GENERALISED LAMBERT W FUNCTION AND ITS APPLICATIONS

CHEW CHUN YONG

DOCTOR OF PHILOSPHY (SCIENCE)

LEE KONG CHIAN FACULTY OF ENGINEERING
AND SCIENCE
UNIVERSITI TUNKU ABDUL RAHMAN

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By

CHEW CHUN YONG

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ABSTRACT

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Chew Chun Yong

The Lambert W function plays a pivotal role in solving exponential equations and finds diverse applications across number theory, probability, statistics, and the physical sciences. Recent research has underscored its significance in simulating random variables from Erlang and negative binomial distributions with a shape parameter of two.

As we extended this work to simulate these distributions with a shape parameter of three, we encountered the equation $(w^2 - r)e^w = z$, for which solutions had not been thoroughly investigated. This impelled our exploration of the generalized Lambert W function.

The objectives of our theses are to scrutinize the application of the Lambert W function in delay differential equations, investigate the solutions of $(w^2-r)e^w = z$, $(w^3+pw+q)e^w = z$, and a more comprehensive form, $P_N(w)e^w = z$, where $P_N(w)$ is a polynomial of degree N. Following these investigations, we implement a MATLAB function to compute solutions of $P_N(w)e^w = z$ across various branches using Halley's method.

The techniques we employ throughout our work encompass Lagrange's inversion method and Taylor series expansion. These methodologies enable us to derive series solutions in different branches and determine suitable initial points for numerical computation.

Within this thesis, we delved into the Lambert W function, presenting an

application in delay differential equations. Subsequently, we explored the branch structures of the solutions to $(w^2-r)e^w = z$. In doing so, we obtained series solutions to $(w^2 - r)e^w = z$ across different branches. Additionally, we delved into more comprehensive equations, like $(w^3 + pw + q)e^w = z$, and extended our exploration to equations such as $P_N(w)e^w = z$. Our research made a pivotal contribution by meticulously examining the intricate branch structures and solutions spread across diverse branches, empowering us with effective methods for computing solutions.

While we have successfully accomplished our objectives, intriguing open questions beckon further exploration in future research:

1. Determine the convergence radius of the series expansions of the $W^{(r)}$ function.
2. Explore series expansions of the r-Lambert function capable of computing solutions across various branches.
3. Investigate series solutions of the equation:

$$\frac{P_N(w)}{Q_M(w)}e^w = z.$$

By achieving our objectives, we've not only enriched the field of mathematics but also laid the groundwork for future research. While our thesis journey concludes here, the exploration of the generalised Lambert W function will persist and thrive.

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APPROVAL SHEET

This thesis entitled "**GENERALISED LAMBERT** W **FUNCTION AND ITS APPLICATIONS**" was prepared by CHEW CHUN YONG and submitted as partial fulfillment of the requirements for the degree of Doctor of Philosophy (Science) at Universiti Tunku Abdul Rahman.

Approved by:

(Dr. Goh Yong Kheng) Date: 12 June 2024
Associate Professor/Supervisor
Department of Mathematical and Actuarial Sciences
Lee Kong Chian Faculty of Engineering and Science
Universiti Tunku Abdul Rahman

(Dr. Tan Sin Leng) Date: 12 June 2024
Associate Professor/Supervisor
Department of Mathematical and Actuarial Sciences
Lee Kong Chian Faculty of Engineering and Science
Universiti Tunku Abdul Rahman

(Prof. Dr. Huang-Nan Huang) Date: 12 June 2024
Professor/Supervisor
Department of Applied Mathematics
Tunghai University, Taiwan

LEE KONG CHIAN FACULTY OF ENGINEERING AND SCIENCE

UNIVERSITI TUNKU ABDUL RAHMAN

Date: 12 June 2024

SUBMISSION OF DISSERTATION

It is hereby certified that **Chew Chun Yong** (ID No: **17UED01058**) has completed this thesis entitled "GENERALISED LAMBERT W FUNCTION AND ITS APPLICATIONS" under the supervision of Dr. Goh Yong Kheng (Supervisor) from the Department of Mathematical and Actuarial Sciences, Lee Kong Chian Faculty of Engineering and Science and Dr. Tan Sin Leng (Co-supervisor) from the Department of Mathematical and Actuarial Sciences, Lee Kong Chian Faculty of Engineering and Science and Prof. Dr. Huang-Nan Huang (External co-supervisor) from the Department of Applied Mathematics, Tunghai University, Taiwan.

I understand that the University will upload softcopy of my thesis in pdf format into UTAR Institutional Repository, which may be made accessible to UTAR community and public.

Yours truly,

Chew Chun Yong

DECLARATION

I **CHEW CHUN YONG** hereby declare that the thesis is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UTAR or other institutions.

(Chew Chun Yong)

Date: 12 June 2024

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CHAPTER 1

INTRODUCTION

The Lambert W function, named after Johann Heinrich Lambert, is a special mathematical function that emerges in various areas of mathematics and engineering. It is defined as:

Definition 1.1. *The Lambert W function is defined to be the inverse function of the following transcendental equation:*

$$we^w = z, \tag{1.1}$$

where $z \in \mathbb{C}$ *and the solutions are denoted as* $w = W_k(z)$ *for* $k \in \mathbb{Z}$*. It is common to denote the principal branch,* $W_0(z)$ *by just* $W(z)$ *when there is ambiguity.*

The Lambert W function holds significant applications in solving exponential equations. It has demonstrated its importance in various fields of study. We present some of its properties and an application of the Lambert W function in delay differential equations (DDE).

In recent years, the Lambert W function has also garnered attention for its role in probability and statistics. In their study, Jiménez and Jodrá (2009) showcased how the quantile functions of both the Erlang and negative binomial distributions, when the shape parameter is set to two, can be articulated using the Lambert W function.

The Erlang distribution is a continuous probability distribution commonly used to model the waiting times between events in a Poisson process. It is characterized by two parameters: the shape parameter k and the rate parameter λ. The probability density function (PDF) of the Erlang distribution is given by:

$$f(y; k, \lambda) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}$$

where $y \geq 0$, k is a positive integer, and $\lambda > 0$. The cumulative distribution function (CDF) of the Erlang distribution is expressed as:

$$F(y; k, \lambda) = 1 - \sum_{t=0}^{k-1} \frac{(\lambda y)^t e^{-\lambda y}}{t!}. \tag{1.2}$$

The Erlang distribution is often employed to model the time required for k events to occur in a system with an average event rate of λ.

The negative binomial distribution is a discrete probability distribution frequently used to model the number of successes in a sequence of independent and identically distributed Bernoulli trials before a specified number of failures occur. It is characterized by two parameters: the number of failures r and the success probability p. The probability mass function (PMF) of the negative binomial distribution is given by:

$$P(N = n; r, p) = \binom{n + r - 1}{n} p^n (1-p)^r$$

where n is a non-negative integer and $p \in (0, 1)$. The CDF of the negative binomial distribution is expressed as:

$$F(n; r, p) = 1 - \sum_{i=0}^{r-1} \binom{r+n}{t} (1-p)^{r+n-t} p^t. \tag{1.3}$$

The negative binomial distribution is often used to model count data with overdispersion, where the variance exceeds the mean. In solving the quantile function using the inverse transform method for both distributions, one would need to solve the inverse function of $f(x) = (x^2 - r)e^x$.

The research aims to explore and elucidate solutions derived from complex equations, beginning with fundamental formulations like $(w^2 - r)e^w = z$. It extends its investigation to more general equations, such as $(w^3 + pw + q)e^w = z$. Based on these findings, we introduce an approach for numerically computing solutions of $P_N(w)e^w = z$, wherein $P_N(w)$ denotes a polynomial with real coefficients of degree N. Through these inquiries, this thesis seeks to achieve the following primary objectives:

1. Examine the application of the Lambert W function in delay differential equations.

2. Explore the series solutions of $(w^2 - r)e^w = z$ in various branches, along with its branch analysis.

3. Explore the solutions of the equation $(w^3 + pw + q)e^w = z$ across various branches to analyse their behaviour and outcomes.

4. Compute the solutions of $P_N(w)e^w = z$ using Halley's method.

In the subsequent chapter of this thesis, we carry out a literature review and delve into the characteristics of the Lambert W function and its application in delay differential equations (DDE) in Chapter 3. Following that, we explore solutions of a more extensive equation:

$$(w^2 - r)e^w = z,$$

where $r \in \mathbb{R}$ and $w, z \in \mathbb{C}$. We conduct an in-depth analysis of branch characteristics and series solutions for this equation, employing Lagrange inversion. Additionally, we investigate its Taylor series expansion around $r = 0$, a technique akin to that proposed by Scott, Fee and Grotendorst (2014). Notably, our contribution distinguishes itself through the detailed branch analysis and the diversity of solutions across various branches, in contrast to the findings presented by Scott, Fee and Grotendorst (2014).

In Chapter 5, we expand the scope of our investigation to include series expansions of the solutions to $(w^3 + pw + q)e^w = z$, where $p, q \in \mathbb{R}$. Towards the conclusion of Chapter 5, we delve into the feasibility of employing Halley's method for the computation of solutions to $P_N(w)e^w = z$.

For this thesis, the notation $\log x$ will be used to represent the natural logarithm, following the convention in the Lambert W function community.

CHAPTER 2

LITERATURE REVIEW

2.1 Early developments

The Lambert W function's origins trace back to 1758 when Lambert examined the transcendental equation (Lambert, 1758):

$$x = q + x^m. \tag{2.1}$$

Lambert's studies led him to find a series solution for Equation (2.1), expressed as

$$x = \frac{q}{p} - \frac{q^m}{p^{m+1}} + m\frac{q^{2m-1}}{p^{2m+1}} - \frac{1}{2}m(3m-2)\frac{q^{3m-2}}{p^{3m+1}} + \dots.$$

This solution converges when $(m-1)^{m-1}p^m > m^m q^{m-1}$.

By change of variable $x \to x^{-\beta}$ and let $q = (\alpha - \beta)\nu$ and $m = \alpha/\beta$, Euler (1783) made contributions by transforming Equation (2.1) into

$$x^\alpha - x^\beta = (\alpha - \beta)\nu x^{\alpha+\beta}, \tag{2.2}$$

Euler's work paved the way for further exploration, leading to the formulation of the Lambert W function and its series solutions.

$$\begin{aligned} x^n &= 1 + n\nu + \frac{1}{2}n(n+\alpha+\beta)\nu^2 \\ &\quad + \frac{1}{6}n(n+\alpha+2\beta)(n+2\alpha+\beta)\nu^3 + \dots . \end{aligned} \tag{2.3}$$

Dividing both sides of Equation (2.2) by $(\alpha - \beta)$,

$$\begin{aligned} \lim_{\beta\to\alpha} \frac{x^\alpha - x^\beta}{\alpha - \beta} &= \lim_{\beta\to\alpha} \nu x^{\alpha+\beta} \\ \log x &= \nu x^\alpha \end{aligned} \tag{2.4}$$

Euler observed that by multiplying Equation (2.4) by α,

$$\begin{aligned} \log x^\alpha &= \alpha\nu x^\alpha \\ \log z &= uz, \end{aligned} \tag{2.5}$$

where $z = x^\alpha$ and $u = \alpha\nu$. This implies that if one can solve Equation (2.4) for $\alpha = 1$ then solutions for $\alpha \neq 0$ can be obtained.

To solve this, Euler started with Equation (2.3) and let $\alpha = \beta = 1$, that leads to

$$\begin{aligned} x^n &= 1 + n\nu + \frac{1}{2}n(n+2)\nu^2 + \frac{1}{6}n(n+3)(n+3)\nu^3 + \dots \\ \frac{x^n - 1}{n} &= \nu + \frac{1}{2}(n+2)\nu^2 + \frac{1}{6}(n+3)^2\nu^3 + \dots \\ \lim_{n\to0} \frac{x^n - x^0}{n - 0} &= \lim_{n\to0} \sum_{k=1}^{\infty} \frac{(n+k)^{k-1}}{k!}\nu^k \\ \log x &= \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!}\nu^k = T(\nu), \end{aligned}$$

where $T(\nu)$ is known as tree function (Flajolet and Sedgewick, 2009, p.127–128) and converges when $|\nu| < \frac{1}{e}$.

Since $x = e^{T(\nu)}$ is the solution of Equation (2.4) when $\alpha = 1$, it also fulfills the equation $T(v) = ve^{T(v)}$. Comparing with Equation (1.1), we know that $T(v) = -W(-v)$, or

$$
\begin{aligned}
W(z) &= -T(-v) \\
&= -\sum_{n=1}^{\infty} \frac{n^{n-1}}{n!}(-z)^n \\
&= z - z^2 + \frac{3}{2}z^3 - \dots .
\end{aligned}
$$

More historical developments of the function have been included in (Corless et al., 1996; Mező, 2022).

Besides tree function, the Lambert W function is also introduced as a rapidly convergent series in (Wright, 1959*a*,*b*; Siewert and Burniston, 1973).

If we apply logarithm on both sides of Equation (1.1), we have

$$
w + \log w = \log z. \tag{2.6}
$$

Its multivalued nature was thoroughly studied by Corless et al. (1996) and Corless and Jeffrey (1996), and its applications have since spanned across various disciplines, including delay differential equation (Asl and Ulsoy, 2003; Ohira and Ohira, 2023), gravitational motion equation (Valluri et al., 2000; Scott et al., 2006), cosmology (Saha and Bamba, 2019; Filali et al., 2024), statistical mechanics Caillol (2003), predator-prey model (Davis, 1962), chemical engineering (Kesisoglou et al., 2021), number theory (Visser, 2018) and probability and statistics (Jodrá, 2010).

Indeed, the Lambert W function's significance has led to discussions about its incorporation into educational curricula. This emphasizes its role in enhancing students' comprehension of mathematics. Corless et al. (1996) demonstrated how the Lambert W function can be effectively introduced and taught in calculus and complex analysis courses through pedagogical examples. Furthermore, its application in theoretical physics education has also been explored by Kazakova and Pisanova (2010), highlighting its relevance in advanced scientific disciplines.

The proposal to include the Lambert W function as a new elementary function in senior secondary and introductory tertiary level mathematics curricula, as suggested by Stewart (2005), reflects its increasing importance and potential impact on mathematical education. By introducing students to this function, educators can provide them with valuable insights into solving complex equations, as well as fostering a deeper appreciation for the broader applications of mathematics in various fields.

2.2 Generalisation of the Lambert W function

Generalisation of Equation (1.1) has also been of interest for a long time. For example, Comtet (1970) studied a generalisation, $y^\alpha e^y = x$, and showed that for $\alpha \in \mathbb{R}$ and x being large, the equation

$$y^\alpha e^y = x$$

has a solution

$$\Phi_\alpha(x) = L_1 - \alpha L_2 + \alpha \sum_{n=1}^{\infty} \frac{\alpha^n}{L_1^n} \sum_{m=1}^{n} (-1)^{n+m} \binom{n}{n-m+1} \frac{L_2^m}{m!},$$

where $L_1 = \log x$ and $L_2 = \log\log x$. In the English translation (Jeffrey et al., 1995), the authors showed that if $\alpha \geq 1$, this series is convergent when $x > (\alpha e)^\alpha$; if $\alpha < 1$, it is convergent when $x > e$.

Siewert and Burniston (1974) studied the solutions of $we^w = a(w+b)$, and other similar equations were also considered in Wright (1960) and Noonburg (1969).

In their work, Cooke and van den Driessche (1986) investigated the equation:

$$P_N(w) + Q_M(w)e^{-w} = 0, \tag{2.7}$$

Here, $P_N(w)$ and $Q_M(w)$ represent polynomials of degree N and M respectively. The stability condition of the solutions is also derived. Similar expressions are also observed in other problems such as molecular physics (Scott et al., 1993), inverse Langevin function, and water waves (Mező and Keady, 2016).

Note that when $N = 1$, $M = 0$, and $N = 0$, $M = 1$, Equation (2.7) can be solved in terms of the Lambert W function.

Example 2.1. *Given $N = 1, M = 0$, the equation takes the form $(aw+b)+ce^{-w} = 0$ where a and c are non-zero. Rewriting it leads to:*

$$\left(w + \frac{b}{a}\right) e^{w+\frac{b}{a}} = -\frac{c}{a} e^{\frac{b}{a}},$$

This representation reveals solutions in the form of $w = W_k\left(-\frac{c}{a}e^{\frac{b}{a}}\right) - \frac{b}{a}$. A similar approach can be applied to solve $c + (aw + b)e^{-w} = 0$, leading to solutions of $w = -W_k\left(\frac{c}{a}e^{-\frac{b}{a}}\right) - \frac{b}{a}$.

The Lambert W function finds its application in diverse mathematical areas, notably in deriving asymptotic approximations for different sequences, including Bell numbers. Lovász in his work (Lovász, 2007, Section 1.14, Problem 9) provided an asymptotic expression for Bell numbers in terms of the Lambert W function:

$$B_n \sim \frac{1}{\sqrt{n}} \left(\frac{n}{W(n)}\right)^{n+\frac{1}{2}} e^{\frac{n}{W(n)-n-1}}.$$

Corcino (Corcino and Corcino, 2013) highlighted the significance of solving the equation

$$xe^x + rx = n \tag{2.8}$$

to derive the asymptotic approximation for r-Bell numbers. These solutions, known as r-Lambert W function ($W_r(n)$), are essential in obtaining these approximations.

It was also shown in (Mező and Baricz, 2017) that Equation (2.8) can be transformed and written in a more generalized form:

$$\frac{w-t}{w-s}e^{w} = a. \tag{2.9}$$

The multi-valued inverse function of Equation (2.9) is denoted as $W\left(\begin{smallmatrix} t \\ s \end{smallmatrix}; a\right)$. The solution is also known as the $(1,1)$-type Lambert W function (Mező, 2022). In their study, they derived the solution in terms of the r-Lambert W function:

$$W\left(\begin{smallmatrix} t \\ s \end{smallmatrix}; a\right) = t + W_{-ae^{-t}}\left(ae^{-t}T\right).$$

with detailed discussions on the branch structure provided in (Mező, 2017).

Equation (2.9) can be generalized to accommodate a higher degree, resulting in the following expression:

$$\frac{(w-t_1)(w-t_2)\ldots(w-t_N)}{(w-s_1)(w-s_2)\ldots(w-s_M)}e^{w} = a. \tag{2.10}$$

This is also referred to as the N *upper* M *lower parameters*, and its solutions are named (N, M)-type Lambert W function, denoted as $W\left(\begin{smallmatrix} t_1 & t_2 & \ldots & t_N \\ s_1 & s_2 & \ldots & s_M \end{smallmatrix}; a\right)$. Mező and Baricz (2017) and Mező (2022) have extensively discussed the case of having $(1,1)$-type Lambert W function.

Interestingly, similar expressions are useful in physical science as well. In solving the gravitational motion equation, Scott et al. (2006) showed that

$$e^{-2wR} = a_0 b_0 (w-r_1)(w-r_2) \tag{2.11}$$

has solutions that can be written as a product of two Lambert W functions:

$$w = -\log\left[a_0 b_0 \frac{W((1+\epsilon)Re^{-r_1(1+\epsilon)R}/a_0)W((1-\epsilon)Re^{-r_2(1-\epsilon)R)}/b_0)}{(1+\epsilon)(1-\epsilon)R^2}\right],$$

where ϵ satisfies

$$r_1 - r_2 = \frac{W((1-\epsilon)Re^{-r_2(1-\epsilon)R)}/b_0)}{(1-\epsilon)R} - \frac{W((1+\epsilon)Re^{-r_1(1+\epsilon)R}/a_0)}{(1+\epsilon)R}.$$

This solution was pointed out by Mező and Baricz (2017) to be not satisfactory.

For real parameters r_1, r_2 and c, the equation

$$e^{-cw} = a_0(w - r_1)(w - r_2)$$

was studied by Scott, Fee and Grotendorst (2014), and they derived a series solution presented below:

$$\begin{aligned} w = r_m + 2\frac{W_0}{c} + \frac{1}{4}\frac{cr_d^2}{W_0(W_0+1)} + \frac{1}{64}\frac{c^3 r_d^4(2W_0^2-1)}{W_0^3(W_0+1)^3} \\ + \frac{1}{1536}\frac{c^5 r_d^6(8W_0^4 - 4W_0^3 - 12W_0^2 + 3)}{W_0^5(W_0+1)^5} \\ + \frac{1}{49152}\frac{c^7 r_d^8(48W_0^6 - 64W_0^5 - 132W_0^4 + 40W_0^3 + 90W_0^2 - 15)}{W_0^7(W_0+1)^7} + O\left(c^9 r_d^{10}\right), \end{aligned}$$

where $r_d = \frac{r_1 - r_2}{2}$, $r_m = \frac{r_1 + r_2}{2}$, and $W_0 = W_0\left(\pm\frac{1}{2}\sqrt{\frac{c^2}{a_0}}e^{-cr_m/2}\right)$ represents the Lambert W function at the principal branch. It was proposed for the case when three real solutions exist, that if two solutions calculated using Taylor series are positive and the third solution is negative, then the third solution can be obtained by letting $w \to -w$, $c \to -c$, $r_i \to -r_i$.

A common technique to obtain series representations of inverse functions is by utilizing Lagrange inversion, which is stated below (Abramowitz and Stegun, 1992, Page 14):

Theorem 2.1. *Let $y = f(x)$, $y_0 = f(x_0)$. Suppose that $f'(x_0) \neq 0$, the inverse* *function is given by*

$$x = x_0 + \sum_{k=1}^{\infty} \frac{(y-y_0)^k}{k!} \left[\frac{d^{k-1}}{dx^{k-1}} \left\{ \frac{x-x_0}{f(x)-y_0} \right\}^k \right]_{x=x_0}.$$

This was utilised by Mugnaini (2014) to study the solutions of

$$(x-a)(x-b) = le^x,$$

where $a, b \in \mathbb{R}$. A series solution that can be written in terms of Bessel polynomials, B_n, is obtained:

$$x = a + \sum_{n=1}^{\infty} \frac{1}{n!n} \left(\frac{nle^a}{a-b} \right)^n B_{n-1} \left(\frac{-2}{n(a-b)} \right).$$

One can obtain another solution by interchanging a and b in the series solution above. However, computing solutions at other branches is not possible with this approach.

Neither Scott et al. nor Mugnaini were able to obtain the radius of convergence for their series solution. It was pointed out that the radius of convergence for $e^{-cw} = a_0(w-a_1)(w-a_2)$ is provided by the magnitude of the critical radius, $r_{d_{\text{crit}}}$ (Scott, Fee and Grotendorst, 2014):

$$r_{d_{\text{crit}}} = \pm \frac{1}{c} \sqrt{2W(-2z_0^2) + W(-2z_0^2)^2}, \tag{2.12}$$

where $z_0 = \frac{1}{2} \frac{c}{\sqrt{a_0}} e^{-c\frac{r_m}{2}}$ and $W(z)$ is the Lambert W function. However, this is not a necessary condition. Other applications of the (N, M)-type Lambert function in biology, ecology, or probability have been discussed in Mező (2022).

There are also studies done on other types of generalisations, such as the matrix Lambert function (Asl and Ulsoy, 2003; Yi et al., 2006; Yi and Ulsoy, 2006;

Yi et al., 2007; Jarlebring and Damm, 2007; Cepeda-Gomez and Michiels, 2015), Lambert–Tsallis W_q function (da Silva and Ramos, 2019; da Silva et al., 2019; Mendes et al., 2022), and the cubic Lambert W function (Corcino and Corcino, 2020).

CHAPTER 3

PROPERTIES AND APPLICATIONS OF THE LAMBERT W FUNCTION

In this chapter, we discuss some of the fundamental properties and series solutions of the Lambert W function. We also include an application of the Lambert W function in eigenvalue assignment of a single delay differential equation in the last section.

3.1 Equations Solvable Using the Lambert W Function

We discuss various types of equations that can be solved in terms of the Lambert W function in this section. Since the equation $p(w)e^{q(w)} = ax + b$ can always be rewritten as $p(w)e^{q(w)} = x'$, without loss of generality, we consider the right-hand side of any Lambert-like equations to be x. Unless specified otherwise, we assume x to be real, and we shall postpone the discussion of complex solutions or solutions in other branches until we discuss the branch structure in Section 3.5.

3.1.1 $(pw + q)e^w = x$

We are interested in solving the equation

$$(pw + q)e^w = x. \tag{3.1}$$

It can be checked that when $p = 0$, we have the solution $w = \log x$. Thus, without loss of generality, we consider the case $p \neq 0$ only.

Since $p \neq 0$, we multiply the equation by $\frac{1}{p}e^{\frac{q}{p}}$ on both sides:

$$\left(w + \frac{q}{p}\right) e^{w+\frac{q}{p}} = \frac{x}{p}e^{\frac{q}{p}}.$$

Making a substitution $w' = w + \frac{q}{p}$ gives

$$w' e^{w'} = \frac{x}{p}e^{\frac{q}{p}}.$$

Using Definition (1.1), we know that the solution is

$$w' = W\left(\frac{x}{p}e^{\frac{q}{p}}\right),$$

or

$$w = W\left(\frac{x}{p}e^{\frac{q}{p}}\right) - \frac{q}{p}. \tag{3.2}$$

3.1.2 $we^{rw+s} = x$

Likewise, the equation

$$we^{rw+s} = x \tag{3.3}$$

can also be solved using the Lambert W function. Solving the case where $r = 0$ is trivial, thus we consider the case where r is nonzero.

By multiplying both sides of the equation by re^{-s}, we obtain

$$rwe^{rw} = w'e^{w'} = rxe^{-s},$$

which implies that the solution is given by

$$w = \frac{1}{r}W(rxe^{-s}). \tag{3.4}$$

3.1.3 $wb^w = x$

Consider a scenario where the base in Equation (1.1) is altered to an arbitrary value $b \neq 1$, resulting in the equation:

$$wb^w = x. \tag{3.5}$$

This equation can be solved using the Lambert W function by initially transforming the base b to e:

$$we^{w \log b} = x.$$

After multiplying both sides of the equation by $\log b$:

$$w \log b e^{w \log b} = x \log b,$$

it becomes apparent that the solution to the above equation is given by:

$$w = \frac{1}{\log b} W(x \log b). \tag{3.6}$$

3.1.4 $(pw + q)b^{rw+s} = x$

Utilising insights from previous sections, we can address a more general equation:

$$(pw + q)b^{rw+s} = x. \tag{3.7}$$

For cases where $p = 0$, $r = 0$, or $b = 1$, the equation can be easily solved. Hence, we need only consider the scenario where p and r are both non-zero, and b is not equal to 1.

To address Equation (3.7), we proceed by converting the base b to e and making a suitable substitution:

$$(pw + q)e^{(rw+s)\log b} = x.$$

Next, implementing another substitution, $w' = (rw + s)\log b$, yields an alternate form that enables further analysis:

$$\left(\frac{pw'}{r\log b} - \frac{ps}{r} + \frac{pq}{r}\right)e^{w'} = x.$$

Upon comparing the aforementioned equation with Equation (3.1), we deduce the following expressions:

$$w' = W\left(\frac{x}{p/(r\log b)}e^{\frac{-ps/r+pq/r}{p/(r\log b)}}\right) - \frac{-ps/r + pq/r}{p/(r\log b)}$$

Simplifying expression above leads to:

$$w' = W\left(\frac{r\log b}{p}xe^{(q-s)\log b}\right) - (q-s)\log b.$$

Thus, the solution to Equation (3.7) is given by:

$$w = \frac{1}{r\log b}W\left(\frac{r\log b}{p}xe^{(q-s)\log b}\right) - \frac{q}{r}. \tag{3.8}$$

3.1.5 $w^m e^{w^n} = x$

When considering all m and n values except zero, the equation

$$w^m e^{w^n} = x \tag{3.9}$$

can be solved by an operation involving raising both sides to the power of $\frac{n}{m}$ and a constant multiplication of $\frac{n}{m}$. This results in:

$$\frac{n}{m}w^n e^{\frac{n}{m}w^n} = \frac{n}{m}x^{\frac{n}{m}},$$

leading to the derived solution:

$$w = \left[\frac{m}{n}W\left(\frac{n}{m}x^{\frac{n}{m}}\right)\right]^{\frac{1}{n}}. \tag{3.10}$$

This solution presents a systematic method to resolve the equation, offering a clear path to find the value of w given specific values of m, n and x.

3.1.6 $pw + b^{rw} = x$

Utilising the result from Section 3.1.3, we could also solve the equation

$$pw + b^{rw} = x, \tag{3.11}$$

where $p, r \neq 0$ and $b \neq 1$.

Start by rescaling the initial term on the left-hand side of the equation to rw, rewriting it as:

$$b^{rw+\frac{r}{p}b^{rw}} = b^{\frac{r}{p}x}.$$

Then, by multiplying both sides by $\frac{r}{p}$ and substituting $w = \frac{r}{p}b^{rw}$, we arrive at:

$$wb^w = \frac{r}{p}b^{\frac{r}{p}x}.$$

Upon comparison with Equation (3.5), the solution can be expressed as:

$$w = \frac{1}{r}\log_b\left[\frac{p}{r\log b}W\left(\frac{r}{p}b^{\frac{r}{p}x}\log b\right)\right]. \tag{3.12}$$

3.1.7 $pw + \log_b rw = x$

Another type of equation that can be solved using the Lambert W function is

$$pw + \log_b rw = x, \tag{3.13}$$

where $p, r \neq 0$ and $b \neq 1$. Exponentiating both sides of the equation:

$$b^{pw+\log_b rw} = rwb^{pw} = b^x.$$

Rewriting this equation, we have

$$pwb^{pw} = \frac{p}{r}b^x.$$

Using the result from Section 3.1.3, we know that the solution is:

$$w = \frac{1}{p \log b} W\left(\frac{p}{r}b^x \log b\right). \tag{3.14}$$

3.1.8 $pw \log_b rw = x$

We could also have the multiplicative type of Equation (3.13) to be solved using the Lambert W function. Consider the equation

$$pw \log_b(rw) = x, \tag{3.15}$$

where p and r are non-zero values, and b is not equal to 1.

Upon substituting $rw = b^x$ into the equation, the manipulation yields:

$$\frac{p}{r}b^x \log_b b^x = \frac{p}{r}xb^x = x.$$

Referencing the methodology outlined in Section 3.1.3, the resulting solution emerges as:

$$w = \frac{1}{r} b^x = \frac{1}{r} b^{\frac{1}{\log b} W\left(\frac{r}{p} x \log b\right)}. \tag{3.16}$$

Note that this implies the solution of the equation

$$(rw)^{pw} = x \tag{3.17}$$

as well. By taking the logarithm on both sides of the equation, we have

$$pw \log rw = \log x = x'.$$

This shows that the solution of Equation (3.17) is

$$w = \frac{1}{r} e^x = \frac{1}{r} e^{W\left(\frac{r}{p} \log x\right)}. \tag{3.18}$$

This solution derived for Equation (3.17) through logarithmic transformation showcases the interrelation between exponential and the Lambert W function.

3.1.9 $(rw)^{(pw)^s} = x$

Following from the previous type of equation, we include a more general form:

$$(rw)^{(pw)^s} = x, \tag{3.19}$$

where $r, p, s \neq 0$.

Applying logarithm on both sides of the equation:

$$(pw)^s \log rw = \log x.$$

We could also rewrite the equation above by multiplying $s\frac{r^s}{p^s}$ on both sides:

$$(rw)^s \log(rw)^s = \frac{r^s}{p^s} \log x^s.$$

This allows us to utilise what we have developed in the previous section. Thus, the solution of Equation (3.19) is:

$$w = \frac{1}{r} \left\{ e^{W\left(\log\left[\frac{r^s}{p^s} \log x^s\right]\right)} \right\}^{\frac{1}{s}}. \tag{3.20}$$

3.2 Basic properties of the Lambert W function

We will discuss some basic properties of the Lambert W function in this section. Some of the interesting values are:

1. $W\left(-\frac{1}{e}\right) = -1$,
2. $W(0) = 0$,
3. $W(1) = \Omega$, which is also known as the omega constant,
4. $W(e) = 1$,
5. $W\left(e^{e+1}\right) = e$.

The omega constant, Ω, is defined as below:

Definition 3.1. *The solution of $we^w = 1$ is $w = W(1) = 0.56714\ldots$. This value is also known as the omega constant, Ω. Thus,*

$$\Omega e^{\Omega} = 1. \tag{3.21}$$

From Definition 3.1, we know that the solution of $we^w = 1$ is the omega constant. However, if we consider complex solutions, we have infinitely many, such as $-1.5339-4.3752i, -1.5339+4.3752i, -2.4016+10.7763i, -2.8536+17.1135i$, which are solutions in branch $k = -1, 1, 2$ and 3, respectively. While Ω is not the only solution, it is the only real solution.

In most of the physical problems, one might be interested in the real solutions when x is real. It has been shown by Scott, Fee, Grotendorst and Zhang (2014) that $e^w = P(w)$, where $P(w)$ is a real coefficient polynomial of degree n, has at most $n + 1$ real roots. This implies that

$$\frac{1}{P(w)} e^w = x$$

has at most $n + 1$ real roots. Using this result, we have the following lemma.

Lemma 3.1. *Given $P(w)$ to be polynomial of degree n, the equation*

$$P(w)e^w = x$$

where $x \neq 0$ has a maximum of $n + 1$ real roots.

Proof. Since $P(w)e^{-w} = x' = \frac{1}{x}$ has at most $n + 1$ real roots, we rewrite the polynomial in terms of $-w$:

$$\begin{aligned} P(w) &= c_0 + c_1 w + c_2 w^2 + \cdots + c_n w^n \\ &= c_0 + (-c_1)(-w) + c_2(-w)^2 + \cdots + (-1)^n c_n (-w)^n. \end{aligned}$$

Let $d_n = (-1)^n c_n$, we have

$$P(w) = d_0 + d_1'(-w) + d_2'(-w)^2 + \cdots + d_n'(-w)^n.$$

Thus, $P(w)e^{-w} = Q(-w)e^{-w} = Q(w')e^{w'} = x'$ has at most $n + 1$ roots. □

Specifically, for the Lambert W function, the sufficient condition for two real solutions is that x must be real and between $-\frac{1}{e}$ and 0.

Example 3.1. *Solutions of the equation* $we^w = -0.3$ *are* $W(-0.3)$ *where* $k = 0, \pm 1, \pm 2, \ldots$. *Since* $-\frac{1}{e} < x = -0.3 < 0$, *we know that this equation has exactly two real solutions, which are* $W_0(-0.3) = -0.4894$ *and* $W_{-1}(-0.3) = -1.7813$.

The values of $W(0.3)$ *can be obtained from mathematical software such as MATLAB, Python, or Maxima. Series solutions of* $W(x)$ *will be discussed in a later section.*

3.2.1 The omega constant, Ω

The omega constant, Ω, is known to be an irrational number as well as a transcendental number. In fact, for any $x \neq 0$ that is algebraic, $W(x)$ is transcendental. This could be proven by the Lindemann–Weierstrass theorem:

Theorem 3.1 (Lindemann-Weierstrass theorem)**.** *If* $\alpha_1, \ldots, \alpha_n$ *are algebraic numbers that are linearly independent over the rational numbers* $\mathbb{Q}$, *then* $e^{\alpha_1}, \ldots, e^{\alpha_n}$ *are algebraically independent over* $\mathbb{Q}$.

Theorem 3.2. *The omega constant,* Ω, *is a transcendental number.*

Proof. Suppose that Ω is algebraic. From Theorem 3.1, e^{Ω} is transcendental. Thus, Ωe^{Ω} is transcendental, which contradicts the fact that $\Omega e^{\Omega} = 1$. □

In fact, it has also been proven by Bronstein et al. (2008) that for any $x \neq 0$ that is algebraic, $W(x)$ is transcendental for all branches. A comparable transcendental characteristic is also identified within the Lambert-Tsallis function, as highlighted by (da Silva and Ramos, 2020). This function is defined as the solution to the equation:

$$W_q(x)e_q^{W_q(x)} = x,$$

where e_q is the q-exponential function.

3.2.2 Linear combination of the Lambert W functions

The linear combination of two Lambert W functions, $aW(x_1) + bW(x_2)$, was discussed in Mező (2022, Chapter 1.2.3). Starting from the relationship $e^{W(x)} = \frac{x}{W(x)}$, we derive the following equality:

$$e^{aW(x_1)} e^{bW(x_2)} = \left[\frac{x_1}{W(x_1)}\right]^a \left[\frac{x_2}{W(x_2)}\right]^b,$$

which can be written as:

$$e^{aW(x_1)+bW(x_2)} = \left[\frac{x_1}{W(x_1)}\right]^a \left[\frac{x_2}{W(x_2)}\right]^b.$$

By multiplying both sides of the equation by $aW(x_1) + bW(x_2)$, we have:

$$\begin{aligned}&[aW(x_1) + bW(x_2)]\, e^{aW(x_1)+bW(x_2)} \\ &\quad = [aW(x_1) + bW(x_2)] \left[\frac{x_1}{W(x_1)}\right]^a \left[\frac{x_2}{W(x_2)}\right]^b.\end{aligned}$$

Utilising the Lambert W function yields the following expression:

$$\begin{aligned}&aW(x_1) + bW(x_2) \\ &\quad = W\left([aW(x_1) + bW(x_2)] \left[\frac{x_1}{W(x_1)}\right]^a \left[\frac{x_2}{W(x_2)}\right]^b\right).\end{aligned} \tag{3.22}$$

We extend this result to the sum of n terms.

Theorem 3.3.

$$\sum_{t=1}^{n} a_t W(x_t) = W\left(\left[\sum_{t=1}^{n} a_t W(x_t)\right] \left[\prod_{t=1}^{n} \left(\frac{x_t}{W(x_t)}\right)^{a_t}\right]\right).$$

Proof. Since:

$$e^{\sum_{t=1}^{n} a_t W(x_t)} = \prod_{t=1}^{n} \left(e^{W(x_t)}\right)^{a_t} = \prod_{t=1}^{n} \left(\frac{x_t}{W(x_t)}\right)^{a_t},$$

we have that:

$$\left\{\sum_{t=1}^{n} a_t W(x_t)\right\} \times e^{\sum_{t=1}^{n} a_t W(x_t)} = \sum_{t=1}^{n} a_t W(x_t) \times \prod_{t=1}^{n} \left(\frac{x_t}{W(x_t)}\right)^{a_t}.$$

Thus, solutions to the equation above can be written in terms of the Lambert W function:

$$\sum_{t=1}^{n} a_t W(x_t) = W\left(\left[\sum_{t=1}^{n} a_t W(x_t)\right]\left[\prod_{t=1}^{n} \left(\frac{x_t}{W(x_t)}\right)^{a_t}\right]\right).$$

□

Using this result, we obtain an identity that involves Ω. Let $a_t = x_t = 1$ for all $t = 1, 2, \ldots, n$, we have:

$$W\left(\frac{n}{\Omega^{n-1}}\right) = n\Omega.$$

We end this section by providing a few interesting identities.

Theorem 3.4.

$$W(x \log x) = \log x \quad \left(x \geq \frac{1}{e}\right)$$

$$W\left(-\frac{\log x}{x}\right) = -\log x \quad (0 \leq x \leq e)$$

Proof. By substituting $n = 1, a_1 = 1, W(x_1) = \log x$ into Equation (3.3) we have the first identity. The second identity can be obtained by taking $n = 1, a_1 = -1, W(x_1) = \log x$.

Note that the range of x in each identity is chosen such that the argument of $W(x)$ is always greater than or equal to $-\frac{1}{e}$. This is to assure the existence of a real solution in the principal branch (Branches of the Lambert W function in

Section 3.5). $\square$

3.3 Derivatives and integrals

3.3.1 Derivatives

To derive the first derivative of $W(x)$, we differentiate the equation $we^w = x$. Since w is a function of x, we will express the equation as $W(x)e^{W(x)} = x$ to avoid any potential confusion in this section.

By differentiating both sides of the equation with respect to x, we obtain

$$\frac{dW(x)}{dx}e^{W(x)} + W(x)e^{W(x)}\frac{dW(x)}{dx} = 1,$$

and we can solve for the first derivative:

$$\frac{dW(x)}{dx} = \frac{e^{-W(x)}}{W(x)+1} = \frac{W(x)}{x[1+W(x)]}. \tag{3.23}$$

It is noteworthy that the Lambert W function is not differentiable at $W(x) = -1$ or $x = -e^{-1}$ and $x = 0$. Higher derivatives can be derived through induction.

Theorem 3.5. *For $n \geq 1$, the n-th derivative of the Lambert W function is*

$$\frac{d^n W(x)}{dx^n} = \frac{e^{-nW(x)}p_n(W(x))}{(1+W(x))^{2n-1}}, \tag{3.24}$$

where $p_{n+1}(w) = -(nw + 3n - 1)p_n(w) + (1+w)p_n'(w)$ and $p_1(w) = 1$.

Proof. It can be checked easily that Equation (3.24) is true for the case $n = 1$.

Next, assuming that Equation (3.24) holds for $n = s$, we proceed to differentiate the s-th derivative:

$$\frac{d^{s+1}W(x)}{dx^{s+1}} = \frac{e^{-sW(x)}W'(x)(1+W_s(x))^{2s-2}}{(1+W(x))^{4s-2}} \times \left[-(sW(x)+3s-1)p_s(W(x)) + (1+W(x))p_s'(W(x))\right],$$

where $p_s'(W(x))$ denotes the derivative of $p_s(W(x))$ with respect to x.

By employing the recurrence relation and $W'(x) = \frac{e^{-W(x)}}{1+W(x)}$, we find

$$\frac{d^{s+1}W(x)}{dx^{s+1}} = \frac{e^{-(s+1)W(x)}p_{s+1}(W(x))}{(1+W_s(x))^{2s+1}}.$$

This concludes the proof. □

Consider the equation

$$W(e^x)e^{W(e^x)} = e^x. \tag{3.25}$$

This equation, along with the analysis of its derivatives, presents an intriguing comparison between the derivatives of $W(e^x)$ and $W(x)$.

Upon differentiating both sides of Equation (3.25), we have

$$e^{W(e^x)}\frac{dW(e^x)}{dx} + W(e^x)e^{W(e^x)}\frac{dW(e^x)}{dx} = e^x.$$

Solving for the first derivative yields:

$$\frac{dW(e^x)}{dx} = \frac{e^x}{e^{W(e^x)}\left\{1+W(e^x)\right\}}. \tag{3.26}$$

As $W(e^x) = \frac{e^x}{1+e^{W(e^x)}}$, we have

$$\frac{dW(e^x)}{dx} = \frac{W(e^x)}{1+W(e^x)}. \tag{3.27}$$

This derivative sheds light on the relationship between the Lambert W function and its derivative, showing a simplified form that contrasts the derivatives of $W(x)$.

Higher derivatives are given in the following theorem:

Theorem 3.6. *For $n \geq 1$, the n-th derivative of $W(e^x)$ can be expressed as follows:*

$$\frac{d^n W(e^x)}{dx^n} = \frac{q_n(W(e^x))}{(1+W(e^x))^{2n-1}}, \tag{3.28}$$

where the initial polynomial $q_1(w)$ is w and $q_{n+1}(w)$ is given by $-(2n-1)wq_n(w) + w(1+w)q_n'(w)$.

Proof. From the first derivative, we deduce that Equation (3.28) holds with $q_1(w) = w$.

Assuming that the expression for the n-th derivative and recurrence relation for $q_n(w)$ are true for all $n \leq s$, we have

$$\frac{d^s W(e^x)}{dx^s} = \frac{q_s(W(e^x))}{(1+W(e^x))^{2s-1}}.$$

Differentiating both sides of the equation with respect to x:

$$\frac{d^{s+1} W(e^x)}{dx^{s+1}} = \frac{q_s'(W(e^x))W(e^x)(1+W(e^x)) - (2s-1)W(e^x)q_s(W(e^x))}{(1+W(e^x))^{2s+1}},$$

where $q_{s+1}(w) = q_s'(W(e^x))W(e^x)(1+W(e^x)) - (2s-1)W(e^x)q_s(W(e^x))$. □

3.3.2 Polynomials $p_n(w)$ and $q_n(w)$

We discuss some properties of $p_n(w)$ and $q_n(w)$ in this section.

Proposition 3.1. *The leading coefficient of $p_n(w)$ is $(-1)^{n-1}(n-1)!$ for $n \geq 1$.*

Proof. Starting from the initial condition where $p_1(w) = 1 = (-1)^0 \times 0!$, we confirm the validity of the statement for $n = 1$. Assuming the statement holds for all $n \leq s$, Referring to the recurrence relation, we recognize that the leading term of $p_{s+1}(w)$ matches the leading term of

$$f(w) = -swp_s(w).$$

Given that the leading term of $p_s(w)$ is $(-1)^{s-1}(s-1)!$, the leading term of $f(w)$ becomes $-s \times (-1)^{s-1}(s-1)! = (-1)^s s!$, which aligns with the leading term of $p_{s+1}(w)$. This consistency confirms the validity of the statement for $n = s+1$. $\square$

In addition to the leading term, we are interested in the constant term or the value of $p_n(0)$. This result is significant as it can be used in the Taylor series expansion of the Lambert W function. Corless et al. (1996) stated this result without providing an explicit proof. Mező (2022, Chapter 1.3.3) expanded on this and presented a comprehensive formula for all coefficients of $p_n(w)$. We can summarise this result as follows:

Proposition 3.2. *The polynomials* $p_n(w) = (-1)^{n-1} \sum_{k=0}^{n-1} \beta_{n,k} x^k$, *where*

$$\beta_{n,k} = \sum_{m=0}^{k} \frac{1}{m!} \binom{2n-1}{k-m} \sum_{q=0}^{m} \binom{m}{q} (-1)^q (q+n)^{m+n-1}.$$

Using Proposition 3.2, it's straightforward to derive the following result:

Proposition 3.3. *For all* $n \geq 1$, $p_n(0) = (-n)^{n-1}$.

Proof. As

$$p_n(w) = (-1)^{n-1} \sum_{k=0}^{n-1} \beta_{n,k} w^k,$$

substituting $w = 0$ into both sides of the equation, we have

$$p_n(0) = (-1)^{n-1} \beta_{n,0} = (-1)^{n-1} \times n^{n-1} = (-n)^{n-1}.$$

$\square$

The second-order Eulerian numbers, which were initially observed by Gessel and Stanley (1978), have been found to have a connection with the Lambert W function as well (Corless et al., 1996). It's important to note that these second-order Eulerian numbers satisfy the subsequent recurrence relation (Graham et al., 1994, Chapter 6.3):

$$\left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle = (k+1) \left\langle\!\!\left\langle {n-1 \atop k} \right\rangle\!\!\right\rangle + (2n-1-k) \left\langle\!\!\left\langle {n-1 \atop k-1} \right\rangle\!\!\right\rangle, \tag{3.29}$$

with an initial condition of

$$\left\langle\!\!\left\langle {0 \atop m} \right\rangle\!\!\right\rangle = [m = 0].$$

It's worth mentioning that the expression $[m = 0]$ is referred to as the Iverson bracket, a generalization of the Kronecker delta. It takes on a value of 1 if the statement enclosed within the bracket is true, and 0 otherwise.

Theorem 3.7. *The polynomials q_n are given by*

$$q_n(w) = \sum_{k=0}^{n-1} \left\langle\!\!\left\langle {n-1 \atop k} \right\rangle\!\!\right\rangle (-1)^k w^{k+1}. \tag{3.30}$$

Proof. This identity can also be proven through induction. It can be easily verified that when $n = 1$, the identity holds true.

Assume that Equation (3.30) holds true for $n = s$. By utilizing the recurrence relation for $q_n(w)$, we obtain

$$\begin{aligned} q_{s+1}(w) &= -(2s-1)wq_s(w) + w(1+w)q_s'(w) \\ &= -(2s-1) \sum_{k=0}^{s-1} \left\langle\!\!\left\langle {s-1 \atop k} \right\rangle\!\!\right\rangle (-1)^k w^{k+2} \\ &\quad + (1+w) \sum_{k=0}^{s-1} (k+1) \left\langle\!\!\left\langle {s-1 \atop k} \right\rangle\!\!\right\rangle (-1)^k w^{k+1}. \end{aligned}$$

By expanding and changing the index, we arrive at

$$\begin{aligned}
q_{s+1}(w) = (2s-1)\sum_{k=1}^{s} \left\langle\!\!\left\langle {s-1 \atop k-1} \right\rangle\!\!\right\rangle (-1)^k w^{k+1} \\
+ \sum_{k=0}^{s-1}(k+1) \left\langle\!\!\left\langle {s-1 \atop k} \right\rangle\!\!\right\rangle (-1)^k w^{k+1} + \sum_{k=1}^{s} k \left\langle\!\!\left\langle {s-1 \atop k-1} \right\rangle\!\!\right\rangle (-1)^{k-1} w^{k+1} \\
= (2s-k-1)\sum_{k=1}^{s} \left\langle\!\!\left\langle {s-1 \atop k-1} \right\rangle\!\!\right\rangle (-1)^k w^{k+1} \\
+ \sum_{k=0}^{s-1}(k+1) \left\langle\!\!\left\langle {s-1 \atop k} \right\rangle\!\!\right\rangle (-1)^k w^{k+1}.
\end{aligned}$$

This can be further simplified by utilizing Equation (3.29) and the fact that

$$\left\langle\!\!\left\langle {s-1 \atop s-1} \right\rangle\!\!\right\rangle = 0.$$

Group the summations together and apply the recurrence relation for second-order Eulerian numbers:

$$\begin{aligned}
q_{s+1}(w) &= w + \sum_{k=1}^{s-1} \left\{ (2s-k-1) \left\langle\!\!\left\langle {s-1 \atop k-1} \right\rangle\!\!\right\rangle + (k+1) \left\langle\!\!\left\langle {s-1 \atop k} \right\rangle\!\!\right\rangle \right\} (-1)^k w^{k+1} \\
&= w + \sum_{k=1}^{s-1} \left\langle\!\!\left\langle {s \atop k} \right\rangle\!\!\right\rangle (-1)^k w^{k+1} \\
&= \sum_{k=0}^{s} \left\langle\!\!\left\langle {s \atop k} \right\rangle\!\!\right\rangle (-1)^k w^{k+1}.
\end{aligned}$$

Thus, the expression for $q_n(w)$ holds true for all $n \geq 1$. □

3.3.3 Integrals

The integral $\int W(x)\,dx$ can be easily evaluated using integration by parts twice:

$$\begin{aligned}\int W(x)\,dx &= xW(x) - \int x\frac{dW(x)}{dx}\,dx + C\\ &= xW(x) - \int W(x)e^{W(x)}\,dW(x) + C\\ &= xW(x) - W(x)e^{W(x)} + e^{W(x)} + C\\ &= x\left[W(x) - 1 + \frac{1}{W(x)}\right] + C.\end{aligned}$$

As the Lambert W function is defined as the inverse function of $we^w = x$, one could also utilize the technique of inverse function integration, which is also a result of integration by parts.

Suppose that $y = f(x)$ and $x = f^{-1}(y)$ are single-valued and continuously differentiable. An extension of the technique mentioned above was presented by Parker in 1955 (Parker, 1955), providing the expression:

$$\int f^n(x)\,dx = xf^n(x) - n\int y^{n-1}f^{-1}(y)\,dy. \tag{3.31}$$

This equation offers a broader perspective on integrating powers of a function $f(x)$ and its inverse $f^{-1}(y)$.

Example 3.2. *Let $y = f(x) = W(x)$. We know that $f^{-1}(y) = x = W(x)e^{W(x)}$. Thus,*

$$\begin{aligned}\int W(x)\,dx &= xW(x) - \int W(x)e^{W(x)}\,dW(x)\\ &= x(\ln x)^2 - 2(ye^y - e^y) + C\\ &= x\left[W(x) - 1 + \frac{1}{W(x)}\right] + C.\end{aligned}$$

We can also derive a more general identity using integration by parts:

$$\int x^n f^m(x)\,dx = \frac{x^{n+1} f^m(x)}{n+1} - \frac{m}{n+1} \int \left[f^{-1}(y)\right]^{n+1} y^{m-1} dy, \tag{3.32}$$

where $n \geq 0$ and $m \geq 1$. This identity can be used to evaluate integrals that contain $W(x)$ easily.

Example 3.3. *Using Equation (3.32), the integral $\int xW(x)\,dx$ can be expressed as:*

$$\frac{x^2 W(x)}{2} - \frac{1}{2} \int W(x)^2 e^{2W(x)}\, dW(x),$$

where the second term can be evaluated using integration by parts:

$$\begin{aligned}\int xW(x)\,dx &= \frac{W(x)^3 e^{2W(x)}}{2} - \left[2W(x)^2 - 2W(x) + 1\right] \frac{e^{2W(x)}}{8} + C\\ &= \frac{1}{2}\left(W(x) - \frac{1}{2}\right)\left(W^2(x) + \frac{1}{2}\right) e^{2W(x)} + C.\end{aligned}$$

By applying integration by parts repeatedly, we obtain the following result:

Lemma 3.2. *For all integers n, m where n is greater than zero, m is greater than 1,*

$$\int x^n e^{mx}\, dx = e^{mx} \sum_{s=0}^{n} \frac{(-1)^s n! x^{n-s}}{(n-s)! m^{s+1}} + C. \tag{3.33}$$

We are now ready to present a more general form of the two examples above.

Theorem 3.8. *For $n \geq 0, m \geq 1$,*

$$\int x^n W^m(x)\,dx = x^{n+1} W^m(x) \left[\frac{1}{n+1} - \frac{m}{n+1} \sum_{s=0}^{n+m} \frac{(-1)^s (n+m)!}{(n+m-s)!(n+1)^{s+1} W^{s+1}(x)}\right] + C. \tag{3.34}$$

Proof. By using Equation (3.32), the left-hand side can be written as

$$\int x^n W^m(x)\,dx = \frac{x^{n+1}W^m(x)}{n+1} - \frac{m}{n+1}\int \left[W(x)e^{W(x)}\right]^{n+1} W^{m-1}(x)dW(x).$$

The integral on the right-hand side is equivalent to

$$\int W^{n+m}(x)e^{(n+1)W(x)}dW(x),$$

and by Lemma 3.2 we have

$$\int x^n W^m(x)\,dx = \frac{x^{n+1}W^m(x)}{n+1} - \frac{me^{(n+1)W(x)}}{n+1}\sum_{s=0}^{n+m}\frac{(-1)^s(n+m)!W^{n+m-s}(x)}{(n+m-s)!(n+1)^{s+1}} + C.$$

By noting that $W(x)e^{W(x)} = x$, the above can be simplified to the desired result. □

Theorem (3.8) allows us to obtain some results presented by other researchers. For example, by taking $n = m = 1$, we recover the result by Corless et al. (1996):

$$\begin{aligned}\int xW(x)\,dx &= \frac{x^2W(x)}{2}\left[1 - \frac{1}{2W(x)} + \frac{1}{2W^2(x)} - \frac{1}{4W^3(x)}\right] + C\\ &= \frac{1}{2}\left(W(x) - \frac{1}{2}\right)\left(W^2(x) + \frac{1}{2}\right)e^{2W(x)} + C\end{aligned}$$

Recall that incomplete Gamma function, $\Gamma(s,z)$, is defined as

$$\Gamma(s,z) = \int_z^{\infty} t^{s-1}e^{-t}\,dt.$$

For $s = 1, 2, 3, \ldots$, the incomplete Gamma function can also be written as

$$\Gamma(s,z) = (s-1)!e^{-z}\sum_{k=0}^{s-1}\frac{z^k}{k!}.$$

Thus, for $m = 1$, the Equation (3.34) can be written

$$\begin{aligned}
\int x^n W(x)\,dx &= x^{n+1}W(x)\left[\frac{1}{n+1} - \frac{1}{n+1}\sum_{s=0}^{n+1}\frac{(-1)^s(n+1)!}{(n-s+1)!(n+1)^{s+1}W^{s+1}(x)}\right] + C \\
&= \frac{x^{n+1}W(x)}{n+1} + n!x^{n+2}e^{-W(x)}\sum_{k=0}^{n+1}\frac{1}{k!(-n-1)^{n-k+2}W^{n-k+2}(x)} + C \\
&= \frac{x^{n+1}W(x)}{n+1} + \frac{(-1)^n(n+1)!e^{(n+1)W(x)}}{(n+1)^{n+3}}\sum_{k=0}^{n+1}\frac{[(-n-1)W(x)]^k}{k!} + C \\
&= \frac{x^{n+1}W(x)}{n+1} + \frac{(-1)^n}{(n+1)^{n+3}}\Gamma(n+2, -(n+1)W(x)) + C,
\end{aligned} \tag{3.35}$$

which matches the result presented in Mező (2022). It is important to note that the result presented in (Mező, 2022) contains a minor typo in the second term of the equation; it should be positive instead of alternating sign.

The Mellin transform of the Lambert W function has been derived in (Mező, 2022), and a nice result in terms of the Gamma function was obtained:

$$\{\mathcal{M}W\}(s) = \frac{(-s)^{-s}}{s}\Gamma(s), \quad \text{for } -1 < \mathrm{Re}(s) < 0. \tag{3.36}$$

While the Mellin transformation can be interpreted as a multiplicative variant of the two-sided Laplace transformation, there is no elegant expression for the Laplace transform of the Lambert W function in terms of well-known functions.

3.4 Relationships with Riemann Zeta Function

In this section, we discuss the relationships between the Lambert W function and the Riemann zeta function. The generalized poly-Bernoulli numbers, $B^{(\mu)}_{n,\geq m}$, is defined as

$$\sum_{n=0}^{\infty} B^{(\mu)}_{n,\geq m}\frac{t^n}{n!} = \frac{Li_\mu\left(E_{m-1}(-t) - e^{-t}\right)}{E_{m-1}(-t) - e^{-t}},$$

where $E_m(t) = \sum_{k=0}^{m}\frac{t^k}{k!}$.

It was proven by Komatsu et al. (2016) that for any $\mu \in \mathbb{C}$ with $\mathrm{Re}(\mu) > 1$, the Riemann zeta function can be expressed as

$$\zeta(\mu) = \sum_{n=0}^{\infty} B_{n,\geq 2}^{(\mu)} \frac{(W_k(-1))^n}{n!}, \tag{3.37}$$

where $k = 0, 1$.

Using the fact that $\zeta(s)$ satisfies the functional equation,

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s), \tag{3.38}$$

we obtained

$$\begin{aligned} 2^{\mu}\pi^{\mu-1} \sin\left(\frac{\pi\mu}{2}\right) \Gamma(1-\mu)\zeta(1-\mu) &= \sum_{n=0}^{\infty} B_{n,\geq 2}^{(\mu)} \frac{(W_k(-1))^n}{n!} \\ \zeta(1-\mu) &= \frac{\sum_{n=0}^{\infty} B_{n,\geq 2}^{(\mu)} \frac{(W_k(-1))^n}{n!}}{2^{\mu}\pi^{\mu-1} \sin\left(\frac{\pi\mu}{2}\right) \Gamma(1-\mu)}, \end{aligned}$$

or

$$\zeta(s) = \frac{2^{s-1}\pi^s}{\sin\left(\frac{\pi(1-s)}{2}\right)\Gamma(s)} \sum_{n=0}^{\infty} B_{n,\geq 2}^{(1-s)} \frac{(W_k(-1))^n}{n!}, \tag{3.39}$$

for $\mathrm{Re}(s) < 0$.

3.5 Branches of the Lambert W function

As presented in Lemma 3.1, we know that the Lambert W function has a maximum of two real roots. From the graph of $W(x)$, we can make the following observations:

1. There is no real solution when $x < -\frac{1}{e}$,

2. There is exactly one real solution when $x \geq 0$, and

3. There are two real solutions when $-\frac{1}{e} < x < 0$.

For $-\frac{1}{e} < x < 0$, the two values of $W(x)$ are defined as values in two different branches, which we denote as $W_0(x)$ and $W_{-1}(x)$. The graph below illustrates the $W_0(x)$ and $W_{-1}(x)$ branches.

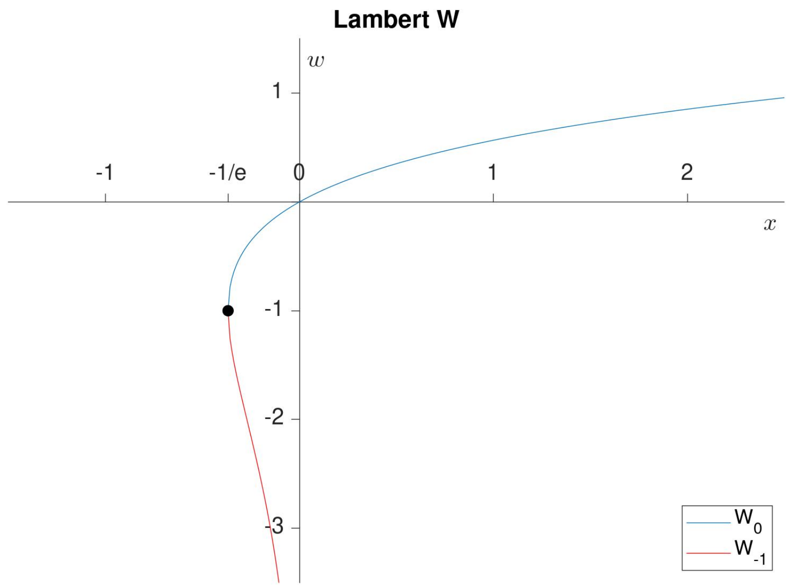

Figure 3.1: Real branches of the $W_k(x)$ function when x is real.

In fact, the Lambert W function has infinitely many branches, which can be seen clearly from the fact that $w + \log w = \log z$, where the logarithm has infinitely many branches. This suggests that the logarithm could be used in the study of branches of the Lambert W function. Jeffrey et al. (1996) showed the following relationship between the Lambert W function and the complex logarithm:

$$W_k(z) + \log W_k(z) = \begin{cases} \log z, & \text{for } k = -1 \text{ and } z \in [-\frac{1}{e}, 0), \\ \log_k z, & \text{otherwise.} \end{cases} \tag{3.40}$$

It is known that the complex logarithm, $w = \log z$, has a branch point at $z = 0$, and its branch cut is defined to be the negative real axis. The figure below shows the branch cut, where the solid line on the negative real axis indicates the closure of the branch cut. This choice of branch cut follows the rule of *counter-*

clockwise continuity.

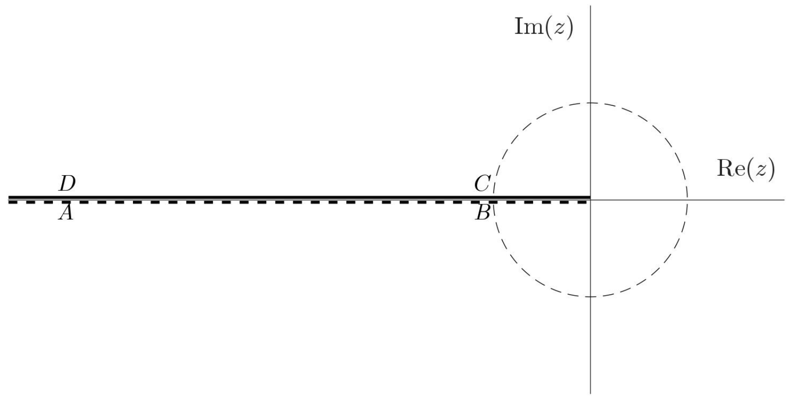

Figure 3.2: z-plane of $w = \log z$.

The ranges (or w-plane) of $w = \log z$ are shown in Figure 3.3. The principal branch is $-\pi < \mathrm{Im}(w) \leq \pi$, and the vertical dashed line represents the range of the dashed circle in Figure 3.2.

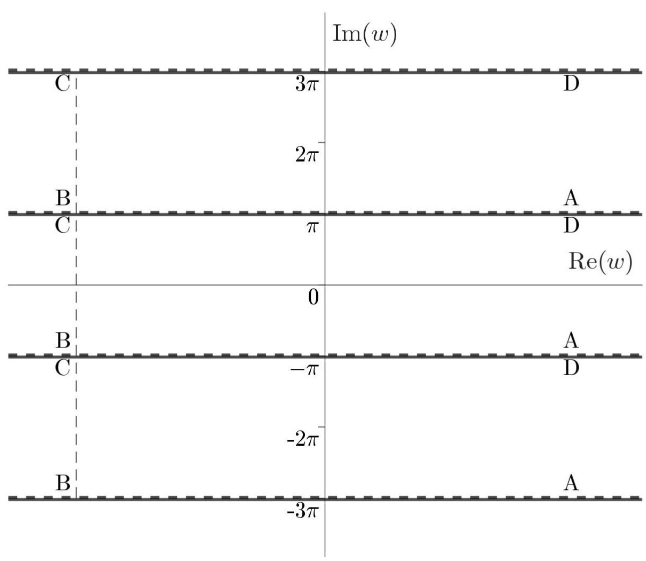

Figure 3.3: The ranges of $w = \log z$.

3.5.1 Branch points and branch cuts

Similar to $\log z$, we need to determine all the branch points of $W_k(z)$ before we study its branch structure.

Note that $f(w) = we^w = z$ and $f'(w) = (1+w)e^w$. Since $f'(-1) = 0$, we know that $w = -1$ or $z = -\frac{1}{e}$ is a branch point. In fact, it is a second-order branch point due to $f''(-1) \neq 0$. Thus, $z = -\frac{1}{e}$ is also a branch point of $W_{-1}(z)$ and $W_1(z)$.

We can also observe that $f(w) \to 0$ as $w \to -\infty$. Therefore, $z = 0$ is another branch point. However, for $z = 0$, possible values for w are 0 and $-\infty$. From Figure 3.1, we can see that only the principal branch contains non-negative real numbers. Thus, we have $W_0(0) = 0$ and $W_k(0) = -\infty$ for all $k \neq 0$.

In summary, the branch points are as follows:

- Principal branch ($W_0(z)$): Branch points at $-\infty$ and $z = -\frac{1}{e}$.
- Branches $W_{-1}(z)$ and $W_1(z)$: Branch points at $-\infty$, $z = -\frac{1}{e}$, and $z = 0$.
- All other branches: Branch points at $-\infty$ and $z = 0$.

The choice of branch cuts is as follows:

- Principal branch: $\left\{z : -\infty < z \leq -\frac{1}{e}\right\}$.
- Branch $W_{-1}(z)$ and $W_1(z)$: $\left\{z : -\infty < z \leq -\frac{1}{e}\right\}$ and $\{z : -\infty < z \leq 0\}$.
- All other branches: $\{z : -\infty < z \leq 0\}$.

All the branch cuts are closed on top to conform with counter-clockwise continuity. Due to the double branch cuts in $W_{-1}(z)$ and $W_1(z)$, there is interesting behaviour around the point $z = -\frac{1}{e}$. This will be discussed further once we complete the discussion of the branch structure in the next section.

3.5.2 Branch structure

To define the boundary curves that partition the w-plane of the Lambert W function, let $w = \xi + i\eta$ and $z = x + iy$. Substituting this into Equation (1.1), we obtain:

$$x = e^{\xi}(\xi \cos\eta - \eta \sin\eta) \tag{3.41}$$

and

$$y = e^{\xi}(\eta \cos \eta + \xi \sin \eta). \tag{3.42}$$

Since the branch cuts are defined to be similar to the complex logarithm, which are on the negative real axis of the z-plane, we can solve Equation (3.42) by setting $y = 0$ and obtain:

$$\eta \cos \eta = -\xi \sin \eta, \tag{3.43}$$

which implies that $\eta = 0$ or $\xi = -\eta \cot \eta$. Therefore

$$w = -\eta \cot \eta + i\eta. \tag{3.44}$$

Note that for the branch cuts to be on the negative real axis of the z-plane, we must have $x < 0$, or equivalently:

$$\xi \cos \eta < \eta \sin \eta. \tag{3.45}$$

The equations (3.43) and (3.45) define the boundary curves that separate the different branches, as indicated in Figure 3.4.

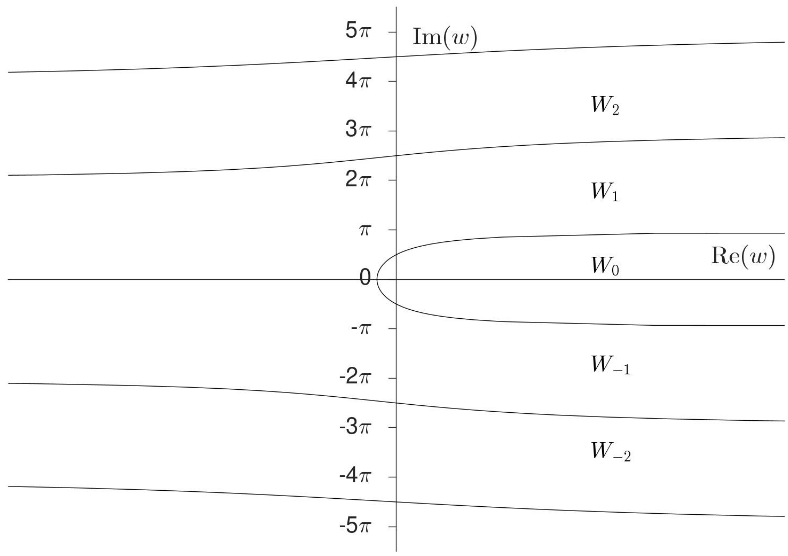

Figure 3.4: Boundaries of branches $W_k(z)$.

Now let's examine the behaviour when we traverse around the branch points. In the subsequent figures, we use thick solid and thick dashed lines to represent the boundaries of branches, where the thick solid lines indicate the closed curves.

Figure 3.5 shows the paths around the branch points $z = -\frac{1}{e}$ and $z = 0$ and their images in $W_{-1}(z)$. Note that the branch $k = -1$ has double branch cuts: $\left\{-\infty < z \leq -\frac{1}{e}\right\}$ and $\{-\infty < z \leq 0\}$.

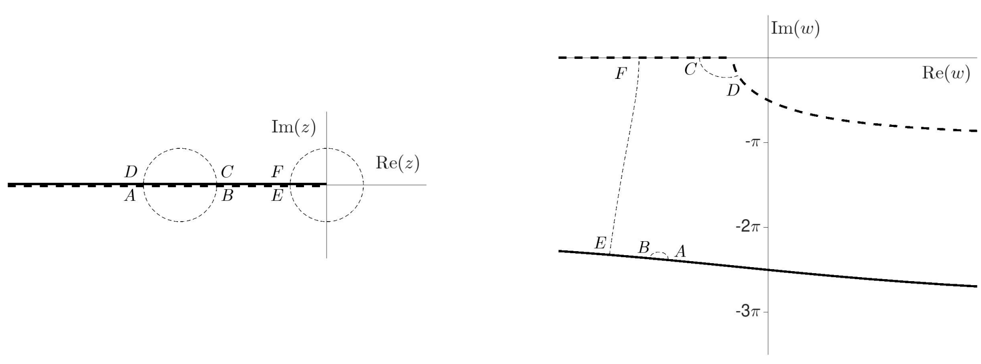

Figure 3.5: Images of AB, CD, and EF in $W_{-1}(z)$.

For the principal branch, as it has only one branch cut from $-\infty$ to $-\frac{1}{e}$, traversing a full circle from $ABCD$ results in a continuous image in $W_0(z)$. Also, since the point $z = 0$ is not a branch point, the image of circle EF forms a closed curve.

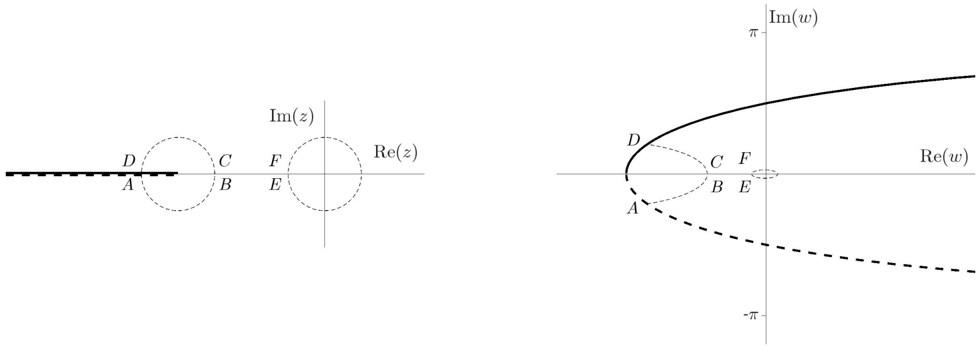

Figure 3.6: Images of AB, CD, and EF in $W_0(z)$.

Similarly, for the branch $k = 1$ with double branch cuts, the images of semi-circles AB, CD, and circle EF are shown in Figure 3.7.

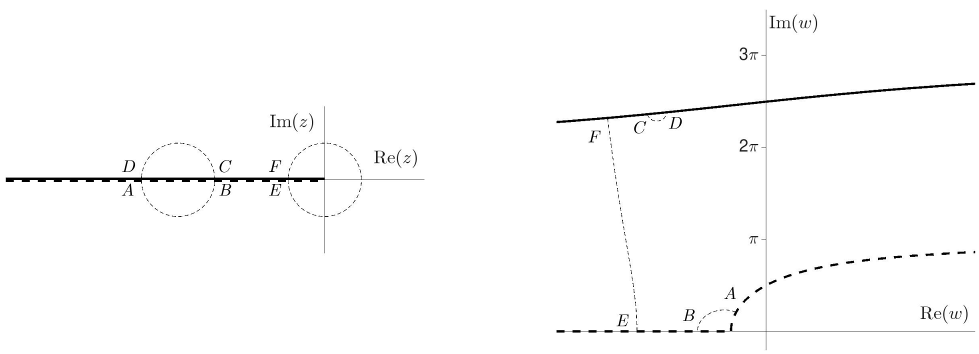

Figure 3.7: Images of AB, CD, and EF in $W_1(z)$.

As for the behaviour when a point traverses along the circle $z = 0.2e^{i\theta}$, the image is shown in Figure 3.8.

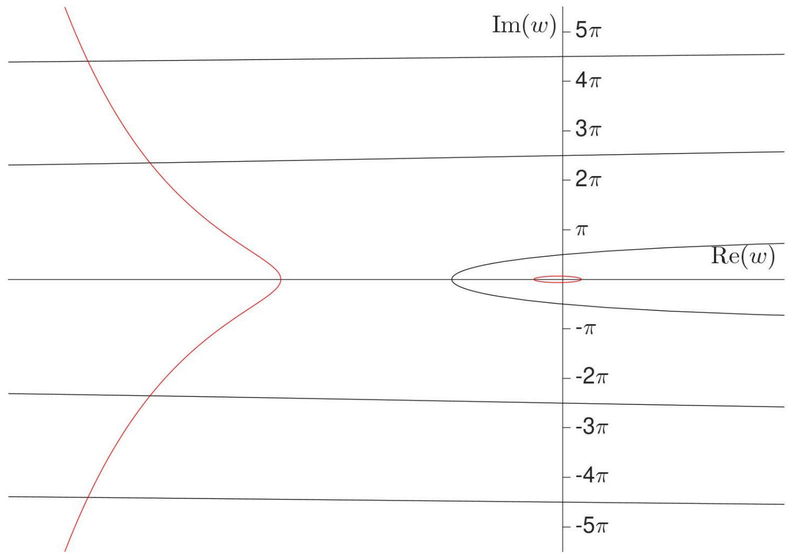

Figure 3.8: Image of $z = 0.2e^{i\theta}$.

However, when the radius of the circle is increased to enclose both branch points, we get a similar image as shown in Figure 3.9.

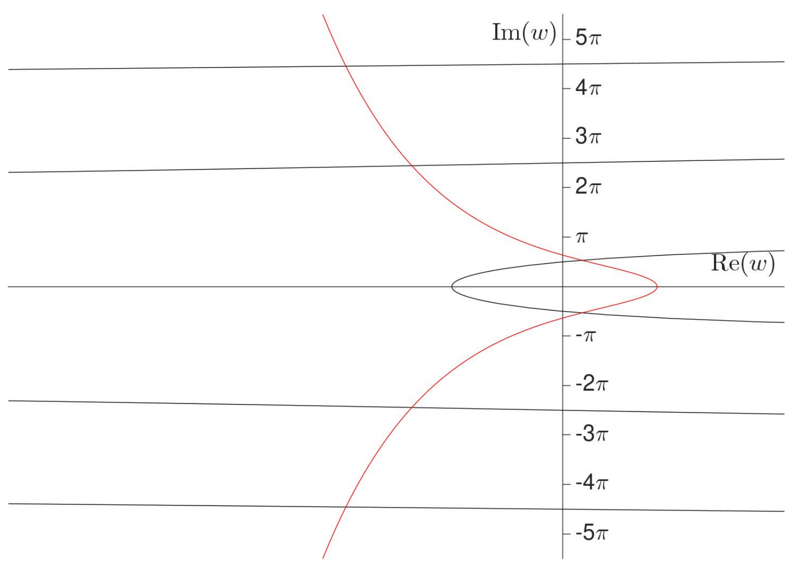

Figure 3.9: Image of $z = 2e^{i\theta}$.

Considering a path centered at the point $z = -\frac{1}{e}$, the image of $z = 0.2e^{i\theta} - \frac{1}{e}$ is illustrated in Figure 3.10.

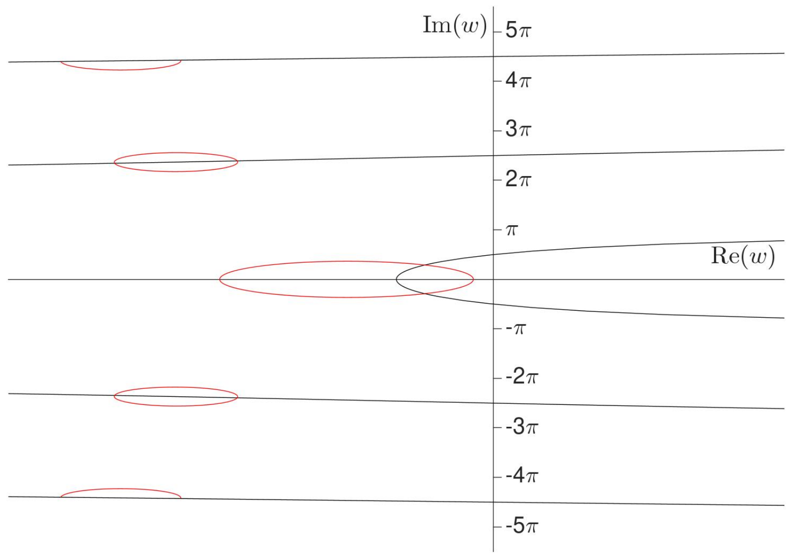

Figure 3.10: Image of $z = 0.2e^{i\theta} - \frac{1}{e}$.

Increasing the radius to enclose both branch points yields an image similar to Figure 3.9.

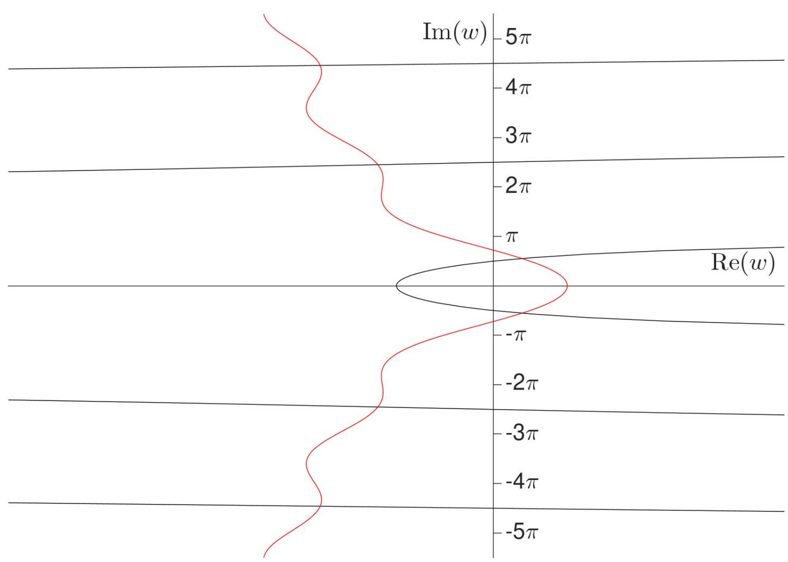

Figure 3.11: Image of $z = 2e^{i\theta} - \frac{1}{e}$.

An important property of the Lambert W function is proved by (Shinozaki, 2008; Huang, 2017):

Lemma 3.3. *Let $z \in \mathbb{C}$. Then $\overline{W_k(z)} = W_{-k}(\overline{z})$ and $\max(\operatorname{Re} W_k(z)) = \operatorname{Re} W_0(z)$ for $k = 0, \pm 1, \pm 2, \ldots$.*

This result implies that the image of the Lambert W function is symmetric about the real axis, and the rightmost eigenvalue lies in $W_0(z)$.

3.6 Series solutions

Consider $z = f(w) = we^w$ and $z_0 = 0$. From $f(w_0) = z_0 = 0$, we deduce that $w_0 = 0$ in the principal branch. Calculating the first derivative, we find $f'(w_0) = 1$. Hence, using Lagrange inversion (Theorem 2.1), we can derive the series expansion of $W_0(z)$:

$$W_0(z) = \sum_{n=1}^{\infty} \frac{z^n}{n!} \left[\frac{d^{n-1}}{dw^{n-1}} \left\{ \frac{w}{we^w} \right\}^n \right]_{w=0} \tag{3.46}$$

Recognizing that the $(n-1)$-th derivative of $\left\{\frac{w}{we^w}\right\}^n$ is $(-n)^{n-1}e^{-nw}$, we further simplify the above equation to:

$$W_0(z) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} z^n. \tag{3.47}$$

From the previous section, we know that the principal branch has a branch point at $z = -\frac{1}{e}$. Therefore, the radius of convergence of Equation (3.47) is bounded by $\frac{1}{e}$. Applying the ratio test to this series, we find

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \left(1 + \frac{1}{n}\right)^{n-1} z \right| < 1 \implies |z| < \frac{1}{e}.$$

The asymptotic expansion at 0 and infinity gives a series representation for $W_k(z)$ that is defined on all non-principal branches (Corless et al., 1996):

$$\begin{aligned} W_k(z) = \log z + 2\pi ik - \log(\log z + 2\pi ik) \\ + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c_{mn} \log^n(\log z + 2\pi ik)(\log z + 2\pi ik)^{-m-n}, \end{aligned} \tag{3.48}$$

where $c_{mn} = \frac{1}{n!}(-1)^m \left[{m+n \atop m+1}\right]$ and $\left[{m+n \atop m+1}\right]$ is a Stirling number of the first kind.

The convergence of this series was studied by Kalugin and Jeffrey (2012), and the authors concluded that for $z \in \mathbb{R}$, the series converges when $z > e$. For $z \in \mathbb{C}$, the series converges when

$$\operatorname{Re} W_m\left(-\frac{\log z}{e}\right) > -1,$$

where $m = -1$ when $-\pi < \operatorname{Arg} z \leq 0$ and $m = 1$ when $0 < \operatorname{Arg} z \leq \pi$.

The series expansion around $z = -\frac{1}{e}$ has also been studied by Corless et al. (1996). Since $z = -\frac{1}{e}$ is a branch point for $W_0(z), W_{-1}(z)$, and $W_1(z)$, the study of the series expansion focuses on these branches only.

It was shown that in these branches, with $p_+ = +\sqrt{2(ez+1)}$, the expansion around $z = -\frac{1}{e}$ is

$$W(z) = -1 + p_+ - \frac{1}{3}p_+^2 + \frac{11}{72}p_+^3 + \cdots = \sum_{l=0}^{\infty} \mu_l p_+^l, \tag{3.49}$$

where

$$\mu_k = \frac{k-1}{k+1}\left(\frac{\mu_{k-2}}{2} + \frac{\alpha_{k-2}}{4}\right) - \frac{\alpha_k}{2} - \frac{\mu_{k-1}}{k+1}$$

and

$$\alpha_k = \sum_{j=2}^{k-1} \mu_j \mu_{k+1-j}, \quad \alpha_0 = 2, \quad \alpha_1 = -1.$$

The initial conditions are $\mu_0 = -1$ and $\mu_1 = 1$.

Corless et al. (1996) further commented that using $p_- = -\sqrt{2(ez+1)}$ in

Equation (3.49) is useful in computing $W_{-1}(z)$ when $\text{Im}(z) \geq 0$. The series with p_- could be used to obtain $W_1(z)$ when $\text{Im}(z) < 0$.

Example 3.4. *Given* $z = -\frac{1}{e} - 0.02i$. *We could obtain the values of* $W_k(z)$ *using any modern software such as Python or MATLAB:*

$$W_1(z) = -1.2282 + 0.2731i.$$

On the other hand, we have $p_- = -0.2332 + 0.2332i$. *Hence, using the first four terms of Equation (3.49), we obtain* $W_1(z) \sim -1.2293 + 0.2733i$.

Similarly, if $z = -\frac{1}{e} + 0.02i$. *We have:*

$$W_{-1}(z) = -1.2282 - 0.2731i.$$

Using $p_+ = -0.2332 - 0.2332i$, *we obtain* $W_{-1}(z) \sim -1.2293 - 0.2733i$.

In practice, one could apply Halley's method to obtain the values of all the branches of W_k. Equation (3.48) could be used to obtain the initial guess for most of the values of z. For the case where z is around $-\frac{1}{e}$, the initial guess for $W_{-1}(z)$, $W_0(z)$, and $W_1(z)$ can be obtained from Equation (3.49). Padé approximation could be used to compute $W_0(z)$ when z is near to 0. When z is not too near to 0 or $-\frac{1}{e}$, we could use a rational approximation.

Lastly, it's worth noting that Mező (2022) has provided a detailed discussion on the series expansion of the Lambert W function.

3.7 Unwinding number

Recall that certain identities valid in the field of real numbers may not hold when considering complex numbers. For example,

$$\log z_1 z_2 \neq \log z_1 + \log z_2, \tag{3.50}$$

where $z_1, z_2 \in \mathbb{C}$. For instance, take $z_1 = z_2 = -i$, then the left-hand side becomes $\log(-1) = \pi i$ and the right hand side is $2\log(-i) = -\pi i$.

The following example illustrates that not all identities in Section 3.1 are valid for all complex numbers.

Example 3.5. *Consider the equation $w + \log w = z$, where $z \in \mathbb{C}$. From Equation (3.14), we know that the solution is*

$$w = W\left(e^z\right).$$

When $z = 2i$, using the Lambert W function in the principal branch, we have $w = W\left(e^{2i}\right) = -2.7308 - 15.0998i$, which can be verified as a solution. However, considering other branches such as $k = -2$, we have $w_{-2} = W_{-2}\left(e^{2i}\right) = -2.1996 - 8.7493i$, but

$$w_{-2} + log(w_{-2}) = 0.0000 - 10.5664i \neq z.$$

Thus, values in other branches may not be valid solution.

The main reason for the invalidity of solutions in Section 3.1 is the presence of the logarithm or n-th root in the equations. Similarly, Equation (1.1) does not imply $w_k + \log w_k = \log z$, where $w_k = W_k(z)$. A correct identity for this is Equation (3.40) that can be derived using the *unwinding number*, $\mathcal{K}(z)$ (Jeffrey et al., 1996).

The unwinding number is defined by

$$\log e^z = z + 2\pi i \mathcal{K}(z). \tag{3.51}$$

With this definition, we obtain the "correct" version of Equation (3.50).

Proposition 3.4. *For $z_1, z_2 \in \mathbb{C}$, we have*

$$\log z_1 z_2 = \log z_1 + \log z_2 + 2\pi i \mathcal{K}(\log z_1 + \log z_2). \tag{3.52}$$

Proof. From Equation (3.51), we know that

$$\log e^{\log z_1 + \log z_2} = \log z_1 + \log z_2 + 2\pi i \mathcal{K}(\log z_1 + \log z_2).$$

The left-hand side can be simplified using the fact that

$$e^{\log z_1 + \log z_2} = e^{\log z_1} e^{\log z_2} = z_1 z_2.$$

Thus, we have the desired identity. □

The unwinding number can also be expressed using the notation of the floor function:

$$\mathcal{K}(z) = \left\lfloor \frac{\pi - \operatorname{Im} z}{2\pi} \right\rfloor. \tag{3.53}$$

A useful property of the unwinding number is

$$\mathcal{K}(z + 2\pi i n) = \mathcal{K}(z) - n. \tag{3.54}$$

This can be proved using Equation (3.53):

$$\begin{aligned}
\mathcal{K}(z + 2\pi i n) &= \left\lfloor \frac{\pi - \operatorname{Im}(z + 2\pi i n)}{2\pi} \right\rfloor \\
&= \left\lfloor \frac{\pi - \operatorname{Im} z - 2\pi n}{2\pi} \right\rfloor \\
&= \left\lfloor \frac{\pi - \operatorname{Im} z}{2\pi} - n \right\rfloor \\
&= \mathcal{K}(z) - n.
\end{aligned}$$

Considering all possible values of $\operatorname{Im} z$, we have

$$\mathcal{K}(z) = \begin{cases} 1, & \text{when } -3\pi < \operatorname{Im} z \le -\pi, \\ 0, & \text{when } -\pi < \operatorname{Im} z \le \pi, \\ -1, & \text{when } \pi < \operatorname{Im} z \le 3\pi, \\ n, & \text{when } (2n-1)\pi < \operatorname{Im} z \le (2n+1)\pi. \end{cases} \tag{3.55}$$

Before we derive Equation (3.40), we present the following result (Jeffrey et al., 1996):

Lemma 3.4. *For $z \neq re^{i\theta}$, we have*

$$\operatorname{Arg} W_k(z) + \operatorname{Im} W_k(z) = \begin{cases} \theta, & \textit{for } k = -1 \textit{ and } -\frac{1}{e} \le z < 0, \\ \theta + 2k\pi, & \textit{otherwise}. \end{cases} \tag{3.56}$$

Proof. Let $W_k(z) = W_k = \xi + i\eta$ and $z = re^{i\theta}$. Substitute into $we^w = z$, we have

$$(\xi + i\eta)e^{\xi + i\eta} = re^{i\theta}.$$

Equating the real and imaginary parts yields

$$r\cos\theta = e^{\xi}(\xi\cos\eta - \eta\sin\eta)$$
$$r\sin\theta = e^{\xi}(\eta\cos\eta + \xi\sin\eta).$$

Assume that W_k is not real (which implies $\eta \neq 0$ and $\theta \neq 0, \pi$), we divide the first equation by the second:

$$\begin{aligned}\cot\theta &= \frac{\xi\cos\eta - \eta\sin\eta}{\eta\cos\eta + \xi\sin\eta}\\ &= \frac{\frac{\xi}{\eta}\cot\eta - 1}{\frac{\xi}{\eta} + \cot\eta}\\ &= \frac{\cot\operatorname{Arg} W_k \cot\eta - 1}{\cot\operatorname{Arg} W_k + \cot\eta}.\end{aligned}$$

Using angle summation formula, the last equality can be further simplified to $\cot\theta = \cot(\operatorname{Arg} W_k + \eta) = \cot(\operatorname{Arg} W_k + \operatorname{Im} W_k)$, which implies

$$\theta + 2n\pi = \operatorname{Arg} W_k + \operatorname{Im} W_k,$$

where $n \in \mathbb{Z}$.

This relation holds for all values of r and θ due to continuity. In order to determine the value of n, we consider the limiting case $|r| \to \infty$. From $we^w = z$, we know that as z grows, e^w is the dominant term, and hence w is asymptotic to $\log z$. Thus,

$$\begin{aligned}\lim_{|r|\to\infty}(\operatorname{Arg} W_k + \operatorname{Im} W_k) &= \lim_{|r|\to\infty}\operatorname{Arg} W_k + \lim_{|r|\to\infty}\eta\\ &= 0 + \lim_{|r|\to\infty}\eta\\ &= \theta + 2k\pi\end{aligned}$$

Thus, $(2k-1)\pi < \operatorname{Arg} W_k + \operatorname{Im} W_k \leq (2k+1)\pi$ holds for all W_k that is not real.

When W_k is real, z must be real, this implies $k = 0$ or $k = -1$. We know that $W_0 \geq -1$, and if W_0 is positive, z must be positive as well, and hence $\operatorname{Arg} W_0 + \operatorname{Im} W_0 = 0 = \theta + 2 \times 0 \times \pi$. If W_0 is negative, then $-\frac{1}{e} \leq z < 0$ ($\theta = \pi$) and $\operatorname{Arg} W_0 + \operatorname{Im} W_0 = \pi = \pi + 2 \times 0 \times \pi$. Thus, Equation (3.56) is true for $k = 0$.

For W_{-1} to be real, z must be real, and satisfy $-\frac{1}{e} \leq z < 0$ ($\theta = \pi$). Also, $W_{-1} \leq -1$ when $-\frac{1}{e} \leq z < 0$, and hence $\operatorname{Arg} W_{-1} + \operatorname{Im} W_{-1} = \pi = \theta$. $\square$

Theorem 3.9.

$$W_k(z) + \log W_k(z) = \begin{cases} \log z, & \textit{for } k = -1 \textit{ and } z \in [-\frac{1}{e}, 0), \\ \log_k z, & \textit{otherwise}. \end{cases}$$

Proof. The equation $W_k e^{W_k} = z$ can be reformulated as follows:

$$\log \left(W_k e^{W_k} \right) = \log z.$$

Utilising Equation (3.52), the left-hand side can be expressed as $\log W_k + \log e^{W_k} + 2\pi i \mathcal{K} \left(\log W_k + \log e^{W_k} \right)$.

By applying Equation (3.52) and Equation (3.54), we obtain

$$\begin{aligned} \log z &= \log W_k + W_k + 2\pi i \mathcal{K}(W_k) + 2\pi i \mathcal{K} \left(\log W_k + W_k + 2\pi i \mathcal{K}(W_k) \right) \\ &= \log W_k + W_k + 2\pi i \mathcal{K}(\omega), \end{aligned}$$

where $\omega = \log W_k + W_k$.

Note that $\operatorname{Im}(\omega) = \operatorname{Im}(\ln |W_k| + \operatorname{Arg}(W_k) i + W_k) = \operatorname{Arg}(W_k) + \operatorname{Im}(W_k)$ and from Lemma 3.4, we have

$$\operatorname{Im}(\omega) = \begin{cases} \theta, & \text{for } k = -1 \text{ and } -\frac{1}{e} \leq z < 0, \\ \theta + 2k\pi, & \text{otherwise}. \end{cases}$$

For the case $k = -1$ and $-\frac{1}{e} \leq z < 0$, we know that $-\pi < \mathrm{Im}(\omega) \leq \pi$. From Equation (3.55), we have $\mathcal{K}(\omega) = 0$. For other cases, the following inequality can be obtained:

$$(2k-1)\pi < \mathrm{Im}(\omega) \leq (2k+1)\pi,$$

which implies $\mathcal{K}(\omega) = -k$. □

3.8 Application in delay differential equation

In recent decades, the stabilisation and control of linear systems with delays have been extensively studied. For example, the assignment of the spectrum (eigenvalues) for linear delay systems was explored in 1978 (Olbrot, 1978). More recently, Asl and Ulsoy (2003) proposed an approach for solving linear time-delay systems using the Lambert W function. As a result, robust stability and related topics for designing feedback controllers have been well established (Yi et al., 2010*b*; Shinozaki and Mori, 2006), along with references therein.

The assignment of eigenvalues for delay systems with a single delay through the Lambert W function was initially developed by Yi et al. (2010*a*). This method aims to assign the rightmost eigenvalue of the delay system to a predefined (desired) location for stabilisation. Unfortunately, in the scalar case, only the real or real part of the rightmost eigenvalue can be assigned. Alternatively, the assignment of a complex eigenvalue to the largest eigenvalue of a scalar single-delay system using a complex feedback gain is not realistic (Shinozaki, 2008). These studies design the controller by providing feedback only on the current state, with no conditions imposed on the value of the desired eigenvalue, ensuring the existence of the feedback controller. On the other hand, although a more general time-delay system can be analysed using the matrix Lambert W function, the approach of computing the rightmost eigenvalue that does not utilise the principal branch contradicts the main proposition of this method (Cepeda-Gomez and Michiels, 2015).

We begin by presenting some simple examples of delay differential equations and discussing the general approach for solving such systems. This chapter will conclude with an exploration of scalar systems with a single delay.

The primary focus lies in deriving the conditions for the existence of a feedback controller related to assigning the rightmost eigenvalue of the system to a desired value. The formula for computing feedback gains for the current and delayed states is then obtained.

3.8.1 Step function

For $t > 0$, consider the following first-order homogeneous DDE with a single delay:

$$x'(t) = ax(t) + a_d x(t-h), \tag{3.57}$$

with initial conditions $x(0) = x_0$ and $x(\tau) = \phi(\tau)$ for $\tau \in [-h, 0)$.

Various tools can be employed to solve this system, including the step function and Laplace transform. The significance of the Lambert W function becomes apparent when we employ the Laplace transform to solve the system.

Given a delay of h units of time, we define $x_p(t)$ as the solution for $t \in [(p-1)h, ph]$, where $p = 1, 2, 3, \ldots$. We then solve the DDE for the interval $t \in [0, h]$:

$$\begin{aligned} x_1'(t) &= ax_1(t) + a_d x_1(t-h) \\ \int_0^t e^{-a\tau}\left(x_1'(\tau) - ax_1(\tau)\right) d\tau &= a_d \int_0^t e^{-a\tau} x_1(\tau - h)\, d\tau \\ x_1(t) &= x_0 e^{at} + a_d \int_0^t e^{a(t-\tau)} \phi(\tau - h)\, d\tau. \end{aligned}$$

Utilising the fact that $x_2(h) = x_1(h)$, we proceed to solve for $x_2(t)$ through the same process. Note that for $t \in [h, 2h]$, $x(t) = x_2(t)$ and $x(t-h) = x_1(t-h)$.

Thus,

$$x_2'(t) = ax_2(t) + a_d x_1(t-h), \quad x_2(h) = x_1(h), \quad t \in [h, 2h]$$
$$\int_h^t e^{-a\tau} \left[x_2'(\tau) - ax_2(\tau)\right] \, d\tau = \int_h^t e^{-a\tau} a_d x_1(\tau - h) \, d\tau$$
$$x_2(t) = x_1(h)e^{a(t-h)} + a_d \int_h^t e^{a(t-\tau)} x_1(\tau - h) d\tau,$$

where $x_1(h) = x_0 e^{ah} + a_d \int_0^h e^{a(h-\tau)} \phi(\tau - h) \, d\tau$.

This process can be repeated indefinitely to find $x_p(t)$ for any positive integer p, each valid for the corresponding time interval $[(p-1)h, ph]$.

3.8.2 Laplace transform

Another approach to solve $x'(t) = ax(t) + a_d x(t-h)$ involves using the Laplace transform. Upon integrating both sides of this DDE, we get

$$\int_0^\infty x'(t)e^{-st} \, dt = \int_0^\infty (ax(t) + a_d x(t-h)) \, e^{-st} \, dt.$$

Evaluating the left-hand side of the equation using integration by parts, we obtain:

$$\int_0^\infty x'(t)e^{-st} \, dt = x(t)e^{-st}\Big|_{t=0}^\infty + s \int_0^\infty x(t)e^{-st} \, dt.$$

Equation above can be rewritten by substituting $X(s) = \int_0^\infty x(t)e^{-st}, dt$:

$$\int_0^\infty x'(t)e^{-st} \, dt = -x(0) + sX(s).$$

For the right-hand side of the equation:

$$
\begin{aligned}
a \int_0^\infty x(t) e^{-st}\, dt + a_d \int_0^\infty x(t-h) e^{-st}\, dt \\
&= aX(s) + a_d e^{-sh} \int_{-h}^\infty x(t) e^{-st}\, dt \\
&= aX(s) + a_d e^{-sh} X(s) + a_d e^{-sh} \int_{-h}^0 x(t) e^{-st}\, dt.
\end{aligned}
$$

Let $\Phi(s) = \int_{-h}^0 x(t) e^{-st}\, dt$ and equating both sides, we derive:

$$
X(s) = \frac{a_d e^{-sh}\Phi(s) + x(0)}{s - a - a_d e^{-sh}}.
$$

Thus, the characteristic equation is given by

$$
\Delta(s) = s - a - a_d e^{-sh} = 0,
$$

and the eigenvalues are the solutions to this transcendental equation. This equation possesses a solution in terms of the Lambert W function, namely

$$
s_k = a + \frac{1}{h} W_k(a_d h e^{-ah}).
$$

Consequently, we can rewrite $X(s)$ as

$$
\begin{aligned}
X(s) &= \frac{x(0)}{\prod_{k=-\infty}^{\infty}(s - s_k)} + \frac{a_d e^{-sh}\Phi(s)}{\prod_{k=-\infty}^{\infty}(s - s_k)} \\
&= x(0) \sum_{k=-\infty}^{\infty} \frac{C_k}{s - s_k} + a_d e^{-sh}\Phi(s) \sum_{k=-\infty}^{\infty} \frac{C_k^I}{s - s_k}.
\end{aligned}
$$

Applying the inverse Laplace transform yields

$$
x(t) = \sum_{k=-\infty}^{\infty} \left(C_k e^{s_k t} x(0) + C_k^I e^{s_k t} \right).
$$

3.8.3 Substitution of $x(t) = Ce^{st}$

From the previous approach, we can observe that the solution takes the form Ce^{st}. In this section, we will directly assume the solution to the DDE is $x(t) = Ce^{st}$. This leads to the following expression:

$$\begin{aligned} Cse^{st} &= Cae^{st} + Ca_d e^{s(t-h)} \\ (sh - ah)e^{sh-ah} &= a_d he^{-ah} \\ s_k &= a + \frac{1}{h} W_k(a_d he^{-ah}). \end{aligned}$$

Consequently, the solution is given by:

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{s_k t}.$$

3.8.4 DDE with exogenous input

In this subsection, we consider system with exogenous input from the environment:

$$\begin{aligned} &x'(t) = ax(t) + a_d x(t-h) + bu(t), \quad t > 0, \\ &x(0) = x_0, \quad t = 0, \\ &x(\tau) = \phi(\tau), \quad \tau \in [-h, 0). \end{aligned} \tag{3.58}$$

The term $u(t)$ represents the exogenous input, a proportional control that is proposed to stabilise the system by providing feedback based on current and delayed states. Let

$$u(t) = kx(t) + k_d x(t-h), \tag{3.59}$$

where $k, k_d \in \mathbb{R}$ are parameters to be designed. This leads to the closed-loop system:

$$x'(t) = (a + bk)x(t) + (a_d + bk_d)x(t-h) = \alpha x(t) + \beta x(t-h), \tag{3.60}$$

where $\alpha = a + bk, \beta = a_d + bk_d$. The characteristic equation becomes

$$s - \alpha - \beta e^{-sh} = 0 \implies s_k = \alpha + \frac{1}{h} W_k \left(\beta h e^{-\alpha h} \right). \tag{3.61}$$

For the case of a system with only input delay ($h > 0$) described by:

$$x'(t) = ax(t) + bu(t - h), \tag{3.62}$$

where $x(0) = x_0$, we apply the state feedback controller $u(t) = kx(t)$. The system becomes $x'(t) = ax(t) + bkx(t - h)$, equivalent to Equation (3.60). Thus, our focus remains on the system given by Equation (3.60).

The stability of a system requires that the real part of the rightmost eigenvalue is negative. As shown in Lemma 3.3, the rightmost eigenvalue is s_0. Hence, we aim to have s_0 be negative and assign s_0 to a desired location, $S_{0,des}$.

This can be achieved by adjusting the real parameters k and k_d. Define $W_0^\alpha = \left\{ S_{0,des} - \frac{1}{h} W_0(z) | z \in \mathbb{C} \right\}$ and choose α from $W_0^\alpha \cap \mathbb{R}$. With α determined, we can compute β as:

$$\beta = \left(S_{0,des} - \alpha \right) e^{S_{0,des} h}.$$

We now analyse the conditions for parameter existence and the existence of feedback gains and controllers.

Let $s = S_{0,des} = u + iv$ for $v > 0$. From characteristic equation, we get

$$u + iv - \alpha = \beta e^{-(u+iv)h}.$$

Equating the real and imaginary parts yields

$$u - \alpha = \beta e^{-uh} \cos(vh) \quad \text{and} \quad v = -\beta e^{-uh} \sin(vh). \tag{3.63}$$

For $\beta \neq 0$ and $\sin vh \neq 0$, we divide the first equation by the second:

$$u - \alpha = -v \cot(vh),$$

which can be rewritten as

$$(S_{0,des} - \alpha)h = (u - \alpha)h + i\eta = -\eta \cot \eta + i\eta,$$

where $\eta = vh$. Comparing this equation with Equation (3.44), we know that $(S_{0,des} - \alpha)h$ lies on the boundary between W_0 and W_1, i.e. $(S_{0,des} - \alpha)h \in W_0(BC)$.

Using Equation (3.63), we can solve for α and β:

$$\beta = -ve^{uh} \csc vh \quad \text{and} \quad \alpha = u + v \cot vh. \tag{3.64}$$

Since $\alpha = a + bk$ and $\beta = a_d + bk_d$, we have:

$$\begin{aligned} k &= (u + v \cot vh - a)/b, \\ k_d &= -(ve^{uh} \csc vh + a_d)/b. \end{aligned} \tag{3.65}$$

We discuss the case when $k = 0$ and $k_d = 0$ separately. These correspond to situations where one of the states is not included in the feedback loop, as described by Equation (3.59). From Equation (3.61), we know that

$$(s_k - \alpha)he^{(s_k - \alpha)h} = \beta he^{-\alpha h},$$

and substituting this into the case $k = 0$ (when the delay state is not used), we have:

$$k_d = \frac{-ve^{uh \csc vh} - a_d}{b} = \frac{\beta - a_d}{b} = \frac{(S_{0,des} - \alpha)e^{S_{0,des}} - a_d}{b}, \tag{3.66}$$

with the condition that $(S_{0,des} - \alpha)h \in W_0(BC)$. When $k_d = 0$, $S_{0,des}$ must be chosen such that $ve^{uh} \csc vh + a_d = 0$ and the current state feedback gain is given by

$$k = \frac{S_{0,des} - \alpha - a_d e^{-S_{0,des} h}}{b}. \tag{3.67}$$

Suppose that $S_{0,des}$ is real. From Equation (3.61), we know that $\beta h e^{-\alpha h} \geq -\frac{1}{e}$, implying $S_{0,des} \geq \alpha - \frac{1}{h}$. Solving for α and β:

$$\alpha \leq S_{0,des} + \frac{1}{h} \quad \text{and} \quad \beta = (S_{0,des} - \alpha) e^{S_{0,des} h}.$$

Using $k = \frac{\alpha - a}{b}$ and $k_d = \frac{\beta - a_d}{b}$, we arrive at

$$\begin{aligned} k &\leq \frac{S_{0,des} - a}{b} + \frac{1}{bh}, \\ k_d &= \frac{\left[S_{0,des} - (a + bk)\right] e^{S_{0,des} h}}{b} - \frac{a_d}{b}. \end{aligned} \tag{3.68}$$

Similar to the case when $S_{0,des}$ is not real, we consider the situation when $k = 0$ and $k_d = 0$ separately. When $k = 0$, the rightmost eigenvalue is assignable if $S_{0,des} \geq a - \frac{1}{h}$, and the delay state feedback gain is still given by Equation (3.66). On the other hand, for $k_d = 0$, we have $S_{0,des} - a_d e^{-S_{0,des} h} = a + bk$, which is always achievable by the feedback gain from Equation (3.67).

The above derivation addresses the existence question on $S_{0,des}$ such that eigenvalue assignment can be performed. We summarise the results in theorem below:

Theorem 3.10. *Suppose the system (3.58) is not an input-delay system, the following statements hold:*

1. *For a given $S_{0,des} = u + iv$, the rightmost eigenvalue of the system (3.60) can be assigned to any desired location via the controller (3.59) with both current and delay state feedback gains defined by Equation (3.65). Furthermore, if the current or delay state is not included in the feedback loop, the must satisfy the*

condition or be such that the associated gain is described by Equation (3.66) or (3.67), respectively.

2. *For a given $S_{0,des} \in \mathbb{R}$, the rightmost eigenvalue of the system (3.60) can be assignable to any desired location $S_{0,des}$ via the controller (3.59) with feedback gains defined by Equation (3.68). Furthermore, if the current or delay state is not included in the feedback loop, $S_{0,des}$ must satisfy the condition*

$$S_{0,des} \geq a - \frac{1}{h},$$

or no constraint such that the associated gain is still described by Equation (3.66) or (3.67), respectively.

Based on this result, we present another two corollaries.

Corollary 3.1. *Suppose the system (3.58) is not an input-delay system and $S_{0,des}, S_{1,des} \in \mathbb{R}$. The following statements hold:*

1. *If $S_{0,des}$ and $S_{1,des} < S_{0,des}$ satisfy*

$$\frac{S_{1,des}he^{-S_{0,des}h} - S_{0,des}he^{-S_{1,des}h}}{e^{-S_{0,des}h} - e^{-S_{1,des}h}} > 1 + \log \frac{S_{1,des}h - S_{0,des}h}{e^{-S_{0,des}h} - e^{-S_{1,des}h}},$$

they are assignable to the rightmost eigenvalue and the eigenvalue in the range of W_{-1}, respectively. The corresponding feedback gains are the described by

$$\begin{aligned} k &= \frac{(S_{1,des} - a)e^{-S_{0,des}h} - (S_{0,des} - a)e^{-S_{1,des}h}}{\left(e^{-S_{0,des}h} - e^{-S_{1,des}h}\right) b}, \\ k_d &= \frac{S_{0,des} - S_{1,des}}{\left(e^{-S_{0,des}h} - e^{-S_{1,des}h}\right) b} - \frac{a_d}{b}. \end{aligned}$$

2. *If the feedback gains:*

$$\begin{aligned} k &= \frac{S_{0,des} - a}{b} + \frac{1}{bh}, \\ k_d &= -\frac{1}{bh}e^{S_{0,des}h} - \frac{a_d}{b}, \end{aligned}$$

is adopted, then the closed loop system (3.60) has $S_{0,des}$ *to be its rightmost eigenvalue with multiplicity* 2.

3. *If the feedback gains*

$$k = \frac{S_{0,des} - a}{b},$$
$$k_d = -\frac{a_d}{b}$$

is adopted, the closed loop system has $S_{0,des}$ *as its eigenvalue and this system becomes delay free.*

Proof. When the desired eigenvalue $S_{0,des} \in \mathbb{R}$, there are two free parameters k and k_d to be determined. Since only one Equation (3.61) needs to hold, hence one more eigenvalue, say $S_{1,des} < S_{0,des} \in \mathbb{R}$, can be assigned. Under this circumstance it follows that

$$S_{0,des} - \alpha = \beta e^{-S_{0,des}h},$$
$$S_{1,des} - \alpha = \beta e^{-S_{1,des}h}.$$

Suppose $S_{0,des} \neq S_{1,des}$, then one obtains

$$\alpha = \frac{S_{1,des}e^{-S_{0,des}h} - S_{0,des}e^{-S_{1,des}h}}{e^{-S_{0,des}h} - e^{-S_{1,des}h}},$$
$$\beta = \frac{S_{0,des} - S_{1,des}}{e^{-S_{0,des}h} - e^{-S_{1,des}h}}.$$

Since $S_{0,des}$ must be assigned to the rightmost eigenvalue of the closed-loop system and suppose that $S_{1,des}$ is located in the range of a certain branch, say the k-th branch of the Lambert W function. Let $z_w = \beta h e^{-\alpha h} \in \mathbb{R}$, then it is obvious that both $W_0(z_w)$ and $W_k(z_w)$ must be real, and hence $-\frac{1}{e} < z_w < 0$ and $k = -1$. Thus,

$$S_{0,des} = \alpha + \frac{1}{h}W_0\left(\beta h e^{\alpha h}\right) \quad \text{and} \quad S_{1,des} = \alpha + \frac{1}{h}W_{-1}\left(\beta h e^{\alpha h}\right).$$

where $-\frac{1}{e} < \beta h e^{\alpha h} < 0$ or $\alpha h > 1 + \log(-\beta h)$. Substituting the expression for α

and β:

$$\frac{S_{1,des}he^{-S_{0,des}h} - S_{0,des}he^{-S_{1,des}h}}{e^{-S_{0,des}h} - e^{-S_{1,des}h}} > 1 + \log \frac{S_{1,des}h - S_{0,des}h}{e^{-S_{0,des}h} - e^{-S_{1,des}h}}. \tag{3.69}$$

Once this condition is satisfied, the feedback controller exists, and the associated gains are described by

$$\begin{aligned} k &= \frac{(S_{1,des} - a)e^{-S_{0,des}h} - (S_{0,des} - a)e^{-S_{1,des}h}}{\left(e^{-S_{0,des}h} - e^{-S_{1,des}h}\right) b}, \\ k_d &= \frac{S_{0,des} - S_{1,des}}{\left(e^{-S_{0,des}h} - e^{-S_{1,des}h}\right) b} - \frac{a_d}{b}. \end{aligned} \tag{3.70}$$

Alternatively we assign two eigenvalues into the same location, that is, $S_{1,des} = S_{0,des}$. We take the limit $S_{1,des} \to S_{0,des}$ in Equation (3.70), we obtain

$$\alpha = S_{0,des} + \frac{1}{h} \quad \text{and} \quad \beta = -\frac{1}{h}e^{S_{0,des}h}.$$

and Equation (3.70) is simplified to

$$\begin{aligned} k &= \frac{S_{0,des} - a}{b} + \frac{1}{bh}, \\ k_d &= -\frac{1}{bh}e^{S_{0,des}h} - \frac{a_d}{b}, \end{aligned} \tag{3.71}$$

with $\beta he^{-\alpha h} = -\frac{1}{e}$.

When $S_{0,des} \in \mathbb{R}$, there is another possibility to design the controller by selecting

$$\begin{aligned} k &= \frac{S_{0,des} - a}{b}, \\ k_d &= -\frac{a_d}{b}. \end{aligned} \tag{3.72}$$

In this way, the closed loop system becomes

$$x'(t) = S_{0,des}x(t)$$

whose solution is given by

$$x(t) = x_0 e^{S_{0,des} t}.$$

This approach can get rid of the delay state by using the feedback controller to compensate for the delay effect. $\square$

Corollary 3.2. *For an input-delay system (3.62), if $S_{0,des} - \alpha$ belongs to the upper boundary on the range of W_0 or $[-1, \infty)$, a real feedback gain k through*

$$k = \frac{(S_{0,des} - a)e^{S_{0,des} h}}{b}$$

is obtained.

Proof. An input-delay system (3.62) is assignable to any complex number $S_{0,des}$ if $(S_{0,des} - a)h \in W_0(BC)$, i.e. $(S_{0,des} - a)he^{S_{0,des} h} = z$, for some real number $z < -\frac{1}{e}$. Then the associated real feedback gain for the controller $u(t) = kx(t)$ is determined by

$$k = \frac{(S_{0,des} - a)e^{S_{0,des} h}}{b}. \tag{3.73}$$

If $S_{0,des}$ is real, it must satisfy

$$S_{0,des} \geq \alpha - \frac{1}{h}$$

and the feedback gain is still given by Equation (3.73) which is the same as the result presented in (Shinozaki, 2008). $\square$

CHAPTER 4

THE $W^{(r)}$ function

4.1 Introduction

Numerous studies have focused on solving the generalised Lambert W function, including the $(2,0)$-type Lambert function. While these works have concentrated on different types of equations, they all stem from variants of the equation:

$$(w - a_1)(w - a_2)e^w = z, \tag{4.1}$$

where w and z are complex while a_1, a_2 are real parameters. This scenario is referred to as the *two upper parameters* case, which can be shown to be equivalent to $(\omega^2 - r^2)e^\omega = z$. In this chapter, we investigate a more general equation:

$$(w^2 - r)e^w = z, \tag{4.2}$$

where $r \in \mathbb{R}$ and $w, z \in \mathbb{C}$.

Drawing inspiration from Scott, Fee and Grotendorst (2014), we present a recursive formula for determining coefficients of the series solution, as well as branch analysis. Throughout the following sections, the solution of Eq. (4.2) in the k-th branch is denoted as $W_k^{(r)}(z)$. Similar to the Lambert W function, $W^{(r)}(z)$ is employed to denote the principal branch when there is no ambiguity.

In the subsequent section, we introduce equations that can be solved using the $W^{(r)}$ function. Following that, we discuss basic properties, derivatives, and integrals. Additionally, we present some series representations of the $W^{(r)}$ function and conduct branch analysis.

4.2 Equations solvable using the $W^{(r)}$ function

In this section, we present some of the equations that can be solved using the $W^{(r)}$ function. Unless otherwise specified, a, b, c, p are real constants.

4.2.1 $(aw^2 + bw + c)e^{pw} = x$

One of the equations that can be solved using $W^{(r)}$ function is

$$(aw^2 + bw + c)e^{pw} = x, \tag{4.3}$$

with $a \neq 0$.

By completing the square, we are able to rewrite equation above as

$$\left\{\left(pw + \frac{pb}{2a}\right)^2 - r\right\} e^{pw} = \frac{p^2 x}{a},$$

where $r = \left(\frac{pb}{2a}\right)^2 - \frac{p^2 c}{a}$. Multiplying both sides with $e^{\frac{pb}{2a}}$, we obtain

$$\left\{\left(pw + \frac{pb}{2a}\right)^2 - r\right\} e^{pw+\frac{pb}{2a}} = \frac{p^2 x}{a} e^{\frac{pb}{2a}}.$$

Thus, the solution is

$$w = \frac{1}{p} W^{(r)}\left(\frac{p^2 x}{a} e^{\frac{pb}{2a}}\right) - \frac{b}{2a}. \tag{4.4}$$

4.2.2 $(w-a)(w-b)e^w = x$

This is also known as the $(2,0)$-type Lambert W function (Mező, 2022), and the solution is denoted as $W\left(\begin{smallmatrix} a & b \end{smallmatrix}; x\right)$. Rewriting $(w-a)(w-b) = w^2 + (-a-b)w + ab$ and using Equation (4.3), we find that the solution is

$$W\left(\begin{smallmatrix} a & b \end{smallmatrix}; x\right) = W^{(r)}\left(xe^{-\frac{a+b}{2}}\right) + \frac{a+b}{2}. \tag{4.5}$$

4.2.3 $(w^2-r)b^w = x$

Similar to the Lambert W function, the equation

$$(w^2-r)b^w = x, \tag{4.6}$$

where $b \neq 0, 1$, can be solved by rewriting b^w as $e^{w \log b}$:

$$\left\{(w \log b)^2 - r (\log b)^2\right\} e^{w \log b} = x (\log b)^2 .$$

Thus, we obtain the base change formula

$$w = \frac{1}{\log b} W^{(r(\log b)^2)}\left(x (\log b)^2\right). \tag{4.7}$$

4.3 Basic properties of the $W^{(r)}$ function

When $r = 0$, Equation (4.2) is reduced to

$$w^2 e^w = x, \tag{4.8}$$

which has solutions that can be expressed in terms of the Lambert W function:

$$W^{(0)}(x) = 2W_k\left(\pm\frac{\sqrt{x}}{2}\right), \tag{4.9}$$

where $k \in \mathbb{Z}$. Since $W_0(x)$ is real when $x \geq -e^{-1}$ and $W_{-1}(x) \in \mathbb{R}$ when $-e^{-1} \leq x < 0$, it is known that Equation (4.8) has at most three real roots. In fact, by Lemma 3.1, we know that Equation (4.2) has at most three real roots for all $r \in \mathbb{R}$.

4.3.1 Real branches

From Lemma 3.1, it is known that $(w^2 - r)e^w = x$ has at most three real solutions. To determine the real branches, the branch points are first determined.

The first derivative of Eq. (4.2),

$$\frac{dw}{dx} = \frac{w^2 - r}{x(w^2 + 2w - r)}, \tag{4.10}$$

suggests that we have three branch points, $x = 0$ and $P_n = (w_n^2 - r)e^{w_n}$ for $n = 1, 2$ with

$$w_n^2 + 2w_n - r = 0.$$

Solving for w_n, we obtain

$$w_n = -1 + (-1)^n \sqrt{1 + r}, \tag{4.11}$$

and

$$P_n = (w_n^2 - r)e^{w_n} = -2w_n e^{w_n}. \tag{4.12}$$

From Eq. (4.11), it can be concluded that P_n are two distinct real points if $r > -1$, repeated real point if $r = -1$ and two distinct complex points if $r < -1$. The figure below shows the real branches for different values of r.

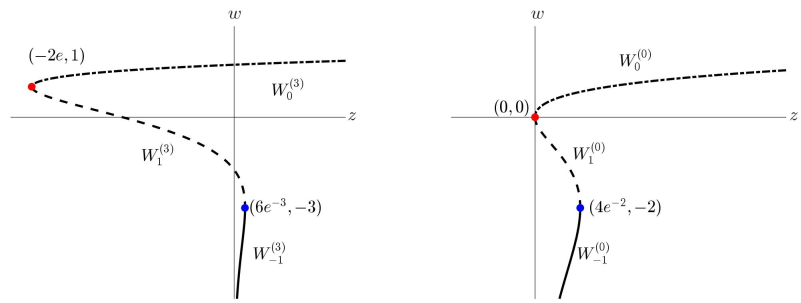

Figure 4.1: Real branches for $r = 3$ and $r = 0$.

These branch points separate the curve into three parts, which are labelled as $W_0^{(r)}, W_1^{(r)}$ and $W_{-1}^{(r)}$. These notations are used to indicate solutions in different branches. A detailed discussion of branch structure will be included in the next section.

For $r = -1$, we know that $P_1 = P_2 = 2e^{-1}$. In this case, there are only two real branches, $W_0^{(-1)}$ and $W_{-1}^{(-1)}$. For $r < -1$, it holds that $P_1, P_2 \in \mathbb{C}$. As the series solution that will be derived later is expressed in terms of $W_k(z_j)$ and $z_j = \frac{\sqrt{x}e^{i\pi j}}{2} = -e^{-1}$ (for $j = 1, 2$) is the branch point for $W_0(z_j)$ and $W_1(z_j)$. Thus, the real branch is separated by the point $x = 4e^{-2}$.

For both cases there is at most one real solution, however, the curve is separated into two parts (branches) as illustrated in figures below.

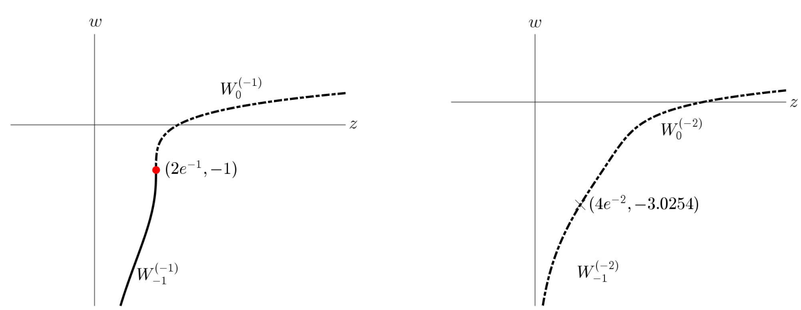

Figure 4.2: Real branches for $r = -1$ and $r = -2$

Theorem 4.1. *The equation* $(w^2 - r)e^w = x$ *has three real solutions when the following conditions are met:*

- $r > -1$, *and*

- $P_2 < x < P_1$,

where $P_n = 2(1 + (-1)^{n+1}\sqrt{1+r})e^{-1+(-1)^n\sqrt{1+r}}$ *for* $n = 1, 2$.

4.3.2 The omega-r constant, Ω_r

The omega-r constant, denoted as Ω_r, is defined as the real solution of the equation $(w^2 - r)e^w = 1$.

Theorem 4.2. *For all* $r \in \mathbb{R}$, $(w^2 - r)e^w = 1$ *has one and only one real solution.*

Proof. According to Theorem 4.1, this equation has at most one real solution when $r \leq -1$.

Since $2x < e^x$ holds for all $x \in \mathbb{R}$, we conclude that for $r > -1$,

$$2(1 + \sqrt{1+r}) < e^{1+\sqrt{1+r}},$$

which further implies

$$P_1 = 2(1 + \sqrt{1+r})e^{-1-\sqrt{1+r}} < 1.$$

Thus, there is at most one real solution for $x \geq 1$. $\square$

Theorem 4.3. *The omega-r constant, denoted as* Ω_r, *is a transcendental number if r is algebraic.*

Proof. Suppose that Ω_r is algebraic. According to Theorem 3.1, e^{Ω_r} is transcendental. Consequently, $(\Omega_r^2 - r)e^{\Omega_r}$ would also be transcendental. However, this contradicts the fact that $(\Omega_r^2 - r)e^{\Omega_r} = 1$. $\square$

It's worth noting that the inverse of the theorem above is not universally true. This can be demonstrated with the counter-example provided below.

Example 4.1. *The number* Ω_{1-e} *is algebraic as* $(\Omega_{1-e}^2 + e - 1)e^{\Omega_{1-e}} = 1$, *which implies* $\Omega_{1-e} = -1$.

Using Equation (4.9), we can express Ω_0 in terms of the Lambert W function:

$$\Omega_0 = 2W_0\left(\frac{1}{2}\right). \tag{4.13}$$

The selection of the principal branch and the positive root here leads to the sole real solution.

4.3.3 Linear combination of the $W^{(r)}$ function

Similar to the Lambert W function, it is possible to obtain a linear combination of $W^{(r)}(x)$. Given that $e^{W^{(r)}(x)} = \frac{x}{W^{(r)}(x)^2 - r}$, the following equality holds:

$$e^{aW^{(r)}(x_1)+bW^{(r)}(x_2)} = \left[\frac{x_1}{W^{(r)}(x_1)^2 - r}\right]^a \left[\frac{x_2}{W^{(r)}(x_2)^2 - r}\right]^b,$$

by multiplying both sides of the equation by $aW(x_1) + bW(x_2)$, we get

$$\begin{aligned}&\left[aW^{(r)}(x_1) + bW^{(r)}(x_2)\right] e^{aW^{(r)}(x_1)+bW^{(r)}(x_2)} \\ &\quad = \left[aW^{(r)}(x_1) + bW^{(r)}(x_2)\right] \left[\frac{x_1}{W^{(r)}(x_1)^2 - r}\right]^a \left[\frac{x_2}{W^{(r)}(x_2)^2 - r}\right]^b,\end{aligned}$$

which implies

$$\begin{aligned}&aW^{(r)}(x_1) + bW^{(r)}(x_2) \\ &\quad = W\left(\left[aW^{(r)}(x_1) + bW^{(r)}(x_2)\right] \left[\frac{x_1}{W^{(r)}(x_1)^2 - r}\right]^a \left[\frac{x_2}{W^{(r)}(x_2)^2 - r}\right]^b\right).\end{aligned}$$

Generalizing this to n terms using the same technique, we have

Theorem 4.4.

$$\sum_{t=1}^{n} a_t W^{(r)}(x_t) = W\left(\left[\sum_{t=1}^{n} a_t W^{(r)}(x_t)\right]\left[\prod_{t=1}^{n}\left(\frac{x_t}{W^{(r)}(x_t)^2 - r}\right)^{a_t}\right]\right). \tag{4.14}$$

By taking $a_t = x_t = 1$ for $t = 1, 2, \ldots, n$ and $r = 0$, we have

$$W\left(\frac{n}{\Omega_0^{2n-1}}\right) = n\Omega_0.$$

Using Equation (4.13), we obtain another identity for the Lambert W function:

$$W\left(\frac{n}{2^{2n-1} W(0.5)^{2n-1}}\right) = 2nW(0.5).$$

4.4 Derivatives and integrals

In this section, we discuss several properties relevant to derivatives and integrals of the $W^{(r)}$ function.

4.4.1 Derivatives

The first derivative is given by Equation (4.10) and can be expressed as

$$\frac{dW^{(r)}(x)}{dx} = \frac{e^{-W^{(r)}(x)}}{W^{(r)}(x)^2 + 2W^{(r)}(x) - r}.$$

Suppose the k-th derivative can be written in the form of

$$\frac{d^k W^{(r)}(x)}{dx^k} = \frac{p_k(W^{(r)}(x))e^{-kW^{(r)}(x)}}{(W^{(r)}(x)^2 + 2W^{(r)}(x) - r)^{2k-1}},$$

The n^{th} derivative of $W^{(r)}$ function derived using induction, is presented below:

$$\frac{d^n W^{(r)}(x)}{dx^n} = \frac{p_n(W^{(r)}(x))e^{-nW^{(r)}(x)}}{(W^{(r)}(x)^2 + 2W^{(r)}(x) - r)^{2n-1}}, \quad \text{for } n \geq 1, \tag{4.15}$$

where $p_{n+1}(w) = -[n(w^2+2w-r)+(2n-1)(2w+2)]p_n(w)+(w^2+2w-r)p_n'(w)$. The initial polynomial $p_1(w) = 1$.

The polynomials p_n appear to be unique and are not recognized in any other context, much like the Lambert W function. This prompts us to examine the derivatives of $W^{(r)}(e^x)$.

Similarly, using induction, we demonstrate that the n^{th} derivative of $W^{(r)}(e^x)$ with respect to x is represented by

$$\frac{d^n W^{(r)}(e^x)}{dx^n} = \frac{q_n(W^{(r)}(e^x))}{(W^{(r)}(e^x)^2 + 2W^{(r)}(e^x) - r)^{2n-1}}, \quad \text{for } n \geq 1, \tag{4.16}$$

where $q_{n+1}(w) = -(2n-1)(2w+2)(w^2-r)q_n(w) + (w^2+2w-r)(w^2-r)q_n'(w)$. The initial polynomials is given by $q_1(w) = w^2 - r$. Interestingly, the coefficients of $q_n(w)$ for general r do not appear to be expressible using any known formula.

4.4.2 Integrals

Similar to the Lambert W function, the integral $\int W^{(r)}(x)\,dx$ can be obtained using integration by parts:

$$\begin{aligned}
\int W^{(r)}(x)\,dx &= xW^{(r)}(x)\,dx - \int x\frac{dW^{(r)}(x)}{dx}dx + C \\
&= xW^{(r)}(x) - \int (W^{(r)}(x)^2 - r)e^{W^{(r)}(x)}\,dW^{(r)}(x) \\
&= \left[W^{(r)}(x) - 1\right]\left[x + 2e^{W^{(r)}(x)}\right] + C.
\end{aligned}$$

We use w to denote $W^{(r)}(x)$ in order to simplify the notation. A more general form can be obtained by using Equation (3.32):

$$
\begin{aligned}
\int x^n w^m \, dx &= \frac{x^{n+1} w^m}{n+1} - \frac{m}{n+1} \int x^{n+1} w^{m-1} \, dw \\
&= \frac{x^{n+1} w^m}{n+1} - \frac{m}{n+1} \int (w^2 - r)^{n+1} e^{(n+1)w} w^{m-1} \, dw \\
&= \frac{x^{n+1} w^m}{n+1} - \frac{m}{n+1} \int \sum_{k=0}^{n+1} \binom{n+1}{k} w^{2n+m-2k+1} (-r)^k e^{(n+1)w} \\
&= \frac{x^{n+1} w^m}{n+1} - \frac{m}{n+1} \sum_{k=0}^{n+1} \binom{n+1}{k} (-r)^k \int w^{2n+m-2k+1} e^{(n+1)w} \, dw
\end{aligned}
$$

The integral on right hand side of the equation can be evaluated using Lemma (3.2) and we arrive at

$$
\begin{aligned}
\int x^n w^m \, dx = \frac{x^{n+1} w^m}{n+1} & \\
& - \frac{m e^{(n+1)w}}{n+1} \sum_{k=0}^{n+1} \left\{ \binom{n+1}{k} (-r)^k \right. \\
& \times \left. \sum_{s=0}^{2n+m-2k+1} \frac{(-1)^s (2n+m-2k+1)! w^{2n+m-2k+1-s}}{(2n+m-2k+1-s)!(n+1)^{s+1}} \right\} + C.
\end{aligned}
$$

4.5 Series solutions

In this section, we explore series solutions of the $W^{(r)}$ function. We derive these series solutions using three distinct approaches: Taylor series expansion at $r = 0$ as proposed by Scott, Fee and Grotendorst (2014), Lagrange inversion, and asymptotic expansions using logarithms and the Lambert W function.

4.5.1 Taylor series at $r = 0$

When r is small or z is significantly larger than r, the term $w^2 e^w$ dominates. As a result,

$$
w \sim 2W_k \left(\frac{\sqrt{z} e^{j\pi i}}{2} \right) = 2W_k(z'), \tag{4.17}
$$

where $j = 0, 1$. Here, $i = \sqrt{-1}$, and $\sqrt{z}$ represents the principal square root of z. As different pairs of (k, j) yield distinct values for w, we use the notation $W_{2k+j}^{(r)}(z)$ to signify the $(2k + j)$-th branch of the solutions.

For $r = 0$, the solutions to Eq. (4.2) are given by

$$w = 2W_k\left(\frac{z^{1/2}}{2}\right) = 2W_k\left(\frac{\sqrt{z}e^{j\pi i}}{2}\right) = W_{2k+j}^{(0)}(z), \tag{4.18}$$

where $k \in \mathbb{Z}$ and $j = 0, 1$.

By differentiating Eq. (4.2) with respect to r, we obtain

$$\frac{dw}{dr} = \frac{1}{w^2 + 2w - r}.$$

Induction reveals that higher-order derivatives are expressed as

$$\frac{d^n w}{dr^n} = \frac{p_n(w, r)}{\left(w^2 + 2w - r\right)^{2n-1}}, \tag{4.19}$$

where the polynomial $p_{n+1}(w, r)$ satisfies

$$p_{n+1}(w, r) = (w^2 + 2w - r)^2 p_n'(w, r) + (2n - 1)(w^2 - r - 2)p_n(w, r) \tag{4.20}$$

for $n \geq 1$. The initial polynomial $p_1(w, r) = 1$ and $p_n'(w, r)$ are derivatives of $p_n(w, r)$ with respect to r. Eq. (4.20) can also be written in a more compact form:

$$(w^2 + 2w - r)^{-2n-1}p_{n+1}(w, r) = \frac{d}{dr}\left[(w^2 + 2w - r)^{-2n+1}p_n(w, r)\right]. \tag{4.21}$$

The derivatives and Eq. (4.18) lead to the following result:

Theorem 4.5. *The Taylor series of* $W^{(r)}_{2k+j}(z)$ *at around* $r = 0$ *is*

$$W^{(r)}_{2k+j}(z) = 2W_k(z_j) + \sum_{n=1}^{\infty} \frac{p_n(2W_k(z_j), 0)}{\left\{4W_k(z_j)^2 + 4W_k(z_j)\right\}^{2n-1}} \frac{r^n}{n!} \tag{4.22}$$

where $k \in \mathbb{Z}$ *and* $z_j = \frac{\sqrt{z}e^{j\pi i}}{2}$ *for* $j = 0, 1$. *The polynomial* $p_{n+1}(w, r)$ *is given by the following recurrence relation:*

$$(w^2 + 2w - r)^{-2n-1} p_{n+1}(w, r) = \frac{d}{dr}\left[(w^2 + 2w - r)^{-2n+1} p_n(w, r)\right].$$

for $n \geq 1$. *The initial polynomial is* $p_1(w, r) = 1$ *and* $p'_n(w, r)$ *are derivatives of* $p_n(w, r)$ *with respect to* r.

Similar to Mugnaini (2014), determining the radius of convergence for the series solutions remains a challenge. Nevertheless, interesting phenomena useful for radius of convergence determination have been observed:

1. The series converges when $\left|\frac{r}{W_k(z_j)}\right|$ is small;
2. If the series converges, the rate of convergence increases as k increases.

4.5.2 Lagrange inversion

As per Lagrange's inversion method (Abramowitz and Stegun, 1992, Page 14), if $x = f(w)$, $x_0 = f(w_0)$, and $f'(w_0) \neq 0$, then the following relation holds:

$$w = w_0 + \sum_{k=1}^{\infty} \frac{(x - x_0)^k}{k!} \lim_{w \to w_0} \left[\frac{d^{k-1}}{dw^{k-1}} \left\{\frac{w - w_0}{f(w) - x_0}\right\}^k\right]. \tag{4.23}$$

For $r = 0$, we have $W^{(0)}_k(x) = 2W_k\left(\pm\frac{\sqrt{x}}{2}\right)$, where $W_k(x)$ is the k-th branch of the Lambert W function. Using Lagrange's inversion method, we can derive the series expansion of w based on Equation (4.23), with $w_0 = 2W_k\left(\pm\frac{\sqrt{x}}{2}\right)$:

$$
\begin{aligned}
W_{-1}^{(r)}(x) &= 2W_k \\
&\quad + \sum_{t=1}^{\infty} \frac{r^t e^{2tW_k}}{t!} \lim_{w\to 2W_k}\left[\frac{d^{t-1}}{dw^{t-1}}\left\{\frac{w-2W_k}{(w^2-r)e^w-(4W^2-r)e^{2W_k}}\right\}^t\right] \\
&= 2W_k + \frac{4W_k^2}{z(4W_k^2+4W_k-r)} - \frac{16W_k^4(4W_k^2+8W_k-r+2)}{z^2(4W_k^2+4W_k-r)^3} + \dots.
\end{aligned}
$$

where $W_k = W_k\left(\pm\frac{\sqrt{x}}{2}\right)$.

4.5.3 Asymptotic expansions

When $|z| \gg r$, we know that w^2e^w is the dominating factor and hence we write $w = W_k^{(r)}(z) = 2W_k + u$, where $W_k = W_k\left(\frac{\pm\sqrt{z}}{2}\right)$ is the k-th branch of the Lambert W function. Substituting to the original equation yields

$$
\begin{aligned}
(4W_k^2 + 2uW_k + u^2 - r)e^{2W_k+u} &= z \\
\left[(2W_k e^{W_k})^2 + (u^2 + 2uW_k - r)e^{2W_k}\right] e^u &= z \\
\left[z + (u^2 + 2uW_k - r)e^{2W_k}\right] e^u &= z \\
\left[\frac{1}{z} + \left(\frac{u^2}{z^2} + 2W_k\frac{u}{z} - \frac{r}{z^2}\right)e^{2W_k}\right] e^u &= \frac{1}{z}.
\end{aligned}
$$

Based on the assumption that $|u| \ll |z|$, we have

$$
\begin{aligned}
\log\left[\frac{1}{z} - \frac{r}{z^2}e^{2W_k}\right] + u &\sim -\log z \\
u &\sim \log z - \log\left[z - re^{2W_k}\right].
\end{aligned}
$$

Hence, we arrive at

$$
w = 2W_k + \log z - \log\left(z - re^{2W_k}\right) + v, \tag{4.24}
$$

where $e^{-v} + r\tau e^{2W_k} - (\sigma + v)^2 \tau e^{2W_k} = 0$ with $\tau = \left(z - re^{2W_k}\right)^{-1}$ and $\sigma = 2W_k + \log z - \log\left(z - re^{2W_k}\right)$.

4.6 Branches of the $W^{(r)}$ function

Among all series solutions derived in previous section, the Taylor series at around $r = 0$ allows us to compute solutions in different branches. In this section, we will consider the branch cuts and branch structure for this series solution.

From Eq. (4.11), we have $P_1, P_2 \in \mathbb{R}$ and $P_2 < P_1$ if $r > -1$. For $r = -1$, $P_1 = P_2 \in \mathbb{R}$. For $r < -1$, we solve Eq. (4.12) and arrive at

$$w_n = W_k\left(-\frac{z_n}{2}\right) \quad \text{and} \quad w_1 = \overline{w_2}. \tag{4.25}$$

This implies P_1, P_2 are complex and $P_1 = \overline{P_2}$. Thus, we will discuss the branch structure in three separate cases: $r > -1$, $r = -1$ and $r < -1$.

4.6.1 Branch structure when $r > -1$

In this case, we know that we have three distinct branch points $z = 0$ and P_1, P_2. From Figure 4.1, we know that P_1 is a branch point of $W_{-1}^{(r)}$ and $W_1^{(r)}$, while P_2 is a branch point of $W_0^{(r)}$ and $W_1^{(r)}$.

It's important to note that P_1 is shared by $W_3^{(r)}$ also. For $z = 0$, it is analytic in the branch $W_0^{(r)}$ and $W_1^{(r)}$, but it is a branch point in all other branches. Thus, we choose the branch cuts as follows:

- $W_{-1}^{(r)}$: $\{z : -\infty < z \le 0\}$, $\{z : 0 \le z < P_1\}$ and $\{z : P_1 \le z < \infty\}$,
- $W_0^{(r)}$: $\{z : -\infty < z \le P_2\}$,
- $W_1^{(r)}$: $\{z : -\infty < z \le P_2\}$ and $\{z : P_1 \le z < \infty\}$,
- $W_3^{(r)}$: $\{z : -\infty < z \le 0\}$, $\{z : 0 \le z < P_1\}$ and $\{z : P_1 \le z < \infty\}$,

- For all $k \neq 0$, $W_{2k}^{(r)}$: $\{z : -\infty < z \leq 0\}$,
- For all $|k| \geq 2$, $W_{2k+1}^{(r)}$: $\{z : -\infty < z \leq 0\}$ and $\{z : 0 < z < \infty\}$.

Note that for all $k \geq 2, j = 1$, the $(2k + j)$-th branch has additional branch cut $\{z : 0 \leq z < \infty\}$ due to our choice of series solutions expressed in terms of the Lambert W function, $W_k\left(\frac{\sqrt{z}e^{j\pi i}}{2}\right)$. When $j = 1$, $\sqrt{z}e^{\pi i}$ maps the positive real axis to the negative real axis, which is the branch cut for $W_k(z)$ where $k \neq 0$. Thus, positive real axis is included as an additional branch cut. This phenomenon is not observed in $(2k)$-th branch as $\sqrt{z}$ does not map to the negative real axis.

Branch structure

For the case $r = 0$, the Equation (4.18) reveals that for each k, there are two solutions which correspond to $j = 0, 1$.

This suggests that the branch structure is similar to that of the Lambert W and the square root function. Specifically, the ranges of the Lambert W function will be doubled due to the coefficient of two in the Equation (4.18), and the resultant ranges will be separated into two distinct ranges due to the square root function. This is illustrated in the figure below.

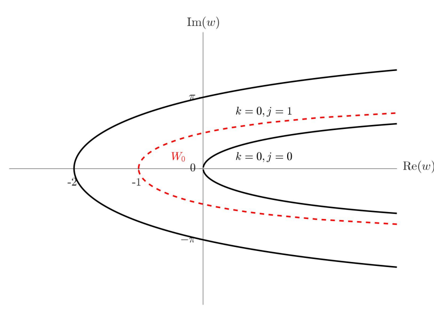

Figure 4.3: Ranges of $W_{2k+j}^{(0)}$ for $k = 0$ and $j = 0, 1$. The dashed line represents the range of the principal branch of the Lambert W function.

The case $k = -1$ is presented in the figure below. Similar to $k = 0$, the range of $j = 0$ is enclosed by ranges of $j = 1$.

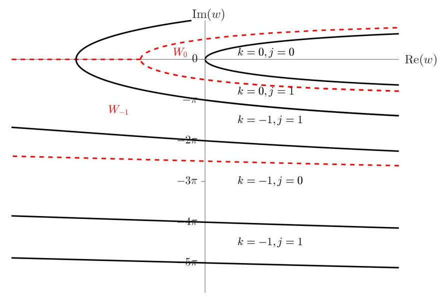

Figure 4.4: Ranges of $W^{(0)}_{2k+j}$ for $k = -1$ and $j = 0, 1$. The region bounded by dashed lines represents the range of W_{-1}.

From Eq. (4.11) and (4.12), we can deduce that $P_1 = 4e^{-2}, P_2 = 0$ are branch points for the real branches. Consequently, $W^{(0)}_0$ has a branch cut $\{z : -\infty < z \leq 0\}$ and $W^{(0)}_1$ has branch cuts $\{z : -\infty < z \leq 0\}$ and $\{z : 4e^{-2} \leq z < \infty\}$.

Figures below illustrate the images of paths encircling around the origin in a counter-clockwise manner with a radius $\rho > P_1 = 4e^{-2}$. The solid line indicates closure.

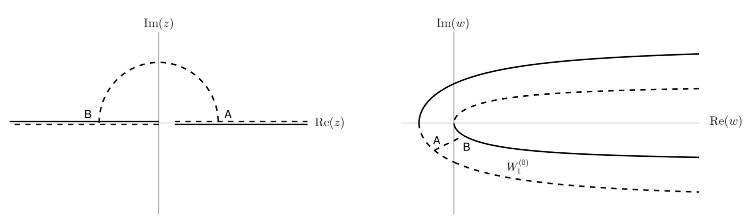

Figure 4.5: $W^{(0)}_1$ and the image of semicircle AB in the w-plane.

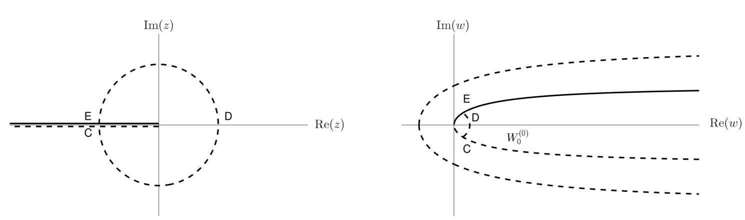

Figure 4.6: $W^{(0)}_0$ and the image of circle CDE in the w-plane.

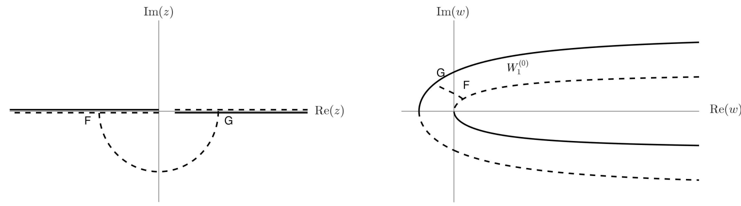

Figure 4.7: $W_1^{(0)}$ and the image of semicircle FG in the w-plane.

Starting in $W_1^{(0)}$, the dashed semicircle in the left side of Figure 4.5 moving counter-clockwise from point A to B in the z-plane corresponds to an image in the w-plane as illustrated on the right side of Figure 4.5. Continuing to move counter-clockwise, we transition into the $W_0^{(0)}$ sheet, as shown in Figure 4.6, and after completing a 2π rotation, we return to the $W_1^{(0)}$ sheet, as shown in Figure 4.7.

In fact, all $W_{2k}^{(0)}$ and $W_{2k+1}^{(0)}$ sheets are interconnected through the branch cut on the negative real axis. Additionally, $W_{2k+1}^{(0)}$ is linked to $W_{2(k+1)+1}^{(0)}$ and $W_{2(k-1)+1}^{(0)}$ through the branch cut on the positive real z-axis. For example, if we traverse beyond point G in Figure 4.7, we transition into the $W_3^{(0)}$ sheet. The following figures illustrate this:

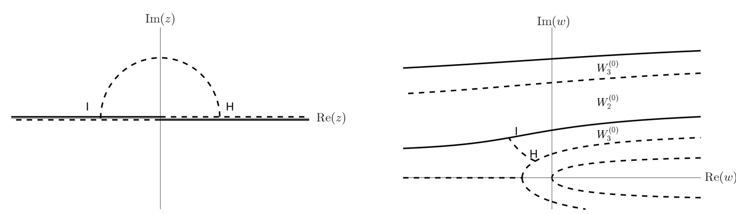

Figure 4.8: $W_3^{(0)}$ and the image of the semicircle HI in the w-plane.

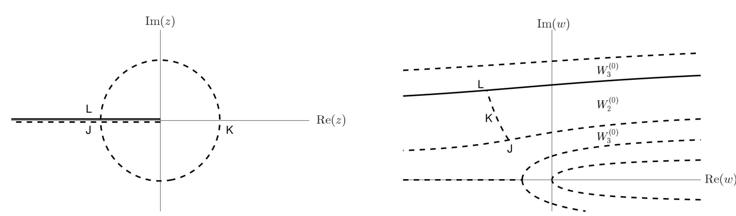

Figure 4.9: $W_2^{(0)}$ and the image of the circle JKL in the w-plane.

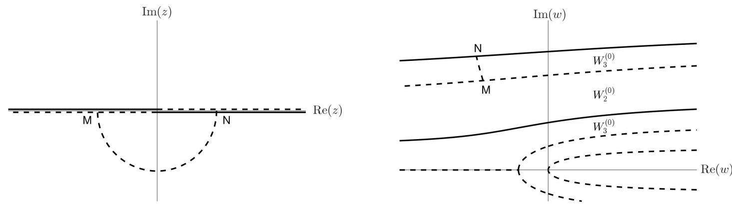

Figure 4.10: $W_3^{(0)}$ and the image of the semicircle MN in the w-plane.

The image of the semicircle with radius $\rho < 4e^{-2}$ in $W_3^{(0)}$ does not connect to $W_1^{(0)}$ (see Figure 4.8 for comparison). This is due to the fact that $W_3^{(0)}$ has two branch cuts on the positive real axis: $\{z : 0 \le z < 4e^{-2}\}$ and $\{z : 4e^{-2} \le z < \infty\}$. $W_{-1}^{(0)}$ exhibits similar behaviour.

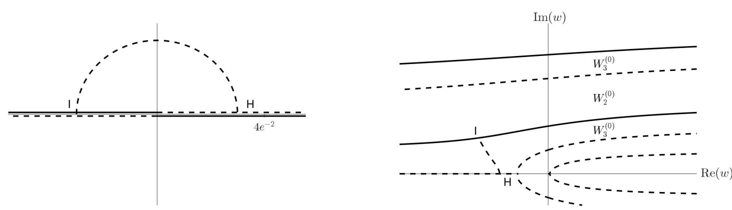

Figure 4.11: $W_3^{(0)}$ and image of semicircle HI in w-plane.

The branch structure for the case $r \neq 0$ ($r > -1$) is akin to the $r = 0$ case, with the distinction that $P_2 \neq 0$ and $P_1 \neq 4e^{-2}$. The figures below portray the images of paths encircling around P_1 in a counter-clockwise manner.

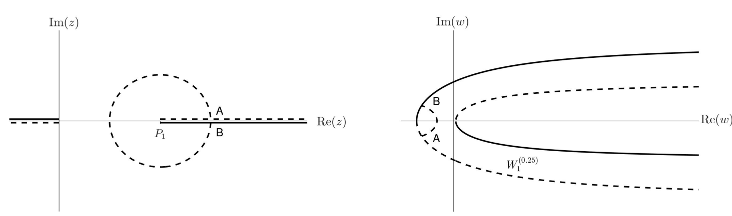

Figure 4.12: Image of circle AB in $W_1^{(0.25)}$.

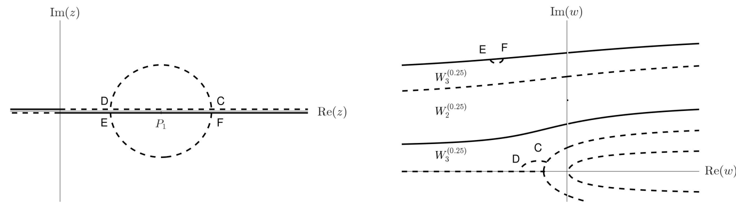

Figure 4.13: Image of semicircles CD and EF in $W_3^{(0.25)}$.

Boundary curves

By substituting $w = \xi + i\eta$ and $z = x + iy$ into Eq. (4.2), we obtain

$$[A\cos\eta - B\sin\eta + i(A\sin\eta + B\cos\eta)]\, e^{\xi} = x + iy,$$

where $A = \xi^2 - \eta^2 - r$ and $B = 2\eta\xi$. As all branch cuts fall on the real axis of z-plane, the boundary curves must satisfy $y = 0$ or $A\sin\eta + B\cos\eta = 0$.

The boundary curves of $W_k^{(r)}$ can be obtained by considering the values of ξ and η that satisfy $A\sin\eta + B\cos\eta = 0$, subject to $A\cos\eta + B\sin\eta$ has the same ranges as all the branch cuts. For instance, in the case of the branch $k = 2$, the conditions $A\cos\eta + B\sin\eta \le 0$ and $A\cos\eta + B\sin\eta \ge 0$ yield the two boundary curves of $k = 2$ branch. Figure 4.14 illustrates the boundary curves for $k = -1, 0, 1$ and $r = 0$.

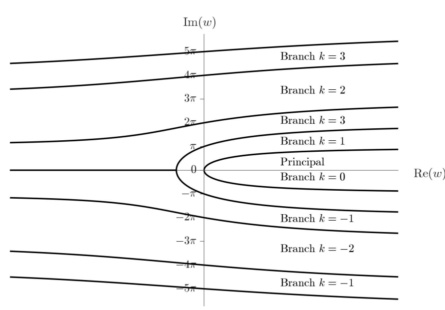

Figure 4.14: Ranges of $W_k^{(0)}$.

4.6.2 Branch cuts when $r = -1$

Similar with the case $r > -1$, we note that $P_1 = P_2 = 2e^{-1}$. Hence, the branch cuts are chosen to be:

- $W_{-1}^{(-1)}$: $\{z : -\infty < z \leq 0\}$, $\{z : 0 \leq z < 2e^{-1}\}$ and $\{z : 2e^{-1} \leq z < \infty\}$,
- $W_{0}^{(-1)}$: $\{z : -\infty < z \leq 2e^{-1}\}$,
- $W_{1}^{(-1)}$: $\{z : -\infty < z \leq 2e^{-1}\}$ and $\{z : 2e^{-1} \leq z < \infty\}$,
- $W_{3}^{(-1)}$: $\{z : -\infty < z \leq 0\}$, $\{z : 0 \leq z < 2e^{-1}\}$ and $\{z : 2e^{-1} \leq z < \infty\}$,
- For all $k \neq 0$, $W_{2k}^{(-1)}$: $\{z : -\infty < z \leq 0\}$,
- For all $|k| \geq 2$, $W_{2k+1}^{(-1)}$: $\{z : -\infty < z \leq 0\}$ and $\{z : 0 < z < \infty\}$.

As $P_1 = P_2$, there will be only two real branches, $W_0^{(-1)}$ and $W_{-1}^{(-1)}$, as presented in Figure 4.2. Since all branch cuts are positioned on the real axis, the boundary curves can be constructed using the same approach as the case $r > -1$. Plotting all the relevant curves of $A \sin \eta + B \cos \eta = 0$, we obtain figure below.

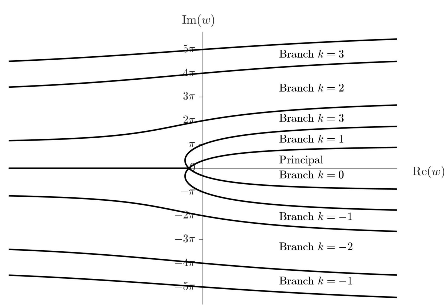

Figure 4.15: Ranges of $W_k^{(-1)}$.

4.6.3 Branch cuts when $r < -1$

Similar to the case $r > -1$, there exist three distinct branch points, namely, $z = 0$ and P_1, P_2. Given that $z = 0$ is branch point of $W_{2k+j}^{(r)}$ for all $k > 0$, we select the branch cuts for $W_{2k+j}^{(r)}$ in a manner analogous to the $r > -1$ case. The exceptions are:

- $k = 0$, $j = 0, 1$,
- $k = -1$, $j = 1$, and
- $k = 1$, $j = 1$.

As P_1, P_2 are complex when $r < -1$, the chosen branch cuts for these cases extend from the point $4e^{-2}$ to P_1 or P_2. We summarise the branch cuts for all branches as follow:

- $W_{-1}^{(r)}$: $\{z : -\infty < z \leq 0\}$, $\{z : 0 \leq z < 4e^{-2}\}$, $\{z : 4e^{-2} \leq z < \infty\}$ and $\{z : \mathrm{Arg}(z) = \mathrm{Arg}(P_1 - 4e^{-2}),\ 4e^{-2} \leq \mathrm{Re}(z) \leq \mathrm{Re}(P_1)\}$,
- $W_{0}^{(r)}$: $\{z : -\infty < z \leq 4e^{-2}\}$, $\{z : \mathrm{Arg}(z) = \mathrm{Arg}(P_1 - 4e^{-2}),\ 4e^{-2} \leq \mathrm{Re}(z) \leq \mathrm{Re}(P_1)\}$ and $\{z : \mathrm{Arg}(z) = \mathrm{Arg}(P_2 - 4e^{-2}),\ 4e^{-2} \leq \mathrm{Re}(z) \leq \mathrm{Re}(P_2)\}$,
- $W_{1}^{(r)}$: $\{z : -\infty < z \leq 4e^{-2}\}$ and $\{z : 4e^{-2} \leq z < \infty\}$,
- $W_{3}^{(r)}$: $\{z : -\infty < z \leq 0\}$, $\{z : 0 \leq z < 4e^{-2}\}$, $\{z : 4e^{-2} \leq z < \infty\}$ and $\{z : \mathrm{Arg}(z) = \mathrm{Arg}(P_2 - 4e^{-2}),\ 4e^{-2} \leq \mathrm{Re}(z) \leq \mathrm{Re}(P_2)\}$,
- For all $k \neq 0$, $W_{2k}^{(r)}$: $\{z : -\infty < z \leq 0\}$,
- For all $|k| \geq 2$, $W_{2k+1}^{(r)}$: $\{z : -\infty < z \leq 0\}$ and $\{z : 0 < z < \infty\}$.

The presence of the disconnected branch $W_{1}^{(r)}$ introduces an additional branch cut (compared to $r \geq -1$) in $W_{-1}^{(r)}$, $W_{0}^{(r)}$, and $W_{3}^{(r)}$. For instance, $W_{-1}^{(r)}$ is connected to five other branches: $W_{3}^{(r)}, W_{0}^{(r)}, W_{1}^{(r)}, W_{-2}^{(r)}$, and $W_{-5}^{(r)}$ (see Figure 4.26). Thus, four branch cuts are necessary.

Branch structure

We will illustrate a circular path that traverse counter-clockwise, centred at $z = 4e^{-2}$ and radius $\rho = 0.3$, from $W_{-1}^{(-2)}$ to $W_0^{(-2)}$ and then to $W_3^{(-2)}$.

All branch cuts are chosen to be closed in a counter-clockwise direction, and all solid lines in the figures below indicate the closure of the branch.

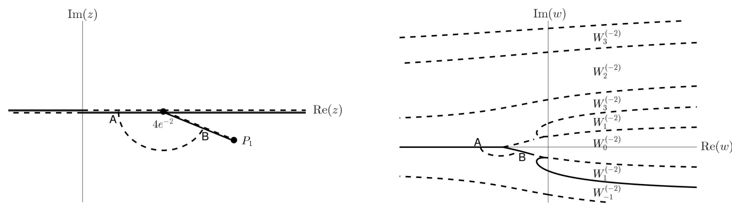

Figure 4.16: Image of AB in $W_{-1}^{(-2)}$.

Continuing the traversal from C to D, we move into the branch $W_0^{(-2)}$. As P_1 and P_2 are complex, and since $W_1^{(-2)}$ is disconnected, we can now move into $W_0^{(-2)}$ without passing through $W_1^{(-2)}$.

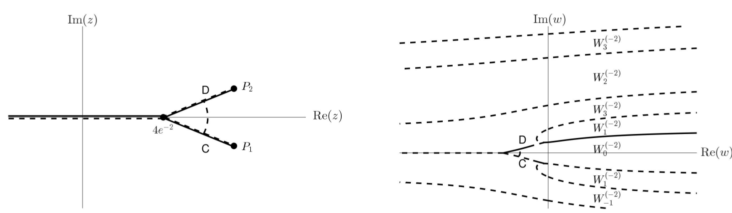

Figure 4.17: Image of CD in $W_0^{(-2)}$.

Continuing on the path EF, we obtain the following figure:

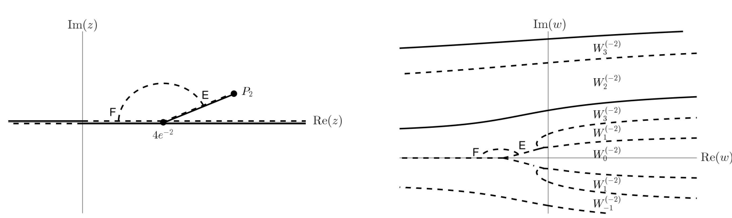

Figure 4.18: Image of EF in $W_3^{(-2)}$.

Figure below shows the images of different paths in $W_3^{(-2)}$:

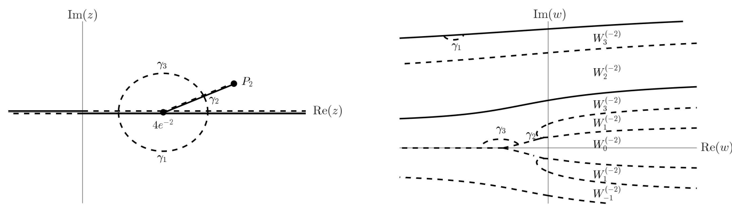

Figure 4.19: Images of γ_i in $W_3^{(-2)}$.

The image of γ_2 is hardly visible, so we present a larger image using a solid red curve. Note that the solid boundary curve is the image of the line from $z = 4e^{-2}$ to P_2.

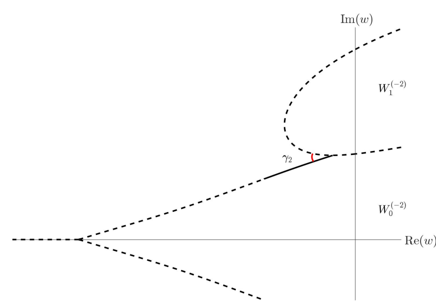

Figure 4.20: Image of γ_2 in $W_3^{(-2)}$.

Now, consider a circle with a radius of $\rho = 2$, centred at $z = 4e^{-2}$. The images in different branches are presented below.

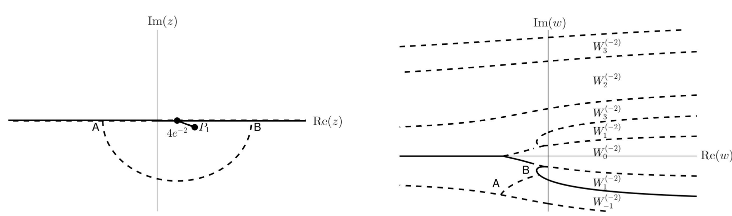

Figure 4.21: Image of AB in $W_{-1}^{(-2)}$.

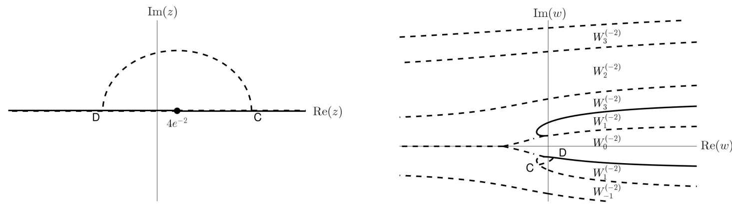

Figure 4.22: Image of CD in $W_1^{(-2)}$.

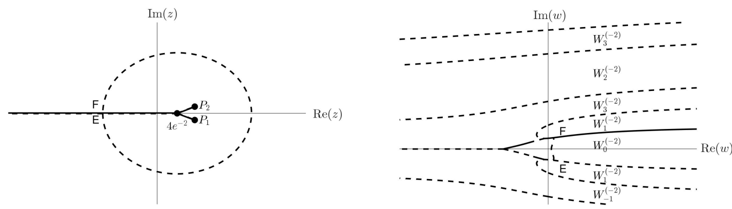

Figure 4.23: Image of EF in $W_0^{(-2)}$.

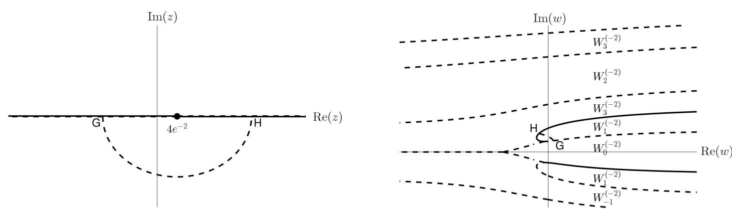

Figure 4.24: Image of GH in $W_1^{(-2)}$.

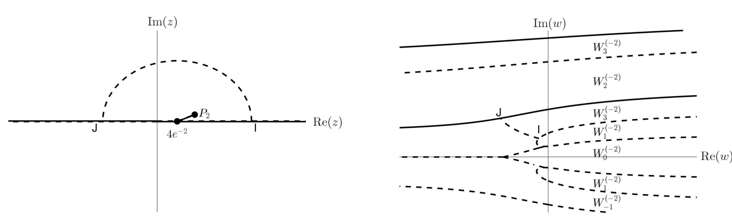

Figure 4.25: Image of IJ in $W_3^{(-2)}$.

Boundary curves

Since all branch cuts of $W_k^{(r)}$, where $k \neq 0, \pm 1, 3$, fall on the real axis, the boundary curves satisfy $A \sin \eta + B \cos \eta = 0$. For $W_0^{(r)}$, there are two branch cuts that are not on the real axis. For these branch cuts, we construct the boundary curves using

$$\mathrm{Arg}[(A\cos\eta - B\sin\eta)e^{\xi} - 4e^{-2} + (A\sin y + B\cos y)e^{\xi}i] = \mathrm{Arg}(z_n - 4e^{-2}),$$

subject to $4e^{-2} \le (A\cos\eta - B\sin\eta)e^{\xi} \le \mathrm{Re}(z_n)$. Here, $A = \xi^2 - \eta^2 - r$ and $B = 2\xi\eta$.

The figure below presents the branch structures for $r = -2$.

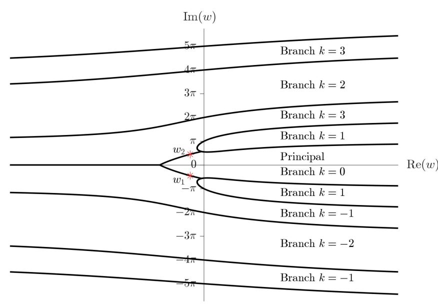

Figure 4.26: Ranges of $W_k^{(-2)}$.

Although there is at most one real solution for $(w^2 - r)e^w = z$ when $r < -1$ (see Figure 4.2), the real axis of the w-plane is still divided into two branches due to our choice of the series solution.

4.7 Applications

4.7.1 Solutions to other exponential-polynomial equations

Theorem 4.5 can be used to obtain series solutions for cases with two upper parameters and two lower parameters.

Corollary 4.1. *For all $a_1, a_2 \in \mathbb{R}$. the series solutions of $(w - a_1)(w - a_2)e^w = z$ is given by*

$$W_{2k+j}\left(\begin{smallmatrix} a_1 & a_2 \end{smallmatrix}; z\right) = \frac{a_1 + a_2}{2} + 2W_k(z'_j) + \sum_{n=1}^{\infty} \frac{p_n(2W_k(z'_j), 0)}{\left\{4W_k(z'_j)^2 + 4W_k(z'_j)\right\}^{2n-1}} \frac{r^n}{n!},$$

where $r = \left(\frac{a_1-a_2}{2}\right)^2$ *and* $z'_j = \frac{\sqrt{ze^{-r_m}}e^{j\pi i}}{2}$.

Proof. Let $r_d = \frac{a_1-a_2}{2}$ and $r_m = \frac{a_1+a_2}{2}$. The equation can be written as

$$\left((w-r_m)^2 - r\right) e^{w-r_m} = ze^{-r_m},$$

where $r = r_d^2$. Thus,

$$w = r_m + W_{2k+j}^{(r)}\left(ze^{-r_m}\right).$$

$\square$

Corollary 4.2. *The solutions of* $\frac{1}{(w-a_1)(w-a_2)}e^w = z$, $z \neq 0$ *is given by*

$$w = -W_{2k+j}\left(\begin{matrix}-a_1 & -a_2\end{matrix}; \frac{1}{z}\right).$$

Proof. It can be shown easily by rewriting the parameters:

$$(w-a_1)(w-a_2)e^{-w} = [-w-(-a_1)][-w-(-a_2)]e^{-w} = \frac{1}{z}$$

which implies the solutions are $w = -W_{2k+j}\left(\begin{matrix}-a_1 & -a_2\end{matrix}; \frac{1}{z}\right)$. $\square$

We consider another example of application in physics (Scott, Fee and Grotendorst, 2014), which arises from the Double Well Dirac Delta Potential model. We wish to solve the equation

$$e^{-cx} = a_0(x-r_1)(x-r_2), \tag{4.26}$$

where $c = 2R$, $a_0 = \frac{1}{\lambda}$, $r_1 = 1$, and $r_2 = \lambda$. Here, R is the internuclear distance and λ is treated as a real perturbative parameter.

Example 4.2. *Let* $\lambda = 0.8$ *and* $R = 2$*. Eq. (4.26) can be written as*

$$(4x - 4)(4x - 3.2)e^{4x} = \frac{64}{5}.$$

Since $-e^{-1} < -z_0 = -0.0489 < 0$*, the equation has three real solutions. Using the series solution, we obtain*

$$x_1 = W_0\left({}^{4\ 3.2}; 64/5\right)/4 = 1.0485,$$

$$x_2 = W_1\left({}^{4\ 3.2}; 64/5\right)/4 = 0.6249,$$

$$x_3 = W_{-1}\left({}^{4\ 3.2}; 64/5\right)/4 = 0.$$

4.7.2 Inverse transform method of Erlang and negative binomial distributions

The inverse transform method is a widely used technique for generating random variables from probability distributions. In this section, we will explore how the $W_k^{(r)}$ function can be utilized to obtain the quantile function of the Erlang and negative binomial distributions with a shape parameter of $k = 3$. Subsequently, we will apply the inverse transform method to simulate random variables based on these quantile functions.

Using the inverse transform method, random variables from the Erlang and negative binomial distributions can be generated through the following steps:

1. Generate a random number u uniformly distributed between 0 and 1.
2. Calculate the quantile function $F^{-1}(u)$ using the appropriate equation involving the $W_k^{(r)}$ function.
3. The resulting value is a random variable that follows the desired distribution.

By employing this approach, we can simulate random variables from the Erlang and negative binomial distributions by leveraging the properties of the $W_k^{(r)}$

function. This method provides an effective and efficient way to generate random samples from these distributions, enabling statistical analyses, simulations, and other applications.

Erlang distribution

For the Erlang distribution with $k = 3$, the quantile function equation is derived from its CDF as follows:

$$1 - \left(1 + \lambda y + \frac{1}{2}\lambda^2 y^2\right) e^{-\lambda y} = u' = 1 - u, \tag{4.27}$$

where y represents the quantile, λ is a parameter, and u is a random variable following a uniform distribution, $U[0, 1]$. By introducing the variable $w = -\lambda y - 1$, the equation can be rewritten as:

$$\left(w^2 + 1\right) e^{w} = x, \tag{4.28}$$

where $x = 2ue^{-1}$. Therefore, we have $r = -1$ and $0 < x < 2e^{-1}$. From the graph of the real branches of $W_k^{(-1)}$ in Figure 4.2, we observe that w lies in the branch $W_{-1}^{(-1)}(x)$. The quantile function can be expressed as:

$$y = -\frac{1}{\lambda}\left[W_{-1}^{(-1)}(2ue^{-1}) + 1\right]. \tag{4.29}$$

Negative binomial distribution

For the negative binomial distribution with $k = 3$, the quantile function can be obtained using the inverse transform method. Let n represent the quantile, $p = 1 - q$ be the probability of success, and u be a random variable.

The quantile function equation can be written as:

$$1 - \sum_{t=0}^{2} \binom{3+n}{t} q^{3+n-t} p^{t} = u' = 1 - u, \tag{4.30}$$

By introducing the variables $\alpha = qp^{-1}$ and defining:

$$w = (n + \alpha + 2.5) \log q, \tag{4.31}$$

the equation can be written as:

$$(w^2 - r)e^{w} = x, \tag{4.32}$$

where $r = (0.25 - \alpha - \alpha^2) \log^2 q$ and $x = 2uq^{\alpha+1.5} p^{-2} \log^2 q$.

Since $w < 0$ and $F(n; 3, p) = 1 - u$ implies that n is inversely proportional to u, we know that $w \propto x = 2uq^{\alpha+1.5} p^{-2} \log^2 q$. The only real branch that is increasing and negative is $W_{-1}^{(r)}$.

Let $u_0 = 1 > u_1 > \cdots > u_{n-1} > u_n > \cdots > 0$ such that

$$w_n = (n + \alpha + 2.5) \log q,$$

where $(w_n^2 - r)e^{w_n} = x_n$ and $x_n = 2u_n q^{\alpha+1.5} p^{-2} \log^2 q$. Thus, for $u_{n-1} < u \leq u_n$, we have $w_n \leq w < w_{n-1}$. Solving Equation (4.31) for inverse transform method, we obtain:

$$n = \left\lceil \frac{W_{-1}^{(r)}(x)}{\log q} - \alpha - 2.5 \right\rceil. \tag{4.33}$$

CHAPTER 5

$(N, 0)$-TYPE LAMBERT FUNCTION

In this chapter, we delve into the study of the series solution of the equation $P_N(w)e^w = z$, where $P_N(w) = \prod_{t=1}^{N}(w - c_t)$, c_t are real numbers, and z is a complex number. This equation holds significant importance in various areas of mathematics and engineering, and its series solution plays a crucial role in solving many practical problems.

The series solution of $P_N(w)e^w = z$, which is also known as the $(N, 0)$-type Lambert function, provides a powerful tool for obtaining approximate solutions and understanding the behaviour of complex functions. By expanding the polynomial $P_N(w)$ in a series form, we can express the equation as an infinite series involving powers of w. This series can then be manipulated and truncated to obtain an approximation of the solution.

The $(N, 0)$-type Lambert function, denoted as $W_k\left(\,{}^{c_1 \ldots c_N}; z\right)$, represents the multi-valued function that arises from the solutions of $P_N(w)e^w = z$. It has significant applications in various fields, including physics, engineering, and mathematical modelling.

By studying the series solution of $P_N(w)e^w = z$ and the $(N, 0)$-type Lambert function, we aim to provide a comprehensive understanding of these powerful techniques and their potential impact on mathematical modelling, scientific analysis, and engineering design. This knowledge will enable researchers, engineers, and practitioners to tackle complex problems, advance their fields, and make informed decisions based on rigorous mathematical foundations.

In the first section, we start with the study of the transformed equation $(w^3 + pw + q)e^w = z$, where p and q are real coefficients and w and z are complex variables. We investigate the series solution of this third-order equation and analyse its properties and behaviour.

Following this, we explore the extension of the series solution method to obtain series expansions for the $(N, 0)$-type Lambert function, where N is greater than three. We analyse the solutions of $P_N(w)e^w = z$ for different values of N and investigate the behaviour of the $(N, 0)$-type Lambert function in various contexts.

Throughout the chapter, we denote the solutions of the transformed equation as $W_k^{(p,q)}(z)$ and the $(N, 0)$-type Lambert function as $W_k\left(\,{}^{c_1 \ldots c_N}\,; z\right)$, where k is the branch index. Similarly, the branch index, k, is omitted to represent the principal branch when there is no ambiguity. By exploring the series solutions and properties of these functions, we aim to provide valuable insights and tools for researchers and practitioners in their mathematical analyses, problem-solving, and decision-making processes.

5.1 The $W^{(p,q)}$ function

Since all cubic equations can be transformed into depressed cubic equation, we, without loss of generality, study the equation

$$(w^3 + pw + q)e^w = z, \tag{5.1}$$

where $p, q \in \mathbb{R}$ and $w, z \in \mathbb{C}$.

5.1.1 Equations solvable using the $W^{(p,q)}$ function

In this section, we present some examples of the equations that can be solved using the $W^{(p,q)}$ function. Unless specified otherwise, a, b, c, d, s are real constants.

Example 5.1. *For $a, s \neq 0$, we rewrite the equation*

$$(aw^3 + bw^2 + cw + d)e^{sw} = x \tag{5.2}$$

by letting $w = \frac{\omega}{s} - \frac{b}{3a}$, we have

$$\left[\omega^3 + \left(\frac{3acs^2 - b^2s^2}{3a^2}\right)\omega + \frac{2b^3s^3 - 9abcs^3 + 27a^2ds^3}{27a^3}\right]e^{\omega} = \frac{s^3x}{a}e^{\frac{bs}{3a}}$$

which leads to

$$w = \frac{\omega}{s} - \frac{b}{3a} = \frac{1}{s}W^{(p,q)}\left(\frac{s^3x}{a}e^{\frac{bs}{3a}}\right) - \frac{b}{3a}, \tag{5.3}$$

where

$$p = \frac{3acs^2 - b^2s^2}{3a^2},$$
$$q = \frac{2b^3s^3 - 9abcs^3 + 27a^2ds^3}{27a^3}.$$

Example 5.2. *The equation*

$$(w-a)(w-b)(w-c)e^{w} = x \tag{5.4}$$

known as three upper parameters equation, can be solved using a similar approach as the previous example. Expanding the polynomial on the left hand side:

$$[w^3 - (a+b+c)w^2 + (ab+ac+bc)w - abc]e^{w} = x.$$

Compare with previous example, we obtain the solution

$$w = W^{(p,q)}\left(xe^{-r_m}\right) + r_m, \tag{5.5}$$

where

$$r_m = \frac{a+b+c}{3}$$
$$p = (r_m - b)(r_m - c) - (r_m - a)^2,$$
$$q = (r_m - a)(r_m - b)(r_m - c).$$

Example 5.3. *For $b \neq 0, 1$, the equation*

$$(w^3 + pw + q)b^w = x \tag{5.6}$$

can be solved in terms of $W^{(p,q)}$ by rewriting b^w as $e^{w \log b}$:

$$\left[(w\log b)^3 + p\log^2 b(w\log b) + q\log^3 b\right] e^{w\log b} = x(\log b)^3.$$

Thus, the solution is

$$w = \frac{1}{\log b} W^{(p\log^2 b, q\log^3 b)}(x\log^3 b) \tag{5.7}$$

5.1.2 Basic properties of the $W^{(p,q)}$ function

Lemma 3.1 indicates that Eq. (5.1) has at most four real solutions, depending on the value of p and q. Figures below illustrate the real branches for different values of p and q.

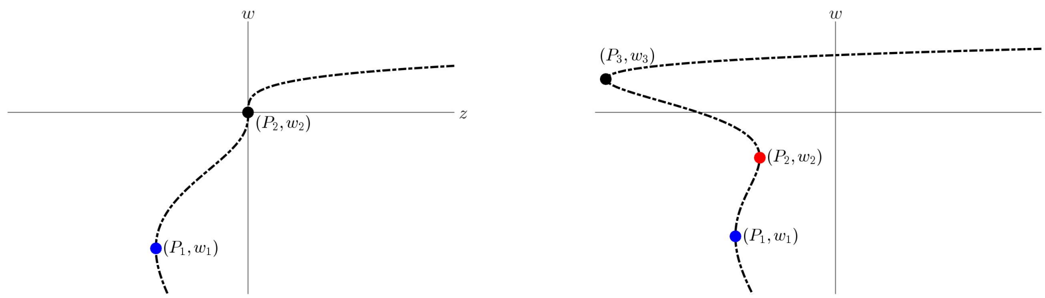

Figure 5.1: Real branches for $p = q = 0$ and $p = 0, q = -2$.

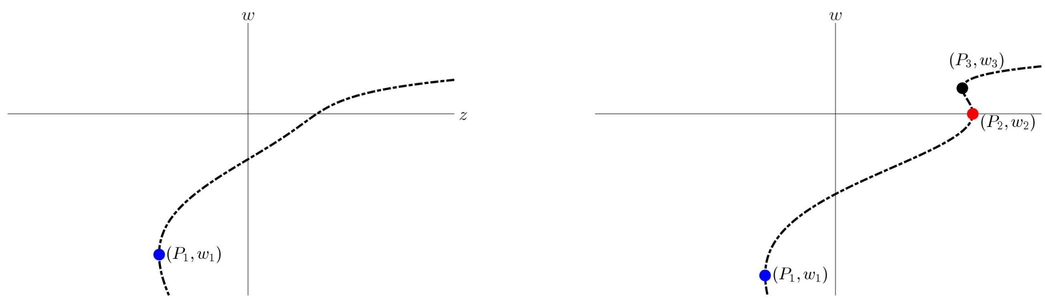

Figure 5.2: Real branches for $p = 0$, $q = 1$ and $p = -2$, $q = 2$.

The values P_n and w_n in figures above are branch points and can be computed by considering the first derivative of Eq. (5.1):

$$\frac{dw}{dx} = \frac{w^3 + pw + q}{x(w^3 + 3w^2 + pw + p + q)} \tag{5.8}$$

Equating the denominator to zero, we obtain $x = 0$ and $w = w_n$ such that $w_n^3 + 3w_n^2 + pw_n + p + q = 0$, for $n = 1, 2, 3$. Thus, branch points are $x = 0$ and $x = P_n$ where

$$P_n = (w_n^3 + pw_n + q)e^{w_n} = (-3w_n^2 - p)e^{w_n}.$$

Thus, Equation (5.1) has four real solutions when P_n are distinct real, at least two of the P_n are negative, and $P_1 < x < \min(0, P_2)$.

In the case $-4p^3 + 36p^2 - 108p - 27q^2 - 108q > 0$, we have three distinct real w_n. Suppose that $w_1 < w_2 < w_3$.

Theorem 5.1. *If $p \geq 0$ and*

$$-4p^3 + 36p^2 - 108p - 27q^2 - 108q > 0,$$

the equation $(w^3 + pw + q)e^w = x$ has four real solutions when $P_1 < x < P_2$.

Proof. The equation $f(w) = w^3 + 3w^2 + pw + p + q$ has three distinct real roots if and only if its discriminant is greater than zero. Simplifying its discriminant, we have

$$-4p^3 + 36p^2 - 108p - 27q^2 - 108q > 0.$$

Since w_n satisfy the equation $w_n^3 + 3w_n^2 + pw_n + p + q = 0$, we know that

$$P_n = (w_n^3 + pw_n + q)e^{w_n} = (-3w_n^2 - p)e^{w_n}.$$

The condition, $p > 0$ ensures that all P_n are negative. Hence, there will be four real solutions when $P_1 < x < P_2$. □

Since the $W^{(p,q)}$ functionhas one extra branch point as compared to the $W^{(r)}$ function, the branch structure is expected to be much more complicated. Thus, branch analysis of $W^{(p,q)}$ will not be conducted in this project.

Similar to the $W^{(r)}$ function, we define $\Omega_{p,q}$ to be the constant such that

$$(\Omega_{p,q}^3 + p\Omega_{p,q} + q)e^{\Omega_{p,q}} = 1.$$

Theorem 5.2. *The $\Omega_{p,q}$ constant is transcendental if p, q are algebraic.*

Proof. Suppose that $\Omega_{p,q}$ is algebraic, then $e^{\Omega_{p,q}}$ is transcendental. On the other hand, the polynomial $\Omega_{p,q}^3 + p\Omega_{p,q} + q$ is algebraic, leads to the contradiction that $(\Omega_{p,q}^3 + p\Omega_{p,q} + q)e^{\Omega_{p,q}}$ is algebraic. □

The following example demonstrates that the omega constant can be algebraic.

Example 5.4. *The constant* $\Omega_{-e,1} = -1$ *is algebraic.*

5.1.3 Series solutions

In this subsection, we derive the series expansion of the $W^{(p,q)}$ function by using the Taylor series.

By differentiating Eq. (5.1) with respect to p and q, and following induction, we obtained:

$$\frac{\partial^{m+n} w}{\partial p^m \partial q^n} = \frac{p_{m,n}(w,p,q)}{(w^3+3w^2+(w+1)p+q)^{2m+2n-1}} \tag{5.9}$$

where $m, n \geq 0$, $m+n \geq 1$. Polynomials $p_{m,n}(w,p,q)$ satisfy:

$$\begin{aligned}
p_{m+1,n}(w,p,q) &= (w^3+3w^2+(w+1)p+q)^2 \frac{\partial}{\partial p} p_{m,n}(w,p,q) \\
&\quad + (2m+2n-1)(w^4+w^3+(p-3)w^2 \\
&\quad + (p+q)(w+1))p_{m,n}(w,p,q), \\
p_{m,n+1}(w,p,q) &= (w^3+3w^2+(w+1)p+q)^2 \frac{\partial}{\partial q} p_{m,n}(w,p,q) \\
&\quad - (2m+2n-1)(w^3-6w+wp+q)p_{m,n}(w,p,q)
\end{aligned}$$

where $p_{1,0}(w,p,q) = -w$ and $p_{0,1}(w,p,q) = -1$.

The Taylor series of Eq. (5.1) can be obtained by using Eq. (5.9) and the fact that

$$W^{(0,0)}_{3k+j}(z) = 3W_k\left(\frac{\sqrt[3]{z}}{3} e^{\frac{2\pi i}{3} j}\right).$$

Theorem 5.3. *The Taylor series of $W_{3k+j}^{(p,q)}(z)$ at around $p = q = 0$ is given by*

$$W_{3k+j}^{(p,q)}(z) = 3W_k(z_j) + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{p_{m,n}(3W_k(z_j), 0, 0)}{\left\{27W_k(z_j)^3 + 27W_k(z_j)^2\right\}^{2m+2n-1}} \frac{p^m q^n}{m!\,n!}, \quad (5.10)$$

where $k \in \mathbb{Z}$ and $z_j = \frac{\sqrt[3]{z}}{3} e^{\frac{2\pi i}{3} j}$ for $j = 0, 1, 2$. The polynomial $p_{m,n}(w, p, q)$ is given by:

$$\begin{aligned}
p_{m+1,n}(w, p, q) &= (w^3 + 3w^2 + (w+1)p + q)^2 \frac{\partial}{\partial p} p_{m,n}(w, p, q) \\
&\quad + (2m + 2n - 1)(w^4 + w^3 + (p-3)w^2 \\
&\quad + (p+q)(w+1)) p_{m,n}(w, p, q), \\
p_{m,n+1}(w, p, q) &= (w^3 + 3w^2 + (w+1)p + q)^2 \frac{\partial}{\partial q} p_{m,n}(w, p, q) \\
&\quad - (2m + 2n - 1)(w^3 - 6w + wp + q) p_{m,n}(w, p, q)
\end{aligned}$$

for $n \geq 1$. The initial polynomials are $p_{0,0}(w, p, q) = 0, p_{1,0}(w, p, q) = -w$ and $p_{0,1}(w, p, q) = -1$.

The following two corollaries lead to series solutions of three upper parameters and three lower parameters.

Corollary 5.1. *The series solutions of $(w - a_1)(w - a_2)(w - a_3)e^w = z$ where $a_i \in \mathbb{R}$ are given by*

$$\begin{aligned}
W_{3k+j}\left(\begin{smallmatrix} a_1 & a_2 & a_3 \end{smallmatrix}; z\right) &= \frac{a_1 + a_2 + a_3}{3} + 3W_k(z'_j) \\
&\quad + \frac{p_{m,n}(3W_k(z'_j), 0, 0)}{\left\{27W_k(z'_j)^3 + 27W_k(z'_j)^2\right\}^{2m+2n-1}} \frac{p^m q^n}{m!\,n!}
\end{aligned}$$

where $p = (r_m - a_2)(r_m - a_3) - (r_m - a_1)^2, q = (r_m - a_1)(r_m - a_2)(r_m - a_3)$ and $z'_j = \frac{\sqrt[3]{ze^{-r_m}}}{3} e^{\frac{2j\pi i}{3}}$.

Proof. By letting $w = v + r_m$ and $r_m = \frac{a_1+a_2+a_3}{3}$, the equation

$$(w - a_1)(w - a_2)(w - a_3)e^w = z, \tag{5.11}$$

where $a_i \in \mathbb{R}$ can be transformed into

$$(v^3 + pv + q)e^v = ze^{-r_m},$$

where

$$p = (r_m - a_2)(r_m - a_3) - (r_m - a_1)^2$$
$$q = (r_m - a_1)(r_m - a_2)(r_m - a_3).$$

Thus,

$$W_{3k+j}\left({a_1\ a_2\ a_3 \atop } ; z\right) = \frac{a_1 + a_2 + a_3}{3} + W_{3k+j}^{(p,q)}(z'),$$

where $z' = ze^{-r_m}$ and we have the desired result. □

Similarly, we obtain the following corollary for three lower parameters.

Corollary 5.2. *The solutions of* $\frac{1}{(w-a_1)(w-a_2)(w-a_3)}e^w = z, z \neq 0$ *are given by*

$$w = -W_k\left({\atop -a_1\ -a_2\ -a_3} ; -\frac{1}{z}\right).$$

Proof. The result follows from rewriting the equation as

$$[-w - (-a_1)][-w - (-a_2)][-w - (-a_3)]e^{-w} = -\frac{1}{z}.$$

□

From Lemma (3.1), Eq. (5.11) has at most four real solutions. The following corollary provides an approach to compute all real solutions.

Corollary 5.3. *If equation* $(w-a_1)(w-a_2)(w-a_3)e^w = z$ *has four real solutions, then these solutions are* $W_k\left(\begin{smallmatrix} a_1 & a_2 & a_3 \end{smallmatrix}; z\right)$, *for* $k = 0, 1, 2$ *and* $W_{-3+j}\left(\begin{smallmatrix} a_1 & a_2 & a_3 \end{smallmatrix}; z\right)$, *where* j *satisfies* $-e^{-1} < z_j' < 0$.

Proof. Since Taylor series of $W_{3t+j}\left(\begin{smallmatrix} a_1 & a_2 & a_3 \end{smallmatrix}; z\right)$ can be expressed in terms of $W_k(z_j')$, and considering the properties of the Lambert W function, three real solutions can be obtained by taking $t = 0$, while the forth real solution can be obtained when $t = -1$ with either $j = 0, 1$ or $j = 2$ such that $-e^{-1} < z_j' < 0$. □

The following example explores a scenario involving four real solutions.

Example 5.5. *Consider the case where* $p = 0.5$ *and* $q = -1$. *Upon inspection, we can verify that*

$$-4p^3 + 36p^2 - 108p - 27q^2 - 108q = 35.5 > 0.$$

Additionally, the branch points are $z = -1.48, -0.83$, *and* $z = -1.11$. *Consequently, there exist four real solutions within the interval* $-1.11 < z < -0.83$.

For example, if we choose $z = -0.9$, *we obtain four real solutions:* $w = -4.90, -0.94, -0.21$, *and* $w = 0.59$.

5.2 Extension to higher degree

Using the transformation $r_m = \sum_{i=1}^{n} a_i/n$ allows one to derive the Taylor series of $W_{nk+j}\left(\begin{smallmatrix} a_1 & \cdots & a_n \end{smallmatrix}; z\right)$, and we obtain

$$W_{nk+j}\left(\begin{smallmatrix} a_1 & \cdots & a_n \end{smallmatrix}; z\right) \sim r_m + nW_k\left(z_0\omega^j\right), \quad \text{for } j = 0, 1, \ldots, n-1 \tag{5.12}$$

where

$$z_0 = \frac{\sqrt[n]{z}}{n} e^{-\frac{r_m}{n}}, \quad \omega = e^{\frac{2\pi i}{n}}.$$

The following example shows the approximated values obtained from Eq. (5.12).

Example 5.6. *Consider the case of five upper parameters, with* $a_1 = 0.2, a_2 = 0.4, a_3 = 0.65, a_4 = 0.7, a_5 = 0.9$ *and* $z = 10$*. Let* W^*_{5m+j} *to be the approximation given by Eq. (5.12), we tabulate the results for* $m = 0, \pm 1$ *in Table 5.1.*

Table 5.1: W^*_{5m+j} for $m = 0, 1, -1$

j	W^*_j	W^*_{5+j}	W^*_{-5+j}
0	1.6984	$-13.7997 + 20.5058i$	$-13.7997 - 20.5058i$
1	$1.1996 + 1.0761i$	$-14.9245 + 27.2609i$	$-12.3235 - 13.4591i$
2	$-0.5159 + 1.3815i$	$-15.8374 + 33.8729i$	$-10.1437 - 5.5206i$
3	$-0.5159 - 1.3815i$	$-10.1437 + 5.5206i$	$-15.8374 - 33.8729i$
4	$1.1996 - 1.0761i$	$-12.3235 + 13.4591i$	$-14.9245 - 27.2609i$

From the computations above, it seems suitable to use Eq. (5.12) as an initial point for numerical computation. For the $W_k\left({}^{a_1 a_2 \ldots a_n}; z\right)$ function, Halley's method is given in the form:

$$w_{n+1} = w_n - \frac{f(w)}{(1+s_1)\left[\prod_{t=1}^{n}(w-a_t)\right]e^w - \frac{f(w)}{2}\left(-\frac{s_2}{1+s_1} + s_1 + 1\right)} \tag{5.13}$$

where $f(w) = \prod_{t=1}^{n}(w-a_t)e^w - z$, $s_1 = \sum_{t=1}^{n}(w-a_t)^{-1}$ and $s_2 = \sum_{t=1}^{n}(w-a_t)^{-2}$.

Table below summarises the results from Halley's method:

Table 5.2: W_{5m+j} for $m = 0, 1, -1$

j	W_j	W_{5+j}	W_{-5+j}
0	1.7195	$-13.7998 + 20.5060i$	$-13.7998 - 20.5060i$
1	$1.2068 + 1.0569i$	$-14.9246 + 27.2610i$	$-12.3237 - 13.4595i$
2	$-0.5332 + 1.3701i$	$-15.8375 + 33.8730i$	$-10.1432 - 5.5221i$
3	$-0.5332 - 1.3701i$	$-10.1432 + 5.5221i$	$-15.8375 - 33.8730i$
4	$1.2068 - 1.0569i$	$-12.3237 + 13.4595i$	$-14.9246 - 27.2610i$

We propose the following conjectures as avenues for further analysis:

Conjecture 1. *The solutions of* $(w-a_1)(w-a_2)\ldots(w-a_n)e^w = z$ *is conjectured to satisfy the following relation:*

$$\lim_{|m|\to\infty} W_{nm+j}\left(\begin{smallmatrix} a_1 & a_2 & \dots & a_n \end{smallmatrix}; z\right) \to \frac{\sqrt[n]{z}}{n} e^{-\frac{a_1+a_2+\dots+a_n}{n^2}} + nW_m\left(z_0 e^{\frac{2\pi i}{n} j}\right),$$

for $j = 1, 2, \ldots, n-1$.

Conjecture 2. *The generalised Lambert W function is conjectured to exhibit conjugate symmetry as follows:*

$$W_{nm+j}\left(\begin{smallmatrix} a_1 & a_2 & \dots & a_n \end{smallmatrix}; z\right) = \overline{W_{-nm+(5-j)}\left(\begin{smallmatrix} a_1 & a_2 & \dots & a_n \end{smallmatrix}; z\right)}.$$

CHAPTER 6

CONCLUSION

In this thesis, we have studied the generalised Lambert W function and made noteworthy contributions to our understanding of these topics. Our journey began with the aim of addressing the central questions and objectives that guided our research.

Our first objective was to investigate the application of the Lambert W function in delay differential equations. We conducted a thorough analysis of scalar systems with single delays, deriving conditions for the existence of feedback controllers capable of assigning desired rightmost eigenvalues to the system.

The second objective involved exploring the series solutions and branch structures of $(w^2 - r)e^w = z$, denoted as the $W^{(r)}$ function. We initiated our analysis with equations solvable using the $W^{(r)}$ function, examining fundamental properties, derivatives, and integrals of the $W^{(r)}$ function. We investigated series expansions of the $W^{(r)}$ function using Taylor series expansion at $r = 0$, Lagrange inversion, and asymptotic expansions involving logarithms and the Lambert W function. Additionally, we studied the branch structures of the $W^{(r)}$ function.

The third objective encompassed an investigation into solutions of $(w^3 + pw + q)e^w = z$ and $P_N(w)e^w$ across distinct branches. Employing an approach similar to that used for $(w^2 - r)e^w = z$, we obtained series solutions for $(w^3 + pw + q)e^w = z$. We also acknowledged the infeasibility of extending this approach to $P_N(w)e^w = z$.

Our fourth and final objective aimed to numerically determine solutions for $P_N(w)e^w = z$. In this pursuit, we first identified appropriate initial points that could be utilized to compute solutions in different branches. We employed Halley's method for numerical computation of solutions until achieving a specified level of precision.

While we have successfully achieved our objectives, there remain open questions that warrant exploration and offer intriguing prospects for future research: Our first objective was to examine the application of the Lambert W function in delay differential equations. We performed detailed analysis on scalar systems with a single delay, to provide the conditions for the existence of a feedback controller related to assigning the rightmost eigenvalue of the system to a desired value.

1. Determining the convergence radius of the series expansions of the $W^{(r)}$ function.
2. Exploring series expansions of the r-Lambert function capable of computing solutions across diverse branches.
3. Investigating series solutions of the equation:

$$\frac{P_N(w)}{Q_M(w)}e^w = z.$$

In accomplishing our objectives, we have not only contributed to the advancement of pure mathematics but have also paved the way for future research directions. The insights gained from our investigations have illuminated new avenues for exploration and opened doors to further discoveries. Although our journey within this thesis has reached its conclusion, the mathematical exploration ignited by our objectives will undoubtedly continue to flourish, inspiring future mathematicians to delve deeper into the realms of abstract mathematical thought.

BIBLIOGRAPHY

Abramowitz, M. and Stegun, I. A., eds, 1992. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Dover Publications, Inc., New York. Reprint of the 1972 edition.

Asl, F. M. and Ulsoy, A. G., 2003. Analysis of a system of linear delay differential equations, *Journal of Dynamic Systems, Measurement, and Control*, 125(2), pp. 215–223.

Bronstein, M., Corless, R. M., Davenport, J. H. and Jeffrey, D. J., 2008. Algebraic properties of the Lambert W function from a result of Rosenlicht and of Liouville, *Integral Transforms Spec. Funct.*, 19(9-10), pp. 709–712.

Caillol, J.-M., 2003. Some applications of the Lambert W function to classical statistical mechanics, *J. Phys. A*, 36(42), pp. 10431–10442.

Cepeda-Gomez, R. and Michiels, W., 2015. Some special cases in the stability analysis of multi-dimensional time-delay systems using the matrix Lambert W function, *Automatica J. IFAC*, 53, pp. 339–345.

Comtet, L., 1970. Inversion de $y^\alpha e^\gamma$ et $y \log^\alpha y$ au moyen des nombres de Stirling, *C. R. Acad. Sci. Paris Sér. A-B*, 270, pp. A1085–A1088.

Cooke, K. L. and van den Driessche, P., 1986. On zeroes of some transcendental equations, *Funkcial. Ekvac.*, 29(1), pp. 77–90.

Corcino, C. B. and Corcino, R. B., 2013. An asymptotic formula for r-Bell numbers with real arguments, *ISRN Discrete Math.*, 2013, p. 7 pages.

Corcino, C. and Corcino, R., 2020. The cubic Lambert w function, *CNU-Journal of Higher Education*, 14, pp. 1–6.

Corless, R. M., Gonnet, G. H., Hare, D. E. G., Jeffrey, D. J. and Knuth, D. E., 1996. On the Lambert W function, *Adv. Comput. Math.*, 5(4), pp. 329–359.

Corless, R. M. and Jeffrey, D. J., 1996. The unwinding number, *ACM SIGSAM Bulletin*, 30(2), pp. 28–35.

da Silva, G. B. and Ramos, R. V., 2019. The Lambert-Tsallis W_q function, *Phys. A*, 525, pp. 164–170.

da Silva, J. L. M., Mendes, F. V. and Ramos, R. V., 2019. Radial basis function network using Lambert-Tsallis W_q function, *Phys. A*, 534, pp. 122168, 9.

da Silva, J. and Ramos, R., 2020. Transcendental numbers and the Lambert-tsallis function, *arXiv preprint arXiv:2004.07101*, .

Davis, H. T., 1962. *Introduction to nonlinear differential and integral equations*, Dover Publications, Inc., New York.

Euler, L., 1783. De serie Lambertina plurimisque eius insignibus proprietatibus, *Acta Acad. Scient. Petropol*, 2, pp. 29–51.

Filali, H., Koussour, M., Bennai, M. and Laamara, R. A., 2024. Holographic F(Q,T) Gravity with Lambert Solution, *Internat. J. Theoret. Phys.*, 63(4), p. Paper No. 104. **URL:** *https://doi.org/10.1007/s10773-024-05643-6*

Flajolet, P. and Sedgewick, R., 2009. *Analytic combinatorics*, Cambridge University Press, Cambridge.

Gessel, I. and Stanley, R. P., 1978. Stirling polynomials, *J. Combinatorial Theory Ser. A*, 24(1), pp. 24–33.

Graham, R. L., Knuth, D. E. and Patashnik, O., 1994. *Concrete mathematics*, second edn, Addison-Wesley Publishing Company, Reading, MA. A foundation for computer science.

Huang, H.-N., 2017. Characteristic roots of a linear scalar system with multiple delays, *in* ‘2017 13th IEEE International Conference on Control & Automation (ICCA)’, IEEE, pp. 341–346.

Jarlebring, E. and Damm, T., 2007. The Lambert W function and the spectrum of some multidimensional time-delay systems, *Automatica*, 43(12), pp. 2124 –2128.

Jeffrey, D. J., Corless, R. M., Hare, D. E. G. and Knuth, D. E., 1995. Sur l’inversion de $y^\alpha e^y$ au moyen des nombres de Stirling associés, *C. R. Acad. Sci. Paris Sér. I Math.*, 320(12), pp. 1449–1452.

Jeffrey, D. J., Hare, D. E. G. and Corless, R. M., 1996. Unwinding the branches of the Lambert W function, *Math. Sci.*, 21(1), pp. 1–7.

Jiménez, F. and Jodrá, P., 2009. On the computer generation of the Erlang and negative binomial distributions with shape parameter equal to two, *Math. Comput. Simulation*, 79(5), pp. 1636–1640.

Jodrá, P., 2010. Computer generation of random variables with Lindley or Poisson-Lindley distribution via the Lambert W function, *Math. Comput. Simulation*, 81(4), pp. 851–859.

Kalugin, G. A. and Jeffrey, D. J., 2012. Convergence in $$ of series for the Lambert w function, *arXiv: Classical Analysis and ODEs*, .

Kazakova, S. and Pisanova, E., 2010. Some applications of the Lambert W-function to theoretical physics education, *in* ‘AIP Conference Proceedings’, Vol. 1203, American Institute of Physics, pp. 1354–1359.

Kesisoglou, I., Singh, G. and Nikolaou, M., 2021. The Lambert function should be in the engineering mathematical toolbox, *Computers & chemical engineering*, 148, p. 107259.

Komatsu, T., Liptai, K. and Mező, I., 2016. Incomplete poly-Bernoulli numbers associated with incomplete Stirling numbers, *Publ. Math. Debrecen*, 88(3-4), pp. 357–368.

Lambert, J. H., 1758. Observationes variae in mathesin puram, *Acta Helvetica*, 3(1), pp. 128–168.

Lovász, L., 2007. *Combinatorial problems and exercises*, second edn, AMS Chelsea Publishing, Providence, RI.

Mendes, F. V., Lima, C. and Ramos, R. V., 2022. Applications of the Lambert–Tsallis Wq function in quantum photonic Gaussian boson sampling, *Quantum Inf. Process.*, 21(6), p. Paper No. 215.

Mező, I., 2017. On the structure of the solution set of a generalized Euler-Lambert equation, *J. Math. Anal. Appl.*, 455(1), pp. 538–553.

Mező, I., 2022. *The Lambert W Function: Its Generalizations and Applications*, CRC Press.

Mező, I. and Baricz, A., 2017. On the generalization of the Lambert W function, *Trans. Amer. Math. Soc.*, 369(11), pp. 7917–7934.

Mező, I. and Keady, G., 2016. Some physical applications of generalized Lambert functions, *European Journal of Physics*, 37(6), p. 065802.

Mugnaini, G., 2014. Generalization of Lambert W function, bessel polynomials and transcendental equations, *arXiv preprint arXiv:1501.00138*, .

Noonburg, V. W., 1969. Roots of a transcendental equation associated with a system of differential-difference equations, *SIAM Journal on Applied Mathematics*, 17(1), pp. 198–205.

Ohira, K. and Ohira, T., 2023. Delay, resonance and the lambert w function, *arXiv preprint arXiv:2301.13448*, .

Olbrot, A., 1978. Stabilizability, detectability, and spectrum assignment for linear autonomous systems with general time delays, *IEEE Transactions on Automatic Control*, 23(5), pp. 887–890.

Parker, F. D., 1955. Classroom Notes: Integrals of Inverse Functions, *Amer. Math. Monthly*, 62(6), pp. 439–440.

Saha, S. and Bamba, K., 2019. The lambert w function: A newcomer in the cosmology class?, *Zeitschrift für Naturforschung A*, 75(1), pp. 23–27.

Scott, T., Babb, J., Dalgarno, A. and Morgan III, J. D., 1993. Resolution of a paradox in the calculation of exchange forces for h+ 2, *Chemical physics letters*, 203(2-3), pp. 175–183.

Scott, T. C., Fee, G. and Grotendorst, J., 2014. Asymptotic series of generalized Lambert W function, *ACM Communications in Computer Algebra*, 47(3/4), pp. 75–83.

Scott, T. C., Fee, G., Grotendorst, J. and Zhang, W., 2014. Numerics of the generalized Lambert w function, *ACM Communications in computer algebra*, 48(1/2), pp. 42–56.

Scott, T. C., Mann, R. and Martinez, II, R. E., 2006. General relativity and quantum mechanics: towards a generalization of the Lambert W function: a generalization of the Lambert W function, *Appl. Algebra Engrg. Comm. Comput.*, 17(1), pp. 41–47.

Shinozaki, H., 2008. Lambert W function approach to stability and stabilization problems for linear time-delay systems, , .

Shinozaki, H. and Mori, T., 2006. Robust stability analysis of linear time-delay systems by Lambert W function: Some extreme point results, *Automatica*, 42(10), pp. 1791–1799.

Siewert, C. and Burniston, E., 1973. Exact analytical solutions of $ze^z = a$, *Journal of Mathematical Analysis and Applications*, 43(3), pp. 626–632.

Siewert, C. and Burniston, E., 1974. Solutions of the equation $ze^z = a(z + b)$, *Journal of Mathematical Analysis and Applications*, 46(2), pp. 329–337.

Stewart, S., 2005. A new elementary function for our curricula?, *Australian Senior Mathematics Journal*, 19(2), pp. 8–26.

Valluri, S. R., Jeffrey, D. J. and Corless, R. M., 2000. Some applications of the Lambert W function to physics, *Canadian Journal Of Physics*, 78(9), pp. 823–831.

Visser, M., 2018. Primes and the Lambert W function, *Mathematics*, 6(4), p. 56.

Wright, E. M., 1959*a*. Solution of the equation $ze^z = a$, *Bull. Amer. Math. Soc.*, 65, pp. 89–93.

Wright, E. M., 1959*b*. Solution of the equation $ze^z = a$, *Proc. Roy. Soc. Edinburgh Sect. A*, 65, pp. 193–203.

Wright, E. M., 1960. Solution of the equation $(pz+q)e^z = rz+s$, *Bull. Amer. Math. Soc.*, 66, pp. 277–281.

Yi, S., Nelson, P. and Ulsoy, A., 2007. Delay differential equations via the matrix Lambert *W* function and bifurcation analysis: application to machine tool chatter, *Mathematical Biosciences and Engineering*, 4(2), p. 355.

Yi, S., Nelson, P. W. and Ulsoy, A. G., 2010*a*. Eigenvalue assignment via the Lambert W function for control of time-delay systems, *J. Vib. Control*, 16(7-8), pp. 961–982.

Yi, S., Nelson, P. W. and Ulsoy, A. G., 2010*b*. *Time-delay systems: analysis and control using the Lambert W function*, World Scientific.

Yi, S. and Ulsoy, A. G., 2006. Solution of a system of linear delay differential equations using the matrix Lambert function, *in* '2006 American Control Conference', IEEE, pp. 6–pp.

Yi, S., Ulsoy, A. G. and Nelson, P. W., 2006. Solution of systems of linear delay differential equations via Laplace transformation, *in* 'Decision and Control, 2006 45th IEEE Conference on', IEEE, pp. 2535–2540.

APPENDIX A

MATLAB CODE

A.1 MATLAB function for the $W^{(r)}$ function

We present the MATLAB code to compute the approximated $W^{(r)}$ function using Equation (4.22).

```
function y=wrlambert(k, r, z)
    % Compute j and 'k'
    j = ceil(k/2 - floor(k/2));
    k = floor(k/2);

    % Compute z_j
    z_j = (-1)^j*z^(1/2)/2;
    Wk = lambertw(k,z_j);

    % First 10 terms
    term(1) =  r/(4*Wk*(Wk + 1));
    term(2) = (r^2*(4*Wk^2 - 2))/(128*Wk^3*(Wk + 1)^3);
    term(3) = -(r^3*(- 32*Wk^4 + 16*Wk^3 + 48*Wk^2 - 12)) ...
        /(6144*Wk^5*(Wk + 1)^5);
    term(4) = (r^4*(384*Wk^6 - 512*Wk^5 - 1056*Wk^4 ...
        + 320*Wk^3 + 720*Wk^2 - 120))/(393216*Wk^7*(Wk + 1)^7);
    term(5) = -(r^5*(- 6144*Wk^8 + 14848*Wk^7 + 23168*Wk^6 ...
        - 22784*Wk^5 - 31680*Wk^4 + 6720*Wk^3 + 13440*Wk^2 ...
        - 1680))/(31457280*Wk^9*(Wk + 1)^9);
    term(6) = (r^6*(122880*Wk^10 - 454656*Wk^9 ...
        - 495616*Wk^8 + 1197568*Wk^7 + 1180160*Wk^6 ...
        - 849408*Wk^5 - 994560*Wk^4 + 161280*Wk^3 ...
        + 302400*Wk^2 - 30240))/(3019898880*Wk^11*(Wk + 1)^11);
    term(7) = -(r^7*(- 2949120*Wk^12 + 15187968*Wk^11 ...
        + 8871936*Wk^10 - 57819136*Wk^9 - 36314112*Wk^8 ...
        + 71454720*Wk^7 + 57666560*Wk^6 - 31868928*Wk^5 ...
        - 33868800*Wk^4 + 4435200*Wk^3 + 7983360*Wk^2 ...
        - 665280))/(338228674560*Wk^13*(Wk + 1)^13);
    term(8) = (r^8*(82575360*Wk^14 - 556793856*Wk^13 ...
        - 15106048*Wk^12 + 2743828480*Wk^11 ...
```

```
        + 624291840*Wk^10 - 5092556800*Wk^9 ...
        - 2574473216*Wk^8 + 3941978112*Wk^7 ...
        + 2848204800*Wk^6 - 1263144960*Wk^5 ...
        - 1261370880*Wk^4 + 138378240*Wk^3 + 242161920*Wk^2 ...
        - 17297280))/(43293270343680*Wk^15*(Wk + 1)^15);
    term(9) = -(r^9*(- 2642411520*Wk^16 + 22356688896*Wk^15 ...
        - 13038649344*Wk^14 - 131221422080*Wk^13 ...
        + 34736865280*Wk^12 + 333501480960*Wk^11 ...
        + 56692719616*Wk^10 - 392708194304*Wk^9 ...
        - 179071119360*Wk^8 + 216340439040*Wk^7 ...
        + 145779701760*Wk^6 - 53690757120*Wk^5 ...
        - 51338327040*Wk^4 + 4843238400*Wk^3 ...
        + 8302694400*Wk^2 - 518918400)) ...
        /(6234230929489920*Wk^17*(Wk + 1)^17);
    term(10) = (r^10*(95126814720*Wk^18 ...
        - 978824724480*Wk^17 + 1218840625152*Wk^16 ...
        + 6366973919232*Wk^15 - 5749812101120*Wk^14 ...
        - 20707596042240*Wk^13 + 5430117171200*Wk^12 ...
        + 34399065014272*Wk^11 + 5758244831232*Wk^10 ...
        - 28845622558720*Wk^9 - 12438287769600*Wk^8 ...
        + 12179973611520*Wk^7 + 7824090193920*Wk^6 ...
        - 2456490516480*Wk^5 - 2274938265600*Wk^4 ...
        + 188194406400*Wk^3 + 317578060800*Wk^2 ...
        - 17643225600))/(997476948718387200*Wk^19*(Wk + 1)^19);

    S = 0;
    for m=1:10
        S = S + term(m);
    end
    y = 2*Wk + S;
end
```

A.2 MATLAB function for $W^{(p,q)}$ function

We present the MATLAB code to compute the approximated $W^{(r)}$ function using Equation (5.10).

```
function y=wpqlambert(k, p, q, z)
t = 5;

% Compute j and 'k'
j = k - 3*floor(k/3);
k = floor(k/3);

% Compute z_j
z_j = z^(1/3)/3*exp(2*pi*1i/3*j);
Wk = lambertw(k,z_j);

% Terms
term(1, 1) = 0;
term(1, 2) = -q/(27*Wk^2*(Wk + 1));
term(1, 3) = (q^2*(9*Wk^2 - 6))/(13122*Wk^5*(Wk + 1)^3);
term(1, 4) = -(q^3*(18*Wk - 486*Wk^2 - 243*Wk^3 ...
    + 162*Wk^4 + 126))/(9565938*Wk^8*(Wk + 1)^5);
term(1, 5) = -(q^4*(540*Wk - 9504*Wk^2 - 8478*Wk^3 ...
    + 7209*Wk^4 + 5832*Wk^5 - 1458*Wk^6 + 1512))...
    /(3099363912*Wk^11*(Wk + 1)^7);
term(2, 1) = -p/(9*Wk*(Wk + 1));
term(2, 2) = (p*q*(9*Wk^2 - 6))/(2187*Wk^4*(Wk + 1)^3);
term(2, 3) = (p*q^2*(15*Wk^2 + 6*Wk^3 - 6*Wk^4 - 4))...
    /(39366*Wk^7*(Wk + 1)^5);
term(2, 4) = (p*q^3*(19926*Wk^2 + 12150*Wk^3 ...
    - 20169*Wk^4 - 11664*Wk^5 + 4374*Wk^6 - 3240))...
    /(774840978*Wk^10*(Wk + 1)^7);
term(2, 5) = (p*q^4*(1172232*Wk^2 + 868968*Wk^3 ...
    - 2108997*Wk^4 - 2029536*Wk^5 + 822312*Wk^6 ...
    + 761076*Wk^7 - 157464*Wk^8 - 136080))...
    /(753145430616*Wk^13*(Wk + 1)^9);
term(3, 1) = -(p^2*(3*Wk + 2))/(486*Wk^2*(Wk + 1)^3);
term(3, 2) = -(p^2*q*(13*Wk + 2*Wk^2 - 6*Wk^3 + 6))...
    /(13122*Wk^5*(Wk + 1)^5);
term(3, 3) = -(p^2*q^2*(4968*Wk - 108*Wk^2 - 7371*Wk^3 ...
    - 2916*Wk^4 + 1458*Wk^5 + 1944))...
    /(57395628*Wk^8*(Wk + 1)^7);
term(3, 4) = -(p^2*q^3*(258228*Wk - 83754*Wk^2 ...
    - 767637*Wk^3 - 429624*Wk^4 + 355752*Wk^5 ...
    + 218700*Wk^6 - 52488*Wk^7 + 91368)) ...
```

```
    /(41841412812*Wk^11*(Wk + 1)^9);
term(3, 5) = -(p^2*q^4*(16201296*Wk - 11139120*Wk^2 ...
    - 77472288*Wk^3 - 47290230*Wk^4 + 69172623*Wk^5 ...
    + 66738492*Wk^6 - 12439656*Wk^7 - 15903864*Wk^8 ...
    + 2361960*Wk^9 + 5342112)) ...
    /(40669853253264*Wk^14*(Wk + 1)^11);
term(4, 1) = -(p^3*(108*Wk + 387*Wk^2 + 594*Wk^3 ...
    + 324*Wk^4 + 18))/(354294*Wk^5*(Wk + 1)^5);
term(4, 2) = -(p^3*q*(120*Wk + 505*Wk^2 + 1005*Wk^3 ...
    + 666*Wk^4 - 243*Wk^5 - 324*Wk^6 + 20)) ...
    /(3188646*Wk^8*(Wk + 1)^7);
term(4, 3) = -(p^3*q^2*(136080*Wk + 630990*Wk^2 ...
    + 1468692*Wk^3 + 961065*Wk^4 - 1288872*Wk^5 ...
    - 1867698*Wk^6 - 262440*Wk^7 + 314928*Wk^8 ...
    + 22680))/(41841412812*Wk^11*(Wk + 1)^9);
term(4, 4) = -(p^3*q^3*(7348320*Wk + 36245880*Wk^2 ...
    + 93871872*Wk^3 + 52440615*Wk^4 - 179218089*Wk^5 ...
    - 279721674*Wk^6 - 41117787*Wk^7 + 124554024*Wk^8 ...
    + 39917124*Wk^9 - 14171760*Wk^10 + 1224720))...
    /(30502389939948*Wk^14*(Wk + 1)^11);
term(4, 5) = -(p^3*q^4*(221760*Wk + 1138240*Wk^2 ...
    + 3188144*Wk^3 + 1271398*Wk^4 - 10555296*Wk^5 ...
    - 17324799*Wk^6 - 1027098*Wk^7 + 15388812*Wk^8 ...
    + 7702776*Wk^9 - 3123036*Wk^10 - 1796256*Wk^11 ...
    + 349920*Wk^12 + 36960)) ...
    /(13556617751088*Wk^17*(Wk + 1)^13);
term(5, 1) = -(p^4*(4752*Wk + 14976*Wk^2 + 24651*Wk^3 ...
    + 20412*Wk^4 + 6804*Wk^5 + 684)) ...
    /(38263752*Wk^6*(Wk + 1)^7);
term(5, 2) = -(p^4*q*(24*Wk + 80*Wk^2 + 1176*Wk^3 ...
    + 11985*Wk^4 + 74268*Wk^5 + 253827*Wk^6 ...
    + 480330*Wk^7 + 504225*Wk^8 + 275562*Wk^9 ...
    + 61236*Wk^10 + 4))/(3099363912*Wk^12*(Wk + 1)^9);
term(5, 3) = -(p^4*q^2*(97200*Wk + 431406*Wk^2 ...
    + 888408*Wk^3 + 1735263*Wk^4 + 39538044*Wk^5 ...
    + 325482462*Wk^6 + 1240698222*Wk^7 ...
    + 2639057274*Wk^8 + 3336858990*Wk^9 ...
    + 2499544170*Wk^10 + 1026744012*Wk^11 ...
    + 178564176*Wk^12 + 16200)) ...
    /(122009559759792*Wk^15*(Wk + 1)^11);
term(5, 4) = -(p^4*q^3*(5715360*Wk + 27886680*Wk^2 ...
    + 70269768*Wk^3 + 29681235*Wk^4 - 39162609*Wk^5 ...
    + 2572939890*Wk^6 + 21803534883*Wk^7 ...
    + 86693222376*Wk^8 + 203237564454*Wk^9 ...
    + 300196849140*Wk^10 + 283351586616*Wk^11 ...
```

```
        + 166316586714*Wk^12 + 55444176648*Wk^13 ...
        + 8035387920*Wk^14 + 952560)) ...
        /(8894496906488368*Wk^18*(Wk + 1)^13);
    term(5, 5) = -(p^4*q^4*(181440*Wk + 926720*Wk^2 ...
        + 2572048*Wk^3 + 928130*Wk^4 - 8839392*Wk^5 ...
        - 9694197*Wk^6 + 95024862*Wk^7 + 756172332*Wk^8 ...
        + 3112999884*Wk^9 + 8002240452*Wk^10 ...
        + 13520247840*Wk^11 + 15325679520*Wk^12 ...
        + 11589087960*Wk^13 + 5623826760*Wk^14 ...
        + 1587237120*Wk^15 + 198404640*Wk^16 + 30240)) ...
        /(395310973621726O8*Wk^21*(Wk + 1)^15);

    S = 0;
    for m=1:t
        for n=1:t
            S = S + term(m,n);
        end
    end
    y = 3*Wk + S;
end
```

A.3 MATLAB function for solving $P_N(w)e^w = z$

We present the MATLAB code to compute the approximated solutions for $P_N(w)e^w = z$ using Equation (5.13) with Equation (5.12) as initial point.

```
function w = initVal(inputArr, z, k)
    % Handle optional input arguments
    if nargin == 3
        %k is given
    else
        k = 0;
    end

    % Compute parameters
    n = numel(inputArr);
    rm = sum(inputArr) / numel(inputArr);
    z0 = z^(1/n)/n*exp(-rm/n);
    omg = exp(2*pi*1i/n);
    j = k - floor(k/n)*n;
    k = floor(k/n);

    % Compute initial value
    w = rm + n*lambertw(k, z0*omg^j);
end

function y = genLambert(inputArr, z, k)
    if nargin == 3
        % k is given
    else
        k = 0;
    end

    % Parameters
    target = 1e-7;
    loopCnt = 0;

    % Initial value
    w = initVal(inputArr, z, k);
    chk = 1;

    while (chk > target) && (loopCnt <= 100)
        e = w - inputArr;
        w1 = prod(e)*exp(w);
        s1 = sum(1./e);
        s2 = sum(1./(e.^2));
```

```
        f = w1 - z;
        f1 = (1 + s1)*w1;
        f2_f1 = -s2/(1+s1) + s1 + 1;

        w = w - f/(f1 - 0.5*f*f2_f1);
        chk = abs(prod(w - inputArr)*exp(w) - z);
        loopCnt = loopCnt + 1;
    end
    y = w;
end
```

APPENDIX B

LIST OF PUBLICATIONS

1. Chew, C. Y. and Goh, Y. K., 2019. Electromagnetic Casimir effect of concentric cylinders in $(D + 1)$-dimensional space–time. *International Journal of Modern Physics A, 34(08)*, p.1950035.

2. Chew, C. Y. and Huang, H. N., 2018. Exact Eigen Value Assignment of Linear Scalar Systems with Single Delay Using Lambert *W* Function. *Advances in Natural and Applied Sciences, 12(8)*, pp.26-33.