting portfolio $\mathbf{b}_1 = \mathbf{b}_{1975}^*$, are $43.2025, 43.2025, 27.2148$ and 39.9419 respectively. These are much higher than the wealths achieved by the BCRP's which are $37.5867, 37.5867, 20.7169$ and 24.6381 for the four data sets respectively. We verify that the results are true from the sufficient condition for the Helmbold universal portfolio to achieve a wealth higher than that of the BCRP in Proposition 4.4(ii). From Tables 3.4 and 2.11, the values of $S_{1975}(max)$ for the CSD universal portfolios, where the initial starting portfolio $\mathbf{b}_1 = \mathbf{b}_{1975}^*$, are $48.3525, 48.3525, 26.5128$ and 32.8167 for data sets D, E, F and G respectively. Again the sufficient condition in Proposition 4.4(i) holds and the values of $S_{1975}(max)$ for the CSD universal portfolios are higher compared to the wealths achieved by the BCRP's.

Before implementing the Mahalanobis universal portfolios on the 10-stock data sets, we consider the following two special symmetric, positive definite matrices $A_3(r)$ and $A_4(r,t)$ that are given by:

$$A_3(r) = \frac{1}{(1-r^2)}\begin{bmatrix} 1 & -r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -r & 1+r^2 & -r & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -r & 1+r^2 & -r & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -r & 1+r^2 & -r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -r & 1+r^2 & -r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -r & 1+r^2 & -r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -r & 1+r^2 & -r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -r & 1+r^2 & -r & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -r & 1+r^2 & -r \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -r & 1 \end{bmatrix} \qquad (4.24)$$

where $0 < r < 1$, and

$$A_4(r,t) = \frac{1}{r(r^2-t^2)}\begin{bmatrix} r^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -rt \\ 0 & r^2-t^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r^2-t^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r^2-t^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r^2-t^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r^2-t^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r^2-t^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & r^2-t^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r^2-t^2 & 0 \\ -rt & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r^2 \end{bmatrix} \qquad (4.25)$$

where $0 < t < r$. The corresponding inverse matrices of $A_3(r)$ and $A_4(r,t)$ are given by

$$A_3^{-1}(r) = \begin{bmatrix} 1 & r & r^2 & r^3 & r^4 & r^5 & r^6 & r^7 & r^8 & r^9 \\ r & 1 & r & r^2 & r^3 & r^4 & r^5 & r^6 & r^7 & r^8 \\ r^2 & r & 1 & r & r^2 & r^3 & r^4 & r^5 & r^6 & r^7 \\ r^3 & r^2 & r & 1 & r & r^2 & r^3 & r^4 & r^5 & r^6 \\ r^4 & r^3 & r^2 & r & 1 & r & r^2 & r^3 & r^4 & r^5 \\ r^5 & r^4 & r^3 & r^2 & r & 1 & r & r^2 & r^3 & r^4 \\ r^6 & r^5 & r^4 & r^3 & r^2 & r & 1 & r & r^2 & r^3 \\ r^7 & r^6 & r^5 & r^4 & r^3 & r^2 & r & 1 & r & r^2 \\ r^8 & r^7 & r^6 & r^5 & r^4 & r^3 & r^2 & r & 1 & r \\ r^9 & r^8 & r^7 & r^6 & r^5 & r^4 & r^3 & r^2 & r & 1 \end{bmatrix} \qquad (4.26)$$

$$A_4^{-1}(r,t) = \begin{bmatrix} r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t \\ 0 & r & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & r & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r & 0 \\ t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r \end{bmatrix}. \tag{4.27}$$

The explicit formulae of the xi-parametric families of $A_3(r)$ and $A_4(r,t)$ universal portfolios $\mathbf{b}_{n+1}$ are

$$\mathbf{b}_{n+1} = \mathbf{b}_n + \frac{\xi}{(\mathbf{b}_n^t\mathbf{x}_n)} C_3(n)\mathbf{x}_n \tag{4.28}$$

and

$$\mathbf{b}_{n+1} = \mathbf{b}_n + \frac{\xi}{(\mathbf{b}_n^t\mathbf{x}_n)} C_4(n)\mathbf{x}_n \tag{4.29}$$

respectively. The matrix $C_3(n) = (c_{ij})$ for $0 < r < 1$ is listed in Appendix A and the matrix $C_4(n) = (c_{ij})$ for $0 < t < r$ is listed in Appendix B and C. Note that ξ is a valid parameter only if $b_{n+1,i} \geq 0$ for all $n = 1,2,3,\dots$ and $i = 1,2,3$, given the initial starting portfolio $\mathbf{b}_1$.

We run the $A_3(r)$ and $A_4(r,t)$ universal portfolios on data sets D, E, F and G with the same initial starting portfolio $\mathbf{b}_1 = (0.1000, 0.1000, \dots, 0.1000)$. From Tables 2.10 and 3.3, the Helmbold universal portfolio performs better than the CSD universal portfolio for data set D and the maximum wealth achieved by the Helmbold universal portfolio is 18.2486. Whereas for data sets E, F and G, the CSD universal portfolios outperform the Helmbold universal portfolios and the maximum wealths achieved by the CSD universal portfolios are $29.1040, 22.3262$ and 25.5834 respectively.

The portfolios $\mathbf{b}_{1976}$ and the maximum wealths $S_{1975}(max)$ achieved by respective ξ's over the range of values of ξ considered are listed in Tables 4.7, 4.8 and 4.9 for nine selected values of r in Table 4.7 and 13 selected pairs of (r, t) in Tables 4.8 and 4.9. For data set D, the $A_3(r)$ universal portfolios for nine selected values of r in Table 4.8 achieve higher wealths of $S_{1975}(max)$ compared with the values of $S_{1975}(max) = 18.2486$ for the Helmbold universal portfolio. In Tables 4.8 and 4.9, the $A_4(b_{n1}, 0.5b_{n1})$ and $A_4(11,10)$ universal portfolios achieve the maximum wealths of $S_{1975}(max) = 20.2585$ and 18.8172 respectively which are higher than the maximum wealth achieved by the Helmbold universal portfolio for data set D. Both the CSD and Helmbold universal portfolios perform better than the $A_3(r)$ universal portfolios for nine selected values of r in Table 4.8 for data set E. For data set E in Tables 4.8 and 4.9, the maximum wealths achieved by $A_4(b_{n1}, 0.5b_{n1})$ and $A_4(11,10)$ universal portfolios are $S_{1975}(max) = 25.6109$ and 23.5094 respectively, and these are higher than the values of $S_{1975}(max) = 22.9859$ for the Helmbold universal portfolio but they do not exceed the values of $S_{1975}(max) = 29.1040$ for the CSD universal portfolio. However, the $A_3(r)$ and $A_4(r, t)$ universal portfolios for selected values of the parameters in Tables 4.7, 4.8 and 4.9 do not generate higher values of maximum wealths over the Helmbold and CSD universal portfolios for data sets F and G.

Table 4.7: The portfolios $\mathbf{b}_{1976}$ and the maximum wealths $S_{1975}(max)$ achieved by respective ξ's within an extended range of ξ by the $A_3(r)$ universal portfolio for selected values of r for data sets D, E, F and G, where $\mathbf{b}_1 = (0.1000, 0.1000, \ldots, 0.1000)$

Data set	r	Smallest ξ	Biggest ξ	Best ξ	$\mathbf{b}_{1976}$	$S_{1975}(max)$
Set D	0.1	-0.0771	0.0663	0.0663	(0.1593, 0.1589, 0.1440, 0.1174, 0.1152, 0.1069, 0.0827, 0.0086, 0.0610, 0.0460)	18.7404
	0.2	-0.0759	0.0649	0.0649	(0.1641, 0.1670, 0.1509, 0.1238, 0.1173, 0.1051, 0.0749, 0.0062, 0.0485, 0.0422)	19.1134
	0.3	-0.0743	0.0631	0.0631	(0.1691, 0.1744, 0.1582, 0.1308, 0.1194, 0.1022, 0.0673, 0.0041, 0.0371, 0.0374)	19.5177
	0.4	-0.0699	0.0612	0.0612	(0.1745, 0.1814, 0.1656, 0.1379, 0.1212, 0.0985, 0.0601, 0.0025, 0.0268, 0.0316)	19.9440
	0.5	-0.0641	0.0589	0.0589	(0.1794, 0.1868, 0.1717, 0.1440, 0.1223, 0.0946, 0.0544, 0.0025, 0.0187, 0.0257)	20.3540
	0.6	-0.0588	0.0579	0.0579	(0.1853, 0.1918, 0.1770, 0.1490, 0.1229, 0.0909, 0.0497, 0.0029, 0.0117, 0.0189)	20.7607
	0.7	-0.0570	0.0599	0.0599	(0.1929, 0.1971, 0.1816, 0.1528, 0.1230, 0.0878, 0.0462, 0.0030, 0.0051, 0.0104)	21.1709
	0.8	-0.0618	0.0651	0.0651	(0.1968, 0.1971, 0.1807, 0.1524, 0.1216, 0.0862, 0.0468, 0.0085, 0.0043, 0.0057)	21.3865
	0.9	-0.0856	0.0893	0.0893	(0.1973, 0.1928, 0.1754, 0.1486, 0.1190, 0.0860, 0.0508, 0.0174, 0.0081, 0.0044)	21.4057
Set E	0.1	-0.0795	0.0785	-0.0795	(0.0360, 0.0396, 0.0564, 0.0883, 0.0933, 0.1014, 0.1214, 0.1346, 0.1503, 0.1787)	21.2555
	0.2	-0.0786	0.0774	-0.0786	(0.0306, 0.0305, 0.0491, 0.0826, 0.0920, 0.1034, 0.1257, 0.1423, 0.1609, 0.1829)	20.8407
	0.3	-0.0773	0.0737	-0.0773	(0.0249, 0.0218, 0.0412, 0.0758, 0.0902, 0.1059, 0.1309, 0.1509, 0.1712, 0.1873)	20.3906
	0.4	-0.0754	0.0697	0.0697	(0.1714, 0.1762, 0.1574, 0.1268, 0.1105, 0.0922, 0.0674, 0.0486, 0.0302, 0.0193)	20.5225
	0.5	-0.0713	0.0657	0.0657	(0.1739, 0.1789, 0.1613, 0.1319, 0.1119, 0.0902, 0.0640, 0.0434, 0.0259, 0.0185)	20.9279
	0.6	-0.0676	0.0625	0.0625	(0.1761, 0.1803, 0.1639, 0.1360, 0.1131, 0.0886, 0.0614, 0.0396, 0.0231, 0.0178)	21.3113
	0.7	-0.0658	0.0614	0.0614	(0.1782, 0.1804, 0.1648, 0.1384, 0.1138, 0.0876, 0.0600, 0.0378, 0.0219, 0.0171)	21.6518
	0.8	-0.0697	0.0658	0.0658	(0.1800, 0.1793, 0.1639, 0.1390, 0.1139, 0.0873, 0.0601, 0.0379, 0.0221, 0.0164)	21.9304
	0.9	-0.0959	0.0915	0.0915	(0.1815, 0.1770, 0.1612, 0.1378, 0.1134, 0.0876, 0.0618, 0.0400, 0.0238, 0.0158)	22.1351
Set F	0.1	-0.0801	0.0914	-0.0801	(0.0478, 0.0450, 0.0518, 0.0719, 0.0852, 0.1012, 0.1411, 0.1623, 0.1464, 0.1473)	12.9063
	0.2	-0.0770	0.0910	-0.0770	(0.0439, 0.0377, 0.0452, 0.0664, 0.0831, 0.1045, 0.1455, 0.1678, 0.1551, 0.1508)	12.6373
	0.3	-0.0735	0.0922	-0.0346	(0.0720, 0.0679, 0.0716, 0.0820, 0.0913, 0.1036, 0.1228, 0.1339, 0.1293, 0.1257)	12.3765
	0.4	-0.0699	0.0917	0.0917	(0.1799, 0.1928, 0.1822, 0.1529, 0.1253, 0.0872, 0.0366, 0.0046, 0.0139, 0.0246)	12.5802
	0.5	-0.0666	0.0848	0.0848	(0.1831, 0.1955, 0.1851, 0.1562, 0.1250, 0.0847, 0.0356, 0.0044, 0.0094, 0.0210)	12.8835
	0.6	-0.0642	0.0800	0.0800	(0.1868, 0.1973, 0.1866, 0.1580, 0.1243, 0.0828, 0.0359, 0.0054, 0.0062, 0.0167)	13.1567
	0.7	-0.0643	0.0786	0.0786	(0.1909, 0.1980, 0.1861, 0.1578, 0.1231, 0.0818, 0.0376, 0.0080, 0.0046, 0.0121)	13.3883
	0.8	-0.0703	0.0847	0.0847	(0.1946, 0.1971, 0.1834, 0.1556, 0.1213, 0.0818, 0.0411, 0.0124, 0.0048, 0.0079)	13.5663
	0.9	-0.0987	0.1189	0.1189	(0.1978, 0.1947, 0.1788, 0.1517, 0.1192, 0.0827, 0.0458, 0.0183, 0.0067, 0.0042)	13.6845

Table 4.7 continued

Data set	r	Smallest ξ	Biggest ξ	Best ξ	$\mathbf{b}_{1976}$	$S_{1975}(max)$
Set G	0.1	-0.0865	0.0867	-0.0865	(0.1143, 0.1085, 0.1519, 0.0744, 0.1468, 0.0689, 0.0243, 0.0580, 0.0928, 0.1602)	15.7198
	0.2	-0.0849	0.0856	-0.0849	(0.1168, 0.1151, 0.1501, 0.0852, 0.1395, 0.0671, 0.0219, 0.0529, 0.0942, 0.1572)	15.6329
	0.3	-0.0851	0.0849	-0.0851	(0.1204, 0.1219, 0.1502, 0.0944, 0.1330, 0.0642, 0.0195, 0.0484, 0.0940, 0.1539)	15.5673
	0.4	-0.0875	0.0847	-0.0875	(0.1254, 0.1292, 0.1519, 0.1022, 0.1270, 0.0608, 0.0171, 0.0441, 0.0923, 0.1500)	15.5261
	0.5	-0.0930	0.0852	-0.0930	(0.1322, 0.1372, 0.1548, 0.1087, 0.1214, 0.0571, 0.0148, 0.0399, 0.0890, 0.1450)	15.5147
	0.6	-0.1037	0.0868	-0.1037	(0.1411, 0.1462, 0.1587, 0.1142, 0.1164, 0.0534, 0.0123, 0.0353, 0.0840, 0.1383)	15.5437
	0.7	-0.1229	0.0925	-0.1229	(0.1520, 0.1558, 0.1626, 0.1188, 0.1120, 0.0506, 0.0110, 0.0309, 0.0772, 0.1290)	15.6159
	0.8	-0.1352	0.1088	-0.1352	(0.1524, 0.1530, 0.1539, 0.1180, 0.1067, 0.0582, 0.0270, 0.0405, 0.0753, 0.1149)	15.5513
	0.9	-0.1940	0.1677	-0.1940	(0.1504, 0.1479, 0.1447, 0.1159, 0.1036, 0.0665, 0.0424, 0.0506, 0.0749, 0.1032)	15.5168

Table 4.8: The portfolios $\mathbf{b}_{1976}$ and the maximum wealths $S_{1975}(max)$ achieved by respective ξ's within an extended range of ξ by the $A_4(r,t)$ universal portfolio for selected values of (r,t) where $r = 2t$ for data sets D, E, F and G, where $\mathbf{b}_1 = (0.1000, 0.1000, \ldots, 0.1000)$

Data set	r	t	Smallest ξ	Biggest ξ	Best ξ	$\mathbf{b}_{1976}$	$S_{1975}(max)$
Set D	0.1	0.05	-1.0908	0.6640	-1.0908	(0.0523, 0.0189, 0.0359, 0.0791, 0.0801, 0.0881, 0.1129, 0.2488, 0.1419, 0.1421)	18.3449
	0.3	0.15	-0.3636	0.2213	-0.3636	(0.0523, 0.0189, 0.0359, 0.0791, 0.0801, 0.0881, 0.1129, 0.2488, 0.1419, 0.1421)	18.3449
	20	10	-0.0054	0.0033	-0.0054	(0.0528, 0.0197, 0.0365, 0.0793, 0.0803, 0.0882, 0.1128, 0.2473, 0.1415, 0.1416)	18.3433
	b_{n1}	$0.5b_{n1}$	-0.9983	0.5344	-0.9983	(0.0556, 0.0825, 0.1241, 0.0645, 0.1065, 0.0787, 0.0768, 0.1891, 0.0848, 0.1374)	20.2585
	b_{n2}	$0.5b_{n2}$	-0.8013	0.7327	-0.1222	(0.0943, 0.0911, 0.0924, 0.0974, 0.0981, 0.0994, 0.1015, 0.1167, 0.1046, 0.1045)	18.1867
	b_{n3}	$0.5b_{n3}$	-0.7303	0.6363	-0.7303	(0.0278, 0.0967, 0.0540, 0.0388, 0.1206, 0.1253, 0.1071, 0.1905, 0.1168, 0.1224)	18.7318
	b_{n4}	$0.5b_{n4}$	-1.2706	0.5152	-1.2706	(0.0799, 0.0672, 0.1311, 0.0649, 0.1032, 0.0567, 0.0661, 0.1915, 0.0835, 0.1560)	20.2515
	b_{n5}	$0.5b_{n5}$	-0.7916	0.8051	-0.0809	(0.0964, 0.0939, 0.0951, 0.0984, 0.0984, 0.0992, 0.1011, 0.1110, 0.1033, 0.1031)	18.1833
	b_{n6}	$0.5b_{n6}$	-0.7937	0.7501	-0.0623	(0.0974, 0.0952, 0.0962, 0.0990, 0.0988, 0.0993, 0.1009, 0.1084, 0.1025, 0.1024)	18.1818
	b_{n7}	$0.5b_{n7}$	-8.3597	0.5808	-3.5099	(0.1343, 0.0846, 0.0691, 0.1140, 0.1174, 0.0482, 0.0087, 0.1520, 0.0899, 0.1818)	18.2697

Table 4.8 continued

Data set	r	t	Smallest ξ	Biggest ξ	Best ξ	$\mathbf{b}_{1976}$	$S_{1975}(max)$
	b_{n8}	$0.5b_{n8}$	-0.5787	1.2565	-0.0780	(0.0966, 0.0934, 0.0952, 0.0988, 0.0982, 0.0990, 0.1014, 0.1109, 0.1034, 0.1031)	18.1832
	b_{n9}	$0.5b_{n9}$	-1.7772	0.6754	-1.7772	(0.0956, 0.0195, 0.0823, 0.1061, 0.0584, 0.0754, 0.0651, 0.2634, 0.0758, 0.1583)	19.7305
	$b_{n,10}$	$0.5b_{n,10}$	-0.7955	0.8888	-0.7955	(0.0371, 0.0368, 0.0603, 0.0738, 0.0830, 0.0929, 0.1233, 0.2282, 0.1457, 0.1189)	18.4615
Set E	0.1	0.05	-0.9573	0.7689	-0.9573	(0.0725, 0.0383, 0.0544, 0.0908, 0.0923, 0.0992, 0.1207, 0.1336, 0.1466, 0.1517)	22.5669
	0.3	0.15	-0.3191	0.2563	-0.3191	(0.0725, 0.0383, 0.0544, 0.0908, 0.0923, 0.0992, 0.1207, 0.1336, 0.1466, 0.1517)	22.5669
	20	10	-0.0047	0.0038	-0.0047	(0.0730, 0.0394, 0.0552, 0.0910, 0.0924, 0.0992, 0.1204, 0.1329, 0.1458, 0.1507)	22.5227
	b_{n1}	$0.5b_{n1}$	-1.2008	0.6545	-1.2008	(0.0564, 0.0892, 0.1460, 0.0814, 0.1115, 0.0723, 0.0756, 0.1746, 0.0794, 0.1136)	25.6109
	b_{n2}	$0.5b_{n2}$	-0.7196	0.8836	-0.7196	(0.0609, 0.0586, 0.0378, 0.0821, 0.1088, 0.1252, 0.1143, 0.1473, 0.1326, 0.1324)	21.9863
	b_{n3}	$0.5b_{n3}$	-0.6343	0.7628	-0.6343	(0.0397, 0.0971, 0.0642, 0.0609, 0.1212, 0.1262, 0.1154, 0.1448, 0.1247, 0.1058)	22.6598
	b_{n4}	$0.5b_{n4}$	-1.3057	0.7107	-1.3057	(0.0720, 0.0673, 0.1356, 0.0715, 0.1082, 0.0580, 0.0733, 0.1854, 0.0910, 0.1378)	24.9203
	b_{n5}	$0.5b_{n5}$	-0.7494	0.8828	-0.7494	(0.0725, 0.0436, 0.0360, 0.0847, 0.0862, 0.1107, 0.1213, 0.1372, 0.1560, 0.1517)	21.4996
	b_{n6}	$0.5b_{n6}$	-0.7959	0.7385	-0.7511	(0.1016, 0.0296, 0.0317, 0.1124, 0.0882, 0.0904, 0.1246, 0.1180, 0.1436, 0.1600)	21.0837
	b_{n7}	$0.5b_{n7}$	-8.5644	0.6542	-2.8352	(0.1102, 0.0647, 0.0603, 0.1192, 0.1098, 0.0538, 0.0270, 0.1808, 0.1243, 0.1498)	21.7422
	b_{n8}	$0.5b_{n8}$	-0.6013	7.7343	-0.6013	(0.0698, 0.0365, 0.0459, 0.0759, 0.0895, 0.1120, 0.1281, 0.1151, 0.1705, 0.1565)	21.5750
	b_{n9}	$0.5b_{n9}$	-1.6284	0.6557	-1.6284	(0.1297, 0.0173, 0.0844, 0.1286, 0.0688, 0.0873, 0.0937, 0.0850, 0.0993, 0.2059)	22.6953
	$b_{n,10}$	$0.5b_{n,10}$	-1.0910	0.7939	-1.0910	(0.0704, 0.0242, 0.0973, 0.1132, 0.0760, 0.0707, 0.1380, 0.1275, 0.1455, 0.1372)	23.9270
Set F	0.1	0.05	-0.9172	0.8678	-0.9172	(0.0706, 0.0480, 0.0538, 0.0746, 0.0868, 0.0983, 0.1412, 0.1631, 0.1419, 0.1219)	13.4338
	0.3	0.15	-0.3057	0.2892	-0.3057	(0.0706, 0.0480, 0.0538, 0.0746, 0.0868, 0.0983, 0.1412, 0.1631, 0.1419, 0.1219)	13.4337
	20	10	-0.0045	0.0043	-0.0045	(0.0711, 0.0490, 0.0547, 0.0751, 0.0870, 0.0983, 0.1404, 0.1619, 0.1410, 0.1215)	13.4128
	b_{n1}	$0.5b_{n1}$	1.3646	0.5700	1.3646	(0.0516, 0.0980, 0.0423, 0.0472, 0.1653, 0.0437, 0.1753, 0.1592, 0.1317, 0.0856)	13.6639
	b_{n2}	$0.5b_{n2}$	-0.8036	1.0218	-0.8036	(0.0454, 0.0576, 0.0781, 0.0684, 0.1109, 0.1024, 0.1417, 0.1690, 0.1371, 0.0895)	13.6128

Table 4.8 continued

Data set	r	t	Smallest ξ	Biggest ξ	Best ξ	$\mathbf{b}_{1976}$	$S_{1975}(max)$
	b_{n3}	$0.5b_{n3}$	-1.3978	0.8223	-1.3978	(0.0716, 0.0471, 0.0355, 0.0481, 0.1070, 0.0730, 0.1406, 0.2203, 0.1407, 0.1160)	13.8168
	b_{n4}	$0.5b_{n4}$	-8.5710	0.6759	-1.8249	(0.0683, 0.0903, 0.0613, 0.0278, 0.1475, 0.0971, 0.1447, 0.1592, 0.1271, 0.0768)	13.4172
	b_{n5}	$0.5b_{n5}$	-0.5808	6.3903	-0.5808	(0.0528, 0.0604, 0.0762, 0.0764, 0.0862, 0.1118, 0.1316, 0.1584, 0.1386, 0.1076)	13.3606
	b_{n6}	$0.5b_{n6}$	-1.4878	0.7295	-1.4878	(0.1036, 0.0604, 0.0642, 0.0346, 0.0949, 0.0495, 0.1385, 0.1625, 0.1380, 0.1538)	12.8251
	b_{n7}	$0.5b_{n7}$	-0.6139	15.5624	-0.6139	(0.0633, 0.0534, 0.0731, 0.0925, 0.0878, 0.1156, 0.1214, 0.1534, 0.1337, 0.1058)	13.4846
	b_{n8}	$0.5b_{n8}$	-0.7385	1.2187	-0.7385	(0.0883, 0.0169, 0.0265, 0.0758, 0.0564, 0.1139, 0.1408, 0.1550, 0.1655, 0.1609)	13.1255
	b_{n9}	$0.5b_{n9}$	-0.8460	0.9255	-0.8460	(0.0820, 0.0387, 0.0563, 0.0939, 0.0762, 0.1092, 0.1365, 0.1546, 0.1235, 0.1292)	13.3314
	$b_{n,10}$	$0.5b_{n,10}$	-1.2778	0.6008	-1.2778	(0.0740, 0.0750, 0.0323, 0.0824, 0.1053, 0.0588, 0.1720, 0.1513, 0.1409, 0.1080)	13.5838
Set G	0.1	0.05	-0.8713	0.8767	-0.8713	(0.1384, 0.0980, 0.1511, 0.0594, 0.1501, 0.0668, 0.0255, 0.0614, 0.0867, 0.1625)	15.6803
	0.3	0.15	-0.2904	0.2922	-0.2904	(0.1384, 0.0980, 0.1511, 0.0594, 0.1501, 0.0668, 0.0256, 0.0614, 0.0867, 0.1625)	15.6802
	20	10	-0.0043	0.0043	-0.0043	(0.1379, 0.0980, 0.1505, 0.0599, 0.1494, 0.0673, 0.0265, 0.0619, 0.0869, 0.1617)	15.6658
	b_{n1}	$0.5b_{n1}$	-1.7963	0.6740	-1.7963	(0.0944, 0.0572, 0.1739, 0.1014, 0.1703, 0.0335, 0.0699, 0.0479, 0.0628, 0.1886)	15.6586
	b_{n2}	$0.5b_{n2}$	-0.5736	5.8736	-0.5736	(0.1508, 0.0925, 0.1512, 0.0430, 0.1408, 0.0797, 0.0236, 0.0655, 0.0854, 0.1675)	15.2276
	b_{n3}	$0.5b_{n3}$	-0.7680	1.0477	-0.7680	(0.1527, 0.0809, 0.1359, 0.0717, 0.1412, 0.0601, 0.0337, 0.0460, 0.1038, 0.1738)	15.5640
	b_{n4}	$0.5b_{n4}$	-1.2149	0.6810	-1.2149	(0.0722, 0.1692, 0.1367, 0.0469, 0.1698, 0.0465, 0.1155, 0.0932, 0.0450, 0.1049)	17.5055
	b_{n5}	$0.5b_{n5}$	-0.5795	14.8871	14.8871	(0.1098, 0.0749, 0.0986, 0.1179, 0.0000, 0.1281, 0.1906, 0.0743, 0.0920, 0.1138)	15.8507
	b_{n6}	$0.5b_{n6}$	-0.7396	0.7859	-0.7396	(0.1332, 0.1035, 0.1447, 0.0806, 0.1388, 0.0689, 0.0168, 0.0710, 0.0888, 0.1538)	15.1810
	b_{n7}	$0.5b_{n7}$	-0.7010	0.7401	-0.7010	(0.0996, 0.1355, 0.1252, 0.0271, 0.1606, 0.1065, 0.0462, 0.1068, 0.0867, 0.1059)	16.3894
	b_{n8}	$0.5b_{n8}$	-0.7562	0.9586	-0.7562	(0.1356, 0.1177, 0.1512, 0.0515, 0.1511, 0.0826, 0.0152, 0.0654, 0.0816, 0.1480)	15.3945
	b_{n9}	$0.5b_{n9}$	-8.7551	0.6915	-2.1902	(0.1012, 0.1685, 0.1427, 0.0910, 0.1618, 0.0572, 0.0560, 0.1009, 0.0228, 0.0980)	15.6721
	$b_{n,10}$	$0.5b_{n,10}$	-0.9780	0.7572	-0.9780	(0.1255, 0.0478, 0.1590, 0.1006, 0.1538, 0.0364, 0.0475, 0.0454, 0.1108, 0.1732)	15.6307

Table 4.9: The portfolios $\mathbf{b}_{1976}$ and the maximum wealths $S_{1975}(max)$ achieved by respective ξ's within an extended range of ξ by the $A_4(r,t)$ universal portfolio for selected values of (r,t) where $r = t + 1$ for data sets D, E, F and G, where $\mathbf{b}_1 = (0.1000, 0.1000, \ldots, 0.1000)$

Data set	r	t	Smallest ξ	Biggest ξ	Best ξ	$\mathbf{b}_{1976}$	$S_{1975}(max)$
Set D	1.05	0.05	-0.0767	0.0636	0.0636	(0.1519, 0.1501, 0.1375, 0.1118, 0.1131, 0.1073, 0.0907, 0.0118, 0.0743, 0.0515)	18.3724
	1.5	0.5	-0.0666	0.0443	0.0203	(0.1171, 0.1228, 0.1173, 0.1055, 0.1059, 0.1033, 0.0958, 0.0595, 0.0883, 0.0846)	18.1783
	11	10	-0.0098	0.0060	-0.0098	(0.0882, 0.0201, 0.0370, 0.0797, 0.0805, 0.0885, 0.1129, 0.2473, 0.1417, 0.1041)	18.8172
	b_{n1}+1	b_{n1}	-0.0780	0.0582	0.0582	(0.1478, 0.1515, 0.1403, 0.1107, 0.1140, 0.1062, 0.0890, 0.0121, 0.0724, 0.0558)	18.4328
	b_{n2}+1	b_{n2}	-0.0748	0.0611	0.0611	(0.1492, 0.1511, 0.1364, 0.1107, 0.1143, 0.1093, 0.0903, 0.0122, 0.0738, 0.0527)	18.3222
	b_{n3}+1	b_{n3}	-0.0736	0.0602	0.0602	(0.1476, 0.1532, 0.1366, 0.1089, 0.1153, 0.1098, 0.0905, 0.0121, 0.0734, 0.0525)	18.3867
	b_{n4}+1	b_{n4}	-0.0784	0.0588	0.0588	(0.1489, 0.1501, 0.1398, 0.1110, 0.1135, 0.1058, 0.0896, 0.0120, 0.0733, 0.0560)	18.3918
	b_{n5}+1	b_{n5}	-0.0751	0.0614	0.0614	(0.1500, 0.1503, 0.1366, 0.1113, 0.1127, 0.1081, 0.0908, 0.0120, 0.0751, 0.0531)	18.3070
	b_{n6}+1	b_{n6}	-0.0761	0.0611	0.0611	(0.1509, 0.1488, 0.1358, 0.1132, 0.1125, 0.1066, 0.0911, 0.0118, 0.0747, 0.0544)	18.2845
	b_{n7}+1	b_{n7}	-0.0769	0.0596	0.0596	(0.1493, 0.1492, 0.1371, 0.1129, 0.1126, 0.1065, 0.0900, 0.0117, 0.0754, 0.0552)	18.3130
	b_{n8}+1	b_{n8}	-0.0735	0.0634	0.0634	(0.1524, 0.1472, 0.1368, 0.1139, 0.1114, 0.1064, 0.0929, 0.0110, 0.0757, 0.0524)	18.3134
	b_{n9}+1	b_{n9}	-0.0762	0.0606	0.0606	(0.1507, 0.1492, 0.1380, 0.1135, 0.1118, 0.1065, 0.0904, 0.0115, 0.0734, 0.0550)	18.3404
	$b_{n,10}$+1	$b_{n,10}$	-0.0728	0.0631	0.0631	(0.1500, 0.1496, 0.1371, 0.1122, 0.1125, 0.1079, 0.0925, 0.0117, 0.0761, 0.0505)	18.3109
Set E	1.05	0.05	-0.079	0.0745	-0.0790	(0.0423, 0.0470, 0.0612, 0.0926, 0.0939, 0.0998, 0.1185, 0.1293, 0.1409, 0.1745)	21.7355
	1.5	0.5	-0.0609	0.0515	-0.0609	(0.0602, 0.0413, 0.0567, 0.0914, 0.0928, 0.0994, 0.1200, 0.1321, 0.1446, 0.1614)	22.2494
	11	10	-0.0099	0.0068	-0.0099	(0.1071, 0.0294, 0.0474, 0.0891, 0.0908, 0.0987, 0.1231, 0.1383, 0.1527, 0.1233)	23.5094
	b_{n1}+1	b_{n1}	-0.0777	0.0694	-0.0777	(0.0404, 0.0511, 0.0678, 0.0916, 0.0960, 0.0981, 0.1153, 0.1333, 0.1362, 0.1703)	22.0048
	b_{n2}+1	b_{n2}	-0.0763	0.0718	-0.0763	(0.0432, 0.0470, 0.0579, 0.0911, 0.0954, 0.1027, 0.1186, 0.1322, 0.1411, 0.1708)	21.8231

Table 4.9 continued

Data set	r	t	Smallest ξ	Biggest ξ	Best ξ	$\mathbf{b}_{1976}$	$S_{1975}(max)$
	b_{n3}+1	b_{n3}	-0.0739	0.0712	-0.0739	(0.0412, 0.0535, 0.0610, 0.0890, 0.0977, 0.1039, 0.1186, 0.1316, 0.1390, 0.1645)	21.9282
	b_{n4}+1	b_{n4}	-0.0793	0.0702	-0.0793	(0.0420, 0.0464, 0.0649, 0.0914, 0.0944, 0.0970, 0.1169, 0.1329, 0.1393, 0.1747)	21.9298
	b_{n5}+1	b_{n5}	-0.0767	0.0717	-0.0767	(0.0452, 0.0449, 0.0577, 0.0917, 0.0927, 0.1007, 0.1195, 0.1306, 0.1436, 0.1734)	21.7684
	b_{n6}+1	b_{n6}	-0.0787	0.0707	-0.0787	(0.0467, 0.0415, 0.0553, 0.0947, 0.0923, 0.0983, 0.1204, 0.1288, 0.1437, 0.1784)	21.7034
	b_{n7}+1	b_{n7}	-0.0796	0.0695	-0.0796	(0.0441, 0.0436, 0.0588, 0.0944, 0.0927, 0.0982, 0.1179, 0.1291, 0.1439, 0.1773)	21.7332
	b_{n8}+1	b_{n8}	-0.0745	0.0741	-0.0745	(0.0473, 0.0439, 0.0583, 0.0906, 0.0929, 0.1013, 0.1205, 0.1281, 0.1454, 0.1718)	21.8006
	b_{n9}+1	b_{n9}	-0.0802	0.0697	-0.0802	(0.0463, 0.0418, 0.0593, 0.0952, 0.0910, 0.0980, 0.1192, 0.1268, 0.1417, 0.1806)	21.7571
	$b_{n,10}$+1	$b_{n,10}$	-0.0770	0.0718	-0.0770	(0.0459, 0.0421, 0.0603, 0.0944, 0.0911, 0.0983, 0.1220, 0.1269, 0.1443, 0.1746)	21.7857
Set F	1.05	0.05	-0.0797	0.0860	-0.0797	(0.0529, 0.0523, 0.0576, 0.0766, 0.0875, 0.0982, 0.1372, 0.1572, 0.1379, 0.1427)	13.1906
	1.5	0.5	-0.0598	0.0586	-0.0598	(0.0633, 0.0491, 0.0547, 0.0750, 0.0869, 0.0982, 0.1401, 0.1616, 0.1408, 0.1303)	13.3523
	11	10	-0.0077	0.0076	-0.0077	(0.0913, 0.0522, 0.0576, 0.0770, 0.0879, 0.0988, 0.1381, 0.1585, 0.1387, 0.0999)	13.4743
	b_{n1}+1	b_{n1}	-0.0788	0.0776	-0.0788	(0.0502, 0.0556, 0.0569, 0.0745, 0.0934, 0.0954, 0.1398, 0.1577, 0.1376, 0.1389)	13.2399
	b_{n2}+1	b_{n2}	-0.0759	0.0827	-0.0759	(0.0512, 0.0529, 0.0597, 0.0760, 0.0900, 0.0987, 0.1381, 0.1585, 0.1382, 0.1366)	13.2386
	b_{n3}+1	b_{n3}	-0.0785	0.0823	-0.0785	(0.0541, 0.0513, 0.0560, 0.0756, 0.0873, 0.0976, 0.1374, 0.1598, 0.1385, 0.1424)	13.2086
	b_{n4}+1	b_{n4}	-0.0788	0.0787	-0.0788	(0.0529, 0.0532, 0.0575, 0.0746, 0.0890, 0.0990, 0.1369, 0.1579, 0.1373, 0.1416)	13.2035
	b_{n5}+1	b_{n5}	-0.0737	0.0872	-0.0737	(0.0535, 0.0527, 0.0598, 0.0765, 0.0871, 0.0999, 0.1372, 0.1583, 0.1387, 0.1363)	13.2431
	b_{n6}+1	b_{n6}	-0.0808	0.0802	-0.0808	(0.0565, 0.0496, 0.0554, 0.0741, 0.0854, 0.0954, 0.1377, 0.1591, 0.1392, 0.1475)	13.1626
	b_{n7}+1	b_{n7}	-0.0735	0.0846	-0.0735	(0.0550, 0.0522, 0.0592, 0.0787, 0.0871, 0.1004, 0.1354, 0.1569, 0.1376, 0.1376)	13.2371
	b_{n8}+1	b_{n8}	-0.0766	0.0855	-0.0766	(0.0579, 0.0482, 0.0539, 0.0762, 0.0838, 0.0991, 0.1380, 0.1580, 0.1408, 0.1441)	13.2058
	b_{n9}+1	b_{n9}	-0.0761	0.0817	-0.0761	(0.0566, 0.0507, 0.0570, 0.0782, 0.0856, 0.0991, 0.1371, 0.1567, 0.1368, 0.1422)	13.1981
	$b_{n,10}$+1	$b_{n,10}$	-0.0767	0.0790	-0.0767	(0.0559, 0.0524, 0.0564, 0.0788, 0.0856, 0.0966, 0.1385, 0.1554, 0.1382, 0.1423)	13.1941

Table 4.9 continued

Data set	r	t	Smallest ξ	Biggest ξ	Best ξ	$\mathbf{b}_{1976}$	$S_{1975}(max)$
Set G	1.05	0.05	-0.0850	0.0838	-0.085	(0.1156, 0.1010, 0.1555, 0.0615, 0.1545, 0.0691, 0.0267, 0.0635, 0.0894, 0.1631)	15.8091
	1.5	0.5	-0.0585	0.0585	-0.0585	(0.1303, 0.0990, 0.1526, 0.0602, 0.1516, 0.0676, 0.0260, 0.0622, 0.0876, 0.1629)	15.7250
	11	10	-0.0077	0.0079	-0.0077	(0.1560, 0.0957, 0.1474, 0.0583, 0.1464, 0.0655, 0.0254, 0.0602, 0.0849, 0.1602)	15.5622
	b_{n1}+1	b_{n1}	-0.0827	0.0785	-0.0827	(0.1189, 0.0980, 0.1559, 0.0642, 0.1533, 0.0675, 0.0275, 0.0615, 0.0882, 0.1651)	15.7546
	b_{n2}+1	b_{n2}	-0.0768	0.0830	-0.0768	(0.1226, 0.0989, 0.1549, 0.0583, 0.1523, 0.0703, 0.0256, 0.0630, 0.0886, 0.1654)	15.7037
	b_{n3}+1	b_{n3}	-0.0793	0.0808	-0.0793	(0.1212, 0.0976, 0.1532, 0.0629, 0.1525, 0.0677, 0.0274, 0.0611, 0.0910, 0.1655)	15.7524
	b_{n4}+1	b_{n4}	-0.0858	0.0791	-0.0858	(0.1110, 0.1067, 0.1563, 0.0595, 0.1579, 0.0662, 0.0321, 0.0652, 0.0861, 0.1591)	16.0125
	b_{n5}+1	b_{n5}	-0.0761	0.0826	-0.0761	(0.1215, 0.0990, 0.1534, 0.0619, 0.1501, 0.0699, 0.0263, 0.0628, 0.0913, 0.1639)	15.6943
	b_{n6}+1	b_{n6}	-0.0789	0.0793	-0.0789	(0.1184, 0.1005, 0.1540, 0.0645, 0.1519, 0.0689, 0.0256, 0.0642, 0.0896, 0.1625)	15.7021
	b_{n7}+1	b_{n7}	-0.0809	0.0796	-0.0809	(0.1104, 0.1064, 0.1531, 0.0569, 0.1568, 0.0749, 0.0274, 0.0695, 0.0890, 0.1557)	15.9454
	b_{n8}+1	b_{n8}	-0.0797	0.0803	-0.0797	(0.1176, 0.1026, 0.1552, 0.0603, 0.1541, 0.0705, 0.0255, 0.0634, 0.0886, 0.1621)	15.7518
	b_{n9}+1	b_{n9}	-0.0831	0.0781	-0.0831	(0.1166, 0.1018, 0.1547, 0.0637, 0.1534, 0.0683, 0.0271, 0.0640, 0.0873, 0.1631)	15.7703
	$b_{n,10}$+1	$b_{n,10}$	-0.0818	0.0792	-0.0818	(0.1173, 0.0970, 0.1562, 0.0633, 0.1553, 0.0662, 0.0283, 0.0618, 0.0916, 0.1631)	15.8382

We observe in Subsection 4.1.1 that the portfolios $\mathbf{b}_{1976}$ and maximum wealths S_{1975} achieved by $A_2(r,t)$ universal portfolios are approximately equal for different pairs of (r,t) satisfying the relationship of $r = gt$, where r, g and t are constants. The values of smallest, largest and best ξ vary according to the respective pairs of (r,t) in some manner. From Table 4.8, we do believe that the $A_4(r,t)$ universal portfolios satisfying the relationship of $r = gt$, where r, g and t are constants, also possess the same behaviour for each particular g.

The BCRP's for data sets D, E, F and G are listed in Table 2.11 and some of the components of BCRP's are zero. We can replace the BCRP's with zero components by the approximate positive BCRP's. We use the notation $\mathbf{b}_{1975}^{\odot}$ for the approximate positive BCRP's. Next, we run the $A_3(r)$ and $A_4(r,t)$ universal portfolios with the initial starting portfolios being the approximate positive BCRP's $\mathbf{b}_{1975}^{\odot}$ given in Table 4.10.

Table 4.10: The best constant rebalanced portfolios $\mathbf{b}_{1975}^{*}$ and the approximate positive best constant rebalanced portfolios $\mathbf{b}_{1975}^{\odot}$ for data sets D, E, F and G

Data set	$\mathbf{b}_{1975}^{*}$	$\mathbf{b}_{1975}^{\odot}$
Set D	(0.5981, 0.4019, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000)	(0.5581, 0.3619, 0.0100, 0.0100, 0.0100, 0.0100, 0.0100, 0.0100, 0.0100, 0.0100)
Set E	(0.5981, 0.4019, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000)	(0.5581, 0.3619, 0.0100, 0.0100, 0.0100, 0.0100, 0.0100, 0.0100, 0.0100, 0.0100)
Set F	(0.4836, 0.3869, 0.1295, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000)	(0.4536, 0.3669, 0.1095, 0.0100, 0.0100, 0.0100, 0.0100, 0.0100, 0.0100, 0.0100)
Set G	(0.0000, 0.0000, 0.0000, 0.1965, 0.0000, 0.0000, 0.5926, 0.2109, 0.0000, 0.0000)	(0.0100, 0.0100, 0.0100, 0.1765, 0.0100, 0.0100, 0.5626, 0.1909, 0.0100, 0.0100)

Tables 4.11 shows the portfolios $\mathbf{b}_{1976}$ and the maximum wealths $S_{1975}(max)$ achieved by respective ξ 's over the range of values of ξ considered for data sets D, E, F and G after 1975 trading days for nine selected $A_3(r)$ universal portfolios where the initial starting portfolios $\mathbf{b}_1 = \mathbf{b}_{1975}^{\odot}$. Similarly, the results for 13 selected $A_4(r,t)$ universal portfolios are listed in Tables 4.12 and 4.13. The values of $S_{1975}(max)$ for selected $A_3(r)$ and $A_4(r,t)$ universal portfolios in Tables 4.11, 4.12 and 4.13 are much lower than the maximum wealths achieved by the Helmbold and CSD universal portfolios where the initial starting portfolios $\mathbf{b}_1 = \mathbf{b}_{1975}^{*}$. Since the initial starting portfolios are replaced by the approximate positive BCRP's

instead of the true BCRP's with zero components, the Proposition 4.4(i) does not hold for the results in Tables 4.11, 4.12 and 4.13. From Tables 4.11, 4.12 and 4.13, we observe that the values of $S_{1975}(max)$ for selected $A_3(r)$ and $A_4(r,t)$ universal portfolios do not exceed the wealths achieved by the BCRP's. However, the values of $S_{1975}(max)$ for selected $A_3(r)$ and $A_4(r,t)$ universal portfolios for data sets D, E, F and G, which are approximately $35, 35, 19$ and 23 respectively, are slightly lower than the wealths achieved by the BCRP's. The performance for selected $A_3(r)$ and $A_4(r,t)$ universal portfolios are considered to be good since it is close to the wealths achieved by the BCRP's, $S^*_{1975} = 37.5867, 37.5867, 20.7169$ and 24.6381 respectively, for data sets D, E, F and G.

Table 4.11: The portfolios $\mathbf{b}_{1976}$ and the maximum wealths $S_{1975}(max)$ achieved by respective ξ's within an extended range of ξ by the $A_3(r)$ universal portfolio for selected values of r for data sets D, E, F and G, where $\mathbf{b}_1 = \mathbf{b}^{\odot}_{1975}$

Data set	r	Smallest ξ	Biggest ξ	Best ξ	$\mathbf{b}_{1976}$	$S_{1975}(max)$
Set D	0.1	-0.0096	0.0066	0.0066	(0.5630, 0.3672, 0.0148, 0.0118, 0.0116, 0.0109, 0.0086, 0.0010, 0.0064, 0.0048)	35.2323
	0.2	-0.0086	0.0065	0.0065	(0.5635, 0.3680, 0.0154, 0.0125, 0.0119, 0.0108, 0.0078, 0.0007, 0.0051, 0.0044)	35.2970
	0.3	-0.0078	0.0064	0.0064	(0.5641, 0.3688, 0.0162, 0.0133, 0.0121, 0.0105, 0.0070, 0.0004, 0.0039, 0.0038)	35.3687
	0.4	-0.0071	0.0062	0.0062	(0.5646, 0.3695, 0.0168, 0.0140, 0.0123, 0.0101, 0.0063, 0.0003, 0.0028, 0.0032)	35.4414
	0.5	-0.0066	0.0060	0.0060	(0.5652, 0.3701, 0.0174, 0.0146, 0.0125, 0.0097, 0.0057, 0.0003, 0.0020, 0.0026)	35.5127
	0.6	-0.0063	0.0059	0.0059	(0.5658, 0.3706, 0.0178, 0.0151, 0.0125, 0.0094, 0.0052, 0.0004, 0.0012, 0.0019)	35.5804
	0.7	-0.0064	0.0061	0.0061	(0.5667, 0.3712, 0.0182, 0.0154, 0.0125, 0.0090, 0.0049, 0.0004, 0.0006, 0.0011)	35.6463
	0.8	-0.0073	0.0067	0.0067	(0.5672, 0.3713, 0.0181, 0.0154, 0.0124, 0.0088, 0.0049, 0.0009, 0.0005, 0.0006)	35.6861
	0.9	-0.0111	0.0092	0.0092	(0.5674, 0.3709, 0.0175, 0.0149, 0.0121, 0.0088, 0.0052, 0.0018, 0.0009, 0.0005)	35.6898

Table 4.11 continued

Data set	r	Smallest ξ	Biggest ξ	Best ξ	$\mathbf{b}_{1976}$	$S_{1975}(max)$
Set E	0.1	-0.0109	0.0079	-0.0109	(0.5513, 0.3547, 0.0036, 0.0085, 0.0089, 0.0099, 0.0126, 0.0142, 0.0162, 0.0201)	35.8853
	0.2	-0.0100	0.0079	-0.0100	(0.5512, 0.3542, 0.0033, 0.0078, 0.0088, 0.0101, 0.0128, 0.0148, 0.0170, 0.0198)	35.7862
	0.3	-0.0093	0.0075	-0.0093	(0.5510, 0.3537, 0.0030, 0.0072, 0.0086, 0.0104, 0.0132, 0.0155, 0.0177, 0.0198)	35.6939
	0.4	-0.0086	0.0071	-0.0086	(0.5507, 0.3534, 0.0027, 0.0065, 0.0085, 0.0106, 0.0136, 0.0161, 0.0183, 0.0197)	35.6044
	0.5	-0.0081	0.0067	0.0067	(0.5645, 0.3692, 0.0163, 0.0133, 0.0114, 0.0093, 0.0067, 0.0045, 0.0028, 0.0020)	35.6085
	0.6	-0.0078	0.0064	0.0064	(0.5648, 0.3694, 0.0165, 0.0137, 0.0115, 0.0091, 0.0064, 0.0041, 0.0025, 0.0019)	35.6736
	0.7	-0.0080	0.0063	0.0063	(0.5651, 0.3694, 0.0165, 0.0139, 0.0116, 0.0090, 0.0063, 0.0040, 0.0024, 0.0018)	35.7308
	0.8	-0.0091	0.0068	0.0068	(0.5655, 0.3694, 0.0164, 0.0140, 0.0116, 0.0089, 0.0062, 0.0040, 0.0024, 0.0017)	35.7783
	0.9	-0.0133	0.0095	0.0095	(0.5657, 0.3693, 0.0161, 0.0138, 0.0115, 0.0089, 0.0064, 0.0042, 0.0025, 0.0017)	35.8132
Set F	0.1	-0.0124	0.0091	-0.0124	(0.4470, 0.3598, 0.1025, 0.0055, 0.0073, 0.0096, 0.0156, 0.0193, 0.0164, 0.0170)	19.9367
	0.2	-0.0124	0.0091	-0.0124	(0.4462, 0.3585, 0.1013, 0.0045, 0.0069, 0.0100, 0.0164, 0.0204, 0.0180, 0.0178)	19.8847
	0.3	-0.0122	0.0093	-0.0122	(0.4454, 0.3572, 0.1002, 0.0035, 0.0065, 0.0105, 0.0172, 0.0214, 0.0195, 0.0187)	19.8251
	0.4	-0.0119	0.0094	-0.0119	(0.4445, 0.3561, 0.0991, 0.0026, 0.0062, 0.0110, 0.0179, 0.0222, 0.0209, 0.0196)	19.7616
	0.5	-0.0117	0.0087	0.0087	(0.4611, 0.3757, 0.1179, 0.0160, 0.0129, 0.0089, 0.0038, 0.0004, 0.0010, 0.0021)	19.8024
	0.6	-0.0117	0.0082	0.0082	(0.4615, 0.3760, 0.1180, 0.0161, 0.0128, 0.0087, 0.0038, 0.0006, 0.0007, 0.0017)	19.8443
	0.7	-0.0123	0.0081	0.0081	(0.4621, 0.3762, 0.1180, 0.0160, 0.0127, 0.0086, 0.0040, 0.0009, 0.0005, 0.0012)	19.8798
	0.8	-0.0144	0.0087	0.0087	(0.4625, 0.3761, 0.1177, 0.0157, 0.0124, 0.0085, 0.0043, 0.0014, 0.0006, 0.0008)	19.9058
	0.9	-0.0220	0.0123	0.0123	(0.4629, 0.3760, 0.1173, 0.0153, 0.0121, 0.0085, 0.0048, 0.0019, 0.0008, 0.0005)	19.9244
Set G	0.1	-0.0111	0.0088	-0.0111	(0.0111, 0.0103, 0.0158, 0.1731, 0.0149, 0.0060, 0.5571, 0.1858, 0.0087, 0.0171)	23.9551
	0.2	-0.0119	0.0087	-0.0119	(0.0115, 0.0111, 0.0160, 0.1741, 0.0145, 0.0057, 0.5560, 0.1850, 0.0088, 0.0174)	23.9683
	0.3	-0.0127	0.0087	-0.0127	(0.0119, 0.0120, 0.0162, 0.1751, 0.0139, 0.0053, 0.5551, 0.1843, 0.0089, 0.0174)	23.9778
	0.4	-0.0135	0.0087	-0.0135	(0.0125, 0.0129, 0.0164, 0.1760, 0.0132, 0.0049, 0.5543, 0.1838, 0.0088, 0.0172)	23.9829
	0.5	-0.0138	0.0089	-0.0138	(0.0130, 0.0136, 0.0163, 0.1768, 0.0124, 0.0048, 0.5543, 0.1838, 0.0087, 0.0163)	23.9703
	0.6	-0.0131	0.0091	-0.0131	(0.0133, 0.0138, 0.0155, 0.1772, 0.0115, 0.0053, 0.5552, 0.1846, 0.0086, 0.0148)	23.9293
	0.7	-0.0130	0.0097	-0.0130	(0.0134, 0.0138, 0.0147, 0.1774, 0.0109, 0.0059, 0.5563, 0.1855, 0.0086, 0.0135)	23.8948
	0.8	-0.0143	0.0114	-0.0143	(0.0134, 0.0135, 0.0139, 0.1774, 0.0104, 0.0066, 0.5575, 0.1864, 0.0085, 0.0123)	23.8681
	0.9	-0.0203	0.0175	-0.0203	(0.0132, 0.0131, 0.0131, 0.1773, 0.0102, 0.0073, 0.5586, 0.1874, 0.0086, 0.0113)	23.8443

Table 4.12: The portfolios $\mathbf{b}_{1976}$ and the maximum wealths $S_{1975}(max)$ achieved by respective ξ's within an extended range of ξ by the $A_4(r,t)$ universal portfolio for selected values of (r,t) where $r = 2t$ for data sets D, E, F and G, where $\mathbf{b}_1 = \mathbf{b}_{1975}^{\odot}$

Data set	r	t	Smallest ξ	Biggest ξ	Best ξ	$\mathbf{b}_{1976}$	$S_{1975}(max)$
Set D	0.1	0.05	-0.1036	0.0667	-0.1036	(0.5550, 0.3549, 0.0034, 0.0080, 0.0078, 0.0085, 0.0109, 0.0237, 0.0134, 0.0142)	35.2542
	0.3	0.15	-0.0345	0.0222	-0.0345	(0.5550, 0.3549, 0.0034, 0.0080, 0.0078, 0.0085, 0.0109, 0.0237, 0.0134, 0.0142)	35.2542
	20	10	-0.0005	0.0003	-0.0005	(0.5551, 0.3552, 0.0036, 0.0081, 0.0079, 0.0086, 0.0109, 0.0232, 0.0133, 0.0141)	35.2515
	b_{n1}	$0.5b_{n1}$	-0.0189	0.0119	-0.0189	(0.5549, 0.3549, 0.0035, 0.0080, 0.0078, 0.0085, 0.0109, 0.0239, 0.0134, 0.0143)	35.2628
	b_{n2}	$0.5b_{n2}$	-0.0284	0.0185	-0.0284	(0.5549, 0.3550, 0.0033, 0.0080, 0.0079, 0.0087, 0.0109, 0.0237, 0.0134, 0.0142)	35.2499
	b_{n3}	$0.5b_{n3}$	-0.7527	0.6268	-0.7527	(0.5516, 0.3623, 0.0051, 0.0036, 0.0121, 0.0124, 0.0103, 0.0189, 0.0113, 0.0125)	35.3441
	b_{n4}	$0.5b_{n4}$	-1.2654	0.5164	-1.2654	(0.5570, 0.3590, 0.0125, 0.0066, 0.0101, 0.0055, 0.0065, 0.0189, 0.0082, 0.0157)	35.5815
	b_{n5}	$0.5b_{n5}$	-0.7812	0.8037	-0.2082	(0.5574, 0.3605, 0.0085, 0.0095, 0.0095, 0.0098, 0.0102, 0.0129, 0.0108, 0.0109)	35.1832
	b_{n6}	$0.5b_{n6}$	-0.7896	0.7439	-0.1538	(0.5577, 0.3608, 0.0089, 0.0098, 0.0096, 0.0097, 0.0102, 0.0121, 0.0105, 0.0106)	35.1809
	b_{n7}	$0.5b_{n7}$	-8.2472	0.5737	-0.4867	(0.5573, 0.3589, 0.0073, 0.0097, 0.0090, 0.0091, 0.0099, 0.0153, 0.0118, 0.0117)	35.1905
	b_{n8}	$0.5b_{n8}$	-0.5709	1.2750	-0.2089	(0.5575, 0.3599, 0.0084, 0.0098, 0.0093, 0.0096, 0.0105, 0.0131, 0.0110, 0.0110)	35.1834
	b_{n9}	$0.5b_{n9}$	-1.8004	0.6688	-1.8004	(0.5589, 0.3551, 0.0078, 0.0104, 0.0058, 0.0075, 0.0061, 0.0255, 0.0070, 0.0159)	35.4871
	$b_{n,10}$	$0.5b_{n,10}$	-0.8234	0.9266	-0.8234	(0.5529, 0.3559, 0.0052, 0.0073, 0.0079, 0.0089, 0.0123, 0.0231, 0.0144, 0.0122)	35.2830
Set E	0.1	0.05	-0.1154	0.0773	-0.1154	(0.5564, 0.3553, 0.0038, 0.0090, 0.0087, 0.0095, 0.0121, 0.0134, 0.0150, 0.0166)	36.0646
	0.3	0.15	-0.0384	0.0257	-0.0384	(0.5564, 0.3553, 0.0038, 0.0090, 0.0087, 0.0095, 0.0121, 0.0134, 0.0150, 0.0166)	36.0638
	20	10	-0.0005	0.0003	-0.0005	(0.5566, 0.3562, 0.0047, 0.0091, 0.0089, 0.0096, 0.0119, 0.0130, 0.0143, 0.0157)	35.9980
	b_{n1}	$0.5b_{n1}$	-0.0206	0.0138	-0.0206	(0.5564, 0.3554, 0.0041, 0.0090, 0.0088, 0.0095, 0.0121, 0.0135, 0.0148, 0.0165)	36.0709
	b_{n2}	$0.5b_{n2}$	-0.0317	0.0214	-0.0317	(0.5563, 0.3554, 0.0037, 0.0089, 0.0088, 0.0097, 0.0121, 0.0135, 0.0150, 0.0165)	36.0617
	b_{n3}	$0.5b_{n3}$	-0.7620	0.7611	-0.7620	(0.5506, 0.3632, 0.0054, 0.0044, 0.0131, 0.0134, 0.0115, 0.0158, 0.0124, 0.0101)	36.1349

Table 4.12 continued

Data set	r	t	Smallest ξ	Biggest ξ	Best ξ	$\mathbf{b}_{1976}$	$S_{1975}(max)$
	b_{n4}	$0.5b_{n4}$	-1.2941	0.7148	-1.2941	(0.5564, 0.3591, 0.0128, 0.0073, 0.0106, 0.0057, 0.0073, 0.0179, 0.0090, 0.0140)	36.3305
	b_{n5}	$0.5b_{n5}$	-0.7715	0.8850	-0.7715	(0.5565, 0.3569, 0.0030, 0.0084, 0.0084, 0.0109, 0.0118, 0.0135, 0.0152, 0.0155)	35.8148
	b_{n6}	$0.5b_{n6}$	-0.7901	0.7380	-0.7901	(0.5594, 0.3552, 0.0025, 0.0113, 0.0086, 0.0087, 0.0122, 0.0116, 0.0140, 0.0164)	35.7333
	b_{n7}	$0.5b_{n7}$	-8.4108	0.6513	-2.5357	(0.5596, 0.3580, 0.0056, 0.0121, 0.0103, 0.0056, 0.0037, 0.0168, 0.0131, 0.0151)	35.8293
	b_{n8}	$0.5b_{n8}$	-0.6079	7.8157	-0.6079	(0.5564, 0.3564, 0.0041, 0.0076, 0.0088, 0.0109, 0.0124, 0.0112, 0.0164, 0.0159)	35.8310
	b_{n9}	$0.5b_{n9}$	-1.6446	0.6566	-1.6446	(0.5622, 0.3550, 0.0078, 0.0127, 0.0068, 0.0086, 0.0089, 0.0084, 0.0092, 0.0203)	35.9868
	$b_{n,10}$	$0.5b_{n,10}$	-1.0776	0.8065	-1.0776	(0.5567, 0.3548, 0.0087, 0.0115, 0.0072, 0.0068, 0.0138, 0.0119, 0.0143, 0.0142)	36.1783
Set F	0.1	0.05	-0.1131	0.0871	-0.1131	(0.4511, 0.3615, 0.1041, 0.0067, 0.0079, 0.0094, 0.0143, 0.0175, 0.0144, 0.0131)	19.9876
	0.3	0.15	-0.0377	0.0290	-0.0377	(0.4511, 0.3615, 0.1041, 0.0067, 0.0079, 0.0094, 0.0143, 0.0175, 0.0144, 0.0131)	19.9876
	20	10	-0.0005	0.0004	-0.0005	(0.4514, 0.3621, 0.1047, 0.0071, 0.0082, 0.0094, 0.0138, 0.0166, 0.0139, 0.0128)	19.9610
	b_{n1}	$0.5b_{n1}$	-0.0250	0.0189	-0.0250	(0.4511, 0.3616, 0.1040, 0.0067, 0.0081, 0.0092, 0.0144, 0.0174, 0.0143, 0.0130)	19.9876
	b_{n2}	$0.5b_{n2}$	-0.0306	0.0238	-0.0306	(0.4510, 0.3615, 0.1042, 0.0067, 0.0080, 0.0094, 0.0143, 0.0175, 0.0144, 0.0130)	19.9883
	b_{n3}	$0.5b_{n3}$	-0.1055	0.0804	-0.1055	(0.4512, 0.3614, 0.1039, 0.0066, 0.0079, 0.0093, 0.0143, 0.0178, 0.0144, 0.0132)	19.9874
	b_{n4}	$0.5b_{n4}$	-8.4457	0.6699	-1.8418	(0.4510, 0.3664, 0.1058, 0.0027, 0.0145, 0.0094, 0.0142, 0.0157, 0.0124, 0.0079)	19.9361
	b_{n5}	$0.5b_{n5}$	-0.6848	6.3537	-0.6848	(0.4482, 0.3631, 0.1071, 0.0068, 0.0081, 0.0113, 0.0133, 0.0171, 0.0141, 0.0109)	19.9784
	b_{n6}	$0.5b_{n6}$	-1.4915	0.7283	-1.4915	(0.4545, 0.3637, 0.1062, 0.0034, 0.0094, 0.0048, 0.0134, 0.0160, 0.0132, 0.0154)	19.8469
	b_{n7}	$0.5b_{n7}$	-0.7098	15.4809	-0.7098	(0.4497, 0.3624, 0.1069, 0.0090, 0.0086, 0.0116, 0.0118, 0.0162, 0.0133, 0.0105)	19.9977
	b_{n8}	$0.5b_{n8}$	-0.8007	1.2154	-0.8007	(0.4536, 0.3588, 0.1016, 0.0072, 0.0047, 0.0111, 0.0136, 0.0157, 0.0164, 0.0174)	19.9146
	b_{n9}	$0.5b_{n9}$	-1.0399	0.9180	-1.0399	(0.4521, 0.3608, 0.1048, 0.0091, 0.0075, 0.0106, 0.0138, 0.0164, 0.0115, 0.0133)	19.9742
	$b_{n,10}$	$0.5b_{n,10}$	-1.3238	0.6053	-1.3238	(0.4519, 0.3649, 0.1025, 0.0082, 0.0099, 0.0056, 0.0168, 0.0151, 0.0137, 0.0113)	19.9750

Table 4.12 continued

Data set	r	t	Smallest ξ	Biggest ξ	Best ξ	$\mathbf{b}_{1976}$	$S_{1975}(max)$
Set G	0.1	0.05	-0.1027	0.0879	-0.1027	(0.0138, 0.0093, 0.0153, 0.1719, 0.0149, 0.0059, 0.5578, 0.1863, 0.0082, 0.0167)	23.9214
	0.3	0.15	-0.0342	0.0293	-0.0342	(0.0138, 0.0093, 0.0153, 0.1719, 0.0149, 0.0059, 0.5578, 0.1863, 0.0082, 0.0167)	23.9211
	20	10	-0.0005	0.0004	-0.0005	(0.0137, 0.0093, 0.0152, 0.1720, 0.0148, 0.0060, 0.5579, 0.1864, 0.0082, 0.0165)	23.9139
	b_{n1}	$0.5b_{n1}$	-1.8170	0.7261	-1.8170	(0.0085, 0.0061, 0.0164, 0.1764, 0.0160, 0.0036, 0.5636, 0.1861, 0.0058, 0.0175)	23.8533
	b_{n2}	$0.5b_{n2}$	-0.7000	5.8475	-0.7000	(0.0161, 0.0087, 0.0160, 0.1687, 0.0142, 0.0074, 0.5567, 0.1866, 0.0076, 0.0180)	23.8607
	b_{n3}	$0.5b_{n3}$	-0.9496	1.0388	-0.9496	(0.0155, 0.0076, 0.0132, 0.1732, 0.0140, 0.0049, 0.5592, 0.1843, 0.0102, 0.0179)	23.9140
	b_{n4}	$0.5b_{n4}$	-0.0586	0.0494	-0.0586	(0.0134, 0.0097, 0.0152, 0.1718, 0.0150, 0.0058, 0.5583, 0.1866, 0.0080, 0.0163)	23.9302
	b_{n5}	$0.5b_{n5}$	-0.7574	2.3452	-0.7574	(0.0159, 0.0093, 0.0154, 0.1710, 0.0127, 0.0072, 0.5559, 0.1857, 0.0102, 0.0168)	23.8813
	b_{n6}	$0.5b_{n6}$	-1.5418	0.8123	-1.5418	(0.0136, 0.0133, 0.0177, 0.1751, 0.0167, 0.0036, 0.5484, 0.1870, 0.0067, 0.0179)	23.9437
	b_{n7}	$0.5b_{n7}$	-0.0181	0.0156	-0.0181	(0.0137, 0.0094, 0.0152, 0.1718, 0.0149, 0.0060, 0.5578, 0.1865, 0.0082, 0.0165)	23.9235
	b_{n8}	$0.5b_{n8}$	-0.0535	0.0462	-0.0535	(0.0138, 0.0094, 0.0154, 0.1718, 0.0150, 0.0060, 0.5577, 0.1863, 0.0081, 0.0166)	23.9197
	b_{n9}	$0.5b_{n9}$	-8.6854	0.7262	-2.1040	(0.0101, 0.0162, 0.0139, 0.1756, 0.0155, 0.0057, 0.5599, 0.1908, 0.0025, 0.0098)	23.8652
	$b_{n,10}$	$0.5b_{n,10}$	-1.3240	0.7513	-1.3240	(0.0112, 0.0021, 0.0168, 0.1781, 0.0160, 0.0009, 0.5613, 0.1842, 0.0115, 0.0180)	23.8989

Table 4.13: The portfolios $\mathbf{b}_{1976}$ and the maximum wealths $S_{1975}(max)$ achieved by respective ξ's within an extended range of ξ by the $A_4(r,t)$ universal portfolio for selected values of (r,t) where $r = t + 1$ for data sets D, E, F and G, where $\mathbf{b}_1 = \mathbf{b}_{1975}^{\odot}$

Data set	r	t	Smallest ξ	Biggest ξ	Best ξ	$\mathbf{b}_{1976}$	$S_{1975}(max)$
Set D	1.05	0.05	-0.0100	0.0063	-0.0100	(0.5514, 0.3549, 0.0033, 0.0081, 0.0078, 0.0086, 0.0109, 0.0239, 0.0135, 0.0176)	35.1818
	1.5	0.5	-0.0069	0.0044	-0.0069	(0.5537, 0.3550, 0.0034, 0.0081, 0.0078, 0.0086, 0.0109, 0.0237, 0.0134, 0.0154)	35.2280
	11	10	-0.0009	0.0006	-0.0009	(0.5581, 0.3552, 0.0037, 0.0081, 0.0079, 0.0086, 0.0108, 0.0231, 0.0133, 0.0113)	35.3102

Table 4.13 continued

Data set	r	t	Smallest ξ	Biggest ξ	Best ξ	$\mathbf{b}_{1976}$	$S_{1975}(max)$
	b_{n1}+1	b_{n1}	-0.0067	0.0042	-0.0067	(0.5539, 0.3549, 0.0034, 0.0080, 0.0078, 0.0085, 0.0109, 0.0238, 0.0134, 0.0153)	35.2352
	b_{n2}+1	b_{n2}	-0.0076	0.0049	-0.0076	(0.5532, 0.3550, 0.0034, 0.0080, 0.0079, 0.0086, 0.0109, 0.0237, 0.0134, 0.0159)	35.2163
	b_{n3}+1	b_{n3}	-0.0104	0.0066	-0.0104	(0.5511, 0.3550, 0.0033, 0.0080, 0.0079, 0.0086, 0.0109, 0.0239, 0.0135, 0.0178)	35.1782
	b_{n4}+1	b_{n4}	-0.0105	0.0066	-0.0105	(0.5511, 0.3549, 0.0034, 0.0080, 0.0078, 0.0085, 0.0109, 0.0240, 0.0135, 0.0179)	35.1784
	b_{n5}+1	b_{n5}	-0.0104	0.0066	-0.0024	(0.5565, 0.3603, 0.0085, 0.0096, 0.0095, 0.0097, 0.0102, 0.0132, 0.0108, 0.0118)	35.1752
	b_{n6}+1	b_{n6}	-0.0103	0.0066	-0.0016	(0.5570, 0.3608, 0.0090, 0.0097, 0.0097, 0.0098, 0.0101, 0.0121, 0.0105, 0.0112)	35.1752
	b_{n7}+1	b_{n7}	-0.0104	0.0066	-0.0027	(0.5563, 0.3601, 0.0083, 0.0095, 0.0094, 0.0096, 0.0102, 0.0136, 0.0109, 0.0120)	35.1753
	b_{n8}+1	b_{n8}	-0.0103	0.0066	-0.0026	(0.5564, 0.3601, 0.0083, 0.0095, 0.0095, 0.0096, 0.0102, 0.0135, 0.0109, 0.0120)	35.1753
	b_{n9}+1	b_{n9}	-0.0104	0.0066	-0.007	(0.5534, 0.3572, 0.0055, 0.0087, 0.0085, 0.0090, 0.0106, 0.0194, 0.0124, 0.0153)	35.1755
	$b_{n,10}$+1	$b_{n,10}$	-0.0103	0.0066	-0.0024	(0.5565, 0.3603, 0.0085, 0.0096, 0.0095, 0.0097, 0.0102, 0.0132, 0.0108, 0.0118)	35.1752
Set E	1.05	0.05	-0.0109	0.0074	-0.0109	(0.5521, 0.3555, 0.0040, 0.0091, 0.0089, 0.0097, 0.0123, 0.0136, 0.0151, 0.0197)	35.9828
	1.5	0.5	-0.0076	0.0051	-0.0076	(0.5549, 0.3554, 0.0040, 0.0090, 0.0088, 0.0096, 0.0122, 0.0135, 0.0150, 0.0177)	36.0306
	11	10	-0.0010	0.0006	-0.0010	(0.5601, 0.3555, 0.0040, 0.0089, 0.0087, 0.0095, 0.0119, 0.0132, 0.0147, 0.0135)	36.1054
	b_{n1}+1	b_{n1}	-0.0074	0.0049	-0.0074	(0.5550, 0.3554, 0.0040, 0.0090, 0.0088, 0.0096, 0.0122, 0.0135, 0.0150, 0.0175)	36.0432
	b_{n2}+1	b_{n2}	-0.0084	0.0057	-0.0084	(0.5542, 0.3554, 0.0039, 0.0090, 0.0088, 0.0097, 0.0122, 0.0135, 0.0150, 0.0181)	36.0210
	b_{n3}+1	b_{n3}	-0.0113	0.0077	-0.0113	(0.5517, 0.3557, 0.0040, 0.0091, 0.0090, 0.0098, 0.0123, 0.0136, 0.0151, 0.0198)	35.9814
	b_{n4}+1	b_{n4}	-0.0114	0.0077	-0.0114	(0.5517, 0.3555, 0.0041, 0.0091, 0.0089, 0.0097, 0.0123, 0.0136, 0.0151, 0.0200)	35.9806
	b_{n5}+1	b_{n5}	-0.0114	0.0077	-0.0114	(0.5517, 0.3555, 0.0040, 0.0091, 0.0089, 0.0097, 0.0123, 0.0136, 0.0152, 0.0200)	35.9772
	b_{n6}+1	b_{n6}	-0.0114	0.0077	-0.0114	(0.5518, 0.3554, 0.0039, 0.0092, 0.0089, 0.0097, 0.0123, 0.0136, 0.0152, 0.0201)	35.9738
	b_{n7}+1	b_{n7}	-0.0114	0.0077	-0.0114	(0.5518, 0.3555, 0.0040, 0.0092, 0.0089, 0.0097, 0.0123, 0.0136, 0.0152, 0.0200)	35.9742
	b_{n8}+1	b_{n8}	-0.0113	0.0078	-0.0113	(0.5518, 0.3555, 0.0040, 0.0091, 0.0089, 0.0097, 0.0123, 0.0135, 0.0152, 0.0199)	35.9764

Table 4.13 continued

Data set	r	t	Smallest ξ	Biggest ξ	Best ξ	$\mathbf{b}_{1976}$	$S_{1975}(max)$
	b_{n9}+1	b_{n9}	-0.0114	0.0077	-0.0114	(0.5518, 0.3555, 0.0040, 0.0092, 0.0089, 0.0097, 0.0123, 0.0135, 0.0151, 0.0201)	35.9745
	$b_{n,10}$+1	$b_{n,10}$	-0.0114	0.0078	-0.0114	(0.5518, 0.3554, 0.0040, 0.0092, 0.0089, 0.0097, 0.0124, 0.0135, 0.0152, 0.0200)	35.9773
Set F	1.05	0.05	-0.0105	0.0086	-0.0105	(0.4486, 0.3616, 0.1042, 0.0068, 0.0080, 0.0094, 0.0142, 0.0173, 0.0143, 0.0155)	19.9602
	1.5	0.5	-0.0074	0.0058	-0.0074	(0.4502, 0.3616, 0.1042, 0.0068, 0.0080, 0.0094, 0.0143, 0.0174, 0.0143, 0.0140)	19.9755
	11	10	-0.0008	0.0007	-0.0008	(0.4535, 0.3627, 0.1053, 0.0074, 0.0084, 0.0095, 0.0134, 0.0158, 0.0134, 0.0107)	19.9510
	b_{n1}+1	b_{n1}	-0.0077	0.0060	-0.0077	(0.4500, 0.3616, 0.1041, 0.0067, 0.0080, 0.0093, 0.0143, 0.0174, 0.0143, 0.0141)	19.9763
	b_{n2}+1	b_{n2}	-0.0081	0.0065	-0.0081	(0.4498, 0.3616, 0.1042, 0.0068, 0.0080, 0.0094, 0.0143, 0.0174, 0.0143, 0.0143)	19.9726
	b_{n3}+1	b_{n3}	-0.0099	0.0081	-0.0099	(0.4489, 0.3617, 0.1042, 0.0068, 0.0080, 0.0094, 0.0142, 0.0173, 0.0143, 0.0152)	19.9613
	b_{n4}+1	b_{n4}	-0.0109	0.0089	-0.0109	(0.4484, 0.3617, 0.1043, 0.0068, 0.0080, 0.0094, 0.0142, 0.0173, 0.0143, 0.0157)	19.9571
	b_{n5}+1	b_{n5}	-0.0108	0.0090	-0.0108	(0.4484, 0.3617, 0.1043, 0.0068, 0.0080, 0.0094, 0.0142, 0.0173, 0.0143, 0.0156)	19.9582
	b_{n6}+1	b_{n6}	-0.0109	0.0089	-0.0109	(0.4484, 0.3617, 0.1042, 0.0068, 0.0080, 0.0094, 0.0142, 0.0173, 0.0143, 0.0158)	19.9554
	b_{n7}+1	b_{n7}	-0.0108	0.0090	-0.0108	(0.4484, 0.3617, 0.1043, 0.0069, 0.0080, 0.0094, 0.0142, 0.0173, 0.0143, 0.0156)	19.9581
	b_{n8}+1	b_{n8}	-0.0108	0.0090	-0.0108	(0.4485, 0.3616, 0.1042, 0.0068, 0.0079, 0.0094, 0.0142, 0.0173, 0.0143, 0.0157)	19.9561
	b_{n9}+1	b_{n9}	-0.0108	0.0090	-0.0108	(0.4485, 0.3617, 0.1043, 0.0069, 0.0080, 0.0094, 0.0142, 0.0172, 0.0142, 0.0157)	19.9560
	$b_{n,10}$+1	$b_{n,10}$	-0.0109	0.0089	-0.0109	(0.4484, 0.3617, 0.1042, 0.0069, 0.0080, 0.0094, 0.0143, 0.0173, 0.0143, 0.0157)	19.9575
Set G	1.05	0.05	-0.0098	0.0084	-0.0098	(0.0113, 0.0096, 0.0156, 0.1722, 0.0152, 0.0062, 0.5581, 0.1866, 0.0085, 0.0167)	23.9365
	1.5	0.5	-0.0068	0.0058	-0.0068	(0.0129, 0.0094, 0.0154, 0.1720, 0.0150, 0.0060, 0.5579, 0.1864, 0.0083, 0.0167)	23.9249
	11	10	-0.0009	0.0007	-0.0009	(0.0157, 0.0090, 0.0149, 0.1718, 0.0145, 0.0058, 0.5577, 0.1862, 0.0080, 0.0162)	23.8979
	b_{n1}+1	b_{n1}	-0.0102	0.0087	-0.0102	(0.0110, 0.0096, 0.0157, 0.1723, 0.0152, 0.0062, 0.5582, 0.1866, 0.0085, 0.0168)	23.9363
	b_{n2}+1	b_{n2}	-0.0101	0.0088	-0.0101	(0.0111, 0.0096, 0.0157, 0.1722, 0.0152, 0.0062, 0.5581, 0.1866, 0.0085, 0.0168)	23.9353
	b_{n3}+1	b_{n3}	-0.0102	0.0087	-0.0102	(0.0111, 0.0096, 0.0157, 0.1722, 0.0153, 0.0062, 0.5581, 0.1866, 0.0085, 0.0168)	23.9378

Table 4.13 continued

Data set	r	t	Smallest ξ	Biggest ξ	Best ξ	$\mathbf{b}_{1976}$	$S_{1975}(max)$
	b_{n4}+1	b_{n4}	-0.0087	0.0075	-0.0087	(0.0118, 0.0096, 0.0155, 0.1721, 0.0151, 0.0061, 0.5581, 0.1866, 0.0084, 0.0166)	23.9332
	b_{n5}+1	b_{n5}	-0.0101	0.0088	-0.0101	(0.0111, 0.0096, 0.0156, 0.1722, 0.0152, 0.0062, 0.5581, 0.1866, 0.0085, 0.0167)	23.9352
	b_{n6}+1	b_{n6}	-0.0102	0.0087	-0.0102	(0.0110, 0.0096, 0.0157, 0.1723, 0.0153, 0.0062, 0.5581, 0.1866, 0.0085, 0.0168)	23.9368
	b_{n7}+1	b_{n7}	-0.0065	0.0056	-0.0065	(0.0130, 0.0094, 0.0153, 0.1720, 0.0150, 0.0060, 0.5579, 0.1865, 0.0083, 0.0166)	23.9246
	b_{n8}+1	b_{n8}	-0.0086	0.0074	-0.0086	(0.0119, 0.0095, 0.0156, 0.1721, 0.0151, 0.0061, 0.5580, 0.1865, 0.0084, 0.0167)	23.9314
	b_{n9}+1	b_{n9}	-0.0102	0.0087	-0.0102	(0.0110, 0.0096, 0.0157, 0.1723, 0.0152, 0.0062, 0.5581, 0.1867, 0.0085, 0.0167)	23.9365
	$b_{n,10}$+1	$b_{n,10}$	-0.0102	0.0087	-0.0102	(0.0110, 0.0095, 0.0157, 0.1723, 0.0153, 0.0062, 0.5582, 0.1866, 0.0085, 0.0167)	23.9384

4.3 The Modified Mahalanobis Universal Portfolio

The Mahalanobis universal portfolio is an additive-update universal portfolio and hence the portfolio vectors can get out of range $[0,1]$ easily. The sufficient condition for portfolio vectors to be within the range $[0,1]$ has been derived in Proposition 4.2(ii). This condition could lead to the ξ parameter interval to be an empty set whenever a zero element occurs in the portfolio vectors. From the last few tables, we observe that the intervals of ξ are relatively small. It might cause the Mahalanobis universal portfolio to behave like a constant rebalanced portfolio for small values of ξ. The modified Mahalanobis universal portfolio is introduced in this section to handle the above difficulties of $\mathbf{b}_{n+1}$ having negative values and ξ being restricted to be small.

The *modified Mahalanobis universal portfolio* is a sequence of universal portfolios $\{\mathbf{b}_{n+1}\}$ generated by (4.2) where the initial starting portfolio $\mathbf{b}_1$ is given, ξ is any real number and $\mathbf{b}_n$ is modified according to (4.30) and (4.31) below if $\mathbf{b}_n$ is not a portfolio vector before applying the update (4.2), for $n = 2,3,\ldots$.

We now state the modification necessary to change $\mathbf{b}_n$ into a portfolio vector. If there exists an invalid portfolio vector $\mathbf{b}_n$ where $b_{ni} < 0$ for some i, then let

$$w_{nk} = b_{nk} - \min_j\{b_{nj}\} \tag{4.30}$$

for all $k = 1,2,\ldots,m$ and the new genuine portfolio vector is given by

$$\mathbf{b}_n = \frac{\mathbf{w}_n}{\sum_{k=1}^{m} w_{nk}}, \tag{4.31}$$

where $\mathbf{w}_n = (w_{nk})$. The portfolio vectors $\mathbf{b}_n$ are remain unchanged if $b_{nk} \geq 0$ for all $k = 1,2,\ldots,m$.

To see why (4.30) and (4.31) lead to genuine portfolio vectors, we consider the following argument. According to (1.1), the portfolio vector $\mathbf{b}_n$ is not genuine if there exists a $b_{ni} < 0$ for some i. Identify the minimum of b_{nj} for $j = 1,2,\ldots,m$, say, $\min_j\{b_{nj}\}$, and hence

$$w_{nk} = b_{nk} - \min_j\{b_{nj}\} \geq 0$$

for all $k = 1,2,\ldots,m$. The following sum

$$\frac{w_{n1}}{\sum_{k=1}^{m} w_{nk}} + \frac{w_{n2}}{\sum_{k=1}^{m} w_{nk}} + \cdots + \frac{w_{nm}}{\sum_{k=1}^{m} w_{nk}} = 1$$

shows that $\mathbf{b}_n$ given by (4.31) is a genuine portfolio vector. The portfolio vectors $\mathbf{b}_n$ that satisfy $b_{nk} \geq 0$ for all $k = 1,2,\ldots,m$ remain unchanged and the next update (4.2) can be applied immediately. This modification allows us to use any value of ξ as the parameter.

4.3.1 Empirical Results

From previous results, the valid intervals of ξ for the selected Mahalanobis universal portfolios are mostly between -1 and 1. Since the Mahalanobis universal portfolios with initial starting portfolio $\mathbf{b}_1 = \mathbf{b}^{\odot}_{1975}$, where $\mathbf{b}^{\odot}_{1975}$ is the approximate positive BCRP, does not outperform the BCRP in the previous section and Proposition 4.4 might not be hold for the modified Mahalanobis universal portfolios, we shall omit the study of $\mathbf{b}_1 = \mathbf{b}^{\odot}_{1975}$ for the latter case. Now, we run the modified $A_3(0.10)$ universal portfolio and five selected modified $A_4(r,t)$ universal portfolios on data sets D, E, F and G with the same initial starting portfolio $\mathbf{b}_1 = (0.1000, 0.1000, \ldots, 0.1000)$ and the parameter ξ is chosen to be 10. The portfolios $\mathbf{b}_{1976}$ and wealths S_{1975} achieved after 1975 trading days where $\xi = 10$ are recorded in Table 4.14.

Table 4.14: The portfolios $\mathbf{b}_{1976}$ and the wealths S_{1975} achieved by the selected modified Mahalanobis universal portfolios for data sets D, E, F and G, where $\xi = 10$ and $\mathbf{b}_1 = (0.1000, 0.1000, \ldots, 0.1000)$

Data set	$A_3(r)$ or $A_4(r,t)$	$\mathbf{b}_{1976}$	S_{1975}
Set D	$A_3(0.10)$	(0.1433, 0.1828, 0.0000, 0.0723, 0.1130, 0.0300, 0.0101, 0.1593, 0.2577, 0.0315)	44.1900
	$A_4(0.10, 0.025)$	(0.1414, 0.1520, 0.0000, 0.0581, 0.0959, 0.1183, 0.1064, 0.1006, 0.0773, 0.1502)	43.3606

Table 4.14 continued

Data set	$A_3(r)$ or $A_4(r,t)$	$\mathbf{b}_{1976}$	S_{1975}
	$A_4(0.10,0.050)$	(0.1478, 0.1498, 0.0000, 0.0575, 0.0940, 0.1162, 0.1051, 0.0992, 0.0768, 0.1536)	43.0123
	$A_4(0.10,0.075)$	(0.1539, 0.1476, 0.0000, 0.0572, 0.0925, 0.1137, 0.1042, 0.0979, 0.0764, 0.1566)	42.0498
	$A_4(0.55,0.050)$	(0.1195, 0.1683, 0.0000, 0.0874, 0.1178, 0.0513, 0.0432, 0.1339, 0.2087, 0.0700)	45.7066
	$A_4(1.05,0.050)$	(0.1351, 0.1911, 0.0000, 0.0744, 0.1218, 0.0343, 0.0079, 0.1564, 0.2748, 0.0043)	44.9506
Set E	$A_3(0.10)$	(0.1336, 0.1667, 0.0000, 0.0683, 0.1084, 0.0304, 0.0080, 0.0789, 0.2391, 0.1665)	54.4666
	$A_4(0.10,0.025)$	(0.1385, 0.1566, 0.0000, 0.0643, 0.1004, 0.1201, 0.1142, 0.1077, 0.0810, 0.1171)	50.2567
	$A_4(0.10,0.050)$	(0.1379, 0.1552, 0.0000, 0.0642, 0.0992, 0.1187, 0.1138, 0.1062, 0.0808, 0.1239)	50.5023
	$A_4(0.10,0.075)$	(0.1371, 0.1541, 0.0000, 0.0641, 0.0982, 0.1174, 0.1134, 0.1051, 0.0805, 0.1301)	50.4402
	$A_4(0.55,0.050)$	(0.1223, 0.1632, 0.0000, 0.0867, 0.1157, 0.0513, 0.0485, 0.0768, 0.2027, 0.1328)	57.3251
	$A_4(1.05,0.050)$	(0.1297, 0.1711, 0.0000, 0.0688, 0.1134, 0.0333, 0.0107, 0.0728, 0.2443, 0.1558)	55.0500
Set F	$A_3(0.10)$	(0.1039, 0.1359, 0.0291, 0.0000, 0.0611, 0.1906, 0.1007, 0.0345, 0.1527, 0.1914)	26.9298
	$A_4(0.10,0.025)$	(0.0552, 0.1049, 0.1344, 0.1138, 0.1144, 0.0672, 0.1646, 0.0474, 0.1063, 0.0919)	23.2058
	$A_4(0.10,0.050)$	(0.0566, 0.1065, 0.1358, 0.1143, 0.1160, 0.0670, 0.1662, 0.0483, 0.1078, 0.0815)	23.0645
	$A_4(0.10,0.075)$	(0.0581, 0.1078, 0.1378, 0.1148, 0.1176, 0.0669, 0.1675, 0.0493, 0.1092, 0.0709)	22.9006
	$A_4(0.55,0.050)$	(0.0923, 0.1358, 0.0137, 0.0388, 0.0537, 0.1905, 0.1547, 0.0000, 0.1380, 0.1825)	25.0662
	$A_4(1.05,0.050)$	(0.0959, 0.1403, 0.0276, 0.0000, 0.0608, 0.2195, 0.0734, 0.0286, 0.1601, 0.1938)	27.4424
Set G	$A_3(0.10)$	(0.2683, 0.1092, 0.1714, 0.0811, 0.0162, 0.0259, 0.0000, 0.1131, 0.0283, 0.1865)	33.6402
	$A_4(0.10,0.025)$	(0.0806, 0.1174, 0.1151, 0.0653, 0.1573, 0.1330, 0.0000, 0.1095, 0.1162, 0.1057)	31.9183
	$A_4(0.10,0.050)$	(0.0806, 0.1195, 0.1158, 0.0639, 0.1581, 0.1360, 0.0000, 0.1096, 0.1174, 0.0992)	31.3930
	$A_4(0.10,0.075)$	(0.0804, 0.1224, 0.1169, 0.0618, 0.1593, 0.1400, 0.0000, 0.1094, 0.1188, 0.0909)	30.7773
	$A_4(0.55,0.050)$	(0.2171, 0.0821, 0.1398, 0.0903, 0.0742, 0.0533, 0.0000, 0.1203, 0.0493, 0.1737)	35.7843
	$A_4(1.05,0.050)$	(0.2822, 0.0827, 0.1758, 0.0767, 0.0076, 0.0357, 0.0000, 0.1247, 0.0103, 0.2044)	34.2477

Next, we compare the wealths achieved by the modified and unmodified universal portfolios displayed in Tables 4.14 and 4.7, 4.8, 4.9 respectively. The wealths achieved by the modified $A_3(0.10)$ universal portfolios where $\xi = 10$ for data sets D, E, F and G are $44.1900, 54.4666, 26.9298$ and 33.6402 respectively. These wealths are much higher than the maximum wealths achieved by the unmodified $A_3(0.10)$

universal portfolios, namely, $S_{1975}(max) = 18.7404, 21.2555, 12.9063$ and 15.7198 respectively in Table 4.7. From Tables 4.14 and 4.8, the values of S_{1975} for the modified $A_4(0.10,0.050)$ universal portfolios where $\xi = 10$ are $43.0123, 50.5023, 23.0645$ and 31.3930 for data sets D, E, F and G respectively, which are higher than the maximum wealths achieved by the unmodified $A_4(0.10,0.050)$ universal portfolios, namely, $S_{1975}(max) = 18.3449, 22.5669, 13.4338$ and 15.6803 respectively. For data sets D, E, F and G, the modified $A_4(1.05,0.050)$ universal portfolios where $\xi = 10$ in Table 4.14 perform better than the unmodified $A_4(1.05,0.050)$ universal portfolios in Table 4.9 in terms of the wealths achieved. The values of S_{1975} for the modified $A_4(1.05,0.050)$ universal portfolios where $\xi = 10$ for the four 10-stock data sets in Table 4.14 are $44.9506, 55.0500, 27.4424$ and 34.2477 respectively. The maximum wealths achieved by the unmodified $A_4(1.05,0.050)$ universal portfolios for the same 10-stock data sets in Table 4.9 are $18.3724, 21.7355, 13.1906$ and 15.8091 respectively.

4.3.2 The Modified Mahalanobis Universal Portfolio with Varying Parameter ξ

To study the dependence of the wealth achieved on the parameter ξ, the modified $A_3(0.10)$ and modified $A_4(0.10,0.050)$ universal portfolios are run on data set G with the same initial starting portfolio. The values of ξ vary from -100 to 100. The portfolios $\mathbf{b}_{1976}$ and the wealths S_{1975} achieved after 1975 trading days are calculated for selected values of ξ and are listed in Tables 4.15 and 4.16. The modified $A_3(0.10)$ universal portfolio can achieve a higher wealth of $S_{1975} = 34.6071$ at $\xi = 5$ compared to $S_{1975} = 33.6402$ at $\xi = 10$. In Table 4.16, a higher wealth of $S_{1975} = 35.8063$ can be obtained by the modified $A_4(0.10, 0.050)$ universal portfolio at $\xi = 50$. In both Tables 4.15 and 4.16, the wealth achieved S_{1975} seems to be increasing when the parameter ξ increases.

Table 4.15: The portfolios $\mathbf{b}_{1976}$ and the wealths S_{1975} achieved by the modified $A_3(0.10)$ universal portfolio for selected values of ξ for data set G, where $\mathbf{b}_1 = (0.1000, 0.1000, \dots, 0.1000)$

ξ	$\mathbf{b}_{1976}$	S_{1975}
-100	(0.0000, 0.0879, 0.0605, 0.1350, 0.1677, 0.1417, 0.1199, 0.0982, 0.1305, 0.0585)	7.0730
-50	(0.0000, 0.0875, 0.0600, 0.1319, 0.1684, 0.1417, 0.1228, 0.0990, 0.1309, 0.0579)	7.0294
-25	(0.0000, 0.0876, 0.0588, 0.1255, 0.1683, 0.1417, 0.1299, 0.0992, 0.1330, 0.0559)	6.8192
-10	(0.0000, 0.0919, 0.0516, 0.1042, 0.1578, 0.1466, 0.1674, 0.0928, 0.1456, 0.0420)	6.5986
-5	(0.0026, 0.1047, 0.0480, 0.0884, 0.1090, 0.1622, 0.2316, 0.0798, 0.1534, 0.0203)	6.5180
-3	(0.0281, 0.1018, 0.0789, 0.1122, 0.0654, 0.1409, 0.2219, 0.0970, 0.1095, 0.0441)	6.5363
-1	(0.1658, 0.0621, 0.0645, 0.1504, 0.0218, 0.0429, 0.2217, 0.0843, 0.0984, 0.0881)	7.0182
-0.1	(0.1166, 0.1079, 0.1502, 0.0768, 0.1451, 0.0651, 0.0280, 0.0571, 0.0932, 0.1600)	15.7968
0.1	(0.0835, 0.1092, 0.0473, 0.1223, 0.0566, 0.1360, 0.1718, 0.1481, 0.0989, 0.0264)	13.6876
1	(0.0780, 0.1163, 0.1139, 0.0739, 0.1582, 0.1287, 0.0000, 0.1038, 0.1158, 0.1114)	31.8554
3	(0.1691, 0.1003, 0.1186, 0.0898, 0.1140, 0.0692, 0.0000, 0.0976, 0.0977, 0.1436)	34.3421
5	(0.2032, 0.0986, 0.1373, 0.0959, 0.0828, 0.0492, 0.0000, 0.1103, 0.0638, 0.1589)	34.6071
10	(0.2683, 0.1092, 0.1714, 0.0811, 0.0162, 0.0259, 0.0000, 0.1131, 0.0283, 0.1865)	33.6402
25	(0.2496, 0.1172, 0.1622, 0.0650, 0.0000, 0.0378, 0.0501, 0.1026, 0.0481, 0.1674)	31.4401
50	(0.2460, 0.1182, 0.1589, 0.0567, 0.0000, 0.0393, 0.0634, 0.1013, 0.0539, 0.1623)	30.7228
100	(0.2468, 0.1181, 0.1583, 0.0508, 0.0000, 0.0388, 0.0689, 0.1020, 0.0551, 0.1613)	29.8822

Table 4.16: The portfolios $\mathbf{b}_{1976}$ and the wealths S_{1975} achieved by the modified $A_4(0.10, 0.050)$ universal portfolio for selected values of ξ for data set G, where $\mathbf{b}_1 = (0.1000, 0.1000, \dots, 0.1000)$

ξ	$\mathbf{b}_{1976}$	S_{1975}
-100	(0.0000, 0.1137, 0.0666, 0.1095, 0.1522, 0.1358, 0.1621, 0.0914, 0.1527, 0.0161)	6.2864
-50	(0.0000, 0.1236, 0.0676, 0.1020, 0.1118, 0.1450, 0.2081, 0.0767, 0.1588, 0.0063)	6.3368
-25	(0.0274, 0.1133, 0.0878, 0.1291, 0.0577, 0.1299, 0.2217, 0.0948, 0.1080, 0.0304)	6.4086
-10	(0.1505, 0.0637, 0.0693, 0.1489, 0.0230, 0.0439, 0.2191, 0.0833, 0.0841, 0.1142)	7.0541
-5	(0.1616, 0.0540, 0.0685, 0.1659, 0.0519, 0.0185, 0.1595, 0.0580, 0.1270, 0.1351)	8.3242
-3	(0.1681, 0.0286, 0.0927, 0.1714, 0.0776, 0.0111, 0.0936, 0.0543, 0.1387, 0.1638)	9.1166
-1	(0.1386, 0.0905, 0.1456, 0.0712, 0.1451, 0.0637, 0.0293, 0.0605, 0.0926, 0.1628)	15.6038
-0.1	(0.1045, 0.0996, 0.1057, 0.0955, 0.1055, 0.0963, 0.0916, 0.0955, 0.0986, 0.1072)	14.7192
0.1	(0.0955, 0.1005, 0.0944, 0.1045, 0.0946, 0.1037, 0.1083, 0.1045, 0.1013, 0.0928)	14.4802
1	(0.0509, 0.1201, 0.0476, 0.1373, 0.0553, 0.1376, 0.1739, 0.1474, 0.1062, 0.0237)	13.8533
3	(0.0416, 0.2139, 0.1103, 0.0037, 0.1231, 0.1615, 0.1403, 0.1561, 0.0310, 0.0185)	24.6439

Table 4.16 continued

ξ	$\mathbf{b}_{1976}$	S_{1975}
5	(0.0343, 0.1744, 0.1216, 0.0284, 0.1550, 0.1653, 0.0642, 0.1403, 0.0640, 0.0525)	28.8928
10	(0.0806, 0.1195, 0.1158, 0.0639, 0.1581, 0.1360, 0.0000, 0.1096, 0.1174, 0.0992)	31.3930
25	(0.1608, 0.0948, 0.1107, 0.0755, 0.1199, 0.0848, 0.0000, 0.0946, 0.1025, 0.1563)	35.6848
50	(0.2132, 0.0799, 0.1284, 0.0859, 0.0751, 0.0533, 0.0000, 0.1134, 0.0539, 0.1969)	35.8063
100	(0.2797, 0.0749, 0.1559, 0.0711, 0.0132, 0.0343, 0.0000, 0.1130, 0.0121, 0.2458)	35.2452

We can conclude that the achievable universal wealth depends on the parameter ξ. From Tables 4.15 and 4.16, it is observed that for some parameters the wealths S_{1975} achieved are less than 10. These wealths achieved are considered low for this set of data. Thus, an improper choice of ξ may lead to a low investment wealth. The advantage of using the modified Mahalanobis universal portfolio is that we can choose any real number to be the parameter ξ without worrying the portfolio vectors will get out of the range $[0,1]$. The modified Mahalanobis universal portfolio ensures that the generated portfolio vectors are always within the range $[0,1]$. Based on the limited data sets studied here, the modified Mahalanobis universal portfolio seems to perform better than the unmodified Mahalanobis universal portfolio.

CHAPTER FIVE

DIRICHLET UNIVERSAL PORTFOLIO OF ORDER ONE

The concept of the Dirichlet-weighted universal portfolio was introduced by Cover and Ordentlich [9]. They focussed their study on two special cases of the Dirichlet-weighted universal portfolios. The authors have shown that these special Dirichlet-weighted universal portfolios have the same asymptotic exponential growth rate of wealth as the best constant rebalanced portfolio (BCRP). The implementation of the Dirichlet-weighted universal portfolio requires processing all the stock data starting from the day of investment until the current time. The implementation time and computer memory requirements for generating the updates of the universal portfolio are growing exponentially in the number of stocks. To save time and computer memory requirements, we propose a new universal portfolio in this chapter that achieves the purpose of saving substantial time and computer memory in its implementation.

5.1 The Alpha-Parametric Family of Dirichlet Universal Portfolio of Order One

We say that a universal portfolio is of order one if the next update depends only on the last-known price-relative data. The Dirichlet universal portfolio of order one is derived from the Dirichlet-weighted universal portfolio where the next update depends only on one day of last-known price-relative data instead of all the past stock data. The theory of universal portfolios of finite order generated by probability distributions is due to Tan [33].

The *Dirichlet probability measure* $\mu(\mathbf{b})$ is defined as

$$d\mu(\mathbf{b}) = \frac{\Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_m)}{\Gamma(\alpha_1)\Gamma(\alpha_2) \ldots \Gamma(\alpha_m)} b_1^{\alpha_1 - 1} b_2^{\alpha_2 - 1} \ldots b_m^{\alpha_m - 1} d\mathbf{b}, \qquad (5.1)$$

where $\alpha_i > 0$ for $i = 1,2, \ldots, m$ and $d\mathbf{b}$ refers to the differential with respect to any $m - 1$ independent variables from $\mathbf{b} = (b_1, b_2, \ldots, b_m) \in B$, where the simplex B of portfolio vector is defined in (1.1).

The *Dirichlet universal portfolio of order one* is a sequence of portfolio vectors $\{\mathbf{b}_{n+1}\}$ generated by the following:

$$b_{n+1,i} = \frac{1}{\left(\sum_{j=1}^{m} \alpha_j + 1\right)} \left[\alpha_i + \frac{\alpha_i x_{ni}}{\left(\sum_{j=1}^{m} \alpha_j x_{nj}\right)}\right], \qquad (5.2)$$

where the initial starting portfolio $\mathbf{b}_1$ and the parameters $\alpha_i > 0$ for $i = 1,2, \ldots, m$ are given.

Note: The parametric vector $\boldsymbol{\alpha} \neq \mathbf{0}$ is a valid parametric vector provided (5.2) is well-defined. This means the vector $\boldsymbol{\alpha}$ can have some zeros but not all zeros.

First, we show that the alpha-parametric family of Dirichlet universal portfolios of order one is a consequence of weighting the current portfolio components b_{ni} by the current daily wealth $\mathbf{b}_n^t \mathbf{x}_n$ with respect to the Dirichlet probability measure $\mu(\mathbf{b})$.

Proposition 5.1 Consider the following portfolio obtained by weighting the current portfolio components b_{ni} by the current daily wealth $\mathbf{b}_n^t\mathbf{x}_n$ with respect to the Dirichlet probability measure $\mu(\mathbf{b})$, namely,

$$b_{n+1,i} = \frac{\int_B \; b_{ni}(\mathbf{b}_n^t\mathbf{x}_n)\, d\mu(\mathbf{b})}{\int_B \; (\mathbf{b}_n^t\mathbf{x}_n)\, d\mu(\mathbf{b})}. \tag{5.3}$$

The universal portfolios (5.3) and (5.2) are equivalent.

Proof. Since the wealth return on day n is defined as $\mathbf{b}_n^t\mathbf{x}_n = \sum_{j=1}^{m} b_{nj}x_{nj}$ in (1.2), let us consider the numerator of $b_{n+1,i}$ in (5.3), that is,

$$\int_B \; b_{ni}\left(\sum_{j=1}^{m} b_{nj}x_{nj}\right) d\mu(\mathbf{b}) \tag{5.4}$$

for $i = 1,2,\ldots,m$. Substitute $d\mu(\mathbf{b})$ in (5.4) by (5.1), we have

$$\frac{\Gamma(\alpha_1+\alpha_2+\cdots+\alpha_m)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_m)}\left\{\sum_{\substack{j=1\\ j\neq i}}^{m} x_{nj}\int_B \; b_{n1}^{\alpha_1-1}\, b_{n2}^{\alpha_2-1}\ldots\right.$$

$$b_{n,i-1}^{\alpha_{i-1}-1} b_{ni}^{\alpha_i} b_{n,i+1}^{\alpha_{i+1}-1}\ldots b_{n,j-1}^{\alpha_{j-1}-1}\, b_{nj}^{\alpha_j} b_{n,j+1}^{\alpha_{j+1}-1}\ldots b_{nm}^{\alpha_m-1} d\mathbf{b}$$

$$\left. + x_{ni}\int_B \; b_{n1}^{\alpha_1-1}\, b_{n2}^{\alpha_2-1}\ldots b_{n,i-1}^{\alpha_{i-1}-1} b_{ni}^{\alpha_i+1} b_{n,i+1}^{\alpha_{i+1}-1}\ldots b_{nm}^{\alpha_m-1} d\mathbf{b}\right\}$$

for $i = 1,2,\ldots,m$. Evaluating the above integral, we obtain

$$\frac{\Gamma(\alpha_1+\alpha_2+\cdots+\alpha_m)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_m)}\left\{\sum_{\substack{j=1\\ j\neq i}}^{m} x_{nj}\frac{\begin{array}{c}\Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_{i-1})\Gamma(\alpha_i+1)\Gamma(\alpha_{i+1})\ldots\\ \Gamma(\alpha_{j-1})\Gamma(\alpha_j+1)\Gamma(\alpha_{j+1})\ldots\Gamma(\alpha_m)\end{array}}{\Gamma(\alpha_1+\alpha_2+\cdots+\alpha_m+2)}\right.$$

$$\left. + x_{ni}\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\ldots\Gamma(\alpha_{i-1})\Gamma(\alpha_i+2)\Gamma(\alpha_{i+1})\ldots\Gamma(\alpha_m)}{\Gamma(\alpha_1+\alpha_2+\cdots+\alpha_m+2)}\right\}$$

for $i = 1,2,\ldots,m$. Cancelling the common factors of numerator and denominator, we have

$$\frac{\Gamma(\alpha_1+\alpha_2+\cdots+\alpha_m)}{\Gamma(\alpha_1+\alpha_2+\cdots+\alpha_m+2)}\left\{\sum_{\substack{j=1\\j\neq i}}^{m} x_{nj}\frac{\Gamma(\alpha_i+1)\Gamma(\alpha_j+1)}{\Gamma(\alpha_i)\Gamma(\alpha_j)}+x_{ni}\frac{\Gamma(\alpha_i+2)}{\Gamma(\alpha_i)}\right\}$$

for $i = 1,2,\dots,m$. Since $\Gamma(\alpha_1+\alpha_2+\cdots+\alpha_m+2) = (\alpha_1+\alpha_2+\cdots+\alpha_m+1)(\alpha_1+\alpha_2+\cdots+\alpha_m)\Gamma(\alpha_1+\alpha_2+\cdots+\alpha_m)$, $\Gamma(\alpha_i+1) = \alpha_i\Gamma(\alpha_i)$, $\Gamma(\alpha_j+1) = \alpha_j\Gamma(\alpha_j)$ and $\Gamma(\alpha_i+2) = (\alpha_i+1)\alpha_i\Gamma(\alpha_i)$, therefore $\int_B b_{ni}(\mathbf{b}_n^t\mathbf{x}_n)\,d\mu(\mathbf{b})$ in (5.4) equals to

$$\frac{\left\{\sum_{\substack{j=1\\j\neq i}}^{m} x_{nj}\alpha_i\alpha_j + x_{ni}(\alpha_i+1)\alpha_i\right\}}{(\alpha_1+\alpha_2+\cdots+\alpha_m+1)(\alpha_1+\alpha_2+\cdots+\alpha_m)} \tag{5.5}$$

for $i = 1,2,\dots,m$. Next, consider the denominator of $b_{n+1,i}$ in (5.3), namely,

$$\int_B \left(\sum_{j=1}^{m} b_{nj}x_{nj}\right)d\mu(\mathbf{b}) \tag{5.6}$$

for $i = 1,2,\dots,m$. Substitute $d\mu(\mathbf{b})$ in (5.6) by (5.1), we have

$$\frac{\Gamma(\alpha_1+\alpha_2+\cdots+\alpha_m)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\dots\Gamma(\alpha_m)}\left\{\sum_{j=1}^{m} x_{nj}\int_B b_{n1}^{\alpha_1-1}\,b_{n2}^{\alpha_2-1}\dots b_{n,j-1}^{\alpha_{j-1}-1}b_{nj}^{\alpha_j}b_{n,j+1}^{\alpha_{j+1}-1}\dots b_{nm}^{\alpha_m-1}d\mathbf{b}\right\}$$

for $i = 1,2,\dots,m$. Evaluating the above integral, we obtain

$$\frac{\Gamma(\alpha_1+\alpha_2+\cdots+\alpha_m)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\dots\Gamma(\alpha_m)}\left\{\sum_{j=1}^{m} x_{nj}\frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\dots\Gamma(\alpha_{j-1})\Gamma(\alpha_j+1)\Gamma(\alpha_{j+1})\dots\Gamma(\alpha_m)}{\Gamma(\alpha_1+\alpha_2+\cdots+\alpha_m+1)}\right\}$$

for $i = 1,2,\dots,m$. Cancelling the common factors of numerator and denominator, we have

$$\frac{\Gamma(\alpha_1+\alpha_2+\cdots+\alpha_m)}{\Gamma(\alpha_1+\alpha_2+\cdots+\alpha_m+1)}\left\{\sum_{j=1}^{m} x_{nj}\frac{\Gamma(\alpha_j+1)}{\Gamma(\alpha_j)}\right\}$$

for $i = 1,2,\ldots,m$. Since $\Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_m + 1) = (\alpha_1 + \alpha_2 + \cdots + \alpha_m)\Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_m)$ and $\Gamma(\alpha_j + 1) = \alpha_j\Gamma(\alpha_j)$, therefore $\int_B (\mathbf{b}_n^t\mathbf{x}_n)\, d\mu(\mathbf{b})$ in (5.6) equals to

$$\frac{1}{(\alpha_1 + \alpha_2 + \cdots + \alpha_m)}\left\{\sum_{j=1}^{m} x_{nj}\,\alpha_j\right\} \qquad (5.7)$$

for $i = 1,2,\ldots,m$. From (5.3), (5.5) and (5.7), the portfolios $b_{n+1,i}$ for $i = 1,2,\ldots,m$ are given by:

$$b_{n+1,i} = \frac{\sum_{\substack{j=1\\ j\neq i}}^{m} x_{nj}\alpha_i\alpha_j + x_{ni}(\alpha_i + 1)\alpha_i}{(\alpha_1 + \alpha_2 + \cdots + \alpha_m + 1)\left(\sum_{j=1}^{m} x_{nj}\,\alpha_j\right)}.$$

The above $b_{n+1,i}$ can be rewritten as

$$b_{n+1,i} = \frac{\sum_{j=1}^{m} \alpha_i\, x_{nj}\alpha_j + x_{ni}\alpha_i}{(\alpha_1 + \alpha_2 + \cdots + \alpha_m + 1)\left(\sum_{j=1}^{m} x_{nj}\,\alpha_j\right)}$$

for $i = 1,2,\ldots,m$ which is equivalent to (5.2). □

5.1.1 Empirical Results

The implementation of the Dirichlet universal portfolio of order one requires much lesser computer memory requirements compared to the Dirichlet-weighted universal portfolio introduced by Cover and Ordentlich [9]. We run the Dirichlet universal portfolio of order one on the four 10-stock data sets designated as D, E, F and G using $\mathbf{b}_1 = (0.1000, 0.1000, \ldots, 0.1000)$. Table 5.1 shows the portfolios $\mathbf{b}_{1976}$ and the wealths S_{1975} achieved by the respective $\boldsymbol{\alpha}$'s after 1975 trading days.

Table 5.1: The portfolios $\mathbf{b}_{1976}$ and the wealths S_{1975} achieved by some selected α's by the Dirichlet universal portfolio of order one for data sets D, E, F and G, where $\mathbf{b}_1 = (0.1000, 0.1000, \ldots, 0.1000)$

Data set	α	$\mathbf{b}_{1976}$	S_{1975}
Set D	(0.1000 0.1000 0.1000 0.1000 0.1000 0.1000 0.1000 0.1000 0.1000 0.1000)	(0.1004, 0.1004, 0.0998, 0.0994, 0.1001, 0.0995, 0.0995, 0.1004, 0.1014, 0.0991)	18.2914
	(1 1 1 1 1 1 1 1 1 1)	(0.1001, 0.1001, 0.1000, 0.0999, 0.1000, 0.0999, 0.0999, 0.1001, 0.1003, 0.0998)	18.1975
	(1000 1000 1000 1000 1000 1000 1000 1000 1000 1000)	(0.1000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000)	18.1767
	(1 2 3 4 5 6 7 8 9 10)	(0.0182, 0.0364, 0.0545, 0.0727, 0.0909, 0.1091, 0.1272, 0.1455, 0.1637, 0.1818)	13.4232
	(1 2 4 8 16 32 64 128 256 512)	(0.0010, 0.0020, 0.0039, 0.0078, 0.0156, 0.0313, 0.0626, 0.1251, 0.2502, 0.5005)	9.0073
	(0.5981 0.4019 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000)	(0.5980, 0.4020, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000)	37.8708
	(10.5981 10.4019 1.0000 1.0000 1.0000 1.0000 1.0000 1.0000 1.0000 1.0000)	(0.3655, 0.3587, 0.0345, 0.0345, 0.0345, 0.0345, 0.0345, 0.0345, 0.0346, 0.0344)	29.7630
Set E	(0.1000 0.1000 0.1000 0.1000 0.1000 0.1000 0.1000 0.1000 0.1000 0.1000)	(0.1003, 0.1003, 0.0997, 0.0993, 0.1000, 0.0994, 0.0994, 0.0998, 0.1013, 0.1005)	20.4556
	(1 1 1 1 1 1 1 1 1 1)	(0.1001, 0.1001, 0.0999, 0.0999, 0.1000, 0.0999, 0.0999, 0.1000, 0.1002, 0.1001)	20.3257
	(1000 1000 1000 1000 1000 1000 1000 1000 1000 1000)	(0.1000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000)	20.2969
	(1 2 3 4 5 6 7 8 9 10)	(0.0182, 0.0364, 0.0545, 0.0727, 0.0909, 0.1091, 0.1272, 0.1454, 0.1637, 0.1818)	15.7208
	(1 2 4 8 16 32 64 128 256 512)	(0.0010, 0.0020, 0.0039, 0.0078, 0.0156, 0.0313, 0.0626, 0.1251, 0.2502, 0.5005)	10.1487
	(0.5981 0.4019 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000)	(0.5980, 0.4020, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000)	37.9554
	(10.5981 10.4019 1.0000 1.0000 1.0000 1.0000 1.0000 1.0000 1.0000 1.0000)	(0.3655, 0.3587, 0.0345, 0.0345, 0.0345, 0.0345, 0.0345, 0.0345, 0.0345, 0.0345)	30.9736
Set F	(0.1000 0.1000 0.1000 0.1000 0.1000 0.1000 0.1000 0.1000 0.1000 0.1000)	(0.0994, 0.1001, 0.0995, 0.0995, 0.0999, 0.1014, 0.0991, 0.0999, 0.1006, 0.1006)	12.4524
	(1 1 1 1 1 1 1 1 1 1)	(0.0999, 0.1000, 0.0999, 0.0999, 0.1000, 0.1003, 0.0998, 0.1000, 0.1001, 0.1001)	12.3834
	(1000 1000 1000 1000 1000 1000 1000 1000 1000 1000)	(0.1000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000)	12.3681
	(1 2 3 4 5 6 7 8 9 10)	(0.0182, 0.0364, 0.0545, 0.0727, 0.0909, 0.1091, 0.1272, 0.1454, 0.1637, 0.1818)	9.7869
	(1 2 4 8 16 32 64 128 256 512)	(0.0010, 0.0020, 0.0039, 0.0078, 0.0156, 0.0313, 0.0626, 0.1251, 0.2502, 0.5005)	7.2812
	(0.4836 0.3869 0.1295 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000 0.0000)	(0.4824, 0.3884, 0.1292, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000, 0.0000)	20.8371
	(10.4836 10.3869 10.1295 1.0000 1.0000 1.0000 1.0000 1.0000 1.0000 1.0000)	(0.2758, 0.2734, 0.2665, 0.0263, 0.0263, 0.0263, 0.0263, 0.0263, 0.0263, 0.0263)	18.1233
Set G	(0.1000 0.1000 0.1000 0.1000 0.1000 0.1000 0.1000 0.1000 0.1000 0.1000)	(0.1014, 0.0999, 0.1006, 0.0995, 0.0991, 0.0995, 0.0998, 0.1001, 0.0995, 0.1006)	14.7060
	(1 1 1 1 1 1 1 1 1 1)	(0.1003, 0.1000, 0.1001, 0.0999, 0.0998, 0.0999, 0.1000, 0.1000, 0.0999, 0.1001)	14.6185
	(1000 1000 1000 1000 1000 1000 1000 1000 1000 1000)	(0.1000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000)	14.5992
	(1 2 3 4 5 6 7 8 9 10)	(0.0182, 0.0364, 0.0546, 0.0727, 0.0909, 0.1091, 0.1273, 0.1455, 0.1636, 0.1819)	14.8733
	(1 2 4 8 16 32 64 128 256 512)	(0.0010, 0.0020, 0.0039, 0.0078, 0.0156, 0.0313, 0.0626, 0.1251, 0.2502, 0.5005)	11.2267

Table 5.1 continued

Data set	$\boldsymbol{\alpha}$	$\mathbf{b}_{1976}$	S_{1975}
	(0.0000 0.0000 0.0000 0.1965 0.0000 0.0000 0.5926 0.2109 0.0000 0.0000)	(0.0000, 0.0000, 0.0000, 0.1959, 0.0000, 0.0000, 0.5926, 0.2116, 0.0000, 0.0000)	24.1179
	(1.0000 1.0000 1.0000 10.1965 1.0000 1.0000 10.5926 10.2109 1.0000 1.0000)	(0.0263, 0.0263, 0.0263, 0.2683, 0.0263, 0.0263, 0.2787, 0.2687, 0.0263, 0.0263)	21.0003

For data sets D, E, F and G, the highest wealths achieved among the selected $\boldsymbol{\alpha}$'s in Table 5.1 are $S_{1975} = 37.8708, 37.9554, 20.8371$ and 24.1179 respectively. These respective best wealths are achieved by $\boldsymbol{\alpha} = \mathbf{b}^*_{1975}$ where $\mathbf{b}^*_{1975}$ is the respective BCRP. For data sets D and E, the wealths achieved by $\boldsymbol{\alpha} = \mathbf{b}^*_{1975}$ for the Dirichlet universal portfolio of order one are much higher than the maximum wealths achieved by the Helmbold and chi-square divergence (CSD) universal portfolios from Tables 5.1, 2.10 and 3.3 when $\mathbf{b}_1 = (0.1000, 0.1000, \dots, 0.1000)$. The wealths achieved by $\boldsymbol{\alpha} = \mathbf{b}^*_{1975}$ for the Dirichlet universal portfolio of order one for data sets D and E are also higher than the wealths achieved by the $A_3(r)$ and $A_4(r,t)$ universal portfolios for selected r , t in Tables 4.7, 4.8 and 4.9 when $\mathbf{b}_1 = (0.1000, 0.1000, \dots, 0.1000)$. For data sets F and G where $\mathbf{b}_1 = (0.1000, 0.1000, \dots, 0.1000)$, the values of S_{1975} for the Dirichlet universal portfolio of order one where $\boldsymbol{\alpha} = \mathbf{b}^*_{1975}$ in Table 5.1 are higher than the value of $S_{1975}(max)$ for the Helmbold universal portfolio in Table 2.10 and the values of the S_{1975} for $A_3(r)$ and $A_4(r,t)$ universal portfolios for selected r, t in Tables 4.7, 4.8 and 4.9. But the CSD universal portfolio performs better than the Dirichlet universal portfolio of order one where $\boldsymbol{\alpha} = \mathbf{b}^*_{1975}$ in terms of the wealth achieved for data sets F and G where $\mathbf{b}_1 = (0.1000, 0.1000, \dots, 0.1000)$.

5.1.2 The Wealths Achieved by the Dirichlet Universal Portfolios of Order One with Different Initial Starting Portfolios

The portfolio $\mathbf{b}_{n+1}$ in (5.2) is constant for the same $\boldsymbol{\alpha}$ and same price-relatives $\{\mathbf{x}_n\}$. In this subsection, we study the dependence of the wealth S_n achieved by the Dirichlet universal portfolio of order one on the initial starting portfolio $\mathbf{b}_1$. The Dirichlet universal portfolio of order one is run on data set G with the selected initial starting portfolios $\mathbf{b}_1$. The parameter $\boldsymbol{\alpha} = (0.1000, 0.1000, \dots, 0.1000)$ is used for all the portfolios. The portfolios $\mathbf{b}_{256}$, $\mathbf{b}_{1121}$, $\mathbf{b}_{1976}$ and the wealths S_{1975}, ${S_{1975}}/{S_1}$ are calculated for selected initial starting portfolios $\mathbf{b}_1$ and listed in Table 5.2. The wealth ${S_{1975}}/{S_1}$ is defined as the wealth achieved after 1975 trading days where the wealth S_1 achieved on the first day is excluded. Since the portfolio $\mathbf{b}_{n+1}$ does not depend on the initial starting portfolio $\mathbf{b}_1$ for $n = 1,2,3,\dots$ from (5.2), it is clear the daily wealth $\mathbf{b}_{n+1}^t\mathbf{x}_{n+1}$ does not depend on the initial starting portfolio $\mathbf{b}_1$ for $n = 1,2,3,\dots$. The wealth $S_n = \prod_{i=1}^{n} \mathbf{b}_i^t\mathbf{x}_i = (\prod_{i=2}^{n} \mathbf{b}_i^t\mathbf{x}_i)S_1$ depends on the initial starting portfolio $\mathbf{b}_1$ through S_1. Hence ${S_n}/{S_1}$ does not depend on $\mathbf{b}_1$ for $n = 2,3,4,\dots$.

Table 5.2: The portfolios $\mathbf{b}_{256}$, $\mathbf{b}_{1121}$, $\mathbf{b}_{1976}$ and the wealths S_{1975}, ${S_{1975}}/{S_1}$ achieved by the Dirichlet universal portfolios of order one for selected initial starting portfolios for data set G, where $\boldsymbol{\alpha} = (0.1000, 0.1000, \ldots, 0.1000)$

$\mathbf{b}_1$	$\mathbf{b}_{256}$	$\mathbf{b}_{1121}$	$\mathbf{b}_{1976}$	S_{1975}	${S_{1975}}/{S_1}$
(0.1000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000, 0.1000)	(0.1002, 0.0997, 0.1002, 0.0999, 0.1002, 0.1005, 0.0995, 0.1005, 0.0993, 0.1002)	(0.0986, 0.0999, 0.0990, 0.1005, 0.1003, 0.1002, 0.1008, 0.1005, 0.0999, 0.1004)	(0.1014, 0.0999, 0.1006, 0.0995, 0.0991, 0.0995, 0.0998, 0.1001, 0.0995, 0.1006)	14.7060	14.6175
(0.0000, 0.0000, 0.0000, 0.1965, 0.0000, 0.0000, 0.5926, 0.2109, 0.0000, 0.0000)	(0.1002, 0.0997, 0.1002, 0.0999, 0.1002, 0.1005, 0.0995, 0.1005, 0.0993, 0.1002)	(0.0986, 0.0999, 0.0990, 0.1005, 0.1003, 0.1002, 0.1008, 0.1005, 0.0999, 0.1004)	(0.1014, 0.0999, 0.1006, 0.0995, 0.0991, 0.0995, 0.0998, 0.1001, 0.0995, 0.1006)	15.0734	14.6175
(0.1500, 0.0500, 0.2000, 0.1200, 0.0800, 0.0900, 0.0800, 0.1000, 0.1000, 0.0300)	(0.1002, 0.0997, 0.1002, 0.0999, 0.1002, 0.1005, 0.0995, 0.1005, 0.0993, 0.1002)	(0.0986, 0.0999, 0.0990, 0.1005, 0.1003, 0.1002, 0.1008, 0.1005, 0.0999, 0.1004)	(0.1014, 0.0999, 0.1006, 0.0995, 0.0991, 0.0995, 0.0998, 0.1001, 0.0995, 0.1006)	14.6997	14.6175
(0.2200, 0.0200, 0.1100, 0.1300, 0.0000, 0.1800, 0.1000, 0.1500, 0.0500, 0.0400)	(0.1002, 0.0997, 0.1002, 0.0999, 0.1002, 0.1005, 0.0995, 0.1005, 0.0993, 0.1002)	(0.0986, 0.0999, 0.0990, 0.1005, 0.1003, 0.1002, 0.1008, 0.1005, 0.0999, 0.1004)	(0.1014, 0.0999, 0.1006, 0.0995, 0.0991, 0.0995, 0.0998, 0.1001, 0.0995, 0.1006)	14.6937	14.6175
(0.0600, 0.0700, 0.0900, 0.2500, 0.0500, 0.0300, 0.2100, 0.1700, 0.0500, 0.0200)	(0.1002, 0.0997, 0.1002, 0.0999, 0.1002, 0.1005, 0.0995, 0.1005, 0.0993, 0.1002)	(0.0986, 0.0999, 0.0990, 0.1005, 0.1003, 0.1002, 0.1008, 0.1005, 0.0999, 0.1004)	(0.1014, 0.0999, 0.1006, 0.0995, 0.0991, 0.0995, 0.0998, 0.1001, 0.0995, 0.1006)	14.7893	14.6175

The wealths S_{1975} achieved for the five different selected initial starting portfolios according the order in Table 5.2 are $14.7060, 15.0734, 14.6997, 14.6937$ and 14.7893 respectively. Whereas, the wealths ${S_{1975}}/{S_1}$ for the five selected initial starting portfolios are constant, equal to 14.6175, that is, ${S_{1975}}/{S_1}$ does not depend on $\mathbf{b}_1$. The portfolios $\mathbf{b}_{256}$, $\mathbf{b}_{1121}$, $\mathbf{b}_{1976}$ for all the selected initial starting portfolios are exactly equal for the same $\boldsymbol{\alpha}$ and data set G. From (5.2), we observe that the portfolio vectors $\{\mathbf{b}_{n+1}\}$ generated by the Dirichlet universal portfolio of order one are independent of the past portfolios vectors whenever $\boldsymbol{\alpha}$ and $\{\mathbf{x}_n\}$ are the same. We can conclude that for Dirichlet universal portfolio of order one where

$\alpha \neq \mathbf{b}_n$ for $n = 1,2,3, \ldots$, the current portfolio vector does not depend on past portfolio vectors. In conclusion, all the daily achievable wealths $\mathbf{b}_n^t \mathbf{x}_n$ do not depend on the initial starting portfolio $\mathbf{b}_1$, except the wealth S_1 achieved on the first day. The accumulated wealth S_n at the end of day n depends on $\mathbf{b}_1$ only through S_1.

5.2 Relationship between the Dirichlet Universal Portfolio of Order One and the CSD Universal Portfolio

In Chapter Four, we know that the CSD universal portfolio belongs to a general class of universal portfolios generated by the Mahalanobis squared divergence. We identify the relationship between the Dirichlet universal portfolio of order one and the CSD universal portfolio in this section.

First, we state the relationship between CSD universal portfolio and Mahalanobis universal portfolio. Consider a Mahalanobis universal portfolio generated by a diagonal matrix $A = D = (d_i^{-1})$ given in (4.19), the Mahalanobis universal portfolio becomes the CSD universal portfolio when $d_i = b_{ni}$ for $i = 1,2, \ldots, m$.

Now, we identify the relationship between the Dirichlet universal portfolio and the CSD universal portfolio. In the previous subsection, we notice that the update portfolio vector generated by the Dirichlet universal portfolio of order one does not depend on the past portfolio vectors whenever $\boldsymbol{\alpha}$ and $\{\mathbf{x}_n\}$ are the same. If the parameters α_i of Dirichlet universal portfolio of order one are such that $\alpha_i = b_{ni}$ for $i = 1,2,\dots,m$, then the Dirichlet universal portfolio of order one (5.2) becomes some type of additive-update universal portfolio, that is next update portfolio vector $\{\mathbf{b}_{n+1}\}$ depends on the last-known portfolio vector $\{\mathbf{b}_n\}$. When $\alpha_i = b_{ni}$ for $i = 1,2,\dots,m$, the Dirichlet universal portfolio of order one becomes the CSD universal portfolio with parameter $\xi = 1/2$. To verify this, consider a Dirichlet universal portfolio of order one (5.2). Let $\alpha_i = b_{ni}$ for $i = 1,2,\dots,m$, that is,

$$b_{n+1,i} = \frac{1}{\left(\sum_{j=1}^{m} b_{nj} + 1\right)}\left[b_{ni} + \frac{b_{ni}x_{ni}}{\left(\sum_{j=1}^{m} b_{nj}x_{nj}\right)}\right]. \tag{5.8}$$

Since $\sum_{j=1}^{m} b_{nj} = 1$ and $\sum_{j=1}^{m} b_{nj}x_{nj} = \mathbf{b}_n^t\mathbf{x}_n$, (5.8) becomes

$$b_{n+1,i} = \frac{1}{2}\left[b_{ni} + \frac{b_{ni}x_{ni}}{(\mathbf{b}_n^t\mathbf{x}_n)}\right]$$

which can be rewritten as

$$b_{n+1,i} = b_{ni}\left[\frac{\frac{1}{2}\mathbf{b}_n^t\mathbf{x}_n + \frac{1}{2}x_{ni}}{\mathbf{b}_n^t\mathbf{x}_n}\right].$$

Since $\frac{1}{2}\mathbf{b}_n^t\mathbf{x}_n = \mathbf{b}_n^t\mathbf{x}_n - \frac{1}{2}\mathbf{b}_n^t\mathbf{x}_n$, the above $b_{n+1,i}$ can be simplified as

$$b_{n+1,i} = b_{ni}\left[1 + \frac{\frac{1}{2}(x_{ni} - \mathbf{b}_n^t\mathbf{x}_n)}{\mathbf{b}_n^t\mathbf{x}_n}\right]. \tag{5.9}$$

Compare (5.9) with the update portfolio vector generated by the CSD universal portfolio (3.2), it is obvious that (5.9) is a CSD universal portfolio with parameter $\xi = 1/2$.

From the above observation, we can say that the Dirichlet universal portfolio of order one where $\boldsymbol{\alpha} = \mathbf{b}_n$ belongs to the class of Mahalanobis additive-update universal portfolios.

CHAPTER SIX

MIXTURE-CURRENT-RUN UNIVERSAL PORTFOLIO

In this chapter we study the problem of combining or mixing two or more universal portfolios with the aim of obtaining a universal portfolio with a better performance over the original portfolios. Sometimes there may be two or more universal portfolios which are close in performance in terms of the wealth returns. It is difficult in this case to choose a single portfolio for use in investment. This difficulty can be avoided by mixing the universal portfolios in some way that extract the advantages of each portfolio to be exploited in a single mixture portfolio. The experiment focusses on running the Helmbold and chi-square divergence (CSD) universal portfolios on four selected stock-price data sets.

6.1 Mixture Universal Portfolio

Mixing two or more types of universal portfolios can be done by introducing parameter weights (p_i) for the respective universal portfolios. The weights are chosen in some way in order to achieve a higher wealth. The performance of the mixture universal portfolio is studied by running the mixture universal portfolio on the four 10-stock data sets that are used in the previous chapter.

Let us consider k universal portfolios which we designate as $\mathbf{b}_n^1, \mathbf{b}_n^2, \ldots, \mathbf{b}_n^k$. The portfolio

$$\mathbf{b}_n = \sum_{i=1}^{k} p_i \mathbf{b}_n^i \tag{6.1}$$

is a *mixture* of the k universal portfolios if the weights (p_i) are chosen such that $0 \le p_i \le 1$ for $i = 1,2,\ldots,k$ and $\sum_{i=1}^{k} p_i = 1$, for $n = 1,2,3,\ldots$. In terms of the wealths achieved, it is the objective to choose the weights $\mathbf{p} = (p_i)$ that maximizes $S_n(\mathbf{p})$ where

$$S_n(\mathbf{p}) = \prod_{j=1}^{n} \left(\sum_{i=1}^{k} p_i \left(\mathbf{b}_j^i\right)^t \mathbf{x}_j \right), \tag{6.2}$$

although this is usually difficult to achieve. Next, we focus our empirical study on mixing two universal portfolios.

We mix the Helmbold and CSD universal portfolios according to (6.1) for the four 10-stock data sets designated as D, E, F and G in this thesis using $\mathbf{b}_1 = (0.1000, 0.1000, \ldots, 0.1000)$ and choosing an appropriate $\mathbf{p} = (p_1, p_2)$. The best parameters η (Helmbold universal portfolio) and ξ (CSD universal portfolio) for the four 10-stock data sets respectively are used in this study. The maximum values of S_{1975} achieved by the Helmbold universal portfolios for data sets D, E, F and G are at $\eta = 0.4138, -2.3639, -9.4444$ and -83.1143 respectively from Table 2.10, whereas the maximum values of S_{1975} achieved by the CSD universal portfolios for data sets D, E, F and G are at $\xi = 0.3769, -2.8760, -4.9553$ and -3.7942 respectively from Table 3.3. The portfolios $\mathbf{b}_{1976}$ and the maximum wealths $S_{1975}(max)$ achieved by chosen $\mathbf{p}$'s after 1975 trading days are listed in Table 6.1. The letter p_1 refers to the weight assigned on the Helmbold universal portfolio and p_2 is the weight assigned on the CSD universal portfolio, where $p_1 + p_2 = 1$. Through experimentation, we find that maximum wealths are achieved by using extreme weights of $p_i = 0$ or 1 for $i = 1,2$. The results are summarised in Table 6.1.

Table 6.1: The portfolios $\boldsymbol{b}_{1976}$ and the maximum wealths $S_{1975}(max)$ achieved by the mixture universal portfolio for data sets D, E, F and G, where $\mathbf{b}_1 = (0.1000, 0.1000, \ldots, 0.1000)$ and the weight vectors (p_1, p_2) achieving the maximum wealths

Data set	(p_1, p_2)	$\mathbf{b}_{1976}$	$S_{1975}(max)$
Set D	(1, 0)	(0.1356, 0.1319, 0.1223, 0.1042, 0.1048, 0.1012, 0.0914, 0.0558, 0.0825, 0.0702)	18.2486
Set E	(0, 1)	(0.0035, 0.0057, 0.0014, 0.0310, 0.0250, 0.0224, 0.0261, 0.6996, 0.0260, 0.1593)	29.1040
Set F	(0, 1)	(0.0003, 0.0001, 0.0000, 0.0000, 0.6796, 0.0000, 0.0008, 0.3062, 0.0003, 0.0127)	22.3262
Set G	(0, 1)	(0.0013, 0.7729, 0.0184, 0.0044, 0.0626, 0.0017, 0.0000, 0.0026, 0.0021, 0.1340)	25.5834

The maximum wealths $S_{1975}(max)$ achieved (6.2) by mixing the Helmbold and CSD universal portfolios for data sets D, E, F and G are $18.2486, 29.1040, 22.3262$ and 25.5834 respectively are shown in Table 6.1. For data set D, the weight p_1 on the Helmbold universal portfolio is 1 whereas the weight p_2 on the CSD universal portfolio is 0. The weights on the Helmbold and CSD universal portfolios are 0 and 1 respectively for the three data sets E, F and G. It seems that the weight is biased towards the portfolio that achieves the higher wealth and the observed maximum wealths achieved by the mixture universal portfolio in Table 6.1 do not exceed the maximum wealths achieved by the individual Helmbold and CSD universal portfolios.

6.2 Mixture-Current-Run Universal Portfolio

The weights for the mixture universal portfolio studied in Section 6.1 are constant throughout the entire period of investment. Mixing two or more types of universal portfolios using time-varying weights is studied in this section. Next, we introduce the mixture-current-run (MCR) universal portfolio that follows the current run of the portfolio that achieves the best single-day wealth return to avoid the uncertainty of predicting the best-performing universal portfolio in the future.

Let $\mathbf{b}_n^1, \mathbf{b}_n^2, \ldots, \mathbf{b}_n^k$ be k universal portfolios. The portfolio

$$\mathbf{b}_n = \sum_{i=1}^{k} p_{ni}\mathbf{b}_n^i \tag{6.3}$$

is a *time-varying mixture* of the k universal portfolios if the weights $\{p_{ni}\}$ where $0 \le p_{ni} \le 1$ for $i = 1,2,\ldots,k$ and $n = 1,2,3,\ldots$, are time-dependent and chosen according to some decision rule. The *MCR* universal portfolio is a portfolio where the weights $\{p_{ni}\}$ are chosen according to the rule that given $\mathbf{x}_n$,

$$p_{n+1,i} = 1 \text{ if } \max_l\{(\mathbf{b}_n^l)^t\mathbf{x}_n\} = \left(\mathbf{b}_n^i\right)^t\mathbf{x}_n \tag{6.4}$$

and

$$p_{n+1,j} = 0 \text{ for all } j \ne i. \tag{6.5}$$

In other words, if the portfolio $\mathbf{b}_n^i$ achieves the maximum single-day wealth on day n, then the MCR portfolio on day $n+1$ is $\mathbf{b}_{n+1} = \mathbf{b}_{n+1}^i$. If

$$\max_l\{(\mathbf{b}_{n+r}^l)^t\mathbf{x}_{n+r}\} = \left(\mathbf{b}_{n+r}^i\right)^t\mathbf{x}_{n+r} \text{ for } r = 0,1,\ldots,s \tag{6.6}$$

for some positive integers r, s, then the ith portfolio in the mixture creates a run on days $n, n+1, \ldots, n+s$. The run of the ith portfolio is terminated if there exists a smallest integer $u > s$ such that

$$\max_l\{(\mathbf{b}_{n+u}^l)^t \mathbf{x}_{n+u}\} = \left(\mathbf{b}_{n+u}^j\right)^t \mathbf{x}_{n+u} \qquad (6.7)$$

where $j \neq i$. The MCR portfolio follows the run of the ith portfolio, namely,

$$\mathbf{b}_{n+r} = \mathbf{b}_{n+r}^i \ \text{ for } \ r = 1,2, \ldots, u \qquad (6.8)$$

for days $n+1, n+2, n+3, \ldots$, until day $n+u$ and changes to the run of the jth portfolio on day $n+u+1$, namely,

$$\mathbf{b}_{n+u+1} = \mathbf{b}_{n+u+1}^j \ \text{ where } \ j \neq i. \qquad (6.9)$$

Proposition 6.1 Let $\mathbf{b}_n$ be a mixture of the k universal portfolios $\mathbf{b}_n^1, \mathbf{b}_n^2, \ldots, \mathbf{b}_n^k$ defined by the weights $\{p_{ni}\}$ and let S_n be the wealth achieved at the end of the nth trading day. Then

$$\prod_{j=1}^{n} \min_l \left\{\left(\mathbf{b}_j^l\right)^t \mathbf{x}_j\right\} \leq S_n \leq \prod_{j=1}^{n} \max_l \left\{\left(\mathbf{b}_j^l\right)^t \mathbf{x}_j\right\}. \qquad (6.10)$$

Proof. Since $\mathbf{b}_j^t \mathbf{x}_j = \sum_{i=1}^m b_{ji} x_{ji}$ and from (6.3), the wealth achieved in a single day j is

$$\begin{aligned}
\mathbf{b}_j^t \mathbf{x}_j &= \left(\sum_{i=1}^{k} p_{ji} \mathbf{b}_j^i\right)^t \mathbf{x}_j \\
&= \sum_{i=1}^{k} p_{ji} \left(\mathbf{b}_j^i\right)^t \mathbf{x}_j \\
&\leq \sum_{i=1}^{k} p_{ji} \max_l \left\{\left(\mathbf{b}_j^l\right)^t \mathbf{x}_j\right\} \\
&= \max_l \left\{\left(\mathbf{b}_j^l\right)^t \mathbf{x}_j\right\}.
\end{aligned}$$

Noting that $S_n = \prod_{j=1}^{n} \mathbf{b}_j^t \mathbf{x}_j$, the upper bound is obtained. In a similar manner, the lower bound follows. □

The MCR universal portfolio attempts to achieve the upper bound in wealth. Let N be a fixed number of trading days. A measure of the performance of a mixture universal portfolio is the pair of coefficients $\left(c_1(\mathbf{b}_1^N, \mathbf{x}_1^N), c_2(\mathbf{b}_1^N, \mathbf{x}_1^N)\right)$ where $\mathbf{b}_1^N = (\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_N)$, $\mathbf{x}_1^N = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N)$,

$$c_1(\mathbf{b}_1^N, \mathbf{x}_1^N) = \left[\prod_{j=1}^{N} \max_l \left\{\left(\mathbf{b}_j^l\right)^t \mathbf{x}_j\right\} - S_N\right], \tag{6.11}$$

$$c_2(\mathbf{b}_1^N, \mathbf{x}_1^N) = \left[S_N - \max_l \left\{\prod_{j=1}^{N} \left(\mathbf{b}_j^l\right)^t \mathbf{x}_j\right\}\right]. \tag{6.12}$$

For a good-performing mixture universal portfolio, $c_1(\mathbf{b}_1^N, \mathbf{x}_1^N)$ is required to be small and $c_2(\mathbf{b}_1^N, \mathbf{x}_1^N)$ is required to be large. Any mixture universal portfolio achieving the upper bound in wealth given by Proposition 6.1 will have $c_1(\mathbf{b}_1^N, \mathbf{x}_1^N) = 0$. It seems that there may be no advantage in using the mixture if $c_2(\mathbf{b}_1^N, \mathbf{x}_1^N)$ is zero or negative. The coefficient $c_2(\mathbf{b}_1^N, \mathbf{x}_1^N)$ measures the extra wealth achieved by the mixture over the best individual wealth among the k universal portfolios in the mixture. However, since the best individual universal portfolio is unknown at the beginning of the investment period, it is still reasonable to use the mixture even though the wealth achieved may be below the best individual wealth for some data sets. We shall show in the next section, it is possible to achieve extra wealth over the best individual wealth by using the MCR universal portfolio.

Consider an MCR universal portfolio defined by (6.3), (6.4) and (6.5). Our next objective is to estimate the probability of achieving the upper bound in wealth given by Proposition 6.1. Suppose that on the nth trading day, the ith universal portfolio in the mixture achieves the maximum daily wealth given the price-relative vector $\mathbf{x}_n$, namely, $\max_l\{(\mathbf{b}_n^l)^t\mathbf{x}_n\} = \left(\mathbf{b}_n^i\right)^t\mathbf{x}_n$. Assume that there is a stochastic process $\{Y_n\}_{n=1}^{\infty}$ generating the symbol u_i on day n if the ith universal portfolio $\mathbf{b}_n^i$ achieves the maximum daily wealth, namely,

$$Y_n = u_i \text{ if } \max_l\{(\mathbf{b}_n^l)^t\mathbf{x}_n\} = \left(\mathbf{b}_n^i\right)^t\mathbf{x}_n. \tag{6.13}$$

Proposition 6.2 Consider an MCR universal portfolio defined by (6.3), (6.4) and (6.5). Let $\{Y_n\}_{n=1}^{\infty}$ be a stochastic process generating the symbols $u_1, u_2, \dots, u_k$ according to (6.13). An estimate of the probability of the wealth of the MCR universal portfolio achieving the upper bound in Proposition 6.1 is $1 - P(\epsilon)$ where $P(\epsilon) = \sum_{i=1}^{k}\sum_{\substack{j=1\\ j\neq i}}^{k} P\left[Y_n = u_i, Y_{n+1} = u_j\right]$. If $\{Y_n\}_{n=1}^{\infty}$ is an ergodic process, the probability $P\left[Y_n = u_i, Y_{n+1} = u_j\right]$ can be estimated by the relative frequency of $\left(u_i, u_j\right)$ in a long sequence of symbols generated by the process.

Proof. The number $P(\epsilon)$ can be considered as the probability of error in the sense that the wealth of the MCR universal portfolio does not achieve the theoretical upper bound. Now

$$P(\epsilon) = \sum_{i=1}^{k} P(\epsilon|Y_n = u_i)P(Y_n = u_i)$$

$$= \sum_{i=1}^{k} P(Y_{n+1} = u_j, j \neq i | Y_n = u_i) P(Y_n = u_i)$$

$$= \sum_{i=1}^{k} \left\{ \sum_{\substack{j=1 \\ j \neq i}}^{k} P(Y_{n+1} = u_j | Y_n = u_i) \right\} P(Y_n = u_i)$$

$$= \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k} P(Y_n = u_i, Y_{n+1} = u_j).$$

If $\{Y_n\}_{n=1}^{\infty}$ is ergodic, the relative frequency of any finite sequence of symbols converges to its probability almost surely. Hence $P(Y_n = u_i, Y_{n+1} = u_j)$ can be estimated by the relative frequency of (u_i, u_j) in a long sequence of generated symbols. □

6.2.1 Empirical Results

For the purpose of comparison, we run the MCR universal portfolio on the same 10-stock data sets designated as D, E, F and G by mixing the Helmbold and CSD universal portfolios. The initial starting portfolios of the Helmbold, CSD and MCR universal portfolios are $\mathbf{b}_1 = (0.1000, 0.1000, \dots, 0.1000)$. The parameters η (Helmbold universal portfolio) and ξ (CSD universal portfolio) were chosen to maximize the respective wealths achieved. The values of the best parameters η for data sets D, E, F and G are $0.4138, -2.3639, -9.4444$ and -83.1143 respectively from Table 2.10, while the values of the best parameters ξ for data sets D, E, F and G are $0.3769, -2.8760, -4{,}9553$ and -3.7942 respectively from Table

3.3. Table 6.2 shows the resulting portfolios $\mathbf{b}_{1976}$ and the wealths S_{1975} achieved by the MCR universal portfolio after 1975 trading days.

Table 6.2: The portfolios $\mathbf{b}_{1976}$ and the wealths S_{1975} achieved by the MCR universal portfolio for data sets D, E, F and G, where $\mathbf{b}_1 = (0.1000, 0.1000, \ldots, 0.1000)$

Data set	$\mathbf{b}_{1976}$	S_{1975}
Set D	(0.1356, 0.1319, 0.1223, 0.1042, 0.1048, 0.1012, 0.0914, 0.0558, 0.0825, 0.0702)	18.2725
Set E	(0.0085, 0.0107, 0.0161, 0.0391, 0.0409, 0.0469, 0.0797, 0.1374, 0.1487, 0.4719)	26.2478
Set F	(0.0000, 0.0000, 0.0000, 0.0000, 0.0004, 0.0003, 0.0371, 0.8604, 0.0272, 0.0745)	19.1401
Set G	(0.0000, 0.0000, 0.0002, 0.0000, 0.0460, 0.0000, 0.0000, 0.0000, 0.0000, 0.9537)	31.9251

The wealths S_{1975} achieved by the MCR universal portfolio by mixing the Helmbold and CSD universal portfolios for data sets D, E, F and G are $18.2725, 26.2478, 19.1401$ and 31.9251 respectively from Table 6.2. For data sets D and G, the wealths S_{1975} achieved by the MCR universal portfolio, namely 18.2725 and 31.9251 are higher than the maximum wealths $S_{1975}(max)$ achieved by the mixture universal portfolio in Section 6.1, namely 18.2486 and 25.5834. For data sets E and F, the values of S_{1975} for the MCR universal portfolio are lower than the values of $S_{1975}(max)$ for the mixture universal portfolio of the previous section.

In Table 6.3, the wealths achieved by the four types of universal portfolios CSD, Helmbold, MCR and best constant rebalanced portfolio (BCRP) are displayed for the four 10-stock data sets based on 1975 trading days. Let $B(\mathbf{b}_1^n, \mathbf{x}_1^n)$ denote the upper bound given by Proposition 6.1, namely

$$B(\mathbf{b}_1^n, \mathbf{x}_1^n) = \prod_{j=1}^{n} \max_l \left\{ \left(\mathbf{b}_j^l\right)^t \mathbf{x}_j \right\}. \tag{6.14}$$

Then $c_1(\mathbf{b}_1^N, \mathbf{x}_1^N)$ defined by (6.11) can be rewritten as:

$$c_1(\mathbf{b}_1^N, \mathbf{x}_1^N) = B(\mathbf{b}_1^N, \mathbf{x}_1^N) - S_N \tag{6.15}$$

for N fixed. In our study $N = 1975$. The estimated probability for the wealth of the MCR universal portfolio to achieve the upper bound $B(\mathbf{b}_1^N, \mathbf{x}_1^N)$ is given by $1 - P(\epsilon)$ in Proposition 6.2. The values of the quantities $B(\mathbf{b}_1^N, \mathbf{x}_1^N)$, $1 - P(\epsilon)$, $c_1(\mathbf{b}_1^N, \mathbf{x}_1^N)$ and $c_2(\mathbf{b}_1^N, \mathbf{x}_1^N)$ are listed in Table 6.3 for the MCR universal portfolios.

Table 6.3: The wealths S_{1975} achieved by the Helmbold, CSD, MCR universal portfolios and BCRP, together with the values of $B(\mathbf{b}_1^N, \mathbf{x}_1^N)$, $1 - P(\epsilon)$, $c_1(\mathbf{b}_1^N, \mathbf{x}_1^N)$ and $c_2(\mathbf{b}_1^N, \mathbf{x}_1^N)$ for data sets D, E, F and G, where $\mathbf{b}_1 = (0.1000, 0.1000, \ldots, 0.1000)$

Set D						
Type	Parameter	S_{1975}	B	$1 - P(\epsilon)$	c_1	c_2
Helmbold	0.4138	18.2486				
CSD	0.3769	18.2431				
MCR		18.2725	19.5243	0.5127	1.2518	0.0239
BCRP		37.5867				
Set E						
Type	Parameter	S_{1975}	B	$1 - P(\epsilon)$	c_1	c_2
Helmbold	-2.3639	22.9859				
CSD	-2.8760	29.1040				
MCR		26.2478	511.5745	0.5117	485.3267	-2.8562
BCRP		37.5867				
Set F						
Type	Parameter	S_{1975}	B	$1 - P(\epsilon)$	c_1	c_2
Helmbold	-9.4444	15.7558				
CSD	-4.9553	22.3262				
MCR		19.1401	2523.3919	0.5187	2504.2518	-3.1861
BCRP		20.7169				
Set G						
Type	Parameter	S_{1975}	B	$1 - P(\epsilon)$	c_1	c_2
Helmbold	-83.1143	19.9357				
CSD	-3.7942	25.5834				
MCR		31.9251	4146.1442	0.5339	4114.2191	6.3417
BCRP		24.6381				

The theoretical upper bound in wealth given by $B(\mathbf{b}_1^N, \mathbf{x}_1^N)$ is in general not achievable by the MCR universal portfolio unless the runs are completely predictable or there is only one run in the whole trading period. Thus, we observe in Table 6.3 that the values of the coefficients $c_1(\mathbf{b}_1^N, \mathbf{x}_1^N)$ are large for data sets E, F and G. The coefficients $c_2(\mathbf{b}_1^N, \mathbf{x}_1^N)$ measures the excess wealth achieved by the MCR universal portfolio over the individual best wealth achieved by either the CSD or the Helmbold universal portfolios. For data sets D and G, the values of $c_2(\mathbf{b}_1^N, \mathbf{x}_1^N)$ are positive and hence the MCR universal portfolios outperform both the CSD and Helmbold universal portfolios. For data set G, the MCR universal portfolio outperforms the best CSD universal portfolio by 6.3417 units of wealth achieved. The values of $c_2(\mathbf{b}_1^N, \mathbf{x}_1^N)$ for data sets E and F are negative. The MCR universal portfolios for these two data sets still achieve higher wealths than the worst Helmbold universal portfolios.

In Figures 6.1, 6.2, 6.3 and 6.4, the three superimposed graphs of the wealths S_n achieved by the Helmbold universal portfolio against the number of trading days n, the wealths S_n achieved by the CSD universal portfolio against the number of trading days n and the wealths S_n achieved by the MCR universal portfolio against the number of trading days n are shown for data sets D, E, F and G respectively, where $\mathbf{b}_1 = (0.1000, 0.1000, \ldots, 0.1000)$. From Figures 6.1 and 6.4, the MCR universal portfolios perform better than both the Helmbold and CSD universal portfolios most of the trading days in terms of wealth S_n achieved for data sets D and G. The values of S_n for the MCR universal portfolio dominate the values of S_n for both the CSD and Helmbold universal portfolios during the trading days $n = 1400$ until $n = 1600$ for data sets E in Figure 6.2.

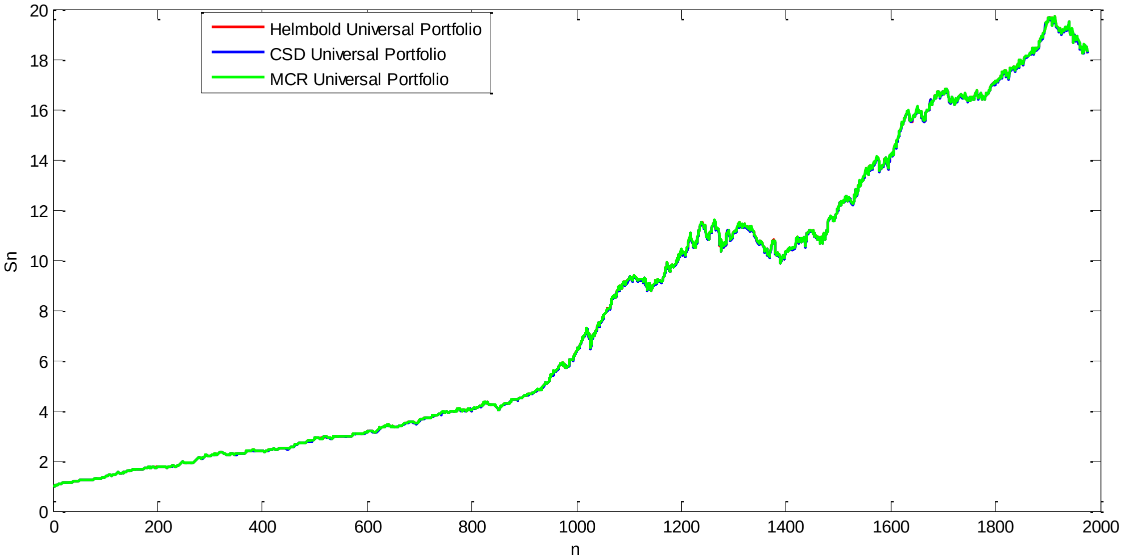

Figure 6.1: Three superimposed graphs of (i)the wealths S_n achieved by the Helmbold universal portfolio against the number of trading days n, (ii)the wealths S_n achieved by the CSD universal portfolio against the number of trading days n and (iii)the wealths S_n achieved by the MCR universal portfolio against the number of trading days n, for data set D, where $\mathbf{b}_1 = (0.1000, 0.1000, \ldots, 0.1000)$

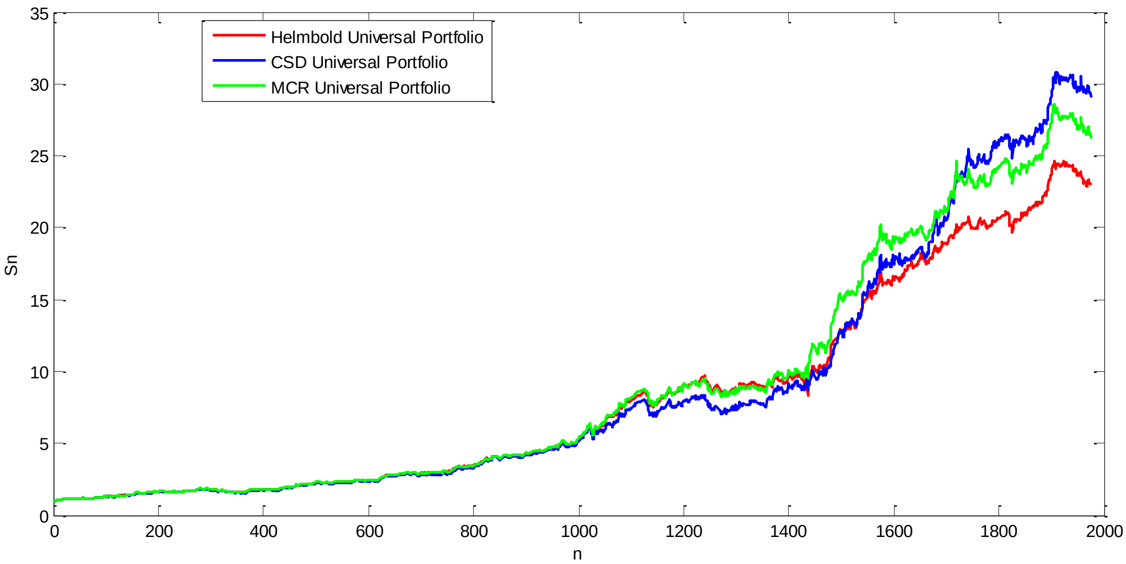

Figure 6.2: Three superimposed graphs of (i)the wealths S_n achieved by the Helmbold universal portfolio against the number of trading days n, (ii)the wealths S_n achieved by the CSD universal portfolio against the number of trading days n and (iii)the wealths S_n achieved by the MCR universal portfolio against the number of trading days n, for data set E, where $\mathbf{b}_1 = (0.1000, 0.1000, \ldots, 0.1000)$

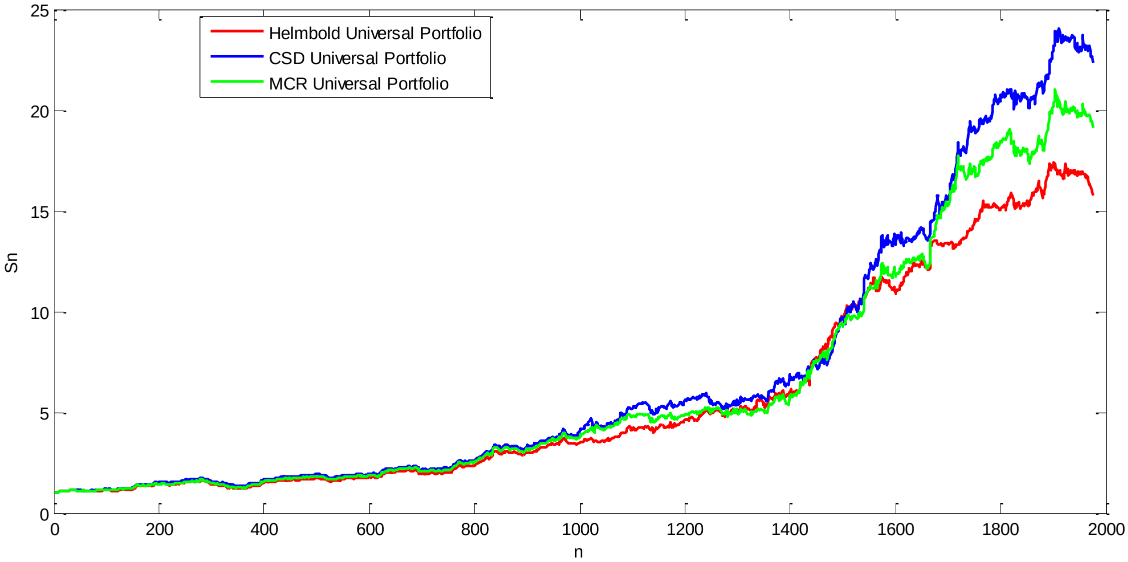

Figure 6.3: Three superimposed graphs of (i)the wealths S_n achieved by the Helmbold universal portfolio against the number of trading days n, (ii)the wealths S_n achieved by the CSD universal portfolio against the number of trading days n and (iii)the wealths S_n achieved by the MCR universal portfolio against the number of trading days n, for data set F, where $\mathbf{b}_1 = (0.1000, 0.1000, \ldots, 0.1000)$

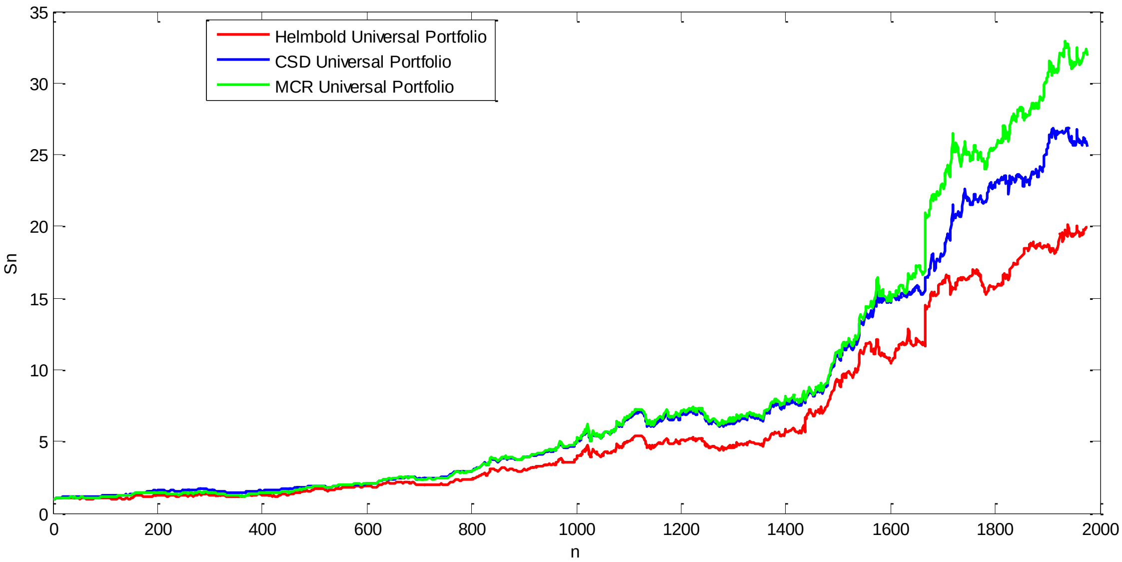

Figure 6.4: Three superimposed graphs of (i)the wealths S_n achieved by the Helmbold universal portfolio against the number of trading days n, (ii)the wealths S_n achieved by the CSD universal portfolio against the number of trading days n and (iii)the wealths S_n achieved by the MCR universal portfolio against the number of trading days n, for data set G, where $\mathbf{b}_1 = (0.1000, 0.1000, \ldots, 0.1000)$

It is worth noting that the Dirichlet-weighted universal portfolios in [9] cannot achieve wealths higher than that of the BCRP. In Table 6.3, the wealths achieved by the CSD universal portfolios are higher than that of the BCRP's in data sets F and G. Furthermore, there is an MCR universal portfolio outperforming the BCRP in data set G. This study concludes with the observation that there are universal portfolios achieving higher wealths than that of the BCRP for certain data sets. The results in this section are reported in Tan and Lim [28].

6.2.2 Application of the Mixture-Current-Run Universal Portfolio in Identifying the Best Current-Run Parameter

We focus our study on mixing two or more types of universal portfolios in the previous sections. We introduce the MCR universal portfolio that follows the current run of the portfolio that achieves the best single-day wealth return. In this section, we discuss the application of the MCR universal portfolio in mixing two or more universal portfolios of the same type to estimate the best-performing parameter corresponding to the run of the best daily wealth.

In Chapters Two, Three and Four, we observe that the parameters η in the Helmbold universal portfolio and ξ in the CSD and the Mahalanobis universal portfolios are important factors influencing the performance of the universal portfolios. An improper choice of η or ξ may lead to a lower investment wealth. It is a difficult task to choose a good parameter at the beginning of the investment period. The MCR universal portfolio can be applied to deal with the above difficulty by mixing two or more universal portfolios of the same type with different values of the scalar parameter by using the better-performing parameter on each trading day.

Let us consider a universal portfolio with the parameter θ defined within a certain range of values, say $\theta_1 \leq \theta \leq \theta_k$. We can form k universal portfolios of the same type using k different values of same scalar parameter. The k different values of same scalar parameter can be obtained by discretizing the range $[\theta_1, \theta_k]$ by $\theta_i = \theta_{i-1} + \frac{\theta_k - \theta_1}{k-1}$ for $i = 2,3, \ldots, k$. Then the MCR universal portfolio is generated by the k universal portfolios of the same type with parameters $\theta_1, \theta_2, \ldots, \theta_k$ will follow the current run of the portfolio which achieves the best single-day wealth return. We can identify the best parameter among the k parameters by keeping track of their performance daily.

The current parameter generating the run of the best single-day wealth is estimated to be the best current-run parameter. In this connection, the most frequent best-current-run parameter throughout the whole investment period may be estimated to be best parameter among the k parameters achieving the highest wealth throughout the investment wealth. This best-performing universal portfolio may be different from the actual best-performing universal portfolio determined from hindsight. It is impossible to determine the latter best-performing universal portfolio at the beginning of the investment period.

Hence it is natural to use the best current-run parameter to achieve a higher investment wealth.

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APPENDIX A

The Matrix $C_3(\boldsymbol{n})$ in (4.28)

$C_3(n) = (c_{ij})$ where

$$c_{ij} = \frac{h_{ij}}{10 + 18r + 16r^2 + 14r^3 + 12r^4 + 10r^5 + 8r^6 + 6r^7 + 4r^8 + 2r^9}.$$

For $i = 1,2,\dots,5$ and $j = 1,2,\dots,10$, h_{ij} are given as

$$h_{11} = (9 + 16r + 13r^2 + 10r^3 + 7r^4 + 4r^5 + r^6 - 2r^7 - 5r^8 - 8r^9 - 9r^{10} - 8r^{11} - 7r^{12} - 6r^{13} - 5r^{14} - 4r^{15} - 3r^{16} - 2r^{17} - r^{18})$$

$$h_{12} = (-1 + 7r + 14r^2 + 11r^3 + 8r^4 + 5r^5 + 2r^6 - r^7 - 4r^8 - 6r^9 - 7r^{10} - 7r^{11} - 6r^{12} - 5r^{13} - 4r^{14} - 3r^{15} - 2r^{16} - r^{17})$$

$$h_{13} = (-1 - 3r + 5r^2 + 12r^3 + 9r^4 + 6r^5 + 3r^6 - 2r^8 - 4r^9 - 5r^{10} - 5r^{11} - 5r^{12} - 4r^{13} - 3r^{14} - 2r^{15} - r^{16})$$

$$h_{14} = (-1 - 3r - 5r^2 + 3r^3 + 10r^4 + 7r^5 + 4r^6 + 2r^7 - 2r^9 - 3r^{10} - 3r^{11} - 3r^{12} - 3r^{13} - 2r^{14} - r^{15})$$

$$h_{15} = (-1 - 3r - 5r^2 - 7r^3 + r^4 + 8r^5 + 6r^6 + 4r^7 + 2r^8 - r^{10} - r^{11} - r^{12} - r^{13} - r^{14})$$

$$h_{16} = (-1 - 3r - 5r^2 - 7r^3 - 9r^4 + 8r^6 + 6r^7 + 4r^8 + 2r^9 + r^{10} + r^{11} + r^{12} + r^{13} + r^{14})$$

$$h_{17} = (-1 - 3r - 5r^2 - 7r^3 - 8r^4 - 9r^5 + 8r^7 + 6r^8 + 4r^9 + 3r^{10} + 3r^{11} + 3r^{12} + 3r^{13} + 2r^{14} + r^{15})$$

$$h_{18} = (-1 - 3r - 5r^2 - 6r^3 - 7r^4 - 8r^5 - 9r^6 + 8r^8 + 6r^9 + 5r^{10} + 5r^{11} + 5r^{12} + 4r^{13} + 3r^{14} + 2r^{15} + r^{16})$$

$$h_{19} = (-1 - 3r - 4r^2 - 5r^3 - 6r^4 - 7r^5 - 8r^6 - 9r^7 + 8r^9 + 7r^{10} + 7r^{11} + 6r^{12} + 5r^{13} + 4r^{14} + 3r^{15} + 2r^{16} + r^{17})$$

$$h_{1,10} = (-1 - 2r - 3r^2 - 4r^3 - 5r^4 - 6r^5 - 7r^6 - 8r^7 - 9r^8 + 9r^{10} + 8r^{11} + 7r^{12} + 6r^{13} + 5r^{14} + 4r^{15} + 3r^{16} + 2r^{17} + r^{18})$$

$$h_{21} = (-1 + 7r + 14r^2 + 11r^3 + 8r^4 + 5r^5 + 2r^6 - r^7 - 4r^8 - 6r^9 - 7r^{10} - 7r^{11} - 6r^{12} - 5r^{13} - 4r^{14} - 3r^{15} - 2r^{16} - r^{17})$$

$$h_{22} = (9 + 14r + 10r^2 + 8r^3 + 5r^4 + 2r^5 - r^6 - 4r^7 - 7r^8 - 8r^9 - 7r^{10} - 6r^{11} - 5r^{12} - 4r^{13} - 3r^{14} - 2r^{15} - r^{16})$$

$$h_{23} = (-1 + 6r + 11r^2 + 8r^3 + 6r^4 + 3r^5 - 3r^7 - 5r^8 - 5r^9 - 5r^{10} - 5r^{11} - 4r^{12} - 3r^{13} - 2r^{14} - r^{15})$$

$$h_{24} = (-1 - 4r + 3r^2 + 9r^3 + 6r^4 + 4r^5 + r^6 - r^7 - 2r^8 - 3r^9 - 3r^{10} - 3r^{11} - 3r^{12} - 2r^{13} - r^{14})$$

$$h_{25} = (-1 - 4r - 7r^2 + r^3 + 7r^4 + 4r^5 + 3r^6 + 2r^7 - r^9 - r^{10} - r^{11} - r^{12} - r^{13})$$

$$h_{26} = (-1 - 4r - 7r^2 - 9r^3 - r^4 + 6r^5 + 5r^6 + 4r^7 + 2r^8 + r^9 + r^{10} + r^{11} + r^{12} + r^{13})$$

$$h_{27} = (-1 - 4r - 7r^2 - 9r^3 - 10r^4 + 7r^6 + 5r^7 + 4r^8 + 3r^9 + 3r^{10} + 3r^{11} + 3r^{12} + 2r^{13} + r^{14})$$

$$h_{28} = (-1 - 4r - 7r^2 - 8r^3 - 8r^4 - 9r^5 + 7r^7 + 5r^8 + 5r^9 + 5r^{10} + 5r^{11} + 4r^{12} + 3r^{13} + 2r^{14} + r^{15})$$

$$h_{29} = (-1 - 4r - 6r^2 - 6r^3 - 7r^4 - 8r^5 - 9r^6 + 7r^8 + 6r^9 + 7r^{10} + 6r^{11} + 5r^{12} + 4r^{13} + 3r^{14} + 2r^{15} + r^{16})$$

$$h_{2,10} = (-1 - 3r - 4r^2 - 5r^3 - 6r^4 - 7r^5 - 8r^6 - 9r^7 + 8r^9 + 7r^{10} + 7r^{11} + 6r^{12} + 5r^{13} + 4r^{14} + 3r^{15} + 2r^{16} + r^{17})$$

$$h_{31} = (-1 - 3r + 5r^2 + 12r^3 + 9r^4 + 6r^5 + 3r^6 - 2r^8 - 4r^9 - 5r^{10} - 5r^{11} - 5r^{12} - 4r^{13} - 3r^{14} - 2r^{15} - r^{16})$$

$$h_{32} = (-1 + 6r + 11r^2 + 8r^3 + 6r^4 + 3r^5 - 3r^7 - 5r^8 - 5r^9 - 5r^{10} - 5r^{11} - 4r^{12} - 3r^{13} - 2r^{14} - r^{15})$$

$$h_{33} = (9 + 14r + 8r^2 + 4r^3 + 2r^4 - 3r^6 - 6r^7 - 7r^8 - 6r^9 - 5r^{10} - 4r^{11} - 3r^{12} - 2r^{13} - r^{14})$$

$$h_{34} = (-1 + 6r + 10r^2 + 5r^3 + 2r^4 - 2r^6 - 4r^7 - 4r^8 - 3r^9 - 3r^{10} - 3r^{11} - 2r^{12} - r^{13})$$

$$h_{35} = (-1 - 4r + 2r^2 + 7r^3 + 3r^4 - r^6 - r^7 - r^8 - r^9 - r^{10} - r^{11} - r^{12})$$

$$h_{36} = (-1 - 4r - 8r^2 - r^3 + 5r^4 + 2r^5 + r^6 + r^7 + r^8 + r^9 + r^{10} + r^{11} + r^{12})$$

$$h_{37} = (-1 - 4r - 8r^2 - 11r^3 - 2r^4 + 6r^5 + 4r^6 + 2r^7 + 2r^8 + 3r^9 + 3r^{10} + 3r^{11} + 2r^{12} + r^{13})$$

$$h_{38} = (-1 - 4r - 8r^2 - 10r^3 - 10r^4 + 7r^6 + 4r^7 + 3r^8 + 4r^9 + 5r^{10} + 4r^{11} + 3r^{12} + 2r^{13} + r^{14})$$

$$h_{39} = (-1 - 4r - 7r^2 - 8r^3 - 8r^4 - 9r^5 + 7r^7 + 5r^8 + 5r^9 + 5r^{10} + 5r^{11} + 4r^{12} + 3r^{13} + 2r^{14} + r^{15})$$

$$h_{3,10} = (-1 - 3r - 5r^2 - 6r^3 - 7r^4 - 8r^5 - 9r^6 + 8r^8 + 6r^9 + 5r^{10} + 5r^{11} + 5r^{12} + 4r^{13} + 3r^{14} + 2r^{15} + r^{16})$$

$$h_{41} = (-1 - 3r - 5r^2 + 3r^3 + 10r^4 + 7r^5 + 4r^6 + 2r^7 - 2r^9 - 3r^{10} - 3r^{11} - 3r^{12} - 3r^{13} - 2r^{14} - r^{15})$$

$$h_{42} = (-1 - 4r + 3r^2 + 9r^3 + 6r^4 + 4r^5 + r^6 - r^7 - 2r^8 - 3r^9 - 3r^{10} - 3r^{11} - 3r^{12} - 2r^{13} - r^{14})$$

$$h_{43} = (-1 + 6r + 10r^2 + 5r^3 + 2r^4 - 2r^6 - 4r^7 - 4r^8 - 3r^9 - 3r^{10} - 3r^{11} - 2r^{12} - r^{13})$$

$$h_{44} = (9 + 14r + 8r^2 + 2r^3 - 2r^4 - 4r^5 - 6r^6 - 6r^7 - 5r^8 - 4r^9 - 3r^{10} - 2r^{11} - r^{12})$$

$$h_{45} = (-1 + 6r + 10r^2 + 4r^3 - r^4 - 4r^5 - 5r^6 - 4r^7 - 2r^8 - r^9 - r^{10} - r^{11})$$

$$h_{46} = (-1 - 4r + 2r^2 + 6r^3 + r^4 - 2r^5 - 3r^6 - 2r^7 + r^9 + r^{10} + r^{11})$$

$$h_{47} = (-1 - 4r - 8r^2 - 2r^3 + 4r^4 + 2r^5 + r^8 + 2r^9 + 3r^{10} + 2r^{11} + r^{12})$$

$$h_{48} = (-1 - 4r - 8r^2 - 11r^3 - 2r^4 + 6r^5 + 4r^6 + 2r^7 + 2r^8 + 3r^9 + 3r^{10} + 3r^{11} + 2r^{12} + r^{13})$$

$$h_{49} = (-1 - 4r - 7r^2 - 9r^3 - 10r^4 + 7r^6 + 5r^7 + 4r^8 + 3r^9 + 3r^{10} + 3r^{11} + 3r^{12} + 2r^{13} + r^{14})$$

$$h_{4,10} = (-1 - 3r - 5r^2 - 7r^3 - 8r^4 - 9r^5 + 8r^7 + 6r^8 + 4r^9 + 3r^{10} + 3r^{11} + 3r^{12} + 3r^{13} + 2r^{14} + r^{15})$$

$$h_{51} = (-1 - 3r - 5r^2 - 7r^3 + r^4 + 8r^5 + 6r^6 + 4r^7 + 2r^8 - r^{10} - r^{11} - r^{12} - r^{13} - r^{14})$$

$$h_{52} = (-1 - 4r - 7r^2 + r^3 + 7r^4 + 4r^5 + 3r^6 + 2r^7 - r^9 - r^{10} - r^{11} - r^{12} - r^{13})$$

$$h_{53} = (-1 - 4r + 2r^2 + 7r^3 + 3r^4 - r^6 - r^7 - r^8 - r^9 - r^{10} - r^{11} - r^{12})$$

$$h_{54} = (-1 + 6r + 10r^2 + 4r^3 - r^4 - 4r^5 - 5r^6 - 4r^7 - 2r^8 - r^9 - r^{10} - r^{11})$$

$$h_{55} = (9 + 14r + 8r^2 + 2r^3 - 4r^4 - 8r^5 - 8r^6 - 6r^7 - 4r^8 - 2r^9 - r^{10})$$

$$h_{56} = (-1 + 6r + 10r^2 + 4r^3 - 2r^4 - 6r^5 - 6r^6 - 4r^7 - 2r^8 + r^{10})$$

$$h_{57} = (-1 - 4r + 2r^2 + 6r^3 + r^4 - 2r^5 - 3r^6 - 2r^7 + r^9 + r^{10} + r^{11})$$

$$h_{58} = (-1 - 4r - 8r^2 - r^3 + 5r^4 + 2r^5 + r^6 + r^7 + r^8 + r^9 + r^{10} + r^{11} + r^{12})$$

$$h_{59} = (-1 - 4r - 7r^2 - 9r^3 - r^4 + 6r^5 + 5r^6 + 4r^7 + 2r^8 + r^9 + r^{10} + r^{11} + r^{12} + r^{13})$$

$$h_{5,10} = (-1 - 3r - 5r^2 - 7r^3 - 9r^4 + 8r^6 + 6r^7 + 4r^8 + 2r^9 + r^{10} + r^{11} + r^{12} + r^{13} + r^{14}).$$

h_{ij} are given as

$$h_{ij} = h_{11-i,11-j}$$

for $i = 6,7,\ldots,10$ and $j = 1,2,\ldots,10$.

APPENDIX B

The Matrix $C_4(n)$ in (4.29) (version 1)

$C_3(n) = (c_{ij})$ where

$$c_{ij} = \frac{h_{ij}}{10r + 2t}.$$

For $i = 1,2,\dots,5$ and $j = 1,2,\dots,10$, h_{ij} are given as

$$h_{11} = (9r^2 - t^2).$$

$$h_{1,10} = (-r^2 + 8rt + t^2).$$

$$h_{1j} = (-r^2 - rt) \qquad \text{for } j = 2,3,\dots,9.$$

$$h_{ij} = (-r^2 - rt) \qquad \text{for } i = 2,3,\dots,5 \text{ and } j = 1,10.$$

$$h_{ii} = (9r^2 + 2rt) \qquad \text{for } i = 2,3,\dots,5.$$

$$h_{ij} = (-r^2) \qquad \text{for } i \neq j,\ i = 2,3,\dots,5 \text{ and } j = 2,3,\dots,9.$$

h_{ij} are given as

$$h_{ij} = h_{11-i,11-j}$$

for $i = 6,7,\dots,10$ and $j = 1,2,\dots,10$.

APPENDIX C

The Matrix $C_4(n)$ in (4.29) (version 2)

$C_3(n) = (c_{ij})$ where

$$c_{ij} = \frac{h_{ij}}{10r + 2t}.$$

For $i = 1,2, \ldots ,5$ and $j = 1,2, \ldots ,10$, h_{ij} are given as

$$h_{11} = (9r^2 - t^2)$$

$$h_{12} = (-r^2 - rt)$$

$$h_{13} = (-r^2 - rt)$$

$$h_{14} = (-r^2 - rt)$$

$$h_{15} = (-r^2 - rt)$$

$$h_{16} = (-r^2 - rt)$$

$$h_{17} = (-r^2 - rt)$$

$$h_{18} = (-r^2 - rt)$$

$$h_{19} = (-r^2 - rt)$$

$$h_{1,10} = (-r^2 + 8rt + t^2)$$

$$h_{21} = (-r^2 - rt)$$

$$h_{22} = (9r^2 + 2rt)$$

$$h_{23} = (-r^2)$$

$$h_{24} = (-r^2)$$

$$h_{25} = (-r^2)$$

$$h_{26} = (-r^2)$$

$$h_{27} = (-r^2)$$

$$h_{28} = (-r^2)$$

$$h_{29} = (-r^2)$$

$$h_{2,10} = (-r^2 - rt)$$

$$h_{31} = (-r^2 - rt)$$

$$h_{32} = (-r^2)$$

$$h_{33} = (9r^2 + 2rt)$$

$$h_{34} = (-r^2)$$

$$h_{35} = (-r^2)$$

$$h_{36} = (-r^2)$$

$$h_{37} = (-r^2)$$

$$h_{38} = (-r^2)$$

$$h_{39} = (-r^2)$$

$$h_{3,10} = (-r^2 - rt)$$

$$h_{41} = (-r^2 - rt)$$

$$h_{42} = (-r^2)$$

$$h_{43} = (-r^2)$$

$$h_{44} = (9r^2 + 2rt)$$

$$h_{45} = (-r^2)$$

$$h_{46} = (-r^2)$$

$$h_{47} = (-r^2)$$

$$h_{48} = (-r^2)$$

$$h_{49} = (-r^2)$$

$$h_{4,10} = (-r^2 - rt)$$

$$h_{51} = (-r^2 - rt)$$

$$h_{52} = (-r^2)$$

$$h_{53} = (-r^2)$$

$$h_{54} = (-r^2)$$

$$h_{55} = (9r^2 + 2rt)$$

$$h_{56} = (-r^2)$$

$$h_{57} = (-r^2)$$

$$h_{58} = (-r^2)$$

$$h_{59} = (-r^2)$$

$$h_{5,10} = (-r^2 - rt).$$

h_{ij} are given as

$$h_{ij} = h_{11-i,11-j}$$

for $i = 6,7, \ldots ,10$ and $j = 1,2, \ldots ,10$.